ON ISING MODEL COUPLED TO RANDOM PLANAR TRIANGULATIONS

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DOCTORAL DISSERTATION

To be presented for public discussion with the permission of the Faculty of Science of the University of Helsinki, in Auditorium P674, Porthania, on the 30th of June, 2020 at 14 o'clock.

Helsinki 2020
ISBN 978-951-51-6264-9 (PDF)
https://ethesis.helsinki.fi

Unigrafia
Helsinki 2020
Abstract

Random planar maps have been widely studied, due to their role as a discretization of the random surfaces in pure Liouville quantum gravity. However, there are still relatively few mathematical studies concerning random planar maps coupled to statistical physics lattice models, which in turn model Liouville quantum gravity coupled with matter fields. A canonical model of that kind is the Ising model, originally introduced in order to describe the behavior of ferromagnetic matter. This thesis improves understanding of the geometry of random planar triangulations coupled to the Ising model at the critical temperature and away from it, starting from the discrete level, via a Markovian exploration process following the interface on an Ising-decorated random planar map. The exploration is called the peeling process. The interface is imposed by a boundary with Dobrushin boundary conditions, consisting of two boundary arcs with opposite spins, which is the most natural setting involving a single non-closed interface.

The first article in this thesis consists of the study of Boltzmann distributed random triangulations coupled to the Ising model on their faces, under Dobrushin boundary conditions along a simple boundary and at the critical values of the coupling constants. It contains the combinatorial solution of the model as well as the construction of the local limits and the proof of the local weak convergence of the Boltzmann measure as the boundary length tends to infinity arc by arc. In the combinatorics part, explicit rational parametrizations for the generating functions are provided, and a method for singularity analysis via rational parametrizations is developed. This allows to find the asymptotics of the partition function. The construction of the local limit is then based on a peeling exploration process, which always chooses a - edge at the junction of the - and + boundaries, thus following the left-most interface. The peeling process naturally defines a two-dimensional perimeter process, whose components correspond to the + and - boundary lengths in the course of the peeling exploration. Scaling limits of the perimeter fluctuations are also provided, which correspond to a pair of 4/3-stable Lévy processes, as well as the scaling limit of the perimeter process of the + boundary as its length tends to infinity.

The second article studies the phase transition around the critical point by examining the model in the high temperature and the low temperature regimes. First, a critical line of the coupling constants is identified, along which the critical behavior of the Boltzmann-Ising triangulations changes at the critical temperature. The rational parametrizations of the first article are also generalized to arbitrary temperatures. Outside the critical point, the critical behavior of the generating functions is shown to correspond to the one of the pure gravity. However, the geometry in the local limit depends on whether the temperature is smaller or greater than the critical one. In the high-temperature regime, the model in the local limit is shown to be reminiscent of the subcritical face percolation. In the low-temperature regime, the local limit is shown to have a bottleneck almost surely, which results from the fact that the peeling process can jump from the origin to the infinity in a single step. The change of geometry is also reflected by a novel order parameter which is directly constructed from the peeling process. Furthermore, this article strengthens the local convergence of the first article to a regime where the two boundary arcs tend to infinity simultaneously. This diagonal rescaling allows to pass to the local limit without an intermediate step, and is more
natural for studying the scaling limit of the interface. The main results related to this are asymptotic formulas for the partition function under the diagonal rescaling as well as a new scaling limit result related to the interface length.

The third article concerns the half-plane version of the Ising model on random triangulations with spins on the vertices. Using the combinatorial results derived by Albenque, Ménard and Schaeffer and the methods introduced in the two first articles of this thesis, the local weak convergence of such triangulations of the disk as the perimeter tends to infinity is shown, and the interface imposed by the Dobrushin boundary condition is studied. As a consequence of this analysis, it is verified that this model in the high-temperature regime resembles the critical site percolation. Moreover, the fact that the interface is a simple curve gives an explicit scaling limit of the interface length, revealing a rather direct connection to the continuum Liouville Quantum Gravity surfaces.
Acknowledgements

First of all, I acknowledge my advisers Antti Kupiainen and Konstantin Izyurov for their support during the years culminating in this thesis. It was back in 2010, shortly after I had just begun my university studies, when I followed a student colloquium talk by Antti about the topic "random geometry". I had been interested in physics research, but my last year in high school directed my interests more towards mathematics. The talk by Antti really sparked my interest towards random surfaces arising from the theory of quantum gravity. Despite detours in mathematical analysis, I redirected my interests towards mathematical physics in 2013 by asking Antti if he could supervise my master's thesis. Kostya kindly reviewed my thesis in 2014, after which it was natural to think about what to do next. Having a solid understanding in discrete random surfaces and being an expert in the two-dimensional Ising model, he helped me to dwell deeper into the world of random planar maps and the Ising model. Antti and Kostya showed a tremendous amount of patience during my first months as a researcher. Their support has continued until now, and I am truly indebted for their influence and encouragement for my academic career.

My co-author Linxiao Chen deserves a huge credit for the collaboration, both ongoing and in the past, which has led to most of the results in this thesis. During the first years of my PhD journey, Linxiao showed me example as a senior PhD student, patiently explained me the key concepts in order to approach the state-of-the-art, as well as introduced me in the French mathematics culture. He has been kind of an academic big brother for me, and his technical skills have made a deep insight into my research.

It was Jérémie Bouttier who first suggested me to collaborate with Linxiao, and he was also the first researcher I met in France. I am indebted to him for that opportunity, as well as the mentorship he provides in the present and the future. I also acknowledge Nicolas Curien for the enthusiasm he showed towards our research, starting from my very first visit in Paris. It is a great honor that he accepted to act as the opponent in my thesis defense. I also thank Timothy Budd and Laurent Ménard for their efforts as the reviewers of this thesis and their encouraging comments.

I acknowledge Sigurður Örn Stefánsson for making my phase transition from a PhD student to a postdoc smooth, by offering me the possibility to work with him in Iceland. I am thankful for the joint research with him and Jakob Björnberg that has provided me with great entertainment and learning experience during the time of the COVID-19 lockdown when most of us have been urged to work at home. It is regrettable that I only had the chance to spent a little time in Reykjavik before the lockdown, and I wish I would have the opportunity to spend a bit more time there before the autumn.

The members of the Mathematical Physics Group at the University of Helsinki, both current and in the past, deserve my sincere thanks for creating a stimulating and an encouraging work environment. I especially want to thank Petri Tuisku for being a great colleague and friend during many academic events and classes. His instructorship in the probability and the mathematical physics courses at the end of my master and the beginning of my PhD provided me with a solid learning experience. The numerous coffee room conversations and conference trips have made this experience to continue many years since then. I also acknowledge Antti Kemppainen and Kalle Kytölä for their lecture courses on mathematical physics and the events organized by them. It is fair to say that they have greatly contributed in building the foundation of my know-how and identity as a probabilist and a mathematical
Likewise, I want to acknowledge all the friends and the colleagues at the Kumpula campus during the almost ten years I have spent there. Certainly all of you would deserve your names mentioned here. I also do not forget my colleagues at Aalto University with whom I have had the privilege to attend many Stochastic Saunas and other seminars. I have also spent more than six months in France during my PhD (if I counted correctly), and naturally met a myriad of people there. I want to thank all my colleagues and friends I had the chance to meet at that time; let us hope our paths will cross in the near future. I also want to thank my colleagues at the University of Iceland for providing me with a truly welcoming environment and great coffee room conversations.

The second last paragraph is devoted to my friends outside the university community who have occasionally reminded me that there is also life somewhere outside there. There are way too many of you to mention here by name, but I bet you will recognize yourself when reading this paragraph.

Lopuksi, haluan kiittää perhettäni kaikesta tuesta vuosien varrella. Olen saanut kasvaa ympäristössä, jossa minua on kuunneltu, arvostettu ja kannustettu. Olen myös saanut tehdä omat valintani opinto- ja urapolullani, mikä ei ole itsestään selvää. Haluan kiittää äitiäni ja isääni siitä rakkaudesta, jolla he ovat minut kasvattaneet. Lisäksi kiitän veljeäni, joka toimi minulle esimerkkinä uravalinnalleni, sekä siskoani, joka puolestaan toimii esimerkkinä siitä, ettei kaikkien tarvitse olla matemaatikkoja. The last sentences here are devoted to Khanh, who has stayed next to me through many years during this PhD journey. Her love and patience have brightened my dark moments and provided me with great happiness.

Joonas Turunen
Helsinki, 2 June 2020
List of included articles

The thesis consists of an introduction and three research articles:


Both of the authors contributed equally to the analysis, computations and writing of the joint articles. Part of Article (I) was included in the thesis of L. Chen [17], corresponding roughly to preliminary versions of Sections 2-4, 6 and the appendices in Article (I). Some results there were further generalized in Article (I).

Article (I) is reprinted under a Creative Commons Attribution 4.0 International License (http://creativecommons.org/licenses/by/4.0/). Articles (II) and (III) are the versions 3 of the respective manuscripts before journal submission.
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0.1 Overview and brief history

The Ising model is a classical model of statistical physics, introduced first time by Lenz in 1920 [38], and solved in its simplest form as a one-dimensional spin chain by his student Ising in his doctoral thesis [30]. It models ferromagnetic matter, such that its definition on the one hand gives a physically sound picture, and on the other hand is simple enough to conduct exact computations. During its one hundred years long history, it has drawn much attention in both theoretical physics and mathematics communities. Indeed, if searched in Google Scholar, the keyword 'Ising model' gives around 418000 results.

