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## The local Tb theorem with rough test functions

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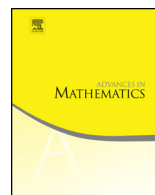
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## ABSTRACT

We prove a local  $Tb$  theorem under close to minimal (up to certain ‘buffering’) integrability assumptions, conjectured by S. Hofmann (El Escorial, 2008): Every cube is assumed to support two non-degenerate functions  $b_Q^1 \in L^p$  and  $b_Q^2 \in L^q$  such that  $1_{2Q}Tb_Q^1 \in L^{q'}$  and  $1_{2Q}T^*b_Q^2 \in L^{p'}$ , with appropriate uniformity and scaling of the norms. This is sufficient for the  $L^2$ -boundedness of the Calderón–Zygmund operator  $T$ , for any  $p, q \in (1, \infty)$ , a result previously unknown for simultaneously small values of  $p$  and  $q$ . We obtain this as a corollary of a local  $Tb$  theorem for the maximal truncations  $T_{\#}$  and  $(T^*)_{\#}$ : for the  $L^2$ -boundedness of  $T$ , it suffices that  $1_Q T_{\#} b_Q^1$  and  $1_Q (T^*)_{\#} b_Q^2$  be uniformly in  $L^0$ . The proof builds on the technique of suppressed operators from the quantitative Vitushkin conjecture due to Nazarov–Treil–Volberg.

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**1. Introduction**

The first  $Tb$  theorems were proved by David, Journé and Semmes [6], and McIntosh and Meyer [15]. Their idea was to characterize the  $L^2$ -boundedness of a singular integral operator  $T$ ,

$$Tf(x) = \int_{\mathbb{R}^d} K(x, y)f(y) dy,$$

where  $K$  is a standard Calderón–Zygmund kernel, by its (and its adjoint’s) action on just one sufficiently non-degenerate function  $b$ . Thus, they generalized the celebrated  $T1$  theorem of David and Journé [5], where this function was required to be  $b \equiv 1$ .

Another significant step in this type of characterizations was taken by Christ [4], who introduced the idea of a *local  $Tb$  theorem*. Rather than testing  $T$  and  $T^*$  on two globally well-behaved (and hence not so easy to find) functions  $b_1$  and  $b_2$ , the operators can be tested against a family of local functions  $b_Q^1$  and  $b_Q^2$ , indexed by the cubes (say)  $Q$  on which they are supported, each of which is only required to satisfy a set of conditions on its ‘own’ cube.

Besides necessary non-degeneracy requirements, Christ’s assumptions on his test functions consisted of the uniform boundedness  $b_Q^1, b_Q^2, Tb_Q^1, T^*b_Q^2 \in L^\infty$ . Weakening these conditions has been a topic of subsequent developments. Nazarov, Treil and Volberg [16] showed (even in the more general context of possibly non-doubling underlying measures in place of the Lebesgue measure) that it suffices to have  $b_Q^1, b_Q^2 \in L^\infty$  and  $Tb_Q^1, T^*b_Q^2 \in \text{BMO}$ , uniformly in  $Q$ . On the other hand, for certain dyadic model operators, Auscher, Hofmann, Muscalu, Tao and Thiele [1] were able to relax these conditions to a substantially lower degree of integrability, namely

$$b_Q^1 \in L^p, \quad b_Q^2 \in L^q, \quad 1_Q Tb_Q^1 \in L^{q'}, \quad 1_Q T^*b_Q^2 \in L^{p'} \tag{1.1}$$

for any  $p, q \in (1, \infty)$ , where the different  $L^r$  norms are appropriately scaled relative to  $|Q|$ . The question then became whether these testing conditions for the model case also suffice for genuine singular integral operators. This was the first of the four open problems on local  $Tb$  theorems formulated by Hofmann during his plenary lectures at the International Conference on Harmonic Analysis and P.D.E. in El Escorial, 2008; it was motivated by possible applications to layer potentials and to free boundary theory (see [8, Section 3.3.1]).

Towards the solution of Hofmann’s problem, the following developments have taken place. First, Hofmann’s [7] positive result concerning the case  $b_Q^i \in L^2$ ,  $T_i b_Q^i \in L^{2+\varepsilon}$ . Next, Auscher and Yang’s [3] elimination of the  $\varepsilon > 0$  by a reduction to the dyadic case. In fact, they settled the result for all ‘large enough’ pairs of exponents  $p, q \in (1, \infty)$ , namely, subject to the sub-duality condition  $1/p + 1/q \leq 1$ . Finally, Auscher and Routin’s [2] work on general pairs  $p, q \in (1, \infty)$ : they gave a direct proof of the sub-duality theorem just mentioned, and obtained a positive result for general exponents under additional side conditions of ‘weak boundedness’ type (but rather more technical than the usual forms of such assumptions). See also [9,12,13] for some related work in the direction of singular integrals with respect to more general underlying measures.

In the paper at hand, we solve a buffered version of Hofmann’s problem for all exponents  $p, q \in (1, \infty)$ ; namely, we strengthen the last two inclusions in (1.1) to

$$b_Q^1 \in L^p, \quad b_Q^2 \in L^q, \quad 1_{2Q} T b_Q^1 \in L^{q'}, \quad 1_{2Q} T^* b_Q^2 \in L^{p'} \tag{1.2}$$

— a condition also present in [2], where further assumptions were imposed. In the sub-duality case  $1/p + 1/q \leq 1$ , the buffered assumption (1.2) already follows from the seemingly weaker (1.1) (cf. Remark 2.11). In a recent work of Martikainen, Mourgoglou and Tolsa [14], the buffering is shown to be redundant in a slightly bigger range of exponents, but their main result for local  $Tb$  theorems [14, Theorem 4.6], while also allowing more general underlying measures, is restricted to antisymmetric kernels,  $K(x, y) = -K(y, x)$ . Our present contribution, while not strictly comparable to [9,12–14], is the most general available form of local  $Tb$  theorems for general (not necessarily antisymmetric) singular integrals with respect to the underlying Lebesgue measure.

We are going to view (1.2) as sufficient conditions for another natural set of assumptions stated in terms of the maximal truncated singular integral

$$T_{\#} f(x) := \sup_{\varepsilon > 0} |T_{\varepsilon} f(x)|, \quad T_{\varepsilon} f(x) := \int_{|x-y|>\varepsilon} K(x, y) f(y) \, dy.$$

We make the assumption that for some  $u \in (1, \infty)$  and  $v \in [0, \infty)$  (sic!), we have

$$b_Q^1, b_Q^2 \in L^u, \quad 1_Q T_{\#} b_Q^1, 1_Q (T^*)_{\#} b_Q^2 \in L^v, \tag{1.3}$$

with appropriate uniform scaling. As we will prove, (1.3) for  $u = v < \min\{p, q, p', q'\}$  is a consequence of (1.1). But (1.3) seems more natural in the sense that (unlike (1.1) in the super-duality case  $1/p + 1/q > 1$ ) it is obviously necessary for the  $L^2$ -boundedness of  $T$ , which implies the  $L^u$ -boundedness of  $T_{\#}$  by classical theory. And (1.3), even with just a uniform  $L^0$ -condition for the images, is also a sufficient condition for the  $L^2$ -boundedness, as a proper  $Tb$  condition should.

Our method of proving this result is new in the context of Hofmann’s problem, although borrowed from other developments in the  $Tb$  circle of ideas, in particular, the

approach to Vitushkin’s conjecture by Nazarov, Treil and Volberg [17]. Namely, we show that it is possible to perturb the rough test functions  $b_Q \in L^u$  so as to obtain better functions  $\tilde{b}_Q \in L^\infty$ , which are still well-behaved under a *suppressed* singular integral

$$T_\Phi f(x) := \int_{\mathbb{R}^d} K_\Phi(x, y) f(y) \, dy, \quad K_\Phi(x, y) := \frac{|x - y|^{2m} K(x, y)}{|x - y|^{2m} + \Phi(x)^m \Phi(y)^m},$$

for a suitably chosen nonnegative Lipschitz function  $\Phi$ . We can then run a local  $Tb$  argument for the suppressed operator  $T_\Phi$  and the bounded test functions  $\tilde{b}_Q$ . Once the boundedness of  $T_\Phi$  has been established, this can be used to construct yet another set of bounded test functions, but now for the original operator  $T$ . Another local  $Tb$  argument with bounded test functions then allows to deduce the boundedness of  $T$  itself.

In the following section, we give a detailed statement of the main theorems and a technical outline of the entire argument, where the main auxiliary propositions are stated without proof. The proofs of these intermediate results are then provided in the subsequent sections.

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**2. Technical outline**

Let  $T$  be a linear operator given by

$$Tf(x) = \int_{\mathbb{R}^d} K(x, y) f(y) \, dy, \tag{2.1}$$

where  $K$  is a Calderón–Zygmund standard kernel:

$$|x - y|^d |K(x, y)| + |x - y|^{d+\alpha} \frac{|K(x, y) - K(x', y)|}{|x - x'|^\alpha} + |x - y|^{d+\alpha} \frac{|K(x, y) - K(x, y')|}{|y - y'|^\alpha} \lesssim 1 \tag{2.2}$$

for all  $x, x', y, y'$  with  $|x - x'| + |y - y'| < \frac{1}{2}|x - y|$  and some fixed  $\alpha \in (0, 1]$ . For convenience, we assume that  $K$  is also bounded, *qualitatively*, so that formulae like (2.1) are meaningful, but this will never be used in the quantitative estimates.

**2.3 Definition** (*Accretive system*). Let  $p \in [1, \infty]$ ,  $u \in [0, \infty]$ . A  $(p, u)$ -accretive system for an operator  $T$  is a family of functions  $b_Q$ , indexed by all dyadic cubes  $Q$ , such that

$$\text{supp } b_Q \subseteq Q, \tag{2.4}$$

$$\int_Q b_Q \, dx = 1, \tag{2.5}$$

$$\left( \int_Q |b_Q|^p \, dx \right)^{1/p} \lesssim 1, \tag{2.6}$$

$$\left( \int_Q |Tb_Q|^u \, dx \right)^{1/u} \lesssim 1, \tag{2.7}$$

with the usual reformulation if  $p$  or  $u$  is  $\infty$ . If  $u = 0$ , we interpret (2.7) as follows: For every  $\varepsilon > 0$ , there exists  $\lambda$  such that for all  $Q$  we have

$$\frac{1}{|Q|} |Q \cap \{|Tb_Q| > \lambda\}| < \varepsilon. \tag{2.8}$$

The family is called a *buffered*  $(p, u)$ -accretive system for  $T$  if (2.7) is replaced by the stronger condition that

$$\left( \int_{2Q} |Tb_Q|^u \, dx \right)^{1/u} \lesssim 1, \tag{2.9}$$

with similar modifications for  $u \in \{0, \infty\}$ .

Addressing a problem posed by Hofmann [8, Section 3.3.1], we prove the following:

**2.10 Theorem.** *Let  $p, q \in (1, \infty)$ . Suppose that there is a buffered  $(p, q')$ -accretive system for  $T$ , and a buffered  $(q, p')$ -accretive system for  $T^*$ . Then  $\|T\|_{L^2 \rightarrow L^2}$  is bounded by a constant depending only on the implied constants in (2.2), (2.6) and (2.9).*

The first theorem of this flavour was proven for so-called perfect dyadic operators [1]. For Calderón–Zygmund operators, prior to our work, it was known in the subduality case:  $1/p + 1/q \leq 1$  [2,3]. For  $1/p + 1/q > 1$ , it had only been verified under additional technical assumptions [2].

**2.11 Remark.** In the subduality case, a  $(p, q')$ -accretive system is automatically buffered, i.e., (2.9) already follows from the other conditions by the following Hardy inequality:

$$\int_{3Q \setminus Q} \left( \int_Q \frac{|f(y)|}{|x-y|^d} dy \right)^u dx \lesssim \|1_Q f\|_{L^u}^u, \quad u \in (1, \infty),$$

whose dualized form is recorded in [2, p. 307]; see also [2, Lemma 2.4] for a more general statement (with proof) in a doubling metric space in place of  $\mathbb{R}^d$ .

Our argument still requires the buffering for general exponents.

We deduce Theorem 2.10 as a corollary to a variant, natural in our opinion, where (2.9) is replaced by an assumption on the maximal truncated operator

$$T_{\#} f(x) := \sup_{\varepsilon > 0} |T_{\varepsilon} f(x)|, \quad T_{\varepsilon} f(x) := \int_{|x-y| > \varepsilon} K(x, y) f(y) dy.$$

Let us first observe that the above conditions on  $T$  imply similar conditions on  $T_{\#}$ :

**2.12 Proposition.** *Suppose that  $b_Q^1$  is a buffered  $(p, q')$  accretive system for  $T$ , and  $b_Q^2$  is a buffered  $(q, p')$  accretive system for  $T^*$ . Then  $b_Q^1$  is also a (buffered)  $(p, u)$  accretive system for  $T_{\#}$ , and  $b_Q^2$  is a (buffered)  $(q, v)$  accretive system for  $(T^*)_{\#}$ , for any  $u < \min\{p, q'\}$  and any  $v < \min\{q, p'\}$ .*

Since  $u < p$ , the  $(p, u)$  accretive system  $b_Q^1$  is automatically buffered by Hardy’s inequality; similarly for  $b_Q^2$ . Thanks to Proposition 2.12, Theorem 2.10 is a consequence of our main result which reads as follows:

**2.13 Theorem (Main theorem).** *Let  $T$  be a Calderón–Zygmund operator, and suppose that for some  $p \in (1, \infty)$  and  $u \in [0, \infty)$  there exist  $(p, u)$ -accretive systems  $b_Q^1$  for  $T_{\#}$  and  $b_Q^2$  for  $(T^*)_{\#}$ . Then  $\|T\|_{L^2 \rightarrow L^2}$  is bounded by a constant depending only on the implied constants in (2.2), (2.6) and (2.7) with  $T_{\#}$  and  $b_Q^1$ , or  $(T^*)_{\#}$  and  $b_Q^2$ , in place of  $T$  and  $b_Q$ .*

**2.14 Remark (Necessity of the conditions).** As in the usual  $Tb$  theorems, the assumptions of Theorem 2.13 are also necessary. Namely, if  $T$  is  $L^2$ -bounded, it follows from standard theory that the maximal truncation  $T_{\#}$  is  $L^p$ -bounded for all  $p \in (1, \infty)$ . Thus, for an  $L^2$ -bounded  $T$ , any function  $b_Q$  with properties (2.4) and (2.6) will also satisfy (2.7) for  $T_{\#}$  in place of  $T$ .

