

**SPARSE DOMINATION THEOREM FOR MULTILINEAR  
SINGULAR INTEGRAL OPERATORS WITH  
 $L^r$ -HÖRMANDER CONDITION**

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ABSTRACT. In this note, we show that if  $T$  is a multilinear singular integral operator associated with a kernel satisfies the so-called multilinear  $L^r$ -Hörmander condition, then  $T$  can be dominated by multilinear sparse operators.

1. INTRODUCTION AND MAIN RESULTS

This note is devoted to obtain a sparse domination formula for a class of multilinear singular integral operators. Dominating Calderón-Zygmund operators by sparse operators starts from Lerner [19], in which paper he obtained the following result,

$$\|Tf(x)\|_X \leq C_T \sup_{\mathcal{D}, \mathcal{S}} \left\| \sum_{Q \in \mathcal{S}} \left( \int_Q |f| \right) \chi_Q(x) \right\|_X,$$

where the supremum is taken over all the sparse families  $\mathcal{S} \subset \mathcal{D}$  (see below for the definition) and all the dyadic grids  $\mathcal{D}$  and  $X$  is an arbitrary Banach function space. Then the  $A_2$  theorem (due to Hytönen [12]) follows as an easy consequence.

Later, this result was refined by a pointwise control independently and simultaneously by Conde-Alonso and Rey [6], and by Lerner and Nazarov [21]. All the results mentioned above require the kernel satisfying the log-Dini condition. Finally, Lacey [17] relaxed the log-Dini condition to Dini condition and then Hytönen, Roncal and Tapiola [14] refined the proof by tracking the precise dependence on the constants. Very recently, Lerner [20] also provided a new proof for this result, which also works for some more general operators. To be precise, Lerner showed the following result:

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**Theorem A.** [20, Theorem 4.2] *Given a sublinear operator  $T$ . Assume that  $T$  is of weak type  $(q, q)$  and the corresponding grand maximal truncated operator  $\mathcal{M}_T$  is of weak type  $(r, r)$ , where  $1 \leq q \leq r < \infty$ . Then, for every compactly supported  $f \in L^r(\mathbb{R}^n)$ , there exists a sparse family  $\mathcal{S}$  such that for a.e.  $x \in \mathbb{R}^n$ ,*

$$|T(f)(x)| \leq c_{n,q,r} (\|T\|_{L^q \rightarrow L^{q,\infty}} + \|\mathcal{M}_T\|_{L^r \rightarrow L^{r,\infty}}) \sum_{Q \in \mathcal{S}} \left( \int_Q |f|^r \right)^{\frac{1}{r}} \chi_Q(x).$$

Let us recall the notations in the above result. We say  $\mathcal{S}$  is a sparse family if for all cubes  $Q \in \mathcal{S}$ , there exists  $E_Q \subset Q$  which are pairwise disjoint and  $|E_Q| \geq \gamma|Q|$ , where  $0 < \gamma < 1$ . For a given operator  $T$ , the so-called ‘‘grand maximal truncated’’ operator  $\mathcal{M}_T$  is defined by

$$\mathcal{M}_T f(x) = \sup_{Q \ni x} \operatorname{ess\,sup}_{\xi \in Q} |T(f \chi_{\mathbb{R}^n \setminus 3Q})(\xi)|.$$

It is shown in [20] that Calderón-Zygmund operators with Dini continuous kernel satisfy the assumption in Theorem A with  $q = r = 1$ . In this short note, we shall give a multilinear analogue of Theorem A. Then as an application, we give a sparse domination formula for multilinear singular integral operators whose kernel  $K(x, y_1, \dots, y_m)$  satisfies the so-called  $m$ -linear  $L^r$ -Hörmander condition

$$K_r := \sup_Q \sup_{x, z \in \frac{1}{2}Q} \sum_{k=1}^{\infty} |2^k Q|^{\frac{m}{r}} \left( \int_{(2^k Q)^m \setminus (2^{k-1} Q)^m} |K(x, y_1, \dots, y_m) - K(z, y_1, \dots, y_m)|^{r'} d\vec{y} \right)^{\frac{1}{r}} < \infty,$$

where  $Q^m = \underbrace{Q \times \dots \times Q}_{m \text{ times}}$  and  $1 \leq r < \infty$ . When  $r = 1$ , the above formula is understood as

$$K_1 := \sup_Q \sup_{x, z \in \frac{1}{2}Q} \sum_{k=1}^{\infty} |2^k Q|^m \operatorname{ess\,sup}_{\vec{y} \in (2^k Q)^m \setminus (2^{k-1} Q)^m} |K(x, y_1, \dots, y_m) - K(z, y_1, \dots, y_m)| < \infty.$$

In [3], Bernicot, Frey and Petermichl showed that a large class of singular non-integral operators can be dominated by sparse operators (actually in the bilinear form sense). Even in the linear case, the  $L^r$ -Hörmander condition is beyond the ‘‘off-diagonal estimate’’ assumption in [3]. We also remark that our assumption is weaker than the assumption (H2) used in [1] (see Proposition 3.3). It is also easy to see that our assumption is weaker than the Dini condition used in [9] (see Proposition 3.2).

