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Simplicial complexes and Lefschetz fixed-point theorem

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Tiivistelmä — Referat — Abstract

We study a subcategory of topological spaces called polyhedrons. In particular, the work focuses on simplicial complexes out of which polyhedrons are constructed. With simplicial complexes we can calculate the homology groups of polyhedrons. These are computationally easier to handle compared to singular homology groups.

We start by introducing simplicial complexes and simplicial maps. We show how polyhedrons and simplicial complexes are related. Simplicial maps are certain maps between simplicial complexes. These can be transformed to piecewise linear maps between polyhedrons. We prove the simplicial approximation theorem which states that for any continuous function between polyhedrons we can find a piecewise linear map which is homotopic to the continuous function.

In section 4 we study simplicial homology groups. We prove that on polyhedrons the simplicial homology groups coincide with singular homology groups. Next we give an algorithm for calculating the homology groups from matrix presentations of boundary homomorphisms. Also examples of these calculations are given for some polyhedrons.

In the last section, we assign an integer called the Lefschetz number for continuous maps from a polyhedron to itself. It is calculated using the induced map between homology groups of the polyhedron. With the help of Hopf trace theorem we can also calculate the Lefschetz number using the induced map between chain complexes of the polyhedron. We prove the Lefschetz fixed-point theorem which states that if the Lefschetz number is not zero, then the continuous function has a fixed-point.

Avainsanat — Nyckelord — Keywords

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1 Introduction

A classical problem in topology is to study whether two spaces are homeomorphic or not. For two spaces to be homeomorphic, there needs to exist a continuous bijection with a continuous inverse between those spaces. We call such a function a homeomorphism. To prove that two spaces are not homeomorphic one has to prove that there does not exist a homeomorphism between them. In a sense we would need to go through all possible maps between the two spaces and show that none of them is a homeomorphism. Approaching the problem that way makes it almost impossible to solve. We can tackle this problem by introducing topological invariants for spaces, like compactness and connectedness. These are properties that are preserved under homeomorphisms. Hence showing that two spaces have different topological invariants can be used to prove that they are not homeomorphic.

For each space we can assign a sequence of abelian groups $H_0, H_1, H_2, \dots$ called the singular homology groups. Each of these groups is a topological invariant in a sense that if two spaces are homeomorphic, then they have isomorphic singular homology groups (corollary 4.11 in [6]). Directly calculating the singular homology groups can be troublesome even for simple spaces like the torus. The difficulty originates from dealing with possibly infinite function spaces. In this thesis we discuss simplicial homology groups which are easier to handle computationally. The downside of this method is that we can only assign these groups to a subcategory of spaces called polyhedrons. Polyhedrons can be considered to be spaces that can be patched up with generalizations of triangles called simplexes.

In topology we also study continuous functions. A function $f\colon X \to X$ is said to have a fixed point if $f(x) = x$ for some $x \in X$. For continuous functions from a polyhedron to itself, we can assign an integer called Lefschetz number. This number depends on the simplicial homology groups of the polyhedron and the induced homomorphism between them. The Lefschetz fixed-point theorem states that if the Lefschetz number is not zero, then the continuous function in question has a fixed point.

In this thesis our main goal is to study simplicial complexes, simplicial homology and the Lefschetz fixed-point theorem. The thesis is constructed as follows: We start with the preliminaries section containing definitions and results used in later sections. In section three we introduce simplicial complexes and construct polyhedrons out of them. We also study simplicial maps. In the fourth section we study simplicial homology groups and show that these coincide with the singular homology groups. In the last section we introduce the Lefschetz number and prove the Lefschetz fixed-point theorem.

The parts about simplicial complexes and simplicial homology are aimed for readers who only have basic knowledge of topology, algebra and simplexes. It is preferred that the reader knows singular homology.

2 Preliminaries

In this section we introduce definitions and results that will be used through out this thesis. The reader should at least skim through this section.

2.1 Topology

Theorem 2.1 (Heine-Borel, Theorem 11.3 in [10])**.** A subset A of some Euclidean space is compact if and only if A is closed and bounded.

Lemma 2.2 (Gluing lemma, Lemma 1.1 in [6])**.** Let X and Y be topological spaces. Assume that X is a finite union of closed sets X_i, $1 \leq i \leq n$. Let $f_i \colon X_i \to Y$ be continuous functions such that $f_i|_{X_i \cap X_j} = f_j|_{X_i \cap X_j}$, for all $i, j \in \{1, \dots, n\}$. Then there is a unique continuous function $f \colon X \to Y$ satisfying $f|_{X_i} = f_i$, for every $i \in \{1, \dots, n\}$.

Definition 2.3. If (X, d) is a metric space and $A \subset X$, then we define the diameter of A to be

$$\operatorname{diam}(A) = \sup\{d(x, y) : x, y \in A\}.$$

Lemma 2.4 (Lebesgue's number lemma, Lemma 27.5 in [4])**.** Let (X, d) be a compact metric space. If $\mathcal{U}$ is an open cover of X, then there exists $\delta > 0$ such that for each subset $A \subset X$ with $\operatorname{diam}(A) < \delta$, there exists an element of $\mathcal{U}$ which covers A.

Lemma 2.5. A finite union of compact sets is compact.

Proof. Let X be a topological space and let $\{C_1, \dots, C_n\}$ be a finite collection of compact subsets of X. Denote $C = \cup_{i=1}^n C_i$ and let $\mathcal{U}$ be an open cover of C. Now $\mathcal{U}$ is also an open cover for each C_i. Because each C_i is compact there exists a finite subcollection $\mathcal{U}_i \subset \mathcal{U}$ such that $\mathcal{U}_i$ is an open cover of C_i for all $i \in \{1, \dots, n\}$. Now the union

$$\bigcup_{i=1}^{n} \bigcup_{U \in \mathcal{U}_i} U$$

is finite and it contains C. Thus C is compact. □

Definition 2.6. A space X is *locally path connected* if for all $x \in X$ and all open neighborhoods U of x there exists an open V with $x \in V \subset U$ such that any two points of V can be joined by a path in U.

Lemma 2.7. If X is locally path connected and $f \colon X \to Y$ is a homeomorphism, then Y is locally path connected.

Proof. Let $y \in Y$ and let $x = f^{-1}(y)$. Since X is locally path connected, there exists an open neighborhood U of x such that for every pair of points in U are connected by a continuous path in U. Because f is an homeomorphism, we have that $f(U)$ is an open neighborhood of y. If $y_1, y_2 \in f(U)$, then there exists a continuous path $\gamma \colon [0, 1] \to U$ between $f^{-1}(y_1)$ and $f^{-1}(y_2)$. Thus $f \circ \gamma \circ f^{-1}$ is a continuous path between y_1 and y_2 in $f(U)$. □

2.2 Algebra

Theorem 2.8 (First isomorphism theorem, Corollary 5.7. in [11])**.** If G and H are groups and $f\colon G \to H$ is a homomorphism, then there exists an isomorphism $G/\ker f \cong \operatorname{im} f$.

Theorem 2.9 (Second isomorphism theorem, Corollary 5.9. in [11])**.** If K and N are subgroups of a group G, with N being normal in G, then

$$K/(N \cap K) \cong (N + K)/N.$$

Lemma 2.10. Let G be an abelian group. If A, B, C, D are subgroups of G such that $A \cap B = 0$, $C \cap D = 0$, $C \leq A$ and $D \leq B$, then

$$\frac{A \oplus B}{C \oplus D} \cong \frac{A}{C} \oplus \frac{B}{D}.$$

Proof. Since G is an abelian group, every subgroup of it is normal. Hence both sides of the isomorphisms are well defined. Define $f\colon (A \oplus B)/(C \oplus D) \to A/C \oplus B/D$ by $a + b + C + D \mapsto a + C + b + D$, where $a \in A$ and $b \in B$. If $c \in C$ and $d \in D$, then $a + c \in A$, $b + d \in B$ and

$$\begin{aligned} f(a + c + b + d + C + D) &= a + c + C + b + d + D = a + C + b + D \\ &= f(a + b + C + D). \end{aligned}$$

Thus f is well defined. Let's show that f is a homomorphism. Let $a, a' \in A$ and $b, b' \in B$. Directly calculating, we get

$$\begin{aligned} &f((a + b + C + D) + (a' + b' + C + D)) \\ &= f(a + a' + b + b' + C + D) \\ &= a + a' + C + b + b' + D \\ &= a + C + b + D + a' + C + b' + D \\ &= f(a + b + C + D) + f(a' + b' + C + D). \end{aligned}$$

Hence f is a homomorphism. If $a+C+b+D \in A/C \oplus B/D$, then $f(a+b+C+D) = a+C+b+D$. Therefore f is a surjection. Suppose $a + b + C + D \in \ker f$ i.e. $f(a + b + C + D) = C + D$. Thus $a + C + b + D = C + D \in A/C \oplus B/D$, which implies that $a \in C$ and $b \in D$. This is the same as $a + b + C + D = C + D \in (A \oplus B)/(C \oplus D)$. Thus, the kernel of f consists of only the identity element. Combining the results, we conclude that f is an isomorphism. □

Theorem 2.11 (Fundamental theorem of finitely generated abelian groups, Theorem 2.2 in [11])**.** If G is a finitely generated abelian group, then we have an isomorphism

$$G \cong \mathbb{Z} \oplus \dots \mathbb{Z} \oplus \mathbb{Z}_{p_1^{k_1}} \oplus \dots \oplus \mathbb{Z}_{p_n^{k_n}},$$

where each p_i is a prime and each k_i is a positive whole number.

2.3 Simplexes

Definition 2.12. A finite set of points $\{p_0, \ldots, p_m\}$ in some Euclidean space $\mathbb{R}^n$ is affine independent if the set $\{p_1 - p_0, \ldots, p_m - p_0\}$ is linearly independent.

In lower dimensions we classify affine independent sets as follow. The pair $\{p_0, p_1\}$ is affine independent if $p_0 \neq p_1$. The set $\{p_0, p_1, p_2\}$ is affine independent if the points are not collinear or in other words we can not draw a straight line through all of the points. Similarly $\{p_0, p_1, p_2, p_3\}$ is affine independent if the points are not coplanar. We need affine independent sets to construct triangles and their different dimensional analogs called simplexes.

Definition 2.13. An m-simplex σ is the convex hull of an affine independent set $\{p_0, \ldots, p_m\} \subset \mathbb{R}^n$. We denote $\sigma = [p_0, \ldots, p_m]$ or by $[p_i]_{i \in I}$, where I is the index set $\{0, \ldots, m\}$.

Sometimes we talk about spanning a simplex by a set of points $\{p_0, \ldots, p_m\}$. In this case the points need not to be affine independent, but the convex hull of the points forms a simplex.

(a) (b)

Figure 1: In (a) the points span a simplex, but in (b) the points do not span a simplex.

In lower dimensions the simplexes can be identified as follows: a 0-simplex is a point, a 1-simplex is a line segment, a 2-simplex is a triangle with the interior included and a 3-simplex is a tetrahedron with the interior included.

Let's go through some definitions and results regarding simplexes.

Definition 2.14. The vertices of $\sigma = [p_0, \ldots, p_m]$ are the points $p_0, \ldots, p_m$ and we define $\mathrm{Vert}(\sigma) = \{p_0, \ldots, p_m\}$.

Lemma 2.15. Let $\sigma = [p_0, \ldots, p_m]$ be a simplex. For every point $x \in \sigma$ there is a unique expression

$$x = \sum_{i=0}^{m} t_i p_i,$$

where $t_i \geq 0$ for all $i \in \{0, \ldots, m\}$ and $\sum_{i=0}^{m} t_i = 1$. We call the points $(t_i)_{i=0}^{m}$ barycentric coordinates.

Proof. See Theorem 2.4 in [6]. □

Lemma 2.16. For the diameter of a simplex, we have the equality

$$\operatorname{diam}([p_0,\dots,p_m]) = \max_{i,j} |p_i - p_j|.$$

Proof. The inequality $\geq$ is obvious because the diameter of $[p_0,\dots,p_m]$ is the supremum of the distance of any two points in $[p_0,\dots,p_m]$. So let's focus on the converse inequality. Let $x, y \in [p_0,\dots,p_m]$. By Lemma 2.15 we can write $y = \sum_{i=0}^m t_i p_i$. Now

$$\begin{aligned}|x-y| &= \left|x - \sum_{i=0}^m t_i p_i\right| = \left|\sum_{i=0}^m t_i x - \sum_{i=0}^m t_i p_i\right| \leq \sum_{i=0}^m t_i |x - p_i| \\ &\leq \sum_{i=0}^m t_i \max_j |x - p_j| \\ &= \max_j |x - p_j|.\end{aligned}$$

Because x and y were arbitrary we conclude from the previous calculation that

$$|x - y| \leq \max_{i,j} |p_i - p_j|$$

which implies

$$\operatorname{diam}([p_0,\dots,p_m]) \leq \max_{i,j} |p_i - p_j|.$$

□

Lemma 2.17. A simplex σ is compact in the ambient Euclidean space.

Proof. First let's show that σ is closed. Let x be an accumulation point of σ and let $(x_n)_{n=1}^\infty$ be a sequence for which $\lim_{n\to\infty} x_n = x$. Now for each integer $n \geq 1$ we have by Lemma 2.15 that $x_n = \sum_{i=0}^m t_{n_i} p_i$ where $t_{n_i} \geq 0$ for all $i \in \{0,\dots,m\}$ and $\sum_{i=0} t_{n_i} = 1$. Thus

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} \sum_{i=0}^m t_{n_i} p_i = \sum_{i=0}^m \lim_{n\to\infty} t_{n_i} p_i$$

where $\lim_{n\to\infty} t_{n_i} \geq \lim_{n\to\infty} 0 = 0$. Now

$$\sum_{i=0}^m \lim_{n\to\infty} t_{n_i} = \lim_{n\to\infty} \sum_{i=0}^m t_{n_i} = \lim_{n\to\infty} 1 = 1.$$

Hence $x \in \sigma$ which implies that σ is closed.

From Lemma 2.16 we have that the diameter of σ is the maximum distance between any two vertices. Because each vertex has a finite distance, we get that the diameter of σ is finite and hence σ is bounded. Now σ is bounded and closed and thus the Heine-Borel theorem implies that σ is compact. □

Definition 2.18. The barycenter of a simplex $\sigma = [p_0, \ldots, p_m]$ is

$$b^\sigma = \frac{1}{m+1} \sum_{i=0}^{m} p_i.$$

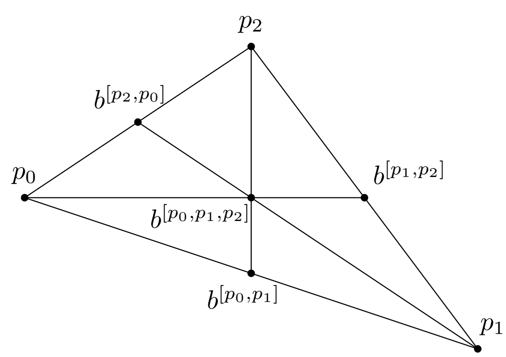


Figure 2: Barycenters of the faces of $[p_0, p_1, p_2]$. Note that $b^{[p_i]} = p_i$ for all $i \in \{0, 1, 2\}$.

Lemma 2.19. For all vertices $p_i \in \mathrm{Vert}(\sigma)$ we have the inequality

$$|b^\sigma - p_i| \leq \frac{m}{m+1} \mathrm{diam}(\sigma).$$

Proof. With direct calculations we get

$$\begin{aligned} |b^\sigma - p_i| &= \left| \frac{1}{m+1} \sum_{j=0}^{m} p_j - p_i \right| = \left| \frac{1}{m+1} \sum_{j=0}^{m} p_j - \frac{1}{m+1} \sum_{j=0}^{m} p_i \right| \\ &= \left| \frac{1}{m+1} \sum_{j=0}^{m} (p_j - p_i) \right| \leq \frac{1}{m+1} \sum_{j=0}^{m} |p_j - p_i| \\ &\leq \frac{1}{m+1} (|p_i - p_i| + m \max_{j \neq i} |p_j - p_i|) \leq \frac{m}{m+1} \mathrm{diam}(\sigma). \end{aligned}$$

□

Definition 2.20. The dimension of the m-simplex σ, denoted by $\dim(\sigma)$, is m.

Definition 2.21. A face of σ is another simplex τ with $\mathrm{Vert}(\tau) \subseteq \mathrm{Vert}(\sigma)$ and we denote $\tau \preceq \sigma$. We say that τ is a proper face if $\mathrm{Vert}(\tau) \subsetneq \mathrm{Vert}(\sigma)$, it is denoted by $\tau \prec \sigma$.

Definition 2.22. The boundary of σ, denoted by $\mathrm{Bd}(\sigma)$, is the union of all of its proper faces, i.e.

$$\sigma = \bigcup_{\tau \prec \sigma} \tau.$$

Definition 2.23. When $m > 0$ the interior of σ is $\mathrm{Int}(\sigma) = \sigma \backslash \mathrm{Bd}(\sigma)$. If $m = 0$ and σ is a 0-simplex, then $\mathrm{Int}(\sigma) = \sigma$.

Remark. Note that if τ is a proper face of σ then $\mathrm{Int}(\tau)$ is not an open subset of σ in the subspace topology.

Lemma 2.24. For every point $x \in \sigma$ there exists only one face τ of σ such that $x \in \mathrm{Int}(\tau)$.

Proof. Fix $x \in \sigma$. By Lemma 2.15 we can write $x = \sum_{i=0}^{m} t_i p_i$ where $t_i \geq 0$ and $\sum_{i=0}^{m} t_i = 1$. Let $J_x = \{j \in \{0, \dots, m\} : t_j > 0\}$ and note that J_x is not empty since $\sum_{i=0}^{m} t_i = 1$. Now $x \in \mathrm{Int}([p_j]_{j \in J_x})$ and $[p_j]_{j \in J_x}$ is a face of σ. □

Definition 2.25. The orientation of the simplex $[p_0, \dots, p_m]$ is a total order of its vertices. An oriented simplex $[p_0, \dots, p_m]$ is the simplex $[p_0, \dots, p_m]$, with the orientation $(p_0, \dots, p_m)$.

Definition 2.26. Define $[p_0, \dots, \hat{p}_i, \dots, p_m]$ to mean a proper face of $[p_0, \dots, p_m]$ where the ith vertex has been omitted.

Definition 2.27. Let σ be an oriented simplex. The induced orientation of a proper face $[p_0, \dots, \hat{p}_i, \dots, p_m]$ is $(p_0, \dots, \hat{p}_i, \dots, p_m)$ if the index i is even and opposite to it otherwise.

Definition 2.28. An affine map $T\colon [p_0, \dots, p_m] \to \mathbb{R}^n$ is a function that satisfies

$$T\Big(\sum_{i=0}^{m} t_i p_i\Big) = \sum_{i=0}^{m} t_i T(p_i).$$

Theorem 2.29. Let $[p_0, \dots, p_m]$ be an m-simplex and let $[q_0, \dots, q_n]$ be an n-simplex and let $f\colon \{p_0, \dots, p_m\} \to [q_0, \dots, q_n]$ be any function. There exists a unique affine map $T\colon [p_0, \dots, p_m] \to [q_0, \dots, q_n]$ such that $T(p_i) = f(p_i)$ for all $i \in \{0, \dots, m\}$.

Proof. See Theorem 2.10 in [6]. □

2.4 Singular homology

In singular homology we assign abelian groups to topological spaces. It is common to state that for $n \geq 0$ we have a functor $H_n\colon \mathbf{Top} \to \mathbf{Ab}$ (theorem 4.10 in [6]) but in detail it is a composition of functors as in the diagram

$$\mathbf{Top} \xrightarrow{S_*} \mathbf{Comp} \xrightarrow{H_n} \mathbf{Ab}.$$

We assume that the reader is familiar with the categories and functors presented in the diagram; see e.g. chapters four and five from [6] for details.

It can be shown that the zeroth homology group of any nonempty space is never trivial. In some cases it helps with the proofs if the zeroth homology group would be trivial if the space is path connected. A solution to this problem is the reduced homology.

Definition 2.30. Let $(S_*(X), \partial)$ be a singular complex of a space X. The augmented singular complex of X is

$$\widetilde{S}_* := \ \dots \longrightarrow S_2(X) \xrightarrow{\partial_2} S_1(X) \xrightarrow{\partial_1} S_0(X) \xrightarrow{\widetilde{\partial}_0} \widetilde{S}_{-1}(X) \longrightarrow 0,$$

where $\widetilde{S}_{-1}(X)$ is the infinite cyclic group generated by the symbol $[\ ]$ and $\widetilde{\partial}_0 \colon S_0(X) \to \widetilde{S}_{-1}(X)$ is defined by $\sum m_x x \mapsto (\sum m_x)[\,]$.

Definition 2.31. For each integer $n \geq 0$, we define the reduced singular homology group of a space X by

$$\widetilde{H}_n(X) = H_n(\widetilde{S}_*(X)).$$

There is also the concept of relative singular homology. The following diagram gives a good illustration what categories and functors are involved in it.

$$\mathbf{Top}^2 \xrightarrow{S_*^2} \mathbf{Comp} \xrightarrow{H_n} \mathbf{Ab}$$

Let's define the category of pairs of topological spaces $\mathbf{Top}^2$. An object is a pair (X, A), where X is a topological space and A is a subspace of X. A morphism $f \colon (X, A) \to (Y, B)$ is a continuous function $f \colon X \to Y$ such that $f(A) \subset B$.

Define the functor $S_*^2 \colon \mathbf{Top}^2 \to \mathbf{Comp}$ as follows. Objects get mapped as $(X, A) \mapsto S_*(X)/S_*(A)$. Here $(S_*(X)/S_*(A), \bar{\partial})$ is a chain complex

$$\dots \longrightarrow S_n(X)/S_n(A) \xrightarrow{\bar{\partial}_n} \dots \xrightarrow{\bar{\partial}_2} S_1(X)/S_1(A) \xrightarrow{\bar{\partial}_1} S_0(X)/S_0(A) \xrightarrow{\bar{\partial}_0} 0,$$

where $\bar{\partial}_n \colon x + S_n(A) \mapsto \partial_n(x) + S_{n-1}(A)$. Morphisms $f \colon (X, A) \to (Y, B)$ get mapped to chain maps $S_*^2(f) \colon S_*(X)/S_*(A) \to S_*(Y)/S_*(B)$ defined by $\gamma + S_n(A) \mapsto f_\#(\gamma) + S_n(B)$, where $\gamma \in S_n(X)$.

Lemma 2.32 (Exact sequence of the triple (X, A, A'))**.** If $A' \subset A \subset X$, then there exists an exact sequence

$$\dots \longrightarrow H_n(A, A') \longrightarrow H_n(X, A') \longrightarrow H_n(X, A) \longrightarrow H_{n-1}(A, A') \longrightarrow \dots.$$

In addition, if there is a commutative diagram

$$\begin{array}{ccccc} (A, A') & \longrightarrow & (X, A') & \longrightarrow & (X, A) \\ \downarrow & & \downarrow & & \downarrow \\ (B, B') & \longrightarrow & (Y, B') & \longrightarrow & (Y, B), \end{array}$$

then there exists a commutative diagram with exact rows

$$\begin{array}{ccccccccc} \dots \longrightarrow & H_n(A, A') & \longrightarrow & H_n(X, A') & \longrightarrow & H_n(X, A) & \longrightarrow & H_{n-1}(A, A') & \longrightarrow \dots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \dots \longrightarrow & H_n(B, B') & \longrightarrow & H_n(Y, B') & \longrightarrow & H_n(Y, B) & \longrightarrow & H_{n-1}(B, B') & \longrightarrow \dots. \end{array}$$

Proof. See Theorem 5.9 in [6]. □

Lemma 2.33. If X is convex, then $\widetilde{H}_n(X) = 0$ for all $n \geq 0$

Proof. Theorem 4.19 in [6] proves the cases $n \geq 1$ and the case $n = 0$ follows from theorems 5.12 and 5.17 in [6]. □

Lemma 2.34. The singular barycentric subdivision $|\mathrm{Sd}|_\# \colon S_*(X) \to S_*(X)$ is a chain map.

Proof. See Lemma 6.12 in [6] □

Lemma 2.35. For all $n \geq 0$, the map $|\mathrm{Sd}|_* = H_n(|\mathrm{Sd}|_\#) \colon H_n(X) \to H_n(X)$ is the identity.

Proof. See Lemma 6.13 in [6] □

Theorem 2.36 (Excision, Theorem 6.17 in [6])**.** Let X_1 and X_2 be subspaces of X with $X = \mathrm{Int}(X_1) \cup \mathrm{Int}(X_2)$. The inclusion $j \colon (X_1, X_1 \cap X_2) \to (X, X_2)$ induces isomorphisms

$$j_* \colon \widetilde{H}_n(X_1, X_1 \cap X_2) \xrightarrow{\sim} \widetilde{H}_n(X, X_2)$$

for all integers $n \geq 0$.

2.5 Diagrams and homology

In this section we have general diagram and homology results. In other words these results are not constrained to singular homology.

Lemma 2.37. For each $n \geq 0$, $H_n \colon \mathbf{Comp} \to \mathbf{Ab}$ is a functor.

Proof. See Theorem 4.10 in [6]. □

Definition 2.38. An exact sequence is a sequence of abelian groups and homomorphisms

$$\dots \longrightarrow G_{n+1} \xrightarrow{\partial_{n+1}} G_n \xrightarrow{\partial_n} G_{n-1} \longrightarrow \dots$$

such that $\operatorname{im} \partial_{n+1} = \ker \partial_n$.