**2.15 Lemma.** *It suffices to prove Theorem 2.13 for  $u = p > 1$ , as long as the implied constants in (2.7) for  $T_{\#}$  and  $(T^*)_{\#}$  contribute linearly to the bound for  $\|T\|_{2 \rightarrow 2}$ .*

**Proof.** Indeed, suppose that the theorem is already proven in this case, and consider the weakest assumptions with  $(p, 0)$ -accretive systems for  $T_{\#}$  and  $(T^*)_{\#}$ . Fix some  $u \in (1, p)$ . Then we have

$$\begin{aligned} \int_Q (T_{\#}b_Q^1)^u dx &= \frac{1}{|Q|} \left( \int_{Q \cap \{T_{\#}b_Q^1 \leq \lambda\}} + \int_{Q \cap \{T_{\#}b_Q^1 > \lambda\}} \right) (T_{\#}b_Q^1)^u dx \\ &\leq \lambda^u + \left( \int_Q (T_{\#}b_Q^1)^p dx \right)^{u/p} \left( \frac{1}{|Q|} |Q \cap \{T_{\#}b_Q^1 > \lambda\}| \right)^{1-u/p} \\ &\leq \lambda^u + \|T_{\#}\|_{p \rightarrow p}^u \left( \int_Q |b_Q^1|^p dx \right)^{u/p} \varepsilon^{1-u/p}, \end{aligned}$$

if  $\lambda = \lambda(\varepsilon)$  is as in (2.8), and hence

$$\begin{aligned} \left( \int_Q (T_{\#}b_Q^1)^u dx \right)^{1/u} &\leq \lambda + C \|T_{\#}\|_{p \rightarrow p} \cdot \varepsilon^{1/u-1/p} \\ &\leq \lambda + C(1 + \|T\|_{2 \rightarrow 2}) \cdot \varepsilon^{1/u-1/p} \leq C_{\delta} + \|T\|_{2 \rightarrow 2} \cdot \delta, \end{aligned}$$

where  $\delta = C\varepsilon^{1/u-1/p}$  can be chosen as small as we like, picking  $\lambda = \lambda(\varepsilon)$ , and therefore  $C_{\delta} = \lambda + C\varepsilon^{1/u-1/p}$  large enough. So in fact  $b_Q^1$  is also a  $(p, u)$ -accretive system (and, a fortiori, a  $(u, u)$ -accretive system) for  $T_{\#}$ , with the implied constant in the last estimate specified above. A similar argument of course works for  $(T^*)_{\#}$  and  $b_Q^2$ . As these bounds contribute linearly to the bound for  $\|T\|_{2 \rightarrow 2}$ , we deduce from the assumed case of Theorem 2.13 that

$$\|T\|_{2 \rightarrow 2} \leq C_{\delta} + c\|T\|_{2 \rightarrow 2} \cdot \delta,$$

and it suffices to fix a small enough  $\delta > 0$  in order to absorb the last term and complete the proof of Theorem 2.13 in general.  $\square$

Let us then discuss the proof of Theorem 2.13 for  $1 < u = p < \infty$ ; we will not pay explicit attention to the linear dependence required by Lemma 2.15, as it will be obvious by inspection. The proof consists of a reduction to the easier case of Hofmann’s conjecture with the help of so-called suppressed operators. For any nonnegative function  $\Phi$  with Lipschitz constant 1, we define

$$T_{\Phi}f(x) := \int K_{\Phi}(x, y)f(y) dy, \quad K_{\Phi}(x, y) := \frac{|x - y|^{2m}K(x, y)}{|x - y|^{2m} + \Phi(x)^m\Phi(y)^m},$$

where  $m \geq d/2$  is fixed. Note that  $T_{\Phi}f = Tf$  if  $\text{supp } f \subseteq \{\Phi = 0\}$ .

Given the assumptions on  $T_{\#}$ , our goal is to construct a better behaved accretive system for the suppressed operator  $T_{\Phi}$ . To achieve this, we need to relax the notion of



an accretive system a little, so as not to demand the supply of test functions for every cube, but only an appropriate subcollection of them.

**2.16 Definition** (*Accretive system on a sparse family*). Let  $Q_0$  be a cube. A *sparse family* of dyadic subcubes of  $Q_0$  is a collection  $\mathcal{Q}$ , containing  $Q_0$ , such that for some  $\tau > 0$  and for all  $Q \in \mathcal{Q}$  we have

$$\left| \bigcup_{\substack{\tilde{Q} \in \mathcal{Q} \\ \tilde{Q} \subsetneq Q}} \tilde{Q} \right| \leq (1 - \tau)|Q|.$$

An  $(\infty, u)$ -accretive system for  $T$  on sparse subcubes of  $Q_0$  is a family of functions  $b_Q$ , indexed by a sparse family  $\mathcal{Q}$  of dyadic subcubes of  $Q_0$ , with the properties (2.4) and the following strengthening of (2.5), (2.6) and (2.7):

$$\begin{aligned} \left| \int_{Q'} b_Q \, dx \right| &\gtrsim 1, \\ \|b_Q\|_\infty &\lesssim 1, \\ \left( \int_{Q'} |Tb_Q|^u \, dx \right)^{1/u} &\lesssim 1, \end{aligned}$$

whenever  $Q' \subseteq Q \in \mathcal{Q}$  is a dyadic subcube, which is not contained in any smaller  $\tilde{Q} \subsetneq Q$  with  $\tilde{Q} \in \mathcal{Q}$ .

Such a system is said to be *strong* if, in addition, we have

$$\left( \int_{Q'} |T(1_{Q'} b_Q)|^u \, dx \right)^{1/u} \lesssim 1 \tag{2.17}$$

under the same conditions on the cubes  $Q' \subseteq Q$ .

It is important for us that the additional condition (2.17) is automatically satisfied in the situations of our interest:

**2.18 Lemma.** *Let  $u \in (1, \infty)$  and suppose that there exists a  $(u', 1)$  accretive system for  $T^*$ . Then any  $(\infty, u)$  accretive system for  $T$  on sparse subcubes of any  $Q_0$  is automatically strong.*

The following proposition provides the key existence result for  $(\infty, u)$  accretive systems:

**2.19 Proposition.** *Suppose that there is a  $(p, p)$  accretive system for  $T_\#$  with  $p > 1$ . Fix a small  $\varrho \in (0, 1)$  and any  $u \in (1, \infty)$ . Then for any cube  $Q_0$ , there exists a nonnegative function  $\Phi$  with Lipschitz constant 1 such that*

$$|\{\Phi > 0\}| \leq \varrho|Q_0|,$$

and there exists an  $(\infty, u)$  accretive system for  $T_\Phi$  on sparse subcubes of  $Q_0$ .

Starting from two  $(p, p)$  accretive systems,  $b_Q^1$  for  $T_\#$  and  $b_Q^2$  for  $(T^*)_\#$ , this gives us two Lipschitz functions  $\Phi_1$  and  $\Phi_2$ , and two  $(\infty, u)$  accretive systems on sparse subcubes —  $\tilde{b}_Q^1$  for  $T_{\Phi_1}$  and  $\tilde{b}_Q^2$  for  $T_{\Phi_2}^*$  —, but we may arrange the construction so that  $\Phi_1 = \Phi_2 =: \Phi$ .

In this case, the two new accretive systems are strong.

Given an accretive system for  $T$  on all dyadic subcubes of  $Q_0$ , it follows from a standard stopping time argument that we can extract a subsystem, which is an accretive system for  $T$  on a sparse family of subcubes of  $Q_0$ . This is typically one of the first steps in the proof of a local  $Tb$  theorem; see [16], for instance. For us, it will be important that it is actually enough to only have an accretive system for the sparse subcubes from the beginning:

**2.20 Proposition (Baby  $Tb$  theorem).** *Let  $T$  be an operator with Calderón–Zygmund kernel, let  $Q_0$  be a cube, let  $t \in (1, \infty)$ , and suppose that there are strong  $(\infty, t)$  accretive systems  $b_Q^1$  for  $T$  and  $b_Q^2$  for  $T^*$ , on sparse subcubes  $\mathcal{Q}_1$  and  $\mathcal{Q}_2$  of  $Q_0$ , respectively. Then*

$$|\langle Tf, g \rangle| \lesssim \|f\|_{s'} \|g\|_{s'} |Q_0|^{1-2/s'} \quad s' \in (\max\{t', 2\}, \infty],$$

for all  $f, g \in L^{s'}(Q_0)$  ( $L^{s'}$  functions supported on  $Q_0$ ).

Note that  $(\infty, t)$  accretive systems are in particular  $(r, r')$  accretive systems for  $r = \max\{4, t'\} < \infty$ ,  $r' = \min\{\frac{4}{3}, t\} \leq t$ . Hence, in principle, we are in the setting of Hofmann’s conjecture in the known case that  $1/p + 1/q = 2/r \leq 1/2 < 1$ . But contrary to the usual, we only assume the presence of (strong) accretive systems on sparse subcubes of  $Q_0$ .

Assuming all the auxiliary results formulated above, we can now give:

**Proof of Theorem 2.13.** By the reduction given by Lemma 2.15, we may assume that for some  $p \in (1, \infty)$ , there are two  $(p, p)$  accretive systems of functions,  $b_Q^1$  for  $T_\#$  and  $b_Q^2$  for  $(T^*)_\#$ . Without loss of generality, we may assume that  $p \in (1, 2)$ .

Fix a cube  $Q_0$ . Then, by Proposition 2.19, there exists a nonnegative function  $\Phi$  with Lipschitz constant 1 such that

$$|\{\Phi > 0\}| \leq \varrho|Q_0|, \tag{2.21}$$

for some fixed  $\varrho \in (0, 1)$  (independent of  $Q_0$ ), and there exist strong  $(\infty, p)$  accretive systems  $\tilde{b}_Q^1$  for  $T_\Phi$  and  $\tilde{b}_Q^2$  for  $T_\Phi^*$  on sparse subcubes of  $Q_0$ .

Thus Proposition 2.20, applied to  $T_\Phi$  in place of  $T$ , implies that

$$|\langle T_\Phi f, g \rangle| \lesssim \|f\|_{s'} \|g\|_{s'} |Q_0|^{1-2/s'} \quad s' \in (p', \infty], \tag{2.22}$$

for all  $f, g \in L^{s'}(Q_0)$ .

Let

$$b_{Q_0} := \frac{|Q_0|}{|Q_0 \cap \{\Phi = 0\}|} 1_{Q_0 \cap \{\Phi = 0\}}.$$

By (2.21), we have  $|Q_0|/|Q_0 \cap \{\Phi = 0\}| \lesssim 1$ , and hence

$$\int_{Q_0} b_{Q_0} \, dx = 1, \quad \|b_{Q_0}\|_\infty \lesssim 1.$$

Since  $\text{supp } b_{Q_0} \subseteq \{\Phi = 0\}$ , we have  $T_\Phi b_{Q_0} = T b_{Q_0}$  and likewise  $T_\Phi^* b_{Q_0} = T^* b_{Q_0}$ . By an application of (2.22) to  $f = b_{Q_0}$  and an arbitrary  $g \in L^{s'}(Q_0)$  of norm 1, we deduce that

$$\left( \int_{Q_0} |T b_{Q_0}|^s \, dx \right)^{1/s} \lesssim |Q_0|^{1/s'} \cdot 1 \cdot |Q_0|^{1-2/s'} = |Q_0|^{1/s}.$$

Similarly, applying (2.22) to  $g = b_{Q_0}$  and an arbitrary  $f \in L^{s'}(Q_0)$  of norm 1, we obtain

$$\left( \int_{Q_0} |T^* b_{Q_0}|^s \, dx \right)^{1/s} \lesssim |Q_0|^{1/s}.$$

The above reasoning applies to any cube  $Q$  in place of  $Q_0$ . Hence, for every  $Q$ , there exists a function  $b_Q$  with

$$\text{supp } b_Q \subseteq Q, \quad \int_Q b_Q \, dx = 1, \quad \|b_Q\|_\infty \lesssim 1, \quad \int_Q |T b_Q|^s \, dx + \int_Q |T^* b_Q|^s \lesssim 1.$$

In other words, there exists an  $(\infty, s)$  accretive system for the original operator  $T$  and its adjoint  $T^*$  on all dyadic cubes. By a standard stopping time construction, for any  $Q_0$ , we can extract  $(\infty, s)$  accretive systems for  $T$  and  $T^*$  on sparse subcubes of  $Q_0$ : Indeed, let  $\mathcal{Q}_1$  consist of the maximal dyadic subcubes  $Q \subseteq Q_0$  that satisfy at least one of the conditions

$$\left| \int_Q b_{Q_0} \, dx \right| \leq \nu < 1, \quad \int_Q |T b_{Q_0}|^s \, dx + \int_Q |T^* b_{Q_0}|^s > N \gg 1,$$

and  $\mathcal{Q}_{1,0} := \{Q \in \mathcal{Q}_1 : |\int_Q b_{Q_0} \, dx| \leq \nu\}$ . Then it is immediate that the opposite estimates hold for all dyadic  $Q \subseteq Q_0$  that are not contained in any  $\tilde{Q} \in \mathcal{Q}_1$ , and

$$\begin{aligned} \sum_{Q \in \mathcal{Q}_1 \setminus \mathcal{Q}_{1,0}} |Q| &\leq \frac{1}{N} \sum_{Q \in \mathcal{Q}_1 \setminus \mathcal{Q}_{1,0}} \left( \int_Q |Tb_{Q_0}|^s dx + \int_Q |T^*b_{Q_0}|^s dx \right) \\ &\leq \frac{1}{N} \left( \int_{Q_0} |Tb_{Q_0}|^s dx + \int_{Q_0} |T^*b_{Q_0}|^s dx \right) \leq \frac{C}{N} |Q_0|, \end{aligned}$$

and

$$\begin{aligned} |Q_0| = \left| \int_{Q_0} b_{Q_0} dx \right| &\leq \sum_{Q \in \mathcal{Q}_{0,1}} \left| \int_Q b_{Q_0} dx \right| + \left| \int_{Q_0 \setminus \bigcup \mathcal{Q}_{0,1}} b_{Q_0} dx \right| \\ &\leq \sum_{Q \in \mathcal{Q}_{0,1}} \nu |Q| + C \left( |Q_0| - \sum_{Q \in \mathcal{Q}_{0,1}} |Q| \right) \end{aligned}$$

so that

$$\frac{1}{|Q_0|} \sum_{Q \in \mathcal{Q}_0} |Q| \leq \frac{1}{|Q_0|} \left( \sum_{Q \in \mathcal{Q}_{0,1}} |Q| + \sum_{Q \in \mathcal{Q}_0 \setminus \mathcal{Q}_{0,1}} |Q| \right) \leq \frac{C-1}{C-\nu} + \frac{C}{N} =: 1 - \tau < 1$$

as soon as  $\nu < 1$  and  $N \gg 1$  is large enough. We then iterate the same construction, starting from each  $Q \in \mathcal{Q}_1$  in place of  $Q_0$  and  $b_Q$  in place of  $b_{Q_0}$ , to obtain a sparse family of cubes  $\mathcal{Q} = \bigcup_{k=0}^\infty \mathcal{Q}_k$ , where  $\mathcal{Q}_0 = \{Q_0\}$  and  $\mathcal{Q}_{k+1} = \bigcup_{Q \in \mathcal{Q}_k} \mathcal{Q}_1(Q)$ . By construction, the family  $\{b_Q\}_{Q \in \mathcal{Q}}$  is an  $(\infty, s)$ -accretive system for both  $T$  and  $T^*$  on the sparse family  $\mathcal{Q}$ .

