Resistive dissipative magnetohydrodynamics from the Boltzmann-Vlasov equation

Gabriel S. Denicol,[1] Etele Molnár,[2,3,*] Harri Niemi,[2,4,5] and Dirk H. Rischke[2,6]

[1]*Instituto de Física, Universidade Federal Fluminense, UFF, Niterói, 24210-346, RJ, Brazil*
[2]*Institut für Theoretische Physik, Johann Wolfgang Goethe–Universität, Max-von-Laue-Street 1, D–60438 Frankfurt am Main, Germany*
[3]*Institute of Physics and Technology, University of Bergen, Allegaten 55, 5007 Bergen, Norway*
[4]*Department of Physics, University of Jyväskylä, P.O. Box 35, FI-40014 University of Jyväskylä, Finland*
[5]*Helsinki Institute of Physics, P.O. Box 64, FI-00014 University of Helsinki, Finland*
[6]*Department of Modern Physics, University of Science and Technology of China, Hefei, Anhui 230026, China*

(Received 5 February 2019; published 28 March 2019)

We derive the equations of motion of relativistic, resistive, second-order dissipative magnetohydrodynamics from the Boltzmann-Vlasov equation using the method of moments. We thus extend our previous work [Phys. Rev. D **98**, 076009 (2018)], where we only considered the nonresistive limit, to the case of finite electric conductivity. This requires keeping terms proportional to the electric field E^{μ} in the equations of motions and leads to new transport coefficients due to the coupling of the electric field to dissipative quantities. We also show that the Navier-Stokes limit of the charge-diffusion current corresponds to Ohm's law, while the coefficients of electrical conductivity and charge diffusion are related by a type of Wiedemann-Franz law.

DOI: 10.1103/PhysRevD.99.056017

I. INTRODUCTION

Second-order theories of relativistic dissipative fluid dynamics play an essential role in understanding the dynamics of ultrarelativistic heavy-ion collisions [1]. Moreover, strong electromagnetic fields are created in noncentral heavy-ion collisions [2–5], which give rise to novel and interesting phenomena in strongly interacting matter, like the chiral magnetic effect [for a review, see Ref. [6] and refs. therein]. In order to describe the evolution of the system, second-order relativistic dissipative fluid dynamics [7,8] needs to be extended to a self-consistent magnetohydrodynamic framework [9,10].

In Ref. [11] the equations of motion of relativistic, nonresistive, second-order dissipative magnetohydrodynamics were derived from the Boltzmann-Vlasov equation. In a nonresistive, i.e., ideally conducting, fluid the electric field is not an independent degree of freedom but is related to the magnetic field by $\mathbf{E} = -\mathbf{v} \times \mathbf{B}$ and therefore can be eliminated from the equations of motion. While this is a common approximation in magnetohydrodynamics, it cannot be realized in a fully consistent manner in a system whose microscopic dynamics is described by the Boltzmann equation. The reason is that the electric conductivity σ_E is a fluid-dynamical transport coefficient and thus, like all other transport coefficients, proportional to the mean free path of the particles. Taking the limit $\sigma_E \to \infty$ while keeping the values of all other transport coefficients finite is inconsistent.

In this paper we will dispense with the assumption of infinite conductivity, and derive the equations of motion of resistive, second-order dissipative magnetohydrodynamics. As in our previous work [11] we assume a single-component system of spin-zero particles with electric charge $\mathbb{q}$ undergoing binary elastic collisions. The fluid-dynamical equations of motion are derived by using the 14-moment approximation in the framework developed in Refs. [8,12,13]. The electric field is now included explicitly, and the resistive magnetohydrodynamic equations of motion contain new terms with new transport coefficients due to the coupling of charged particles to the electric field.

The electric conductivity σ_E is defined through Ohm's law of magnetohydrodynamics, $\mathbb{J}^{\mu}_{\rm ind} = \sigma_E E^{\mu}$, where $\mathbb{J}^{\mu}_{\rm ind}$ is the charge current induced by the electric field E^{μ}. We will show that the electric conductivity is related to the thermal conductivity κ, giving rise to a type of Wiedemann-Franz law, $\sigma_E \equiv \mathbb{q}^2 \kappa / T$, where T is the temperature of matter.

The paper is organized as follows. In Sec. II we recall the equations of motion of magnetohydrodynamics. In Sec. III we derive the infinite set of equations of motion for the irreducible moments up to tensor-rank two of the deviation of the single-particle distribution function from local equilibrium. Section IV is devoted to truncating this infinite system applying the 14-moment approximation, to obtain the equations for resistive, second-order dissipative

*Extreme Light Infrastructure–Nuclear Physics (ELI-NP), Horia Hulubei Institute for Nuclear Physics (IFIN-HH), Reactorului Strasse 30, Bucharest-Magurele RO-077125, Romania.

magnetohydrodynamics. The Navier-Stokes limit of these equations is discussed in Sec. V. The last section contains a summary of this work.

We adopt natural Heaviside-Lorentz units $\hbar = c = k_B = \epsilon_0 = \mu_0 = 1$, and the Minkowski space-time metric $g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. The fluid four-velocity is $u^\mu = \gamma(1, \mathbf{v})^T$, with $\gamma = (1 - \mathbf{v}^2)^{-1/2}$ and normalization $u^\mu u_\mu \equiv 1$, while in the local rest (LR) frame of the fluid, $u^\mu_{LR} = (1, \mathbf{0})^T$. The rank-two projection operator onto the three-space orthogonal to u^μ is defined as $\Delta^{\mu\nu} \equiv g^{\mu\nu} - u^\mu u^\nu$. For any four-vector, A^μ, we define its projection onto the three-dimensional subspace orthogonal to u^μ as $A^{\langle\mu\rangle} \equiv \Delta^\mu_\nu A^\nu$. A straightforward generalization is the symmetric and traceless projection tensor of rank-2ℓ, denoted by $\Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}$, such that the irreducible projections are $A^{\langle\mu_1\cdots\mu_\ell\rangle} \equiv \Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell} A^{\nu_1\cdots\nu_\ell}$ [14]. As an example, the rank-four symmetric and traceless projection operator is defined as $\Delta^{\mu\nu}_{\alpha\beta} \equiv \frac{1}{2}(\Delta^\mu_\alpha \Delta^\nu_\beta + \Delta^\mu_\beta \Delta^\nu_\alpha) - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}$.

The four-momentum k^μ of particles is normalized to their rest mass squared, $k^\mu k_\mu = m_0^2$. The energy of a particle in the LR frame of the fluid is defined as $E_{\mathbf{k}} \equiv k^\mu u_\mu$ and coincides with the on-shell energy $k^0 = \sqrt{\mathbf{k}^2 + m_0^2}$. The three-momentum of particles, $\mathbf{k}$, is defined through the orthogonal projection of the four-momenta, $k^{\langle\mu\rangle} \equiv \Delta^\mu_\nu k^\nu$, in the LR frame. The comoving derivative of a quantity A is denoted by an overdot, i.e., $\dot{A} \equiv u^\mu \partial_\mu A$, while the three-space gradient is $\nabla_\nu A \equiv \Delta^\alpha_\nu \partial_\alpha A$, hence in the LR frame they reduce to the usual time and spatial derivatives $\partial_t A$ and $\boldsymbol{\nabla} A$. Furthermore, we use the decomposition $\partial_\mu u_\nu = u_\mu \dot{u}_\nu + \frac{1}{3}\theta \Delta_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu}$, where we define the expansion scalar, $\theta \equiv \nabla_\mu u^\mu$, the shear tensor $\sigma^{\mu\nu} \equiv \nabla^{\langle\mu} u^{\nu\rangle} = \frac{1}{2}(\nabla^\mu u^\nu + \nabla^\nu u^\mu) - \frac{1}{3}\theta\Delta^{\mu\nu}$, and the vorticity $\omega^{\mu\nu} \equiv \frac{1}{2}(\nabla^\mu u^\nu - \nabla^\nu u^\mu)$.

II. EQUATIONS OF MOTION OF MAGNETOHYDRODYNAMICS

The equations of motion of magnetohydrodynamics are [see Eqs. (24) and (25) of Ref. [11]]

$$\partial_\mu \mathbb{J}^\mu_f = 0, \tag{1}$$

$$\partial_\nu T^{\mu\nu} = -F^{\mu\lambda} \mathbb{J}_{\mathrm{ext},\lambda}. \tag{2}$$

Here,

$$\mathbb{J}^\mu_f = \mathbb{n}_f u^\mu + \mathbb{V}^\mu_f \tag{3}$$

is the electric-charge four-current of the fluid, where $\mathbb{n}_f = u_\nu \mathbb{J}^\nu_f$ is the electric-charge density and $\mathbb{V}^\mu_f = \Delta^\mu_\nu \mathbb{J}^\nu_f$ is the electric-charge diffusion current. The electric-charge four-current is related to the particle four-current N^μ_f by $\mathbb{J}^\mu_f \equiv \mathbb{q} N^\mu_f$. Similarly, $\mathbb{n}_f = \mathbb{q} n_f$, where n_f is the particle density in the fluid, and $\mathbb{V}^\mu_f = \mathbb{q} V^\mu_f$, where V^μ_f is the particle diffusion current. For the sake of generality, we have also added a source term from an external charge current $\mathbb{J}^\mu_{\mathrm{ext}}$ in the energy-momentum equation (2).