In dimension one, Ising showed that the model has no phase transition. Roughly speaking, this means that there is no such temperature, in which the qualitative behavior of the model spontaneously changes. Mistakenly, he also claimed that this should imply the non-existence of a phase transition in higher dimensions. However, in 1936, Peierls showed that there is a spontaneous magnetization in low temperatures in dimension at least two [46], after which the interest to the Ising model sparked. Finally in 1944, Onsager found an exact solution of the model in two-dimensional square grid, also showing the existence of a critical temperature, where a spin ordering phase transition occurs [45].

Since then, the research on the two-dimensional Ising model has remained active until today. Some culmination points have been the identification of conformally invariant scaling limits of various observables of the model, and proofs of the convergences of these observables suitably rescaled toward the aforementioned scaling limits. As a highlight, we mention the celebrated fermionic observables by Smirnov [52] which led to the breakthroughs that suitably rescaled Ising interfaces convergence towards Schramm-Loewner evolutions with parameter $\kappa = 3$, which are families of conformally invariant random curves. We do not, however, study directly conformally invariant random curves in this thesis. Rather, we will find some scaling limits which have connections to them.

All the above results concern the Ising model on a regular, i.e. fixed, lattice. In the 1980s, the study of the Ising model was extended to random lattices in the works of Kazakov [31], and Boulatov and Kazakov [15]. In the same decade, there was also active research on Liouville Conformal Field Theory (LCFT), also known as Liouville Quantum Gravity (LQG), initiated in the work of Polyakov [47]. A central part of this model of two-dimensional quantum gravity is to understand how typical random metrics on two-dimensional surfaces look like. In the simplest setting, the surface is just the 2-sphere $S^2$, and the random metric is uniform. Of course, the latter has much ambiguity. One possibility to study uniformly distributed discretized surfaces, and try to identify their scaling limit. The uniform case falls within the universality class of pure gravity. Pure gravity is already rather well understood, and we do not intend to give a literature review of it. If, however, the discrete random surfaces are coupled with statistical physics lattice models, their scaling limits are conjectured to fall into different universality classes. One example is obtained when coupling discrete random surfaces with the Ising model, and studying its scaling limits. This is called coupling gravity with matter, where in this case the matter field is the Ising spin field. Therefore, the theory of LQG in the general framework of quantum gravity is one justification why one should be interested in studying the Ising model on the random surfaces.

Another motivation for the study of the Ising model on quantum gravity, also related to the above story, is the fact that many properties of statistical mechanics models on regular

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1 As checked on 14 April 2020
planar lattices are conjectured to be reconciled with the corresponding properties on random lattices via the so-called KPZ relations, due to Knizhnik, Polyakov and Zamolodchikov [33]. These involve at least relations of certain scaling exponents of observables related to the models, and also relations of other critical exponents with the parameters of the underlying conformal field theory. One example of such a relation is the relation between the string susceptibility exponent and the central charge of the CFT, which is introduced later in this thesis. It is often easier to compute a critical exponent in the quantum gravity setting, and then pass it to the Euclidean setting via the KPZ relation.

0.2 Ising model on a graph

0.2.1 Definitions

Let $G = (V, E)$ be a finite graph, where $V$ is the set of vertices and $E$ the set of edges. That is, $V = \{v_1, v_2, \ldots, v_n\}$ is a finite set of points and $E$ is a set consisting of pairs of points $\{v_i, v_j\}$, where possibly $i = j$. The latter means that the graph may contain loops, i.e. edges with the same endpoint. In graph theory, these objects are usually called multigraphs.

Assign to each vertex $v \in V$ a spin $\sigma_v \in \{-1, +\}$, so that the graph can also be viewed as bicolored. Let $\sigma = (\sigma_v)_{v \in V}$ be called a spin configuration. It is an element of the space $\Omega_G := \{-1, 1\}^V$. If $B \in \mathbb{R}$, the Hamiltonian is the function defined on $\Omega_G$ as

$$H_{G,B}(\sigma) = -\sum_{\{v,w\} \in E} \sigma_v \sigma_w - B \sum_{v \in V} \sigma_v.$$ 

It can be viewed as the energy of the spin configuration $\sigma$ on $G$, where $B$ represents an external magnetic field.

To add randomness in the picture, we define a probability measure on the space $\Omega_G$ which favors the spins of adjacent vertices to coincide. According to statistical mechanics principles, we also wish the measure to favor minimal energy configurations. The Boltzmann distribution is known to be the unique measure satisfying the above properties. It is defined as follows: First, we define the partition function

$$Z(G, \beta, B) := \sum_{\sigma \in \Omega_G} e^{-\beta H_{G,B}(\sigma)}$$

where $\beta \in \mathbb{R}$ is a constant. It is essentially the moment generating function of the random system, and thus encodes a considerable amount of information. On the other hand, it is simply the normalizing constant of the Boltzmann distribution, which is defined as follows:

$$\mathbb{P}_{G,\beta,B}(\sigma) = \frac{e^{-\beta H_{G,B}(\sigma)}}{Z(G, \beta, B)}.$$ 

The measure (2) models the mutual interaction of adjacent spins as well as the interaction of a spin with the external magnetic field $B$. In other words, there are only nearest neighbor interactions between different spins, and the interaction caused by the external field. More precisely, the measure favors $+$ spins if $B > 0$, and there is spin symmetry if $B = 0$. Moreover, the interaction is ferromagnetic if $\beta > 0$ and antiferromagnetic if $\beta < 0$. If $\beta = 0$, there is no interaction between the spins. The parameter $\beta$ can be viewed as the
inverse of the temperature, which means that the model is ferromagnetic precisely when the temperature is positive and finite. In the sequel, we only consider the ferromagnetic Ising model, and sometimes compare it to the infinite temperature case \( \beta = 0 \), which coincides with percolation models.

It is sometimes fruitful to replace the external magnetic field \( B \) by certain boundary conditions. Informally this means the following: we set \( B = 0 \), and choose a cycle of pairwise adjacent vertices together with the edges incident to them, in which we fix a sequence of spins, and sample the remaining spin configuration on an area restricted by the cycle from the Boltzmann measure conditional on the fixed sequence. The cycle is called the boundary, and the sequence of spins on it the boundary condition. If the cycle is a discrete Jordan curve, we call the boundary simple. In this thesis, we mainly consider monochromatic and Dobrushin boundary conditions on a simple boundary. They are defined as follows: for monochromatic boundary, the spin sequence is fixed to be all + or −, whereas a Dobrushin boundary consists of one + and one − boundary segment. The precise definitions are presented later in the context of each of the models considered. Moreover, we also briefly use free boundary conditions (Article (I)): in this case, the boundary spins are not fixed.

### 0.2.2 Phase transition

From the experiments on some ferromagnetic matter, one expects the magnetic behavior to depend on the temperature of the system. One can think about a piece of iron for instance. More precisely, if the temperature is low enough, the system should be ferromagnetic, and in high temperatures paramagnetic. The phase transition describes how the qualitative behavior of the system changes at some non-zero temperature. This can be verified analytically for the Ising model in dimensions at least two.

For simplicity, we consider the square lattice in dimension \( d \geq 2 \). In \( d = 1 \), Ising showed that the model does not have a phase transition, and it is ferromagnetic in any positive temperature. He also mistakenly claimed that there should be no phase transition in \( d \geq 2 \), which was later shown to be false in a series of works culminated in the celebrated results of Onsager [45]. Here, we define the phase transition rigorously, in the case \( B = 0 \).

For \( n \in \mathbb{N} \), consider the hypercubes \( \Lambda_n := ([-n, n] \cap \mathbb{Z})^d \) and the Ising model on the vertices on it. Here, the hypercube has its canonical nearest-neighbor structure. Let \( \mathbb{P}_{\beta, \ast} \) be the weak limit of the measures \( \mathbb{P}_{\Lambda_n, \beta, 0} \) equipped with + boundary conditions as \( n \to \infty \), also known as the thermodynamic limit. We define the magnetization as \( M_\ast(\beta) = \mathbb{E}_{\beta, \ast}(\sigma_0) \), where \( \sigma_0 \) is the spin of some fixed vertex and \( \mathbb{E}_{\beta, \ast} \) is the expectation related to \( \mathbb{P}_{\beta, \ast} \). Then, the critical inverse temperature can be defined as

\[
\beta_c := \inf\{\beta > 0 : M_\ast(\beta) > 0\} = \sup\{\beta > 0 : M_\ast(\beta) = 0\}.
\]

The definition indeed makes sense, since the function \( \beta \to M_\ast(\beta) \) can be shown to be increasing and non-negative. We say that the model exhibits a spin ordering phase transition if \( 0 < \beta_c < \infty \). In this case, the function \( \beta \mapsto M_\ast(\beta) \) defines an order parameter. That is, \( M_\ast(\beta) = 0 \) if \( \beta < \beta_c \), and \( M_\ast(\beta) > 0 \) if \( \beta > \beta_c \). In general, the behavior of the model at and around the critical temperature \( \beta = \beta_c \) is harder to analyze than far away from it.