Given that there also exist  $(\infty, s)$  accretive systems, and thus a fortiori  $(s', 1)$  accretive systems, for the adjoints of both operators, we conclude from Lemma 2.18 that both extracted systems are strong.

Another application of Proposition 2.20, to the operator  $T$  itself, shows that

$$|\langle Tf, g \rangle| \lesssim \|f\|_{r'} \|g\|_{r'} |Q_0|^{1-2/r'} \quad r' \in (s', \infty],$$

for all  $f, g \in L^{r'}(Q_0)$ , for any cube  $Q_0$ . We apply this to  $f = 1_{Q_0}$  and an arbitrary  $g \in L^{r'}(Q_0)$  of norm 1, and to  $g = 1_{Q_0}$  and an arbitrary  $f \in L^{r'}(Q_0)$  of norm 1, to deduce that

$$\left( \int_{Q_0} |T1_{Q_0}|^r dx \right)^{1/r} \lesssim |Q_0|^{1/r}, \quad \left( \int_{Q_0} |T^*1_{Q_0}|^r dx \right)^{1/r} \lesssim |Q_0|^{1/r}.$$

But this brings us to the setting of the well-known standard local  $T1$  theorem (see e.g. [8, Theorem 1.5 and Remark 1.7]), which gives us the desired bound  $\|T\|_{L^2 \rightarrow L^2} \lesssim 1$ . This completes the proof.  $\square$

**3. Preparatory estimates; proofs of Proposition 2.12 and Lemma 2.18**

We show how to obtain the testing conditions for the maximal truncated operator  $T_{\#}$  from testing conditions for  $T$ , and provide some auxiliary results on the suppressed operators  $T_{\Phi}$  for the subsequent sections.

**3.1 Lemma.** *Let  $p \in [1, \infty]$  and  $u \in (1, \infty]$ . If  $b_Q^1$  is a buffered  $(p, u)$  accretive system for  $T$ , then (2.9) improves to the global estimate*

$$\|Tb_Q^1\|_{L^u} \lesssim |Q|^{1/u}.$$

**Proof.** By (2.9), it only remains to estimate  $1_{(2Q)^c}Tb_Q^1$ . But

$$\begin{aligned} \|1_{(2Q)^c}Tb_Q^1\|_{L^u} &= \left\| x \mapsto \int_Q K(x, y)b_Q^1(y) \, dy \right\|_{L^u((2Q)^c)} \\ &\leq \int_Q \|x \mapsto K(x, y)\|_{L^u((2Q)^c)} |b_Q^1(y)| \, dy \lesssim \int_Q |Q|^{-1/u'} |b_Q^1(y)| \, dy \lesssim |Q|^{1/u}, \end{aligned}$$

by a straightforward estimate of the  $L^u((2Q)^c)$  norm of  $x \mapsto |K(x, y)| \lesssim |x - y|^{-d}$  for fixed  $y \in Q$  and variable  $x \in (2Q)^c$ .  $\square$

We have the following version of Cotlar’s lemma:

**3.2 Lemma.** *Suppose that there is a buffered  $(q, v)$  accretive system for  $T^*$ . Then*

$$T_{\#}f \lesssim M_{q'}(Tf) + M_v f.$$

**3.3 Remark.** The classical Cotlar inequality states that  $T_{\#}f \lesssim M(Tf) + Mf$  assuming that  $T$  is already bounded on  $L^2$ . Thus, trying to use this estimate in an attempt to prove the  $L^2$ -boundedness of  $T$  would result in a circular argument. While we have assumed the  $L^2$ -boundedness of  $T$  qualitatively, an inspection of the proof of Cotlar’s inequality shows that  $\|T\|$  enters quantitatively into the estimate; in fact, a more precise form would be

$$T_{\#}f \lesssim M(Tf) + (1 + \|T\|_{L^1 \rightarrow L^{1,\infty}})Mf,$$

which is of no use to us. So the point of Lemma 3.2 is to establish a version of Cotlar’s inequality without using the boundedness of  $T$ , but only the existence of an accretive system for  $T^*$ .

**Proof of Lemma 3.2.** Fix  $x_0$  and  $\varepsilon > 0$ . Let  $Q$  be the unique dyadic cube containing  $x_0$  and of diameter in the range  $[\frac{1}{8}\varepsilon, \frac{1}{4}\varepsilon]$ ; thus  $B(x_0, c_d\varepsilon) \subset 2Q \subset B(x_0, \varepsilon)$ . For all  $x \in Q$ ,

we have

$$T_\varepsilon f(x_0) = T_\varepsilon f(x_0) - T(f1_{(2Q)^c})(x) + Tf(x) - T(f1_{2Q})(x),$$

where

$$\begin{aligned} |T_\varepsilon f(x_0) - T(f1_{(2Q)^c})(x)| &= \left| \int_{|y-x_0|>\varepsilon} K(x_0, y)f(y) \, dy - \int_{(2Q)^c} K(x, y)f(y) \, dy \right| \\ &\leq \int_{(2Q)^c} |K(x_0, y) - K(x, y)||f(y)| \, dy + \int_{\substack{|y-x_0|\leq\varepsilon \\ y\in(2Q)^c}} |K(x_0, y)||f(y)| \, dy \\ &\lesssim \int_{|y-x_0|>c_d\varepsilon} \frac{\varepsilon^\alpha}{|y-x_0|^{d+\alpha}} |f(y)| \, dy + \int_{c_d\varepsilon<|y-x_0|<\varepsilon} \frac{1}{\varepsilon^d} |f(y)| \, dy \lesssim Mf(x_0). \end{aligned}$$

Thus

$$\begin{aligned} T_\varepsilon f(x_0) &= \int_Q b_Q^2 \, dx \cdot T_\varepsilon f(x_0) \\ &= \mathcal{O}(Mf(x_0)) + \int_Q b_Q^2 \cdot Tf \, dx - \frac{1}{|Q|} \int T^* b_Q^2 \cdot f1_{2Q} \, dx \\ &\lesssim Mf(x_0) + \left( \int_Q |b_Q^2|^q \right)^{1/q} \cdot \left( \int_Q |Tf|^{q'} \, dx \right)^{1/q'} \\ &\quad + \left( \int_{2Q} |T^* b_Q^2|^v \, dx \right)^{1/v} \left( \int_{2Q} |f|^{v'} \, dx \right)^{1/v'} \\ &\lesssim Mf(x_0) + M_{q'}(Tf)(x_0) + M_{v'} f(x_0). \quad \square \end{aligned}$$

**3.4 Lemma.** *Let  $b_Q^1$  be a buffered  $(p, q')$  accretive system for  $T$ , and  $b_Q^2$  a buffered  $(q, p')$  accretive system for  $T^*$ . Then*

$$|Q|^{-1/u} \|1_{2Q} T_\# b_Q^1\|_{L^u} \lesssim |Q|^{-1/s} \|1_{2Q} T_\# b_Q^1\|_{L^{s,\infty}} \lesssim 1, \quad u < s := \min\{p, q'\}.$$

*In particular,  $b_Q^1$  is a buffered  $(p, u)$  accretive system for  $T_\#$  for all  $u < \min\{p, q'\}$ .*

**Proof.** We apply Lemma 3.2, the quasi-triangle inequality and the monotonicity in the exponent of the weak norms, the boundedness of the maximal operators  $M_p : L^p \rightarrow L^{p,\infty}$  and  $M_{q'} : L^{q'} \rightarrow L^{q',\infty}$ , Lemma 3.1, and finally the assumptions on  $b_Q^1$  to deduce

$$\begin{aligned}
 |Q|^{-1/s} \|1_{2Q} T_{\#} b_Q^1\|_{L^{s,\infty}} &\lesssim |Q|^{-1/s} \|1_{2Q} [M_{q'}(Tb_Q^1) + M_p b_Q^1]\|_{L^{s,\infty}} \\
 &\lesssim |Q|^{-1/q'} \|1_{2Q} M_{q'}(Tb_Q^1)\|_{L^{q',\infty}} + |Q|^{-1/p} \|1_{2Q} M_p b_Q^1\|_{L^{p,\infty}} \\
 &\lesssim |Q|^{-1/q'} \|Tb_Q^1\|_{L^{q'}} + |Q|^{-1/p} \|b_Q^1\|_{L^p} \lesssim 1. \quad \square
 \end{aligned}$$

Note that we have now completed the proof of Proposition 2.12, and we conclude the section with:

**Proof of Lemma 2.18.** Let  $b_Q^1$ ,  $Q \in \mathcal{Q}$ , be an  $(\infty, u)$  accretive system for  $T$  on sparse subcubes of  $Q_0$ , and let  $b_Q^2$  be a  $(u', 1)$  accretive system for  $T^*$  (on all cubes). We need to show that  $b_Q^1$  is strong. To this end, fix some  $Q' \subseteq Q \in \mathcal{Q}$ , where  $Q'$  is not contained in any smaller  $Q'' \in \mathcal{Q}$ . We start with

$$\begin{aligned}
 \|1_{Q'} T(1_{Q'} b_Q^1)\|_u &= \|1_{Q'} T b_Q^1 - 1_{Q'} T(1_{3Q' \setminus Q} b_Q^1) - 1_{Q'} T(1_{(3Q')^c} b_Q^1)\|_u \\
 &\lesssim |Q'|^{1/u} + \|1_{3Q' \setminus Q} b_Q^1\|_u + |Q'|^{1/u} \|1_{Q'} T(1_{(3Q')^c} b_Q^1)\|_\infty
 \end{aligned} \tag{3.5}$$

where the first term was estimated directly from the definition of an  $(\infty, u)$  accretive system, and the second term by Hardy’s inequality. Moreover, the  $L^\infty$  bound implies that

$$\|1_{3Q' \setminus Q} b_Q^1\|_u \lesssim |Q'|^{1/u},$$

so it only remains to estimate the third term.

For this task, we call to our service the  $(u', 1)$  accretive system  $b_Q^2$ , which we have for every dyadic cube, so in particular for  $Q'$ . Let  $x \in Q'$ , and write

$$\begin{aligned}
 T(1_{(3Q')^c} b_Q^1)(x) &= \int_{Q'} [T(1_{(3Q')^c} b_Q^1)(x) - T(1_{(3Q')^c} b_Q^1)(y)] b_{Q'}^2(y) \, dy \\
 &\quad + \frac{1}{|Q'|} \langle T(1_{(3Q')^c} b_Q^1), b_{Q'}^2 \rangle \\
 &= \int_{Q'} \int_{(3Q')^c} [K(x, z) - K(y, z)] b_Q^1(z) \, dz \, b_{Q'}^2(y) \, dy \\
 &\quad + \frac{1}{|Q'|} \left( \langle T b_Q^1, b_{Q'}^2 \rangle - \langle T(1_{3Q' \setminus Q} b_Q^1), b_{Q'}^2 \rangle - \langle 1_{Q'} b_Q^1, T^* b_{Q'}^2 \rangle \right).
 \end{aligned}$$

The double integral is estimated by

$$\int_{Q'} \int_{(3Q')^c} \frac{\ell(Q')^\alpha}{|x - z|^{d+\alpha}} \, dz \, |b_{Q'}^2(y)| \, dy \lesssim \int_Q |b_{Q'}^2(y)| \, dy \lesssim 1,$$

and we also have the bounds

$$\begin{aligned} |\langle Tb_Q^1, b_{Q'}^2 \rangle| &\lesssim \|1_{Q'} T b_Q^1\|_u \|b_{Q'}^2\|_{u'} \lesssim |Q'|^{1/u} |Q'|^{1/u'} = |Q'|, \\ |\langle T(1_{3Q' \setminus Q'} b_Q^1), b_{Q'}^2 \rangle| &\lesssim \|1_{3Q' \setminus Q'} b_Q^1\|_u \|b_{Q'}^2\|_{u'} \lesssim |Q'| \quad \text{by Hardy's inequality, and} \\ |\langle 1_{Q'} b_Q^1, T^* b_{Q'}^2 \rangle| &\lesssim \|1_Q b_Q^1\|_\infty \|T^* b_{Q'}^2\|_1 \lesssim 1 \cdot |Q'| = |Q'|, \end{aligned}$$

by the fact that  $b_Q^2$  is a  $(u', 1)$ -accretive system for  $T^*$ . Thus

$$\|T(1_{(3Q')^c} b_Q^1)\|_\infty \lesssim 1,$$

which, substituted to (3.5), completes the proof that  $\|1_{Q'} T(1_{Q'} b_Q^1)\|_u \lesssim |Q'|^{1/u}$ .  $\square$

#### 4. Modified test functions; proof of Proposition 2.19

Throughout this Section 4, we work under the assumptions of Proposition 2.19. In particular, for some  $p > 1$ , we assume that  $b_Q^1$  is a  $(p, p)$  accretive system for the maximal truncated operator  $T_\#$  and, where relevant,  $b_Q^2$  is a similar system for  $(T^*)_\#$ . Given these rough test functions for  $T_\#$ , we want to construct better test functions for  $T_\Phi$ , where  $\Phi$  is suitably chosen. We start with some preparations.

##### 4.1. Generalities on suppressed operators

For completeness, we detail the following observation, which is a routine extension of a special case appearing in [17, Sec. I].