The total energy-momentum tensor of the system is given by

$$T^{\mu\nu} \equiv T^{\mu\nu}_{\mathrm{em}} + T^{\mu\nu}_f. \tag{4}$$

It consists of an electromagnetic contribution which, for nonpolarizable, nonmagnetizable fluids, reads [15–17]

$$T^{\mu\nu}_{\mathrm{em}} \equiv -F^{\mu\lambda} F^\nu_{\ \lambda} + \frac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}. \tag{5}$$

Here,

$$F^{\mu\nu} \equiv E^\mu u^\nu - E^\nu u^\mu + \epsilon^{\mu\nu\alpha\beta} u_\alpha B_\beta, \tag{6}$$

is the Faraday tensor, which we have decomposed in terms of the fluid four-velocity u^μ, as well as the electric and magnetic field four-vectors $E^\mu \equiv F^{\mu\nu} u_\nu$ and $B^\mu \equiv \frac{1}{2}\epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} u_\nu$, respectively, with $\epsilon^{\mu\nu\alpha\beta}$ being the Levi-Civita tensor.

The second part of the energy-momentum tensor (4) is the contribution from the fluid. For a nonpolarizable, nonmagnetizable fluid it reads

$$T^{\mu\nu}_f \equiv \varepsilon u^\mu u^\nu - P\Delta^{\mu\nu} + 2W^{(\mu} u^{\nu)} + \pi^{\mu\nu}, \tag{7}$$

where we defined the energy density $\varepsilon \equiv T^{\mu\nu}_f u_\mu u_\nu$, the isotropic pressure $P \equiv -\frac{1}{3} T^{\mu\nu}_f \Delta_{\mu\nu}$, the energy-momentum diffusion current $W^\mu \equiv \Delta^\mu_\alpha T^{\alpha\beta}_f u_\beta$, and the shear-stress tensor $\pi^{\mu\nu} \equiv \Delta^{\mu\nu}_{\alpha\beta} T^{\alpha\beta}_f$.

Maxwell's equations read [9]

$$\partial_\mu F^{\mu\nu} = \mathbb{J}^\nu, \qquad \epsilon^{\mu\nu\alpha\beta} \partial_\mu F_{\alpha\beta} = 0, \tag{8}$$

where $\mathbb{J}^\mu \equiv \mathbb{J}^\mu_f + \mathbb{J}^\mu_{\mathrm{ext}}$ is the total electric charge four-current. These equations imply that

$$\partial_\nu T^{\mu\nu}_{\mathrm{em}} = -F^{\mu\lambda} \mathbb{J}_\lambda. \tag{9}$$

From this and Eq. (2) follows that the energy-momentum tensor of the fluid satisfies [18]

$$\partial_\nu T^{\mu\nu}_f = F^{\mu\lambda} \mathbb{J}_{f,\lambda}. \tag{10}$$

III. EQUATIONS OF MOTION FOR THE IRREDUCIBLE MOMENTS

The relativistic Boltzmann equation coupled to an electromagnetic field [14,15], the so-called Boltzmann-Vlasov equation reads,

$$k^\mu \partial_\mu f_{\mathbf{k}} + \mathbb{q} F^{\mu\nu} k_\nu \frac{\partial}{\partial k^\mu} f_{\mathbf{k}} = C[f], \tag{11}$$

where $f_{\mathbf{k}}$ is the single-particle distribution function, $C[f]$ is the usual collision term in the Boltzmann equation, see e.g., Eq. (54) of Ref. [11].

A state of local thermal equilibrium is specified by a single-particle distribution function of the form [19]

$$f_{0\mathbf{k}} = [\exp(\beta_0 E_{\mathbf{k}} - \alpha_0) + a]^{-1}, \tag{12}$$

with $\alpha_0 = \mu\beta_0$, where μ is the (in general space-time dependent) chemical potential and $\beta_0 = 1/T$ the (space-time dependent) inverse temperature, while $a = \pm 1$ for fermions/bosons and $a \to 0$ for Boltzmann particles.

Unless α_0, β_0, and u^μ are constants (i.e., independent of space-time coordinates, such that equilibrium is global instead of local), the distribution function $f_{0\mathbf{k}}$ is not a solution of the Boltzmann equation (11). However, it is a convenient starting point to derive the equations of motion for dissipative fluid dynamics using the method of moments [8,13]. To this end, one decomposes

$$f_{\mathbf{k}} = f_{0\mathbf{k}} + \delta f_{\mathbf{k}}, \tag{13}$$

where $\delta f_{\mathbf{k}}$ is the deviation of the solution $f_{\mathbf{k}}$ of the Boltzmann equation from the local-equilibrium distribution function $f_{0\mathbf{k}}$. In the following, we will use the notation

$$\langle \cdots \rangle \equiv \int dK \cdots f_{\mathbf{k}}, \qquad \langle \cdots \rangle_0 \equiv \int dK \cdots f_{0\mathbf{k}},$$
$$\langle \cdots \rangle_\delta \equiv \int dK \cdots \delta f_{\mathbf{k}}, \tag{14}$$

where $dK \equiv g d^3\mathbf{k}/[(2\pi)^3 k^0]$ is the Lorentz-invariant measure in momentum space and g is the degeneracy factor of the state with momentum $\mathbf{k}$. From Eq. (13) follows immediately that $\langle \cdots \rangle = \langle \cdots \rangle_0 + \langle \cdots \rangle_\delta$.

The particle four-current and the energy-momentum tensor of the fluid are given as the following moments of $f_{\mathbf{k}}$,

$$N_f^\mu \equiv \langle k^\mu \rangle, \qquad T_f^{\mu\nu} \equiv \langle k^\mu k^\nu \rangle, \tag{15}$$

and, consequently, we identify the fluid-dynamical variables introduced in the previous section as, $n_f = \langle E_{\mathbf{k}} \rangle$, $V_f^\mu = \langle k^{\langle\mu\rangle} \rangle$, $\varepsilon = \langle E_{\mathbf{k}}^2 \rangle$, $P = -\frac{1}{3}\langle \Delta^{\mu\nu} k_\mu k_\nu \rangle$, $W^\mu = \langle E_{\mathbf{k}} k^{\langle\mu\rangle} \rangle$, $\pi^{\mu\nu} = \langle k^{\langle\mu} k^{\nu\rangle} \rangle$. For reasons of symmetry, $\langle E_{\mathbf{k}}^r k^{\langle\mu_1} \cdots k^{\mu_n\rangle} \rangle_0 \equiv 0$ for $n \geq 1$, thus $V_f^\mu = \langle k^{\langle\mu\rangle} \rangle_\delta$, $W^\mu = \langle E_{\mathbf{k}} k^{\langle\mu\rangle} \rangle_\delta$, $\pi^{\mu\nu} = \langle k^{\langle\mu} k^{\nu\rangle} \rangle_\delta$.

Now, following Refs. [8,13] we define the symmetric and traceless irreducible moments of $\delta f_{\mathbf{k}}$,

$$\rho_r^{\mu_1\cdots\mu_n} \equiv \langle E_{\mathbf{k}}^r k^{\langle\mu_1} \cdots k^{\mu_n\rangle} \rangle_\delta. \tag{16}$$

Note that the tensors $k^{\langle\mu_1} \cdots k^{\mu_\ell\rangle}$ are irreducible with respect to Lorentz transformations that leave the fluid 4-velocity invariant and form a complete and orthogonal set [14]. In terms of the irreducible moments (16) the corrections to the equilibrium values of particle density, n_{f0}, energy density, ε_0, and isotropic pressure, P_0, are $\delta n_f \equiv n_f - n_{f0} = \rho_1$, $\delta\varepsilon \equiv \varepsilon - \varepsilon_0 = \rho_2$, and $\Pi \equiv P - P_0 = (\rho_2 - m_0^2\rho_0)/3$. The particle and energy-momentum diffusion currents orthogonal to the fluid velocity are $V_f^\mu = \rho_0^\mu$ and $W^\mu = \rho_1^\mu$, while the shear-stress tensor is $\pi^{\mu\nu} = \rho_0^{\mu\nu}$.

So far, the local equilibrium state introduced in Eq. (12) has not been defined: the equilibrium variables α_0, β_0, and u^μ must be properly specified in the context of the Boltzmann equation. The first step is to define temperature and chemical potential by introducing matching conditions, $n_f = n_{f0}(\alpha_0, \beta_0)$ and $\varepsilon = \varepsilon_0(\alpha_0, \beta_0)$. These conditions define α_0 and β_0 such that the particle density and energy density of the system are identical to those of a local equilibrium state characterized by $f_{0\mathbf{k}}$. This implies $\rho_1 = \rho_2 = 0$. For the sake of completeness, we shall continue with the derivation of the equations of motion for the irreducible moments without specifying the fluid four-velocity. In this way, the equations of motion derived in this paper can be made compatible with any definition of u^μ.