Definition 2.39. A short exact sequence is an exact sequence of the form

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0.$$

Theorem 2.40. If

$$0 \longrightarrow S'_* \xrightarrow{i} S_* \xrightarrow{p} S''_* \longrightarrow 0$$

is a short exact sequence of chain complexes, then there exists a long exact sequence

$$\dots \longrightarrow H_{n+1}(S''_*) \xrightarrow{d} H_n(S'_*) \xrightarrow{i_*} H_n(S_*) \xrightarrow{p_*} H_n(S''_*) \longrightarrow \dots,$$

where the homomorphisms d are called the connecting homomorphisms.

Proof. See Theorem 5.6 in [6]. □

Theorem 2.41 (Naturality of the Connecting Homomorphism, Theorem 5.7 in [6])**.** If there exists a commutative diagram of complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S'_* & \xrightarrow{i} & S_* & \xrightarrow{p} & S''_* & \longrightarrow & 0 \\ & & \downarrow f' & & \downarrow f & & \downarrow f'' & & \\ 0 & \longrightarrow & T'_* & \xrightarrow{i} & T_* & \xrightarrow{p} & T''_* & \longrightarrow & 0, \end{array}$$

then there is a commutative diagram with exact rows:

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & H_{n+1}(S''_*) & \xrightarrow{d} & H_n(S'_*) & \xrightarrow{i_*} & H_n(S_*) & \xrightarrow{p_*} & H_n(S''_*) & \longrightarrow & \dots \\ & & \downarrow f''_* & & \downarrow f'_* & & \downarrow f_* & & \downarrow f''_* & & \\ \dots & \longrightarrow & H_{n+1}(T''_*) & \xrightarrow{\delta} & H_n(T'_*) & \xrightarrow{i_*} & H_n(T_*) & \xrightarrow{p_*} & H_n(T''_*) & \longrightarrow & \dots. \end{array}$$

Definition 2.42. A short exact sequence

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0$$

is called a split exact sequence or equivalently we say that the sequence splits, if there exists a homomorphism $s\colon C \to B$ with $ps = \mathrm{id}_C$.

Lemma 2.43. If we have a split exact sequence

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0,$$

then $B \cong \operatorname{im} i \oplus C$.

Proof. Let's start by showing $B = \ker p + \operatorname{im} s$. Let $b \in B$. We have an expression $b = b - sp(b) + sp(b)$. We can calculate

$$p(b - sp(b)) = p(b) - psp(b) = p(b) - p(b) = 0.$$

Thus $b - sp(b) \in \ker p$. Also, we have $sp(b) \in \operatorname{im} s$. Hence $B = \ker p + \operatorname{im} s$.

Next we show $\ker p \cap \operatorname{im} s = 0$. Assume there exists $b \in B$ with $p(b) = 0$ and there exists $c \in C$ such that $b = s(c)$. Thus $ps(c) = p(b) = 0$. But $ps(c) = c$, which implies $c = 0$ and therefore $b = 0$. Hence $\ker p \cap \operatorname{im} s = 0$ and with the previous result, we have $B = \ker p \oplus \operatorname{im} s$.

Because $ps = \mathrm{id}_C$, it follows that s is an injection. Thus $s\colon C \to \operatorname{im} s$ is an isomorphism. Since $ps = \mathrm{id}_C$, also $p|\colon \operatorname{im} s \to C$ must be an isomorphism. Hence $B \cong \ker p \oplus C$. Last, since the sequence is exact, we get $B \cong \operatorname{im} i \oplus C$. □

Lemma 2.44. If

$$\dots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{h_n} B_n \xrightarrow{p_n} C_n \xrightarrow{d_n} A_{n-1} \longrightarrow \dots,$$

is an exact sequence and every $h_n\colon A_n \to B_n$ is an isomorphism, then $C_n = 0$ for all n.

Proof. Because the given sequence is exact and every h_n is an isomorphism, we get for every $n \in \mathbb{Z}$

$$\operatorname{im} d_n = \ker h_{n-1} = 0$$

and

$$\ker p_n = \operatorname{im} h_n = B_n$$

Thus, we have that $d_n \colon C_n \to 0$ is a surjection. It is also an injection since $\ker d_n = \operatorname{im} p_n = 0$. □

Lemma 2.45. If we have an exact sequence

$$\dots \longrightarrow B_{n+1} \longrightarrow 0 \longrightarrow A_n \xrightarrow{h_n} B_n \longrightarrow 0 \longrightarrow A_{n-1} \longrightarrow \dots,$$

then $h_n \colon A_n \to B_n$ is an isomorphism for all n.

Proof. Because the seqence is exact we have

$$\operatorname{im} h_n = \ker(B_n \to 0) = B_n$$

and

$$\ker h_n = \operatorname{im}(0 \to A_n) = 0.$$

Thus every h_n is an isomorphism. □

Theorem 2.46 (Five lemma, Theorem 5.10 in [6])**.** Consider the commutative diagram

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & A_5 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\ B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & B_5, \end{array}$$

where the rows are exact.

i) If f_2 and f_4 are surjections and f_5 is an injection, then f_3 is a surjection.

ii) If f_2 and f_4 are injections and f_1 is a surjection, then f_3 is an injection.

iii) If f_1, f_2, f_4 and f_5 are isomorphisms, then f_3 is an isomorphism.

Lemma 2.47 (Barratt-Whitehead)**.** Consider the commutative diagram

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & A_n & \xrightarrow{i_n} & B_n & \xrightarrow{p_n} & C_n & \xrightarrow{d_n} & A_{n-1} & \longrightarrow & \dots \\ & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle g_n} & & \downarrow{\scriptstyle h_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\ \dots & \longrightarrow & A'_n & \xrightarrow{j_n} & B'_n & \xrightarrow{q_n} & C'_n & \longrightarrow & A'_{n-1} & \longrightarrow & \dots, \end{array}$$

where the rows are exact and every h_n is an isomorphism. There exists an exact sequence

$$\dots \longrightarrow A_n \xrightarrow{(i_n, f_n)} B_n \oplus A'_n \xrightarrow{g_n - j_n} B'_n \xrightarrow{d_n h_n^{-1} q_n} A_{n-1} \longrightarrow \dots.$$

Proof. See Lemma 6.2 in [6]. □

Definition 2.48. A *contracting homotopy* of a chain complex (S_*, ∂) is a sequence of homomorphisms $P = \{P_n : S_n \to S_{n+1}\}$ such that for all $n \in \mathbb{Z}$,

$$\partial_{n+1} P_n + P_{n-1} \partial_n = \mathrm{Id}_{S_n}.$$

Lemma 2.49. If a chain complex (S_*, ∂) has a contracting homotopy, then $H_n(S_*) = 0$ for all n.

Proof. See corollary 5.4 in [6]. □

3 Simplicial complexes

In this section we study simplicial complexes, polyhedrons and simplicial maps. We mainly introduce concepts and results which are used later on in this thesis. The concepts presented should give fairly good understanding on simplicial complexes but for further study we recommend the book [3].

3.1 Simplicial complexes

Definition 3.1. A simplicial complex K is a finite collection of simplexes in some Euclidean space such that:

(1) Every face of a simplex of K is in K,

(2) Every intersection of two simplexes in K is either empty or a face of both of them.

Remark. We are only focusing on finite collections of simplexes.

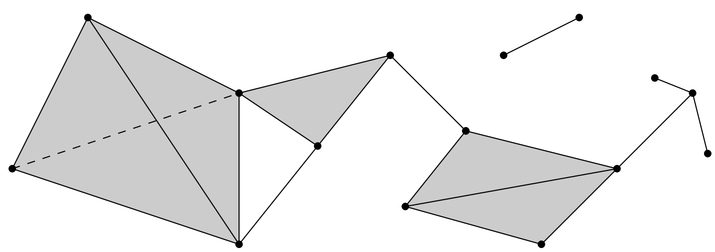


Figure 3: Low-dimensional example of a simplicial complex. It contains 1 3-simplex, 7 2-simplexes, 20 1-simplexes and 15 0-simplexes.

A way to think about simplicial complexes is that we attach simplexes together. The condition (2) of definition 3.1 only restricts how we can attach the simplexes. In Figure 4 there are examples where the second condition is not satisfied. In (a) the intersection between the inner 2-simplex and the outer 2-simplex is not a face of the outer 2-simplex, in (b) the intersection with the 1-simplex and the 2-simplex is not a face of either of them, and in (c) the intersection of the two 2-simplexes is not a face of the 2-simplex on the right.

It can be tedious to use the definition of a simplicial complex to prove that a given collection of simplexes is a simplicial complex. The difficulty comes from showing that the intersection of two simplexes is actually a face of either one of the simplexes. Hence, we introduce an equivalent definition of a simplicial complex which can be useful for proving that a collection of simplexes forms a simplicial complex. For example, it should be intuitive that any simplex and its proper faces form a simplicial complex. Proving it requires some work, but the following Lemma makes most of the work and we can prove the claim in just few steps as is done in Corollary 3.3.

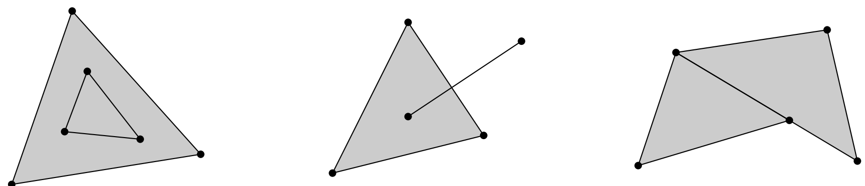


Figure 4: Examples of simplexes that do not satisfy the second requirement for simplicial complexes.

Lemma 3.2. A collection K of simplexes is a simplicial complex if and only if

(1) every face of a simplex of K is in K,

(2') distinct simplexes of K have mutually disjoint interiors, i.e. if $\sigma, \tau \in K$ and $\sigma \neq \tau$, then $\mathrm{Int}(\sigma) \cap \mathrm{Int}(\tau) = \emptyset$.

Proof. The first condition of definition 3.1 of a simplicial complex is the same as the first condition of the lemma. Thus, we need to only focus on the second conditions.

Assume first that K is a simplicial complex. Let $\sigma, \tau \in K$ be distinct such that $\mathrm{Int}(\sigma) \cap \mathrm{Int}(\tau) \neq \emptyset$. We will show that this leads to a contradiction. Denote $\upsilon = \sigma \cap \tau$. By the definition of a simplicial complex υ is a common face of both σ and τ. If υ were a proper face of σ then that would imply $\upsilon \subset \mathrm{Bd}(\sigma)$ but the assumption yields $\upsilon \cap \mathrm{Int}(\sigma) \neq \emptyset$. Hence it must be that $\upsilon = \sigma$. Using similar arguments, we get that $\upsilon = \tau$. Hence $\sigma = \upsilon = \tau$. This is a contradiction because we have assumed that $\sigma \neq \tau$. Thus it must be that $\mathrm{Int}(\sigma) \cap \mathrm{Int}(\tau) = \emptyset$.

Now let's show the converse implication. Let $\sigma, \tau \in K$ such that $\sigma \cap \tau \neq \emptyset$. We will show that the intersection is a face of both σ and τ. Let $\upsilon = [p_{j_0}, \dots, p_{j_n}]$ where $p_{j_0}, \dots, p_{j_n}$ are the common vertices of σ and τ. We want to show that $\upsilon = \sigma \cap \tau$. Now $\sigma \cap \tau$ is convex as an intersection of two convex sets and $\{p_{j_0}, \dots, p_{j_n}\} \subset \sigma \cap \tau$. Because υ is the smallest convex set containing $\{p_{j_0}, \dots, p_{j_n}\}$ we get that $\upsilon \subset \sigma \cap \tau$. For the reverse inclusion, let $x \in \sigma \cap \tau$. By Lemma 2.24 there exist faces $\sigma' \prec \sigma$ and $\tau' \prec \tau$ such that $x \in \mathrm{Int}(\sigma') \cap \mathrm{Int}(\tau')$. The condition (2') implies that $\sigma' = \tau'$ and hence the vertices of σ' are in τ. By the definition of υ we get that $\mathrm{Vert}(\sigma') \subset \mathrm{Vert}(\upsilon)$. Thus σ' is a face of υ which implies that $x \in \upsilon$. Hence $\upsilon \supset \sigma \cap \tau$ and we have shown that $\upsilon = \sigma \cap \tau$. Now $\mathrm{Vert}(\upsilon) \subset \mathrm{Vert}(\sigma)$ implies that υ is a face of σ and $\mathrm{Vert}(\upsilon) \subset \mathrm{Vert}(\tau)$ implies that υ is a face of τ. Thus K is a simplicial complex. □

Corollary 3.3. Let $\sigma = [p_0, \dots, p_m]$ be an m-simplex. If K is the collection of faces of σ, then K is a simplicial complex.

Proof. Condition (1) of Lemma 3.2 is directly satisfied by the definition of K. For condition (2'), Lemma 2.24 states that every point of σ belongs to the interior of only one face. This implies that given two distinct faces of σ, the intersection of their interiors is empty. Thus Lemma 3.2 implies that K is a simplicial complex. □

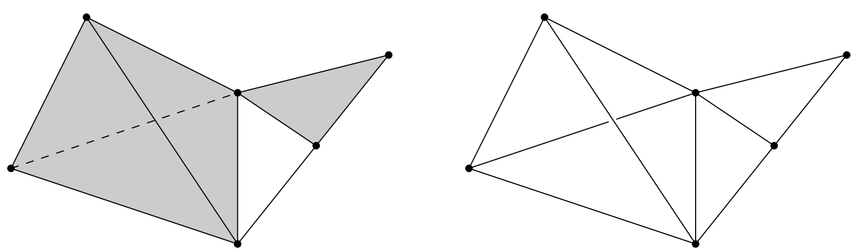
3.2 Subcomplex

We can construct a simplicial complex out of a given simplicial complex by taking a subcollection of simplexes. We need to only account the first requirement for simplicial complexes when taking the subcollection since the simplexes already satisfy the second condition.

Definition 3.4. Let K be a simplicial complex and let $L \subset K$. If L contains all of the faces of its elements, then L is called a subcomplex of K.

If L is a subcomplex of a simplicial complex K, then L contains all of the faces of its simplexes. Hence the first condition of a simplicial complex is satisfied. Now take any two simplexes of L. Because both simplexes are also in the simplicial complex K, the intersection of them is either empty or a common face. Thus L satisfies the second condition and therefore it is a simplicial complex.

Definition 3.5. Let K be a simplicial complex. The n-skeleton $K^{(n)}$ of K consists of all of the simplexes of K that are of dimension n or less.



(a) (b)

Figure 5: In (a) we have a simplicial complex K and in (b) we have $K^{(1)}$.

Lemma 3.6. Let K be a simplicial complex. The n-skeleton $K^{(n)}$ is a subcomplex of K for all integers $n \geq 0$.

Proof. Let $\sigma \in K^{(n)}$. Now it must be that $\dim(\sigma) = m \leq n$. Because the faces of σ have a dimension less than or equal to m, it follows that all the faces are in $K^{(n)}$. Thus $K^{(n)}$ is a subcomplex of K. □

Corollary 3.7. Let $\sigma = [p_0, \ldots, p_m]$ be an m-simplex. If K is the collection of the proper faces of σ, then K is a simplicial complex.

Proof. Corollary 3.3 states that the faces of σ form a simplicial complex. Denote that simplicial complex by L. Now K is the $m-1$-skeleton of L i.e. $K = L^{(m-1)}$. Lemma 3.6 implies that K is a subcomplex and hence a simplicial complex. □

3.3 Polyhedra

In this section we focus on topological spaces that can be constructed out of simplicial complexes. Simplicial complex itself does not contain any topology but we can assign it a topology as in the following definition.

Definition 3.8. If K is a simplicial complex, then the underlying space $|K|$ is the union of all simplexes of K, i.e.

$$|K| = \bigcup_{\sigma \in K} \sigma.$$

Lemma 3.9. The underlying space $|K|$ of a simplicial complex K is a metric space.

Proof. Each simplex of K is a subset of some Euclidean space. Thus, we conclude that $|K|$ as a finite union of simplexes can be embedded to some Euclidean space. Since Euclidean spaces are metric spaces, we can assign the induced metric to $|K|$. □

The underlying spaces form a small class of topological spaces. Hence, we account for topological spaces that are the same as underlying spaces in a topological sense.

Definition 3.10. A topological space X is called a polyhedron if there exist a simplicial complex K and a homeomorphism $f : |K| \to X$. We call K a triangulation of X.

In Figure 4 we showed collections of simplexes that do not satisfy the second condition of simplicial complexes. Nevertheless, we can consider them as polyhedrons, we just have to find triangulations for them. In Figure 6 we give possible triangulations for the spaces.

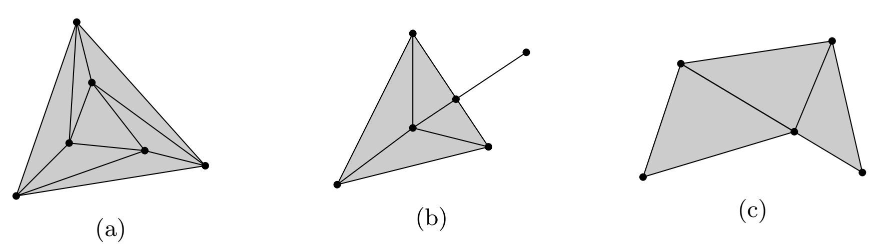


Figure 6: Example triangulations for spaces presented in Figure 4.

In general, the problem usually is that we are given a topological space and we want to show that it is a polyhedron. We prove this by finding a triangulation for the space. We can eliminate some topological spaces that are not polyhedrons by showing that they do not satisfy topological properties of polyhedrons.

Lemma 3.11. The underlying space $|K|$ of a simplicial complex K is compact.

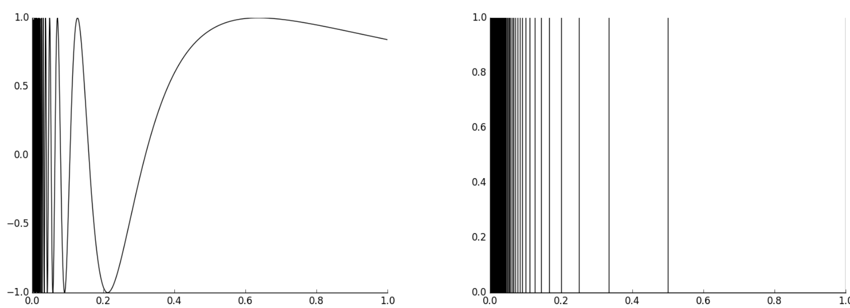
Proof. Since $|K|$ is the union of finitely many compact sets, Lemma 2.5 implies that $|K|$ is compact. □

By Lemma 3.11, every non-compact topological space is not a polyhedron. For example, $\mathbb{R}^n$ is not a polyhedron for any integer $n \geq 1$.

If K is a simplicial complex, then the underlying space $|K|$ is locally path connected. The proof for this claim goes as follows. Fix a point $x \in |K|$. Since the simplexes of K are compact, there exists

$$\epsilon = \min\{d(x, \sigma) : \sigma \in K, x \notin \sigma\} > 0.$$

Thus every open ball $B(x, r)$, where $r < \epsilon$, is path connected. This implies that $|K|$ is locally path connected. By Lemma 2.7, we conclude that polyhedrons are locally path connected. Thus, the topologist's sine curve and the comb space are not polyhedrons.



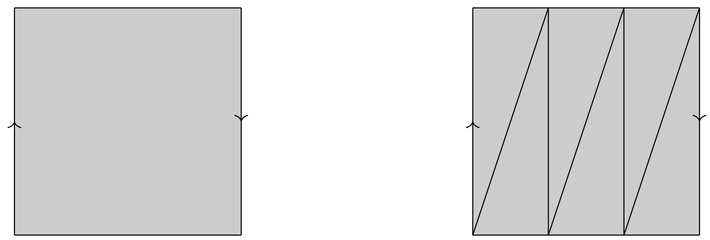
(a) The topologists sine curve

(b) The comb space

Figure 7: Examples of spaces that are not locally path connected.

Let's see some examples of polyhedra. Showing that a function is a homeomorphism can be very tedious. In this thesis we have omitted rigorous proofs for homeomorphisms in the following examples. For spaces with simple triangulations, we argue lightly about the triangulations. For spaces that can be drawn, we present pictures of their triangulations in which the homeomorphisms should be obvious from the pictures.

For the closed n-ball $\mathbb{D}^n$ we assign the triangulation to be the simplicial complex consisting of the faces of any n-simplex. The n-sphere is a polyhedron with the simplicial complex consisting of the proper faces of an n-simplex. With the following spaces we have to be a bit more careful with the triangulations. The Möbius band is a polyhedron with the triangulation presented in Figure 8 (b).



(a) (b)

Figure 8: The unit square in (a), where the sides are identified according to the arrows, is homeomorphic to the Möbius band. In (b) we have a valid triangulation for the Möbius band consisting of 6 2-simplexes, 12 1-simplexes and 6 0-simplexes.

Note that we must be careful with the simplexes touching the identified edges. For example if we tried to triangulate the Möbius band with the simplexes presented in Figure 9, the construction is not a simplicial complex. This is because $[p_0, p_3, p_4] \cap [p_1, p_4, p_5] = \{[p_4], [p_0]\}$, where we used the fact that the two sides are identified and we have $p_0 = p_5$. Now $\{[p_4], [p_0]\}$ is not a face of any simplex, we conclude that the second requirement for a simplicial complex is not satisfied. Notice that in Figure 8 (b) we do not run in to the same problem and it should be clear that the simplexes form a simplicial complex.

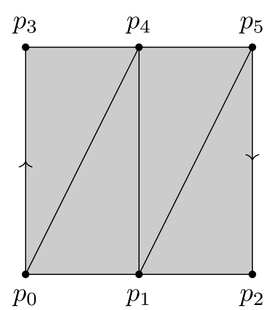
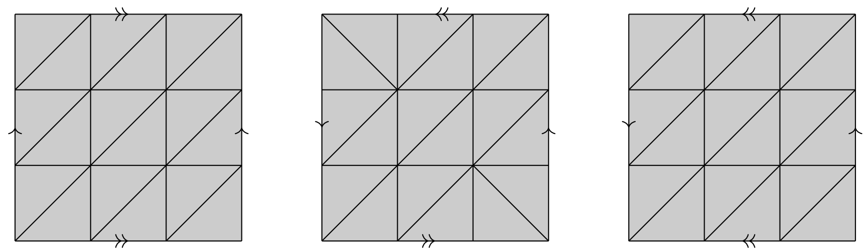


Figure 9: Example of a collection of simplexes that do not form a simplicial complex.

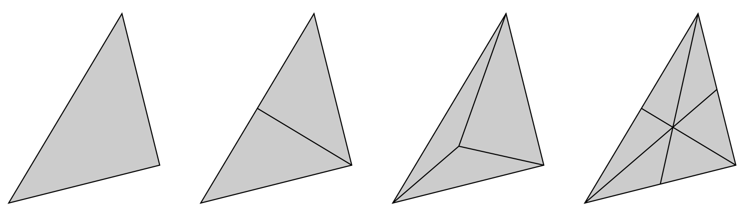


(a) Torus (b) Real projective plane (c) Klein bottle

Figure 10: By identifying opposite edges of a unit square we can form the spaces in (a), (b) and (c). One identifies double headed arrows with each other and one headed arrows with each other. The triangulations presented are valid which should be immediate from the pictures.

3.4 Barycentric subdivision

There is a very nice property of simplicial complexes that for each simplex in the simplicial complex we can subdivide the simplex into smaller simplexes. We can subdivide a simplex almost arbitrarily and in Figure 11 we show a few ways of subdividing a 2-simplex. Our focus is on barycentric subdivision. Compared to other subdivisions, barycentric subdivision is rigid in the sense that it gives us a well defined algorithm for subdividing the simplexes. Each simplex with the same dimension will be subdivided in the same manner and it turns out that we can place an upper bound for the sizes of the new simplexes. The upper bound gets smaller the more we subdivide. Indeed the real power of barycentric subdivision comes when we use it over and over again because then we can end up having arbitrarily small simplexes.



(a) (b) (c) (d)

Figure 11: Three different subdivisions of the 2-simplex in (a). The subdivisions of (b) and (c) were made up, but in (d) we have barycentric subdivision.

Definition 3.12. Let K be a simplicial complex. The barycentric subdivision of K, denoted by $\mathrm{Sd}(K)$, is the simplicial complex with vertices

$$\mathrm{Sd}(K)^{(0)} = \bigcup_{\sigma \in K} \{b^{\sigma}\}$$

and with simplexes $[b^{\sigma_1}, \ldots, b^{\sigma_n}]$ where σ_i are simplexes of K and σ_i is a proper face of σ_j for $i < j$.