Now to state our main result, we need a multilinear analogue of grand maximal truncated operator. Given an operator  $T$ , define

$$\mathcal{M}_T(f_1, \dots, f_m)(x) = \sup_{Q \ni x} \operatorname{ess\,sup}_{\xi \in Q} |T(f_1, \dots, f_m)(\xi) - T(f_1 \chi_{3Q}, \dots, f_m \chi_{3Q})(\xi)|,$$

note that we don't require  $T$  to be multi-sublinear. Given a cube  $Q_0$ , for  $x \in Q_0$ , we also define a local version of  $\mathcal{M}_T$  by

$$\mathcal{M}_{T,Q_0}(f_1, \dots, f_m)(x) = \sup_{Q \ni x, Q \subset Q_0} \operatorname{ess\,sup}_{\xi \in Q} |T(f_1 \chi_{3Q_0}, \dots, f_m \chi_{3Q_0})(\xi) - T(f_1 \chi_{3Q}, \dots, f_m \chi_{3Q})(\xi)|.$$

Our first result reads as follows

**Theorem 1.1.** *Assume that  $T$  is bounded from  $L^q \times \dots \times L^q$  to  $L^{q/m, \infty}$  and  $\mathcal{M}_T$  is bounded from  $L^r \times \dots \times L^r$  to  $L^{r/m, \infty}$ , where  $1 \leq q \leq r < \infty$ . Then, for compactly supported functions  $f_i \in L^r(\mathbb{R}^n)$ ,  $i = 1, \dots, m$ , there exists a sparse family  $\mathcal{S}$  such that for a.e.  $x \in \mathbb{R}^n$ ,*

$$|T(f_1, \dots, f_m)(x)| \leq c_{n,q,r} (\|T\|_{L^q \times \dots \times L^q \rightarrow L^{q/m, \infty}} + \|\mathcal{M}_T\|_{L^r \times \dots \times L^r \rightarrow L^{r/m, \infty}}) \times \sum_{Q \in \mathcal{S}} \prod_{i=1}^m \left( \int_Q |f_i|^r \right)^{\frac{1}{r}} \chi_Q(x).$$

As a consequence, we show the following

**Theorem 1.2.** *Let  $T$  be a multilinear singular integral operator which is bounded from  $L^r \times \dots \times L^r$  to  $L^{r/m, \infty}$  and its kernel satisfies the  $m$ -linear  $L^r$ -Hörmander condition. Then, for compactly supported functions  $f_i \in L^r(\mathbb{R}^n)$ ,  $i = 1, \dots, m$ , there exists a sparse family  $\mathcal{S}$  such that for a.e.  $x \in \mathbb{R}^n$ ,*

$$|T(f_1, \dots, f_m)(x)| \leq c_{n,r} (\|T\|_{L^r \times \dots \times L^r \rightarrow L^{r/m, \infty}} + K_r) \times \sum_{Q \in \mathcal{S}} \prod_{i=1}^m \left( \int_Q |f_i|^r \right)^{\frac{1}{r}} \chi_Q(x).$$

We would like to remark that recently a different approach to sparse domination of singular integrals not relying on weak endpoint bounds for grand maximal functions has been developed by many authors, see [5, 8, 15, 18].

In the next section, we shall give a proof for Theorems 1.1 and 1.2. And in Section 3, we will give some remarks about the  $m$ -linear  $L^r$ -Hörmander condition.

## 2. PROOF OF THEOREMS 1.1 AND 1.2

The proof of Theorem 1.1 will follow the same idea used in [20], but with proper changes to make it suit for the multilinear case. For simplicity, we only prove both results in the case of  $m = 2$ , and the general case can be proved similarly. First, we prove the following lemma.

**Lemma 2.1.** *Suppose  $T$  is bounded from  $L^q \times L^q \rightarrow L^{q/2, \infty}$ ,  $1 \leq q < \infty$ . Then for a.e.  $x \in Q_0$ ,*

$$|T(f_1 \chi_{3Q_0}, f_2 \chi_{3Q_0})(x)| \leq c_n \|T\|_{L^q \times L^q \rightarrow L^{q/2, \infty}} |f_1(x) f_2(x)| + \mathcal{M}_{T,Q_0}(f_1, f_2)(x).$$

*Proof.* Suppose that  $x \in \text{int}Q_0$  and let  $x$  be a point of approximate continuity of  $T(f_1\chi_{3Q_0}, f_2\chi_{3Q_0})$  (see e.g. [10, p.46]). Then for every  $\varepsilon > 0$ , the sets

$$E_s(x) := \{y \in B(x, s) : |T(f_1\chi_{3Q_0}, f_2\chi_{3Q_0})(y) - T(f_1\chi_{3Q_0}, f_2\chi_{3Q_0})(x)| < \varepsilon\}$$

satisfy  $\lim_{s \rightarrow 0} \frac{|E_s(x)|}{|B(x, s)|} = 1$ . Denote by  $Q(x, s)$  the smallest cube centered at  $x$  and containing  $B(x, s)$ . Let  $s > 0$  be so small that  $Q(x, s) \subset Q_0$ . Then for a.e.  $y \in E_s(x)$ ,

$$\begin{aligned} |T(f_1\chi_{3Q_0}, f_2\chi_{3Q_0})(x)| &< |T(f_1\chi_{3Q_0}, f_2\chi_{3Q_0})(y)| + \varepsilon \\ &\leq |T(f_1\chi_{3Q(x, s)}, f_2\chi_{3Q(x, s)})(y)| + \mathcal{M}_{T, Q_0}(f_1, f_2)(x) + \varepsilon. \end{aligned}$$

