From QCD to Neutron Stars and Back
Probing the Fundamental Properties of Dense Matter

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DOCTORAL DISSERTATION

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So che molti diranno questa essere opra inutile.

— Leonardo da Vinci
Abstract

The first theoretical attempts to study neutron stars — the immensely dense remnants of massive stars — were conducted in the 1930s, but it took nearly 40 years for the first one to be detected. Ever since, these fascinating objects have been the subject of significant interest; both nuclear and particle physicists and astronomers have tried to understand their micro- and macro-scale properties. Regardless, the composition of neutron-star cores has, to large extent, remained unknown. This is somewhat surprising because the underlying microscopic theory — quantum chromodynamics — has been available for several decades.

Because it is impossible to describe the structure of neutron stars using current ab initio techniques, other kinds of approaches need to be exploited. In this thesis, state-of-the-art nuclear and particle theory calculations were utilized to restrict the dense-matter equation of state at low and high densities, respectively. Between these two limits, there exists a region, where the equation of state needs to be approximated by employing various interpolation tools. Our analysis has revealed that both the mass-radius curve of neutron stars and the underlying equation of state can be efficiently constrained by making use of the latest astronomical observations — such as the tidal-deformability measurement from the gravitational-wave event GW170817. We have also shown that there exists convincing evidence that the most massive neutron stars have deconfined quark matter in their cores assuming that the equation of state is not very extreme.

In addition, we have taken some first steps towards the realistic implementation of the gauge/gravity duality in neutron-star physics. This method allows one to investigate strongly coupled quantum systems using simpler gravity-based setups. This approach has already led to several promising results in many fields, from condensed matter physics to the study of quark-gluon plasma. It is, therefore, a worthy candidate to become a fruitful framework to examine neutron-star physics, complementing the current nuclear and particle theory methods.

It is expected that the improvement of theoretical calculations together with new, more precise observations will likely resolve the equation of state within a decade or two. Moreover, this progress will eventually disclose, whether quark matter resides inside the heaviest neutron stars in existence. The development of holographic tools may also open up new and powerful ways to study matter at its most extreme densities.
Acknowledgments

Most notably, I want to thank my supervisors Aleksi Kurkela and Aleksi Vuorinen for their help and support. It is vitally important for a young scientist to get assistance as well as freedom to express oneself — but within limits, of course. In my opinion, Aleksis have succeeded to provide an excellent environment for me to evolve and flourish. And not to mention, they have offered me this great opportunity to study fascinating physics.

To become a better researcher, one needs both guidance and external, critical feedback. Therefore, I am particularly thankful that Jürgen Schaffner-Bielich has been willing to be my opponent who reviews this academic dissertation. Moreover, I cannot help but express my gratitude to the pre-examiners Eduardo S. Fraga and Heikki Mäntysaari whose feedback has been valuable. I also wish to thank the members of my grading committee: Mark Hindmarsh and Kimmo Tuominen.

In (theoretical) physics, modern research is usually not done by individual scientists but by a cooperative team of specialists. Hence, I cannot exaggerate the role of my collaborators! Every single research project has taught me something new and useful, from physical phenomena to self-criticism. I would especially like to thank Niko Jokela because his perpetual demand to understand physics behind all the calculations has not only made me a better physicist but improved my general analytics skills as well. Likewise, I gratefully acknowledge the crucial help provided by Tyler Gorda and Joonas Näätänen.

Although the aid of mentors and collaborators is essential, I could not have finished this thesis without the financial support from the Finnish Cultural Foundation and my supervisor Aleksi Vuorinen. This has made it possible for me to work as a full-time scientist, and I have really enjoyed this wonderful chance even though the ride has occasionally been bumpy.

I maintain that a person needs to have free-time activities to be fully functional day in, day out. Thus, I believe that I would not have stayed sane without my friends and my fellow doctoral students. Many thanks to all of them! However, I would like to separately thank the 11:30 lunch group and my colleagues from the infamous office A313 — namely Arttu, Eemeli, Jarkko, Jere, Joonas, Juha-Matti, Kalle, Matti, Miika, Sara, Tommi, Tuomas, and Vera-Maria. In addition, my friend Ville deserves a special mention. Our monthly tea session has always been one of the highlights of a month. Many thanks to the members of Kaljakerho as well.

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List of included papers

This thesis is based on the following publications:

I Gravitational-wave constraints on the neutron-star-matter equation of state
   E. Annala, T. Gorda, A. Kurkela, and A. Vuorinen

II Evidence for quark-matter cores in massive neutron stars

III Holographic compact stars meet gravitational wave constraints
   E. Annala, C. Ecker, C. Hoyos, N. Jokela, D. Rodríguez Fernández, and A. Vuorinen

In all the papers, the authors are listed alphabetically according to the particle physics convention.