The definition can be a bit fussy so let's break it down with an example of a simplicial complex K consisting of the faces of a 2-simplex $\sigma = [p_0, p_1, p_2]$. The barycenters of the faces of σ can be seen from Figure 2 in the preliminaries section. The set of vertices or the 0-simplexes of $\mathrm{Sd}(K)$ are all the barycenters i.e.

$$\mathrm{Sd}(K)^{(0)} = \{p_0, p_1, p_2, b^{[p_0,p_1]}, b^{[p_1,p_2]}, b^{[p_2,p_0]}, b^{[p_0,p_1,p_2]}\}.$$

Note here that the vertices of σ are also in the subdivision of K. Simplexes of K with dimension greater than zero are not in $\mathrm{Sd}(K)$. For example $[p_0, p_1] = [b^{[p_0]}, b^{[p_1]}] \notin \mathrm{Sd}(\sigma)$ because we require $[p_0]$ to be a proper face of $[p_1]$ which clearly is not the case. On the other hand $[p_0, b^{[p_0,p_1]}] = [b^{[p_0]}, b^{[p_0,p_1]}] \in \mathrm{Sd}(\sigma)$ since $[p_0] \prec [p_0, p_1]$. It is important that we require this because otherwise we would end up with simplexes overlapping each other in such a way that the second requirement of simplicial complexes is not fulfilled.

Notice that $\mathrm{Sd}(K)$ is not a subcomplex of K unless $K = K^{(0)}$. This is because K does not contain the barycenters. Also if $\sigma \in K$ and σ is not a 0-simplex, then $\sigma \notin \mathrm{Sd}(K)$. Thus it follows that K is not a subcomplex of $\mathrm{Sd}(K)$ if $K \neq K^{(0)}$.

Lemma 3.13. If K is a simplicial complex, then $|\mathrm{Sd}(K)| = |K|$.

Proof. Let's first show the inclusion $|\mathrm{Sd}(K)| \subset |K|$. Let $[b^{\sigma_0}, \ldots, b^{\sigma_m}]$ be a simplex of $\mathrm{Sd}(K)$, where each σ_i is a simplex of K and $\sigma_i \prec \sigma_j$ whenever $i < j$. Thus $\sigma_i \prec \sigma_m$ for all $i \in \{0, \ldots, m-1\}$. This implies that the barycenters b^{σ_i} are in σ_m for all $i \in \{0, \ldots, m\}$. Hence the convex hull $[b^{\sigma_0}, \ldots, b^{\sigma_m}]$ is also within σ_m and $|K|$. Thus $|\mathrm{Sd}(K)| \subset |K|$.

For the converse inclusion, let $x \in |K|$. There exists $[p_0, \ldots, p_m] \in K$ such that $x \in [p_0, \ldots, p_m]$. By Lemma 2.15, we can write $x = \sum_{i=0}^{m} t_i p_i$. We will show that we can find barycentric coordinates for x in $\mathrm{Sd}(K)$. First, let's reorder the barycentric coordinates in K. Let $\pi : \{0, \ldots, m\} \to \{0, \ldots, m\}$ be a permutation such that $t_{\pi(0)} \geq t_{\pi(1)} \geq \cdots \geq t_{\pi(m)} \geq 0$. Now let's transform the barycentric coordinates using the matrix equation

$$\begin{bmatrix} t_{\pi(m)} \\ t_{\pi(m-1)} \\ \vdots \\ t_{\pi(1)} \\ t_{\pi(0)} \end{bmatrix} = \begin{bmatrix} \frac{1}{m+1} & 0 & \ldots & 0 & 0 \\ \frac{1}{m+1} & \frac{1}{m} & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{m+1} & \frac{1}{m} & \ldots & \frac{1}{2} & 0 \\ \frac{1}{m+1} & \frac{1}{m} & \ldots & \frac{1}{2} & 1 \end{bmatrix} \cdot \begin{bmatrix} s_m \\ s_{m-1} \\ \vdots \\ s_1 \\ s_0 \end{bmatrix},$$

where $s_0, s_1, \ldots, s_m$ will be our barycentric coordinates in $\mathrm{Sd}(K)$. Using the transformation we can calculate

$$\begin{aligned}
x =&t_{\pi(0)}p_{\pi(0)} + t_{\pi(1)}p_{\pi(1)} + \dots + t_{\pi(m)}p_{\pi(m)} \\
=&(\frac{1}{m+1}s_m + \dots + s_0)p_{\pi(0)} + (\frac{1}{m+1}s_m + \dots + \frac{1}{2}s_1)p_{\pi(1)} + \dots \\
&+ \frac{1}{m+1}s_m p_{\pi(m)} \\
=&s_0 p_{\pi(0)} + s_1\frac{1}{2}(p_{\pi(0)} + p_{\pi(1)}) + \dots + s_m\frac{1}{m+1}(p_{\pi(0)} + \dots + p_{\pi(m)}) \\
=&s_0 b^{[p_{\pi(0)}]} + s_1 b^{[p_{\pi(0)},p_{\pi(1)}]} + \dots + s_m b^{[p_{\pi(0)},\dots,p_{\pi(m)}]}.
\end{aligned}$$

Let's show that each s_i is greater or equal to zero. We can solve the matrix equation by using the inverse of the matrix. We get

$$\begin{bmatrix} s_m \\ s_{m-1} \\ s_{m-2} \\ \vdots \\ s_2 \\ s_1 \\ s_0 \end{bmatrix} = \begin{bmatrix} m+1 & 0 & 0 & \dots & 0 & 0 \\ -m & m & 0 & \dots & 0 & 0 \\ 0 & -(m-1) & (m-1) & \dots & 0 & 0 \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & & \dots & 2 & 0 \\ 0 & 0 & & \dots & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} t_{\pi(m)} \\ t_{\pi(m-1)} \\ t_{\pi(m-2)} \\ \vdots \\ t_{\pi(2)} \\ t_{\pi(1)} \\ t_{\pi(0)} \end{bmatrix}.$$

Thus

$$s_m = (m+1)\underbrace{t_{\pi(m)}}_{\geq 0} \geq 0$$

and

$$s_i = (i+1)(\underbrace{t_{\pi(i)} - t_{\pi(i+1)}}_{\geq 0}) \geq 0$$

for integers $0 \leq i \leq m-1$. Last we calculate directly

$$\begin{aligned}
\sum_{i=0}^{m} s_i &= (m+1)t_{\pi(m)} + \sum_{i=0}^{m-1}(i+1)(t_{\pi(i)} - t_{\pi(i+1)}) \\
&= (m+1-m)t_{\pi(m)} + (m-(m-1))t_{\pi(m-1)} + \dots + t_{\pi(0)} \\
&= t_{\pi(m)} + t_{\pi(m-1)} + \dots + t_{\pi(0)} = 1.
\end{aligned}$$

Hence $s_0, s_1, \dots, s_m$ are the barycentric coordinates for x in $\mathrm{Sd}(K)$. This implies $|K| \subset |\mathrm{Sd}(K)|$. □

Definition 3.14. If K is a simplicial complex, then

$$\dim(K) = \max\{\dim(\sigma) : \sigma \in K\}$$

and

$$\mathrm{mesh}(K) = \max\{\mathrm{diam}(\sigma) : \sigma \in K\}.$$

Lemma 3.15. If K is a simplicial complex and $n = \dim(K)$, then

$$\text{mesh}\big(\text{Sd}(K)\big) \leq \left(\frac{n}{n+1}\right)\text{mesh}(K).$$

Proof. Let $\sigma = [p_0, \ldots, p_m] \in K$ and let $\tau \in \text{Sd}(K)$ be an m-simplex such that $\tau \subset \sigma$. Lemma 2.16 states that the diameter of a simplex is the longest length between any two of it's vertices. Thus, in order to show

$$\text{diam}(\tau) \leq \frac{m}{m+1}\text{diam}(\sigma),$$

we need to show that for any two vertices $q, r \in \text{Vert}(\tau)$, we have

$$|q - r| \leq \frac{m}{m+1}\text{diam}(\sigma).$$

Let's divide the solution to two distinct cases. First the bound between the barycenter of σ and any other vertex of τ. Let $b^\sigma \in \text{Vert}(\tau)$ be the barycenter of σ and let $q \in \text{Vert}(\tau)$ be arbitrary. Because $q \in \sigma$, Lemma 2.15 implies that we can write $q = \sum_{i=0}^m t_i p_i$, where each $t_i \geq 0$ and $\sum_{i=0}^m t_i = 1$. Using Lemma 2.19 we get the inequality

$$\begin{aligned} |b^\sigma - q| &= \left|b^\sigma - \sum_{i=0}^m t_i p_i\right| = \left|\sum_{i=0}^m t_i(b^\sigma - p_i)\right| \leq \frac{m}{m+1}\text{diam}(\sigma)\sum_{i=0}^m t_i \\ &= \frac{m}{m+1}\text{diam}(\sigma). \end{aligned}$$

Because q was arbitrary we conclude that the distance between the barycenter of σ and any other vertex of τ is bounded by $\frac{m}{m+1}\text{diam}(\sigma)$.

For the second case we show the bound for the distance between any two vertices of τ in which neither one is the barycenter of σ. Let $q, r \in \text{Vert}(\tau)$ such that neither q nor r is the barycenter of σ. Now both q and r belong to the boundary of σ. There exists a proper face σ' of σ with dimension $m' < m$ and that q or r is the barycenter of σ'. We can choose $q = b^{\sigma'}$ i.e. q is the barycenter of σ'. Because σ was arbitrary, we conclude from the previous case that

$$|b^{\sigma'} - r| \leq \frac{m'}{m'+1}\text{diam}(\sigma') \leq \frac{m'}{m'+1}\text{diam}(\sigma) \leq \frac{m}{m+1}\text{diam}(\sigma).$$

Thus

$$\begin{aligned} \text{mesh}(\text{Sd}(K)) &= \max\{\text{diam}(\tau) : \tau \in \text{Sd}(K)\} \\ &\leq \max\{\frac{\dim(\sigma)}{\dim(\sigma)+1}\text{diam}(\sigma) : \sigma \in K\} \\ &\leq \max\{\frac{n}{n+1}\text{diam}(\sigma) : \sigma \in K\} \\ &= \left(\frac{n}{n+1}\right)\text{mesh}(K). \end{aligned}$$

□

3.5 Orientation

For simplexes we defined the orientation to be the total order of their vertices. For simplicial complexes we think of orientation simplex wise.

Definition 3.16. If K is a simplicial complex, then an orientation for it is a partial order of $\mathrm{Vert}(K)$ such that restricting to the vertices of a simplex in K yields a total order. We call K an oriented simplicial complex, if we have assigned an orientation for it.

The reason for defining the orientation this way will come apparent in the sections of simplicial homology where we calculate the homology groups.

3.6 Simplicial maps

In this section we focus on functions between simplicial complexes. We first introduce the simplicial maps and the we extend to piecewise linear map which is a continuous function between the underlying spaces.

Definition 3.17. Let K and L be simplicial complexes. A simplicial map $\psi\colon K^{(0)} \to L^{(0)}$ is a function such that if $\{p_0, \dots, p_m\}$ spans a simplex in K then $\{\psi(p_0), \dots, \psi(p_m)\}$ spans a simplex in L.

Remark. The definition allows multiple vertices to be mapped to the same vertex.

Lemma 3.18. Let K and L be simplicial complexes. There are only finitely many simplicial maps from K to L.

Proof. Because simplicial complexes are finite, we have that K is a finite domain and L is a finite image. Thus there can be only finitely many maps from K to L and hence there are only finitely many simplicial maps from K to L. □

Definition 3.19. Let $\psi\colon K^{(0)} \to L^{(0)}$ be a simplicial map. Let $x \in |K|$ and note that there exists a simplex $\sigma = [p_0, \dots, p_m] \in K$ such that $x = \sum_{i=0}^{m} t_i p_i \in \sigma$ where all of the t_i are uniquely determined by Lemma 2.15. We define a piecewise linear map $|\psi|\colon |K| \to |L|$ as

$$|\psi|(x) = |\psi|\Big(\sum_{i=0}^{m} t_i p_i\Big) = \sum_{i=0}^{m} t_i \psi(p_i).$$

Lemma 3.20. Let $\psi\colon K^{(0)} \to L^{(0)}$ be a simplicial map. The piecewise linear map $|\psi|\colon |K| \to |L|$ is well defined and continuous.

Proof. By the definition of a piecewise linear map 3.19, $|\psi|$ is well defined when restricted to a simplex of K. To make $|\psi|$ well defined on the whole domain $|K|$ we need to verify that if two simplexes of K overlap then $|\psi|$ gives the same value. Let $\sigma, \tau \in K$ and suppose $\rho = \sigma \cap \tau \neq \emptyset$. By the definition of a simplicial complex 3.1, ρ is a face of both of σ and τ and hence a simplex of K as well.

Now for $x \in \rho$ we can write $x = \sum_{i=0}^{m} t_i p_i$ where $t_i \geq 0$ for all $i \in \{0, \ldots, m\}$, $\sum_{i=0}^{m} t_i = 1$ and p_i are the vertices of ρ i.e. $\rho = [p_0, \ldots, p_m]$. Now

$$|\psi||_\sigma(x) = \sum_{i=0}^{m} t_i \psi(p_i) = |\psi||_\tau(x).$$

Thus $|\psi|$ agrees on overlapping simplexes and $|\psi|$ is well defined on the whole domain $|K|$.

By Theorem 2.29, $|\psi|$ is continuous when restricted to any simplex of K. As we just shown $|\psi|$ agrees on in the intersection of two simplexes and because simplexes are closed in $|K|$, the Gluing Lemma implies that $|\psi|$ is continuous on the whole domain $|K|$. □

3.7 Simplicial approximation

In this section our focus is on forming a piecewise linear map from a continuous function. If we are given a continuous function $f\colon |K| \to |L|$ then the first attempt would be to try to find a simplicial map $\psi\colon K^{(0)} \to L^{(0)}$ such that $f = |\psi|$. Unfortunately there are continuous functions which do not equal any piecewise linear map. For example take $|K| = |L| = [0, 1]$ and $f(x) = x^2$. Then f has a continuous derivative but every piecewise linear map $|\psi|$ has a continuous derivative if and only if the derivative of it is constant. Since $f'(x) = 2x$ is not constant, we get that $f \neq |\psi|$ for all piecewise linear maps $|\psi|\colon |K| \to |L|$. Note that the argument did not depend on the triangulations $|K|$ and $|L|$ and hence there does not exist any triangulations for which this would be true.

The next best thing is to look for a piecewise linear map $|\psi|\colon |K| \to |L|$ which would be homotopic to any given continuous function $f\colon |K| \to |L|$. This approach has the problem that there can be continuous functions which are not homotopic to any piecewise linear map given a certain triangulation of the domain. For example, take $\mathbb{S}^1$. Let K and L be arbitrary triangulations of $\mathbb{S}^1$. Now $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$ implies that there are infinitely many nonhomotopic continuous functions $f\colon \mathbb{S}^1 \to \mathbb{S}^1$. By Lemma 3.18 there are only finitely many piecewise linear maps $|\psi|\colon |K| \to |L|$. Thus there exists some continuous $f\colon \mathbb{S}^1 \to \mathbb{S}^1$ such that f is not homotopic to any piecewise linear map $|\psi|\colon |K| \to |L|$.

We have shown that for any given triangulation of a polyhedron there can be continuous functions which are nonhomotopic to all piecewise linear maps. The point of this section is to prove that if $f\colon X \to Y$ is a continuous function between polyhedrons, then there exist triangulations of X and Y and a piecewise linear map $|\psi|\colon X \to Y$ such that $|\psi|$ is homotopic to f. The idea is to focus on regions near vertices and use barycentric subdivision to achieve small enough simplexes in the domain. For regions near vertices we have the following definition.

Definition 3.21. Let K be a simplicial complex and let $p \in \mathrm{Vert}(K)$. The star of p, denoted by $\mathrm{St}(p, K)$, is

$$\mathrm{St}(p, K) = \bigcup_{\substack{\sigma \in K \\ p \in \mathrm{Vert}(\sigma)}} \mathrm{Int}(\sigma) \subset |K|.$$

Remark. If the domain of the star is obvious from context we can just write the star as $\mathrm{St}(p)$.

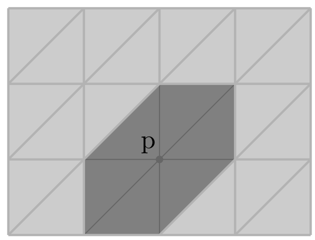


Figure 12: The darker region presents $\mathrm{St}(p)$. Note that the lines emitting from p belong to $\mathrm{St}(p)$ except for the other end point.

Lemma 3.22. Let K be a simplicial complex. The union of all stars of vertices of K is an open cover of $|K|$.

Proof. If K consists of only one simplex, then $K = \{[p]\}$, where p is a point. Now there is only one star $\mathrm{St}(p)$ and clearly $\mathrm{St}(p) = \{p\} = |K|$ and hence the stars cover the space. Since the only possible topology on $|K|$ is the trivial topology, we conclude that $\mathrm{St}(p)$ is open in $|K|$. Thus the stars of the vertices of K form an open cover.

Now assume K consists of more than one simplex. First, let's show that the stars are open. Let $p \in K^{(0)}$ and take arbitrary $x \in \mathrm{St}(p)$. For every simplex $\sigma \in K$ we have $d(x, \sigma) \geq 0$, where the metric d is induced by the ambient Euclidean space. By compactness of $\{x\}$ and σ we have $d(x, \sigma) = 0$ if and only if $x \in \sigma$. Because K consists of more than one simplex but of finitely many of them, there exists

$$\epsilon = \min\{d(x, \sigma) > 0 : \sigma \in K\}.$$

Note that the minimum is taken over all simplexes for which the distance between the simplex and the point x is greater than 0. Now $B(x, \epsilon/2)$ is the open ball in the ambient Euclidean space for which

$$B(x, \epsilon/2) \cap |K| \subset \mathrm{St}(p).$$

Thus $\mathrm{St}(p)$ is open.

Let $x \in |K|$. If x is a vertex of K, then $x \in \mathrm{St}(x)$ and hence x is an element of the cover. Suppose that x is not a vertex. Let $L = \{\sigma \in K : x \in \sigma\}$. Notice that L is not empty because $x \in |K|$, and L is finite because K is finite. Because the intersection of any two simplexes of K is a simplex in K, there exists

$$\tau = \bigcap_{\sigma \in L} \sigma \in K.$$

If $x \notin \mathrm{Int}(\tau)$, then $x \in \rho$ where ρ is a proper face of τ. But ρ should also be in L by the definition of L. The definition of τ implies that $\rho = \tau$, which is not true because is ρ is a proper face of τ. Thus $x \in \mathrm{Int}(\tau)$ and if p is a vertex of τ then $x \in \mathrm{St}(p)$. Hence x is in the cover of the stars. Because x was arbitrary it follows that the stars cover $|K|$. □

Lemma 3.23. Let K be a simplicial complex and let $p_0, \ldots, p_m \in K^{(0)}$. Then $\{p_0, \ldots, p_m\}$ spans a simplex of K if and only if $\cap_{i=0}^m \mathrm{St}(p_i) \neq \emptyset$.

Proof. First assume that $\{p_0, \ldots, p_m\}$ spans a simplex $\sigma \in K$. Now $\mathrm{Int}(\sigma) \subset \mathrm{St}(p_i)$ for all $i \in \{0, \ldots, m\}$. Hence

$$\mathrm{Int}(\sigma) \subset \bigcap_{i=0}^{m} \mathrm{St}(p_i).$$

Because the interior of any simplex is not empty, it follows that the intersection is not empty.

Now suppose that $\cap_{i=0}^m \mathrm{St}(p_i) \neq \emptyset$. By Lemma 3.2 the interiors of simplexes of K are mutually disjoint and they cover $|K|$. Because the stars are the union of interiors of simplexes and the intersection $\cap_{i=0}^m \mathrm{St}(p_i)$ is not empty, we get that there exists a simplex $\sigma \in K$ such that

$$\mathrm{Int}(\sigma) \subset \bigcap_{i=0}^{m} \mathrm{St}(p_i).$$

Now if for some $j \in \{0, \ldots, m\}$ we have $p_j \notin \mathrm{Vert}(\sigma)$, then $\mathrm{Int}(\sigma) \not\subset \mathrm{St}(p_j)$ and the interior of σ is not in the intersection of stars which is a contradiction. Thus $p_j \in \mathrm{Vert}(\sigma)$ for all $j \in \{0, \ldots, m\}$. Hence $\{p_0, \ldots, p_m\}$ spans a face of σ. Because faces are simplexes and K must contain every face of σ, it follows that $\{p_0, \ldots, p_m\}$ spans a simplex of K. □

Lemma 3.24. Let K be a simplicial complex and let $p \in K^{(0)}$. Then

$$\mathrm{diam}\big(\mathrm{St}(p)\big) \leq 2\mathrm{mesh}(K)$$

Proof. Let $x, y \in \mathrm{St}(p)$ and $\sigma, \tau \in K$ such that $x, p \in \sigma$ and $y, p \in \tau$. Then

$$|x - y| \leq |x - p| + |p - y| \leq \mathrm{diam}(\sigma) + \mathrm{diam}(\tau) \leq 2\mathrm{mesh}(K).$$

□

Lemma 3.25. If $\psi \colon K^{(0)} \to L^{(0)}$ is a simplicial map, then for all $p \in K^{(0)}$

$$|\psi|\big(\mathrm{St}(p)\big) \subset \mathrm{St}\big(\psi(p)\big).$$

Proof. Let $x \in \mathrm{St}(p)$. If $x = p$, then $|\psi|(x) = \psi(x) \in \mathrm{St}\big(\psi(p)\big)$. Suppose $x \neq p$. Then $x \in \mathrm{Int}(\sigma)$ for some $\sigma = [p_0, \dots, p_m] \in K$. Because x is in the star of p, there exists $i \in \{0, \dots, m\}$ such that $p_i = p$. We can write $x = \sum_{j=0}^{m} t_j p_j$ where $t_j \geq 0$ and $\sum_{j=0}^{m} t_j = 1$. Now

$$|\psi|(x) = \sum_{j=0}^{m} t_j \psi(p_j) = \sum_{j \neq i} t_j \psi(p_j) + t_i \psi(p)$$

and hence $|\psi|(x) \in \mathrm{St}\big(\psi(p)\big)$. Because $x \in \mathrm{St}(p)$ was arbitrary, the claim follows. $\square$

Definition 3.26. Let $\psi \colon K^{(0)} \to L^{(0)}$ be a simplicial map and let $f \colon |K| \to |L|$ be continuous. We say that ψ is a simplicial approximation to f, if for all vertices p of K,

$$f\big(\mathrm{St}(p)\big) \subset \mathrm{St}\big(\psi(p)\big).$$

Not every continuous function f between polyhedrons $|K|$ and $|L|$ exhibits a simplicial approximation. For example in Figure 13, there can not be a simplicial approximation to f. This is because $f(\mathrm{St}\big(p_0)\big) = f([p_0, p_1] \backslash \{p_1\})$ is not included in any star of the codomain.

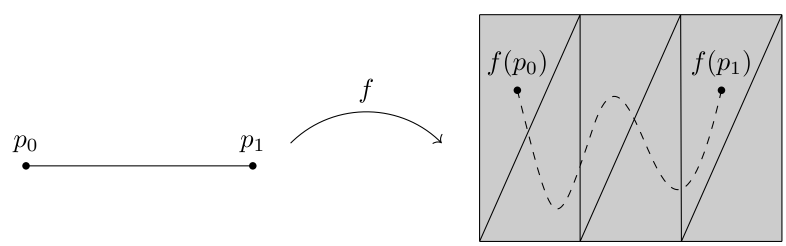


Figure 13: A continuous path f that does not have a simplicial approximation. The domain consists of faces of $[p_0, p_1]$ and the codomain consists of 6 2-simplexes, 13 1-simplexes and 8 0-simplexes.

Fortunately there is a work around in finding a simplicial approximation to f. If we apply the barycentric subdivision on the domain twice, then there exists a simplicial approximation ψ to f which can be viewed in Figure 14. Indeed one can check that $f\big(\mathrm{St}(p_i)\big) \subset \mathrm{St}\big(\psi(p_i)\big)$, for all $i \in \{0, \dots, 4\}$. The map ψ is a simplicial map, since for the non-trivial cases of 1-simplexes, $\{\psi(p_i), \psi(p_{i+1})\}$ spans a simplex of the codomain for all $[p_i, p_{i+1}]$, where $i \in \{0, 1, 2, 3\}$.

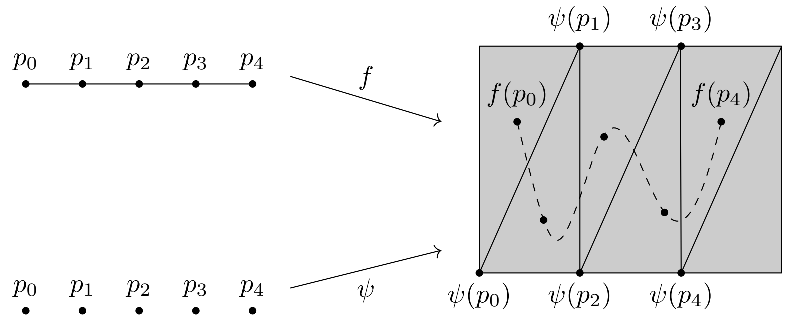


Figure 14: Example of a simplicial map ψ that is a simplicial approximation to f.

Simplicial approximation is not always unique which is apparent from the Figure 14; we could define a map ϕ such that

$$\begin{aligned}\phi(p_0) &= \psi(p_1)\\ \phi(p_1) &= \psi(p_1)\\ \phi(p_2) &= \psi(p_2)\\ \phi(p_3) &= \psi(p_3)\\ \phi(p_4) &= \psi(p_3).\end{aligned}$$

The fact that ϕ is a simplicial map and a simplicial approximation is immediate from the definitions, when observing where the vertices of the domain are mapped.