It follows that

$$\begin{aligned} &|T(f_1\chi_{3Q_0}, f_2\chi_{3Q_0})(x)| \\ &\leq \text{ess inf}_{y \in E_s(x)} |T(f_1\chi_{3Q(x, s)}, f_2\chi_{3Q(x, s)})(y)| + \mathcal{M}_{T, Q_0}(f_1, f_2)(x) + \varepsilon \\ &\leq |E_s(x)|^{-\frac{2}{q}} \|T(f_1\chi_{3Q(x, s)}, f_2\chi_{3Q(x, s)})\|_{L^{q/2, \infty}} + \mathcal{M}_{T, Q_0}(f_1, f_2)(x) + \varepsilon \\ &\leq \|T\|_{L^q \times L^q \rightarrow L^{\frac{q}{2}, \infty}} \frac{1}{|E_s(x)|^{2/q}} \prod_{i=1}^2 \left( \int_{3Q(x, s)} |f_i(y)|^q dy \right)^{\frac{1}{q}} + \mathcal{M}_{T, Q_0}(f_1, f_2)(x) + \varepsilon. \end{aligned}$$

Assuming additionally that  $x$  is a Lebesgue point of  $|f_1|^q$  and  $|f_2|^q$  and letting subsequently  $s \rightarrow 0$  and  $\varepsilon \rightarrow 0$  will conclude the proof.  $\square$

Now we are ready to prove Theorem 1.1.

*Proof of Theorem 1.1.* Fix a cube  $Q_0 \subset \mathbb{R}^n$ . We shall prove the following recursive inequality,

(2.2)

$$|T(f_1\chi_{3Q_0}, f_2\chi_{3Q_0})(x)|\chi_{Q_0} \leq c_{n, T} \langle |f_1|^r \rangle_{3Q_0}^{\frac{1}{r}} \langle |f_2|^r \rangle_{3Q_0}^{\frac{1}{r}} + \sum_j |T(f_1\chi_{3P_j}, f_2\chi_{3P_j})(x)|\chi_{P_j},$$

where  $P_j$  are disjoint dyadic subcubes of  $Q_0$ , i.e.  $P_j \in \mathcal{D}(Q_0)$  and moreover,  $\sum_j |P_j| \leq \frac{1}{2}|Q_0|$ . Observe that for arbitrary pairwise disjoint cubes  $P_j \in \mathcal{D}(Q_0)$ , we have

$$\begin{aligned} &|T(f_1\chi_{3Q_0}, f_2\chi_{3Q_0})(x)|\chi_{Q_0} \\ &= |T(f_1\chi_{3Q_0}, f_2\chi_{3Q_0})(x)|\chi_{Q_0 \setminus \cup_j P_j} + \sum_j |T(f_1\chi_{3Q_0}, f_2\chi_{3Q_0})(x)|\chi_{P_j} \\ &\leq |T(f_1\chi_{3Q_0}, f_2\chi_{3Q_0})(x)|\chi_{Q_0 \setminus \cup_j P_j} + \sum_j |T(f_1\chi_{3P_j}, f_2\chi_{3P_j})(x)|\chi_{P_j} \\ &\quad + \sum_j |T(f_1\chi_{3Q_0}, f_2\chi_{3Q_0})(x) - T(f_1\chi_{3P_j}, f_2\chi_{3P_j})(x)|\chi_{P_j}. \end{aligned}$$

Hence, in order to prove the recursive claim, it suffices to show that one can select pairwise disjoint cubes  $P_j \in \mathcal{D}(Q_0)$  with  $\sum_j |P_j| \leq \frac{1}{2}|Q_0|$  and such

that for a.e.  $x \in Q_0$ ,

$$\begin{aligned} & \sum_j |T(f_1 \chi_{3Q_0}, f_2 \chi_{3Q_0})(x) - T(f_1 \chi_{3P_j}, f_2 \chi_{3P_j})(x)| \chi_{P_j} \\ & + |T(f_1 \chi_{3Q_0}, f_2 \chi_{3Q_0})(x)| \chi_{Q_0 \setminus \cup_j P_j} \leq c_{n,T} \langle |f_1|^r \rangle_{3Q_0}^{\frac{1}{r}} \langle |f_2|^r \rangle_{3Q_0}^{\frac{1}{r}}. \end{aligned}$$