Equations (1) and (10) lead to equations of motion for α_0, β_0, and u^μ:

$$\dot{\alpha}_0 = \frac{1}{D_{20}}[-J_{30}(n_{f0}\theta + \partial_\mu V_f^\mu) + J_{20}(\varepsilon_0 + P_0 + \Pi)\theta + J_{20}(\partial_\mu W^\mu - W^\mu \dot{u}_\mu - \pi^{\mu\nu}\sigma_{\mu\nu}) + J_{20}\mathfrak{q}E^\mu V_{f,\mu}], \tag{17}$$

$$\dot{\beta}_0 = \frac{1}{D_{20}}[-J_{20}(n_{f0}\theta + \partial_\mu V_f^\mu) + J_{10}(\varepsilon_0 + P_0 + \Pi)\theta + J_{10}(\partial_\mu W^\mu - W^\mu \dot{u}_\mu - \pi^{\mu\nu}\sigma_{\mu\nu}) + J_{10}\mathfrak{q}E^\mu V_{f,\mu}], \tag{18}$$

and

$$\dot{u}^\mu = \frac{1}{\varepsilon_0 + P_0}\left[\frac{n_{f0}}{\beta_0}(\nabla^\mu\alpha_0 - h_0\nabla^\mu\beta_0) - \Pi\dot{u}^\mu + \nabla^\mu\Pi - \frac{4}{3}W^\mu\theta - W_\nu(\sigma^{\mu\nu} - \omega^{\mu\nu}) - \dot{W}^\mu - \Delta^\mu_\nu\partial_\kappa\pi^{\kappa\nu}\right]$$
$$+ \frac{1}{\varepsilon_0 + P_0}[\mathfrak{q}n_{f0}E^\mu - \mathfrak{q}Bb^{\mu\nu}V_{f,\nu}], \tag{19}$$

where $h_0 \equiv (\varepsilon_0 + P_0)/n_{f0}$ is the enthalpy per particle in equilibrium and the thermodynamic integrals J_{nq} and D_{nq} are defined in Appendix A. Note that these equations extend Eqs. (70)–(72) of Ref. [11] by terms proportional to the electric field E^μ.[1]

For a given fluid four-velocity u^μ, the equations of motion (1) and (10) only specify five of the 14 independent variables α_0, β_0, Π, V_f^μ, W^μ, and $\pi^{\mu\nu}$. In order to close the system of equations of motion, one needs to specify additional equations of motion that can be provided by a suitable truncation of the infinite set of equations of motion for the irreducible moments $\rho_r^{\mu_1\cdots\mu_\ell}$. The latter equations are obtained by calculating the comoving derivative $\dot{\rho}_r^{\langle\mu_1\cdots\mu_\ell\rangle} \equiv \Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell} u^\alpha \partial_\alpha \rho_r^{\nu_1\cdots\nu_\ell}$, using the Boltzmann equation (11), for details see Refs. [8,11,13]. For the irreducible moments of tensor-rank zero one obtains

$$\dot{\rho}_r - C_{r-1} = \alpha_r^{(0)}\theta + \frac{G_{3r}}{D_{20}}\partial_\mu V_f^\mu - \frac{G_{2r}}{D_{20}}\partial_\mu W^\mu + \frac{\theta}{3}\left[m_0^2(r-1)\rho_{r-2} - (r+2)\rho_r - 3\frac{G_{2r}}{D_{20}}\Pi\right] + \left(r\rho_{r-1}^\mu + \frac{G_{2r}}{D_{20}}W^\mu\right)\dot{u}_\mu - \nabla_\mu\rho_{r-1}^\mu + \left[(r-1)\rho_{r-2}^{\mu\nu} + \frac{G_{2r}}{D_{20}}\pi^{\mu\nu}\right]\sigma_{\mu\nu} - \frac{G_{2r}}{D_{20}}\mathfrak{q}E_\nu V_f^\nu - (r-1)\mathfrak{q}E_\nu\rho_{r-2}^\nu. \tag{20}$$

This equation is very similar to Eq. (75) of Ref. [11] except for the terms proportional to W^μ and the last two terms which constitute the contributions from the electric field.

The equation of motion for irreducible moments of tensor-rank one reads

$$\dot{\rho}_r^{\langle\mu\rangle} - C_{r-1}^{\langle\mu\rangle} = \alpha_r^{(1)}\nabla^\mu\alpha_0 - \alpha_r^h\dot{W}^\mu + r\rho_{r-1}^{\mu\nu}\dot{u}_\nu - \frac{1}{3}\nabla^\mu(m_0^2\rho_{r-1} - \rho_{r+1}) - \Delta_\alpha^\mu(\nabla_\nu\rho_{r-1}^{\alpha\nu} + \alpha_r^h\partial_\kappa\pi^{\kappa\alpha}) + \frac{1}{3}[m_0^2(r-1)\rho_{r-2}^\mu - (r+3)\rho_r^\mu - 4\alpha_r^h W^\mu]\theta + (r-1)\rho_{r-2}^{\mu\nu\lambda}\sigma_{\nu\lambda} + \frac{1}{5}\sigma^{\mu\nu}[2m_0^2(r-1)\rho_{r-2,\nu} - (2r+3)\rho_{r,\nu} - 5\alpha_r^h W_\nu] + (\rho_{r,\nu} + \alpha_r^h W_\nu)\omega^{\mu\nu} + \frac{1}{3}[m_0^2 r\rho_{r-1} - (r+3)\rho_{r+1} - 3\alpha_r^h\Pi]\dot{u}^\mu + \alpha_r^h\nabla^\mu\Pi - \alpha_r^h\mathfrak{q}Bb^{\mu\nu}V_{f,\nu} - \mathfrak{q}Bb^{\mu\nu}\rho_{r-1,\nu} + (\alpha_r^h n_{f0} + \beta_0 J_{r+1,1})\mathfrak{q}E^\mu + \frac{1}{3}[(r+2)\rho_r - m_0^2(r-1)\rho_{r-2}]\mathfrak{q}E^\mu - (r-1)\rho_{r-2}^{\mu\nu}\mathfrak{q}E_\nu. \tag{21}$$

Here we introduced a new dimensionless antisymmetric tensor $b^{\mu\nu} \equiv -\epsilon^{\mu\nu\alpha\beta}u_\alpha b_\beta$, where the unit four-vector in the direction of the magnetic field and orthogonal to u^μ is $b^\mu \equiv \frac{B^\mu}{B}$, with $b^\mu b_\mu = -1$ and $B \equiv \sqrt{-B^\mu B_\mu}$ [21]. Equation (21) differs from Eq. (76) of Ref. [11] by the last three terms taking into account the electric field, as well as by the additional terms proportional to W^μ.

Finally, the equation of motion for the irreducible moments of tensor-rank two is

$$\dot{\rho}_r^{\langle\mu\nu\rangle} - C_{r-1}^{\langle\mu\nu\rangle} = 2\alpha_r^{(2)}\sigma^{\mu\nu} + \frac{2}{15}[m_0^4(r-1)\rho_{r-2} - m_0^2(2r+3)\rho_r + (r+4)\rho_{r+2}]\sigma^{\mu\nu} + \frac{2}{5}\dot{u}^{\langle\mu}[m_0^2 r\rho_{r-1}^{\nu\rangle} - (r+5)\rho_{r+1}^{\nu\rangle}] - \frac{2}{5}[\nabla^{\langle\mu}(m_0^2\rho_{r-1}^{\nu\rangle} - \rho_{r+1}^{\nu\rangle})] + r\rho_{r-1}^{\mu\nu\gamma}\dot{u}_\gamma - \Delta_{\alpha\beta}^{\mu\nu}\nabla_\lambda\rho_{r-1}^{\alpha\beta\lambda} + (r-1)\rho_{r-2}^{\mu\nu\lambda\kappa}\sigma_{\lambda\kappa} + 2\rho_r^{\lambda\langle\mu}\omega_\lambda^{\nu\rangle} + \frac{1}{3}[m_0^2(r-1)\rho_{r-2}^{\mu\nu} - (r+4)\rho_r^{\mu\nu}]\theta + \frac{2}{7}[2m_0^2(r-1)\rho_{r-2}^{\kappa\langle\mu} - (2r+5)\rho_r^{\kappa\langle\mu}]\sigma_\kappa^{\nu\rangle} - 2\mathfrak{q}Bb^{\alpha\beta}\Delta_{\alpha\kappa}^{\mu\nu}g_{\lambda\beta}\rho_{r-1}^{\kappa\lambda} + 2\mathfrak{q}E^{\langle\mu}\rho_r^{\nu\rangle} - (r-1)\Delta_{\alpha\beta}^{\mu\nu}\left[\mathfrak{q}E_\lambda\rho_{r-2}^{\alpha\beta\lambda} + \frac{2}{5}\mathfrak{q}E^{(\alpha}(m_0^2\rho_{r-2}^{\beta)} - \rho_r^{\beta)})\right]. \tag{22}$$

This equation differs from Eq. (77) of Ref. [11] by the last two terms, which constitute the contributions from a nonvanishing electric field.