**4.1 Lemma.** *Let  $K$  be a Calderón–Zygmund standard kernel,  $\Phi$  be a non-negative function with Lipschitz constant 1, and  $m \geq d/2$ . Then the suppressed kernel  $K_\Phi$  is also a Calderón–Zygmund standard kernel with the same Hölder exponent  $\alpha \in (0, 1]$ .*

**Proof.** We need to verify (2.2) for  $K_\Phi$  in place of  $K$ . Let us denote

$$L_\Phi(x, y) := \frac{|x - y|^{2m}}{|x - y|^{2m} + \Phi(x)^m \Phi(y)^m}$$

so that  $K_\Phi = L_\Phi K$ . It is immediate that  $0 \leq L_\Phi \leq 1$ , and hence  $|K_\Phi(x, y)| \leq |K(x, y)| \lesssim |x - y|^{-d}$ . Concerning smoothness, we have

$$L_\Phi(x, y) - L_\Phi(x', y) = \frac{|x - y|^{2m} \Phi(x')^m \Phi(y)^m - |x' - y|^{2m} \Phi(x)^m \Phi(y)^m}{[|x - y|^{2m} + \Phi(x)^m \Phi(y)^m][|x' - y|^{2m} + \Phi(x')^m \Phi(y)^m]} =: \frac{A}{B},$$

where

$$A = (|x - y|^{2m} - |x' - y|^{2m}) \Phi(x')^m \Phi(y)^m + |x' - y|^{2m} (\Phi(x')^m - \Phi(x)^m) \Phi(y)^m =: A_1 + A_2.$$

We need to estimate these for  $|x - x'| \leq \frac{1}{2}|x - y|$ , in which case  $|x - y| \approx |x' - y|$  and



$$\left| \frac{A_1}{B} \right| \lesssim \frac{|x - x'| |x - y|^{2m-1} \Phi(x')^m \Phi(y)^m}{|x - y|^{2m} \times \Phi(x')^m \Phi(y)^m} = \frac{|x - x'|}{|x - y|}.$$

Concerning  $A_2$ , we first observe that

$$|\Phi(x')^m - \Phi(x)^m| = |\Phi(x') - \Phi(x)| \sum_{k=0}^{m-1} \Phi(x')^{m-1-k} \Phi(x)^k \leq |x' - x| \sum_{k=0}^{m-1} \Phi(x')^{m-1-k} \Phi(x)^k$$

by the Lipschitz assumption on  $\Phi$ , and hence

$$|A_2| \leq |x' - y|^{2m} |x' - x| \left( \sum_{k=0}^{m-1} (\Phi(x') \Phi(y))^{m-1-k} (\Phi(x) \Phi(y))^k \right) \Phi(y).$$

Another observation is that  $|x - y|^{2m} + \Phi(x)^m \Phi(y)^m \approx (|x - y|^2 + \Phi(x) \Phi(y))^m$  and, again by the Lipschitz assumption on  $\Phi$ , we have  $\Phi(x) \geq \Phi(y) - |x - y|$  and hence

$$\begin{aligned} |x - y|^2 + \Phi(x) \Phi(y) &\geq |x - y|^2 + \Phi(y)^2 - |x - y| \Phi(y) = (|x - y| - \Phi(y))^2 + |x - y| \Phi(y) \\ &\geq |x - y| \Phi(y). \end{aligned}$$

Thus, for any  $k = 0, 1, \dots, m - 1$ ,

$$\begin{aligned} B &\approx [ |x - y|^2 + \Phi(x) \Phi(y) ]^{1+k+(m-1-k)} [ |x' - y|^2 + \Phi(x') \Phi(y) ]^{(m-1-k)+(k+1)} \\ &\geq |x - y| \Phi(y) \times (\Phi(x) \Phi(y))^k \times |x - y|^{2(m-1-k)} \times (\Phi(x') \Phi(y))^{m-1-k} \times |x' - y|^{2(k+1)} \\ &\approx \Phi(y) \times (\Phi(x) \Phi(y))^k \times (\Phi(x') \Phi(y))^{m-1-k} \times |x' - y|^{2m+1} \end{aligned}$$

and hence  $|A_2/B| \lesssim |x' - x|/|x' - y| \approx |x' - x|/|x - y|$ . Together with the earlier estimates, we have checked that

$$|L_\Phi(x, y) - L_\Phi(x', y)| \lesssim \frac{|x - x'|}{|x - y|}, \quad |x - x'| \leq \frac{1}{2} |x - y|.$$

Thus, for the same range of the variables  $x, x', y$ ,

$$\begin{aligned} |K_\Phi(x, y) - K_\Phi(x', y)| &\leq |L_\Phi(x, y) - L_\Phi(x', y)| K(x, y) + |L_\Phi(x', y)| |K(x, y) - K(x', y)| \\ &\lesssim \frac{|x - x'|}{|x - y|} \frac{1}{|x - y|^d} + 1 \cdot \frac{|x - x'|^\alpha}{|x - y|^{d+\alpha}} \lesssim \frac{|x - x'|^\alpha}{|x - y|^{d+\alpha}}, \quad \alpha \in (0, 1]. \end{aligned}$$

This proves the smoothness estimate in (2.2) in the first variable, and the case of the second variable is entirely analogous, since  $L_\Phi$  itself is symmetric, and the assumptions on  $K$  are symmetric in  $x$  and  $y$ .  $\square$

**4.2 Lemma.** *For any nonnegative function  $\Phi$  with Lipschitz constant 1, we have*

$$|T_\Phi f| \lesssim T_\# f + Mf.$$

**Proof.**

$$\begin{aligned} T_{\Phi}f(x) &= \int K_{\Phi}(x, y)f(y) \, dy \\ &= \int_{|x-y| \leq \frac{1}{2}\Phi(x)} K_{\Phi}(x, y)f(y) \, dy \\ &\quad + \int_{|x-y| > \frac{1}{2}\Phi(x)} (K_{\Phi}(x, y) - K(x, y))f(y) \, dy + T_{\frac{1}{2}\Phi(x)}f(x) \end{aligned}$$

For  $|x - y| \leq \frac{1}{2}\Phi(x)$ , we have  $\Phi(y) \geq \Phi(x) - |x - y| \geq \frac{1}{2}\Phi(x)$ , hence

$$|K_{\Phi}(x, y)| \leq |K(x, y)| \frac{|x - y|^{2m}}{(\frac{1}{2}\Phi(x)^2)^m} \lesssim \frac{|x - y|^{2m-d}}{\Phi(x)^{2m}},$$

and thus

$$\begin{aligned} \left| \int_{|x-y| \leq \frac{1}{2}\Phi(x)} K_{\Phi}(x, y)f(y) \, dy \right| &\lesssim \sum_{j=1}^{\infty} \int_{2^{-j-1}\Phi(x) < |x-y| \leq 2^{-j}\Phi(x)} \frac{(2^{-j}\Phi(x))^{2m-d}}{\Phi(x)^{2m}} |f(y)| \, dy \\ &\lesssim \sum_{j=1}^{\infty} 2^{-2jm} \int_{|x-y| \leq 2^{-j}\Phi(x)} |f(y)| \, dy \lesssim Mf(x). \end{aligned}$$

And for  $|x - y| > \frac{1}{2}\Phi(x)$ , we have  $\Phi(y) \leq \Phi(x) + |x - y| < 3|x - y|$ , so that

$$|K_{\Phi}(x, y) - K(x, y)| = |K(x, y)| \frac{(\Phi(x)\Phi(y))^m}{|x - y|^{2m} + (\Phi(x)\Phi(y))^m} \lesssim \frac{1}{|x - y|^d} \frac{\Phi(x)^m |x - y|^m}{|x - y|^{2m}},$$

and hence

$$\begin{aligned} &\left| \int_{|x-y| > \frac{1}{2}\Phi(x)} (K_{\Phi}(x, y) - K(x, y))f(y) \, dy \right| \\ &\lesssim \sum_{j=0}^{\infty} \int_{2^{j-1}\Phi(x) < |x-y| \leq 2^j\Phi(x)} \frac{\Phi(x)^m}{(2^j\Phi(x))^{d+m}} |f(y)| \, dy \\ &\lesssim \sum_{j=0}^{\infty} 2^{-jm} \int_{|x-y| \leq 2^j\Phi(x)} |f(y)| \, dy \lesssim Mf(x). \end{aligned}$$

Finally, it is clear that  $|T_{\frac{1}{2}\Phi(x)}f(x)| \leq T_{\#}f(x)$ .  $\square$

**4.3 Lemma.** *Let  $b_Q^1$  be a (buffered)  $(p, u)$  accretive system for  $T_{\#}$ , where  $p > 1$ . Then it is also a (buffered)  $(p, s)$  accretive system for any  $T_{\Phi}$ , where  $s = \min(p, u)$ .*

**Proof.** Let  $\alpha \in \{1, 2\}$  according to whether the system is buffered ( $\alpha = 2$ ) or not ( $\alpha = 1$ ). By Lemma 4.2 and the boundedness of  $M$  on  $L^p$ , we have

$$\begin{aligned} |Q|^{-1/s} \|1_{\alpha Q} T_{\Phi} b_Q^1\|_{L^s} &\lesssim |Q|^{-1/s} \|1_{\alpha Q} [T_{\#} b_Q^1 + M b_Q^1]\|_{L^s} \\ &\leq |Q|^{-1/u} \|1_{\alpha Q} T_{\#} b_Q^1\|_{L^u} + |Q|^{-1/p} \|1_{\alpha Q} M b_Q^1\|_{L^p} \lesssim 1. \quad \square \end{aligned}$$

4.2. *First step of the modification and key estimates*

We turn to the actual construction of the modified test functions  $\tilde{b}_Q^1$ .

Consider a fixed cube  $Q_0$ , and abbreviate  $b := b_{Q_0}^1$ . Let  $\mathcal{B}_1 = \mathcal{B}_1(Q_0)$  be the maximal dyadic subcubes  $Q \subseteq Q_0$  with  $f_Q(Mb + T_{\#}b)^p \gg 1$ , and  $\Omega := \bigcup_{Q \in \mathcal{B}_1} Q$ . Let

$$\tilde{b} := \tilde{b}_{Q_0}^1 := 1_{\Omega^c} b + \sum_{Q \in \mathcal{B}_1} \langle b \rangle_Q 1_Q = b - \sum_{Q \in \mathcal{B}_1} (b - \langle b \rangle_Q) 1_Q =: b - \sum_{Q \in \mathcal{B}_1} d_Q \tag{4.4}$$

be (essentially) the good part of the Calderón–Zygmund decomposition of  $b$ .

**4.5 Lemma.** *If  $\Phi$  is a 1-Lipschitz function with*

$$\Phi(x) \geq \sup_{Q \in \mathcal{B}_1} \text{dist}(x, (3Q)^c), \tag{4.6}$$

then

$$\begin{aligned} |T_{\Phi}(b - \tilde{b})(x)| &\leq \sum_{Q \in \mathcal{B}_1} |T_{\Phi} d_Q(x)| \lesssim \sum_{Q \in \mathcal{B}_1} \left( \frac{\ell(Q)}{\ell(Q) + |x - c_Q|} \right)^{d+\alpha} \\ &=: \sum_{Q \in \mathcal{B}_1} \phi_Q(x) =: e_{Q_0}^1(x), \end{aligned}$$

where for all  $u \in [1, \infty)$ ,

$$\|e_{Q_0}^1\|_{L^u} \lesssim |Q_0|^{1/u}.$$

**Proof.** Recall that  $K_{\Phi}$  is a Calderón–Zygmund standard kernel by Lemma 4.1. Thus, for  $x \in (2Q)^c$ ,

$$|T_{\Phi} d_Q(x)| = \left| \int_Q [K_{\Phi}(x, y) - K_{\Phi}(x, c_Q)] d_Q(y) dy \right| \lesssim \frac{\ell(Q)^\alpha}{|x - c_Q|^{d+\alpha}} \|d_Q\|_1 \lesssim \left( \frac{\ell(Q)}{|x - c_Q|} \right)^{d+\alpha}.$$

For  $x \in 2Q, y \in Q$ , we have  $\text{dist}(x, (3Q)^c), \text{dist}(y, (3Q)^c) \gtrsim \ell(Q)$ , hence  $\Phi(x)\Phi(y) \gtrsim \ell(Q)^2$ , and therefore

$$|K_{\Phi}(x, y)| \lesssim \frac{|x - y|^{2m-d}}{\ell(Q)^{2m}} \lesssim \frac{1}{\ell(Q)^d} \tag{4.7}$$

provided that  $m \geq d/2$ . Thus

$$|T_{\Phi}d_Q(x)| = \left| \int_Q K_{\Phi}(x, y)d_Q(y) dy \right| \lesssim \frac{1}{|Q|} \int_Q |d_Q(y)| dy \lesssim 1.$$

Altogether we have

$$|T_{\Phi}d_Q(x)| \lesssim \left( \frac{\ell(Q)}{\ell(Q) + |x - c_Q|} \right)^{d+\alpha} =: \phi_Q(x).$$

By duality, for a suitable  $g \geq 0$  with  $\|g\|_{u'} = 1$ ,

$$\begin{aligned} \left\| \sum_{Q \in \mathcal{B}_1} \phi_Q \right\|_u &= \int g \sum_{Q \in \mathcal{B}_1} \phi_Q \lesssim \sum_{Q \in \mathcal{B}_1} |Q| \inf_Q Mg \leq \int Mg \sum_{Q \in \mathcal{B}_1} 1_Q \\ &\leq \|Mg\|_{u'} \left\| \sum_{Q \in \mathcal{B}_1} 1_Q \right\|_u, \end{aligned}$$

where  $\|Mg\|_{u'} \lesssim \|g\|_{u'} = 1$  by the maximal inequality, and

$$\left\| \sum_{Q \in \mathcal{B}_1} 1_Q \right\|_u = \left( \sum_{Q \in \mathcal{B}_1} |Q| \right)^{1/u} \leq |Q_0|^{1/u}$$

by the disjointness of the cubes  $Q \in \mathcal{B}_1$ .  $\square$

**4.8 Lemma.** *If  $\Phi$  is a 1-Lipschitz function with*

$$\Phi(x) \geq \sup_{Q \in \mathcal{B}_1} \text{dist}(x, (3Q)^c), \tag{4.9}$$

then

$$1_{Q_0}|T_{\Phi}b| \lesssim 1$$

**Proof.** We have

$$1_{Q_0}|T_{\Phi}b| = 1_{Q_0 \setminus \Omega}|T_{\Phi}b| + \sum_{Q \in \mathcal{B}_1} 1_Q|T_{\Phi}b|,$$

where

$$1_{Q_0 \setminus \Omega}|T_{\Phi}b| \lesssim 1_{Q_0 \setminus \Omega}(T_{\#}b + Mb) \lesssim 1.$$

For any  $x, y \in Q \in \mathcal{B}_1$ , we find that

$$\begin{aligned} |T_\Phi b(x) - T_\Phi b(y)| &= \left| \int [K_\Phi(x, z) - K_\Phi(y, z)]b(z) \, dz \right| \\ &\lesssim \int_{2Q} \frac{1}{\ell(Q)^d} |b(z)| \, dz + \int_{(2Q)^c} \frac{\ell(Q)^\alpha}{|z - x|^{d+\alpha}} |b(z)| \, dz \lesssim \inf_Q Mb \leq \int_Q Mb \lesssim 1 \end{aligned}$$

by the maximality of the cubes  $Q$ . Thus

$$\sup_Q |T_\Phi b| \lesssim 1 + \inf_Q |T_\Phi b| \lesssim 1 + \inf_Q (T_\# b + Mb) \leq 1 + \int_Q (T_\# b + Mb) \lesssim 1,$$

again by the maximality of the cubes  $Q$ .  $\square$

We want to interpret the new function  $\tilde{b}_{Q_0}^1$ , and similarly constructed functions for subcubes of  $Q_0$ , as test functions for the operator  $T_\Phi$ . Note that the choice of  $\Phi$  will be fixed only after a stopping time construction, by which we construct the remaining functions  $\tilde{b}_Q^1$ . Before we fix this choice, it is important that all the estimates are valid for any  $\Phi$  with the property (4.9).