Motivated in using the barycentric subdivision for finding a simplicial approximation, we present the simplicial approximation theorem.

Theorem 3.27 (Simplicial approximation theorem)**.** Let K and L be simplicial complexes and let $f\colon |K| \to |L|$ be continuous. There exists an integer $n \geq 1$ and a simplicial approximation $\psi\colon \mathrm{Sd}^n(K) \to L$ to f.

Proof. Let $\mathcal{U} = \{\mathrm{St}(q) : q \in L^{(0)}\}$ be the set of stars of vertices of L. By Lemma 3.22, $\mathcal{U}$ is an open cover of $|L|$. Because f is continuous we get that $\mathcal{V} = \{f^{-1}(\mathrm{St}(q)) : q \in L^{(0)}\}$ is an open cover of $|K|$. Let's subdivide K in such a way that each star in the subdivision of K is entirely contained in at least one member of the cover $\mathcal{V}$. Lemmas 3.9 and 3.11 state that $|K|$ is a compact metric space. Thus the Lebesgue's number Lemma gives us a real number $\delta > 0$ such that any subset of $|K|$ with diameter less than δ, is contained in at least one of the elements of $\mathcal{V}$. By Lemma 3.15, for large enought n we have $\mathrm{mesh}(\mathrm{Sd}^n(K)) < \delta/2$. Lemma 3.24 implies $\mathrm{diam}(\mathrm{St}(p)) < \delta$ for every

$p \in \mathrm{Sd}^n(K)^{(0)}$. Thus the Lebesgue's number Lemma implies that for every vertex $p \in \mathrm{Sd}^q(K)^{(0)}$ there exists $q \in L^{(0)}$ such that

$$\mathrm{St}(p) \subset f^{-1}\big(\mathrm{St}(q)\big).$$

Define $\psi \colon \mathrm{Sd}^q(K)^{(0)} \to L^{(0)}$ as $\psi(p) = q$ where $q \in L^{(0)}$ and such that the above inclusion applies. The Lebesgue's number Lemma guarantees that such $q \in L^{(0)}$ exists but there can be many vertices of L for which this is true. We define ψ by choosing one such vertex of L for which the equation holds. We will show that ψ is a simplicial approximation to f. For all $p \in K^{(0)}$ we have

$$f(\mathrm{St}(p)) \subset \mathrm{St}(q) = \mathrm{St}(\psi(p)).$$

Thus the inclusion requirement is satisfied and it remains to show that ψ is a simplicial map. Assume $\{p_0, \dots, p_m\}$ spans a simplex of K. Lemma 3.23 implies that

$$\bigcap_{i=0}^{m} \mathrm{St}(p_i) \neq \emptyset.$$

Thus

$$\emptyset \neq f\left(\bigcap_{i=0}^{m} \mathrm{St}(p_i)\right) \subset \bigcap_{i=0}^{m} f\big(\mathrm{St}(p_i)\big) \subset \bigcap_{i=0}^{m} \mathrm{St}\big(\psi(p)\big).$$

It follows from Lemma 3.23 that $\{\psi(p_0), \dots, \psi(p_m)\}$ spans a simplex of L. Hence ψ is a simplicial map. $\square$

Theorem 3.28. If $\psi \colon K^{(0)} \to L^{(0)}$ is a simplicial approximation to $f \colon |K| \to |L|$, then $|\psi|$ is homotopic to f.

Proof. Let $x \in |K|$. Thus $x \in \mathrm{Int}(\sigma)$ for some unique $\sigma = [p_0, \dots, p_m] \in K$ and $f(x) \in \mathrm{Int}(\tau)$ for some unique $\tau \in L$. We show that $|\psi|(x) \in \tau$. If τ is a vertex of L, then $f(x) = \tau = \psi(\sigma) = |\psi|(x)$. Suppose that τ is not a vertex of L. Notice that $x \in \mathrm{St}(p_i)$ for all $i \in \{0, \dots, m\}$. Because ψ is a simplicial approximation to f, we get

$$f(x) \in f\big(\mathrm{St}(p_i)\big) \subset \mathrm{St}\big(\psi(p_i)\big)$$

for all $i \in \{0, \dots, m\}$. Thus $f(x)$ is in the interior of a simplex that has $\{\psi(p_0), \dots, \psi(p_m)\}$ as vertices. Because τ was the unique simplex for which $f(x) \in \mathrm{Int}(\tau)$ it must be that $\psi(p_i) \in \mathrm{Vert}(\tau)$ for all $i \in \{0, \dots, m\}$. Thus $\{\psi(p_0), \dots, \psi(p_m)\}$ span a face of τ. Now by Lemma 2.15 we can write $x = \sum_{i=0}^{m} t_i p_i$ where $t_i > 0$ and $\sum_{i=0}^{m} t_i = 1$ and hence

$$|\psi|(x) = \sum_{i=0}^{m} t_i \psi(p_i) \in \tau.$$

Thus each $x \in |K|$ is mapped by f to an interior of a simplex τ and $|\psi|$ maps x to the simplex τ. Define $F\colon |L| \times I \to |L|$ as

$$F(x,t) = tf(x) + (1-t)|\psi|(x).$$

The homotopy F is well defined because $f(x)$ and $|\psi|(x)$ are both in the same simplex for all $x \in |K|$ and so the line segment between $f(x)$ and $|\psi|(x)$ is also in the same simplex. Now F is continuous as a product and sum of continuous functions. Last, note that

$$\begin{aligned} F(x,0) &= |\psi|(x) \\ F(x,1) &= f(x). \end{aligned}$$

Thus we have shown that $|\psi|$ and f are homotopic. □

4 Homology

4.1 Simplicial homology

In this section we will be working with the diagram

$$\mathscr{K} \xrightarrow{C_*} \mathbf{Comp} \xrightarrow{H_n} \mathbf{Ab}.$$

In the diagram we have the category of simplicial complexes $\mathscr{K}$. The objects are simplicial complexes and morphisms are simplicial maps.

The rest of this section we focus on the functors C_* and H_n.

Definition 4.1. Let K be an oriented simplicial complex and $n \geq 0$ be an integer. Define $C_n(K)$ to be the abelian group with the following presentation:

- Generators

  All $n+1$-tuples $(p_0, \ldots, p_n)$ such that every $p_i \in K^{(0)}$ and $\{p_0, \ldots, p_n\}$ spans a simplex of K.

- Relations

  (1) If a vertex is repeated i.e. $p_i = p_j$ for some $i \neq j$, then

  $$(p_0, \ldots, p_n) = 0.$$

  (2) If π is a permutation of $\{0, \ldots, n\}$, then

  $$(p_0, \ldots, p_n) - \operatorname{sgn}(\pi)(p_{\pi(0)}, \ldots, p_{\pi(n)}) = 0.$$

The elements of $C_n(K)$ are called n-chains and $C_n(K)$ is called the group of n-chains.

Let's introduce some notation. Instead of denoting an n-chain corresponding to an $n+1$-tuple by $(p_0, \ldots, p_n)$, we use the notation $\langle p_0, \ldots, p_n \rangle$. Also for simplexes $\sigma = [p_0, \ldots, p_n]$ we assign $\langle \sigma \rangle = \langle p_0, \ldots, p_n \rangle$. Since we defined each $C_n(K)$ to be abelian, we will use additive notation to describe n-chains. Thus instead of writing an element as $\langle \sigma_1 \rangle^2 \langle \sigma_2 \rangle \langle \sigma_3 \rangle^{-3}$, we write $2\langle \sigma_1 \rangle + \langle \sigma_2 \rangle - 3\langle \sigma_3 \rangle$.

Lemma 4.2. Let K be an oriented simplicial complex with $m = \dim(K)$. For $0 \leq n \leq m$, the group of n-chains $C_n(K)$ is a free abelian group with a basis set consisting of $n+1$-tuples $(p_0, \ldots p_n)$ such that $[p_0, \ldots, p_n]$ is an oriented n-simplex of K. The group of n-chains is trivial for $n > m$.

Proof. Let $n \geq 0$. Define B to be the set containing all $n+1$-tuples $(p_0, \ldots, p_n) \subset K^{(0)}$ such that $\{p_0, \ldots, p_n\}$ spans a simplex of K. Let $G_n = \langle B \rangle$. Denote by R_n the set of words consisting of the relators in definition 4.1. Note $R_n \subset G_n$. Thus we have $C_n(K) \cong G_n / N(R_n)$, where $N(R_n)$ is the normal closure of R_n. Let's first prove that $N(R_n) = R_n$. It suffices to show that R_n is a subgroup of

G_n, since every subgroup of an abelian group is normal. If π is the identity permutation of $\{0,\dots,n\}$ and $\{p_0,\dots,p_n\}$ spans an n-simplex of K, then we have $0 = (p_0,\dots,p_n) - (p_0,\dots,p_n) = (p_0,\dots,p_n) - \operatorname{sgn}(\pi)(p_{\pi(0)},\dots,p_{\pi(n)}) \in R_n$. Thus R_n contains the neutral element. Let $n_1\sigma_1 + \cdots + n_k\sigma_k \in R_n$. Thus $-n_k\sigma_k - \cdots - n_1\sigma_1 \in R_n$ and $n_1\sigma_1 + \cdots + n_k\sigma_k - n_k\sigma_k - \cdots - n_1\sigma_1 = 0$. Therefore, each element of R_n has an inverse. Last, let $w_1, w_2 \in R_n$. Now w_1w_2 consists of only the relators presented in definition 4.1. Thus $w_1w_2 \in R_n$. Combining the results, we conclude that R_n is a normal subgroup of G_n. Thus, we have $C_n(K) \cong G_n/R_n$.

Let $0 \le n \le m$ and let's define a new basis for G_n. Let B_1 consist of $n+1$-tuples $(p_0,\dots,p_n)$ such that $[p_0,\dots,p_n]$ is an oriented n-simplex of K, B_2 consists of $n+1$-tuples with repeated vertices and B_3 consists of the elements $(p_0,\dots,p_n) - \operatorname{sgn}(\pi)(p_{\pi(0)},\dots,p_{\pi(n)})$, where π is a permutation of $\{0,\dots,n\}$ and $\{p_0,\dots,p_n\}$ spans an n-simplex of K. There is a simple one-to-one correspondence between B and $B_1 \cup B_2 \cup B_3$

$$(p_0,\dots p_n) \leftrightarrow \begin{cases} (p_0,\dots,p_n), \text{if } [p_0,\dots,p_n] \text{ is an oriented } n\text{-simplex of } K \\ (p_0,\dots,p_n), \text{if } p_i = p_j, \text{ for some } i \neq j \\ (p_{\pi(0)},\dots,p_{\pi(n)}) - \operatorname{sgn}(\pi)(p_0,\dots,p_n), \text{otherwise.} \end{cases}$$

This implies that there exists a isomorphism such that $G_n \cong \langle B_1\rangle \oplus \langle B_2\rangle \oplus \langle B_3\rangle$. We have $R_n = \langle B_2\rangle \oplus \langle B_3\rangle$. With the help of Lemma 2.10, we get

$$C_n(K) \cong G_n/R_n \cong \frac{\langle B_1\rangle \oplus \langle B_2\rangle \oplus \langle B_3\rangle}{\langle B_2\rangle \oplus \langle B_3\rangle} \cong \frac{\langle B_1\rangle}{0} \oplus \frac{\langle B_2\rangle}{\langle B_2\rangle} \oplus \frac{\langle B_3\rangle}{\langle B_3\rangle} \cong \langle B_1\rangle.$$

For $n > m$, every $n+1$-tuple $(p_0,\dots,p_n) \subset K^{(0)}$ consists of at least one repeated vertex. Thus $C_n(K) \cong G_n/R_n = G_n/G_n = 0$. □

Lemma 4.3. For every integer $n \ge 0$, the group of simplicial n-chains is finitely generated.

Proof. Every simplicial complex consists of only finitely many simplexes. Hence the basis is finite which implies that the group of n-chains is finitely generated. □

Definition 4.4. Let K be an oriented simplicial complex and $\dim(K) = m$. For $1 \le n \le m$, we define the boundary operator $\partial_n \colon C_n(K) \to C_{n-1}(K)$ for basis elements as

$$\partial_n^K(\langle p_0,\dots,p_n\rangle) = \sum_{i=0}^{n} (-1)^i \langle p_0,\dots,\hat{p}_i,\dots,p_n\rangle$$

and extend by linearity. For $n > m$, the boundary operator ∂_n^K is the trivial homomorphism.

Remark. We leave the sub- and superscript out of ∂_n^K if there is no confusion about the domain.

Definition 4.5. If K is an oriented simplicial complex, then we define the simplicial chain complex of K to be the sequence

$$\dots \xrightarrow{\partial} C_n(K) \xrightarrow{\partial} C_{n-1}(K) \xrightarrow{\partial} \dots \xrightarrow{\partial} C_0(K) \longrightarrow 0.$$

In short, we denote it as $(C_*(K), \partial)$ or just $C_*(K)$.

Lemma 4.6. If K is an oriented simplicial complex, then the simplicial chain complex $C_*(K)$ is a chain complex.

Proof. Every $C_n(K)$ is an abelian group. The fact that the boundary operators are homomorphisms, follows directly from the definition, since we gave the formula of ∂ for basis elements and we extended the map by linearity. Hence we need to only prove that $\partial\partial = 0$. Let's go through the three different cases separately. First, $0\partial_1 = 0$. For the second case, let $n > \dim(K)$ be an integer. Thus ∂_n is the trivial homomorphism and we conclude that $\partial_{n-1}\partial_n = 0$. For the last case, suppose $1 \leq n \leq \dim(K)$ is an integer. Directly calculating for a basis element $\langle p_0, \dots, p_n \rangle$, we get

$$\begin{aligned}\partial\partial\langle p_0, \dots, p_n\rangle &= \partial\Bigg[\sum_{i=0}^{n}(-1)^i\langle p_0, \dots, \hat{p}_i, \dots, p_n\rangle\Bigg] \\ &= \sum_{i=0}^{n}(-1)^i\Bigg[\sum_{j=0}^{i-1}(-1)^j\langle p_0, \dots, \hat{p}_j, \dots, \hat{p}_i, \dots, p_n\rangle \\ &\qquad + \sum_{j=i}^{n}(-1)^j\langle p_0, \dots, \hat{p}_i, \dots, \hat{p}_j, \dots, p_n\rangle\Bigg].\end{aligned}$$

Let's show that the terms cancel out in the brackets. Fix the integers $0 \leq i < j \leq n$. Now there are two ways to end up with $\langle p_0, \dots, \hat{p}_i, \dots, \hat{p}_j, \dots, p_n\rangle$: we can first remove p_i and after that p_j or we could first remove p_j and after that p_i. Let's focus on the coefficients of these cases separately. If we first remove p_i, then the index of p_j is actually $j-1$ because $i < j$. Thus for this case we have the coefficient $(-1)^i(-1)^{j-1}$. On the other hand if we remove p_j first, then the index of p_i stays the same and hence the coefficient is $(-1)^j(-1)^i$. Focusing only on the terms in the sum where we removed p_i and p_j, we have

$$\begin{aligned}&(-1)^i(-1)^{j-1}\langle p_0, \dots, \hat{p}_i, \dots, \hat{p}_j, \dots, p_n\rangle \\ &+ (-1)^j(-1)^i\langle p_0, \dots, \hat{p}_i, \dots, \hat{p}_j, \dots, p_n\rangle \\ =&(-1)^i(-1)^j((-1)^{-1}+1)\langle p_0, \dots, \hat{p}_i, \dots, \hat{p}_j, \dots, p_n\rangle = 0.\end{aligned}$$

The case $i > j$ is shown similarly. Because i and j were arbitrary, this applies to all the terms. Thus $\partial\partial = 0$. □

Notice that if K is an oriented simplicial complex with $n = \dim(K)$, then chain complex can be reduced to the sequence

$$0 \longrightarrow C_n(K) \xrightarrow{\partial} C_{n-1}(K) \xrightarrow{\partial} \dots \xrightarrow{\partial} C_1(K) \xrightarrow{\partial} C_0(K) \longrightarrow 0.$$

Definition 4.7. Let K and L be oriented simplicial complexes with $\dim(K) = m$. If $\psi\colon K^{(0)} \to L^{(0)}$ is a simplicial map, then for each integer $0 \leq n \leq m$ we define $\psi_\#\colon C_n(K) \to C_n(L)$ for basis elements by

$$\psi_\#(\langle p_0, \ldots, p_n \rangle) = \langle \psi(p_0), \ldots, \psi(p_n) \rangle$$

and extend by linearity. For the cases $n = -1$ and $n > m$, we define $\psi_\#\colon C_n(K) \to C_n(L)$ to be the trivial map.

Lemma 4.8. If $\psi\colon K^{(0)} \to L^{(0)}$ is a simplicial map, then $\psi_\#\colon C_*(K) \to C_*(L)$ is a chain map.

Proof. We need to show $\psi_\# \partial_n^K = \partial_n^L \psi_\#$ for all $n \geq 0$. Since both ∂_n and $\psi_\#$ are extended by linearity, it suffices to show the result for only an arbitrary basis element $\langle p_0, \ldots, p_n \rangle$. With direct calculations, we get

$$\begin{aligned} \psi_\# \partial \langle p_0, \ldots, p_m \rangle &= \psi_\# \sum_{i=0}^{m} (-1)^i \langle p_0, \ldots, \hat{p}_i, \ldots, p_m \rangle \\ &= \sum_{i=0}^{m} (-1)^i \langle \psi(p_0), \ldots, \widehat{p_i}, \ldots, \psi(p_m) \rangle \\ &= \sum_{i=0}^{m} (-1)^i \langle \psi(p_0), \ldots, \widehat{\psi(p_i)}, \ldots, \psi(p_m) \rangle \\ &= \partial \langle \psi(p_0), \ldots, \psi(p_m) \rangle \\ &= \partial \psi_\# \langle p_0, \ldots, p_m \rangle. \end{aligned}$$

□

Theorem 4.9. We have a functor $C_*\colon \mathscr{K} \to \mathbf{Comp}$.

Proof. For each simplicial complex $K \in \mathrm{obj}(\mathscr{K})$, we have shown in Lemma 4.6 that $C_*(K)$ is a chain complex. Lemma 4.8 states that for each simplicial map $\psi\colon K^{(0)} \to L^{(0)}$, the map $C_*(\psi) := \psi_\#$ is a chain map. For the simplicial map $\mathrm{id}\colon K^{(0)} \to L^{(0)}$, we get

$$\mathrm{id}_\# \sum_i n_i \langle \sigma_i \rangle = \sum_i n_i \langle \sigma_i \rangle.$$

Thus the identity morphism is preserved. Let $\psi\colon K^{(0)} \to L^{(0)}$ and $\phi\colon L^{(0)} \to M^{(0)}$ be simplicial maps. For a basis element, we can calculate

$$\begin{aligned} (\phi\psi)_\# \langle p_0, \ldots, p_n \rangle &= \langle \phi\psi p_0, \ldots, \phi\psi p_n \rangle \\ &= \phi_\# \langle \psi p_0, \ldots, \psi p_n \rangle \\ &= \phi_\# \psi_\# \langle p_0, \ldots, p_n \rangle \end{aligned}$$

Since the chain maps are extended by linearity, we conclude that the composition property of a functor is satisfied. Thus C_* is a functor. □

Definition 4.10. Let K be an oriented simplicial complex. The group of simplicial n-cycles is $Z_n(K) = \ker \partial_n$, the group of simplicial n-boundaries is $B_n(K) = \operatorname{im} \partial_{n+1}$ and the nth simplicial homology group is

$$H_n(K) = Z_n(K)/B_n(K).$$

Remark. The quotient $Z_n(K)/B_n(K)$ is well defined for all integers $n \geq 0$, since $\partial_{n+1}\partial_n = 0$ implies $B_n(K) \subset Z_n(K)$.

Lemma 4.11. If K is an oriented simplicial complex, then every $H_n(K)$ is finitely generated.

Proof. By Lemma 4.3 the simplicial chain complexes are finitely generated. Thus $Z_n(K)$ is finitely generated and it's quotient $Z_n(K)/B_n(K) = H_n(K)$ is finitely generated. □

Since the homology groups of simplicial complexes are finitely generated, the fundamental theorem of finitely generated abelian groups classifies these groups to be of the form

$$\mathbb{Z} \oplus \ldots \mathbb{Z} \oplus \mathbb{Z}_{p_1^{k_1}} \oplus \cdots \oplus \mathbb{Z}_{p_n^{k_n}}.$$

We call the rank of the free part the Betti number and $p_i^{k_i}$ are the torsion coefficients.

In singular homology we have the functor $H_n\colon \mathbf{Top} \to \mathbf{Ab}$ which is essentially a composition $\mathbf{Top} \to \mathbf{Comp} \to \mathbf{Ab}$. We use a similar functor for simplicial complexes. All of these functors are denoted by H_n but the domain will always be mentioned so that we know which functor we are working with.

Theorem 4.12. For each $n \geq 0$, $H_n\colon \mathscr{K} \to \mathbf{Ab}$ is a functor.

Proof. For objects K of $\mathscr{K}$ we assign the abelian group $H_n(K)$ as in definition 4.10. If $\psi\colon K^{(0)} \to L^{(0)}$ is a simplicial map, then we define

$$\psi_* = H_n(\psi)\colon H_n(K) \to H_n(L)$$

by

$$z + B_n(K) \mapsto \psi_\#(z) + B_n(L),$$

where $z \in Z_n(K)$. Let's make sure that ψ_* is well defined. First let's show that the image of ψ_* is in $H_n(L)$. Take $z \in Z_n(K)$ i.e. $\partial z = 0$. Using Lemma 4.8 we get

$$\partial \psi_\# z = \psi_\# \partial z = \psi_\# 0 = 0.$$

Thus $\psi_\# z \in Z_n(L)$ and we conclude $\psi_\#(Z_n(K)) \subset Z_n(L)$. Let $b \in B_n(K)$. There exists some $c \in C_{n+1}(K)$ such that $b = \partial c$. Again using Lemma 4.8 we get

$$\psi_\# b = \psi_\# \partial c = \partial \psi_\# c \in C_n(L).$$

Hence $\psi_\# b \in B_n(L)$ and we conclude $\psi_\#(B_n(K)) \subset B_n(L)$. Let's show that ψ_* is independent on the choice of representative of $B_n(K)$. Let $z \in Z_n(K)$ and $b \in B_n(K)$. We can calculate

$$\begin{aligned}\psi_*(z+b+B_n(L)) &= \psi_\#(z+b)+B_n(L)\\ &= \psi_\#(z)+\psi_\#(b)+B_n(L)\\ &= \psi_\#(z)+B_n(L)\\ &= \psi_*(z+B_n(L)).\end{aligned}$$

Thus ψ_* is well defined.

Let's show that $\psi_*\colon H_n(K) \to H_n(L)$ is a homomorphism. Fix the elements $z + B_n(K), w + B_n(K) \in H_n(K)$. With direct calculations, we get

$$\begin{aligned}\psi_*(z+B_n(K)+w+B_n(K)) &= \psi_*(z+w+B_n(K))\\ &= \psi_\#(z+w)+B_n(L)\\ &= \psi_\#(z)+\psi_\#(w)+B_n(L)\\ &= \psi_\#(z)+B_n(L)+\psi_\#(w)+B_n(L)\\ &= \psi_*(z+B_n(K))+\psi_*(w+B_n(K)).\end{aligned}$$

Since 0 is the identity element of $C_n(K)$, it follows that $0 + B_n(K) = B_n(K)$ is the identity element of $H_n(K)$. Because $\psi_\#$ is a homomorphism, we get

$$\psi_*(B_n(K)) = \psi_\#(0) + B_n(L) = 0 + B_n(L).$$

Hence ψ_* is a homomorphism.

Let's show that H_n preserves the identity mapping. Let K be an oriented simplicial complex. For the simplicial map $\mathrm{id}\colon K^{(0)} \to L^{(0)}$, we have

$$\mathrm{id}_*(z + B_n(K)) = \mathrm{id}_\#(z) + B_n(K) = z + B_n(K),$$

where $z \in Z_n(K)$. For the composition property, let $\psi\colon K^{(0)} \to L^{(0)}$ and $\phi\colon L^{(0)} \to M^{(0)}$ be simplicial maps. Directly calculating, we get

$$\begin{aligned}(\phi\psi)_*(z+B_n(K)) &= (\phi\psi)_\#(z)+B_n(M)\\ &= \phi_\#\psi_\#(z)+B_n(M)\\ &= \phi_*(\psi_\#(z)+B_n(L))\\ &= \phi_*\psi_*(z+B_n(K)).\end{aligned}$$

Combining everything, we conclude that H_n is a functor. □

4.2 Reduced simplicial homology

In this section we discuss about reduced simplicial homology groups which are similar constructs as the reduced singular homology groups.

Definition 4.13. Let K be an oriented simplicial complex. Let $\widetilde{C}_{-1}(K)$ be the infinite cyclic group generated by the symbol $\langle\ \rangle$ and define $\widetilde{\partial}_0 \colon C_0(K) \to \widetilde{C}_{-1}(K)$ by

$$\widetilde{\partial}_0\Big(\sum_{p \in K^{(0)}} n_p \langle p \rangle\Big) = \Big(\sum_{p \in K^{(0)}} n_p\Big)\langle\ \rangle.$$

The augmented simplicial chain complex $(\widetilde{C}_*(K), \widetilde{\partial})$ is the sequence

$$\dots \longrightarrow \widetilde{C}_n(K) \xrightarrow{\widetilde{\partial}_n} \widetilde{C}_{n-1}(K) \xrightarrow{\widetilde{\partial}_{n-1}} \dots \xrightarrow{\widetilde{\partial}_1} \widetilde{C}_0(K) \xrightarrow{\widetilde{\partial}_0} \widetilde{C}_{-1}(K) \longrightarrow 0,$$

where $\widetilde{\partial}_n = \partial_n$ for $n \geq 1$ and $\widetilde{C}_n(K) = C_n(K)$ for $n \geq 0$.