[1]Note that terms proportional to W^μ also did not appear in Ref. [11], since the equations derived in that reference employed the Landau frame [20], where u^μ is defined as an eigenvector of the energy-momentum tensor, $u_\mu T^{\mu\nu} = \varepsilon u^\nu$, leading to $W^\mu = 0$.

In Eqs. (20)–(22), α_r^h, $\alpha_r^{(\ell)}$, and G_{ij} are thermodynamic coefficients, which are explicitly given in Appendix A, while the linearized collision integral is defined as

$$\begin{aligned} \mathcal{C}_{r-1}^{\langle\mu_1\cdots\mu_\ell\rangle} &\equiv \Delta^{\mu_1\cdots\mu_\ell}_{\nu_1\cdots\nu_\ell}\int dK\, E_{\mathbf{k}}^{r-1}k^{\nu_1}\cdots k^{\nu_\ell}C[f] \\ &= -\sum_{n=0}^{N_\ell}\mathcal{A}_{rn}^{(\ell)}\rho_n^{\mu_1\cdots\mu_\ell}, \end{aligned} \tag{23}$$

where the coefficient $\mathcal{A}_{rn}^{(\ell)} \sim \lambda_{\mathrm{mfp}}$ contains time scales proportional to the mean free path of the particles. Note that the last equality of the above equation is obtained using the moment expansion of the single-particle distribution function first introduced in Ref. [8], which, for the sake of completeness, is listed in Appendix A.

IV. EQUATIONS OF MOTION IN THE 14-MOMENT APPROXIMATION

In order to obtain a closed system of fluid-dynamical equations of motion, we now truncate the infinite set (20)–(22) of equations of motion for the irreducible moments. The simplest and most widely used truncation is the so-called 14-moment approximation [7]. First, all irreducible tensor moments $\rho_r^{\mu_1\cdots\mu_\ell}$ for $\ell > 2$ are explicitly set to zero in Eqs. (21)–(22). Second, the remaining scalar ρ_r, vector ρ_r^μ, and rank-2 tensor moments $\rho_r^{\mu\nu}$ are expressed as linear combinations of the lowest-order moments $\rho_0 \equiv -3\Pi/m_0^2$, $\rho_0^\mu \equiv V_f^\mu$, $\rho_1^\mu \equiv W^\mu$, and $\rho_0^{\mu\nu} \equiv \pi^{\mu\nu}$, i.e., in terms of quantities appearing in $\mathbb{J}_f^\mu$ and $T_f^{\mu\nu}$, cf. Eqs. (3), (7). The relations affecting this truncation are Eqs. (A5)–(A7).

Equation (20) then leads to an equation of motion for the bulk viscous pressure

$$\begin{aligned} \tau_\Pi\dot{\Pi} + \Pi = {}& -\zeta\theta - \delta_{\Pi\Pi}\Pi\theta + \lambda_{\Pi\pi}\pi^{\mu\nu}\sigma_{\mu\nu} - \ell_{\Pi V}\nabla_\mu V_f^\mu - \tau_{\Pi V}V_f^\mu\dot{u}_\mu - \lambda_{\Pi V}V_f^\mu\nabla_\mu\alpha_0 \\ & - \ell_{\Pi W}\nabla_\mu W^\mu - \tau_{\Pi W}W^\mu\dot{u}_\mu - \lambda_{\Pi W}W^\mu\nabla_\mu\alpha_0 - \delta_{\Pi VE}\mathfrak{q}V_f^\nu E_\nu - \delta_{\Pi WE}\mathfrak{q}W^\nu E_\nu. \end{aligned} \tag{24}$$

Similarly, from Eq. (21) we obtain an equation for the particle- and energy-diffusion currents,

$$\begin{aligned} \tau_V\dot{V}_f^{\langle\mu\rangle} - \tau_V h_0^{-1}\dot{W}^{\langle\mu\rangle} + V_f^\mu - h_0^{-1}W^\mu = {}& \kappa\nabla^\mu\alpha_0 - \tau_V V_{f,\nu}\omega^{\nu\mu} - \delta_{VV}V_f^\mu\theta - \lambda_{VV}V_{f,\nu}\sigma^{\mu\nu} \\ & + \tau_V h_0^{-1}W_\nu\omega^{\nu\mu} - \delta_{WW}W^\mu\theta - \lambda_{WW}W_\nu\sigma^{\mu\nu} - \ell_{V\Pi}\nabla^\mu\Pi + \ell_{V\pi}\Delta^{\mu\nu}\nabla_\lambda\pi_\nu^\lambda + \tau_{V\Pi}\Pi\dot{u}^\mu \\ & - \tau_{V\pi}\pi^{\mu\nu}\dot{u}_\nu + \lambda_{V\Pi}\Pi\nabla^\mu\alpha_0 - \lambda_{V\pi}\pi^{\mu\nu}\nabla_\nu\alpha_0 - \delta_{VB}\mathfrak{q}Bb^{\mu\nu}V_{f,\nu} - \delta_{WB}\mathfrak{q}Bb^{\mu\nu}W_\nu \\ & + \delta_{VE}\mathfrak{q}E^\mu + \delta_{V\Pi E}\mathfrak{q}\Pi E^\mu + \delta_{V\pi E}\mathfrak{q}\pi^{\mu\nu}E_\nu. \end{aligned} \tag{25}$$

The equation of motion for the shear-stress tensor follows from Eq. (22),

$$\begin{aligned} \tau_\pi\dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = {}& 2\eta\sigma^{\mu\nu} + 2\tau_\pi\pi_\lambda^{\langle\mu}\omega^{\nu\rangle\lambda} - \delta_{\pi\pi}\pi^{\mu\nu}\theta - \tau_{\pi\pi}\pi^{\lambda\langle\mu}\sigma_\lambda^{\nu\rangle} + \lambda_{\pi\Pi}\Pi\sigma^{\mu\nu} \\ & - \tau_{\pi V}V_f^{\langle\mu}\dot{u}^{\nu\rangle} + \ell_{\pi V}\nabla^{\langle\mu}V_f^{\nu\rangle} + \lambda_{\pi V}V_f^{\langle\mu}\nabla^{\nu\rangle}\alpha_0 - \tau_{\pi W}W^{\langle\mu}\dot{u}^{\nu\rangle} + \ell_{\pi W}\nabla^{\langle\mu}W^{\nu\rangle} + \lambda_{\pi W}W^{\langle\mu}\nabla^{\nu\rangle}\alpha_0 \\ & - \delta_{\pi B}\mathfrak{q}Bb^{\alpha\beta}\Delta^{\mu\nu}_{\alpha\kappa}g_{\lambda\beta}\pi^{\kappa\lambda} + \delta_{\pi VE}\mathfrak{q}E^{\langle\mu}V_f^{\nu\rangle} + \delta_{\pi WE}\mathfrak{q}E^{\langle\mu}W^{\nu\rangle}. \end{aligned} \tag{26}$$

The coefficients appearing in these equations are listed in Appendix B.

Note that Eq. (25) represents the relaxation equation for the heat flow defined by

$$q^\mu \equiv W^\mu - h_0 V_f^\mu. \tag{27}$$

In case we choose the local rest frame following Landau's picture (which imposes $W^\mu \equiv 0$), the heat flow is simply given in terms of the particle diffusion alone, $q^\mu = -h_0 V_f^\mu$. On the other hand, choosing the rest frame according to Eckart's picture (which requires $V_f^\mu \equiv 0$), leads to a heat flow that is solely given by the flow of energy and momentum, $q^\mu = W^\mu$. Since the relaxation equations (24)–(26) contain both diffusive quantities, the equations of motion derived in this paper are consistent with either choice of local rest frame.

The coefficients proportional to the electric field in the equation for the bulk viscous pressure are

$$\begin{aligned} \delta_{\Pi VE} &= \frac{m_0^2}{3\mathcal{A}_{00}^{(0)}}\left(\mathcal{F}_{20}^{(1)} - \frac{G_{20}}{D_{20}} - \frac{\beta_0}{h_0}\frac{\partial\mathcal{F}_{10}^{(1)}}{\partial\beta_0}\right), \\ \delta_{\Pi WE} &= \frac{m_0^2}{3\mathcal{A}_{00}^{(0)}}\left(\mathcal{F}_{21}^{(1)} - \frac{\beta_0}{h_0}\frac{\partial\mathcal{F}_{11}^{(1)}}{\partial\beta_0}\right). \end{aligned} \tag{28}$$

The coefficients proportional to the electric field in the equation for the particle-diffusion current are

TABLE I. The coefficients for the diffusion equation for a Boltzmann gas with constant cross section in the ultrarelativistic limit, in the 14-moment approximation, with $\tau_{00}^{(1)} = \tau_V$.