For any such  $\Phi$ , we have by Lemmas 4.5 and 4.8 that

$$1_{Q_0} |T_\Phi \tilde{b}_{Q_0}^1| \leq 1_{Q_0} |T_\Phi b_{Q_0}^1| + 1_{Q_0} |T_\Phi (\tilde{b}_{Q_0}^1 - b_{Q_0}^1)| \lesssim 1 + e_{Q_0}^1. \tag{4.10}$$

### 4.3. Different stopping conditions

We refer to the maximal dyadic subcubes  $Q \subseteq Q_0$  with  $\int_Q (Mb_{Q_0}^1 + T_\# b_{Q_0}^1)^p \geq C/\delta$  as its **primary stopping cubes**. They satisfy

$$\sum_{Q \in \mathcal{B}_1(Q_0)} |Q| = |\{M^d (Mb_{Q_0}^1 + T_\# b_{Q_0}^1)^p > C/\delta\}| \leq \frac{1}{C/\delta} \int (Mb_{Q_0}^1 + T_\# b_{Q_0}^1)^p \leq \delta |Q_0|,$$

where in the last step we use the standing assumption of Proposition 2.19 (which is in force throughout this Section 4) that  $b_{Q_0}^1$  is a  $(p, p)$  accretive system for the maximal truncated operator  $T_\#$ . The function  $e_{Q_0}^1$  depends on these cubes, and thus on  $\delta$ ; however, the bound  $\|e_{Q_0}^1\|_u \lesssim |Q|^{1/u}$ , guaranteed by Lemma 4.5, is independent of these parameters.

The **secondary stopping cubes** of  $Q_0$  are defined as the maximal dyadic subcubes  $Q \subseteq Q_0$  that satisfy either of the following conditions: For a given  $u \in (1, \infty)$ , which we take at least as large as  $\max\{p, p'\}$ ,

$$\int_Q (e_{Q_0}^1)^u > \frac{1}{\varepsilon}, \tag{4.11}$$

or

$$\left| \int_Q \tilde{b}_{Q_0}^1 \right| \leq \eta. \tag{4.12}$$

The measure of the cubes in (4.11) is at most

$$|\{M^d(1_{Q_0}(e_{Q_0}^1)^u) > 1/\varepsilon\}| \leq \varepsilon \int_{Q_0} (e_{Q_0}^1)^u \lesssim \varepsilon |Q_0|.$$

For the cubes in (4.12), we compute

$$\begin{aligned} |Q_0| &= \left| \int b_{Q_0}^1 \right| = \left| \int \tilde{b}_{Q_0}^1 \right| \\ &\leq \sum_Q \left| \int \tilde{b}_{Q_0}^1 \right| + \int_{Q_0 \setminus \bigcup Q} |\tilde{b}_{Q_0}^1| \\ &\leq \sum_Q \eta |Q| + |Q_0 \setminus \bigcup Q|^{1/p'} \|\tilde{b}_{Q_0}^1\|_p \\ &\leq \eta |Q_0| + C \left( |Q_0| - \sum_Q |Q| \right)^{1/p'} |Q_0|^{1/p}, \end{aligned}$$

by using  $\|\tilde{b}_{Q_0}^1\|_p \lesssim \|b_{Q_0}^1\|_p \lesssim |Q_0|^{1/p}$  in the last step. From here one can solve

$$\sum |Q| \leq \left\{ 1 - \left( \frac{1-\eta}{C} \right)^{p'} \right\} |Q_0|.$$

Altogether, taking  $\eta < 1$  and  $\varepsilon$  sufficiently small, the measure of the secondary stopping cubes is at most a fraction  $(1 - \tau) < 1$  of  $|Q_0|$ .

#### 4.4. Iteration of the stopping conditions

We iterate the following algorithm, starting from an arbitrary but fixed dyadic cube  $Q_0$ .

- We choose the primary stopping cubes  $\mathcal{B}_1 = \mathcal{B}_1(Q_0)$  of  $Q_0$ , and the secondary stopping cubes  $\mathcal{C}_1 = \mathcal{C}_1(Q_0)$  of  $Q_0$  as explained above.
- Assuming that the collections  $\mathcal{B}_k$  and  $\mathcal{C}_k$  are already constructed, for every  $Q \in \mathcal{C}_k$ , we choose (using  $b_Q^1$ ) the primary stopping cubes  $\mathcal{B}_1(Q)$  of  $Q$ , which determine the functions  $\tilde{b}_Q^1$  and  $e_Q^1$ , and then (using  $\tilde{b}_Q^1$  and  $e_Q^1$ ), the secondary stopping cubes  $\mathcal{C}_1(Q)$  of  $Q$ . We let

$$\mathcal{B}_{k+1} := \bigcup_{Q \in \mathcal{C}_k} \mathcal{B}_1(Q), \quad \mathcal{C}_{k+1} := \bigcup_{Q \in \mathcal{C}_k} \mathcal{C}_1(Q).$$

By iterating the bounds  $\sum_{Q' \in \mathcal{B}_1(Q)} |Q'| \leq \delta |Q|$  and  $\sum_{Q' \in \mathcal{C}_1(Q)} |Q'| \leq (1 - \tau) |Q|$ , we arrive at

$$\sum_{Q \in \mathcal{C}_k(Q_0)} |Q| \leq (1 - \tau)^k |Q_0|, \quad \sum_{Q \in \mathcal{B}_k(Q_0)} |Q| \leq \delta \sum_{Q \in \mathcal{C}_{k-1}(Q_0)} |Q| \leq \delta (1 - \tau)^{k-1} |Q_0|,$$

where we interpret  $\mathcal{C}_0(Q_0) := \{Q_0\}$ . Hence the measure of all primary stopping cubes satisfies

$$\sum_{k=1}^{\infty} \sum_{Q \in \mathcal{B}_k(Q_0)} |Q| \leq \sum_{k=1}^{\infty} \delta (1 - \tau)^{k-1} |Q_0| = \frac{\delta}{\tau} |Q_0|.$$

The parameter  $\delta$  can be made small independently of  $\tau$ , and hence we can make the fraction  $\delta/\tau$  as small as we like.

Then we can define

$$\Phi(x) := \sup \left\{ \text{dist}(x, (3Q)^c) : Q \in \bigcup_{k=1}^{\infty} \mathcal{B}_k \right\}.$$

It follows that

$$|\{ \Phi > 0 \}| = \left| \bigcup_{k=1}^{\infty} \bigcup_{Q \in \mathcal{B}_k} 3Q \right| \leq 3^d \frac{\delta}{\tau} |Q_0| =: \varrho |Q_0|, \tag{4.13}$$

where the fraction  $\varrho$  can be made arbitrarily small.

*4.5. Construction summary; completion of the proof of Proposition 2.19*

Given a cube  $Q_0$ , we find the stopping cubes  $\bigcup_{k=1}^{\infty} \mathcal{B}_k(Q_0)$  and  $\bigcup_{k=1}^{\infty} \mathcal{C}_k(Q_0)$ . Let us also call  $Q_0$  itself a stopping cube, and denote  $\mathcal{C}_0(Q_0) := \{Q_0\}$ . The stopping cubes determine the function  $\Phi$ . For each  $Q \in \mathcal{C}(Q_0) := \bigcup_{k=0}^{\infty} \mathcal{C}_k(Q_0)$ , there is a function  $\tilde{b}_Q^1$ , the good part of the Calderón–Zygmund decomposition of  $b_Q^1$ , thus

$$\|\tilde{b}_Q^1\|_{\infty} \lesssim 1.$$

For every  $Q \in \mathcal{C}(Q_0)$ , we can apply the estimate (4.10) for  $Q$  in place of  $Q_0$ . Indeed, it suffices to check that the chosen  $\Phi$  satisfies the analogue of (4.9) with  $Q \in \mathcal{C}(Q_0)$  in place of  $Q_0$ , namely, that

$$\Phi(x) \geq \sup_{Q' \in \mathcal{B}_1(Q)} \text{dist}(x, (3Q')^c).$$

But this is clear from the definition of  $\Phi$ , since  $\mathcal{B}_1(Q) \subseteq \mathcal{B}(Q_0)$  for all  $Q \in \mathcal{C}(Q_0)$ , and  $\Phi$  is the supremum over this larger set. Thus, by (4.10) applied to  $Q$  in place of  $Q_0$ , we have

$$|T_\Phi \tilde{b}_Q^1| \lesssim 1 + e_Q^1.$$

If  $Q' \subseteq Q \in \mathcal{C}(Q_0)$  is not contained in any smaller  $Q'' \in \mathcal{C}(Q_0)$ , equivalently, not in any  $Q'' \in \mathcal{C}_1(Q)$ , then, by the construction of the secondary stopping cubes  $\mathcal{C}_1(Q)$ , this means that

$$\int_{Q'} |T_\Phi \tilde{b}_Q^1|^u \lesssim \int_{Q'} (1 + e_Q^1)^u \lesssim 1, \tag{4.14}$$

and

$$\left| \int_{Q'} \tilde{b}_Q^1 \right| \gtrsim 1, \quad Q' \subseteq Q \in \mathcal{C}(Q_0), \quad Q' \not\subseteq Q'' \in \mathcal{C}_1(Q).$$

(We are suppressing the dependence on the parameters  $\varepsilon, \eta$ , since they are now fixed and no longer relevant to us.) Recall also that

$$\sum_{Q' \in \mathcal{C}_1(Q)} |Q'| \leq (1 - \tau)|Q|.$$

Summa summarum, associated to every secondary stopping cube  $Q \in \mathcal{C}(Q_0)$ , there is a non-degenerate test function  $\tilde{b}_Q^1 \in L^\infty(Q)$  such that  $1_Q T_\Phi \tilde{b}_Q^1 \in L^u(Q)$ , with the correct normalization. Moreover,

$$|\{\Phi > 0\}| \leq \varrho|Q_0|.$$

Of course, starting from the original test functions  $b_Q^2$  and  $T^*$  in place of  $T$ , we can similarly produce new test functions  $\tilde{b}_Q^2 \in L^\infty(Q)$  with  $1_Q T_\Phi^* \tilde{b}_Q^2 \in L^u(Q)$  for another set of secondary stopping cube  $Q \in \mathcal{C}^2(Q_0)$ . To have the same  $\Phi$  both for  $T$  and  $T^*$ , we should define

$$\Phi(x) := \sup\{\text{dist}(x, (3Q)^c) : Q \in \bigcup_{k=1}^\infty (\mathcal{B}_k^1 \cup \mathcal{B}_k^2)\},$$

where  $\mathcal{B}_k^1$  and  $\mathcal{B}_k^2$  are the primary stopping cubes related to  $b_Q^1$  and  $b_Q^2$ , respectively. Clearly, this still satisfies the bound (4.13), with at most twice the original constant  $\varrho$ , which we can make arbitrarily small in any case.

To check that the  $(\infty, u)$  accretive system  $\tilde{b}_Q^1$  for  $T_\Phi$ , on sparse subcubes of  $Q_0$ , is strong, by Lemma 2.18, we only need to check that there is a  $(u', 1)$  accretive system for  $T_\Phi^*$  on all dyadic cubes. And we claim that such a system is provided by the original  $(p, p)$  accretive system  $b_Q^2$  for  $(T^*)_\#$ , whose existence is contained in the very assumptions of Proposition 2.19 under which we are working in this Section 4. By Lemma 4.3,  $b_Q^2$  is also a  $(p, p)$  accretive system for  $T_\Phi^*$ , and thus a  $(u', 1)$  accretive system provided that  $u' \leq p$ .



But we can choose  $u$  as large as we like, hence  $u'$  as close to 1 as we like, and then this condition is clearly fulfilled.

This completes the proof of Proposition 2.19.

**5. The baby  $Tb$  theorem; proof of Proposition 2.20**

Let us denote the reference cube in which we operate by  $Q^0$ , as we will need the notation  $Q_0$  for another purpose below. Let us first deal with just one accretive system  $b_Q$  defined on a sparse family  $\mathcal{Q}$ ; later on, these results will be applied to both  $b_Q^1$  on  $\mathcal{Q}_1$  and  $b_Q^2$  on  $\mathcal{Q}_2$ . We also refer to the members of  $\mathcal{Q}$  as *stopping cubes*. For every  $Q \subseteq Q^0$ , let  $Q^a$  be the minimal stopping cube that contains  $Q$ . Then

$$\left| \int_Q b_{Q^a} dx \right| \gtrsim 1, \quad \int_Q |b_{Q^a}|^t dx \lesssim 1, \quad \int_Q |Tb_{Q^a}|^t dx \lesssim 1,$$

for some exponent  $t \in (1, \infty)$  as in the assumptions of Proposition 2.20.

We start by recalling the adapted martingale difference framework for a local  $Tb$  theorem from [16] and [9,11]. (Also the subsequent analysis borrows from these papers, even when this is not always stated. On the other hand, we take the opportunity to simplify several details, as we are in the simpler case of the Lebesgue measure, rather than a non-doubling one; this allows us to work with the fixed system of standard dyadic cubes, instead of their random translation.)