By our assumption,  $\mathcal{M}_T$  is bounded from  $L^r \times L^r$  to  $L^{r/2, \infty}$ . Therefore, there is some sufficient large  $c_n$  such that the set

$$\begin{aligned} E := & \{x \in Q_0 : |f_1(x)f_2(x)| > c_n \langle |f_1|^r \rangle_{3Q_0}^{\frac{1}{r}} \langle |f_2|^r \rangle_{3Q_0}^{\frac{1}{r}}\} \\ & \cup \{x \in Q_0 : \mathcal{M}_{T, Q_0}(f_1, f_2)(x) > c_n \|\mathcal{M}_T\|_{L^r \times L^r \rightarrow L^{r/2, \infty}} \langle |f_1|^r \rangle_{3Q_0}^{\frac{1}{r}} \langle |f_2|^r \rangle_{3Q_0}^{\frac{1}{r}}\} \end{aligned}$$

will satisfy  $|E| \leq \frac{1}{2^{n+2}}|Q_0|$ . The Calderón-Zygmund decomposition applied to the function  $\chi_E$  on  $Q_0$  at height  $\lambda = \frac{1}{2^{n+1}}$  produces pairwise disjoint cubes  $P_j \in \mathcal{D}(Q_0)$  such that

$$\frac{1}{2^{n+1}}|P_j| \leq |P_j \cap E| < \frac{1}{2}|P_j|$$

and  $|E \setminus \cup_j P_j| = 0$ . It follows that  $\sum_j |P_j| \leq \frac{1}{2}|Q_0|$  and  $P_j \cap E^c \neq \emptyset$ . Therefore,

$$\begin{aligned} & \operatorname{ess\,sup}_{\xi \in P_j} |T(f_1 \chi_{3Q_0}, f_2 \chi_{3Q_0})(\xi) - T(f_1 \chi_{3P_j}, f_2 \chi_{3P_j})(\xi)| \\ & \leq c_n \|\mathcal{M}_T\|_{L^r \times L^r \rightarrow L^{r/2, \infty}} \langle |f_1|^r \rangle_{3Q_0}^{\frac{1}{r}} \langle |f_2|^r \rangle_{3Q_0}^{\frac{1}{r}}. \end{aligned}$$

On the other hand, by Lemma 2.1, for a.e.  $x \in Q_0 \setminus \cup_j P_j$ , we have

$$\begin{aligned} |T(f_1 \chi_{3Q_0}, f_2 \chi_{3Q_0})(x)| & \leq c_n (\|T\|_{L^q \times L^q \rightarrow L^{q/2, \infty}} + \|\mathcal{M}_T\|_{L^r \times L^r \rightarrow L^{r/2, \infty}}) \\ & \quad \times \langle |f_1|^r \rangle_{3Q_0}^{\frac{1}{r}} \langle |f_2|^r \rangle_{3Q_0}^{\frac{1}{r}}. \end{aligned}$$

Therefore, combining the estimates we arrive at (2.2) with

$$c_{n,T} \approx \|T\|_{L^q \times L^q \rightarrow L^{q/2, \infty}} + \|\mathcal{M}_T\|_{L^r \times L^r \rightarrow L^{r/2, \infty}}.$$

Now with (2.2), the rest of the argument is the same as that in [20] and we complete the proof.  $\square$

Next we turn to prove Theorem 1.2

*Proof of Theorem 1.2.* It suffices to prove that  $\mathcal{M}_T$  is bounded from  $L^r \times L^r$  to  $L^{r/2, \infty}$ . Indeed, let  $x, x', \xi \in Q \subset \frac{1}{2} \cdot 3Q$ . We have

$$\begin{aligned} (2.3) \quad & |T(f_1, f_2)(\xi) - T(f_1 \chi_{3Q}, f_2 \chi_{3Q})(\xi)| \\ & = \left| \iint_{(\mathbb{R}^n)^2 \setminus (3Q)^2} K(\xi, y_1, y_2) f_1(y_1) f_2(y_2) dy_1 dy_2 \right| \\ & \leq \left| \iint_{(\mathbb{R}^n)^2 \setminus (3Q)^2} (K(\xi, y_1, y_2) - K(x', y_1, y_2)) f_1(y_1) f_2(y_2) dy_1 dy_2 \right| \\ & \quad + |T(f_1, f_2)(x')| + |T(f_1 \chi_{3Q}, f_2 \chi_{3Q})(x')|. \end{aligned}$$

By the bilinear  $L^r$ -Hörmander condition, we have

$$\begin{aligned}
& \left| \iint_{(\mathbb{R}^n)^2 \setminus (3Q)^2} (K(\xi, y_1, y_2) - K(x', y_1, y_2)) f_1(y_1) f_2(y_2) dy_1 dy_2 \right| \\
& \leq \sum_{k=1}^{\infty} \iint_{(2^k 3Q)^2 \setminus (2^{k-1} 3Q)^2} |K(\xi, y_1, y_2) - K(x', y_1, y_2)| \cdot |f_1(y_1)| \cdot |f_2(y_2)| dy_1 dy_2 \\
& \leq \sum_{k=1}^{\infty} \left( \iint_{(2^k 3Q)^2 \setminus (2^{k-1} 3Q)^2} |K(\xi, y_1, y_2) - K(x', y_1, y_2)|^{r'} dy_1 dy_2 \right)^{\frac{1}{r'}} \\
& \quad \times \left( \iint_{(2^k 3Q)^2} |f_1(y_1) f_2(y_2)|^r dy_1 dy_2 \right)^{\frac{1}{r}} \\
& = \sum_{k=1}^{\infty} |2^k 3Q|^{\frac{2}{r}} \left( \iint_{(2^k 3Q)^2 \setminus (2^{k-1} 3Q)^2} |K(\xi, y_1, y_2) - K(x', y_1, y_2)|^{r'} dy_1 dy_2 \right)^{\frac{1}{r'}} \\
& \quad \times \left( \frac{1}{|2^k 3Q|^2} \iint_{(2^k 3Q)^2} |f_1(y_1) f_2(y_2)|^r dy_1 dy_2 \right)^{\frac{1}{r}} \\
& \leq K_r (\mathcal{M}(|f_1|^r, |f_2|^r)(x))^{\frac{1}{r}}.
\end{aligned}$$