Remark. Since most of the groups and homomorphisms are the same as in the simplicial chain complex, we can write the sequence of augmented simplicial chain complex as

$$\dots \longrightarrow C_n(K) \xrightarrow{\partial} C_{n-1}(K) \xrightarrow{\partial} \dots \xrightarrow{\partial} C_0(K) \xrightarrow{\widetilde{\partial}_0} \widetilde{C}_{-1}(K) \longrightarrow 0.$$

With the augmented simplicial chain complex, we allow the simplicial complex to be the empty set. With this in mind, we can interpret the emptyset to be the unique -1-simplex, which also acts as a generator for $\widetilde{C}_{-1}$.

Lemma 4.14. The augmented simplicial chain complex of an oriented simplicial complex K is a chain complex.

Proof. For a basis element $\langle p_0, p_1 \rangle \in C_1(K)$, we have

$$\widetilde{\partial}_0 \partial_1 \langle p_0, p_1 \rangle = \widetilde{\partial}_0(\langle p_1 \rangle - \langle p_0 \rangle) = \langle\ \rangle - \langle\ \rangle = 0.$$

Rest of the cases are proved in Lemma 4.6 □

Definition 4.15. If K is an oriented simplicial complex, then the reduced simplicial homology groups are defined for every integer $n \geq 0$ as

$$\widetilde{H}_n(K) = H_n(\widetilde{C}_*(K)).$$

Now that we have constructed the reduced simplicial homology groups, we show the connection between them and the simplicial homology groups.

Lemma 4.16. If K is a nonempty oriented simplicial complex, then

$$H_n(K) \cong \begin{cases} \widetilde{H}_n(K), & \text{if } n \geq 1 \\ \widetilde{H}_0(K) \oplus \mathbb{Z}, & \text{if } n = 0. \end{cases}$$

Proof. For $n \geq 1$ we directly get

$$H_n(K) = \ker \partial_n / \operatorname{im} \partial_{n+1} = \ker \widetilde{\partial}_n / \operatorname{im} \widetilde{\partial}_{n+1} = \widetilde{H}_n(K).$$

For the case $n = 0$, let's show that we have a split exact sequence

$$0 \longrightarrow \ker \widetilde{\partial}_0 \xrightarrow{\ i\ } \ker \partial_0 \xrightarrow{\ \widetilde{\partial}_0\ } \widetilde{C}_{-1}(K) \longrightarrow 0.$$

Since $\ker \widetilde{\partial}_0 \subset \ker \partial_0 = C_0(K)$, we assign i to be the inclusion homomorphism. Let's prove that the sequence is exact. Since $0 \to \widetilde{H}_0(K)$ is the trivial homomorphism and i is injective, we have

$$\operatorname{im}(0 \to \ker \widetilde{\partial}_0) = 0 = \ker i.$$

Since i is an inclusion, we have $\operatorname{im} i = \ker \widetilde{\partial}_0$. Last, because K is not empty, there exists $[p] \in K$. Thus if $n\langle\ \rangle \in \widetilde{C}_{-1}(K)$, then $\widetilde{\partial}_0(n\langle p\rangle) = n\langle\ \rangle$. Therefore $\widetilde{\partial}_0$ is a surjection and we have

$$\operatorname{im} \widetilde{\partial}_0 = \widetilde{C}_{-1}(K) = \ker(\widetilde{C}_{-1}(K) \to 0).$$

Thus the sequence is exact.

Let's show that the seqeuence splits. Let $[p] \in K$. Define $s\colon \widetilde{C}_{-1}(K) \to \ker \partial_0$ by $\langle\ \rangle \mapsto \langle p\rangle$ and extending by linearity. If $n\langle\ \rangle \in \widetilde{C}_{-1}(K)$, then

$$\widetilde{\partial}_0 s(n\langle\ \rangle) = \widetilde{\partial}_0(n\langle p\rangle) = n\langle\ \rangle.$$

Thus the sequence splits.

Last, with the help of Lemma 2.10, we get

$$\begin{aligned} H_0(K) = \ker \partial_0 / \operatorname{im} \partial_1 &\cong \frac{\ker \widetilde{\partial}_0 \oplus \operatorname{im} \widetilde{\partial}_0}{\operatorname{im} \partial_1} \cong \frac{\ker \widetilde{\partial}_0 \oplus \operatorname{im} \widetilde{\partial}_0}{\operatorname{im} \partial_1 \oplus 0} \\ &\cong \frac{\ker \widetilde{\partial}_0}{\operatorname{im} \widetilde{\partial}_1} \oplus \frac{\operatorname{im} \widetilde{\partial}_0}{0} \\ &\cong \widetilde{H}_0(K) \oplus \mathbb{Z}. \end{aligned}$$

□

The previous Lemma implies that the zeroth homology group is never trivial because there is the extra $\mathbb{Z}$. Since every set contains the empty set, this extra $\mathbb{Z}$ can be interpreted to be generated by the empty set.

Lemma 4.17. Let $\sigma = [p_0, \dots, p_m]$ be an oriented m-simplex and let K be the oriented simplicial complex consisting of the faces of σ. We have

$$\widetilde{H}_n(K) = 0$$

for all integers $n \geq 0$.

Proof. Let's find a contracting homotopy

$$h = \{h_n \colon \widetilde{C}_n(K) \to \widetilde{C}_{n+1}(K) | n \geq -2\}$$

such that

$$\widetilde{\partial}_{n+1} h_n + h_{n-1} \widetilde{\partial}_n = \mathrm{id}_{\widetilde{C}_n(K)}$$

for all $n \geq -1$. Define $h_{-2} \colon 0 \to C_{-1}(K)$ to be the trivial homomorphism. Next, define $h_{-1} \colon C_{-1}(K) \to C_0(K)$ for the basis element as $\langle\ \rangle \mapsto \langle p_0 \rangle$ and extending by linearity. For $n \geq 0$, define $h_n \colon \widetilde{C}_n(K) \to \widetilde{C}_{n+1}(K)$ for basis elements as $\langle p_{i_0}, \ldots, p_{i_n} \rangle \mapsto \langle p_0, p_{i_0}, \ldots, p_{i_n} \rangle$ and extending by linearity.

Let's show that each homomorphism of h is a contracting homomorphism. Because the homomorphisms are extended by linearity, we need to show the result for only the basis elements. For the case -1, we have

$$\widetilde{\partial}_0 h_{-1} \langle\ \rangle + h_{-2} \widetilde{\partial}_{-1} \langle\ \rangle = \widetilde{\partial}_0 \langle p_0 \rangle = \langle\ \rangle = \mathrm{id}_{\widetilde{C}_{-1}(K)} \langle\ \rangle.$$

If $n \geq 0$, then

$$\begin{aligned} h_{n-1} \widetilde{\partial}_n \langle p_{i_0}, \ldots, p_{i_n} \rangle &= h_{n-1} \sum_{j=0}^{n} (-1)^j \langle p_{i_0}, \ldots, \hat{p}_{i_j}, \ldots, p_{i_n} \rangle \\ &= \sum_{j=0}^{n} (-1)^j \langle p_0, p_{i_0}, \ldots, \hat{p}_{i_j}, \ldots, p_{i_n} \rangle \end{aligned}$$

and

$$\begin{aligned} \widetilde{\partial}_{n+1} h_n \langle p_{i_0}, \ldots, p_{i_n} \rangle &= \widetilde{\partial}_{n+1} \langle p_0, p_{i_0}, \ldots, p_{i_n} \rangle \\ &= \sum_{j=0}^{n+1} (-1)^j \langle p_0, p_{i_0}, \ldots, \hat{p}_{i_j}, \ldots, p_{i_n} \rangle \\ &= \langle p_{i_0}, \ldots, p_{i_n} \rangle + \sum_{j=1}^{n+1} (-1)^j \langle p_0, p_{i_0}, \ldots, \hat{p}_{i_j}, \ldots, p_{i_n} \rangle. \end{aligned}$$

Thus

$$h_{n-1} \widetilde{\partial}_n \langle p_{i_0}, \ldots, p_{i_n} \rangle + \widetilde{\partial}_{n+1} h_n \langle p_{i_0}, \ldots, p_{i_n} \rangle = \langle p_{i_0}, \ldots, p_{i_n} \rangle$$

and we conclude that h is a contracting homomorphism. Hence Lemma 2.49 implies that $\widetilde{H}_n(K) = 0$, for all $n \geq 0$. □

4.3 Relative simplicial homology

In this section we introduce the relative simplicial homology groups. These correspond to parts of simplicial complexes to be compressed to a point like in the case of relative singular homology.

Definition 4.18. Define $\mathscr{K}^2$ to be the category of pairs of simplicial complexes. The objects are pairs (K, K'), where K is a simplicial complex and K' is a subcomplex of K. A morphism $(K, K') \to (L, L')$ is a simplicial map $\psi\colon K^{(0)} \to L^{(0)}$ such that $\psi(K'^{(0)}) \subset L'^{(0)}$.

Definition 4.19. Let K be a simplicial complex and K' be a subcomplex of K. The quotient complex $(C_*(K)/C_*(K'), \overline{\partial})$ is the sequence

$$\cdots \longrightarrow \frac{C_n(K)}{C_n(K')} \xrightarrow{\overline{\partial}_n} \frac{C_{n-1}(K)}{C_{n-1}(K')} \xrightarrow{\overline{\partial}_{n-1}} \cdots \xrightarrow{\overline{\partial}_1} \frac{C_0(K)}{C_0(K')} \longrightarrow 0,$$

where $\overline{\partial}_n\colon \gamma + C_n(K') \mapsto \partial_n\gamma + C_{n-1}(K')$.

Remark. Notice that the map $\overline{\partial}_n$ is well defined, since if $\langle p_0, \ldots, p_n\rangle$ is a basis element of $C_n(K')$, then for all integers $0 \leq i \leq n$, we have $\langle p_0, \ldots, \hat{p}_i, \ldots, p_n\rangle \in C_{n-1}(K')$.

Definition 4.20. If K' is a subcomplex of a simplicial complex K, then the nth relative simplicial homology group is defined as

$$H_n(K, K') = H_n(C_*(K)/C_*(K')).$$

If $K' = \{p\}$, where $p \in K^{(0)}$, then we denote the relative homology groups as $H_n(K, p)$.

Remark. We define the relative reduced homology group as

$$\widetilde{H}_n(K, K') = H_n(\widetilde{C}_*(K)/\widetilde{C}_*(K')).$$

Lemma 4.21. If K is a simplicial complex and K' is a nonempty subcomplex of K, then

$$\widetilde{H}_n(K, K') \cong H_n(K, K')$$

for all $n \geq 0$.

Proof. For $n \geq 1$ the boundary maps $\overline{\partial}_n$ and $\overline{\overline{\partial}}_n$ are the same. Hence $\widetilde{H}_n(K, K') \cong H_n(K, K')$ for $n \geq 1$. Consider the ends of the quotient complexes

$$\cdots \xrightarrow{\overline{\partial}_2} \frac{C_1(K)}{C_1(K')} \xrightarrow{\overline{\partial}_1} \frac{C_0(K)}{C_0(K')} \longrightarrow 0$$

and

$$\cdots \xrightarrow{\overline{\partial}_2} \frac{C_1(K)}{C_1(K')} \xrightarrow{\overline{\partial}_1} \frac{C_0(K)}{C_0(K')} \xrightarrow{\overline{\overline{\partial}}_0} \frac{\widetilde{C}_{-1}(K)}{\widetilde{C}_{-1}(K')} \longrightarrow 0.$$

Since $\widetilde{C}_{-1}(K)$ and $\widetilde{C}_{-1}(K')$ have $\langle\,\rangle$ as the only basis element, we deduce that

$$\frac{\widetilde{C}_{-1}(K)}{\widetilde{C}_{-1}(K')} = 0.$$

Thus $\overline{\overline{\partial}}_0 = \overline{\partial}_0$ is the trivial map and it implies $\widetilde{H}_0(K, K') \cong H_0(K, K')$. $\square$

Lemma 4.22. If K is an oriented simplicial complex and $p \in K^{(0)}$, then

$$\widetilde{H}_n(K, p) \cong \widetilde{H}_n(K)$$

for all n.

Proof. Consider the short sequence of chain complexes

$$0 \longrightarrow \widetilde{C}_*(p) \xrightarrow{\ i\ } \widetilde{C}_*(K) \xrightarrow{\ j\ } \widetilde{C}_*(K)/\widetilde{C}_*(p) \longrightarrow 0,$$

where i and j are the natural inclusion maps. Since i is an injection, j is a surjection and $\operatorname{im} i = C_*(p) = \ker j$, the sequence is exact. Thus Theorem 2.40 states that there exists an exact sequence

$$\ldots \longrightarrow \widetilde{H}_n(p) \longrightarrow \widetilde{H}_n(K) \longrightarrow \widetilde{H}_n(K,p) \longrightarrow \widetilde{H}_{n-1}(p) \longrightarrow \ldots$$

$$\ldots \longrightarrow \widetilde{H}_0(p) \longrightarrow \widetilde{H}_0(K) \longrightarrow \widetilde{H}_0(K,p) \longrightarrow 0.$$

Lemma 4.17 states that every $\widetilde{H}_n(p)$ is trivial. Thus every third group in the sequence is trivial. Hence Lemma 2.45 implies $\widetilde{H}_n(K) \cong \widetilde{H}_n(K,p)$ for all n. $\square$

4.4 Comparison to singular homology

In this section we show that the singular homology groups for polyhedrons are the same as the simplicial homology groups of their triangulations. We construct certain Mayer-Vietoris sequences in both simplicial and singular homology. We connect the two using a natural map from simplicial to singular chains. In the end this map induces isomorphisms between the homologies. The main result is proved for reduced homologies and hence we prove all the intermediary results also for reduced homologies.

Lemma 4.23. If K is an oriented simplicial complex with subcomplexes K_1 and K_2 such that $K_1 \cap K_2 \neq \emptyset$, then

$$\widetilde{C}_*(K_1 \cap K_2) = \widetilde{C}_*(K_1) \cap \widetilde{C}_*(K_2).$$

Proof. Let's first show that $K_1 \cap K_2$ is an oriented simplicial complex. If $\sigma \in K_1 \cap K_2$, then by the first condition of a simplicial complex, the proper faces of σ are in K_1 and K_2 as well. Hence the proper faces are in the intersection. If $\sigma, \tau \in K_1 \cap K_2$, then $\sigma, \tau \in K_1$. Because K_1 is a simplicial complex, the intersection $\sigma \cap \tau$ is either empty or a face of both of them. Let the simplexes of $K_1 \cap K_2$ be oriented the same way they were oriented in K_1 or K_2. Thus $K_1 \cap K_2$ is an oriented simplicial complex.

Let's show the inclusion $\widetilde{C}_n(K_1 \cap K_2) \subset \widetilde{C}_n(K_1) \cap \widetilde{C}_n(K_2)$ for $n \geq 0$. Fix a basis element $\langle\sigma\rangle \in \widetilde{C}_n(K_1 \cap K_2)$. Thus $\sigma \in K_1 \cap K_2$. This implies $\langle\sigma\rangle \in C_n(K_1)$

and $\langle\sigma\rangle \in C_n(K_2)$. Hence $\langle\sigma\rangle \in C_n(K_1) \cap C_n(K_2)$. We deduce that every n-chain in $C_n(K_1 \cap K_2)$ is also in $C_n(K_1) \cap C_n(K_2)$. For the reverse inclusion let $\langle\sigma\rangle \in C_n(K_1) \cap C_n(K_2)$ be a basis element. Thus $\sigma \in K_1$ and $\sigma \in K_2$. Therefore $\sigma \in K_1 \cap K_2$ and $\langle\sigma\rangle \in C_n(K_1 \cap K_2)$. This implies that any n-chain in $C_n(K_1) \cap C_n(K_2)$ belongs to $C_n(K_1 \cap K_2)$. For the case $n = -1$, each $\widetilde{C}_{-1}(K_1 \cap K_2), \widetilde{C}_{-1}(K_1)$ and $\widetilde{C}_{-1}(K_2)$ have $\langle\ \rangle$ as the only basis element. Hence the equality is immediate. ☐

Lemma 4.24 (Simplicial excision)**.** Let K_1 and K_2 be subcomplexes of K such that $K = K_1 \cup K_2$. The inclusion $(K_1, K_1 \cap K_2) \hookrightarrow (K, K_2)$ induces isomorphisms

$$\widetilde{H}_n(K_1, K_1 \cap K_2) \xrightarrow{\sim} \widetilde{H}_n(K, K_2),$$

for all $n \geq 0$

Proof. Consider the diagram

$$\frac{\widetilde{C}_n(K_1)}{\widetilde{C}_n(K_1 \cap K_2)} \xrightarrow{\ i\ } \frac{\widetilde{C}_n(K_1)}{\widetilde{C}_n(K_1) \cap \widetilde{C}_n(K_2)} \xrightarrow{\ j\ } \frac{\widetilde{C}_n(K_1) + \widetilde{C}_n(K_2)}{\widetilde{C}_n(K_2)} \xrightarrow{\ k\ } \frac{\widetilde{C}_n(K)}{\widetilde{C}_n(K_2)}.$$

Let's define each map individually from left to right and show that they are inclusions and isomorphisms. Lemma 4.23 implies that $\widetilde{C}_n(K_1) \cap \widetilde{C}_n(K_2) = \widetilde{C}_n(K_1 \cap K_2)$. Hence we let i to be the identity map. Define j to be the isomorphism from the second isomorphism theorem i.e. $j\colon \gamma_1 + \widetilde{C}_n(K_1) \cap \widetilde{C}_n(K_2) \mapsto \gamma_1 + \widetilde{C}_n(K_2)$, where $\gamma_1 \in \widetilde{C}_n(K_1)$. The way j is defined makes it an inclusion map.

Last, let's show that we can define k to be the identity map. Let $\gamma_1 \in \widetilde{C}_n(K_1)$ and $\gamma_2 \in \widetilde{C}_n(K_2)$. Now both γ_1 and γ_2 are in $\widetilde{C}_n(K)$, since each simplex of K_1 or K_2 is a simplex of K. Hence $\widetilde{C}_n(K_1) + \widetilde{C}_n(K_2) \subset \widetilde{C}_n(K)$ and we can define

$$k\colon \gamma_1 + \gamma_2 + \widetilde{C}_n(K_2) = \gamma_1 + \widetilde{C}_n(K_2) \mapsto \gamma_1 + \widetilde{C}_n(K_2),$$

where $\gamma_1 \in \widetilde{C}_n(K_1)$ and $\gamma_2 \in \widetilde{C}_n(K_2)$. Let's show that k is a surjection. Let $\sum_i m_i\langle\sigma_i\rangle + \widetilde{C}_n(K_2) \in \widetilde{C}_n(K)/\widetilde{C}_n(K_2)$. Thus each $\sigma_i \in K_1 \backslash K_2$. We can calculate $k(\sum_i m_i\langle\sigma_i\rangle + \widetilde{C}_n(K_2)) = \sum_i m_i\langle\sigma_i\rangle + \widetilde{C}_n(K_2)$. Hence k is a surjection and an isomorphism.

Combining the results, we get that kji is an inclusion and an isomorphism. Thus $\widetilde{H}_n(kji)\colon \widetilde{H}_n(K_1, K_1 \cap K_2) \xrightarrow{\sim} \widetilde{H}_n(K, K_2)$ for all $n \geq 0$. ☐

Lemma 4.25 (Simplicial Mayer-Vietoris)**.** Let K_1 and K_2 be subcomplexes of a simplicial complex K such that $K_1 \cup K_2 = K$ and $K_1 \cap K_2 \neq \emptyset$. There exists an exact sequence

$$\ldots \longrightarrow \widetilde{H}_n(K_1 \cap K_2) \longrightarrow \widetilde{H}_n(K_1) \oplus \widetilde{H}_n(K_2) \longrightarrow \widetilde{H}_n(K) \longrightarrow \widetilde{H}_{n-1}(K_1 \cap K_2) \longrightarrow \ldots.$$

Proof. Since $K_1 \cap K_2$, there exists a vertex $p \in K_1 \cap K_2$. Consider the commutative diagram

$$\begin{array}{ccccc}
(K_1 \cap K_2, p) & \xrightarrow{\ i\ } & (K_1, p) & \xrightarrow{\ r\ } & (K_1, K_1 \cap K_2) \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\
(K_2, p) & \xrightarrow{\ j\ } & (K, p) & \xrightarrow{\ s\ } & (K, K_2),
\end{array}$$

where all the maps are simplicial maps and inclusions. The diagram induces a commutative diagram of chain complexes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widetilde{C}_*(K_1 \cap K_2)/\widetilde{C}_*(p) & \xrightarrow{i_\#} & \widetilde{C}_*(K_1)/\widetilde{C}_*(p) & \xrightarrow{r_\#} & \widetilde{C}_*(K_1)/\widetilde{C}_*(K_1 \cap K_2) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_\#} & & \downarrow{\scriptstyle g_\#} & & \downarrow{\scriptstyle h_\#} & & \\
0 & \longrightarrow & \widetilde{C}_*(K_2)/\widetilde{C}_*(p) & \xrightarrow{j_\#} & \widetilde{C}_*(K)/\widetilde{C}_*(p) & \xrightarrow{s_\#} & \widetilde{C}_*(K)/\widetilde{C}_*(K_2) & \longrightarrow & 0.
\end{array}$$

The rows are exact which follows directly from the the definition of the maps. Lemma 4.22 and Theorem 2.41 imply that there exists a commutative diagram with exact rows

$$\begin{array}{ccccccccccc}
\ldots & \longrightarrow & \widetilde{H}_n(K_1 \cap K_2) & \xrightarrow{i_*} & \widetilde{H}_n(K_1) & \xrightarrow{r_*} & \widetilde{H}_n(K_1, K_1 \cap K_2) & \xrightarrow{d} & \widetilde{H}_{n-1}(K_1 \cap K_2) & \longrightarrow & \ldots \\
 & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle h_*} & & \downarrow{\scriptstyle f_*} & & \\
\ldots & \longrightarrow & \widetilde{H}_n(K_2) & \xrightarrow{j_*} & \widetilde{H}_n(K) & \xrightarrow{s_*} & \widetilde{H}_n(K, K_2) & \xrightarrow{\delta} & \widetilde{H}_{n-1}(K_2) & \longrightarrow & \ldots.
\end{array}$$

Lemma 4.24 states that every h_* is an isomorphism. Thus, using the Barratt-Whitehead Lemma (Lemma 2.47) we get an exact sequence

$$\ldots \to \widetilde{H}_n(K_1 \cap K_2) \xrightarrow{(i_*, f_*)} \widetilde{H}_n(K_1) \oplus \widetilde{H}_n(K_2) \xrightarrow{g_* - j_*} \widetilde{H}_n(K) \xrightarrow{dh_*^{-1}q_*} \widetilde{H}_{n-1}(K_1 \cap K_2) \to \ldots.$$

□

Lemma 4.26. Let K be an oriented simplicial complex and let $\sigma \in K$ be of highest dimension i.e. $\dim(\sigma) = \dim(K)$. Define $K_1 = K \backslash \{\sigma\}$ and $K_2 = \{\sigma' \preceq \sigma\}$. There exists an exact Mayer-Vietoris sequence in singular homology

$$\ldots \longrightarrow \widetilde{H}_n(|K_1 \cap K_2|) \longrightarrow \widetilde{H}_n(|K_1|) \oplus \widetilde{H}_n(|K_2|) \longrightarrow \widetilde{H}_n(|K|) \longrightarrow \widetilde{H}_{n-1}(|K_1 \cap K_2|) \longrightarrow \ldots.$$

Proof. We will first prove that

$$\widetilde{H}_n(|K_2|, |K_1| \cap |K_2|) \cong \widetilde{H}_n(|K|, |K_1|)$$

for all n. Let $x \in \operatorname{Int}(\sigma)$. Consider the commutative diagram

$$\begin{array}{ccccc}
(|K_1| \cap |K_2|, \emptyset) & \longrightarrow & (|K_2|, \emptyset) & \longrightarrow & (|K_2|, |K_1| \cap |K_2|) \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle f} \\
(|K_2| \backslash \{x\}, \emptyset) & \longrightarrow & (|K_2|, \emptyset) & \longrightarrow & (|K_2|, |K_2| \backslash \{x\}) \\
\downarrow{\scriptstyle j} & & \downarrow{\scriptstyle l} & & \downarrow{\scriptstyle g} \\
(|K| \backslash \{x\}, \emptyset) & \longrightarrow & (|K|, \emptyset) & \longrightarrow & (|K|, |K| \backslash \{x\}) \\
{\scriptstyle k}\uparrow & & {\scriptstyle \mathrm{id}}\uparrow & & {\scriptstyle h}\uparrow \\
(|K_1|, \emptyset) & \longrightarrow & (|K|, \emptyset) & \longrightarrow & (|K|, |K_1|),
\end{array}$$

where all the maps are inclusions. Lemma 2.32 implies that there is a commutative diagram with exact rows

$$\begin{array}{ccccccccccc}
\ldots \longrightarrow & \widetilde{H}_n(|K_1| \cap |K_2|) & \longrightarrow & \widetilde{H}_n(|K_2|) & \longrightarrow & \widetilde{H}_n(|K_2|, |K_1| \cap |K_2|) & \longrightarrow & \widetilde{H}_{n-1}(|K_1| \cap |K_2|) & \longrightarrow \ldots \\
& \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle i_*} & \\
\ldots \longrightarrow & \widetilde{H}_n(|K_2| \backslash \{x\}) & \longrightarrow & \widetilde{H}_n(|K_2|) & \longrightarrow & \widetilde{H}_n(|K_2|, |K_2| \backslash \{x\}) & \longrightarrow & \widetilde{H}_{n-1}(|K_2| \backslash \{x\}) & \longrightarrow \ldots \\
& \downarrow{\scriptstyle j_*} & & \downarrow{\scriptstyle l_*} & & \downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle j_*} & \\
\ldots \longrightarrow & \widetilde{H}_n(|K| \backslash \{x\}) & \longrightarrow & \widetilde{H}_n(|K|) & \longrightarrow & \widetilde{H}_n(|K|, |K| \backslash \{x\}) & \longrightarrow & \widetilde{H}_{n-1}(|K| \backslash \{x\}) & \longrightarrow \ldots \\
& {\scriptstyle k_*}\uparrow & & {\scriptstyle \mathrm{id}}\uparrow & & {\scriptstyle h_*}\uparrow & & {\scriptstyle k_*}\uparrow & \\
\ldots \longrightarrow & \widetilde{H}_n(|K_1|) & \longrightarrow & \widetilde{H}_n(|K|) & \longrightarrow & \widetilde{H}_n(|K|, |K_1|) & \longrightarrow & \widetilde{H}_{n-1}(|K_1|) & \longrightarrow \ldots .
\end{array}$$

Let's show that f_*, g_*, h_* and k_* are isomorphisms starting from f_*. Note that $|K_1| \cap |K_2|$ is a deformation retract of $|K_2| \backslash \{x\}$. Hence the inclusion $i \colon |K_1| \cap |K_2| \hookrightarrow |K_2| \backslash \{x\}$ is a homotopy equivalence and $i_* \colon \widetilde{H}_n(|K_1| \cap |K_2|) \to \widetilde{H}_n(|K_2| \backslash \{x\})$ is an isomorphism. The five Lemma implies that f_* is an isomorphism. Next let's show g_* is an isomorphism. Notice that $|K| = \operatorname{Int}(|K_2|) \cup \operatorname{Int}(|K| \backslash \{x\})$. Theorem 2.36 states that the inclusion $g \colon (|K_2|, |K_2| \backslash \{x\}) \hookrightarrow (|K|, |K| \backslash \{x\})$ induces isomorphisms

$$g_* \colon \widetilde{H}_n(|K_2|, |K_2| \backslash \{x\}) \xrightarrow{\sim} \widetilde{H}_n(|K|, |K| \backslash \{x\})$$

for every n. Last, let's show that h_* and k_* are isomorphisms. Previously we declared that $|K_1| \cap |K_2|$ is a deformation retract of $|K_2| \backslash \{x\}$. Using similar arguments we conclude that $(|K_1|, \emptyset)$ is a deformation retract of $(|K| \backslash \{x\}, \emptyset)$. Thus the inclusion k is a homotopy equivalence and k_* is an isomorphism. The five Lemma implies that h_* is an isomorphism.