We have the expectation (averaging) operators

$$\mathbb{E}_Q^b f := 1_Q b_{Q^a} \frac{\langle f \rangle_Q}{\langle b_{Q^a} \rangle_Q}, \quad \mathbb{E}_Q := 1_Q \langle f \rangle_Q,$$

and the difference operators

$$\begin{aligned} \mathbb{D}_Q^b f &:= \sum_{i=1}^{2^d} \mathbb{E}_{Q_i}^b f - \mathbb{E}_Q^b f \\ &= \sum_{i=1}^{2^d} \left( 1_{Q_i} b_{Q_i^a} \frac{1}{\langle b_{Q_i^a} \rangle_{Q_i}} - 1_Q b_{Q^a} \frac{|Q_i|}{|Q| \langle b_{Q^a} \rangle_Q} \right) \langle f \rangle_{Q_i} =: \sum_{i=1}^{2^d} \phi_{Q,i} \langle f \rangle_{Q_i}, \end{aligned}$$

where the  $i$ -sum goes through the dyadic children  $Q_i$  of  $Q$ . A direct computation shows that

$$(\mathbb{D}_Q^b)^2 f = \mathbb{D}_Q^b f - \sum_{i=1}^{2^d} 1_{Q_i} \left( \frac{\langle b_{Q^a} \rangle_{Q_i}}{\langle b_{Q_i^a} \rangle_{Q_i}} b_{Q_i^a} - b_{Q^a} \right) \frac{\langle f \rangle_Q}{\langle b_{Q^a} \rangle_Q} =: \mathbb{D}_Q^b f - \omega_Q \langle f \rangle_Q.$$

Note that  $1_{Q_i} \omega_Q$  is nonzero only if  $Q_i^a \neq Q^a$ , i.e., only if  $Q_i$  is a stopping cube. Thus  $\omega_Q$  is nonzero only if  $Q$  has at least one stopping child. Combining the above formulae, we get

$$\mathbb{D}_Q^b f = (\mathbb{D}_Q^b)^2 f + \omega_Q \langle f \rangle_Q = \sum_{i=1}^{2^d} \phi_{Q,i} \langle \mathbb{D}_Q^b f \rangle_{Q_i} + \omega_Q \langle f \rangle_Q =: \sum_{i=0}^{2^d} \phi_{Q,i} \langle \mathbb{D}_{Q,i}^b f \rangle_{Q_i}, \tag{5.1}$$

where  $Q_0 := Q$ ,  $\phi_{Q,0} := \omega_Q$  and

$$\mathbb{D}_{Q,i}^b f := \begin{cases} \mathbb{D}_Q^b f, & \text{if } i = 1, \dots, 2^d, \\ \mathbb{E}_Q f, & \text{if } i = 0 \text{ and } Q \text{ has a stopping child,} \\ 0, & \text{if } i = 0 \text{ and } Q \text{ does not have any stopping children.} \end{cases}$$

We have the following important estimates. The  $L^2$  case is from [16], and its generalization to  $L^r$  from [11]. (Both these papers deal with more general non-doubling situations, the latter even vector-valued—a generality that we do not consider here.)

**5.2 Proposition.** *For all  $r \in (1, \infty)$ ,  $i = 0, 1, \dots, 2^d$ , and  $f \in L^r$ , we have*

$$\left\| \left( \sum_Q |\mathbb{D}_{Q,i}^b f|^2 \right)^{1/2} \right\|_r + \left\| \left( \sum_Q |(\mathbb{D}_{Q,i}^b)^* f|^2 \right)^{1/2} \right\|_r \lesssim \|f\|_r.$$

In particular,

$$\left( \sum_Q \|\mathbb{D}_{Q,i}^b f\|_2^2 \right)^{1/2} + \left( \sum_Q \|(\mathbb{D}_{Q,i}^b)^* f\|_2^2 \right)^{1/2} \lesssim \|f\|_2.$$

For every  $f \in L^r(Q^0)$ , we have the decomposition

$$f = \mathbb{E}_{Q^0} f + \sum_Q \mathbb{D}_Q^b f.$$

Here and below, we apply the implicit convention that all summations over  $Q$  (or other variables designating cubes) always carry the restriction  $Q \subseteq Q^0$ , in addition to possible explicit summation conditions. To simplify writing, we redefine  $\mathbb{D}_{Q^0}^b f$  as  $\mathbb{E}_{Q^0} f + \mathbb{D}_{Q^0}^b f$ , so that we can drop the first term from the sum above. Thus

$$\langle Tf, g \rangle = \sum_{Q,R} \langle T\mathbb{D}_Q^{b_1} f, \mathbb{D}_R^{b_2} g \rangle = \sum_{\substack{Q,R \\ \ell(Q) \leq \ell(R)}} + \sum_{\substack{Q,R \\ \ell(Q) > \ell(R)}}.$$

By symmetry, it suffices to estimate the first half. This we reorganize as follows:

$$\begin{aligned} \sum_{\substack{Q,R \\ \ell(Q) \leq \ell(R)}} \langle T\mathbb{D}_Q^{b_1} f, \mathbb{D}_R^{b_2} g \rangle &= \sum_{k=0}^{\infty} \sum_{m \in \mathbb{Z}^d} \sum_R \sum_{\substack{Q \subseteq R + \ell(R)m \\ \ell(Q) = 2^{-k} \ell(R)}} \langle T\mathbb{D}_Q^{b_1} f, \mathbb{D}_R^{b_2} g \rangle \\ &= \sum_{k=0}^{\infty} \sum_{m \in \mathbb{Z}^d} \sum_R \langle T\mathbb{D}_{R+\ell(R)m}^{b_1,k} f, \mathbb{D}_R^{b_2} g \rangle, \end{aligned}$$

where  $(S := R + \ell(R)m)$

$$\mathbb{D}_S^{b_1,k} f := \sum_{\substack{Q \subseteq S \\ \ell(Q)=2^{-k}\ell(S)}} \mathbb{D}_Q^{b_1} f.$$

Below, we will also use the notation  $\mathbb{D}_{S,i}^{b_1,k}$  (with additional subscript  $i \in \{0, 1, \dots, 2^d\}$ ) similarly defined with  $\mathbb{D}_{Q,i}^{b_1}$  on the right as well.

We now turn to the estimation of the above series in various parts. For some of them, we will achieve domination by the usual  $L^2$  bound

$$\|f\|_2 \|g\|_2 \leq |Q^0|^{1/2-1/s'} \|f\|_{s'} |Q^0|^{1/2-1/s'} \|g\|_{s'} = |Q^0|^{1-2/s'} \|f\|_{s'} \|g\|_{s'},$$

but the weaker right-hand side, as in the statement of Proposition 2.20, will appear in some cases.

### 5.1. Disjoint cubes

We further analyse the part of the  $m$  sum with  $m \neq 0$ , thus  $Q \subseteq R + m\ell(R)$  is disjoint from  $R$ . By (5.1), we can expand

$$\begin{aligned} \langle T\mathbb{D}_Q^{b_1} f, \mathbb{D}_R^{b_2} g \rangle &= \sum_{i,j} \langle \mathbb{D}_{Q,i}^{b_1} f \rangle_{Q_i} \langle T\phi_{Q,i}^1, \phi_{R,j}^2 \rangle \langle \mathbb{D}_{R,j}^{b_2} g \rangle_{R_j}, \quad \text{hence, summing in } Q, \\ \langle T\mathbb{D}_S^{b_1,k} f, \mathbb{D}_R^{b_2} g \rangle &= \sum_{i,j} \iint_{R \times S} \mathbb{D}_{S,i}^{b_1,k} f(y) K_{R,S}^{i,j;k}(x,y) \mathbb{D}_{R,j}^{b_2} g(x) \, dx \, dy, \quad \text{where} \\ K_{R,S}^{i,j;k}(x,y) &:= \sum_{\substack{Q \subseteq S \\ \ell(Q)=2^{-k}\ell(S)}} \frac{1_{Q_i}(y)}{|Q_i|} \langle T\phi_{Q,i}^1, \phi_{R,j}^2 \rangle \frac{1_{R_j}(x)}{|R_j|}. \end{aligned}$$

**5.3 Lemma.** *For  $m \neq 0$ , we have*

$$\|K_{R,R+\ell(R)m}^{i,j;k}\|_{L^2(\mathbb{R}^d \times \mathbb{R}^d)} \lesssim 2^{-k\alpha} |m|^{-d-\alpha},$$

when the Hölder exponent in the Calderón–Zygmund condition (2.2) is  $\alpha \in (0, \frac{1}{2})$ .

Since we can always decrease the Hölder exponent of the Calderón–Zygmund kernel, we will henceforth assume that  $\alpha < \frac{1}{2}$ .

**Proof.** Note that  $Q \neq Q^0$  in this sum. Indeed,  $\ell(Q) \leq \ell(R) \leq \ell(Q^0)$  so the only way that we could have  $Q = Q^0$  is  $\ell(Q) = \ell(R) = \ell(Q^0)$ , and then (since  $Q^0$  is the only cube of sidelength  $\ell(Q^0)$ ),  $Q = R = Q^0$ . But then  $m = 0$ , a contradiction. Thus  $\mathbb{D}_Q^{b_1}$  is always given by the original definition, i.e., without the addition of  $\mathbb{E}_{Q^0}^{b_1}$ , and then all the  $\phi_{Q,i}^1$  have mean zero. Hence we can write

$$\begin{aligned} \langle T\phi_{Q,i}^1, \phi_{R,j}^2 \rangle &= \iint_{R \times Q} K(x, y) \phi_{Q,i}^1(y) \phi_{R,j}^2(x) \, dx \, dy \\ &= \iint_{R \cap 3Q \times Q} K(x, y) \phi_{Q,i}^1(y) \phi_{R,j}^2(x) \, dx \, dy \\ &\quad + \iint_{R \setminus (3Q) \times Q} [K(x, y) - K(x, c_Q)] \phi_{Q,i}^1(y) \phi_{R,j}^2(x) \, dx \, dy, \end{aligned}$$

and then estimate

$$|\langle T\phi_{Q,i}^1, \phi_{R,j}^2 \rangle| \lesssim \iint_{R \cap 3Q \times Q} \frac{dx \, dy}{|x - y|^d} + \iint_{R \setminus (3Q) \times Q} \frac{\ell(Q)^\alpha}{|x - c_Q|^{d+\alpha}} \, dx \, dy.$$

If  $|m|_\infty > 1$ , the first term vanishes, and estimating the second term we get

$$|\langle T\phi_{Q,i}^1, \phi_{R,j}^2 \rangle| \lesssim \frac{\ell(Q)^\alpha}{\text{dist}(Q, R)^{d+\alpha}} |Q| |R| \lesssim 2^{-k\alpha} |m|^{-d-\alpha} |Q|,$$

and then  $\|K_{R, R+\ell(R)m}^{i,j;k}\|_\infty \lesssim 2^{-k\alpha} |m|^{-d-\alpha} / |R|$ .

If  $|m|_\infty = 1$ , then the first term is nonzero (and then bounded by  $C|Q|$ ) only if  $\text{dist}(Q, R) = 0$ , while the second term is estimated by

$$\frac{\ell(Q)^\alpha |Q|}{\max\{\ell(Q), \text{dist}(Q, R)\}^\alpha}.$$

So altogether we have

$$|\langle T\phi_{Q,i}^1, \phi_{R,j}^2 \rangle| \lesssim \left(1 + \frac{\text{dist}(Q, R)}{\ell(Q)}\right)^{-\alpha} |Q|,$$

and then

$$|K_{R, R+\ell(R)m}^{i,j;k}(x, y)| \lesssim \frac{1_{R(x)}}{|R|} \sum_{\substack{Q \subseteq R+\ell(R)m \\ \ell(Q)=2^{-k}\ell(R)}} \left(1 + \frac{\text{dist}(Q, R)}{\ell(Q)}\right)^{-\alpha} 1_Q(y).$$

The number of the cubes  $Q$  with  $\text{dist}(Q, R) = n\ell(Q)$ , ( $n = 0, 1, \dots, 2^k - 1$ ), is  $\mathcal{O}(2^{k(d-1)})$ , while each of them has measure  $|Q| = 2^{-kd}|R|$ . Hence

$$\|K_{R, R+\ell(R)m}^{i,j;k}\|_{L^2(\mathbb{R}^d \times \mathbb{R}^d)}^2 \lesssim \frac{1}{|R|} \sum_{n=0}^{2^k-1} 2^{k(d-1)} \cdot (1+n)^{-2\alpha} \cdot 2^{-kd}|R| \approx 2^{-2k\alpha}. \quad \square$$

(We note that it is the last step of the previous computation that would lead to a slightly different conclusion for  $\alpha \geq \frac{1}{2}$ .)

Lemma 5.3 is enough to estimate the part of the series with  $m \neq 0$ ; indeed

$$\begin{aligned} & \sum_{m \neq 0} \sum_{k=0}^{\infty} \sum_R |\langle T \mathbb{D}_{R+\ell(R)m}^{b_1,k} f, \mathbb{D}_R^{b_2} g \rangle| \\ & \leq \sum_{m \neq 0} \sum_{k=0}^{\infty} \sum_R \sum_{i,j} \|K_{R,R+\ell(R)m}^{i,j;k}\|_2 \|\mathbb{D}_{R+\ell(R)m}^{b_1,k} f\|_2 \|\mathbb{D}_R^{b_2} g\|_2 \\ & \lesssim \sum_{m \neq 0} |m|^{-d-\alpha} \sum_{k=0}^{\infty} 2^{-\alpha k} \sum_R \|\mathbb{D}_{R+\ell(R)m}^{b_1,k} f\|_2 \|\mathbb{D}_R^{b_2} g\|_2, \end{aligned}$$

where

$$\begin{aligned} & \sum_R \|\mathbb{D}_{R+\ell(R)m}^{b_1,k} f\|_2 \|\mathbb{D}_R^{b_2} g\|_2 \\ & \leq \left( \sum_R \|\mathbb{D}_{R+\ell(R)m}^{b_1,k} f\|_2^2 \right)^{1/2} \left( \sum_R \|\mathbb{D}_R^{b_2} g\|_2^2 \right)^{1/2} \\ & = \left( \sum_Q \|\mathbb{D}_Q^{b_1} f\|_2^2 \right)^{1/2} \left( \sum_R \|\mathbb{D}_R^{b_2} g\|_2^2 \right)^{1/2} \lesssim \|f\|_2 \|g\|_2, \end{aligned}$$

and

$$\sum_{m \neq 0} |m|^{-d-\alpha} \sum_{k=0}^{\infty} 2^{-\alpha k} \lesssim 1.$$

5.2. *Nested cubes*

We are left with the part with  $m = 0$ , that is,

$$\sum_{k=0}^{\infty} \sum_R \langle T \mathbb{D}_R^{b_1,k} f, \mathbb{D}_R^{b_2} g \rangle = \sum_R \langle T \mathbb{D}_R^{b_1} f, \mathbb{D}_R^{b_2} g \rangle + \sum_R \sum_{Q \subsetneq R} \langle T \mathbb{D}_Q^{b_1} f, \mathbb{D}_R^{b_2} g \rangle.$$

**5.4 Lemma.** *We have*

$$\begin{aligned} \sum_R \sum_{Q \subsetneq R} \langle T \mathbb{D}_Q^{b_1} f, \mathbb{D}_R^{b_2} g \rangle &= \sum_Q \langle T \mathbb{D}_Q^{b_1} f, b_{Q^{a,2}}^2 \rangle \frac{\langle g \rangle_Q}{\langle b_{Q^{a,2}}^2 \rangle_Q} - \langle T f, b_{Q^0}^2 \rangle \frac{\langle g \rangle_{Q^0}}{\langle b_{Q^0}^2 \rangle_{Q^0}} \\ &+ \sum_{k=0}^{\infty} \sum_R \sum_{\substack{S \subsetneq R \\ \ell(S)=\ell(R)/2}} \sum_{j=0}^{2^d} \langle T \mathbb{D}_S^{b_1,k} f, 1_{S^c} \psi_{R,j;S}^2 \rangle \langle \mathbb{D}_{R,j}^{b_2} g \rangle_{R_j}, \end{aligned} \tag{5.5}$$

for some bounded functions  $\psi_{R,j;S}^2$ .