Author’s contribution

I The author updated the used code — originally written by A. Kurkela — together with
   A. Kurkela and T. Gorda. Furthermore, the author also contributed by coanalyzing the
   interpolation data with the other coauthors of the article.

II The preliminary study with bitropic equations of state was conducted by the author unac-
   companied. Moreover, the author wrote the code that generates and analyses spectrally in-
   terpolated equations of state used in the final work. The author also substantially assisted
   the general analysis, especially by collecting and inspecting realistic hadronic neutron-star
   models.

III The numerical calculations of the macroscopic properties of the found compact-star sol-
  utions were independently carried out by the author. These included solving the mass-
   radius relationships and checking the stability of the given solutions as well as generating
   the corresponding I-Love-Q relations.

In addition, the author took part in writing and revising all publications.
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Chapter 1

Introduction

Neutron stars — the remnant cores of massive stars [1] — are some of the most fascinating objects in the Universe. Excluding black holes, they are the most compact celestial bodies being even denser than atomic nuclei [1]. A typical neutron star is a bit more massive than the Sun having a mass of around 1.4 solar masses (\( M_\odot \)) [2, 3] and the corresponding radius is approximately twelve kilometers (see e.g. [4, 5] or articles I, II). The first gravitational-wave observation of two merging neutron stars [6] and the corresponding electromagnetic counterpart [7] started a new era in the field of physics and astronomy in 2017. This means that novel information about the macroscopic and the microscopic properties of neutron stars is now available. These data have not only improved our comprehension of neutron-star physics but also broadened our understanding of the strong interaction — one of the four fundamental forces of Nature.

A neutron star is a battlefield where gravity fights against the strong interaction relentlessly. That is to say, attractive gravity is trying to squeeze a neutron star into a black hole, but the pressure generated by degenerate neutrons together with repulsive strong force prevents this from happening — as long as densities are not too high. For most of their lives, neutron stars are relatively cold systems — particularly from the perspective of nuclear and particle physics — with temperature \( T \lesssim 10^8 \text{ K} \sim 10 \text{ keV} \). [1, 8] Therefore, they are ideal — and maybe even the only — places to scope the phase diagram of strongly-interacting matter with high density and low temperature.

The fundamental theoretical framework to study the strong force is quantum chromodynamics (QCD). Starting from the well-known Lagrangian, one should be able to solve the equation of state (EoS) for dense QCD matter at zero temperature. However, this task is interesting \textit{per se} since the lack of suitable first-principle tools forces scientists to look for other kinds of approaches. Fortunately, perturbation theory (e.g. [9–12]) can, for example, be used to probe the EoS in some density regions but the interval corresponding to the densities of neutron-star cores is still unknown.

The first object of this thesis is to investigate this undetermined part of the zero-temperature phase diagram of QCD. Simultaneously, some characteristic features of neutron stars are inspected because these two topics are closely connected. Numerous research articles have for
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long studied these themes; for instance, the upper limit of the neutron-star maximum mass (e.g. [13–19]) and the matter content of neutron-star cores (e.g. [20–24]) have been under intense scrutiny. To be more specific, this topic has three focus points that can be phrased as questions:

i) How does the knowledge about the high-density behavior of the underlying EoS restrict neutron-star properties?

ii) How do astronomical observations constrain the general shape of the dense-matter EoS at zero temperature?

iii) Do neutron stars contain deconfined quark matter in their cores?

In order to be able to answer these questions, we need to choose a suitable approach to estimate the unknown part of the EoS. Typically, either model calculations (e.g. [25–30]) or extrapolation techniques (see e.g. [31–35]) have been used to fulfill this demand. In articles I and II, an unorthodox — but not totally unheard of (see [4, 36]) — strategy has been adopted: interpolation between familiar low- and high-density limits. In this way, we can take advantage of the state-of-the-art information of high-energy physics — i.e. perturbative-QCD (pQCD) calculations [11, 12] — to probe the EoS of cold and dense matter. Furthermore, we are also able to investigate the key properties of neutron stars.

In this thesis, we will also examine the possibility to replace the conventional pQCD approach with a holographic one where computations are performed using the gauge/gravity duality (see e.g. [37]). In this manner, challenging microphysical calculations can be carried out using a simpler gravity-based system. The greatest benefit of this method is its first-principle nature which allows to determine the wanted EoS effectively. Although the gauge/gravity duality for QCD is not known at the moment, this approach has shown promising results in several fields of physics (see e.g. [38–40]). This development has led to an increasing interest in exploitation of the duality in neutron-star research in recent years (see e.g. [41–47]).