κ	$\tau_V[\lambda_{\text{mfp}}]$	$\delta_{VV}[\tau_V]$	$\delta_{WW}[\tau_V]$	$\lambda_{VV}[\tau_V]$	$\lambda_{WW}[\tau_V]$	$\lambda_{V\pi}[\tau_V]$	$\ell_{V\pi}[\tau_V]$	$\tau_{V\pi}[\tau_V]$	$\delta_{VB}[\tau_V]$	$\delta_{WB}[\tau_V]$	$\delta_{VE}[\tau_V]$	$\delta_{V\pi E}[\tau_V]$
$3/(16\sigma)$	$9/4$	1	$-\beta_0/3$	$3/5$	$-\beta_0/4$	$\beta_0/20$	$\beta_0/20$	$\beta_0/20$	$5\beta_0/12$	$-\beta_0^2/12$	$P_0\beta_0^2/12$	0

TABLE II. The coefficients for the shear-stress equation for a Boltzmann gas with constant cross section in the ultrarelativistic limit, in the 14-moment approximation, with $\tau_{00}^{(2)} = \tau_\pi$.

η	$\tau_\pi[\lambda_{\text{mfp}}]$	$\delta_{\pi\pi}[\tau_\pi]$	$\tau_{\pi\pi}[\tau_\pi]$	$\lambda_{\pi V}[\tau_\pi]$	$\ell_{\pi V}[\tau_\pi]$	$\tau_{\pi V}[\tau_\pi]$	$\lambda_{\pi W}[\tau_\pi]$	$\ell_{\pi W}[\tau_\pi]$	$\tau_{\pi W}[\tau_\pi]$	$\delta_{\pi B}[\tau_\pi]$	$\delta_{\pi VE}[\tau_\pi]$	$\delta_{\pi WE}[\tau_\pi]$
$4/(3\sigma\beta_0)$	$5/3$	$4/3$	$10/7$	0	0	0	0	$2/5$	2	$2\beta_0/5$	$8/5$	0

$$\delta_{VE} = \frac{1}{\mathcal{A}_{00}^{(1)}}(-n_{f0}h_0^{-1} + \beta_0 J_{11}), \tag{29}$$

$$\delta_{V\Pi E} = -\frac{1}{m_0^2\mathcal{A}_{00}^{(1)}}\left(2 + m_0^2\mathcal{F}_{20}^{(1)} - m_0^2\frac{\beta_0}{h_0}\frac{\partial\mathcal{F}_{10}^{(0)}}{\partial\beta_0}\right),$$
$$\delta_{V\pi E} = \frac{1}{\mathcal{A}_{00}^{(1)}}\left(\mathcal{F}_{20}^{(2)} - \frac{\beta_0}{h_0}\frac{\partial\mathcal{F}_{10}^{(2)}}{\partial\beta_0}\right), \tag{30}$$

and the coefficient coupling W^μ to the magnetic field is

$$\delta_{WB} = \frac{\mathcal{F}_{11}^{(1)}}{\mathcal{A}_{00}^{(1)}}. \tag{31}$$

Finally, the coefficients proportional to the electric field in the equation for the shear-stress tensor are

$$\delta_{\pi VE} = \frac{2}{5\mathcal{A}_{00}^{(2)}}\left(4 + m_0^2\mathcal{F}_{20}^{(1)} - m_0^2\frac{\beta_0}{h_0}\frac{\partial\mathcal{F}_{10}^{(1)}}{\partial\beta_0}\right),$$
$$\delta_{\pi WE} = \frac{2m_0^2}{5\mathcal{A}_{00}^{(2)}}\left(\mathcal{F}_{21}^{(1)} - \frac{\beta_0}{h_0}\frac{\partial\mathcal{F}_{11}^{(1)}}{\partial\beta_0}\right). \tag{32}$$

The thermodynamic integral $\mathcal{F}_{rn}^{(\ell)}$ is defined in Eq. (A4).

In the limit of a massless Boltzmann gas with constant cross section σ, $J_{nq} \equiv I_{nq} = \frac{(n+1)!}{2(2q+1)!!}\beta_0^{2-n}P_0$, and hence $\mathcal{A}_{00}^{(1)} = 4/(9\lambda_{\text{mfp}})$, $\mathcal{A}_{00}^{(2)} = 3/(5\lambda_{\text{mfp}})$, where $\lambda_{\text{mfp}} = 1/(n_0\sigma)$ is the mean free path of the particles. In the massless limit, $m_0 = 0$, the coefficients $\delta_{\Pi VE} = \delta_{\Pi WE} = 0$, while $\delta_{V\Pi E}$ formally diverges $\sim 1/m_0^2$. However, the bulk viscous pressure is $\Pi = -m_0^2\rho_0/3$, which cancels this divergence, and the remaining term is $\sim\rho_0 E^\mu$. Now, E^μ is of order one in gradients (see below and Ref. [22]), while ρ_0 is actually of second order, since the coefficient $\alpha_r^{(0)}$ in the Navier-Stokes term in Eq. (20) vanishes in the massless limit for all r. Thus, the respective term is of third order in gradients and, for this reason, we neglect it in the massless limit.

In Table I we list the $m_0 = 0$ values of those coefficients in Eq. (25), which are not proportional to Π.

Similarly, in Table II we list the $m_0 = 0$ values of those coefficients in Eq. (26), which are not proportional to Π.

V. NAVIER-STOKES LIMIT, OHMIC CURRENT, AND WIEDEMANN–FRANZ LAW

In the Navier-Stokes limit, all second-order terms are discarded from the relaxation equations (24)–(26). We employ the power-counting advertised in Ref. [22], i.e., E^μ is of order one, i.e., of the same order as gradients of α_0, β_0, and u^μ, or of the same order as the dissipative quantities Π, V_f^μ, W^μ, and $\pi^{\mu\nu}$. On the other hand, the magnetic field is of order zero, like other thermodynamic quantities. For the bulk viscous pressure and shear-stress tensor, this ultimately leads to $\Pi = -\zeta\theta$ and $\pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} - \delta_{\pi B}\mathfrak{q}Bb^{\alpha\beta}\Delta^{\mu\nu}_{\alpha\kappa}g_{\lambda\beta}\pi^{\kappa\lambda}$, see the discussion in Sec. IV. B of Ref. [11], where these equations have already been analyzed.

However, for the Navier-Stokes limit of the diffusion currents, the electric field has a non-negligible impact. For the sake of simplicity and comparison to Ref. [11], we work in the Landau frame, where $W^\mu = 0$. To first order, the particle-diffusion current becomes

$$V_f^\mu = \kappa\nabla^\mu\alpha_0 + \delta_{VE}\mathfrak{q}E^\mu - \delta_{VB}\mathfrak{q}Bb^{\mu\nu}V_{f,\nu}. \tag{33}$$

The Ohmic induction current is given by the second term of Eq. (33) (after multiplying by $\mathfrak{q}$),

$$\mathbb{J}^\mu_{\text{ind}} \equiv \sigma_E E^\mu, \tag{34}$$

with the electric conductivity

$$\sigma_E \equiv \mathfrak{q}^2\delta_{VE}. \tag{35}$$

As originally noted by Einstein [23], the electric conductivity and the particle-diffusion coefficient must be related by

$$\sigma_E = \mathbb{q}^2 \beta_0 \kappa, \tag{36}$$

which is the kinetic-theory version of the famous Wiedemann–Franz law. For the massless Boltzmann gas, the validity of this relation can be easily checked using the relation $\delta_{VE} = \frac{3}{16} n_{f0} \beta_0 \lambda_{\text{mfp}}$ and the fact that $\kappa = \frac{3}{16} n_{f0} \lambda_{\text{mfp}}$ [8]. As noted in Ref. [24], this relation must also hold for a different reason: in a state of constant T and u^μ and in the absence of dissipation, an electric field induces a charge-density gradient such that (in our conventions for metric and chemical potential),

$$\nabla^\mu \alpha_0 = -\mathbb{q} \beta_0 E^\mu. \tag{37}$$

This relation can also be found from the second-order transport equation (25), setting all dissipative quantities to zero, which leads to the condition $\kappa \nabla^\mu \alpha_0 = -\delta_{VE} \mathbb{q} E^\mu$. This relation together with Eq. (37) then confirms the Einstein relation (36).