**Proof.** For  $Q \subsetneq R$ , let  $R_Q$  be the unique subcube of  $R$  that contains  $Q$ . Then

$$\mathbb{D}_R^{b_2} g = 1_{R_Q} \mathbb{D}_R^{b_2} g + 1_{R_Q^c} \mathbb{D}_R^{b_2} g,$$

where further (we temporarily drop the superscript 2)

$$1_{R_Q} \mathbb{D}_R^b g = (1 - 1_{R_Q^c}) \left( b_{R_Q^a} \frac{\langle g \rangle_{R_Q}}{\langle b_{R_Q^a} \rangle_{R_Q}} - b_{R^a} \frac{\langle g \rangle_R}{\langle b_{R^a} \rangle_R} \right)$$

and

$$\begin{aligned} & b_{R_Q^a} \frac{\langle g \rangle_{R_Q}}{\langle b_{R_Q^a} \rangle_{R_Q}} - b_{R^a} \frac{\langle g \rangle_R}{\langle b_{R^a} \rangle_R} \\ &= \frac{b_{R_Q^a}}{\langle b_{R_Q^a} \rangle_{R_Q}} \left( \langle g \rangle_{R_Q} - \langle b_{R^a} \rangle_{R_Q} \frac{\langle g \rangle_R}{\langle b_{R^a} \rangle_R} \right) + \left( \frac{\langle b_{R^a} \rangle_{R_Q}}{\langle b_{R_Q^a} \rangle_{R_Q}} b_{R_Q^a} - b_{R^a} \right) \frac{\langle g \rangle_R}{\langle b_{R^a} \rangle_R} \quad (5.6) \\ &= \frac{b_{R_Q^a}}{\langle b_{R_Q^a} \rangle_{R_Q}} \langle \mathbb{D}_R^b g \rangle_{R_Q} + \left( \frac{\langle b_{R^a} \rangle_{R_Q}}{\langle b_{R_Q^a} \rangle_{R_Q}} b_{R_Q^a} - b_{R^a} \right) \frac{\langle \mathbb{D}_{R,0}^b g \rangle_R}{\langle b_{R^a} \rangle_R}. \end{aligned}$$

On the last line we observed that the function in the parentheses is zero unless  $R_Q$  is a stopping cube (i.e.,  $R_Q^a = R_Q \neq R^a$ ), and thus we may replace  $g$  by  $\mathbb{D}_{R,0}^b g$  inside the average on  $R$ .

Substituting back, it follows that

$$\begin{aligned} \mathbb{D}_R^b g &= b_{R_Q^a} \frac{\langle g \rangle_{R_Q}}{\langle b_{R_Q^a} \rangle_{R_Q}} - b_{R^a} \frac{\langle g \rangle_R}{\langle b_{R^a} \rangle_R} \\ &+ 1_{R_Q^c} \left\{ \mathbb{D}_R^b g - \frac{b_{R_Q^a}}{\langle b_{R_Q^a} \rangle_{R_Q}} \langle \mathbb{D}_R^b g \rangle_{R_Q} - \left( \frac{\langle b_{R^a} \rangle_{R_Q}}{\langle b_{R_Q^a} \rangle_{R_Q}} b_{R_Q^a} - b_{R^a} \right) \frac{\langle \mathbb{D}_{R,0}^b g \rangle_R}{\langle b_{R^a} \rangle_R} \right\} \quad (5.7) \\ &= \left( b_{R_Q^a} \frac{\langle g \rangle_{R_Q}}{\langle b_{R_Q^a} \rangle_{R_Q}} - b_{R^a} \frac{\langle g \rangle_R}{\langle b_{R^a} \rangle_R} \right) + 1_{R_Q^c} \sum_{j=0}^{2^d} \psi_{R,j;R_Q}^2 \langle \mathbb{D}_{R,j}^b g \rangle_{R_j}, \end{aligned}$$

where (recalling the formula  $\mathbb{D}_R^b g = \sum_{j=0}^{2^d} \phi_{R,j} \langle \mathbb{D}_{R,j}^b g \rangle_{R_j}$ )

$$\psi_{R,j;S}^2 := \phi_{R,j} - \begin{cases} \left( \langle b_{R^a} \rangle_S / \langle b_{S^a} \rangle_S \cdot b_{S^a} - b_{R^a} \right) / \langle b_{R^a} \rangle_R, & \text{if } j = 0, \\ b_{S^a} / \langle b_{S^a} \rangle_S, & \text{if } j \geq 1 \text{ and } R_j = S, \\ 0. & \text{else} \end{cases}$$

are bounded functions.

Pairing with  $T\mathbb{D}_Q^{b_1} f$ , we obtain (changing the summation order, and observing that  $Q$  is the smallest  $R_Q$ , as well as the telescoping cancellation)

$$\begin{aligned} & \sum_Q \sum_{R \supsetneq Q} \left\langle T\mathbb{D}_Q^{b_1} f, b_{R^{a,2}}^2 \frac{\langle g \rangle_{R_Q}}{\langle b_{R^{a,2}}^2 \rangle_{R_Q}} - b_{R^{a,2}}^2 \frac{\langle g \rangle_R}{\langle b_{R^{a,2}}^2 \rangle_R} \right\rangle \\ &= \sum_Q \langle T\mathbb{D}_Q^{b_1} f, b_{Q^{a,2}}^2 \rangle \frac{\langle g \rangle_Q}{\langle b_{Q^{a,2}}^2 \rangle_Q} - \sum_Q \langle T\mathbb{D}_Q^{b_1} f, b_{Q^0}^2 \rangle \frac{\langle g \rangle_{Q^0}}{\langle b_{Q^0}^2 \rangle_{Q^0}}, \end{aligned}$$

where the first term on the right arises from  $R_Q = Q$  when  $R \supsetneq Q$  takes its minimal value, the dyadic parent of  $Q$ , while the second term arises from the maximal value  $R = Q^0$  coming from the implicit summation condition  $R \subseteq Q^0$ . Since the initial cube  $Q^0$  is defined to be a stopping cube, we have  $(Q^0)^{a,2} = Q^0$ . Recalling that  $f = \sum_Q \mathbb{D}_Q^{b_1} f$ , the second term on the right simplifies to

$$\sum_Q \langle T\mathbb{D}_Q^{b_1} f, b_{Q^0}^2 \rangle \frac{\langle g \rangle_{Q^0}}{\langle b_{Q^0}^2 \rangle_{Q^0}} = \langle Tf, b_{Q^0}^2 \rangle \frac{\langle g \rangle_{Q^0}}{\langle b_{Q^0}^2 \rangle_{Q^0}}.$$

For the other term on the right of (5.7), we introduce the auxiliary summation variable  $S := R_Q$ , regroup the summation according to the relative size of  $Q$  and  $S$ , and recall the notation

$$\mathbb{D}_S^{b_1,k} f := \sum_{\substack{Q \subseteq S \\ \ell(Q)=2^{-k}\ell(S)}} \mathbb{D}_Q^{b_1} f$$

to arrive at the asserted formula.  $\square$

The middle term on the right of (5.5) is estimated by the assumption that  $b_Q^2$  is a (strong)  $(\infty, t)$  accretive system for  $T^*$  on sparse subcubes of  $Q^0$ , namely

$$\begin{aligned} |\langle Tf, b_{Q^0}^2 \rangle| &= |\langle f, T^* b_{Q^0}^2 \rangle| \\ &= |Q^0| \left| \int_{Q^0} f \cdot T^* b_{Q^0}^2 \, dx \right| \leq |Q^0| \left( \int_{Q^0} |f|^{t'} \, dx \right)^{1/t'} \left( \int_{Q^0} |T^*(b_{Q^0}^2)|^t \, dx \right)^{1/t}, \end{aligned}$$

and hence (recalling that  $s' > t' > 1$ )

$$\begin{aligned} \left| \langle Tf, b_{Q^0}^2 \rangle \frac{\langle g \rangle_{Q^0}}{\langle b_{Q^0}^2 \rangle_{Q^0}} \right| &\lesssim |Q^0| \left( \int_{Q^0} |f|^{t'} \, dx \right)^{1/t'} \int_{Q^0} |g| \, dx \\ &\leq |Q^0| \left( \int_{Q^0} |f|^{s'} \, dx \right)^{1/s'} \left( \int_{Q^0} |g|^{s'} \, dx \right)^{1/s'} = |Q^0|^{1-2/s'} \|f\|_{s'} \|g\|_{s'}, \end{aligned}$$

which agrees with the right hand side of the bound in Proposition 2.20.

The last summand in (5.5) can be written as

$$\begin{aligned} \langle T\mathbb{D}_S^{b_1,k} f, 1_{S^c}\psi_{R,j;S}^2 \rangle \langle \mathbb{D}_{R,j}^{b_2} g \rangle_{R_j} &= \sum_{\substack{Q \subseteq S; i=0,\dots,2^d \\ \ell(Q)=2^{-k}\ell(S)}} \langle \mathbb{D}_{Q,i}^{b_1} f \rangle_{Q_i} \langle T\phi_{Q,i}^1, 1_{S^c}\psi_{R,j;S}^2 \rangle \langle \mathbb{D}_{R,j}^{b_2} g \rangle_{R_j} \\ &= \sum_{i=0}^{2^d} \iint \mathbb{D}_{S,i}^{b_1,k} f(y) \tilde{K}_{R,S}^{i,j;k}(x,y) \mathbb{D}_{R,j}^{b_2} g(x) \, dx \, dy, \quad \text{where} \\ \tilde{K}_{R,S}^{i,j;k}(x,y) &:= \sum_{\substack{Q \subseteq S \\ \ell(Q)=2^{-k}\ell(S)}} \frac{1_{Q_i}(y)}{|Q_i|} \langle T\phi_{Q,i}^1, 1_{S^c}\psi_{R,j;S}^2 \rangle \frac{1_{R_j}(x)}{|R_j|}. \end{aligned}$$

Just as in Lemma 5.3 (case  $|m|_\infty = 1$ ) we check that:

**5.8 Lemma.**

$$\|\tilde{K}_{R,S}^{i,j;k}\|_{L^2(\mathbb{R}^{2d})} \lesssim 2^{-k\alpha}, \quad \alpha < \frac{1}{2}.$$

**Proof.** We first estimate the coefficients  $\langle T\phi_{Q,i}^1, 1_{S^c}\psi_{R,j;S}^2 \rangle$ . Using the mean zero property of  $\phi_{Q,i}^1$ ,

$$\begin{aligned} \langle T\phi_{Q,i}^1, 1_{S^c}\psi_{R,j;S}^2 \rangle &= \iint_{S^c \times Q} K(x,y) \phi_{Q,i}^1(y) \psi_{R,j;S}^2(x) \, dx \, dy \\ &= \iint_{S^c \cap 3Q \times Q} K(x,y) \phi_{Q,i}^1(y) \psi_{R,j;S}^2(x) \, dx \, dy \\ &\quad + \iint_{S^c \setminus (3Q) \times Q} [K(x,y) - K(x,c_Q)] \phi_{Q,i}^1(y) \psi_{R,j;S}^2(x) \, dx \, dy. \end{aligned}$$

By the boundedness of both  $\phi_{Q,i}^1$  and  $\psi_{R,j;S}^2$ , and the fact that  $Q \subseteq S \subseteq R$  so that  $S^c \subseteq Q^c$ , we have

$$|\langle T\phi_{Q,i}^1, 1_{S^c}\psi_{R,j;S}^2 \rangle| \lesssim \iint_{(3Q \setminus Q) \times Q} \frac{dx \, dy}{|x-y|^d} + \iint_{S^c \setminus (3Q) \times Q} \frac{\ell(Q)^\alpha}{|x-c_Q|^{d+\alpha}} \, dx \, dy.$$

The first term is bounded by  $C|Q|$  and the second is estimated as

$$\begin{aligned} \int_{S^c \setminus (3Q)} \frac{\ell(Q)^\alpha |Q|}{|x-c_Q|^{d+\alpha}} \, dx &\approx \int_{\max\{\ell(Q), \text{dist}(Q, S^c)\}}^\infty \frac{\ell(Q)^\alpha |Q|}{t^{d+\alpha}} t^{d-1} \, dt \\ &\approx \frac{\ell(Q)^\alpha |Q|}{\max\{\ell(Q), \text{dist}(Q, S^c)\}^\alpha}. \end{aligned}$$



So altogether we have

$$|\langle T\phi_{Q,i}^1, 1_{S^c}\psi_{R,j;S}^2 \rangle| \lesssim \left(1 + \frac{\text{dist}(Q, S^c)}{\ell(Q)}\right)^{-\alpha} |Q|,$$

and then

$$|\tilde{K}_{R,S}^{i,j;k}(x,y)| \lesssim \frac{1_R(x)}{|R|} \sum_{\substack{Q \subseteq S \\ \ell(Q)=2^{-k}\ell(S)}} \left(1 + \frac{\text{dist}(Q, S^c)}{\ell(Q)}\right)^{-\alpha} 1_Q(y).$$

The number of the cubes  $Q$  with  $\text{dist}(Q, S^c) = n\ell(Q)$ , ( $n = 0, 1, \dots, 2^k - 1$ ), is  $\mathcal{O}(2^{k(d-1)})$ , while each of them has measure  $|Q| = 2^{-kd}|S|$ . Recalling also that  $\ell(S) = \ell(R)/2$ , we have

$$\begin{aligned} \|\tilde{K}_{R,S}^{i,j;k}\|_{L^2(\mathbb{R}^d \times \mathbb{R}^d)}^2 &\lesssim \frac{1}{|R|} \sum_{n=0}^{2^k-1} 2^{k(d-1)} \cdot (1+n)^{-2\alpha} \cdot 2^{-kd}|S| \\ &= 2^{-k} \sum_{n=1}^{2^k} n^{-2\alpha} \approx 2^{-2k\alpha}, \quad \alpha < \frac{1}{2}. \quad \square \end{aligned}$$

With the help of Lemma 5.8, the last sum on the right of (5.5) can now be estimated by

$$\begin{aligned} &\sum_{k=0}^{\infty} \sum_R \sum_{\substack{S \subseteq R \\ \ell(S)=\ell(R)/2}} \sum_j |\langle T\mathbb{D}_S^{b_1,k} f, 1_{S^c}\psi_{R,j;S}^2 \rangle \langle \mathbb{D}_{R,j}^{b_2} g \rangle_{R_j}| \\ &\lesssim \sum_{k=0}^{\infty} \sum_R \sum_{\substack{S \subseteq R \\ \ell(S)=\ell(R)/2}} \sum_{j,i} 2^{-k\alpha} \|\mathbb{D}_{S,i}^{b_1,k} f\|_2 \|\mathbb{D}_{R,j}^{b_2} g\|_2 \\ &\lesssim \sum_{k=0}^{\infty} \sum_R \sum_{j,i} 2^{-k\alpha} \|\mathbb{D}_{R,i}^{b_1,k+1} f\|_2 \|\mathbb{D}_{R,j}^{b_2} g\|_2 \lesssim \|f\|_2 \|g\|_2, \end{aligned}$$

which completes this part of the estimate.