Another way to characterize the phase transition is via non-analyticity of certain thermodynamic quantities. One canonical choice of a such is the free energy. In the two-dimensional
Figure 1 – Simulations of the Ising model on the two-dimensional square lattice. The first picture is at $\beta < \beta_c$, second at $\beta = \beta_c$ and the third at $\beta > \beta_c$. The phase transition is clearly visible as a symmetry breaking of the spin clusters: In the high-temperature phase, the spins are disordered, the spin clusters are finite and the behavior of the system is percolation-like. In the low-temperature regime, the spins form large clusters. At $\beta = \beta_c$, the interfaces between the clusters have a non-trivial geometry. The simulations are conducted with the open source EJS 2d Ising model Java applet by Wolfgang Christian.

setting above, it can be defined as

$$F(\beta) := -\frac{1}{\beta_c} \lim_{n \to \infty} \frac{1}{n^2} \log Z(\Lambda_n, \beta, 0).$$

Onsager showed that $F$ is real analytic outside $\beta_c$, but exhibits a second order discontinuity at $\beta = \beta_c$, meaning that the second derivative of $F$ diverges at $\beta_c$. Thus, we say that the phase transition in $d = 2$ is of second order.

Onsager, based on the work of Kramers and Wannier [34], managed actually to show the phase transition for any $d \geq 2$. In $d = 2$, it happens around the critical inverse temperature $\beta_c = \frac{1}{2} \log(1 + \sqrt{2})$. The proof is based on a one-to-one correspondence between the spin configurations and the interface configurations, where an interface consists of dual edges separating two different spins. In dimension two, the interfaces are just discrete paths. At $\beta = \beta_c$, they actually turn out to have conformally invariant scaling limits, which are well predicted by the conformal field theory (CFT). The next section presents some of these beautiful results, which also serve as a great inspiration for some of the main results in this thesis.

0.2.3 Interfaces and their scaling limits

In this section, we present some celebrated results of modern mathematical physics, which deal with conformally invariant scaling limits of the Ising model at the critical temperature $\beta = \beta_c$. The existence of the scaling limit was first postulated by Belavin, Polyakov and Zamolodchikov in terms of the correlation functions [8]. Although there are numerous observables which are rigorously shown to have conformally invariant scaling limits predicted by the CFT such as spin correlations, we only restrict our interest to the interfaces in this thesis.

First, we consider the Ising model on the vertices of a square lattice restricted to some finite simply connected domain and with Dobrushin boundary conditions. More formally,
(a) An Ising spin configuration with Dobrushin boundary conditions on the two-dimensional square lattice at the critical point $\beta = \beta_c$.

(b) A realization of the SLE(3), which is the scaling limit of the critical Ising interface between the spin clusters as the lattice mesh tends to zero.

Figure 2 – These two simulations depict how the SLE(3) curve arises as a scaling limit of an Ising interface generated by Dobrushin boundary conditions. Courtesy of Vincent Beffara.

This means the following: Let $\delta > 0$ be a scaling parameter and $\Omega_\delta$ be a simply connected domain in the distance-rescaled square lattice $\delta \mathbb{Z}$, and let $a_\delta$ and $b_\delta$ be two distinct vertices on the boundary of $\Omega_\delta$. Equip the vertices on the boundary segment from $a_\delta$ to $b_\delta$ counterclockwise with spin $+$ and the segment clockwise with spin $-$. This Dobrushin boundary condition generates a simple curve on the dual lattice from $a_\delta$ to $b_\delta$ such that the curve has spins $-$ immediately on the left and spins $+$ on the right. We call this curve the (spin) interface and denote it by $\gamma_\delta$. Actually, there might be several choices of such a curve, but we can choose the leftmost of them.

Now assume that $\Omega_\delta, a_\delta$ and $b_\delta$ approximate a bounded simply connected domain $\Omega \subset \mathbb{C}$ and its two distinct boundary points $a$ and $b$ (more precisely, two degenerate prime ends), respectively, in a reasonable sense as $\delta \to 0$. See [16] for a more specific statement. Then ([16], Theorem 1), the law of $\gamma_\delta$ converges weakly towards a continuous random curve $\gamma$ from $a$ to $b$ in $\Omega$ as $\delta \to 0$. The law of $\gamma$ is the chordal Schramm-Loewner evolution of parameter $\kappa = 3$ (in short, SLE(3)). The law of SLE($\kappa$) is characterized by its domain Markov property and conformal invariance, together with the driving term $\sqrt{\kappa} B_t$ via the Loewner’s equation, where $(B_t)_{t \geq 0}$ is the standard one-dimensional Brownian motion and $\kappa \geq 0$. See [32] or [49] for a precise definition and the basic properties of the SLE-curves. One should note that the randomness of SLE($\kappa$) is generated by the Brownian motion and the winding of the curve by the parameter $\kappa$. In particular, if $0 \leq \kappa \leq 4$, the SLE($\kappa$) is a simple random curve. The proof of the above convergence relies on the celebrated fermionic observable by Smirnov [52].

The above result concerns the interface separating the two large spin clusters. In the thermodynamic limit $n \to \infty$, where $n$ is the size of the box $\Lambda_n$ in the previous section, this interface is infinite. There are also analogs of the above interface convergence for the interfaces encircling finite spin clusters. A simplest setting is the following: Instead of Dobrushin boundary conditions, we consider now $+$ boundary conditions on some discrete domain. Then one can explore a realization of the Ising model on such a domain from the
boundary until one discovers a vertex with spin $-\uparrow$, after which one can follow the interface encircling the aforementioned vertex together with all the adjacent vertices of spin $-\uparrow$. It can be shown that such an Ising model only has at most one infinite spin cluster, and thus all the above interfaces are finite. Benoist and Hongler showed in [9] that the collection of such interfaces, after a suitable rescaling as in the previous paragraph, converge towards an ensemble of random continuum loops, called the Conformal Loop Ensemble with parameter $\kappa = 3$ (CLE(3) for short). In general, the CLE shares the driving parameter $\kappa$ with SLE($\kappa$) and enjoys similar conformal invariance. For more general properties of the CLE, see [51].

0.3 Random discrete surfaces and two-dimensional Liouville Quantum Gravity

In the previous section, the Ising model was defined on a regular lattice, i.e. as a probability distribution on a fixed graph. In this section, we assume the lattice is also sampled from a probability distribution. In the simplest setting, we consider planar graphs of a fixed size embedded in the two-dimensional sphere $S^2$ (or some other Riemann surface), and sample such a graph together with its embedding uniformly among all choices of its corresponding size. Such an idea origins from the discretization of the space-time in two dimensional quantum gravity and string theory, where a widely regarded starting point is considered to be the seminal work of Polyakov [47]. Mathematically, the corresponding art of discretization of random Riemann surfaces can be handled via the theory of random maps.

0.3.1 Random planar maps

Let $G = (V, E)$ be a finite (multi)graph, where loops and multiple edges between the vertices are allowed. We extend the edges to correspond continuous images of an interval into $S^2$ in a natural way. We say that $G$ is properly embedded in the sphere $S^2$ if the interiors of the edges do not intersect with one another and with the vertices, and each face is homeomorphic to a disk. Then $G$ is a planar map if it is a connected properly embedded graph in $S^2$ viewed up to orientation-preserving homeomorphisms of $S^2$. For combinatorial reasons, we usually distinguish one oriented edge, or a corner, of a map and call it the root. The planar map with a distinguished root is called a rooted map. See Figure 3 for two different maps which have the same graph embedded on the sphere in two different ways.

We will denote a generic planar map as $M$ in the sequel. It can be both seen as a combinatorial and a topological object. A central identity for the combinatorics of planar maps is the Euler’s formula. Let $V(M)$, $E(M)$ and $F(M)$ be the sets of vertices, edges and faces of $M$, respectively. Then, we have the Euler’s identity for the cardinalities of the

Figure 3 – An example of two different planar maps, whose graphs are isomorphic.
above sets at genus zero as

$$|V(M)| + |F(M)| - |E(M)| = 2.$$ 

(4)

If we consider maps with face degrees bounded above by some constant and fix another one of the above three cardinalities to be \(n\), then the corresponding set \(\mathbb{M}_n\) is finite. In this case, one could sample \(M \in \mathbb{M}_n\) uniformly at random. This is the simplest generic example of a random planar map. For example, we might consider planar triangulations with \(n\) faces, and pick one uniformly at random.

One can relax the above assumption of uniformicity, and define in some cases a Boltzmann distribution on maps as follows: Assume that \(\mathbf{q} = (q_n)_{n \geq 0}\) is a non-negative sequence. Let \(\mathbb{M}\) be the set of rooted maps, and define for \(M \in \mathbb{M}\)

$$W^\mathbf{q}(M) = \prod_{f \in F(M)} q_{\text{deg}(f)}$$

(5)

and the generating function of maps by

$$Z^\mathbf{q} = \sum_{M \in \mathbb{M}} W^\mathbf{q}(M).$$

(6)

If the sequence \(\mathbf{q}\) is such that \(Z^\mathbf{q} < \infty\), then one can define the Boltzmann distribution

$$\mathbb{P}^\mathbf{q}(M) = \frac{W^\mathbf{q}(M)}{Z^\mathbf{q}}.$$ 

(7)

The study of this distribution is particularly interesting for certain collections of bipartite planar maps, but we do not study them further here. One may have a closer look on them in [21] and the references therein. One may have a closer look on them in [21] and the references therein. Instead, we assume that the sequence \(\mathbf{q}\) is bounded, and in particular, that \(q_k = t \cdot \delta_{k,3}\). Then, the measure (7) is the law of the Boltzmann triangulation.