Note that in the presence of a magnetic field Eq. (34) no longer holds [25]. Using Eq. (21) in the Navier-Stokes approximation we obtain

$$\rho_r^\mu = \kappa_r^{\mu\nu} \nabla_\nu \alpha_0 + \delta_r^{\mu\nu} \mathbb{q} E_\nu, \tag{38}$$

hence the conductivity tensor can be defined similarly to Eq. (36)

$$\sigma_{E,r}^{\mu\nu} = \mathbb{q}^2 \delta_r^{\mu\nu}. \tag{39}$$

The rank-two tensor coefficients may be decomposed in the direction parallel and orthogonal to the magnetic field in terms of the projection operators, $b^\mu b^\nu$, $\Xi^{\mu\nu} \equiv \Delta^{\mu\nu} + b^\mu b^\nu$, and the tensor $b^{\mu\nu}$ as

$$\kappa_r^{\mu\nu} = \kappa_{r\perp} \Xi^{\mu\nu} - \kappa_{r||} b^\mu b^\nu - \kappa_{r\times} b^{\mu\nu}, \tag{40}$$

$$\delta_r^{\mu\nu} = \delta_{r\perp} \Xi^{\mu\nu} - \delta_{r||} b^\mu b^\nu - \delta_{r\times} b^{\mu\nu}. \tag{41}$$

In order to calculate the transport coefficients $\kappa_r^{\mu\nu}$ or $\delta_r^{\mu\nu}$, we will follow the inversion procedure of Ref. [11], hence in the 14-moment approximation ($N_1 = 1$), setting $\nabla^\mu \alpha_0 = 0$ we get

$$\delta_{0||} = \frac{\beta_0 \alpha_r^{(1)}}{\mathcal{A}_{r0}^{(1)}}, \qquad \delta_{0\perp} = \delta_{0||} \left[1 + \left(\mathbb{q} B \frac{\mathcal{F}_{1-r,0}^{(1)} + \alpha_r^h}{\mathcal{A}_{r0}^{(1)}} \right)^2 \right]^{-1},$$

$$\delta_{0\times} = \delta_{0\perp} \mathbb{q} B \frac{\mathcal{F}_{1-r,0}^{(1)} + \alpha_r^h}{\mathcal{A}_{r0}^{(1)}}. \tag{42}$$

Comparing with Eqs. (101) of Ref. [11], we conclude that

$$\delta_{0||} = \beta_0 \kappa_{0||}, \qquad \delta_{0\perp} = \beta_0 \kappa_0, \qquad \delta_{0\times} = \beta_0 \kappa_{0\times}, \tag{43}$$

confirming that Eq. (36) also holds in tensorial form, $\sigma_{E,r}^{\mu\nu} = \mathbb{q}^2 \beta_0 \kappa_r^{\mu\nu}$, and irrespective of the limit of a massless Boltzmann gas.

VI. CONCLUSIONS AND OUTLOOK

Based on the moment expansion of the Boltzmann equation for a single-component gas of charged spin-zero particles coupled to an electromagnetic field, we have derived the equations of motion of resistive, second-order dissipative magnetohydrodynamics in the 14-moment approximation. New transport coefficients appear due to the coupling to the electric field. We computed these coefficients in the limit of a massless Boltzmann gas. We analyzed the Navier-Stokes limit of the dissipative quantities and recovered Ohm's law. We found that the electrical conductivity and the particle diffusion satisfy the well-known Einstein relation, which constitutes a type of Wiedemann-Franz law.

In future studies, one should address the generalization to particles with nonzero spin. Then, particles have a microscopic dipole moment which generates nonvanishing macroscopic magnetization and polarization fields [17,24]. The spin of the particles also gives rise to spin-vorticity coupling terms, which leads to the so-called chiral vortical effect [26]. This may necessitate an extension of the standard fluid-dynamical conservation laws by an equation of motion for the macroscopic spin tensor [27,28].

ACKNOWLEDGMENTS

The authors acknowledge enlightening discussion with G. Moore. E. M. acknowledges the warm hospitality of the Department of Physics of the University of Jyväskylä, where part of this work was done. This work was supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) through the Collaborative Research Center CRC-TR 211 "Strong-interaction matter under extreme conditions"—Project No. 315477589—TRR 211. G. S. D. thanks for Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) for financial support. E. M. is supported by the Bundesministerium für Bildung und Forschung (BMBF) and by the Research Council of Norway, (NFR) Project No. 255253/F50. H. N. is supported by the Academy of Finland, Project No. 297058. D. H. R. is partially supported by the High-end Foreign Experts Project No. GDW20167100136 of the State Administration of Foreign Experts Affairs of China.

APPENDIX A: SOME USEFUL FORMULAS

Following Refs. [8,13], we recall that the single-particle distribution function $f_{\mathbf{k}}$ can be expanded around $f_{0\mathbf{k}}$ as,

$$f_{\mathbf{k}} = f_{0\mathbf{k}} + f_{0\mathbf{k}}(1 - a f_{0\mathbf{k}}) \sum_{\ell=0}^{\infty} \sum_{n=0}^{N_\ell} \rho_n^{\mu_1 \cdots \mu_\ell} k_{\langle \mu_1} \cdots k_{\mu_\ell \rangle} \mathcal{H}_{\mathbf{k}n}^{(\ell)}, \tag{A1}$$

where the coefficient $\mathcal{H}_{\mathbf{k}n}^{(\ell)}$ is a polynomial in energy and defined as,

$$\mathcal{H}_{\mathbf{k}n}^{(\ell)} = \frac{(-1)^\ell}{\ell! J_{2\ell,\ell}} \sum_{i=n}^{N_\ell} \sum_{m=0}^{i} a_{in}^{(\ell)} a_{im}^{(\ell)} E_{\mathbf{k}}^m. \tag{A2}$$

The coefficients $a_{ij}^{(\ell)}$ are calculated via the Gram-Schmidt orthogonalization procedure and are expressed in terms of thermodynamic integrals J_{nq}, for more details see e.g., Ref. [8].

Any irreducible moment of arbitrary order r and tensor rank ℓ can always be expressed as a linear combination of irreducible moments of all orders n and the same tensor rank,

$$\rho_r^{\mu_1\cdots\mu_\ell} = \sum_{n=0}^{N_\ell} \rho_n^{\mu_1\cdots\mu_\ell} \mathcal{F}_{-r,n}^{(\ell)}, \tag{A3}$$

where

$$\mathcal{F}_{rn}^{(\ell)} = \frac{\ell!}{(2\ell+1)!!} \int dK\, E_{\mathbf{k}}^{-r} \mathcal{H}_{\mathbf{k}n}^{(\ell)} (\Delta^{\alpha\beta} k_\alpha k_\beta)^\ell f_{0\mathbf{k}} (1 - a f_{0\mathbf{k}}). \tag{A4}$$

In the 14-moment approximation the above expressions simplify considerably, hence using Eq. (A3) with the summation limits $N_0 = 2$, $N_1 = 1$, $N_2 = 0$ for different tensor rank, we obtain the following relations,

$$\rho_r \equiv \sum_{n=0,\neq 1,2}^{N_0} \rho_n \mathcal{F}_{-r,n}^{(0)} = -\frac{3}{m_0^2} \Pi \mathcal{F}_{-r,0}^{(0)} \equiv -\frac{3}{m_0^2} \Pi \frac{J_{r0} D_{30} + J_{r+1,0} G_{23} + J_{r+2,0} D_{20}}{J_{20} D_{20} + J_{30} G_{12} + J_{40} D_{10}}, \tag{A5}$$

$$\rho_r^\mu \equiv \sum_{n=0}^{N_1} \rho_n^\mu \mathcal{F}_{-r,n}^{(1)} = V_f^\mu \mathcal{F}_{-r,0}^{(1)} + W^\mu \mathcal{F}_{-r,1}^{(1)} \equiv V_f^\mu \frac{J_{r+2,1} J_{41} - J_{r+3,1} J_{31}}{D_{31}} + W^\mu \frac{-J_{r+2,1} J_{31} + J_{r+3,1} J_{21}}{D_{31}}, \tag{A6}$$

$$\rho_r^{\mu\nu} \equiv \sum_{n=0}^{N_2} \rho_n^{\mu\nu} \mathcal{F}_{-r,n}^{(2)} = \pi^{\mu\nu} \mathcal{F}_{-r,0}^{(2)} \equiv \pi^{\mu\nu} \frac{J_{r+4,2}}{J_{42}}. \tag{A7}$$

Note that Eqs. (A5)–(A7) are the same as Eqs. (115)–(117) of Ref. [11], except for Eq. (A6), which now contains a term proportional to W^μ when compared to Eq. (116) of Ref. [11]. Furthermore, for $r, n \geq 0$, $\mathcal{F}_{-r,n}^{(\ell)} = \delta_{rn}$, however Eqs. (A5)–(A7) are to be used for irreducible moments not only with positive but also with negative r given by

$$\rho_{-r}^{\mu_1\cdots\mu_\ell} = \sum_{n=0}^{N_\ell} \rho_n^{\mu_1\cdots\mu_\ell} \mathcal{F}_{rn}^{(\ell)}. \tag{A8}$$

Truncating the sum as in Eqs. (A5)–(A7), the coefficients of Eq. (A8) can be written similarly as in Eq. (67) of Ref. [8],

$$\rho_{-r} = -\frac{3}{m_0^2} \gamma_r^{(0)} \Pi + \mathcal{O}(\mathrm{Kn}), \tag{A9}$$

$$\rho_{-r}^\mu = \gamma_r^{V(1)} V_f^\mu + \gamma_r^{W(1)} W^\mu + \mathcal{O}(\mathrm{Kn}), \tag{A10}$$

$$\rho_{-r}^{\mu\nu} = \gamma_r^{(2)} \pi^{\mu\nu} + \mathcal{O}(\mathrm{Kn}). \tag{A11}$$

In Ref. [8] the coefficients $\gamma_r^{(\ell)}$ were calculated explicitly in the Landau frame, hence $\gamma_r^{V(1)} \equiv \gamma_r^{(1)}$, while $\gamma_r^{W(1)}$ is a new coefficient in the Eckart frame. Note that, in the 14-moment approximation, $\gamma_r^{(0)} \equiv \mathcal{F}_{r0}^{(0)}$, $\gamma_r^{V(1)} \equiv \mathcal{F}_{r0}^{(1)}$, $\gamma_r^{W(1)} \equiv \mathcal{F}_{r1}^{(1)}$, $\gamma_r^{(2)} \equiv \mathcal{F}_{r0}^{(2)}$.