Then by taking  $L^{r/4}$  average over  $x' \in Q$  on both side of (2.3) we obtain

$$\begin{aligned}
& |T(f_1, f_2)(\xi) - T(f_1 \chi_{3Q}, f_2 \chi_{3Q})(\xi)| \\
& \leq K_r (\mathcal{M}(|f_1|^r, |f_2|^r)(x))^{\frac{1}{r}} + \left( \frac{1}{|Q|} \int_Q |T(f_1, f_2)(x')|^{\frac{r}{4}} dx' \right)^{\frac{4}{r}} \\
& \quad + \left( \frac{1}{|Q|} \int_Q |T(f_1 \chi_{3Q}, f_2 \chi_{3Q})(x')|^{\frac{r}{4}} dx' \right)^{\frac{4}{r}} \\
& \leq K_r (\mathcal{M}(|f_1|^r, |f_2|^r)(x))^{\frac{1}{r}} + M_{r/4}(T(f_1, f_2))(x) \\
& \quad + c_r \|T(f_1 \chi_{3Q}, f_2 \chi_{3Q})\|_{L^{\frac{r}{2}, \infty}(Q, \frac{dx'}{|Q|})} \\
& \leq K_r (\mathcal{M}(|f_1|^r, |f_2|^r)(x))^{\frac{1}{r}} + M_{r/4}(T(f_1, f_2))(x) \\
& \quad + c_r \|T\|_{L^r \times L^r \rightarrow L^{r/2, \infty}} \left( \frac{1}{|Q|} \int_{3Q} |f_1|^r \right)^{\frac{1}{r}} \cdot \left( \frac{1}{|Q|} \int_{3Q} |f_2|^r \right)^{\frac{1}{r}} \\
& \leq (K_r + c_{n,r} \|T\|_{L^r \times L^r \rightarrow L^{r/2, \infty}}) (\mathcal{M}(|f_1|^r, |f_2|^r)(x))^{\frac{1}{r}} + M_{r/4}(T(f_1, f_2))(x),
\end{aligned}$$

where the bilinear maximal function  $\mathcal{M}$  is defined as

$$\mathcal{M}(f, g)(x) = \sup_{Q \ni x} \left( \frac{1}{|Q|} \int_Q |f_1| \right) \left( \frac{1}{|Q|} \int_Q |f_2| \right).$$

So we conclude that

$$\begin{aligned}
\mathcal{M}_T(f_1, f_2)(x) & \leq (K_r + c_{n,r} \|T\|_{L^r \times L^r \rightarrow L^{r/2, \infty}}) (\mathcal{M}(|f_1|^r, |f_2|^r)(x))^{\frac{1}{r}} \\
& \quad + M_{r/4}(T(f_1, f_2))(x).
\end{aligned}$$

It is obvious that  $(\mathcal{M}(|f_1|^r, |f_2|^r)(x))^{\frac{1}{r}}$  is bounded from  $L^r \times L^r$  to  $L^{r/2, \infty}$ . On the other hand, since it is well-known that  $M_{r/4}$  is bounded from  $L^{r/2, \infty}$  to  $L^{r/2, \infty}$ , by the assumption that  $T$  is bounded from  $L^r \times L^r$  to  $L^{r/2, \infty}$ , we obtain

$$\|M_{r/4}(T(f_1, f_2))\|_{L^{r/2, \infty}} \leq c_{n,r} \|T\|_{L^r \times L^r \rightarrow L^{r/2, \infty}} \|f_1\|_{L^r} \|f_2\|_{L^r}.$$

Therefore,  $\mathcal{M}_T$  is bounded from  $L^r \times L^r \rightarrow L^{r/2, \infty}$  and the desired result follows from Theorem 1.1.  $\square$

### 3. SOME REMARKS

In this section, we give some remarks about the  $L^r$ -Hörmander condition and some applications of our main result. It is well known that the Hörmander condition

$$\sup_{x, z \in \mathbb{R}^n} \int_{|y-x| > 2|x-z|} |K(x, y) - K(z, y)| dy < \infty$$

is not sufficient for

$$(3.1) \quad \int_{\mathbb{R}^n} T(f)^p w(x) dx \leq C \int_{\mathbb{R}^n} M_r(f)^p w(x) dx, \quad w \in A_\infty$$

for any  $r \geq 1$  and  $0 < p < \infty$  (see [25, Theorem 3.1]). So we cannot expect the sparse domination theorem for singular integral operators with this kernel. This is because once we have

$$|T(f)(x)| \leq C_{T,r} \sum_{Q \in \mathcal{S}} \left( \int_Q |f|^r \right)^{\frac{1}{r}} \chi_Q(x)$$

for some  $r \geq 1$ , then (3.1) holds for  $p = 1$  (see [13, Lemma 4.1]) and therefore for all  $0 < p < \infty$  (see [7, Theorem 1.1]). Then it is reasonable to consider somewhat stronger condition such as our  $L^r$ -Hörmander condition. For more background, see [16, 25, 26].