One of such studies has been included in this thesis (paper III). This article considers a simple holographic model that describes the quark-matter EoS at zero temperature. This model has been used to compute several macroscopic observables that are then compared with known observational and theoretical evidence. For instance, one especially interesting feature is the I-Love-Q relationship (see [48–50]) which is believed to connect several important neutron-star variables together EoS-insensitively.

This thesis is divided into six chapters, and the basics of neutron-star physics will be discussed first (Chapter 2). This includes a brief review of the structure of neutron stars, and several important quantities describing neutron stars are introduced as well. In contrast, Chapter 3 contains foundational information about the strong interaction dealing with topics from the QCD phase diagram to the usage of the holographic duality in the study of QCD. In Chapter 4, the essential interpolation tools are presented, and the major findings of the thesis are summarized
in Chapter 5. Finally, this dissertation will be wrapped up with concluding remarks and future prospects in Chapter 6.

**Notation**

Throughout this work, the so-called natural units are used, unless otherwise stated. This means that the speed of light in vacuum $c$, the reduced Planck constant $\hbar$, and the Boltzmann constant $k_B$ are set to unity. Likewise, the permittivity of free space $\varepsilon_0$ is fixed so that $4\pi\varepsilon_0 = 1$. Moreover, the mostly-minus, i.e. $(+, -, -, -)$, sign convention for the metric tensor will be used.
Chapter 2

Neutron Stars

Neutron stars† are extremely compact objects with average densities close to the typical value of a heavy atomic nucleus, \( n_s \approx 0.16 \text{ baryons/fm}^3 \). However, it has been hypothesized that the density can even be ten times larger at the center of the maximally-massive neutron star. On the other hand, the (solid) surface region of a neutron star is notably thinner, around \( 10^{-16} \text{ fm}^{-3} \). [1] Due to this enormous density difference, neutron stars serve as laboratories of several microphysical phenomena, or one may see them as femtoscopes — macroscopic tools that can be used to probe femto-scale phenomena.

Historically, the first person to suggest that neutron-star-like objects could exist was L. Landau already in 1931 [51]. His proposal was even before the discovery of the neutron that happened a year later [52, 53]. After this finding, W. Baade and F. Zwicky formulated a similar kind of — but more realistic — neutron-star hypothesis [54]. (See [55] for background information.) However, it took over 30 more years to detect the first neutron star [56]. This observation was, nevertheless, so groundbreaking that A. Hewish — one of the cofinders — coreceived the 1974 Nobel Prize in Physics for the discovery [57].

It has been estimated that our home galaxy alone, the Milky Way, contains a billion neutron stars [58]. Most of the observed neutron stars are pulsars — highly magnetized rotating neutron stars [59]. The name pulsar derives from the fact that these objects emit pulsing electromagnetic radiation, i.e. they act like lighthouses. The emission of photons itself is generated by fast-moving charged particles in the magnetic field while the pulsation is a consequence of the fact that the rotational and magnetic axes are not parallel. [1, 60] To date, about 2,800 pulsars have been found [61, 62], and hence, it is not by any means a surprise that the first detected neutron star [56] is a pulsar.

Even in a typical case, the strength of the magnetic field of a pulsar is notably strong — about \( 10^7 \) to \( 10^9 \text{ T} \), or about 0.01 to 1 MeV. This is a consequence of Gauss’s law for magnetism which ensures that the magnetic flux conserves. In some extreme situations, the magnetic field can be

†In this work, the term neutron star has been used to describe both purely hadronic objects and ones with exotic cores. The latter class is often known as hybrid stars, and in this thesis, this term mainly refers to neutron stars with quark cores and hadronic crusts.
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as high as $10^{10}$ to $10^{11}$ T, or 10 to 100 MeV$^2$, and these special manifestations of pulsars are known as magnetars. [63] On the other hand, the rotational speeds of pulsars are also huge. This is because neutron stars are supernova remnants and angular momentum is conserved during the collapse of the core. The utmost examples of such neutron stars are millisecond pulsars whose rotational periods are of the order of one millisecond. [1] The highest known spin frequency is 716 Hz [64] but it has been estimated that submillisecond pulsars could also exist [1].