The usual thermodynamic integrals are defined in local equilibrium such that,

$$I_{nq} \equiv \frac{(-1)^q}{(2q+1)!!} \int dK\, E_{\mathbf{k}}^{n-2q} (\Delta^{\alpha\beta} k_\alpha k_\beta)^q f_{0\mathbf{k}}, \tag{A12}$$

$$J_{nq} \equiv \frac{(-1)^q}{(2q+1)!!} \int dK\, E_{\mathbf{k}}^{n-2q} (\Delta^{\alpha\beta} k_\alpha k_\beta)^q f_{0\mathbf{k}} (1 - a f_{0\mathbf{k}}). \tag{A13}$$

Here we also recall the following coefficients appearing in Eqs. (20)–(22),

$$\alpha_r^{(0)} \equiv (1-r) I_{r1} - I_{r0} - \frac{n_{f0}}{D_{20}} (h_0 G_{2r} - G_{3r}), \tag{A14}$$

$$\alpha_r^{(1)} \equiv J_{r+1,1} - h_0^{-1} J_{r+2,1}, \tag{A15}$$

$$\alpha_r^{(2)} \equiv I_{r+2,1} + (r-1) I_{r+2,2}, \tag{A16}$$

$$\alpha_r^h \equiv -\frac{\beta_0}{\varepsilon_0 + P_0} J_{r+2,1}, \tag{A17}$$

and

$$D_{nq} \equiv J_{n+1,q} J_{n-1,q} - J_{nq}^2, \tag{A18}$$

$$G_{nm} \equiv J_{n,0} J_{m,0} - J_{n-1,0} J_{m+1,0}. \tag{A19}$$

In the limit of a massless Boltzmann gas with constant cross section, $J_{nq} \equiv I_{nq} = \frac{(n+1)!}{2(2q+1)!!} \beta_0^{2-n} P_0$, and thus $\alpha_0^h = -h_0^{-1} = -\beta_0/4$ as well $\alpha_1^h = 1$, hence the coefficients of interest are

$$\gamma_1^{V(1)} \equiv \mathcal{F}_{10}^{(1)} = \frac{2\beta_0}{3}, \qquad \gamma_2^{V(1)} \equiv \mathcal{F}_{20}^{(1)} = \frac{\beta_0^2}{2}, \tag{A20}$$

$$\gamma_1^{W(1)} \equiv \mathcal{F}_{11}^{(1)} = -\frac{\beta_0^2}{12}, \qquad \gamma_2^{W(1)} \equiv \mathcal{F}_{21}^{(1)} = -\frac{\beta_0^3}{12}, \tag{A21}$$

$$\gamma_1^{(2)} \equiv \mathcal{F}_{10}^{(2)} = \frac{\beta_0}{5}, \qquad \gamma_2^{(2)} \equiv \mathcal{F}_{20}^{(2)} = \frac{\beta_0^2}{20}. \tag{A22}$$

Also note that in the derivation of the relaxation equations we have expressed the proper-time derivative and spatial derivative of the coefficients $\gamma_r^{(\ell)}$ by the following formulas

$$\dot{\gamma}_r^{(\ell)} = \frac{n_0}{D_{20}}\left[\left(J_{20}\frac{\partial \gamma_r^{(\ell)}}{\partial \alpha_0} + J_{10}\frac{\partial \gamma_r^{(\ell)}}{\partial \beta_0}\right)h_0 - \left(J_{30}\frac{\partial \gamma_r^{(\ell)}}{\partial \alpha_0} + J_{20}\frac{\partial \gamma_r^{(\ell)}}{\partial \beta_0}\right)\right]\theta, \tag{A23}$$

$$\nabla^\mu \gamma_r^{(\ell)} = \left(\frac{\partial \gamma_r^{(\ell)}}{\partial \alpha_0} + h_0^{-1}\frac{\partial \gamma_r^{(\ell)}}{\partial \beta_0}\right)\nabla^\mu \alpha_0 - \beta_0 \frac{\partial \gamma_r^{(\ell)}}{\partial \beta_0}\dot{u}^\mu + \frac{\beta_0}{h_0}\frac{\partial \gamma_r^{(\ell)}}{\partial \beta_0}\mathfrak{q}E^\mu. \tag{A24}$$

These equations follow from Eqs. (17)–(19) neglecting terms proportional to the dissipative fields and/or their derivatives.

APPENDIX B: TRANSPORT COEFFICIENTS

1. Coefficients of the bulk equation

Noting that $\tau_{00}^{(0)} = 1/\mathcal{A}_{00}^{(0)}$, the transport coefficients found in the relaxation equation for the bulk viscosity, Eq. (24), up to terms $N_2 = 0, \neq 1, 2$, are

$$\delta_{\Pi\Pi} = \frac{\tau_{00}^{(0)}}{3}\left(2 + m_0^2\gamma_2^{(0)} - m_0^2\frac{G_{20}}{D_{20}}\right), \qquad \lambda_{\Pi\pi} = \frac{m_0^2}{3}\tau_{00}^{(0)}\left(\gamma_2^{(2)} - \frac{G_{20}}{D_{20}}\right). \tag{B1}$$

Furthermore, the coefficients proportional to V^μ are

$$\ell_{\Pi V} = -\frac{m_0^2}{3}\tau_{00}^{(0)}\left(\gamma_1^{V(1)} - \frac{G_{30}}{D_{20}}\right), \qquad \tau_{\Pi V} = \frac{m_0^2}{3}\tau_{00}^{(0)}\left(\beta_0\frac{\partial \gamma_1^{V(1)}}{\partial \beta_0} - \frac{G_{30}}{D_{20}}\right), \tag{B2}$$

$$\lambda_{\Pi V} = -\frac{m_0^2}{3}\tau_{00}^{(0)}\left(\frac{\partial \gamma_1^{V(1)}}{\partial \alpha_0} + h_0^{-1}\frac{\partial \gamma_1^{V(1)}}{\partial \beta_0}\right), \tag{B3}$$

while the new coefficients proportional to W^μ are

$$\ell_{\Pi W} = -\frac{m_0^2}{3}\tau_{00}^{(0)}\left(\gamma_1^{W(1)} + \frac{G_{20}}{D_{20}}\right), \qquad \tau_{\Pi W} = \frac{m_0^2}{3}\tau_{00}^{(0)}\left(\beta_0\frac{\partial \gamma_1^{W(1)}}{\partial \beta_0} + 2\frac{G_{20}}{D_{20}}\right), \tag{B4}$$

$$\lambda_{\Pi W} = -\frac{m_0^2}{3}\tau_{00}^{(0)}\left(\frac{\partial \gamma_1^{W(1)}}{\partial \alpha_0} + h_0^{-1}\frac{\partial \gamma_1^{W(1)}}{\partial \beta_0}\right). \tag{B5}$$

The coefficients of the electric field are

$$\delta_{\Pi VE} = \frac{m_0^2}{3}\tau_{00}^{(0)}\left(\gamma_2^{V(1)} - \frac{G_{20}}{D_{20}} - \frac{\beta_0}{h_0}\frac{\partial \gamma_1^{V(1)}}{\partial \beta_0}\right), \qquad \delta_{\Pi WE} = \frac{m_0^2}{3}\tau_{00}^{(0)}\left(\gamma_2^{W(1)} - \frac{\beta_0}{h_0}\frac{\partial \gamma_1^{W(1)}}{\partial \beta_0}\right). \tag{B6}$$

2. Coefficients of the diffusion equation

Similarly with $\tau_{00}^{(1)} = 1/\mathcal{A}_{00}^{(1)}$, the transport coefficients found in Eq. (25) are

$$\ell_{V\Pi} = \tau_{00}^{(1)}(h_0^{-1} - \gamma_1^{(0)}), \qquad \tau_{V\Pi} = \tau_{00}^{(1)}\left(h_0^{-1} - \beta_0\frac{\partial \gamma_1^{(0)}}{\partial \beta_0}\right), \qquad \lambda_{V\Pi} = \tau_{00}^{(1)}\left(\frac{\partial \gamma_1^{(0)}}{\partial \alpha_0} + h_0^{-1}\frac{\partial \gamma_1^{(0)}}{\partial \beta_0}\right). \tag{B7}$$