Combining the results, we get a commutative diagram

$$\begin{array}{ccccccccccc}
\ldots \longrightarrow & \widetilde{H}_n(|K_1| \cap |K_2|) & \longrightarrow & \widetilde{H}_n(|K_2|) & \longrightarrow & \widetilde{H}_n(|K_2|, |K_1| \cap |K_2|) & \longrightarrow & \widetilde{H}_{n-1}(|K_1| \cap |K_2|) & \longrightarrow \ldots \\
& \downarrow{\scriptstyle k_*^{-1} j_* i_*} & & \downarrow{\scriptstyle l_*} & & \downarrow{\scriptstyle h_*^{-1} g_* f_*} & & \downarrow{\scriptstyle k_*^{-1} j_* i_*} & \\
\ldots \longrightarrow & \widetilde{H}_n(|K_1|) & \longrightarrow & \widetilde{H}_n(|K|) & \longrightarrow & \widetilde{H}_n(|K|, |K_1|) & \longrightarrow & \widetilde{H}_{n-1}(|K_1|) & \longrightarrow \ldots,
\end{array}$$

where the rows are exact and $h_*^{-1}g_*f_*$ are isomorphisms. Thus the Barratt-Whitehead Lemma implies that there exists a Mayer-Vietoris sequence

$$\ldots \longrightarrow \widetilde{H}_n(|K_1 \cap K_2|) \longrightarrow \widetilde{H}_n(|K_1|) \oplus \widetilde{H}_n(|K_2|) \longrightarrow \widetilde{H}_n(|K|) \longrightarrow \widetilde{H}_{n-1}(|K_1 \cap K_2|) \longrightarrow \ldots .$$

□

Lemma 4.27. Let K be an oriented simplicial complex with $\dim(K) = n$. There exists a chain map $j\colon \widetilde{C}_*(K) \to \widetilde{S}_*(|K|)$, where each $j_m\colon \widetilde{C}_m(K) \to \widetilde{S}_m(|K|)$ is an injection.

Proof. Define $j_{-1}\colon C_{-1}(K) \to S_{-1}(|K|)$ for the basis element as $\langle\ \rangle \mapsto [\ ]$ and extend it by linearity. Extending by linearity makes j_{-1} to a homomorphism. Let's show that j_{-1} is an injection. Suppose $j_{-1}(n\langle\ \rangle) = 0$ for some $n \in \mathbb{Z}$. Thus $n[\ ] = 0$ which is true if and only if $n = 0$. Hence the kernel of j_{-1} is the trivial group and we conclude that j_{-1} is an injection.

Let $m \geq 0$ be an integer. Define $j_m\colon C_m(K) \to S_m(|K|)$ for basis elements as $\langle\sigma\rangle = \langle p_0, \ldots, p_m\rangle \mapsto \hat{\sigma}$, where $\hat{\sigma}\colon \Delta^m \to |K|$ is the affine map $\sum t_i e_i \mapsto \sum t_i p_i$, and extend it by linearity. Similarly as before j_m is a homomorphism and to show injectivity we need to show that the kernel is the trivial group. Suppose $j_m(\sum_i n_i\langle\sigma_i\rangle) = 0$ for some $n_i \in \mathbb{Z}$. Thus $\sum_i n_i\hat{\sigma} = 0$ which is true if and only if $n_i = 0$ for all i. Hence the kernel of j_m is the trivial group and j_m is an injection. □

Theorem 4.28. If K is an oriented simplicial complex, then the chain map $j\colon \widetilde{C}_*(K) \to \widetilde{S}_*(|K|)$ induces isomorphisms between the homology groups i.e. for all n,

$$j_*\colon \widetilde{H}_n(K) \xrightarrow{\sim} \widetilde{H}_n(|K|).$$

Proof. We prove the result by induction on the number N of simplexes in K. Since the reduced complex recognizes the empty set as the unique -1-simplex, the simplicial complex consisting of one simplex is $K = \{\emptyset\}$. Thus $C_n(K) = 0$ and therefore $\widetilde{H}_n(K) = 0$ for $n \geq 0$. Similarly, we have $S_n(|K|) = 0$ for $n \geq 0$. Thus $\widetilde{H}_*(K) = 0 = \widetilde{H}_*(|K|)$.

Let $N > 1$. For the inductive hypothesis, assume that the result holds for simplicial complexes with $1, 2, \ldots, N-1$ simplexes. Fix a simplex $\sigma \in K$ of highest dimension i.e. $\dim(\sigma) = \dim(K)$. Define $K_1 = K\setminus\{\sigma\}$ and $K_2 = \{\sigma' \preceq \sigma\}$. Note that both K_1 and K_2 are oriented simplicial subcomplexes in the induced orientation. Lemmas 4.25 and 4.26 give us the Mayer-Vietoris sequences for homologies of simplicial complexes and their underlying spaces. We connect the two using maps induced by j and we end up with the commutative diagram

$$\begin{array}{ccccccccc}
\ldots \longrightarrow & \widetilde{H}_n(K_1 \cap K_2) & \longrightarrow & \widetilde{H}_n(K_1) \oplus \widetilde{H}_n(K_2) & \longrightarrow & \widetilde{H}_n(K) & \longrightarrow & \widetilde{H}_{n-1}(K_1 \cap K_2) & \longrightarrow \ldots \\
& \downarrow & & \downarrow & & \downarrow{\scriptstyle j_*} & & \downarrow & \\
\ldots \longrightarrow & \widetilde{H}_n(|K_1 \cap K_2|) & \longrightarrow & \widetilde{H}_n(|K_1|) \oplus \widetilde{H}_n(|K_2|) & \longrightarrow & \widetilde{H}_n(|K|) & \longrightarrow & \widetilde{H}_{n-1}(|K_1 \cap K_2|) & \longrightarrow \ldots .
\end{array}$$

In order to show that $j_*\colon \widetilde{H}_n(K) \xrightarrow{\sim} \widetilde{H}_n(|K|)$, we will show that every other vertical map is an isomorphism in the above diagram. First, $K_1 \cap K_2$ contains less than N simplexes. Hence the inductive hypothesis implies

$$\widetilde{H}_*(K_1 \cap K_2) \xrightarrow{\sim} \widetilde{H}_*(|K_1 \cap K_2|).$$

Now the other isomorphism. Because K_2 consists of the faces of a simplex, Lemma 4.17 implies

$$\widetilde{H}_*(K_1) \oplus \underbrace{\widetilde{H}_*(K_2)}_{0} \cong \widetilde{H}_*(K_1).$$

Since $|K_2| = \sigma$, Lemma 2.33 implies that

$$\widetilde{H}_*(|K_1|) \oplus \underbrace{\widetilde{H}_*(|K_2|)}_{0} \cong \widetilde{H}_*(|K_1|).$$

Since K_1 contains $N-1$ simplexes, the inductive hypothesis implies

$$\widetilde{H}_*(K_1) \xrightarrow{\sim} \widetilde{H}_*(|K_1|).$$

Hence the five Lemma implies that $\widetilde{H}_*(K) \xrightarrow{\sim} \widetilde{H}_*(|K|)$. □

Corollary 4.29. If K is an oriented simplicial complex, then

$$H_*(K) \cong H_*(|K|).$$

Proof. The claim is immediate for $n \geq 1$. For $n = 0$, using Lemma 2.33, we get

$$H_0(K) \cong \widetilde{H}_0(K) \oplus \mathbb{Z} \cong \widetilde{H}_0(|K|) \oplus \mathbb{Z} \cong H_0(|K|).$$

□

Corollary 4.30 (Alexander-Veblen)**.** If X is a polyhedron with triangulations K and K', then $H_*(K) \cong H_*(K')$.

Proof. By Theorem 4.28, we have

$$H_*(K) \cong H_*(|K|) \cong H_*(K').$$

□

4.5 Calculating homology groups

Since simplicial homology groups and singular homology groups coincide, it suffices to calculate the simplicial homology groups. This is a computationally easier task since the simplicial chains are finitely generated and most of the singular chains are not. In this section we introduce a method for calculating the simplicial homology groups. Our goal is to take the boundary homomorphisms and convert them to matrix form. In the matrix form, we apply row and column operators to achieve the Smith normal form. In that form, there is a nice interpretation of cycles, boundaries and weak boundaries. With these it is possible to calculate the Betti numbers and the torsion coefficients. Last we apply the method to calculate the homology groups for some of the polyhedra that were presented in section three.

Definition 4.31. A Smith normal form is an $n \times m$ matrix N over $\mathbb{Z}$ such that

$$N = \left[\begin{array}{ccc|c} b_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & b_k & \\ \hline & 0 & & 0 \\ & & & \end{array}\right],$$

where the top left submatrix is diagonal with every $b_i \geq 1$ and $b_j | b_{j+1}$ for $1 \leq j \leq k-1$. The rest of the submatrices are zero matrices.

Theorem 4.32 (The reduction algorithm)**.** There exists an algorithm which transforms any given non-zero matrix A over $\mathbb{Z}$ to a normal form using elementary row and column operations.

Proof. As the proof we give the algorithm. Suppose our matrix is

$$A = \begin{bmatrix} a_{11} & \dots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nm} \end{bmatrix}.$$

Define the minimal entry of A to be the element which has the smallest non-zero absolute value. In other words if a_{ij} is a minimal entry of A, then $|a_{ij}| \leq |a_{st}|$ for all s, t. Our first goal is to modify the matrix A in such a way that the minimal entry of it divides every non-zero element of A. Suppose a_{ij} is our minimal element that does not divide a non-zero element a_{kj} in its column. Since $\mathbb{Z}$ is an Euclidean domain, when consider $\mathbb{Z}$ as a ring with the standard multiplication (example after definition 3.8 in [11]), there exists $q, r \in \mathbb{Z}$ such that

$$a_{kj} = qa_{ij} + r,$$

where $0 < |r| < |a_{ij}|$. Then we replace the row k by subtracting q times the row i from row k i.e. $a_{kl} = a_{kl} - qa_{il}$ for all $l \in \{1, \dots, m\}$. The point of this is to replace the element a_{kj} by $r = a_{kj} - qa_{ij}$. Since $|r| < |a_{ij}|$, the minimal entry has changed and is at most $|r|$. Next we go back and look for the new minimal entry and make sure that it divides every other element in its column.

$$\begin{bmatrix} a_{i1} & \dots & a_{ij} & \dots & a_{im} \\ \vdots & & \vdots & & \vdots \\ a_{k1} & \dots & a_{kj} & \dots & a_{km} \end{bmatrix} \rightsquigarrow \begin{bmatrix} a_{i1} & \dots & a_{ij} & \dots & a_{im} \\ \vdots & & \vdots & & \vdots \\ a_{k1} - qa_{i1} & \dots & r & \dots & a_{km} - qa_{im} \end{bmatrix}$$

Now that the minimal entry a_{ij} divides every other element in its column, we focus on the row of the minimal entry. If a_{ij} does not divide some element a_{ik} in its row, then we apply the previous solution modified for rows. After the modification, we go back to beginning.

$$\begin{bmatrix} a_{1j} & \dots & a_{1k} \\ \vdots & & \vdots \\ a_{ij} & \dots & a_{ik} \\ \vdots & & \vdots \\ a_{nj} & \dots & a_{nk} \end{bmatrix} \rightsquigarrow \begin{bmatrix} a_{1j} & \dots & a_{1k} - qa_{1j} \\ \vdots & & \vdots \\ a_{ij} & \dots & r \\ \vdots & & \vdots \\ a_{nj} & \dots & a_{nk} - qa_{nj} \end{bmatrix}$$

We have a minimal entry a_{ij} which divides each element in its column and row. Suppose that there is some element a_{st}, where $s \neq i$ and $t \neq j$, which is not divisible by a_{ij}. Since a_{ij} divides a_{sj}, there exists $q, r \in \mathbb{Z}$ such that $a_{sj} = qa_{ij} + r$ with $0 < |r| < a_{ij}|$. We can replace the row s by summing up q times row i and the row s. This will result in $a_{sj} = 0$. Next we replace the row i by summing up rows i and s. This will result in that a_{ij} fails to divide an element in it's row. To be exact a_{ij} does not divide the element row i and column t. Thus we go back to the step where we make sure that the minimal entry divides every other element in its row.

$$\begin{bmatrix} a_{ij} & \dots & a_{it} \\ \vdots & & \vdots \\ a_{sj} & \dots & a_{st} \end{bmatrix} \rightsquigarrow \begin{bmatrix} a_{ij} & \dots & a_{it} \\ \vdots & & \vdots \\ 0 & \dots & a_{st} + qa_{it} \end{bmatrix} \rightsquigarrow \begin{bmatrix} a_{ij} & \dots & a_{st} + (q+1)a_{it} \\ \vdots & & \vdots \\ 0 & \dots & a_{st} + qa_{it} \end{bmatrix}$$

We apply the previous steps as long as we achieve a minimal entry a_{ij} that divides every non-zero element in A. Let's move the element a_{ij} to the top left corner. We do this by switching rows 1 and i with each other and then we switch columns 1 and j with each other. Thus we end up with a matrix where the minimal entry is in the position 11.

$$\begin{bmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1m} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \dots & a_{ij} & \dots & a_{im} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \dots & a_{nj} & \dots & a_{nm} \end{bmatrix} \rightsquigarrow \begin{bmatrix} a_{i1} & \dots & a_{ij} & \dots & a_{im} \\ \vdots & & \vdots & & \vdots \\ a_{11} & \dots & a_{1j} & \dots & a_{1m} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \dots & a_{nj} & \dots & a_{nm} \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} a_{ij} & \dots & a_{i1} & \dots & a_{im} \\ \vdots & & \vdots & & \vdots \\ a_{1j} & \dots & a_{11} & \dots & a_{1m} \\ \vdots & & \vdots & & \vdots \\ a_{nj} & \dots & a_{n1} & \dots & a_{nm} \end{bmatrix}$$

Since the minimal entry divides all other non-zero elements, we can make all the entries in its row and column zeros. Next we repeat the process with the submatrix of A that does not consist of the first row and column.

$$\begin{bmatrix} a_{ij} & \dots & a_{i1} & \dots & a_{im} \\ \vdots & & \vdots & & \vdots \\ a_{1j} & \dots & a_{11} & \dots & a_{1m} \\ \vdots & & \vdots & & \vdots \\ a_{nj} & \dots & a_{n1} & \dots & a_{nm} \end{bmatrix} \rightsquigarrow \begin{bmatrix} a_{ij} & \dots & a_{i1} & \dots & a_{im} \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & a_{11} - q_1 a_{11} & \dots & a_{1m} - q_1 a_{im} \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & a_{n1} - q_n a_{n1} & \dots & a_{nm} - q_n a_{nm} \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} a_{ij} & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & a_{11} - q_1 a_{11} & \dots & a_{1m} - q_1 a_{im} \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & a_{n1} - q_n a_{n1} & \dots & a_{nm} - q_n a_{nm} \end{bmatrix}$$

We repeat this process until the smaller matrix is either a zero matrix or it disappears. In this stage our matrix is in normal form and we have $a_{11}|a_{22}|\dots|a_{kk}$, where the elements are the only non-zero elements of the transformed matrix. □

As part of this thesis the algorithm is implemented Python and it can be found at `https://github.com/asiekkinen/py-smith-normal-form`

Next we focus on how this applies to simplicial chain complexes and boundary homomorphisms. Since each simplicial chain group C_p is finitely generated and free, we can present all of the boundary homomorphisms as matrices. Each column corresponds to a oriented p-simplex and each row presents an oriented $p-1$-simplex.

Definition 4.33. Let (C_*, ∂) be a chain complex. For each $p \in \mathbb{Z}$ we define the pth group of *weak boundaries* W_p to consist of elements $c_p \in C_p$ such that there exists some non-zero integer λ for which $\lambda c_p \in B_p$.

Remark. Notice that by choosing $\lambda = 1$, we conclude that $B_p \subset W_p$.

Lemma 4.34. If (C_*, ∂) is a chain complex then for each integer p, the group of weak boundaries W_p is a subgroup of C_p

Proof. The group of weak boundaries contains the neutral element 0 since $0 = \partial_{p+1} 0 \in B_p$. Let $c, d \in W_p$ with $n, m \in \mathbb{Z} \backslash 0$ such that $nc, md \in B_p$. Thus $nm(c + d) = mnc + nmd \in B_p$. Hence $c + d \in W_p$. Let $c \in W_p$ with $n \in \mathbb{Z} \backslash 0$ such that $nc \in B_p$. Now $n(-c) = -nc \in B_p$. Hence $-c \in W_p$ and we conclude that W_p is a subgroup of C_p. □

Lemma 4.35. If (C_*, ∂) is a chain complex, then for all integers p, we have

$$B_p \triangleleft W_p \triangleleft Z_p \triangleleft C_p.$$

Proof. Let's first go through the inclusions. The first inclusion is already explained in the remark of definition 4.33. For the second inclusion let $w \in W_p$ with non-zero integer λ such that $\lambda w = \partial_{p+1} c$, where $c \in C_{p+1}$. Thus $\partial_p \lambda w = \partial_p \partial_{p+1} c = 0$. Since the boundary operator is extended by linearity, we conclude that $w \in Z_p$. Last inclusion follows directly from the fact that $Z_p = \ker \partial_p$.

By definition C_p is an abelian group. Every subgroup of C_p is normal. Lemma 4.34 states that W_p is a group. Since the images and kernels of homomoprhims are subgroups, B_p and Z_p are groups. □

Lemma 4.36. Let (C_*, ∂) be a chain complex and $p \in \mathbb{Z}$. If $e_1, \ldots, e_n$ is a basis for C_p and $e'_1, \ldots, e'_m$ is a basis for C_{p-1}, relative to which $\partial_p \colon C_p \to C_{p-1}$ has the normal form

$$\begin{array}{c} \\ e'_1 \\ \vdots \\ e'_k \\ e'_{k+1} \\ \vdots \\ e'_m \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & \cdots & e_k & e_{k+1} & \cdots & e_n \end{array} \\ \left[\begin{array}{ccc|c} b_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & b_k & \\ \hline & & & \\ & 0 & & 0 \\ & & & \end{array}\right] \end{array},$$

then the following hold:

1. $e_{k+1}, \ldots, e_n$ is a basis for Z_p
2. $b_1 e'_1, \ldots, b_k e'_k$ is a basis for B_{p-1}
3. $e'_1, \ldots, e'_k$ is a basis for W_{p-1}

Proof. For all the cases supposes that $c_p = \sum_{i=1}^{n} a_i e_i$ is a p-chain. Let's start with the case 1. The boundary of c_p is

$$\partial_p c_p = \sum_{i=1}^{k} a_i b_i e'_i.$$

Now c_p is a cycle if and only if $a_i = 0$ for $i \in \{1, \ldots, k\}$. Thus $e_{k+1}, \ldots, e_n$ is a basis for Z_p.

Next cases 2 and 3. Since the boundary of c_p was $\sum_{i=0}^{k} a_i b_i e'_i$ and each $b_i \neq 0$, we conclude that $b_1 e'_1, \ldots, b_k e'_k$ is a basis for B_{p-1}. From this, it follows that $e'_i \in W_{p-1}$ for $i \in \{1, \ldots, k\}$. Let's prove that $e'_1, \ldots, e'_k$ is a basis for W_{p-1}. Let $c_{p-1} = \sum_{i=0}^{m} d_i e'_i$ be a $(p-1)$-chain and suppose $c_{p-1} \in W_{p-1}$. Thus there exists some non-zero λ such that

$$\lambda \sum_{i=0}^{m} d_i e'_i = \lambda c_{p-1} = \partial c_p = \sum_{i=1}^{k} a_i b_i e'_i.$$

This implies that $\lambda d_i = 0$ for $i \in \{k+1, \ldots, m\}$ and thus $d_i = 0$ for $i \in \{k+1, \ldots, m\}$. Therefore $e'_1, \ldots, e'_k$ is a basis for W_{p-1}. □

Lemma 4.37. If (C_*, ∂) be a chain complex, then for all $p \in \mathbb{Z}$ we have

$$Z_p(C_*)/W_p(C_*) \cong H_p(C_*)/T_p(C_*),$$

where $T_p(C_*)$ is the torsion subgroup of $H_p(C_*)$. In addition $Z_p(C_*)/W_p(C_*)$ is free.

Proof. Consider the natural projection

$$Z_p \to H_p(C_*) \to H_p(C_*)/T_p(C_*).$$

In other words elements get mapped as $z \mapsto (z + B_p) + T_p(C_*)$. Let's show that the kernel of the projection is W_p. Let $w \in W_p$ with non-zero integer λ such that $\lambda w \in B_p$. Now $\lambda(w + B_p) = \lambda w + B_p = B_p$ which implies that $w + B_p \in T_p(C_*)$. For the reverse inclusion let c be in the kernel of the natural projection. Thus $c \mapsto (c + B_p) + T_p(C_*) = T_p(C_*)$ which implies that $c + B_p$ is a torsion element. Hence there exists some non-zero integer λ such that $\lambda(c + B_p) = c + B_p$. But $\lambda(c + B_p) = \lambda c + B_p = c + B$ implies that $c \in W_p$. Now that we have shown that the kernel is W_p, the first isomorphism theorem states that we have an isomorphism $Z_p/W_p \cong H_p(C_*)/T_p(C_*)$.

Since $H_p(C_*)/T_p(C_*)$ is free, the isomorphism implies that $Z_p(C_*)/W_p(C_*)$ is also free. □

Theorem 4.38 (Standard bases for free chain complexes)**.** Let (C_*, ∂) be a chain complex such that each C_p is free and of finite rank. For each p, there exists subgroups U_p, V_p, W_p of C_p such that

$$C_p = U_p \oplus V_p \oplus W_p,$$

where W_p is the group of weak p-boundaries, $\partial_p(U_p) \subset W_{p-1}$, $\partial(V_p) = 0$, and $\partial_p(W_p) = 0$. In addition, there exists bases for U_p and W_p such that $\partial_p \colon U_p \to W_p$ has a diagonal matrix presentation of the form

$$\begin{bmatrix} b_1 & & 0 \\ & \ddots & \\ 0 & & b_k \end{bmatrix},$$

where every $b_i > 0$ and $b_1 | b_2 | \dots | b_k$.