Now we shall briefly show that our conditions are weaker than Dini condition, which is used in [9] by Damián, Hormozi and the author. Recall that the Dini condition is defined by

$$\begin{aligned} & |K(x+h, y_1, \dots, y_m) - K(x, y_1, \dots, y_m)| + |K(x, y_1+h, \dots, y_m) - K(x, y_1, \dots, y_m)| \\ & + \dots + |K(x, y_1, \dots, y_m+h) - K(x, y_1, \dots, y_m)| \\ & \leq \frac{1}{(\sum_{i=1}^m |x-y_i|)^{2n}} \omega \left( \frac{|h|}{\sum_{i=1}^m |x-y_i|} \right), \end{aligned}$$

whenever  $|h| \leq \frac{1}{2} \max\{|x-y_i| : i = 1, \dots, m\}$ , where  $\omega$  is increasing,  $\omega(0) = 0$  and  $\|\omega\|_{\text{Dini}} = \int_0^1 \omega(t) dt/t < \infty$ .

**Proposition 3.2.**  *$m$ -linear Dini condition implies  $m$ -linear  $L^r$ -Hörmander condition.*

*Proof.* Again, we just prove the case  $m = 2$ . It is obvious that we just require regularity in the  $x$  variable. Fix  $x, z \in \frac{1}{2}Q$ . Since  $|x - z| < \frac{1}{2}\sqrt{n}\ell(Q)$ , for  $k > \log_2(1 + 4\sqrt{n})$  and  $(y_1, y_2) \in (2^k Q)^2 \setminus (2^{k-1} Q)^2$ , we have  $|x - z| \leq \frac{1}{2} \max\{|x - y_1|, |x - y_2|\}$ . Therefore,

$$\begin{aligned} & \sum_{k > \log_2(1+4\sqrt{n})} |2^k Q|^{\frac{2}{r}} \left( \int_{(2^k Q)^2 \setminus (2^{k-1} Q)^2} |K(x, y_1, y_2) - K(z, y_1, y_2)|^{r'} d\vec{y} \right)^{\frac{1}{r'}} \\ & \leq \sum_{k > \log_2(1+4\sqrt{n})} |2^k Q|^{\frac{2}{r}} \cdot w\left(\frac{2\sqrt{n}}{2^k - 1}\right) \cdot \frac{1}{(2^{k-2} - \frac{1}{4})^{2n} \ell(Q)^{2n}} \cdot |2^k Q|^{\frac{2}{r}} \\ & \lesssim_n \sum_{k > \log_2(1+4\sqrt{n})} w\left(\frac{4\sqrt{n}}{2^k}\right) \lesssim_n \|\omega\|_{\text{Dini}}. \end{aligned}$$

It remains to consider those  $1 \leq k \leq \log_2(1 + 4\sqrt{n})$ . We should be careful because we don't assume any size condition. In this case, since

$$\max\{|y - y_1|, |y - y_2|\} \geq \frac{1}{4}\ell(Q), \quad \forall y \in \frac{1}{2}Q \text{ and } (y_1, y_2) \in (2^k Q)^2 \setminus (2^{k-1} Q)^2,$$

we select  $4\lceil\sqrt{n}\rceil$  points  $x_1, \dots, x_{4\lceil\sqrt{n}\rceil}$  in the segment between  $x$  and  $z$  such that

$$|x - x_1|, |x_i - x_{i+1}|, |x_{4\lceil\sqrt{n}\rceil} - z| \leq \frac{1}{8}\ell(Q), \quad i = 1, \dots, 4\lceil\sqrt{n}\rceil - 1.$$

For convenience, denote  $x_0 = x$  and  $x_{4\lceil\sqrt{n}\rceil+1} = z$ . Then we have

$$\begin{aligned} |K(x, y_1, y_2) - K(z, y_1, y_2)| & \leq \sum_{i=0}^{4\lceil\sqrt{n}\rceil} |K(x_i, y_1, y_2) - K(x_{i+1}, y_1, y_2)| \\ & \leq c_n \omega\left(\frac{1}{2}\right) \ell(Q)^{-2n}. \end{aligned}$$

Consequently,

$$\begin{aligned} & \sum_{1 \leq k \leq \log_2(1+4\sqrt{n})} |2^k Q|^{\frac{2}{r}} \left( \int_{(2^k Q)^2 \setminus (2^{k-1} Q)^2} |K(x, y_1, y_2) - K(z, y_1, y_2)|^{r'} d\vec{y} \right)^{\frac{1}{r'}} \\ & \lesssim_n \omega\left(\frac{1}{2}\right) \lesssim \|\omega\|_{\text{Dini}}. \end{aligned}$$

This completes the proof.  $\square$

Next we will show that  $L^r$ -Hörmander condition is also weaker than the regularity assumption used in [1] (which was originally introduced in [2]). Recall that the regularity assumption in [1] reads as follows:

**(H2):** There exists  $\delta > n/r$  so that

$$\left( \int_{S_{j_m}(Q)} \cdots \int_{S_{j_1}(Q)} |K(x, y_1, \dots, y_m) - K(z, y_1, \dots, y_m)|^{r'} d\vec{y} \right)^{\frac{1}{r'}}$$



$$\leq C \frac{|x-z|^{m(\delta-n/r)}}{|Q|^{m\delta/n}} 2^{-m\delta j_0}$$

for all cubes  $Q$ , all  $x, z \in \frac{1}{2}Q$  and  $(j_1, \dots, j_m) \neq (0, \dots, 0)$ , where  $j_0 = \max\{j_i\}_{1 \leq i \leq m}$  and  $S_j(Q) = 2^j Q \setminus 2^{j-1} Q$  if  $j \geq 1$ , otherwise,  $S_j(Q) = Q$ .