Neutron stars are part of the so-called compact-star class that contains extremely dense celestial bodies. Currently, this group holds two other kinds of objects: black holes and white dwarfs.∗ [60] All of these bodies† are the end products of stellar evolution. In other words, when a (main-sequence) star runs out of fuel — hydrogen, it will gravitationally collapse forming these super-dense objects. If the initial star is light ($M_{\text{init}} \lesssim 8M_\odot$), the hydrogenless configuration will first turn into a red giant creating a white dwarf in time. A more massive primary star will produce a red supergiant that will eventually explode as a core-collapse supernova. It is believed that if $M_{\text{init}} \lesssim 25M_\odot$, then the core of the exploded supergiant will not collapse into a black hole, but it will form a (proto-)neutron star‡ instead. [8, 65]

In addition to the known compact objects, some exotic and hypothetical compact-star candidates have also been proposed. Maybe the most well-known ones are quark stars that consist entirely of stable quark matter. Hence, strange stars, the most studied subtype of quark stars, are made of strange-quark matter — containing deconfined up, down, and strange quarks. This kind of matter might be a more stable state than normal nuclear matter as the famous hypothesis by Bodmer [66] and Witten [67] suggests. It has also been proposed that a strange star might need to have a thin nuclear crust [68]. Moreover, multiple research articles (see e.g. [28, 29, 69–75]) have investigated so-called twin-star configurations. If one considers certain dense-matter EoS models (with or without a strong first-order phase transition), one may encounter a new stable branch — known as the third family — after the neutron-star one (see e.g. Fig. 18 of [76]). These hybrid-star-like bodies — twin stars — are very alike to neutron stars having similar masses and radii [76].

In this chapter, different properties of neutron stars will be explored. This is important because these features can be used to estimate the EoS of QCD at zero temperature together with interpolation techniques. At first, some known details about the neutron-star structure will be discussed, and the relationship between the dense-matter EoS and the mass-radius relation of neutron stars will be examined. Second, the tidal response — the tidal deformability — of a neutron star will be presented. This quantity is especially interesting because it provides a way to probe the interior of a neutron star via gravitational waves. Furthermore, Section 2.3

∗A black hole is defined as an area in the spacetime from which nothing can escape due to its extreme density. By contrast, white dwarfs and neutron stars are similar because their masses are about the same. However, the radius of a white dwarf is hundreds of times larger. Additionally, white dwarfs are held together by the electron degeneration pressure — instead of the neutron degeneration pressure and repulsive strong force. [60]

†One should notice that black holes have other possible formation pathways as well.

‡A proto-neutron star is a hot and short-lived initial state of a neutron star formed after the collapse. This kind of object is opaque for neutrinos unlike its end product. [1]
2.1 Structure

As shown in Fig. 2.1, a neutron star consists of three major parts: the atmosphere, crust, and core. The outermost part is the atmosphere which is made up of hot plasma that forms a layer, up to several centimeters thick, around the solid surface (see e.g. the review of [77]). Beneath this region, there exists a well-understood solid crust and a much bigger — but still uncharted — (liquid) core. [1] In terms of neutron-star masses and radii, the core plays the most important role (see Fig. 2.1). Hence, it is essential to figure out its structure before we can truly comprehend the nature of neutron stars.

In this section, we first describe the structure equations derived from general relativity (GR). These equations are used to connect the EoS with the macroscopic structure of neutron stars. This analysis is followed by a discussion of the layer structure of a neutron star.

Figure 2.1: *Left panel:* Rough schematic structure of a heavy neutron star with the main subregions. An estimation of the thicknesses of the main layers has been included as well. Note that the illustration is not to scale. *Right panel:* Pressure (blue line) and mass (red) profiles of a maximally-massive neutron star with total (gravitational) mass of about 2.3 solar masses and a radius of 11.3 kilometers. These curves are plotted against the distance from the center of the star and the corresponding baryon number density. The vertical, dotted line represents the crust-core interface located at around $0.5n_s \approx 0.08/\text{fm}^3$. Secondly, the solid line demonstrates the density of $2n_s \approx 0.32/\text{fm}^3$ which is often associated with the starting point of the inner core.

is a short introduction to universal relations — EoS-insensitive connections between various macroscopic neutron-star variables. Finally, the masses and radii of neutron stars are briefly inspected in the last two sections of this chapter.
2.1.1 Tolman–Oppenheimer–Volkoff equations

Even though neutron stars can rotate with extremely high frequencies, it is often enough to examine a static solution where spherical symmetry applies. We can also further assume that neutron stars are made of a perfect fluid — an idealized model with zero viscosity and thermal conductivity. This simple approach allows to describe the properties of the fluid straightforwardly using the pressure $p$ and energy density $\varepsilon$ alone. These resources help us derive the structure equations — better known as the Tolman–Oppenheimer–Volkoff (TOV) equations — using GR [78].