Proof. Fix $p \in \mathbb{Z}$. Let's prove that there exists some subgroup V_p of C_p such that $Z_p \cong V_p \oplus W_p$. Lemma 4.37 implies that Z_p/W_p is torsion free and finitely generated. Thus there exists a basis $c_1 + W_p, \dots, c_m + W_p$ for it. Also there exists a basis $d_1, \dots, d_n$ for W_p. It follows that $c_1, \dots, c_m, d_1, \dots, d_n$ is a basis for Z_p. Hence by choosing V_p to be the free abelian group with basis $c_1, \dots, c_m$ we get

$$Z_p \cong V_p \oplus W_p.$$

By Theorem 4.32, there exists bases $e_1, \dots, e_n$ for C_p and $e'_1, \dots, e'_m$ for C_{p-1} such that the matrix presentation of $\partial_p \colon C_p \to C_{p-1}$ has the normal form

$$\begin{array}{c} \\ e'_1 \\ \vdots \\ e'_k \\ e'_{k+1} \\ \vdots \\ e'_m \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & \dots & e_k & e_{k+1} & \dots & e_n \end{array} \\ \left[\begin{array}{ccc|ccc} b_1 & & 0 & & & \\ & \ddots & & & 0 & \\ 0 & & b_k & & & \\ \hline & & & & & \\ & 0 & & & 0 & \\ & & & & & \end{array}\right] \end{array},$$

where every $b_i \geq 1$ and $b_1 | b_2 | \dots | b_k$. By Lemma 4.36 $e_{k+1}, \dots, e_n$ is a basis for Z_p. Define U_p to be the group with basis $e_1, \dots, e_k$. Thus we have

$$C_p = U_p \oplus Z_p \cong U_p \oplus V_p \oplus W_p.$$

The normal form and Lemma 4.36 imply that $\partial_p(U_p) \subset W_{p-1}$, $\partial_p(V_p) = 0$, $\partial_p(W_p) = 0$, and $\partial_p|_{U_p} \colon U_p \to W_p$ has the matrix presentation

$$\begin{bmatrix} b_1 & & 0 \\ & \ddots & \\ 0 & & b_k \end{bmatrix}.$$

□

Theorem 4.39. Let K be an oriented simplicial complex. Using the normal forms of boundary homomorphisms ∂ we can calculate the homology groups of K as follows. The pth Betti number is the number of zero columns in ∂_p minus the number of non-zero rows of ∂_{p+1}. The pth torsion coefficients are the elements in ∂_{p+1} which are greater than one.

Proof. By Theorem 4.38, we have the decomposition

$$C_p(K) = U_p \oplus V_p \oplus W_p,$$

where $Z_p = V_p \oplus W_p$ is the group of p-cycle and W_p is the group of weak p-boundaries. Thus we have

$$H_p(K) = Z_p/B_p \cong V_p \oplus (W_p/B_p) \cong (Z_p/W_p) \oplus (W_p/B_p).$$

Lemma 4.37 states that $Z_p/W_p \cong H_p/T_p$ and thus it must be that $W_p/B_p \cong T_p$. We conclude that Z_p/W_p represent the Betti numbers and W_p/B_p the torsion coefficients. Let's first show a way to calculate the Betti numbers. Theorem 4.38 states that every matrix presentation of a boundary homomorphism can be linearly transformed to a Smith normal form. Thus with the help of Lemma 4.36, we have that the rank of Z_p is the number of zero columns and the rank of W_{p-1} is the number of non-zero rows. Thus the pth Betti number is the number of zero columns in the Smith normal form of ∂_p minus the number of non-zero rows in the Smith normal form of ∂_{p+1}.

For the torsion coefficients, we have an isomorphism

$$W_{p-1}/B_{p-1} \cong \mathbb{Z}/b_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/b_k\mathbb{Z}.$$

Thus the pth torsion coefficients are those b_i which are greater than 1 in ∂_{p+1}. □

Let's use this algorithm to calculate reduced homology groups for polyhedrons introduced in the section of simplicial complexes. First we need to orient the triangulations. We will use the following orientation scheme. Label each vertex with p_i, where i goes through the integers starting from 0. Orient each simplex in such a way that the labels of the vertices are in increasing order. For example, $[p_{i_0}, p_{i_1}, \ldots, p_{i_n}]$ is an oriented simplex of some simplicial complex with orientation $p_{i_0} < p_{i_1} < \cdots < p_{i_n}$ if and only if $i_0 < i_1 < \cdots < i_n$. When we construct matrices of the boundary homomorphism, we also order the simplexes in them as well. In the matrix, the oriented p-simplexes are ordered from left to right and the oriented $p-1$-simplex are ordered from top to bottom. In the matrices we write each $[p_{i_0}, p_{i_1}, \ldots, p_{i_k}]$ as $i_0 i_1 \ldots i_k$ and the ordering is based on these labels e.g. $014 < 123$ since 14 is less than 123. We only calculate the Smith normal forms of boundary homomorphism, where the domain is not the trivial group. The exact steps for transforming the matrices to the Smith normal form are not presented.

4.5.1 Disk

The disk $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ has a triangulation consisting of the faces of a 2-simplex $[p_0, p_1, p_2]$. Thus the matrix presentations and the Smith normal forms of the boundary homomorphisms are

$$\partial_2 = \begin{array}{c} \\ {\scriptstyle 01} \\ {\scriptstyle 02} \\ {\scriptstyle 12} \end{array} \begin{array}{c} {\scriptstyle 012} \\ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \end{array} \rightsquigarrow \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

$$\partial_1 = \begin{array}{c} \\ {\scriptstyle 0} \\ {\scriptstyle 1} \\ {\scriptstyle 2} \end{array} \begin{array}{c} {\scriptstyle 01 \quad 02 \quad 12} \\ \begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \end{array} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and

$$\widetilde{\partial}_0 = {\scriptstyle \langle\rangle} \begin{array}{c} {\scriptstyle 0 \quad 1 \quad 2} \\ \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \end{array} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

boundary	number of zero columns	number of non-zero rows	diagonal elements greater than 1
∂_3	0	0	-
∂_2	0	1	-
∂_1	1	2	-
$\widetilde{\partial}_0$	2	1	-

Table 1: Disk

From Table 1 we can see that the are no diagonal elements greater than one. Hence Theorem 4.39 implies homology groups do not have torsion coefficients. Also Theorem 4.39 implies that the Betti numbers are

$$B_n = \begin{cases} 2-2=0, & \text{if } n=0 \\ 1-1=0, & \text{if } n=1 \\ 0, & \text{otherwise.} \end{cases}$$

Thus the reduced homology groups for the disk are all trivial.

4.5.2 Circle

The circle $\{(x,y) \in \mathbb{R}^2 : x^2+y^2=1\}$ has a triangulation consists of the proper faces of a 2-simplex $[p_0, p_1, p_2]$. Hence the matrix presentations and the Smith normal forms of the boundary homomorphisms are

$$\partial_1 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 01 & 02 & 12 \end{array} \\ \begin{bmatrix} -1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \end{array} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$\widetilde{\partial}_0 = \langle\rangle \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \end{array} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

boundary	number of zero columns	number of non-zero rows	diagonal elements greater than 1
∂_2	0	0	-
∂_1	1	2	-
$\widetilde{\partial}_0$	2	1	-

Table 2: Circle

From Table 2 we can deduce that there are no torsion coefficients and the Betti numbers are

$$B_n = \begin{cases} 2-2=0, & \text{if } n=0 \\ 1-0=1, & \text{if } n=1 \\ 0, & \text{otherwise.} \end{cases}$$

Thus the reduced homology groups for the circle are

$$\widetilde{H}_n(\mathbb{S}^1) \cong \begin{cases} \mathbb{Z}, & \text{if } n=1 \\ 0, & \text{otherwise.} \end{cases}$$

4.5.3 Möbius band

For the Möbius band we use the triangulation presented in Figure 15. The matrix presentations and Smith normal forms of the boundary homomorphisms can be found in Appendix A.1.

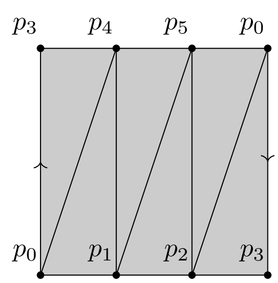


Figure 15: Oriented triangulation of the Möbius band.

From Table 3 we can deduce that there are no torsion coefficients and the Betti numbers are

$$B_n = \begin{cases} 5-5=0, & \text{if } n=0 \\ 7-6=1, & \text{if } n=1 \\ 0, & \text{otherwise.} \end{cases}$$

Thus the reduced homology groups for the Möbius band are

$$\widetilde{H}_n \cong \begin{cases} \mathbb{Z}, & \text{if } n=1 \\ 0, & \text{otherwise.} \end{cases}$$

4.5.4 Torus

We use the triangulation for the torus presented in Figure 16. The matrix presentations and Smith normal forms of the boundary homomorphisms can be found in Appendix A.2.

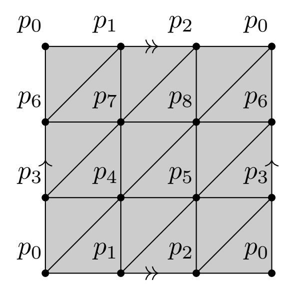


Figure 16: Oriented triangulation of the torus.

From Table 4 we can deduce that there are no torsion coefficients and the Betti numbers are

$$B_n = \begin{cases} 8 - 8 = 0, & \text{if } n = 0 \\ 19 - 17 = 2, & \text{if } n = 1 \\ 1 - 0 = 1, & \text{if } n = 2 \\ 0, & \text{otherwise.} \end{cases}$$

Thus the reduced homology groups for torus are

$$\widetilde{H}_n(\mathbb{T}^2) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}, & \text{if } n = 1 \\ \mathbb{Z}, & \text{if } n = 2 \\ 0, & \text{otherwise.} \end{cases}$$

4.5.5 Real projective plane

We use the triangulation for the Real projective place presented in Figure 17. The matrix presentations and Smith normal forms of the boundary homomorphisms can be found in Appendix A.3.

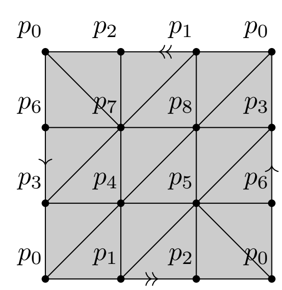


Figure 17: Oriented triangulation of the real projective plane.

From Table 4 we can see that torsion coefficient is 2 for ∂_2. The Betti numbers are

$$B_n = \begin{cases} 8 - 8 = 0, & \text{if } n = 0 \\ 18 - 18 = 0, & \text{if } n = 1 \\ 0, & \text{otherwise.} \end{cases}$$

So all the Betti numbers are zero. Thus the reduced homology groups for real projective plane are

$$\widetilde{H}_n(\mathbb{RP}^2) \cong \begin{cases} \mathbb{Z}_2, & \text{if } n = 1 \\ 0, & \text{otherwise.} \end{cases}$$

4.5.6 Klein bottle

For Klein bottle we use the triangulation presented in Figure 18. The matrix presentations and Smith normal forms of the boundary homomorphisms can be found in Appendix A.4.

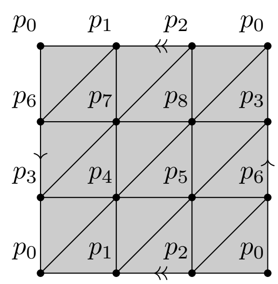


Figure 18: Oriented triangulation of the Klein bottle.

From Table 6 we can see that torsion coefficient is 2 for ∂_2. The Betti numbers are

$$B_n = \begin{cases} 8 - 8 = 0, & \text{if } n = 0 \\ 19 - 18 = 1, & \text{if } n = 1 \\ 0, & \text{otherwise.} \end{cases}$$

Thus the reduced homology groups for Klein bottle are

$$\widetilde{H}_n \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}_2, & \text{if } n = 1 \\ 0, & \text{otherwise.} \end{cases}$$

5 Lefschetz fixed-point theorem

The Lefschetz fixed-point theorem gives us a sufficient condition under which a continuous map $f\colon X \to X$ from a polyhedron to itself has a fixed point. As a reminder, f has a fixed point if $f(x) = x$ for some $x \in X$. We introduce the Lefschetz number which depends on the induced map f_* and hence depends on the homology groups of the space. If the Lefschetz number is not zero, then the function has a fixed point. Calculating the Lefschetz number can be troublesome but the Hopf trace theorem gives us a way to calculate it for simplicial maps.

5.1 Lefschetz number

Definition 5.1. For an $n \times n$ square matrix $A = (a_{ij})$ we define the trace of A by

$$\operatorname{tr}(A) = \sum_{i=1}^{n} a_{ii}.$$

If G is a finitely generated free abelian group, $f\colon G \to G$ is a homomorphism and A is a matrix presentation of f, then

$$\operatorname{tr}(f) = \operatorname{tr}(A).$$

Lemma 5.2. Let G be a finitely generated free abelian group. If $f\colon G \to G$ is a homomorphism and $i\colon G \to G$ is an isomorphism, then

$$\operatorname{tr}(i^{-1}fi) = \operatorname{tr}(f).$$

Proof. Let A and B be matrix presentations of f and i respectively. Thus

$$\operatorname{tr}(i^{-1}fi) = \operatorname{tr}(B^{-1}(AB)) = \operatorname{tr}((AB)B^{-1}) = \operatorname{tr}(A) = \operatorname{tr}(f).$$

□

Lemma 5.3. If $f\colon G \to G$ is a homomorphism from a finitely generated free abelian group to itself, then the trace of f is independent on the choice of basis.

Proof. We can change the basis of G by an isomorphism $i\colon G \to G$. We can write f in a different basis as $i^{-1}fi$. It follows from Lemma 5.2 that $\operatorname{tr}(i^{-1}fi) = \operatorname{tr}(f)$. □

Let K be an oriented simplicial complex and let $\phi\colon C_*(K) \to C_*(K)$ be a chain map. Since each $C_n(K)$ is finitely generated we can define the trace for ϕ. We denote it by $\operatorname{tr}(\phi, C_n(K))$. We can not directly assign the matrix presentation for $\phi_*\colon H_n(K) \to H_n(K)$ since $H_n(K)$ may have torsion subgroups. We fix this by defining the trace for the torsion free part of $H_n(K)$. In other words, we can define $\operatorname{tr}(\phi_*, H_n(K)/T_n(K))$, where $T_n(K)$ is the torsion subgroup of $H_n(K)$.

Definition 5.4. Let K be an oriented simplicial complex. The Lefschetz number for a chain map $\phi\colon C_*(K) \to C_*(K)$ is defined as

$$\lambda(\phi) = \sum_n (-1)^n \operatorname{tr}(\phi, C_n(K)).$$

For a continuous function $f\colon X \to X$, where X is a polyhedron, the Lefschetz number is

$$\lambda(f) = \sum_n (-1)^n \operatorname{tr}(f_*, H_n(X)/T_n(X)).$$

Notice that the Lefschetz number is invariant under homotopies. This is because if $f, g\colon X \to X$ are homotopic, then $f_* = g_*$.

5.2 Hopf trace theorem

Lemma 5.5. If $\phi\colon C_* \to C_*$ is a chain map, then

$$\phi(Z_p) \subset Z_p \qquad \phi(B_p) \subset B_p \qquad \phi(W_p) \subset W_p$$

for all p.

Proof. If $z \in Z_p$, then

$$\partial_p \phi(z) = \phi \partial_p(z) = \phi(0) = 0.$$

Thus $\phi(Z_p) \subset Z_p$.

Let $b \in B_p$ and $c \in C_{p+1}$ be such that $b = \partial_{p+1} c$. Thus,

$$\phi(b) = \phi(\partial_{p+1} c) = \partial_{p+1} \phi(c)$$

which implies $\phi(B_p) \subset B_p$.

Let $w \in W_p$ and $\lambda \in \mathbb{Z}$ be such that $\lambda w = \partial_{p+1} c$ where $c \in C_{p+1}$. Thus,

$$\lambda \phi(w) = \phi(\lambda w) = \phi(\partial_{p+1} c) = \partial_{p+1} \phi(c)$$

which implies $\phi(W_p) \subset W_p$. □

Lemma 5.6. Let G be a free abelian group of finite rank and let H be a subgroup of G such that G/H is free abelian. If $\phi\colon G \to G$ is a homomorphism such that $\phi(H) \subset H$, then

$$\operatorname{tr}(\phi, G) = \operatorname{tr}(\phi', G/H) + \operatorname{tr}(\phi'', H),$$

where ϕ' and ϕ'' are the homomorphisms induced by ϕ.

Proof. Let $\alpha_1 + H, \ldots, \alpha_m + H$ be a basis for G/H and $\beta_1, \ldots, \beta_n$ be a basis for H. It follows that $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n$ is a basis for G. Let $A = a_{ij}$ and $B = b_{ij}$ be the matrices of ϕ' and ϕ'' relative to these basis. Thus we have

$$\phi'(\alpha_j + H) = \sum_i a_{ij}(\alpha_i + H)$$

and

$$\phi''(\beta_j) = \sum_i b_{ij}\beta_i.$$

Hence ϕ has a matrix presentation of the form

$$\begin{array}{c} \\ \alpha_1 \\ \vdots \\ \alpha_m \\ \beta_1 \\ \vdots \\ \beta_n \end{array}
\begin{array}{c}
\begin{array}{cccccc} \alpha_1 & \cdots & \alpha_m & \beta_1 & \cdots & \beta_n \end{array} \\
\left[\begin{array}{ccc|ccc}
a_{11} & \dots & a_{1m} & & & \\
\vdots & \ddots & \vdots & & 0 & \\
a_{m1} & \dots & a_{mm} & & & \\ \hline
 & & & b_{11} & \dots & b_{1n} \\
 & * & & \vdots & \ddots & \vdots \\
 & & & b_{n1} & \dots & b_{nn}
\end{array}\right]
\end{array}.$$

Thus

$$\operatorname{tr}(\phi, G) = \operatorname{tr}(A) + \operatorname{tr}(B) = \operatorname{tr}(\phi', G/H) + \operatorname{tr}(\phi'', H).$$

□

Theorem 5.7 (Hopf trace theorem)**.** If K is an oriented simplicial complex and if $\phi\colon C_*(K) \to C_*(K)$ is a chain map, then

$$\sum_n (-1)^n \operatorname{tr}(\phi, C_n(K)) = \sum_n (-1)^n \operatorname{tr}(\phi_*, H_n(K)/T_n(K)).$$

Proof. If the quotient groups C_p/Z_p and Z_p/W_p are free, then it follows from Lemma 5.6 that

$$\operatorname{tr}(\phi, C_p) = \operatorname{tr}(\phi_1, C_p/Z_p) + \operatorname{tr}(\phi_2, Z_p/W_p) + \operatorname{tr}(\phi_3, W_p),$$

where the maps ϕ_i on the right side of the equation are induced by ϕ. Each quotient group is justified by Lemma 4.35. Let's compute each term on the right side individually.

Consider the boundary homomorphism $\partial_p\colon C_p \to C_{p-1}$. It has an image B_{p-1} and kernel Z_p. The first isomorphism theorem implies that there exists an isomorphism $\bar{\partial}_p\colon C_p/Z_p \xrightarrow{\sim} B_{p-1}$ induced by ∂_p. Thus C_p/Z_p is free group. Since ϕ is a chain map, it commutes with ∂_p and $\bar{\partial}_p$. Thus for $\phi_1' = \bar{\partial}_p\phi_1\bar{\partial}_p^{-1}$, we have

$$\operatorname{tr}(\phi_1', B_{p-1}) = \operatorname{tr}(\bar{\partial}_p\phi_1\bar{\partial}_p^{-1}, B_{p-1}) = \operatorname{tr}(\phi_1, C_p/Z_p).$$

In Lemma 4.37 we proved that the projection $\pi\colon Z_p \to H_p/T_p$ induces an isomorphism $\widetilde{\pi}\colon Z_p/W_p \to H_p/T_p$. Thus Z_p/W_p is a free group. For $z \in Z_p$ we can calculate

$$\begin{aligned}\phi_{2*}\widetilde{\pi}(z + W_p) &= \phi_{2*}\big((z + B_p) + T_p\big) = \big(\phi_2(z) + B_p\big) + T_p = \widetilde{\pi}(\phi(z) + W_p) \\ &= \widetilde{\pi}\phi_2(z + W_p).\end{aligned}$$

This implies that ϕ and $\widetilde{\pi}$ commute. Hence for $\phi_2' = \widetilde{\pi}\phi_2\widetilde{\pi}^{-1}$ we have

$$\operatorname{tr}(\phi_2', H_p/T_p) = \operatorname{tr}(\widetilde{\pi}\phi_2\widetilde{\pi}^{-1}, H_p/T_p) = \operatorname{tr}(\phi_2, Z_p/W_p).$$

Last, we show that

$$\operatorname{tr}(\phi_3, W_p) = \operatorname{tr}(\phi_3, B_p).$$

Since W_p is a free abelian group and B_p is a subgroup of W_p, it follows that W_p has a basis $\alpha_1, \dots, \alpha_n$ such that for some $m_1, \dots, m_k \in \mathbb{Z}\backslash\{0\}$ the elements $m_1\alpha_1, \dots, m_k\alpha_k$ form a basis for B_p. Since W_p/B_p is a torsion group, it must be that $k = n$. Let $A = a_{ij}$ and $B = b_{ij}$ be matrix presentations of ϕ_3 and $\phi_3|_{B_p}$ respectively. Thus

$$\phi_3(\alpha_j) = \sum_i a_{ij}\alpha_i$$
$$\phi_3(m_j\alpha_j) = \sum_i b_{ij}m_i\alpha_i.$$

Multiplying the first equation with m_j, we get

$$m_j\phi_3(\alpha_j) = \sum_i a_{ij}m_j\alpha_i = \sum_i b_{ij}m_i\alpha_i.$$

This implies $a_{ij}m_j = b_{ij}m_i$ $a_ii = b_ii$ for all i and j. In particular, $a_ii = b_ii$ for all i. Thus $\operatorname{tr}(\phi, W_p) = \operatorname{tr}(\phi, B_p)$.

By combining the results, we get

$$\operatorname{tr}(\phi, C_p) = \operatorname{tr}(\phi_1', B_{p-1}) + \operatorname{tr}(\phi_2', H_p/T_p) + \operatorname{tr}(\phi_3, B_p).$$

By multiplying both sides of the equation with $(-1)^p$ and summing over all p, the extra terms cancel out and we get

$$\sum_p (-1)^p \operatorname{tr}(\phi, C_p) = \sum_p (-1)^p \operatorname{tr}(\phi_2', H_p/T_p).$$

□

5.3 Lefschetz fixed-point theorem

Definition 5.8. Let K be an oriented simplicial complex. We define the simplicial subdivision operator $\operatorname{Sd}_{\#}^m \colon C_*(K) \to C_*(\operatorname{Sd}^m(K))$ for the basis element $\langle p_0, \dots, p_n\rangle \in C_n(K)$ by

$$\langle p_0, \dots, p_n\rangle \mapsto \sum_{i_0=0}^{n} \sum_{i_1 \notin \{i_0\}} \cdots \sum_{i_n \notin \{i_0, \dots, i_{n-1}\}} \operatorname{sgn}(\pi_{i_0 i_1 \dots i_n})\langle p_{i_0}, b_{i_0 i_1}, \dots b_{i_0 \dots i_n}\rangle,$$

where $\operatorname{sgn}(\pi_{i_0 \dots i_n})$ is the sign of the permutation to the order $i_j < i_{j+1}$, each $b_{i_0 \dots i_k}$ is the barycenter of $[p_{i_0}, \dots, p_{i_k}]$, and extend it by linearity.