Both the uniform and the Boltzmann distributions have interesting limits in distribution, respectively. First, one can let the size of the map, e.g. the number of faces, tend to infinity, which yields the concept of local limit. This correspond to the thermodynamic limit in physics. There are many ways to pass to the local limit. One is to consider the map conditional on one large distinguished face, called the exterior face, and take the limit of the Boltzmann distribution as the perimeter of this face tends to infinity. Indeed, this case is the main focus in this thesis. One can also consider the random map as a compact metric space endowed with the graph distance, and study the convergence of such pairs where the distance is suitably rescaled. In this case, if the convergence takes place in distribution with respect to the Gromov-Hausdorff topology of the compact metric spaces, one encounters a random compact metric space in the limit. This is called the scaling limit. For uniformly distributed maps with sufficiently regular (e.g. bounded) face degrees, a unique scaling limit exists, called the Brownian map. See the celebrated convergence results of Le Gall [36] and Miermont [41], as well as a more recent generalization to *regular* degree sequences on bipartite maps by Marzouk [40]. If one allows degree sequences with polynomial heavy tails, one encounters different scaling limits. These are called stable maps, and their uniqueness is still conjectured. See [37] and [39] for more.

Since most of the results in this thesis concern local limits, we introduce them in greater detail.
Local limit. The local limit of uniform random planar maps of the sphere was first time introduced by Angel and Schramm [6], in the case of triangulations. Later, many variants of the local limit were discovered, also for more general lattices. In this thesis, we are mostly interested in the Uniform Infinite Triangulation of the Half-Plane (UIHPT), studied first by Angel and Curien [4], [5].

We start with specifying what is meant by a "local limit". Let \( \mathcal{M} \) be some collection of maps. If \( m, m' \in \mathcal{M} \), we define the local distance by

\[
d_{\text{loc}}(m, m') = 2^R\]

where \( R = \sup\{r \geq 0 : [m]_r = [m']_r\} \)

(8)

where \([m]_r\) denotes the combinatorial ball (i.e. the ball w.r.t. the graph distance) of radius \( r \) around the root vertex of \( m \). It is well-known that the metric space \((\mathcal{M}, d)\) is complete and separable in the regular cases (see [6] in the case when \(\mathcal{M}\) is the set of triangulations).

If \( M_n \) is a sequence of random maps on \( \mathcal{M} \), then a map \( M \) is the local limit of \((M_n)_{n \geq 0}\) if \( M_n \overset{d_{\text{loc}}}{\longrightarrow}_{n \to \infty} M \) in distribution with respect to the local distance \( d_{\text{loc}} \). We denote this convergence by \( M_n \overset{d_{\text{loc}}}{\longrightarrow}_{n \to \infty} M \). It should be noted that this convergence is equivalent with saying

\[
P([M_n]_r = b) \overset{n \to \infty}{\longrightarrow} P([M]_r = b)
\]

for all \( r \geq 0 \) and balls \( b \) of radius \( r \).

We restrict now our attention to triangulations. Thus, let \( \mathcal{M} = \mathcal{T} \) be the set of all triangulations, and \( \mathcal{T}_n \subset \mathcal{T} \) the set of those with \( n \) vertices. Angel and Schramm proved in [6] that if \( \mathcal{T}_n \) is uniform on \( \mathcal{T}_n \) then \( \mathcal{T}_n \overset{d_{\text{loc}}}{\longrightarrow}_{n \to \infty} \mathcal{T} \), where \( \mathcal{T} \) is called the Uniform Infinite Planar Triangulation (UIPT). It is an infinite random triangulation of the sphere \( S^2 \), meaning that all of its faces are triangles. Moreover, it is one-ended, meaning that the complement of any finite subgraph of \( \mathcal{T} \) has exactly one infinite connected complement.

Angel also formulated a way to explore the UIPT in [3], in order to study volume growth and percolation on it. The idea of the exploration process, called the peeling process, was first introduced by Watabiki [54] in the context of string theory. Later, pioneered by Angel and Curien [5], the peeling process became a central tool to study infinite random maps of the half-plane. Moreover, it has been proven to be efficient in the study of some statistical mechanics lattice models on random maps. Some examples are various percolation models, already discussed in [3] and [4], and later generalized eg. by Richier [48]. Another example is the Ising model, the topic of this thesis.

From now on, we concentrate on triangulations of the half-plane. We also restrict our attention to so-called Type I triangulations, which allow loops and multiple edges, and thus correspond to embedded multigraphs. We draw our attention to the uniform triangulation of the \( p \)-gon with \( n \) interior vertices, denoted by \( \mathcal{T}_{p,n} \). In this case, among others, the following local convergence was shown in [5]:

\[
\mathcal{T}_{p,n} \overset{d_{\text{loc}}}{\longrightarrow}_{n \to \infty} \mathcal{T}_{p,\infty} \overset{d_{\text{loc}}}{\longrightarrow}_{p \to \infty} \mathcal{T}_\infty,
\]

(9)

where \( \mathcal{T}_{p,\infty} \) is the UIPT of the \( p \)-gon and \( \mathcal{T}_\infty \) the UIPT of the half-plane, called the UIHPT. Both of them are one-ended infinite random triangulations; the former can be embedded in the \( p \)-gon, and the latter in the (upper) half-plane. Moreover, the Boltzmann distribution on the triangulations of the \( p \)-gon, defined as (7) and denoted by \( \mathbb{P}_p \), also converges locally:

\[
\mathbb{P}_p \overset{d_{\text{loc}}}{\longrightarrow}_{p \to \infty} \mathbb{P}_\infty,
\]

where \( \mathbb{P}_\infty \) is the law of the UIHPT \( \mathcal{T}_\infty \).
The UIHPT enjoys a particularly simple **spatial Markov property**, which is in some sense a discrete analog of the domain Markov property of the SLE. It is roughly described as follows: if $G \subset T_\infty$ is a simply connected finite sub-triangulation of $T_\infty$ containing the root edge, then the complement $T_\infty \setminus G$ is independent of $G$ and also has the law $\mathbb{P}_\infty$. The spatial Markov property also has its counterparts for the Boltzmann triangulations with a finite boundary, in which case the boundary length does not stay constant, but rather evolves as a Markov chain. More precisely, the spatial Markov property is actually equivalent to the peeling process.

The peeling process of a Boltzmann triangulation with a boundary of length $p$ can be described as follows: A peeling algorithm chooses a boundary edge, which is then deleted. This operation reveals the triangle incident to the deleted boundary edge. If the third vertex of that triangle is an interior vertex, then the resulting unexplored triangulation is a Boltzmann triangulation with boundary length $p + 1$ and with one interior vertex less. The other possibility is that the third vertex is in the boundary, which splits the triangulation $T_p$ into two Boltzmann triangulations. Then, one of those triangulation is chosen for the new unexplored region, and the other one is filled with a finite Boltzmann triangulation. This procedure is then iterated, yielding a stochastic process consisting of a sequence of growing submaps of $T_p$.

Formally, consider the set of symbols $\mathcal{S} = \{\mathrm{C}, \mathrm{L}_k, \mathrm{R}_k : k \geq 0\}$. The generating function \((6)\) has now the form $Z_p(t) = \sum_{n \geq 0} w_{p,n} t^n$, where $w_{p,n} := |T_{p,n}|$ and $T_{p,n}$ denotes the set of triangulations of the $p$-gon with $n$ vertices. It is called the **partition function** of $T_p$. Now by the definition of the Boltzmann distribution, the first step of the peeling process of $T_p$, denoted by $\mathcal{S}_1$, can be seen as a probability distribution on $\mathcal{S}$ as follows:

$$
\mathbb{P}_p(\mathcal{S}_1 = \mathrm{C}) = \frac{Z_{p+1}}{Z_p}, \quad \mathbb{P}_p(\mathcal{S}_1 = \mathrm{L}_k) = \mathbb{P}_p(\mathcal{S}_1 = \mathrm{R}_k) = \frac{Z_{p-k}Z_{k+1}}{Z_p} \quad (k \geq 0) \quad (10)
$$

where $k$ is truncated at some cutoff in order to cover every equivalent step only once; for example, $\mathrm{L}_p$ and $\mathrm{R}_p$ represent the same step. The reason why the peeling is defined in an infinite set $\mathcal{S}$ is that we are eventually interested in the limit $p \to \infty$ of the peeling process.