**Proposition 3.3.** *Assumption (H2) implies  $m$ -linear  $L^r$ -Hörmander condition.*

*Proof.* Again, we just prove the bilinear case. Observe that

$$\begin{aligned} (2^j Q)^2 \setminus (2^{j-1} Q)^2 &= (2^j Q \setminus 2^{j-1} Q)^2 \cup (2^j Q \times (2^j Q \setminus 2^{j-1} Q)) \\ &\quad \cup ((2^j Q \setminus 2^{j-1} Q) \times 2^j Q) \\ &= (S_j(Q))^2 \cup (\cup_{l \leq j} S_l(Q) \times S_j(Q)) \cup (\cup_{l \leq j} S_j(Q) \times S_l(Q)). \end{aligned}$$

By triangle inequality, we have

$$\begin{aligned} &\sum_{j=1}^{\infty} |2^j Q|^{\frac{2}{r}} \left( \int_{(2^j Q)^2 \setminus (2^{j-1} Q)^2} |K(x, y_1, y_2) - K(z, y_1, y_2)|^{r'} d\vec{y} \right)^{\frac{1}{r'}} \\ &\leq \sum_{j=1}^{\infty} |2^j Q|^{\frac{2}{r}} \left( \int_{(S_j(Q))^2} |K(x, y_1, y_2) - K(z, y_1, y_2)|^{r'} d\vec{y} \right)^{\frac{1}{r'}} \\ &\quad + \sum_{j=1}^{\infty} |2^j Q|^{\frac{2}{r}} \sum_{l=0}^j \left( \int_{S_j(Q) \times S_l(Q)} |K(x, y_1, y_2) - K(z, y_1, y_2)|^{r'} d\vec{y} \right)^{\frac{1}{r'}} \\ &\quad + \sum_{j=1}^{\infty} |2^j Q|^{\frac{2}{r}} \sum_{l=0}^j \left( \int_{S_l(Q) \times S_j(Q)} |K(x, y_1, y_2) - K(z, y_1, y_2)|^{r'} d\vec{y} \right)^{\frac{1}{r'}} \\ &\lesssim \sum_{j=1}^{\infty} |2^j Q|^{\frac{2}{r}} \frac{1}{|Q|^{2/r}} 2^{-2\delta j} (1+j) \\ &< \infty, \end{aligned}$$

the last inequality holds due to  $\delta > n/r$ . This completes the proof.  $\square$

**Remark 3.4.** *For  $r > 1$ , let  $T$  be a linear Fourier multiplier with Hörmander condition with parameter  $n/2 < s < n$  (see (3.6) in below for the definition). It is shown in [16, Theorem 3] that  $T$  is not bounded on  $L^p(w)$  for some  $w \in A_p$  when  $p < n/s$  or  $p > (\frac{n}{s})'$ . This means (H2) (which is a consequence of the assumption described as above, see [1]) and therefore the  $L^r$ -Hörmander condition are not sufficient for the Dini condition.*

*For  $r = 1$ , recall that we only need the regularity on the  $x$ -variable, in this sense,  $L^1$ -Hörmander condition is strictly weaker than the Dini condition. However, the full regularity in the Dini condition is to ensure the weak end-point boundedness. In fact, we can show that there is only a tiny difference between  $L^1$ -Hörmander condition and Dini-condition in the  $x$ -variable. To*

see this, define

$$\omega^{x,z}(t) := \sup_{\vec{y}: \frac{t}{2} \leq \frac{|x-z|}{\sum_{i=1}^2 |x-y_i|} \leq t} |K(x, y_1, y_2) - K(z, y_1, y_2)| \left( \sum_{i=1}^2 |x - y_i| \right)^{2n}.$$

Then

$$\begin{aligned} & \sup_{x,z} \sum_{k=1}^{\infty} \omega^{x,z}(2^{-k}) \\ & \lesssim \sup_Q \sup_{x,z \in \frac{1}{2}Q} \sum_{k=1}^{\infty} |2^k Q|^2 \operatorname{ess\,sup}_{\vec{y} \in (2^k Q)^2 \setminus (2^{k-1} Q)^2} |K(x, y_1, y_2) - K(z, y_1, y_2)| \\ & \leq K_1, \end{aligned}$$

where  $\frac{1}{2}Q$  is a cube which contains both  $x$  and  $z$  with  $\ell(\frac{1}{2}Q) = \|x - z\|_{\infty}$ . However, the Dini condition in the  $x$ -variable can be written as the following

$$\sum_{k=1}^{\infty} \sup_{x,z} \omega^{x,z}(2^{-k}) < \infty.$$

It is hard to find an example to differentiate these two conditions.