Starting from a general static, spherically symmetric metric we may formulate the line element
\[ ds^2 = A(r)dt^2 - B(r)dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2, \]
where $t$ is the time coordinate while $r$, $\theta$, and $\phi$ denote the spherical spatial coordinates. The functions $A$ and $B$ are defined to be positive. Using the line element, we may calculate the Einstein tensor
\[ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R, \]
where $R_{\mu\nu}$ is the Ricci tensor, $R$ is the scalar curvature, and $g_{\mu\nu}$ is the spacetime metric given by Eq. (2.1). It can be shown that the diagonal elements of the Einstein tensor are the only nontrivial ones in the case of Eq. (2.1). For our purposes, the first two of them are the most useful ones
\[ G_{tt} = \frac{A}{B^2r^2} \left( r \frac{dB}{dr} + B^2 - B \right), \quad G_{rr} = \frac{1}{A^2r^2} \left( r \frac{dA}{dr} - A B + A \right). \]

In GR, the Einstein field equations are used to describe relations between the matter content and the structure of spacetime:
\[ G_{\mu\nu} = 8\pi G T_{\mu\nu}. \]
Here, $G$ is the gravitational constant with the value of $6.70883(15) \times 10^{-39}$ GeV$^{-2}$ [80] and $T_{\mu\nu}$ is the energy-momentum tensor. If we examine the outside of a spherical object where the space is empty, the energy-momentum tensor is equal to zero. In other words, the Einstein tensor is also zero, and hence,
\[ A(r) = C \left( 1 + \frac{D}{r} \right), \quad B(r) = \left( 1 + \frac{D}{r} \right)^{-1}, \quad \text{if } r \geq R. \]
Here, $R$ is the radius of the spherical star. Demanding that the metric is asymptotically flat and that the outcome agrees with known Newtonian results, one finds out that $C = 1$ and $D = -2GM$. Here, $M$ is the (gravitational) mass of the star.

Inside the star, the energy-momentum tensor is nonzero. Using the perfect-fluid assumption, it can be written as
\[ T_{\mu\nu} = (\varepsilon + p)u_\mu u_\nu - pg_{\mu\nu}, \]
\[ ^1 \text{In our setup, the value of the cosmological constant } \Lambda_C \text{ is negligible (see e.g. [79]) and is therefore omitted.} \]
where \( u_\mu = \left( A^{1/2}, 0, 0, 0 \right) \) is the fluid’s four-velocity. Combining this and the Einstein field equations with Eqs. (2.3), we end up with
\[
\frac{dA}{dr} = \frac{A}{r} \left[ \left( 8\pi G r^2 p + 1 \right) B - 1 \right], \quad B(r) = \left[ 1 - \frac{8\pi G}{r} \int_0^r d\tilde{r} \hat{r}^2 \varepsilon \left( \tilde{r} \right) \right]^{-1}, \quad \text{if } r \leq R. \tag{2.7}
\]
If the second equation is matched with its counterpart, Eq. (2.5), at the surface \((r = R)\), one finds out that the mass can be expressed as
\[
M = 4\pi \int_0^R d\tilde{r} \hat{r}^2 \varepsilon \left( \tilde{r} \right). \tag{2.8}
\]
As an alternative, one could also use a generalized, interpolated form
\[
m(r) := 4\pi \int_0^r d\tilde{r} \hat{r}^2 \varepsilon \left( \tilde{r} \right). \tag{2.9}
\]
By insisting that energy and momentum are conserved, \( T_{\mu\nu;}^; = 0 \), one may see that
\[
\frac{dp}{dr} = -p + \varepsilon \frac{dA}{2A} \frac{dA}{dr}. \tag{2.10}
\]
Finally, if this formula is combined with Eqs. (2.7) and (2.9), the TOV equations can be constructed as
\[
\frac{dp}{dr} = -\frac{G}{r^2} \left( p + \varepsilon \right) \left( m + 4\pi r^3 p \right) \left( 1 - \frac{2Gm}{r} \right)^{-1}, \tag{2.11a}
\]
\[
\frac{dm}{dr} = 4\pi r^2 \varepsilon. \tag{2.11b}
\]
In order to be able to integrate these differential equations, we need to define boundary conditions. Naturally, the mass function \( m(r) \) is required to be zero at the center of the star, and likewise, the pressure needs to vanish at the edge of the star. Moreover, one has to set an initial value for the pressure at \( r = 0 \).