Indeed, it is known that there exists a critical value $t = t_c > 0$ such that $Z_p(t_c) \sim_{p \to \infty} C \cdot t_c^{-p} p^{-5/2}$. Using these asymptotics, it is not hard to see that the distribution \((10)\) has a limit as $p \to \infty$ which defines a probability distribution on $\mathcal{S}$. Iterating this in a consistent way, the peeling process of the UIHPT is constructed. Moreover, one can see that actually the UIHPT itself can be constructed from the peeling process, which is defined as a weak limit.

![Figure 4 – A schematic explanation of the peeling decomposition.](image)
of the peeling process on the finite random triangulations. Namely, the peeling algorithm
which chooses an edge from the boundary to continue the peeling with can be chosen such
that it explores any neighborhood of the origin roughly in distance layers, and then the
UIHPT can be defined as a growing sequence of this kind of neighborhoods.

Another thing which makes the peeling process powerful is the fact that it can explore
geometric structures on random triangulations tracking their size. For example, Angel used
it to study the growth rate of balls and hulls in the UIPT [3]. Another example is to
study the geometric properties of a statistical physics lattice model on the UIPT or its
variants. A canonical example is a percolation model on the UIHPT, for which Angel and
Curien computed various critical exponents in [4] and showed the critical parameters. In
particular, a suitably chosen peeling process follows the interface between two percolation
clusters, which is particularly clear in a site-percolated triangulation. Using this, some
scaling estimates for the interface length and the volume of the finite cluster containing the
root vertex is deduced, among other things.

In this thesis, the ideas in this paragraph are generalized in order to study triangulations
decorated with the Ising model. More precisely, various local limits of the Boltzmann dis-
tribution of Ising-decorated triangulations of a polygon are constructed and studied using
a peeling process. All of the peeling processes are chosen such that they closely follow the
Ising interface generated by the Dobrushin boundary conditions, and are actually analogous
to the ones used for percolation interfaces. The biggest differences lie in the combinatorics
of the models, and in the geometric properties of the peeling processes. For example, the
peeling decomposition of an Ising-decorated triangulation needs to take into account two
different colors and the colors of nearest neighbors, and the Markov chains associated with
the peeling process have different and more complicated behaviour.

0.3.2 Liouville quantum gravity

Besides scaling limits of random planar maps, there is another approach for the theory of
random (continuum) surfaces, called the Liouville Quantum Gravity, or LQG for short. Here
we restrict ourself to two-dimensional Riemann surfaces, and in particular to the Riemann
sphere $S^2$. In higher generality, LQG it is actually a conformal field theory, originating from
Polyakov’s study of the bosonic string theory [47]. From the CFT point of view, we refer to
the article [22] with a comprehensive study on the Riemann sphere.

However, for the purpose of this thesis, we choose the approach of Duplantier and
Sheffield [24] which is particularly motivated by the geometry of random surfaces, and
also easier to relate to scaling limits of random planar maps. They consider the special case
of the Liouville Quantum Gravity where the Liouville cosmological constant is set to zero,
called the critical LQG. In this case, the LQG is defined as follows:

Let $D \subset \mathbb{C}$ be smooth domain, and $h$ an instance of the free boundary Gaussian free
field on it; see the definition eg. in [48]. We would like to define a random measure on $D$ as

$$\mu_h = e^{\gamma h(z)} dz$$  \hspace{1cm} (11)$$

where $\gamma \in [0, 2)$ is a parameter and $dz$ denotes the Lebesgue measure on $D$. One should
note that $h$ is a random tempered distribution which should not be treated as a function, so
the exponentiation is not really well-defined. However, a suitable regularization procedure
will solve the problem. Among the several possibilities, we choose the following: Let $h_\epsilon(z)$
be the average of $h$ on the circle of radius $\epsilon$ centered at $z \in D$. Then, one can define the LQG measure (11) as
$$\mu_h := \lim_{\epsilon \to 0} \epsilon^2/2 e^{\gamma h_\epsilon(z)} dz$$
where the limit is taken weakly on compact subsets of the space of measures on $D$. The measure (12) is the area measure of the two-dimensional LQG random sphere, which has a conformal structure parametrized by $D$. For a boundary point $z \in \partial D$, the circle average $h_\epsilon(z)$ is restricted on $D \cap \partial B_\epsilon(z)$, and the quantum boundary length measure is defined as
$$\nu_h := \lim_{\epsilon \to 0} \epsilon^{\gamma^2/4} e^{\gamma^2 h_\epsilon(z)/2} dz.$$ (13)

Let $Q := \frac{2}{\gamma} + \frac{\gamma}{2}$, $D$ and $\tilde{D}$ two complex domains, and $\psi : \tilde{D} \to D$ a conformal mapping. Define
$$\tilde{h} = h \circ \psi + Q \log |\psi'|,$$ (14)
in the sense of distributions. Then it can be shown that $\tilde{h}$ is a GFF on $\tilde{D}$, and that the LQG measures $\mu_h$ and $\nu_h$ satisfy $\mu_h(A) = \mu_h(\psi(A))$ and $\nu_h(A) = \mu_h(\psi(A))$ for $A \subset \tilde{D}$ or $A \subset \partial \tilde{D}$, respectively. Thus, the mapping $\psi$ is a coordinate change, which defines an equivalence relation by
$$(D, h) \leftrightarrow (\psi^{-1}(D), \tilde{h}) = (\tilde{D}, \tilde{h}).$$ (15)

We call an equivalence class of pairs $(D, h)$ defined by (15) a quantum surface. In this thesis, we are primarily interested in the following two quantum surfaces, whose more precise definitions can be found in [23]:

- **Thick quantum wedge** with parameter $\alpha \geq 0$: This is a "half-plane like" quantum surface in the following sense: Any quantum surface $(\tilde{D}, \tilde{h})$ with two marked boundary points can be conformally mapped via (15) to the upper half-plane $\mathbb{H}$ with marked points 0 and $\infty$ and an instance of the GFF $h$, such that each bounded neighborhood of 0 has finite mass and each neighborhood of $\infty$ infinite mass with respect to the LQG measures (12) and (13). In order the quantum wedge to be "thick", it requires $\alpha \leq Q$, which is the Seiberg’s bound in the LQG (see [22]).

- **Quantum disk**: It is a quantum surface $(D, h)$ equipped with two boundary points $-\infty, \infty$ which are sampled uniformly and independently from the boundary length measure (13) given $(D, h)$. We are particularly interested in the $(a, b)$-length quantum disk as defined in [2]: the disk has two marked boundary points $x, y$ such that the quantum boundary length of the disk is conditioned to be $a + b$, and the counterclockwise arc from $x$ to $y$ has length $a$.

### 0.4 Coupling gravity with matter

In Section 0.3.1, we concentrated on the Boltzmann distribution on triangulations of the sphere or of a disk. In either case, where the chosen set of triangulations is denoted by $T$, recall that the distribution has the form
$$\mathbb{P}(T) = \frac{t^{\left|T\right|}}{Z(t_c)}$$ where $Z(t) = \sum_{T \in T} t^{\left|T\right|},$ (16)
$|T|$ is the size (e.g. the number of faces) of $T$ and $t_c$ is the radius of convergence of the generating function $Z(t)$, known to positive and finite and realizing the finiteness of $Z$. In this case, we write $t_c = e^{-\mu_0}$ where $\mu_0$ is the critical value of the Liouville coupling constant. For a more thorough account, see [22]. The measure (16) translates to

$$P_{\mu_0, \sqrt{3}}(T) := \frac{e^{-\mu_0|T|}}{Z_{\mu_0, \sqrt{3}}}$$

where $\frac{3}{8}$ is the LQG parameter $\gamma$ corresponding the pure gravity and $Z_{\mu_0, \sqrt{3}} = Z(t_c)$. More generally, the LQG measure (12) is defined for a general parameter $\gamma \in [\sqrt{2}, 2]$. The parameter $\gamma$ is related to the underlying conformal field theory (CFT) of central charge $c \in [-2, 1]$ according to the relation $c = 25 - 6Q^2 = 25 - 6(\gamma/2 + 2/\gamma)$. One notes that various lattice models of statistical mechanics fall within these ranges of parameters. Thus, if $T$ is a fixed triangulation, let $Z_{\gamma}(T)$ be the partition function of the lattice model on the vertices of $T$ which falls within the CFT determined by $\gamma$. Now the discretized quantum gravity coupled with the matter field corresponding to $\gamma$ is defined via the Boltzmann measure

$$P_{\mu_0, \gamma}(T) := \frac{e^{-\mu_0|T|}Z_{\gamma}(T)}{Z_{\mu_0, \gamma}}$$

where $Z_{\mu_0, \gamma}$ is the normalizing constant of the probability distribution. Some examples include the uniform spanning tree (UST, $\gamma = \sqrt{2}$), the discrete Gaussian Free Field (GFF, $\gamma = 2$) and the $O(n)$ loop models (any $\gamma \in [\sqrt{2}, 2]$). Then, there is the critical Ising model coupled to gravity, corresponding to $\gamma = \sqrt{3}$, which is studied in this thesis. It is conjectured that these discrete models equipped with a suitable conformal structure converge to the corresponding continuum LQG models, defined in Section 0.3.2, in the scaling limit; see [35] for more details. Much of the progress in the pure gravity case $\gamma = \sqrt{8/3}$ is obtained in the series of work by Miller and Sheffield [44, 43, 42]. The other values of $\gamma$ still remain rather mysterious. One of the aims in this thesis is to develop some discrete framework in order to approach the question of the scaling limit for $\gamma = \sqrt{3}$.