**Remark 3.5.** We claim that actually the  $L^r$ -Hörmander condition is strictly weaker than (H2). Indeed, (H2) is essentially of Hölder type while  $L^r$  Hörmander condition is essentially of Dini type. To prove our claim we borrow the example from [25]. And to make things easier we only consider the linear case in one dimension. Define

$$K(x) = |x - 4|^{-\frac{1}{r'}} \left( \log \frac{e}{|x - 4|} \right)^{-\frac{1+\beta}{r'}} \chi_{\{3 < x < 5\}}(x).$$

It is easy to check that  $K \in L^{r'} \cap L^1$ . Then  $T : f \rightarrow K * f$  is bounded on  $L^p$  for all  $1 \leq p \leq \infty$ . It is already proved in [25] that  $K$  satisfies the  $L^r$ -Hörmander condition. Define

$$K_{\ell}(x) = \begin{cases} K(x), & x \in \bigcup_{k=0}^{2^{\ell+1}-1} (3 + \frac{k}{2^{\ell}}, 3 + \frac{3k+1}{3 \cdot 2^{\ell}}], \\ 0, & \text{otherwise.} \end{cases}$$

Similar argument as that in [25] shows that  $K_{\ell}$  satisfies the  $L^r$ -Hörmander condition uniformly, i.e.  $\sup_{\ell} (K_{\ell})_r < \infty$ . Let  $x = 0$ ,  $z = 2^{-\ell-1}$  and  $I_{\ell} = [0, 2^{-\ell}]$ . We need to analyze

$$\left( \int_{S_j(I_{\ell})} |K_{\ell}(y-x) - K_{\ell}(y-z)|^{r'} dy \right)^{\frac{1}{r'}}.$$

Observe that only  $j = \ell + 2$  and  $j = \ell + 3$  are non-zero terms. We have

$$\left( \int_{2^{\ell+2}I_{\ell} \setminus 2^{\ell+1}I_{\ell}} |K_{\ell}(y) - K_{\ell}(y - 2^{-\ell-1})|^{r'} dy \right)^{\frac{1}{r'}}$$

$$\begin{aligned}
&\geq \left( \sum_{k=0}^{2^\ell-1} \int_{3+\frac{k}{2^\ell}}^{3+\frac{3k+1}{3 \cdot 2^\ell}} |K_\ell(y)|^{r'} dy \right)^{\frac{1}{r'}} \\
&\gtrsim \|K\|_{L^{r'}} \gtrsim 2^{(\delta-\frac{1}{r})\ell} \|K\|_{L^{r'}} \frac{|x-z|^{(\delta-\frac{1}{r})}}{|I_\ell|^\delta} 2^{-(\ell+2)\delta}.
\end{aligned}$$

This shows that there exist a sequence of operators  $T_\ell : f \rightarrow K_\ell * f$ , whose kernels satisfy  $L^r$  Hörmander condition uniformly. However, since  $\delta > \frac{1}{r}$ , the constant  $C$  in (H2) tends to infinity when  $\ell \rightarrow \infty$ , which means these two conditions cannot be equivalent. In other words,  $L^r$ -Hörmander condition is strictly weaker than (H2).

In the end, we give an application of our result for multilinear Fourier multipliers:

$$T(f_1, \dots, f_m)(x) = \int_{\mathbb{R}^{mn}} a(\vec{y}) e^{2\pi i x \cdot (\sum_{i=1}^m y_i)} \prod_{i=1}^m \widehat{f}(y_i) d\vec{y},$$

it is shown in [2] that multilinear Mihlin condition implies the assumption (H2). However, for the multilinear Hörmander condition ([11]), i.e.,

$$(3.6) \quad \sup_{R>0} \|a(R\xi) \chi_{\{1<|\xi|<2\}}\|_{H^s(\mathbb{R}^{mn})} < \infty, \quad \frac{mn}{2} < s \leq mn,$$

which is weaker than multilinear Mihlin condition, it is unknown. Very recently, Chaffee, Torres and Wu [4] showed that (3.6) implies the multilinear  $L^r$ -Hörmander condition with  $r = mn/s$ . Therefore, Theorem 1.2 also applies to multilinear Fourier multipliers whose symbols satisfy (3.6).

As a consequence of our sparse domination theorem, we can give quantitative weighted bounds for them. We have

**Theorem 3.7.** *Let  $T$  satisfy the assumption in Theorem 1.2, then*

$$\|T\|_{L^{p_1}(w_1) \times \dots \times L^{p_m}(w_m) \rightarrow L^p(v_{\vec{w}})} \leq c_{m,n,T,\vec{p}} [\vec{w}]_{A_{\vec{p}/r}}^{\max\{1, \max_i \frac{(p_i/r)'}{p}\}}.$$

Here

$$[\vec{w}]_{A_{\vec{p}/r}} := \sup_Q \left( \int_Q \prod_{i=1}^m w_i^{\frac{p_i}{r}} \right) \prod_{i=1}^m \left( \int_Q w_i^{-\frac{r}{p_i-r}} \right)^{\frac{p(p_i-r)}{p_i r}}.$$

For the proof, we refer the readers to [1]. One can also follow the maximal function trick used in [9] and then utilize the result in [22]. We can also obtain the  $A_p$ - $A_\infty$  type bounds, see [23, 9] for details. As we have discussed in the above, all these estimates apply to the multilinear Fourier multipliers with symbols satisfying (3.6). Notice that the qualitative result was obtained by the author and Sun in [24].

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