2.1.2 Atmosphere

The outermost part of a neutron star is the atmosphere which is typically a layer of gaseous plasma. Its thickness depends on the age of the neutron star and it varies from millimeters (cold and old) to tens of centimeters (hot and young) \([1, 77] \). In some rare cases, when the magnetic field of a very cold neutron star is extremely strong, the gaseous phase can however condensate forming a liquid or solid surface \([77] \).\(^\dagger\)

\(^\dagger\)Here, we have used the shorthand notation for the covariant derivative of the energy momentum tensor \( T_{\mu\nu;} = T_{\mu\nu}^{\mu\nu} + \Gamma^\sigma_{\mu\nu} T^{\mu\nu} + \Gamma^\mu_{\mu\nu} T^{\nu\nu} \). The metric connections, or the Christoffel symbols, \( \Gamma^\sigma_{\mu\nu} \) can be formulated as \( \Gamma^\sigma_{\mu\nu} = g^\sigma_{\rho\nu} \left( g_{\sigma\nu,\mu} + g_{\sigma\mu,\nu} + g_{\mu\nu,\sigma} \right) / 2 \) while \( g_{\mu\nu,\sigma} := \partial g_{\mu\nu} / \partial x^\sigma \).\(^\dagger\)

\(^\dagger\)This should not be confused with the liquid region between the (outer) crust and the gaseous atmosphere, called the ocean. The properties of this relatively thick (~100 m) layer are important if one desires to obtain information about the interior temperature of a neutron star \([81] \). Because we assume \( T = 0 \) throughout this thesis, we are not going to further discuss the features of the ocean.
Most of the time, the atmosphere consists mainly of either hydrogen or helium (or mixture of thereof) \[82, 83\]. Therefore, the chemical composition of an individual neutron star has to be determined by observations \[77\]. Because the atmosphere is responsible for emitting thermal radiation which can be used to study several properties of the surface layers of neutron stars such as magnetic fields and surface temperature \[1, 77\], atmosphere modeling is an excessively studied field nowadays (see e.g. \[84–87\]).

The atmosphere also has another important role if one is trying to measure the radius (and mass) of a neutron star. If we assume that a neutron star in concern is a perfect blackbody whose surface temperature is uniformly distributed, then we can construct a naïve and simple model to probe the statement (see also \[77, 83\]).

An observer, infinitely distant from the object in question, observes the redshifted (or apparent) luminosity. According to the Stefan-Boltzmann law, this luminosity of a perfect blackbody is

\[
L_\infty = A_\infty \sigma_{SB} T_{\text{eff}, \infty}^4,
\]

(2.12)

where \( A_\infty \) and \( T_{\text{eff}, \infty} \) are the apparent area and effective surface temperature of the object whereas \( \sigma_{SB} \) is the Stefan-Boltzmann constant whose value is equal to \( \pi^2/60 \). These apparent quantities take the redshift effect into account. As a result, the nonredshifted values of the luminosity and effective temperature are larger than the apparent ones, for example. On the other hand, this apparent luminosity may be written as

\[
L_\infty = 4\pi D^2 F_{\text{bol}},
\]

(2.13)

where \( D \) is the distance to the object and \( F_{\text{bol}} \) is the total (bolometric) flux density. Combining these equations, the apparent radius can be solved so that

\[
R_{\infty}^2 = \frac{D^2 F_{\text{bol}}}{\sigma_{SB} T_{\text{eff}, \infty}^4},
\]

(2.14)

where we have used the fact that \( A_\infty = 4\pi R_{\infty}^2 \). \[77, 83\]

According to Wien’s displacement law, the peak energy of a photon from a blackbody is

\[
E_{\text{peak}} = \left[ 3 + W \left( -\frac{3}{e^3} \right) \right] T_{\text{eff}, \infty} \approx 2.8 T_{\text{eff}, \infty},
\]

(2.15)

when the frequency distribution is considered \[88\]. Here, \( W \) and \( e \) are the Lambert W function and Euler’s number, respectively. This implies that the observed effective temperature \( T_{\text{eff}, \infty} \) can be determined by measuring the blackbody spectrum. If the total flux density \( F_{\text{bol}} \) and the distance \( D \) are known, we can connect the (actual) radius \( R \) and the mass \( M \) of the observed neutron star. To be able to do so, we need to introduce a connection between the actual and observed radius of the neutron star,

\[
R_{\infty} = R(1 + z),
\]

(2.16)

*Here, it is also reasonable to ignore the possible effects of the interstellar medium and other similar complications.
where the cosmological redshift $z$ is given by $z = (1 - 2C)^{-\frac{1}{2}} - 1$. Here, $C$ is the compactness of the star that is defined as $C := GM/R$. Now, the relation can be formulated as

$$M = \frac{R}{2G} \left( 1 - \frac{\sigma_{SB}T_{\text{eff},\infty}}{D^2 F_{\text{bol}}R^2} \right),$$

for example. [83]

Alternatively stated, one can relate the mass and radius of a neutron star utilizing this type of a simple observational setup. Naturally, realistic frameworks demand more sophisticated modelling. The current mass-radius results are, however, consistent with current theoretical predictions (see e.g. Extended Data Fig. 5 of [II]). Unfortunately, the precisions of these observations are not optimal (see also Section 2.5).