0.4.1 Random planar maps coupled to the Ising model

Now we consider the case $\gamma = \sqrt{3}$ and the critical Ising model coupled to random triangulations. We consider first triangulations which are equipped with spins on the vertices. Then, the measure (18) gets the form

$$P_{\mu_0, \sqrt{3}}(T) := \frac{e^{-\mu_0|T|}e^{\beta_c\sum_{\{v,w\} \in E(T)} \sigma_v \sigma_w}}{Z_{\mu_0, \sqrt{3}}}$$

This measure can naturally be extended to any inverse temperature $\beta > 0$, but only at $\beta = \beta_c$ the scaling limit is expected to be a CFT.

When the spins are on the vertices of the triangulation, the interfaces are simple discrete curves, and their scaling limits are expected to fall within the SLE(3)/CLE(3) paradigm on the $\sqrt{3}$-Liouville quantum gravity surfaces. For integrability reasons, it is also sometimes reasonable to put the spins on the faces of the triangulation instead of the vertices. In other words, this means that the spins lie on the vertices of the dual graph of the triangulation,
and the edges are the dual edges. This is done eg. in the work [15] and the expository article [29]. It should be noted that in this case, the interfaces are no more simple curves, but rather families of curves, which are still expected to converge towards the SLE/CLE in the scaling limit. The scaling limit results of this thesis will partly confirm this hypothesis.

0.4.2 Historical notes

As noted before, the study of the two-dimensional Liouville Quantum gravity was initiated by Polyakov in 1981 [47], and sparked the interest of wider community via the celebrated KPZ formula relating various Euclidean and quantum critical exponents of statistical physics lattice models by Knizhnik, Polyakov and Zamolodchikov [33]. In between these works, Boulatov and Kazakov studied the Ising model on so-called dynamical planar lattices, which are equivalent to random planar maps introduced later by the combinatorics community. More precisely, in [31] Kazakov studied the model with zero magnetic field coupled to random quadrangulations, finding an explicit solution of the partition function using matrix integrals, as well as showed the existence of a third-order phase transition in terms of the non-analyticity of the free energy. These results were generalized to triangulations together with Boulatov in [15], where the model was also studied with an external magnetic field. Various critical exponents were also computed. Since then, the study has been extended to the combinatorics of the $q$-Potts model as well as the Ising model on more general Riemann surfaces [20], [27], [25]. An enumerative combinatorics approach via the theory of combinatorial invariants applying to the Ising model has been more recently developed by Bernardi and Bousquet-Mélou [10]. Based on this work and on [1], Albenque, Ménard and Schaeffer recently showed the local convergence of the triangulations of the sphere coupled with the Ising model at arbitrary temperature [1]. It is still worth mentioning the works on the $O(n)$ loop model on random planar maps popularized by Borot, Bouttier and Guitter and studied by various authors since then ([14], [13], [12], [11], [18]). In particular, the aforementioned works reveal some interesting connections to the CLE on the LQG surfaces.

0.5 Contributions of this thesis

0.5.1 The models considered in this thesis

In this thesis, we consider random planar triangulations of a simple polygon coupled with the Ising model on either its faces or its vertices. The approach to this problem is combinatorial, and thus also the notation follows the convention of some prior combinatorial literature (e.g. [10]). More precisely, the underlying planar map consists of one exterior face with a simple boundary, and otherwise triangular faces with loops and multiple edges allowed. The map is either rooted on a corner of the exterior face (if the spins are on the faces) or a boundary edge (if the spins are on the vertices). In each of the case, there is an oriented root edge determined such that the exterior face lies on its right, as well as the root vertex $\rho$ which is the starting point of the oriented root edge. This vertex is also known as the origin.

The exterior face is equipped with Dobrushin boundary conditions, which means the following: If the spin configuration lies on the faces, the boundary edges are assigned a sequence of spins of the form $+\cdots + - \cdots -$ in the counter-clockwise order starting from the root vertex. More precisely, one could think that the external face is divided into two faces.
of fixed spins $+$ and $-$, respectively. If the spins lie on the vertices, then the sequence of spins is assigned to the boundary vertices of the exterior face such that the root edge is oriented from a $-$ vertex to a $+$ vertex.

A triangulation $t$ together with an Ising spin configuration on either its faces or its vertices is represented by a pair $(t, \sigma)$ where $\sigma \in \{+, -\}^{S(t)}$ where the set $S(t)$ is either the set of faces $F(t)$ or the set of vertices $V(t)$. In the case when the spins are on vertices, an edge $e$ of $t$ is said to be monochromatic if the spins on both of its endpoints are the same. If the spins are on faces, then the edge is monochromatic if its dual edge is monochromatic in the dual graph. When $e$ is a boundary edge, this definition takes into account the boundary condition as well. By an abuse of notation, we consider the information about the boundary condition to be contained in the coloring $\sigma$, and denote by $E(t, \sigma)$ the set of monochromatic edges in $(t, \sigma)$.

Let $p$ and $q$ be the numbers of $+$ and of $-$ spins in the boundary condition, respectively. Then we call $(t, \sigma)$ an Ising-triangulation of the $(p, q)$-gon or alternatively a bicolored triangulation of the $(p, q)$-gon. Depending on the context, we either talk about vertex-bicolored or face-bicolored triangulations. We denote by $BT_{p,q}$ the set of all bicolored triangulation of the $(p, q)$-gon. The elements of $BT_{p,q}$ are enumerated by the partition function

$$z_{p,q}(t, \nu) = \sum_{(t, \sigma) \in BT_{p,q}} \nu^{|E(t, \sigma)|} \nu^{R(t)}$$

where $\nu > 0$ is the coupling constant for the Ising model, $t$ is a parameter that controls the volume of the triangulation either by the number of faces (spins on faces) or by the number of edges (spins on vertices) and $R(t)$ is either $F(t)$ (spins on faces) or $E(t)$ (spins on vertices). In both of the cases in this thesis, the connection to the measure (19) is via the identity $\nu = e^{2\beta}$ where $\beta$ is the usual Ising inverse temperature. In particular, $0 < \nu < 1$ corresponds to the antiferromagnetic regime and $1 < \nu < \infty$ the ferromagnetic regime. Moreover, both of the models have a unique temperature where the spin ordering phase transition occurs: for spins on the faces, $\nu_c = 1 + 2\sqrt{7}$, and for spins on the vertices, $\nu_c = 1 + \frac{1}{\sqrt{7}}$. These are mutually related via the Kramers-Wannier duality; see Article (I) for more details.
In each of the cases, we denote the set of all Ising-triangulations of the \((p, q)\)-gon by \(BT_{p,q}\). When \(z_{p,q}(t, \nu) < \infty\), we define the Boltzmann distribution on \(BT_{p,q}\) by

\[
P^{t,\nu}_{p,q}(t, \sigma) = \frac{t^{[R(t)]_{L^m(t,\nu)}}}{z_{p,q}(t, \nu)}
\]

for all \((t, \sigma) \in BT_{p,q}\). A random variable of law \(P^{t,\nu}_{p,q}\) will be called a Boltzmann Ising-triangulation of the \((p, q)\)-gon. In this thesis, we are primarily interested in the asymptotic properties of these distributions when \(p, q \to \infty\). In particular, we derive results on the asymptotic behavior of the partition function \(z_{p,q}(t, \nu)\) at various temperatures \(\nu\) and show that the Boltzmann distributions (20) have local limits in distribution as \(p, q \to \infty\) for suitable parameter values \(\nu\) and \(t\) in various regimes of \(p, q\). Moreover, the local limits are amenable for many explicit computations related to the Ising interface.

0.5.2 Combinatorial results

The partition functions \(z_{p,q}\) can be encoded via the generating functions

\[
Z_q(u; t, \nu) = \sum_{p=0}^{\infty} z_{p,q}(t, \nu) u^p
\]

and

\[
Z(u, v; t, \nu) = \sum_{p,q \geq 0} z_{p,q}(t, \nu) u^p v^q = \sum_{q=0}^{\infty} Z_q(u; t, \nu) v^q,
\]

where by convention \(z_{0,0} = 1\) if the spins are on faces, and \(0\) otherwise. From Articles (I) ([19]) and (II) ([20]), the following explicit expressions are obtained when the spins are on the faces:

**Theorem 1** (Theorem 1 in (II)). For \(\nu \geq 1\), \(Z(u, v; t, \nu)\) satisfies the parametric equation

\[
t^2 = \hat{T}(S, \nu), \quad t \cdot u = \hat{U}(H; S, \nu), \quad t \cdot v = \hat{U}(K; S, \nu)
\]

and

\[
Z(u, v; t, \nu) = \hat{Z}(H, K; S, \nu),
\]

where \(\hat{T}, \hat{U}\) and \(\hat{Z}\) are rational functions with explicit expressions.

In addition, rational parametrizations for \(Z_0(u; t, \nu)\) and \(Z_1(u; t, \nu)\) are found as well. When the spins are on the vertices, Article (III) ([53]) provides similar parametrizations for the high-temperature regime \(\nu \in (1, \nu_c)\) and the critical temperature \(\nu = \nu_c\). This result is an indication that it might be justified to call the Ising model on random triangulations *exactly solvable*, a term often encountered when the partition function of a statistical mechanics model can be explicitly expressed using known functions.