### 5.3. The paraproduct

The other part in (5.5), which still requires attention, is

$$\begin{aligned} \sum_R |\langle T\mathbb{D}_R^{b_1} f, b_{R^{a,2}}^2 \rangle \frac{\langle g \rangle_R}{\langle b_{R^{a,2}}^2 \rangle_R}| &\lesssim \sum_R | \langle (\mathbb{D}_R^{b_1})^2 f + \omega_R^1 \mathbb{D}_{R,0}^{b_1} f, T^* b_{R^{a,2}}^2 \rangle \langle g \rangle_R | \\ &\leq \sum_R |\langle \mathbb{D}_R^{b_1} f, (\mathbb{D}_R^{b_1})^* T^* b_{R^{a,2}}^2 \rangle \langle g \rangle_R| + \sum_R |\langle \mathbb{D}_{R,0}^{b_1} f, \omega_R^1 T^* b_{R^{a,2}}^2 \rangle \langle g \rangle_R|. \end{aligned}$$

Using estimates of the type

$$\sum_R |\langle \phi_R, \psi_R \rangle| \leq \sum_R \int |\phi_R| |\psi_R| \leq \int \left( \sum_R |\phi_R|^2 \right)^{1/2} \left( \sum_R |\psi_R|^2 \right)^{1/2}$$

and then Hölder’s inequality, we may continue with

$$\begin{aligned} &\leq \left\| \left( \sum_R |\mathbb{D}_R^{b_1} f|^2 \right)^{1/2} \right\|_{L^{s'}} \left\| \left( \sum_R |(\mathbb{D}_R^{b_1})^* T^* b_{R^{a,2}}^2 \langle g \rangle_R|^2 \right)^{1/2} \right\|_{L^s} \\ &\quad + \left\| \left( \sum_R |\mathbb{D}_{R,0}^{b_1} f|^2 \right)^{1/2} \right\|_{L^{s'}} \left\| \left( \sum_R |\omega_R^1 T^* b_{R^{a,2}}^2 \langle g \rangle_R|^2 \right)^{1/2} \right\|_{L^s} \\ &\lesssim \|f\|_{L^{s'}} \|g\|_{L^s} \sup_S |S|^{-1/s} \left\| \left( \sum_{R \subseteq S} |(\mathbb{D}_R^{b_1})^* T^* b_{R^{a,2}}^2|^2 \right)^{1/2} \right\|_{L^s} \\ &\quad + \|f\|_{L^{s'}} \|g\|_{L^s} \sup_S |S|^{-1/s} \left\| \left( \sum_{R \subseteq S} |\omega_R^1 T^* b_{R^{a,2}}^2|^2 \right)^{1/2} \right\|_{L^s}, \end{aligned}$$

where in the last step we used Proposition 5.2 and the following  $L^s$  version of the Carleson embedding theorem from [10, Theorem 8.2]:

**5.9 Proposition.**

$$\left\| \left( \sum_R |\theta_R \langle g \rangle_R|^2 \right)^{1/2} \right\|_{L^s} \lesssim \|g\|_{L^s} \sup_S |S|^{-1/s} \left\| \left( \sum_{R \subseteq S} |\theta_R|^2 \right)^{1/2} \right\|_{L^s}, \quad s \in (1, 2].$$

It remains to estimate the two Carleson norms above.

**5.10 Lemma.**

$$\left\| \left( \sum_{R \subseteq S} |(\mathbb{D}_R^{b_1})^* T^* b_{R^{a,2}}^2|^2 \right)^{1/2} \right\|_{L^s} \lesssim |S|^{1/s}.$$

**Proof.** We reorganise the summation as

$$\sum_{R \subseteq S} = \sum_{\substack{R \subseteq S \\ R^{a,2} = S^{a,2}}} + \sum_{\substack{P \subseteq S \\ P^{a,2} = P}} \sum_{\substack{R \subseteq P \\ R^{a,2} = P}} = \sum_{k=0}^{\infty} \sum_{P \in \mathcal{P}^k(S)} \sum_{\substack{R \subseteq P \\ R^{a,2} = P^{a,2}}}, \tag{5.11}$$

where  $\mathcal{P}^0(S) := \{S\}$ ,  $\mathcal{P}^1(S)$  consists of the maximal  $P \subsetneq S$  with  $P = P^{a,2}$ , and recursively  $\mathcal{P}^{k+1}(S) := \bigcup_{G \in \mathcal{P}^k(S)} \mathcal{P}^1(G)$ . Since all  $P \in \mathcal{P}^k(S)$  are disjoint for a fixed  $k$ , we get

$$\begin{aligned} & \left\| \left( \sum_{R \subseteq S} |(\mathbb{D}_R^{b_1})^* T^* b_{R^{a,2}}^2|^2 \right)^{1/2} \right\|_{L^s} \\ & \leq \sum_{k=0}^{\infty} \left( \sum_{P \in \mathcal{P}^k(S)} \left\| \left( \sum_{\substack{R \subseteq P \\ R^{a,2} = P^{a,2}}} |(\mathbb{D}_R^{b_1})^* T^* b_{P^{a,2}}^2|^2 \right)^{1/2} \right\|_{L^s} \right)^{1/s}, \end{aligned}$$

and then by the square function estimate from Proposition 5.2,

$$\lesssim \sum_{k=0}^{\infty} \left( \sum_{P \in \mathcal{P}^k(S)} \|1_P T^* b_{P^{a,2}}^2\|_{L^s} \right)^{1/s} =: A.$$

Next, using Hölder’s inequality (recall that  $s' > t'$ , hence  $s < t$ ) and the assumption of Proposition 2.20 that  $b_Q^2$  is  $(\infty, t)$  accretive for  $T^*$  on the stopping cubes,

$$\|1_P T^* b_{P^{a,2}}^2\|_{L^s} \leq |P|^{1/s-1/t} \|1_P T^* b_{P^{a,2}}^2\|_{L^t} \lesssim |P|^{1/s}.$$

Therefore

$$A \lesssim \sum_{k=0}^{\infty} \left( \sum_{P \in \mathcal{P}^k(S)} |P| \right)^{1/s} \leq \sum_{k=0}^{\infty} \left( (1-\tau)^{k-1} |S| \right)^{1/s} \lesssim |S|^{1/s}. \quad \square$$

**5.12 Lemma.**

$$\left\| \left( \sum_{R \subseteq S} |\omega_R^1 T^* b_{R^{a,2}}^2|^2 \right)^{1/2} \right\|_{L^s} \lesssim |S|^{1/s}.$$

**Proof.** This is proven by a similar splitting but, instead of the square function estimate for  $(\mathbb{D}_R^{b_1})^*$ , using

$$\begin{aligned} \left\| \left( \sum_{R \subseteq P} |\omega_R^1 T^* b_{P^{a,2}}^2|^2 \right)^{1/2} \right\|_{L^s} &= \left\| \left( \sum_{R \subseteq P} |\omega_R^1|^2 \right)^{1/2} |T^* b_{P^{a,2}}^2| \right\|_{L^s} \\ &\leq \left\| \left( \sum_{R \subseteq P} |\omega_R^1|^2 \right)^{1/2} \right\|_{L^u} \|1_P T^* b_{P^{a,2}}^2\|_{L^t}, \quad \frac{1}{s} = \frac{1}{u} + \frac{1}{t}. \end{aligned}$$

Then, using a decomposition as in (5.11) but in terms of the stopping cubes on the  $b_1$ -side rather than the  $b_2$ -side,

$$\begin{aligned} \left\| \left( \sum_{R \subseteq P} |\omega_R^1|^2 \right)^{1/2} \right\|_{L^s} &\lesssim \left\| \left( \sum_{\substack{R \subseteq P \\ R^{a,1} = R}} 1_R \right)^{1/2} \right\|_{L^u} \leq \sum_{k=0}^{\infty} \left( \sum_{G \in \mathcal{P}^k(P)} |G| \right)^{1/u} \\ &\leq \sum_{k=0}^{\infty} \left( (1-\tau)^{k-1} |P| \right)^{1/u} \lesssim |P|^{1/u}. \end{aligned}$$

Since  $\|1_P T^* b_{P^{a,2}}^2\|_{L^t} \lesssim |P|^{1/t}$ , we get the asserted bound  $|P|^{1/u} |P|^{1/t} = |P|^{1/s}$ .  $\square$

5.4. *The diagonal*

It only remains to estimate

$$\begin{aligned} \sum_R |\langle T \mathbb{D}_R^{b_1} f, \mathbb{D}_R^{b_2} g \rangle| &\leq \sum_R \sum_{i,j=1}^{2^d} |\langle T(1_{R_i} \mathbb{D}_R^{b_1} f), 1_{R_j} \mathbb{D}_R^{b_2} g \rangle| \\ &\lesssim \sum_R \sum_{i,j:i \neq j} \|1_{R_i} \mathbb{D}_R^{b_1} f\|_2 \|1_{R_j} \mathbb{D}_R^{b_2} g\|_2 \\ &\quad + \sum_R \sum_j |\langle T(1_{R_j} \mathbb{D}_R^{b_1} f), 1_{R_j} \mathbb{D}_R^{b_2} g \rangle|, \end{aligned}$$

where the unequal subcubes were estimated by Hardy’s inequality, and this part is readily bounded by  $\|f\|_2 \|g\|_2$ . For the final part, we write, as in (5.6),

$$1_{R_j} \mathbb{D}_R^b f = \frac{1_{R_j} b_{R_j^a}}{\langle b_{R_j^a} \rangle_{R_j}} \langle \mathbb{D}_R^b f \rangle_{R_j} + 1_{R_j} \left( \frac{\langle b_{R^a} \rangle_{R_j}}{\langle b_{R_j^a} \rangle_{R_j}} b_{R_j^a} - b_{R^a} \right) \frac{\langle \mathbb{D}_{R,0}^b f \rangle_R}{\langle b_{R^a} \rangle_R},$$

and similarly for  $1_{R_j} \mathbb{D}_R^{b_2} g$ . Thus

$$\begin{aligned} &|\langle T(1_{R_j} \mathbb{D}_R^{b_1} f), 1_{R_j} \mathbb{D}_R^{b_2} g \rangle| \\ &\lesssim \left( \sum_{i,h \in \{0,j\}} |\langle T(1_{R_j} b_{R_i^{a,1}}^1), 1_{R_j} b_{R_h^{a,2}}^2 \rangle| \right) \left( \sum_{i,h \in \{0,j\}} |\langle \mathbb{D}_{R,i}^{b_1} f \rangle_{R_i}| |\langle \mathbb{D}_{R,h}^{b_2} g \rangle_{R_h}| \right), \end{aligned}$$

and it remains to check that the first term is dominated by  $|R|$ , for then e.g.

$$|R|^{1/2} |\langle \mathbb{D}_{R,i}^{b_1} f \rangle_{R_i}| \lesssim \| \mathbb{D}_{R,i}^{b_1} f \|_2,$$

and it is easy to conclude by Proposition 5.2.

If  $i = j$ , then

$$|\langle T(1_{R_j} b_{R_i^{a,1}}^1), 1_{R_j} b_{R_h^{a,2}}^2 \rangle| \leq \|1_{R_j} T(1_{R_j} b_{R_j^{a,1}}^1)\|_1 \|b_{R_h^{a,2}}^2\|_\infty \lesssim |R_j| \cdot 1 \lesssim |R|$$

directly from the definition of strong accretivity (with  $Q' = R_j$  and  $Q = R_j^{a,1}$ ).

If  $i = 0$ , then

$$\begin{aligned} |\langle T(1_{R_j} b_{R_i^{a,1}}^1), 1_{R_j} b_{R_h^{a,2}}^2 \rangle| &= |\langle T(1_R b_{R^{a,1}}^1), 1_{R_j} b_{R_h^{a,2}}^2 \rangle - \langle T(1_{R \setminus R_j} b_{R^{a,1}}^1), 1_{R_j} b_{R_h^{a,2}}^2 \rangle| \\ &\lesssim \|1_R T(1_R b_{R^{a,1}}^1)\|_1 \|b_{R_h^{a,2}}^2\|_\infty + \|1_{R \setminus R_j} b_{R^{a,1}}^1\|_2 \|1_{R_j} b_{R_h^{a,2}}^2\|_2 \\ &\lesssim |R| \cdot 1 + |R|^{1/2} |R|^{1/2} \lesssim |R|, \end{aligned}$$

where we used Hardy's inequality for the second term and the strong accretivity (with  $Q' = R$  and  $Q = R^{a,1}$ ) to the first one.

This completes the proof of the “baby  $Tb$  theorem”, Proposition 2.20. Since this was the last missing ingredient, the proofs of the Main Theorem 2.13 and Theorem 2.10 are also finished.

## 6. Concluding remark

We have completed the proofs of Theorems 2.10 and 2.13, dealing with the boundedness of singular integral operators on  $\mathbb{R}^d$  with the Lebesgue measure. Using the dyadic cubes of Christ [4] in place of the Euclidean dyadic cubes, these results extend to a metric space with a doubling measure without difficulty. In particular, a Hardy inequality is also valid for Christ's dyadic cubes, as observed by Auscher and Routin [2]. A version for non-doubling measures is recently due to Martikainen, Mouroglou and Tolsa [14], but only for antisymmetric kernels.

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