The coefficients proportional to $\pi^{\mu\nu}$ are

$$\ell_{V\pi} = \tau_{00}^{(1)}(h_0^{-1} - \gamma_1^{(2)}), \qquad \tau_{V\pi} = \tau_{00}^{(1)}\left(h_0^{-1} - \beta_0\frac{\partial \gamma_1^{(2)}}{\partial \beta_0}\right), \qquad \lambda_{V\pi} = \tau_{00}^{(1)}\left(\frac{\partial \gamma_1^{(2)}}{\partial \alpha_0} + h_0^{-1}\frac{\partial \gamma_1^{(2)}}{\partial \beta_0}\right). \tag{B8}$$

The coefficients proportional to V_f^μ are

$$\delta_{VV} = \tau_{00}^{(1)}\left(1 + \frac{m_0^2}{3}\gamma_2^{V(1)}\right), \qquad \lambda_{VV} \equiv \frac{1}{5}\tau_{00}^{(1)}(3 + 2m_0^2\gamma_2^{V(1)}), \tag{B9}$$

while similarly the coefficients proportional to W^μ are

$$\delta_{WW} = \frac{\tau_{00}^{(1)}}{3}(-4h_0^{-1} + m_0^2\gamma_2^{W(1)}), \qquad \lambda_{WW} = \tau_{00}^{(1)}\left(-h_0^{-1} + \frac{2m_0^2}{5}\gamma_2^{W(1)}\right). \tag{B10}$$

The coefficients due to the magnetic field are

$$\delta_{VB} = \tau_{00}^{(1)}(-h_0^{-1} + \gamma_1^{V(1)}), \qquad \delta_{WB} = \tau_{00}^{(1)}\gamma_1^{W(1)}, \tag{B11}$$

while the new coefficients due to the electric field are

$$\delta_{VE} = \tau_{00}^{(1)}(-n_0 h_0^{-1} + \beta_0 J_{11}), \tag{B12}$$

$$\delta_{V\Pi E} = -\frac{1}{m_0^2}\tau_{00}^{(1)}\left(2 + m_0^2\gamma_2^{(0)} - m_0^2\frac{\beta_0}{h_0}\frac{\partial\gamma_1^{(0)}}{\partial\beta_0}\right), \qquad \delta_{V\pi E} = \tau_{00}^{(1)}\left(\gamma_2^{(2)} - \frac{\beta_0}{h_0}\frac{\partial\gamma_1^{(2)}}{\partial\beta_0}\right). \tag{B13}$$

3. Coefficients of the shear-stress equation

Finally, using $\tau_{00}^{(2)} = 1/\mathcal{A}_{00}^{(2)}$, the transport coefficients found in Eq. (26) are

$$\delta_{\pi\pi} = \tau_{00}^{(2)}\left(\frac{4}{3} + \gamma_2^{(2)}\frac{m_0^2}{3}\right), \qquad \tau_{\pi\pi} = \frac{2\tau_{00}^{(2)}}{7}(5 + 2m_0^2\gamma_2^{(2)}), \qquad \lambda_{\pi\Pi} = \frac{2}{5}\tau_{00}^{(2)}(3 + m_0^2\gamma_2^{(0)}). \tag{B14}$$

Furthermore we have the coefficients of V_f^μ,

$$\ell_{\pi V} = -\frac{2m_0^2}{5}\tau_{00}^{(2)}\gamma_1^{V(1)}, \qquad \lambda_{\pi V} = -\frac{2m_0^2}{5}\tau_{00}^{(2)}\left(\frac{\partial\gamma_1^{V(1)}}{\partial\alpha_0} + h_0^{-1}\frac{\partial\gamma_1^{V(1)}}{\partial\beta_0}\right), \qquad \tau_{\pi V} = -\frac{2m_0^2}{5}\tau_{00}^{(2)}\beta_0\frac{\partial\gamma_1^{V(1)}}{\partial\beta_0}, \tag{B15}$$

and the new coefficients of W^μ,

$$\ell_{\pi W} = \frac{2}{5}\tau_{00}^{(2)}(1 - m_0^2\gamma_1^{W(1)}), \qquad \lambda_{\pi W} = -\frac{2m_0^2}{5}\tau_{00}^{(2)}\left(\frac{\partial\gamma_1^{W(1)}}{\partial\alpha_0} + h_0^{-1}\frac{\partial\gamma_1^{W(1)}}{\partial\beta_0}\right), \qquad \tau_{\pi W} = 2\tau_{00}^{(2)}\left(1 - \frac{m_0^2}{5}\beta_0\frac{\partial\gamma_1^{W(1)}}{\partial\beta_0}\right). \tag{B16}$$

The new coefficient due to the magnetic and electric fields are

$$\delta_{\pi B} = 2\tau_{00}^{(2)}\gamma_1^{(2)}, \tag{B17}$$

and

$$\delta_{\pi VE} = \frac{2\tau_{00}^{(2)}}{5}\left(4 + m_0^2\gamma_2^{V(1)} - m_0^2\frac{\beta_0}{h_0}\frac{\partial\gamma_1^{V(1)}}{\partial\beta_0}\right), \qquad \delta_{\pi WE} = \frac{2m_0^2\tau_{00}^{(2)}}{5}\left(\gamma_2^{W(1)} - \frac{\beta_0}{h_0}\frac{\partial\gamma_1^{W(1)}}{\partial\beta_0}\right). \tag{B18}$$

[1] U. Heinz and R. Snellings, Ann. Rev. Nucl. Part. Sci. **63**, 123 (2013).
[2] V. Skokov, A. Y. Illarionov, and V. Toneev, Int. J. Mod. Phys. A **24**, 5925 (2009).
[3] W. T. Deng and X. G. Huang, Phys. Rev. C **85**, 044907 (2012).
[4] K. Tuchin, Phys. Rev. C **88**, 024911 (2013).
[5] A. Bzdak and V. Skokov, Phys. Lett. B **710**, 171 (2012).
[6] X.-G. Huang, Rep. Prog. Phys. **79**, 076302 (2016).
[7] W. Israel and J. M. Stewart, Ann. Phys. (N.Y.) **118**, 341 (1979).
[8] G. S. Denicol, H. Niemi, E. Molnar, and D. H. Rischke, Phys. Rev. D **85**, 114047 (2012); **91**, 039902(E) (2015).
[9] S. R. de Groot and L. G. Suttorp, *Foundations of Electrodynamics* (North-Holland Publishing Company, Amsterdam, 1972).
[10] L. Rezzolla and O. Zanotti, *Relativistic Hydrodynamics* (Oxford University Press, New York, 2013).
[11] G. S. Denicol, X. G. Huang, E. Molnár, G. M. Monteiro, H. Niemi, J. Noronha, D. H. Rischke, and Q. Wang, Phys. Rev. D **98**, 076009 (2018).
[12] G. S. Denicol, T. Koide, and D. H. Rischke, Phys. Rev. Lett. **105**, 162501 (2010).
[13] G. S. Denicol, E. Molnár, H. Niemi, and D. H. Rischke, Eur. Phys. J. A **48**, 170 (2012).
[14] S. R. de Groot, W. A. van Leeuwen, and C. G. van Weert, *Relativistic Kinetic Theory—Principles and Applications* (North Holland, Amsterdam, 1980).

[15] C. Cercignani and G. M. Kremer, *The Relativistic Boltzmann Equation: Theory and Applications* (Birkhäuser, Basel, 2002).
[16] J. D. Barrow, R. Maartens, and C. G. Tsagas, Phys. Rep. **449**, 131 (2007).
[17] W. Israel, Gen. Relativ. Gravit. **9**, 451 (1978).
[18] C. Eckart, Phys. Rev. **58**, 919 (1940).
[19] F. Jüttner, Ann. Phys. (N.Y.) **339**, 856 (1911); Z. Phys. **47**, 542 (1928).
[20] L. D. Landau and E. M. Lifshitz, *Fluid Dynamics*, 2nd ed. (Butterworth-Heinemann, Oxford, 1987).
[21] X.-G. Huang, A. Sedrakian, and D. H. Rischke, Ann. Phys. (Amsterdam) **326**, 3075 (2011).
[22] J. Hernandez and P. Kovtun, J. High Energy Phys. 05 (2017) 001.
[23] P. B. Arnold, G. D. Moore, and L. G. Yaffe, J. High Energy Phys. 11 (2000) 001.
[24] P. Kovtun, J. High Energy Phys. 07 (2016) 028.
[25] J. D. Bekenstein and E. Oron, Phys. Rev. D **18**, 1809 (1978).
[26] D. T. Son and P. Surowka, Phys. Rev. Lett. **103**, 191601 (2009).
[27] W. Florkowski, B. Friman, A. Jaiswal, and E. Speranza, Phys. Rev. C **97**, 041901 (2018).
[28] F. Becattini, W. Florkowski, and E. Speranza, Phys. Lett. B **789**, 419 (2019).