The idea of the proof of the above theorem is to write functional equations for the generating functions \(Z(u, v; t, \nu)\) and \(Z_0(u; t, \nu)\) via the recursive peeling decomposition, and then manipulate them to genus zero algebraic equations. Then, some explicit rational parametrizations can be found using computer algebra, and these parametrizations can be further simplified using conformal mappings.

It can be shown that all the partition functions \(z_{p,q}(t, \nu)\) have the same radius of convergence, denoted by \(t_c(\nu)\), when \(\nu > 1\). This corresponds to the ferromagnetic Ising model. The pair \((\nu, t_c(\nu))\) is called the *critical line*. When restricted to this regime, the asymptotic analysis of the partition functions in Theorem 1 conducted in Articles (I) and (II) yield the following asymptotics of the partition function for the spins on faces:

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Theorem 2 (Theorem 2 in (II)). For any fixed $\nu > 1$ and $0 < \lambda_{\text{min}} < \lambda_{\text{max}} < \infty$, we have

\[
\begin{align*}
    u_c(\nu)^q \cdot z_{p,q}(\nu) &= \frac{a_p(\nu)}{\Gamma(-\alpha_0)} \cdot q^{-(\alpha_0+1)} + O\left(q^{-(\alpha_0+1+\delta)}\right) \quad \text{for each fixed } p \geq 0, \\
    u_c(\nu)^p \cdot a_p(\nu) &= \frac{b(\nu)}{\Gamma(-\alpha_1)} \cdot p^{-(\alpha_1+1)} + O\left(p^{-(\alpha_1+1+\delta)}\right), \\
    u_c(\nu)^{p+q} \cdot z_{p,q}(\nu) &= \frac{b(\nu) \cdot c(q/p)}{\Gamma(-\alpha_0)\Gamma(-\alpha_1)} \cdot p^{-(\alpha_2+2)} + O\left(p^{-(\alpha_2+2+\delta)}\right) \quad \text{while } q/p \in [\lambda_{\text{min}}, \lambda_{\text{max}}],
\end{align*}
\]

where the exponents $\alpha_i$, $\delta$ and the scaling function $c(\lambda)$ only depend on the phase of the model, and are given by

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<th>$\alpha_0$</th>
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<th>$\alpha_2$</th>
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</tr>
<tr>
<td>$\nu \in (1, \nu_c)$</td>
<td>$3/2$</td>
<td>$-1$</td>
<td>$1/2$</td>
<td>$1/2$</td>
</tr>
</tbody>
</table>

$c(\lambda) = \begin{cases} \lambda^{-5/2} & (\nu > \nu_c) \\
\frac{4}{3} \int_0^{\infty} (1 + r)^{-7/3}(\lambda + r)^{-7/3} dr & (\nu = \nu_c) \\
(1 + \lambda)^{-5/2} & (1 < \nu < \nu_c). \end{cases}$

Here, $u_c(\nu)$, $a_p(\nu)$ (for $p \geq 0$) and $b(\nu)$ are analytic functions of $\nu$ on $(1, \nu_c)$ and $(\nu_c, \infty)$, respectively, and $u_c(\nu)$ is continuous at $\nu = \nu_c$. The parameters $u_c(\nu)$, $b(\nu)$ and the generating function $A(u; \nu) := \sum_p a_p(\nu)u^p$ have explicit rational parametrizations.

The proofs leading to the asymptotics comprise generalizations of the singularity analysis methods in [28], such that the singularity analysis of the generating functions is conducted via rational parametrizations. The core theory of this is presented in Article (I). More precisely, the idea is to show the uniqueness of the dominant singularity of the generating functions and to find the parameter corresponding to the dominant singularity, as well as to analyse the singularity structure of the rational parametrization near a domain containing the origin inside and the parameter of the dominant singularity on the boundary. After the singularity structure of the rational parametrization is understood, it can be expanded around the parameter of the dominant singularity. This brings the asymptotic analysis back to the framework of [28] for algebraic singularities.

A counterpart of Theorem 2 is also shown for the model with spins on the vertices when $1 < \nu \leq \nu_c$, as done in Article (III). The theorem has two main points. On the one hand, it provides us with a system of critical exponents for the perimeter of a large Boltzmann-Ising triangulation of a disk, as given in the table. These exponent only depend on the phase of the Ising model determined by $\nu$, and they match with the predictions from statistical physics; see eg. [11] in the case of the $O(n)$ loop models. On the other hand, it provides explicit temperature-dependent constants which will be useful in doing explicit computations on the local limits of (20).

### 0.5.3 Local limits

In this thesis, the local distance of maps, as introduced in Section 0.3.1, is generalized to bicolored maps as follows. The local distance between bicolored triangulations (or actually any bicolored maps) is defined by

\[
d_{\text{loc}}((t, \sigma), (t', \sigma')) = 2^{-R} \quad \text{where} \quad R = \sup \{ r \geq 0 : [t, \sigma]_r = [t', \sigma']_r \}
\]
and \([t, \sigma]_r\) denotes the ball of radius \(r\) around the origin in \((t, \sigma)\) (w.r.t. the graph distance) which takes into account the spins of the faces or the vertices. The set \(\mathcal{B}^{T}\) of finite bicolored triangulations of a polygon is a metric space under \(d_{10c}\). Let \(\mathcal{B}^{T}\) be its Cauchy completion. We call an element of \(\mathcal{B}^{T}\) a bicolored triangulation of the half plane if it is one-ended and its external face has infinite degree. Indeed, such a triangulation has a proper embedding in the upper half plane without accumulation points and such that the boundary coincides with the real axis. The set of all bicolored triangulations of the half plane is denoted by \(\mathcal{B}^{T}\). Moreover, denote by \(\mathcal{B}^{T}_{(2)}\) the set of bicolored triangulations with an infinite boundary and exactly two ends (as defined in [7, 14.2]).

In Article (I), in the case of face-decorated Ising triangulations, the local convergence of (20) is shown at \(\nu = \nu_c\) if \(q \to \infty\) and \(p \to \infty\) one after the other. Moreover, it contains the construction of the local limit \(\mathbb{P}^{\nu}_{\infty} \equiv \mathbb{P}^{\nu}_{\nu_c}\). This is generalized to any \(\nu > 1\) in Article (II), as well as to a diagonal regime where \(p, q \to \infty\) simultaneously at a comparable speed. Article (III) contains the proof of similar local convergence results when the spins lie on the vertices and when \(\nu \in (1, \nu_c]\).

**Theorem 3** (Theorem 4 in (II) and Theorem 2 in (III)). Consider face-decorated Ising triangulations. Then for every \(\nu > 1\), there exist probability distributions \(\mathbb{P}^{\nu}_p\) and \(\mathbb{P}^{\nu}_{\infty}\), such that

\[
\mathbb{P}^{\nu}_{p,q} \xrightarrow{q \to \infty} \mathbb{P}^{\nu}_p \xrightarrow{p \to \infty} \mathbb{P}^{\nu}_{\infty}
\]

locally in distribution. Moreover, \(\mathbb{P}^{\nu}_p\) is supported on \(\mathcal{B}^{T}_{\infty}\) for all \(\nu > 1\), whereas \(\mathbb{P}^{\nu}_{\infty}\) is supported on \(\mathcal{B}^{T}_{\infty}\) when \(1 < \nu \leq \nu_c\) and on \(\mathcal{B}^{T}_{(2)}\) when \(\nu > \nu_c\). In addition, for \(0 < \lambda' \leq 1 \leq \lambda < \infty\), we have

\[
\mathbb{P}^{\nu}_{p,q} \xrightarrow{p,q \to \infty} \mathbb{P}^{\nu}_{\infty} \quad \text{when} \quad \frac{q}{p} \in [\lambda', \lambda]
\]

locally in distribution.

The latter result is also proven for vertex-decorated triangulations for all \(1 < \nu \leq \nu_c\), and the former at \(\nu = \nu_c\).

The constructions of the local limits and the proofs of the local convergences are all based on the analysis of suitably chosen peeling processes of the corresponding Boltzmann Ising-triangulations. More precisely, weak limits of the peeling processes of the finite Boltzmann Ising-triangulations are obtained, which are used both to construct the local limits and to encode the local convergence. Many of the arguments are based on probability estimates of the Markov chains encoding the perimeter variations of the triangulations. In particular, the local limit at the critical temperature \(\nu = \nu_c\) has a non-trivial geometry, and the proof of the local convergence in this case deserves much attention in Articles (I) and (II). The following section explains these ideas a bit more in detail.

**0.5.4 Phase transition**

Theorem 2 showed that the models exhibit different critical perimeter exponents depending on if \(\nu \in (1, \nu_c), \nu = \nu_c\) or \(\nu > \nu_c\). Also, Theorem 2 tells that there is a spontaneous change of geometry when \(\nu = \nu_c\), from the high-temperature regime \(1 < \nu < \nu_c\) to the low
temperatures $\nu > \nu_c$. Indeed, since the local limit at $\nu > \nu_c$ is two-ended, it necessarily contains a finite "bottleneck".