### 2.1.3 Crust

Beneath the gaseous atmosphere and the liquid ocean, there exists a hundreds-of-meters thick solid layer — the crust (see Fig. 2.1). This region can roughly be divided into two parts. The outermost one shares its EoS with white dwarfs, and therefore, the main source of resistance against gravity is coming from the degeneracy pressure of mostly relativistic electrons. Due to electron capture (see next subsection), the neutron concentration of these ions increases with increasing pressure. In other words, neutron-rich nuclei become energetically more favorable states at greater depths. [1, 81, 89]

The interface between the inner and outer crust is known as the neutron-drip point, and it happens at a density of about $4 \times 10^{11}$ g cm$^{-3}$. After this point, the neutronization of nuclei continues so that neutrons start to leak out of the nuclei. In other words, matter does not just contain electrons and neutron-richer nuclei but free neutrons as well. This process will go on until nuclei finally fall apart. [1, 81, 89] This limit corresponds to the crust-core interface which is located at around $0.1n_s$ to $0.5n_s$ [90, 91]. Here, $n_s \approx 0.16$ fm$^{-3}$ (see e.g. [92, 93]) is the nuclear saturation density, i.e., the (number) density where the average energy per particle for symmetric nuclear matter is minimized.

It has also been hypothesized [94] (see e.g. [1, 89, 95] for further details) that there would be a thin, liquid mantle containing superfluid matter right before the crust-core interface. Here, matter could contain several nonspherical nucleus states that, due to their appearance, are called nuclear pasta.

The EoS of the crust region is relatively well known. Up to a density of about $6 \times 10^{10}$ g cm$^{-3}$, this EoS can be determined using experimental nuclear data [89, 96]. Higher densities, where nucleons are too neutron rich to be produced with current technology, can also be probed using extrapolation techniques [89]. In various studies (e.g. [32, 74, 97] or papers I–III), the classical results of [98, 99] have been used to model the crust because the mass-radius is relatively insensitive to the precise form of the crust EoS (cf. Fig. 2.1; see however [100]) — especially if one compares these extrapolation effects to the uncertainties of the simultaneous mass-radius
measurements (see Section 2.5). However, the tidal deformability (see the next section) may need a more in-depth treatment, see for instance [101]. Thus, the exact behavior of the crust EoS is still an important research topic (see e.g. [102, 103]).

2.1.4 Core

The core of a neutron star is a far less understood region than the crust. Yet, almost all of the mass and most of the volume of a neutron star is located inside it (see Fig. 2.1). Hence, much of modern research is especially interested in its detailed study.

As in the case of the crust, it is typical that the core is divided into outer and inner parts. The outer one mainly consists of free neutrons (n) — as the name suggests. Because of beta-equilibrium and charge-neutrality conditions, a core cannot be modeled using neutrons alone but a small amount of free protons (p) and electrons (e) have to be included as well. It is worthwhile to mention that muons, heavier charged leptons, may also appear with increasing density. Thus, purely hadronic neutron-star matter is often called as npeµ matter. [1]

In this context, beta equilibrium refers to a state where $\beta^-$ decay

$$n \rightarrow p + e + \bar{\nu}_e$$

is in static equilibrium with its relevant inverse process, called electron capture:

$$p + e \rightarrow n + \nu_e.$$  

Here, $\nu_e$ and $\bar{\nu}_e$ are the electron neutrino and its antiparticle, respectively. In other words, the reaction rate should be the same for both processes so that matter remains charge neutral. Especially in the case of the inner crust, this analysis can be generalized so that the equilibrium concerns all similar reactions driven by the weak interaction. In summary, we simply demand that the neutron-star matter is in the lowest possible charge-neutral energy state. [8, 104]

Scientists have for long been trying to solve the EoS of the core utilizing multiple different methods. Various nuclear models — both phenomenological (e.g. [105–108]) and ab initio (e.g. [26, 109–111]) ones — have been created to extend the known low-density calculations into high densities to model the core region. Recent theoretical studies, such as [9, 35, 112, 113], have been able to reliably probe the outer core EoS up to $n_s$ to $2n_s$ (see Section 3.2 as well). At the same time, nuclear physics experiments have also probed various saturation parameters. At the moment, the most prominent ones are the symmetry energy $S(n_s)$, i.e. the energy difference per particle between nuclear and neutron matter at $n_s$, and its derivative

$$\mathcal{L} := 3n_s \frac{dS(n_B)}{dn_B} \bigg|_{n_s},$$  \hspace{1cm} (2.18)

but their values are still poorly known (see e.g. [114–117]).