Lemma 5.9. The simplicial subdivision operator $\mathrm{Sd}_{\#}\colon C_*(K) \to C_*(\mathrm{Sd}(K))$ is a chain map

Proof. Let $\langle p_0, \dots, p_n \rangle$ be a basis element of $C_n(K)$. We can calculate

$$\partial \mathrm{Sd}_{\#}\langle p_0, \dots, p_n\rangle = \partial \sum_{i_0=0}^{n} \sum_{i_1 \notin \{i_0\}} \cdots \sum_{i_n \notin \{i_0, \dots, i_{n-1}\}} \mathrm{sgn}(\pi_{i_0 i_1 \dots i_n}) \langle p_{i_0}, b_{i_0 i_1}, \dots b_{i_0 \dots i_n}\rangle$$

$$= \sum_{i_0=0}^{n} \sum_{i_1 \notin \{i_0\}} \cdots \sum_{i_n \notin \{i_0, \dots, i_{n-1}\}} \mathrm{sgn}(\pi_{i_0 i_1 \dots i_n}) \sum_{j=0}^{n} (-1)^j \langle p_{i_0}, b_{i_0 i_1}, \dots, \widehat{b_{i_0 \dots i_j}}, \dots, b_{i_0 \dots i_n}\rangle.$$

There are terms in the sums that cancel out. Let $j < n$. Consider the extract of the sums, where we omit the jth term

$$\mathrm{sgn}(\pi_{i_0 \dots i_j i_{j+1} \dots i_n})(-1)^j \langle p_{i_0}, \dots \widehat{b_{i_0 \dots i_j}}, b_{i_0 \dots i_j i_{j+1}}, \dots, b_{i_0 \dots i_j i_{j+1} \dots i_n}\rangle$$
$$+ \mathrm{sgn}(\pi_{i_0 \dots i_{j+1} i_j \dots i_n})(-1)^j \langle p_{i_0}, \dots \widehat{b_{i_0 \dots i_{j+1}}}, b_{i_0 \dots i_{j+1} i_j}, \dots, b_{i_0 \dots i_{j+1} i_j \dots i_n}\rangle.$$

In the latter term we have only swapped the indexes i_j and i_{j+1} with each other. Notice that in the basis elements, only the omitted vertex is different i.e $b_{i_0 \dots i_{j-1} i_j} \neq b_{i_0 \dots i_{j-1} i_{j+1}}$. Since the different vertexes are omitted, we conclude that the basis elements are the same. Because we swapped i_j with i_{j+1}, we have for the signs of the permutations

$$\mathrm{sgn}(\pi_{i_0 \dots i_j i_{j+1} \dots i_n}) = (-1)\mathrm{sgn}(\pi_{i_0 \dots i_{j+1} i_j \dots i_n}).$$

This implies that the terms cancel out. Thus if we omit a term which is not the last i.e. we do not omit the barycenter of $[p_0, \dots, p_n]$, then those terms cancel out. Therefore we are left with

$$\sum_{i_0=0}^{n} \sum_{i_1 \notin \{i_0\}} \cdots \sum_{i_n \notin \{i_0, \dots, i_{n-1}\}} \mathrm{sgn}(\pi_{i_0 i_1 \dots i_n})(-1)^n \langle p_{i_0}, b_{i_0 i_1}, \dots, b_{i_0 \dots i_{n-1}}\rangle.$$

On the other hand we can calculate

$$\mathrm{Sd}_{\#}\partial\langle p_0, \dots, p_n\rangle = \mathrm{Sd}_{\#} \sum_{k=0}^{n} \langle p_0, \dots, \hat{p}_k, \dots, p_n\rangle$$

$$= \sum_{k=0}^{n} \sum_{i_0 \notin \{k\}} \cdots \sum_{i_{n-1} \notin \{k, i_0, \dots, i_{n-2}\}} (-1)^k \mathrm{sgn}(\pi_{i_0 \dots i_{n-1}}) \langle p_{i_0}, b_{i_0 i_1}, \dots, b_{i_0 \dots i_{n-1}}\rangle$$

□

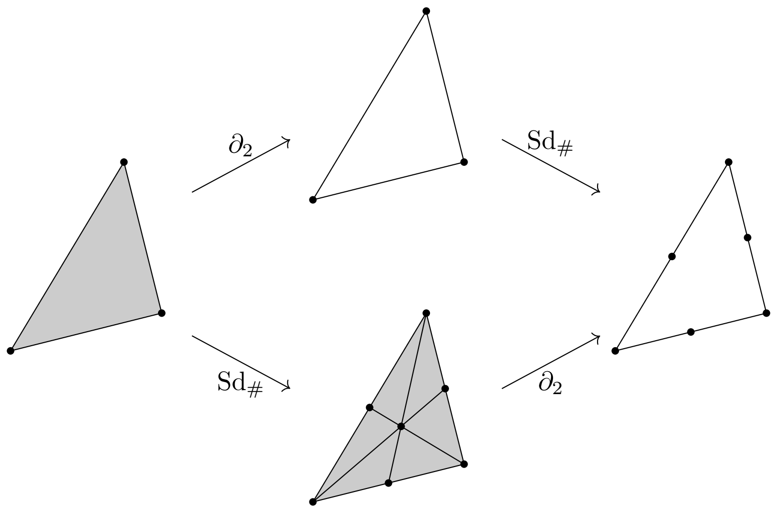


Figure 19: Illustration of how the boundary and subdivision homomorphisms commute in a 2-simplex.

Lemma 5.10. If $f\colon |K| \to |K|$ is a continuous map from a polyhedron to itself and $\psi\colon \mathrm{Sd}^m(K) \to K$ is a simplicial approximation to f, then

$$\lambda(f) = \lambda(\psi_\# \mathrm{Sd}^m_\#).$$

Proof. For every $n \geq 0$, we have a commutative diagram

$$\begin{array}{ccc} C_n(K) & \xrightarrow{\quad j \quad} & S_n(|K|) \\ \Big\downarrow{\scriptstyle \mathrm{Sd}^m_\#} & & \Big\downarrow{\scriptstyle |\mathrm{Sd}^m|_\#} \\ C_n(\mathrm{Sd}^m(K)) & \xrightarrow{\ j\ } & S_n(|\mathrm{Sd}^m(K)|) \\ \Big\downarrow{\scriptstyle \psi_\#} & & \Big\downarrow{\scriptstyle |\psi|_\#} \\ C_n(K) & \xrightarrow{\quad j \quad} & S_n(|K|), \end{array}$$

where j is the map from Lemma 4.27. Note that j was defined for augmented chain complexes, but $C_n = \widetilde{C}_n$ and $S_n = \widetilde{S}_n$ for $n \geq 0$. The commutativity follows directly from the definitions of the maps. We can calculate

$$\begin{aligned}\lambda(f) &= \sum_n (-1)^n \mathrm{tr}(f_*, H_n(|K|)/T_n(|K|) \\ &= \sum_n (-1)^n \mathrm{tr}(|\psi|_* |\mathrm{Sd}^m|_*, H_n(|K|)/T_n(|K|)) \\ &= \sum_n (-1)^n \mathrm{tr}(|\psi|_* |\mathrm{Sd}^m|_* j_* j_*^{-1}, H_n(|K|)/T_n(|K|)) \\ &= \sum_n (-1)^n \mathrm{tr}(j_* \psi_* \mathrm{Sd}_*^m j_*^{-1}, H_n(|K|)/T_n(|K|)) \\ &= \sum_n (-1)^n \mathrm{tr}(\psi_* \mathrm{Sd}_*^m, H_n(K)/T_n(K)) \\ &= \sum_n (-1)^n \mathrm{tr}(\psi_\# \mathrm{Sd}_\#^m, C_n(K)) \\ &= \lambda(\psi_\# \mathrm{Sd}_\#^m).\end{aligned}$$

The first equality is the definition. The second follows from the facts that $|\psi|$ is homotopic to f by Theorem 3.28 and Lemma 2.35 implies that $|\mathrm{Sd}^{m'}|_*$ is the identity. By Theorem 4.28, j_* is an isomorphism and hence the third equality follows. The fourth follows the commutative diagram before and the fact that H_n is a functor. Since j_* is an isomorphism, the fifth equality follows from Lemma 5.2. In the last equality we used the Hopf trace theorem. □

Let's motivate the Lefschetz fixed-point theorem by an example. Let K be the simplicial complex consisting of the proper faces of a 2-simplex $[p_0, p_1, p_2]$. Let $f\colon |K| \to |K|$ be the continuous map that traverses the boundary of a triangle with twice the speed. Analogous version with the circle would be the function $e^{it} \mapsto e^{i2t}$. By the simplicial approximation theorem, there exists a subdivision $\mathrm{Sd}^n(K)$ and a simplicial approximation $\psi\colon \mathrm{Sd}^n(K) \to K$ to f. Choose $n = 1$. The simplicial map $\psi\colon \mathrm{Sd}(K) \to K$

$$\begin{aligned}&\psi(p_0) = p_0, \\ &\psi(b^{[p_0,p_1]}) = p_1, \\ &\psi(p_1) = p_2, \\ &\psi(b^{[p_1,p_2]}) = p_0, \\ &\psi(p_2) = p_1, \\ &\psi(b^{[p_0,p_2]}) = p_2\end{aligned}$$

is a simplicial approximation to f, which can be seen by showing the inclusions $f(\mathrm{St}(p, \mathrm{Sd}(K)) \subset \mathrm{St}(\psi(p), K)$, where $p \in \mathrm{Sd}(K)^{(0)}$. Now $|\psi|$ has a fixed point, $|\psi|(p_0) = p_0$. Let's calculate the Lefschetz number for $|\psi|$. Lemma 5.10 implies that $\lambda(|\psi|) = \lambda(\psi_\# \mathrm{Sd}_\#)$. Thus we have matrices of $\psi_\# \mathrm{Sd}(K)$ for C_0 as

$$\begin{array}{c} \\ \langle p_0\rangle \\ \langle p_1\rangle \\ \langle p_2\rangle \end{array} \begin{array}{c} \begin{array}{ccc} \langle p_0\rangle & \langle p_1\rangle & \langle p_2\rangle \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \cdot \begin{array}{c} \\ \langle p_0\rangle \\ \langle p_1\rangle \\ \langle p_2\rangle \end{array} \begin{array}{c} \begin{array}{ccc} \langle p_0\rangle & \langle p_1\rangle & \langle p_2\rangle \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array} = \begin{array}{c} \\ \langle p_0\rangle \\ \langle p_1\rangle \\ \langle p_2\rangle \end{array} \begin{array}{c} \begin{array}{ccc} \langle p_0\rangle & \langle p_1\rangle & \langle p_2\rangle \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

and for C_1 as

$$\begin{array}{c} \\ \langle p_0,p_1\rangle \\ \langle p_1,p_2\rangle \\ \langle p_0,p_2\rangle \end{array} \begin{array}{c} \begin{array}{cccccc} \langle p_0,b_{01}\rangle & \langle b_{01},p_1\rangle & \langle p_1,b_{12}\rangle & \langle b_{12},p_2\rangle & \langle p_2,b_{02}\rangle & \langle b_{02},p_0\rangle \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$\cdot \begin{array}{c} \\ \langle p_0,b_{01}\rangle \\ \langle b_{01},p_1\rangle \\ \langle p_1,b_{12}\rangle \\ \langle b_{12},p_2\rangle \\ \langle p_2,b_{02}\rangle \\ \langle b_{02},p_0\rangle \end{array} \begin{array}{c} \begin{array}{ccc} \langle p_0,p_1\rangle & \langle p_1,p_2\rangle & \langle p_0,p_2\rangle \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$= \begin{array}{c} \\ \langle p_0,p_1\rangle \\ \langle p_1,p_2\rangle \\ \langle p_0,p_2\rangle \end{array} \begin{array}{c} \begin{array}{ccc} \langle p_0,p_1\rangle & \langle p_1,p_2\rangle & \langle p_0,p_2\rangle \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{array}.$$

Thus

$$\lambda(|\psi|) = (-1)^0 \cdot 1 + (-1)^1 \cdot 2 = -1.$$

The thing to note here is that the Lefschetz number is non-zero and $|\psi|$ has a fixed point.

Theorem 5.11 (Lefschetz fixed-point theorem)**.** Let K be an oriented simplicial complex and let $f\colon |K| \to |K|$ be continuous. If $\lambda(f) \neq 0$, then f has a fixed point.

Proof. Assume that f does not have a fixed point. We will prove that $\lambda(f) = 0$. First we show that there exists $m \geq 0$ such that, for all $p \in \mathrm{Sd}^m(K)^{(0)}$,

$$f(\overline{\mathrm{St}(p, \mathrm{Sd}^m(K))}) \cap \overline{\mathrm{St}(p, \mathrm{Sd}^m(K))} = \emptyset.$$

Since f has no fixed points and $|K|$ is compact, there exists

$$\epsilon = \min\{|x - f(x)| : x \in |K|\} > 0.$$

Also, the compactness of $|K|$ implies that f is uniformly continunuous. Hence we can choose $\delta > 0$ such that whenever $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon/3$. Let $\gamma = \min\{\delta, \epsilon/2\}$. Thus for all subsets $A \subset |K|$ with diameter less than γ, we have $A \cap f(A) = \emptyset$. Lemmas 3.15 and 3.24 give us the inequalities for all $m \geq 0$ and $p \in \mathrm{Sd}^m(K)^{(0)}$

$$\mathrm{diam}(\mathrm{St}(p)) \leq 2\mathrm{mesh}(\mathrm{Sd}^m(K)) \leq 2(\frac{\dim(K)}{\dim(K)+1})^m \mathrm{mesh}(K).$$

Thus choosing m large enough, we get

$$\mathrm{diam}(\mathrm{St}(p)) < \gamma$$

for all $p \in \mathrm{Sd}^m(K)^{(0)}$.

By the simplicial approximation theorem, there exists $m' \geq m$ and a simplicial approximation $\psi\colon \mathrm{Sd}^{m'}(K)^{(0)} \to \mathrm{Sd}^m(K)^{(0)}$ to f. Let's show that if $\tau \in \mathrm{Sd}^{m'}(K)$ and $\sigma \in \mathrm{Sd}^m(K)$ are simplexes such that $\tau \subset \sigma$, then $\psi(\tau) \neq \sigma$. Assume the contrary i.e. $\psi(\tau) = \sigma$. Let p be a vertex of τ and note that $\psi(p)$ is a vertex of σ. Since $\tau \subset \sigma$, we get that $p \in \overline{\mathrm{St}(\psi(p), \mathrm{Sd}^m(K))}$ which implies

$$f(p) \in f(\overline{\mathrm{St}(\psi(p), \mathrm{Sd}^m(K))}).$$

On the other hand, simplicial approximation states that

$$f(\mathrm{St}(p, \mathrm{Sd}^{m'}(K))) \subset \mathrm{St}(\psi(p), \mathrm{Sd}^m(K)),$$

which implies

$$f(p) \in \overline{\mathrm{St}(\psi(p), \mathrm{Sd}^m(K))}.$$

Thus we have

$$f(p) \in \overline{\mathrm{St}(\psi(p), \mathrm{Sd}^m(K))} \cap f(\overline{\mathrm{St}(\psi(p), \mathrm{Sd}^m(K))}) = \emptyset,$$

which is a contradiction.

By Lemma 5.10, we have

$$\lambda(f) = \lambda(\psi_{\#}\mathrm{Sd}_{\#}^{m'-m}).$$

Let's compute a trace of $\psi_{\#}\mathrm{Sd}_{\#}^{m'-m}$ for $C_n(\mathrm{Sd}^m(K))$. Let A be a matrix presentation of $\psi_{\#}\mathrm{Sd}_{\#}^{m'-m}$ for the usual basis of $C_n(\mathrm{Sd}^m(K))$ consisting of oriented n-simplexes. Let $\langle\sigma\rangle \in C_n(\mathrm{Sd}^m(K))$ be a basis element. The chain $\mathrm{Sd}_{\#}^{m'-m}(\langle\sigma\rangle)$ is a linear combination of basis elements $\langle\tau\rangle$ of $\mathrm{Sd}^{m'}(K)$ such that $\tau \subset \sigma$. By the previous result we have $\psi(\tau) \neq \sigma$. Thus $\psi_{\#}\mathrm{Sd}_{\#}^{m'-m}(\langle\sigma\rangle)$ is a linear combination that does not contain $\langle\sigma\rangle$. Therefore all diagonal elements of A equal 0. Hence

$$\mathrm{tr}(\psi_{\#}\mathrm{Sd}_{\#}^{m'-m}, C_n(\mathrm{Sd}^m(K))) = \mathrm{tr}(A) = 0$$

which implies

$$\lambda(f) = \lambda(\psi_{\#}\mathrm{Sd}_{\#}^{m'-m}) = 0.$$

□

Notice that, if the Lefschetz number of a continuous map is zero, then the map can have a fixed point. For example take K to be simplicial complex consisting of the proper faces of a 2-simplex. Let our continuous map be the identity map $\mathrm{Id}\colon |K| \to |K|$. Notice that the simplicial identity map $\mathrm{Id}\colon K \to K$ is a simplicial approximation to $\mathrm{Id}\colon |K| \to |K|$. Every point of $|K|$ is a fixed point of identity, but

$$\begin{aligned}\lambda(\mathrm{Id}) &= \sum_{n=0}^{1}(-1)^n \mathrm{tr}(\mathrm{Id}, H_n(|K|)/T_n(|K|)) \\ &= (-1)^0\mathrm{tr}(\mathrm{Id}, \mathbb{Z}) + (-1)^1\mathrm{tr}(\mathrm{Id}, \mathbb{Z}) \\ &= 1 - 1 = 0.\end{aligned}$$

Definition 5.12. A space X is said to have the fixed point property if every continuous function $f\colon X \to X$ has a fixed point.

Corollary 5.13. Every path connected polyhedron X for which $H_n(X)$ is finite for $n > 0$, has the fixed point property.

Proof. Let $f\colon X \to X$ be continuous. For $n > 0$ we have $H_n(X)/T_n(X) = 0$ and hence $\mathrm{tr}(f_*, H_n(X)/T_n(X)) = 0$. We need to show that $\mathrm{tr}(f_*, H_0(X)/T_0(X)) = 0$. Let $x_0, x_1 \in X$ be arbitrary. By Theorem 4.14(i) in [6] we have that $x_0 + B_0(X) = x_1 + B_0(X)$ is a generator of $H_0(X)$. Thus

$$f_*(x + B_0(X)) = f(x) + B_0(X) = x + B_0(X)$$

which implies that $\mathrm{tr}(f_*) = 1$. Therefore $\lambda(f) = 1 \neq 0$ and the Lefschetz fixed-point theorem implies that f has a fixed point. □

The following corollary is an extension of the Brouwer fixed-point theorem.

Corollary 5.14. Every contractible polyhedron has the fixed point property.

Proof. Let X be a contractible polyhedron. Since X is contractible, we have

$$H_n(X) = \begin{cases} \mathbb{Z}, & \text{if } n = 0 \\ 0, & \text{otherwise.} \end{cases}$$

By Corollary 5.13, X has the fixed point property. □

Even though every contractible polyhedron has the fixed point property, it is not true for every compact contractible space. Let's construct such a space. Let X be the space of a circle and a spiral converging into the circle from the outside. Now the cone of X is contractible and compact but it does not have the fixed point property (theorem 21 in [9]).

Corollary 5.15. The real projective plane $\mathbb{R}P^2$ has the fixed point property.

In section 4.5.5 we showed that

$$H_n(\mathbb{RP}^2) = \begin{cases} \mathbb{Z}, & n = 0 \\ \mathbb{Z}_2, & n = 1 \\ 0, & \text{otherwise.} \end{cases}$$

Thus Corollary 5.13 implies that $\mathbb{RP}^2$ has the fixed point property.

A Appendix

A.1 Möbius band

$$\partial_2 = \begin{array}{c} \\ 01 \\ 02 \\ 03 \\ 04 \\ 05 \\ 12 \\ 14 \\ 15 \\ 23 \\ 25 \\ 34 \\ 45 \end{array} \begin{array}{c} \begin{array}{cccccc} 014 & 023 & 025 & 034 & 125 & 145 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\partial_1 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccccccccc} 01 & 02 & 03 & 04 & 05 & 12 & 14 & 15 & 23 & 25 & 34 & 45 \end{array} \\ \begin{bmatrix} -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\widetilde{\partial}_0 = \langle\rangle \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

boundary	number of zero columns	number of non-zero rows	diagonal elements greater than 1
∂_3	0	0	-
∂_2	0	6	-
∂_1	7	5	-
$\widetilde{\partial}_0$	5	1	-

Table 3: Möbius band

A.2 Torus

$\partial_2 =$

	014	016	023	028	034	068	125	127	145	167	235	278	347	356	367	458	478	568
01	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
02	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
03	0	0	−1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
04	−1	0	0	0	−1	0	0	0	0	0	0	0	0	0	0	0	0	0
06	0	−1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
08	0	0	0	−1	0	−1	0	0	0	0	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0
14	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	−1	0	−1	0	0	0	0	0	0	0	0	0
16	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
17	0	0	0	0	0	0	0	−1	0	−1	0	0	0	0	0	0	0	0
23	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
25	0	0	0	0	0	0	1	0	0	0	−1	0	0	0	0	0	0	0
27	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0
28	0	0	0	1	0	0	0	0	0	0	0	−1	0	0	0	0	0	0
34	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0
35	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0
36	0	0	0	0	0	0	0	0	0	0	0	0	0	−1	1	0	0	0
37	0	0	0	0	0	0	0	0	0	0	0	0	−1	0	−1	0	0	0
45	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0
47	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0
48	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	−1	−1	0
56	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1
58	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	−1
67	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0
68	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1
78	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0

$$
\rightsquigarrow \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
\partial_1 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}
\begin{array}{c}
\begin{array}{ccccccccccccccccccccccccccc}
01 & 02 & 03 & 04 & 06 & 08 & 12 & 14 & 15 & 16 & 17 & 23 & 25 & 27 & 28 & 34 & 35 & 36 & 37 & 45 & 47 & 48 & 56 & 58 & 67 & 68 & 78
\end{array} \\
\begin{bmatrix}
-1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1
\end{bmatrix}
\end{array}
$$

$$
\rightsquigarrow
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$\widetilde{\partial}_0 =_{\langle\rangle} \begin{array}{c} \begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

boundary	number of zero columns	number of non-zero rows	diagonal elements greater than 1
∂_3	0	0	-
∂_2	1	17	-
∂_1	19	8	-
$\widetilde{\partial}_0$	8	1	-

Table 4: Torus

A.3 Real projective plane

$\partial_2 =$

	014	018	025	027	034	038	056	067	125	127	145	178	347	356	358	367	458	478
01	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
02	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
03	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0
04	−1	0	0	0	−1	0	0	0	0	0	0	0	0	0	0	0	0	0
05	0	0	−1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
06	0	0	0	0	0	0	−1	1	0	0	0	0	0	0	0	0	0	0
07	0	0	0	−1	0	0	0	−1	0	0	0	0	0	0	0	0	0	0
08	0	−1	0	0	0	−1	0	0	0	0	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0
14	1	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0	−1	0	−1	0	0	0	0	0	0	0
17	0	0	0	0	0	0	0	0	0	−1	0	1	0	0	0	0	0	0
18	0	1	0	0	0	0	0	0	0	0	0	−1	0	0	0	0	0	0
25	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
27	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0
34	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0
35	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0
36	0	0	0	0	0	0	0	0	0	0	0	0	0	−1	0	1	0	0
37	0	0	0	0	0	0	0	0	0	0	0	0	−1	0	0	−1	0	0
38	0	0	0	0	0	1	0	0	0	0	0	0	0	0	−1	0	0	0
45	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0
47	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1
48	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	−1	−1
56	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	0	0	0
58	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0
67	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0
78	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1

$$\rightsquigarrow \begin{bmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{bmatrix}$$

$$
\partial_1 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}
\begin{array}{ccccccccccccccccccccccccccc}
01 & 02 & 03 & 04 & 05 & 06 & 07 & 08 & 12 & 14 & 15 & 17 & 18 & 25 & 27 & 34 & 35 & 36 & 37 & 38 & 45 & 47 & 48 & 56 & 58 & 67 & 78 \\
\hline
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1
\end{array}
$$

$$
\rightsquigarrow \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$\widetilde{\partial}_0 =_{\langle\rangle} \begin{array}{c} \begin{matrix} \scriptstyle 0 & \scriptstyle 1 & \scriptstyle 2 & \scriptstyle 3 & \scriptstyle 4 & \scriptstyle 5 & \scriptstyle 6 & \scriptstyle 7 & \scriptstyle 8 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

boundary	number of zero columns	number of non-zero rows	diagonal elements greater than 1
∂_3	0	0	-
∂_2	0	18	2
∂_1	18	8	-
$\widetilde{\partial}_0$	8	1	-

Table 5: Real projective plane

A.4 Klein bottle

$\partial_2 =$

	014	016	026	028	034	038	125	127	145	167	256	278	347	356	358	367	458	478
01	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
02	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
03	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0
04	−1	0	0	0	−1	0	0	0	0	0	0	0	0	0	0	0	0	0
06	0	−1	−1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
08	0	0	0	−1	0	−1	0	0	0	0	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0
14	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	−1	0	−1	0	0	0	0	0	0	0	0	0
16	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
17	0	0	0	0	0	0	0	−1	0	−1	0	0	0	0	0	0	0	0
25	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0
26	0	0	1	0	0	0	0	0	0	0	−1	0	0	0	0	0	0	0
27	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0
28	0	0	0	1	0	0	0	0	0	0	0	−1	0	0	0	0	0	0
34	0	0	0	0	1	0	0	0	0	0	0	0	1	0	0	0	0	0
35	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0
36	0	0	0	0	0	0	0	0	0	0	0	0	0	−1	0	1	0	0
37	0	0	0	0	0	0	0	0	0	0	0	0	−1	0	0	−1	0	0
38	0	0	0	0	0	1	0	0	0	0	0	0	0	0	−1	0	0	0
45	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0
47	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1
48	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	−1	−1
56	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0
58	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0
67	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0
78	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1

$$
\rightsquigarrow \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
\begin{array}{c}
\\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8
\end{array}
\begin{array}{c}
\begin{array}{ccccccccccccccccccccccccccc}
01 & 02 & 03 & 04 & 06 & 08 & 12 & 14 & 15 & 16 & 17 & 25 & 26 & 27 & 28 & 34 & 35 & 36 & 37 & 38 & 45 & 47 & 48 & 56 & 58 & 67 & 78
\end{array} \\
\begin{bmatrix}
-1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1
\end{bmatrix}
\end{array}
$$

$$
\rightsquigarrow
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$\widetilde{\partial}_0 =_{\langle\rangle} \begin{array}{c} \\ \end{array} \overset{\begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{matrix}}{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}}$$

$$\rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

boundary	number of zero columns	number of non-zero rows	diagonal elements greater than 1
∂_3	0	0	-
∂_2	0	18	2
∂_1	19	8	-
$\widetilde{\partial}_0$	8	1	-

Table 6: Klein bottle

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