Both of the results indicate symmetry breakings at $\nu = \nu_c$, which are caused by the spin ordering phase transition. In [15], it was already found out that the free energy of the Ising model on dynamical triangulations has a third order non-analyticity. This is re-derived in Article (II). Thus, the model exhibits a third order phase transition. Figure 6 sketches the phase diagram of the model as the graph of the critical line $(\nu, t_c(\nu))$.

In Article (II), some order parameters are also found, which are associated with the peeling process of the Boltzmann Ising-triangulations. The core idea of this peeling process is to start exploring from the root edge, whose deletion preserves the Dobrushin boundary since it is bichromatic. Iterating the process from the following bichromatic edge on the boundary, or from the closest monochromatic edge if the boundary is monochromatic, yields an exploration which closely follows the interface. If the spins are put on the vertices, the peeling process explores the interface precisely, as discussed in Article (III).

More precisely, the exploration process reveals one triangle adjacent to the interface at each step, and swallows a finite number of other triangles if the revealed triangle separates the unexplored part in two pieces. Formally, the peeling process is defined as an increasing sequence of explored maps $(\mathfrak{e}_n)_{n \geq 0}$. A general definition of $\mathfrak{e}_n$ in the various cases is presented in Article (II).

The peeling process is also encoded by a sequence of peeling events $(S_n)_{n \geq 1}$ taking values
in a countable set of symbols, where $S_n$ indicates the position of the triangle revealed at time $n$ relative to the explored map $e_{n-1}$ and together with its spin. The law of the sequence $(S_n)_{n \geq 1}$ can be written down fairly easily and one can perform explicit computations with it. We denote by $P_{\nu}^{p,q}$ the law of the sequence $(S_n)_{n \geq 1}$ under $P_{\nu}^{p,q}$, where the $\nu$ is omitted when it is clear from the context. Moreover, we denote by $(P_n, Q_n)$ the boundary condition of the unexplored map at time $n$ and by $(X_n, Y_n)$ its variations, $X_n = P_n - P_0$ and $Y_n = Q_n - Q_0$. Now $(X_n, Y_n)$ is actually a deterministic function of the peeling events $(S_k)_{1 \leq k \leq n}$ with a well-defined limit when $p, q \to \infty$. This allows us to define the law of the process $(X_n, Y_n)_{n \geq 0}$ under $P_{\infty} = \lim_{p,q \to \infty} P_{\nu}^{p,q}$ despite the fact that $P_n = Q_n = \infty$ almost surely in the limit.

Then, under $P_{\infty}$, the process $(X_n, Y_n)_{n \geq 0}$ is a two-dimensional random walk. It was proven in Article (I) for the corresponding expectations of the increments that when $\nu = \nu_c$, $E_{\infty}(X_1) = E_{\infty}(Y_1) = \mu := \frac{1}{4\sqrt{7}} > 0$, telling that the interface drifts towards the infinity. Considering the temperature $\nu$ as a variable, this drift defines an order parameter:

**Proposition 4** (Proposition 5 in (II)). Let $O(\nu) := E_{\infty}^\nu((X_1 + Y_1) 1_{|X_1|\wedge|Y_1| < \infty})$. Then

$$O(\nu) = \begin{cases} 0, & \text{if } 1 < \nu < \nu_c \\ f(\nu), & \text{if } \nu \geq \nu_c, \end{cases}$$

where $f : [\nu_c, \infty) \to \mathbb{R}$ is a continuous, strictly increasing function such that $f(\nu_c) = 2\mu > 0$ and $\lim_{\nu \to \infty} f(\nu) < \infty$ exists. Moreover, for $1 < \nu < \nu_c$, we have the drift condition $E_{\infty}^\nu(X_1) = -E_{\infty}^\nu(Y_1) > 0$.

The above proposition entails that the perimeter variations associated to the peeling process at arbitrary temperature $\nu > 1$ define a discontinuous order parameter. The geometric interpretation of the proposition is that when $\nu < \nu_c$, the interface behaves like the interface of subcritical face percolation on the UIHPT, whereas for $\nu \geq \nu_c$, the interface drifts to the infinity the faster the lower the temperature is. For more interpretations, see Article (II), Section 6.3.

### 0.5.5 Scaling limits related to the interface

In this section, we consider the peeling process and the interface at $\nu = \nu_c$. Recall that in this case we expect it to converge, suitably rescaled and with a suitable conformal embedding to the Riemann sphere, to an SLE(3) on a LQG surface. Although the conjectured convergence of the Ising model on random planar maps towards the LQG($\sqrt{3}$) still remains mysterious, the results presented in this section provide a small glimpse towards that direction. In other words, it can be seen as a small bridge between the discrete and the continuum LQG with parameter $\sqrt{3}$.

For $m \geq 0$, let $T_m := \inf\{n \geq 0 : \min\{P_n, Q_n\} \leq m\}$, which can be seen as the first hitting time of the interface in a neighborhood of the infinity. Then in Articles (I) and (II), the following result is proven:
Theorem 5 (Proposition 11 in (I) and Theorem 6 in (II)). Let $\nu = \nu_c$. For all $m \in \mathbb{N}$, the jump time $T_m$ has the following scaling limit:

$$\forall t > 0, \quad \lim_{p,q \to \infty} \mathbb{P}_{p,q} (\mu T_m > tp) = \int_t^\infty (1 + s)^{-7/3}(\lambda + s)^{-7/3}ds$$

where the limit is taken such that $q/p \to \lambda \in (0, \infty)$. In particular, for $\lambda = 1$,

$$\lim_{p,q \to \infty} \mathbb{P}_{p,q} (T_m > tp) = (1 + \mu t)^{-11/3}.$$

Moreover,

$$\lim_{p \to \infty} \mathbb{P}_p (T_m > tp) = (1 + \mu t)^{-4/3}$$

where $\mathbb{P}_p$ is the weak limit of $\mathbb{P}_{p,q}$ as $q \to \infty$.

The proof of the above result is based on an observation that with high probability, the perimeter process $P_n$ is of order $p$ before the random time $T_m$ at which it jumps to a neighborhood of zero in a single big jump. This, in turn, follows from the fact that the perimeter fluctuations $(X_n)_{n \geq 0}$ and $(Y_n)_{n \geq 0}$ are heavy-tailed random walks under $\mathbb{P}_\infty$. This was first observed in (I) under $\mathbb{P}_p$ and generalized to $\mathbb{P}_{p,q}$ in (II).

It is tempting to conjecture that the random time $T_m$ could be replaced by the length of the main interface which is generated by the Dobrushin boundary conditions. Unfortunately, it is hard to control the interface length inside finite Boltzmann Ising-triangulations which fill in the finite regions swallowed by the peeling exploration, if the spins are on the faces. See Article (I), Section 6, for more explanation.

The above problem does not exist when the spins lie on the vertices of the triangulation. Let $\eta_{p,q}$ denote the length of the interface between the edges $\rho$ and $\rho^l$ in a vertex-decorated Ising-triangulation $(t, \sigma)$ sampled from $\mathbb{P}_{p,q}$, where $\rho$ represents the root edge and $\rho^l$ the other bichromatic edge on the boundary. Similarly, let $\eta_p$ be the length of the interface in $(t, \sigma)$ sampled from $\mathbb{P}_p$. One of the main result of Article (III) is the following:

Theorem 6 (Theorem 4 in (III)). Let $\nu = \nu_c$, and $\mu := \frac{11 - 5\sqrt{7}}{12\sqrt{7} - 48} > 0$. Then

$$\forall t > 0, \quad \lim_{p,q \to \infty} \mathbb{P}_{p,q} (\mu \eta_{p,q} > tp) = \int_t^\infty (1 + s)^{-7/3}(\lambda + s)^{-7/3}ds$$

where the limit is taken such that $q/p \to \lambda \in (0, \infty)$. In particular, for $\lambda = 1$,

$$\lim_{p,q \to \infty} \mathbb{P}_{p,q} (\eta_{p,q} > tp) = (1 + \mu t)^{-11/3}.$$

Moreover,

$$\forall t > 0, \quad \lim_{p \to \infty} \mathbb{P}_p (\eta_p > tp) = (1 + \mu t)^{-4/3}.$$

The first of the aforementioned limit laws can be seen as the law of the length $L$ of the gluing interface of two $\sqrt{3}$-LQG quantum disks with two marked boundary points, respectively, having quantum boundary lengths $1 + L$ and $L + \lambda$, respectively. This is explained in detail in (II), Section 8.1. The latter of the limit laws gives the length $L$ of the gluing interface of a quantum disk of boundary length $1 + L$ glued together with a thick quantum wedge. This is explained more detailed in (I), Section 6. One should note that, as explained in (I), the miraculous exponent $4/3$ in the above limit law should not be a coincidence, but rather is related to the Liouville parameter $\gamma = \sqrt{3}$, or alternatively the SLE parameter $\kappa = 3$. These interpretations explain why the exponents in $4/3$ and $11/3$ in the asymptotics of $z_{p,q}(t_c, \nu_c)$ coincide with the exponents in the limit laws. Their combinatorial interpretations, however, still remain unknown.