The inner core can be defined to start when matter begins to deviate significantly from the npe(µ) model, although it is not known how and when this happens. Hence, it is often practical
to assume that this region simply starts at around $2n_s$. Nowadays, there are three different types of exotic candidates to model the matter region: hyperons, different meson condensates and quarks. One has to note that it is still possible that these types of exotic phases do not exist inside a neutron star, or conversely, all of them may even coexist. In spite of this fact, the circumstances are exceptional because the density can be over $10n_s$ at the very center of a neutron star. [1]

A hyperon is generally speaking a baryon that contains one or more strange valence quarks — the quarks that define the quantum properties of different hadrons. At the end of the 1950s, it was suggested [20, 118] that hypernuclear matter could occur inside the cores of neutron stars due to high pressure even though hyperons are highly unstable particles in vacuum. Hyperonic models, however, contain several problems. First, hyperon-hyperon and hyperon-nucleon interactions are poorly known processes [1]. Secondly, typical hyperonic models produce relatively light neutron stars (e.g. [110, 119]) that disagree with the observations. As will be discussed in Section 2.4, any maximum-mass configuration has to be heavier than about $2M_\odot$ (see [120–122]). Nonetheless, some modern models have overcome the issue (e.g. [109, 123]). Furthermore, data from the first gravitational-wave observation of a binary-neutron-star merger [6, 34] (see also the next section) disfavor many existing hypernuclear EoSs [124]. Despite these problems, hyperonic models are still not completely ruled out.

It has been proposed that charged mesons could form a Bosen-Einstein condensate in a super-saturated medium. Therefore, this kind of matter may appear in neutron-star cores because the attractive interactions diminish the effective masses of the mesons. Traditionally, the lightest negatively charged mesons — pions $\pi^-$ and kaons $K^-$ — have been the primary condensate candidates [8, 125], but the possibility that negatively charged rho mesons $\rho^-$ might form a condensate state, has been studied lately (e.g. [30, 126, 127]). The existence of different meson condensates would soften the EoS leading to smaller maximum mass which would be inconsistent with observations, as in the case of the hyperons. Besides, the lack of knowledge of the high-density behavior prevents us to make any solid statement about the meson-condensate hypothesis. [125]

It is also possible that the inner core is made up of deconfined quark matter. High-density perturbative calculations suggest that hadronic matter eventually undergoes a phase transition (or a smooth crossover) to the quark phase, but it is not known at which density this transition happens (see Section 3.1.3). Hence, it is plausible that a neutron star may have a quark core — from massive to negligible. Numerous studies have examined this interesting possibility (e.g. [28, 74, 128, 129]) including article II.

*Quark cores should not be confused with (strange-)quark stars that belong to the class of hypothetical compact stars.
2.2 Tidal deformations

An object that experiences the tidal force of another object will deform. The susceptibility of deform is often measured using dimensionless quantities that are called Love numbers (named after A. E. H. Love) [130]. For nonrotating bodies, two important types of Love numbers exist. The first set contains gravitational Love numbers which characterize the deformation of the gravitational potential as measured by the mass-multiple moments (see the next subsection for details). The second class includes Love numbers $h_l$ that, correspondingly, describe the surficial deformations. [131] Here, the subscript $l$ tells the order of the multiple moment term, such that the zeroth-order ($l = 0$) term represents monopole moment, the first-order ($l = 1$) one is the dipole term, et cetera.

Generally speaking, neutron stars are not entirely static objects, and therefore, rotational effects sometimes have to be taken in account when one studies tidal deformations. Then, one needs to define additional, so-called rotational-tidal Love numbers that are induced by the coupling between the angular momentum of the star and the external tidal field [132–135]. Every order, both the rotating and nonrotational-tidal Love numbers can be divided into parity-odd (magnetic) and -even (electric) components. This means that every even- (odd-)parity part has its corresponding gravitoelectric (-magnetic) field. [131]

The tidal Love numbers of black holes are somewhat trivial quantities to study due to the relatively simple structure of the black holes. Hence, it can easily be shown that these gravitational Love numbers are actually zero in this case [132, 136]. If we focus on the nonrotating configurations, we may also notice that in the Newtonian limit the magnetic tidal-Love numbers disappear [136, 137]. In other words, the Newtonian Love numbers correspond to the electric-type ones derived from GR in the weak-field limit.

It has been proven in [131] that the gravitational and surficial Love numbers are not necessarily independent but are, in fact, connected if bodies of perfect fluid (cf. the assumptions of the TOV equations) are considered:

$$h_l = \Gamma_1 + 2\Gamma_2 k_{l}^{el},$$

(2.19)

where $k_{l}^{el}$ is the (electric or even-parity) Love number related to the quadrupole moment of the concerned star. In the Newtonian limit, the functions $\Gamma_1$ and $\Gamma_2$ are both equal to one, but in the general case, these functions are linear combinations of the ordinary hypergeometric functions $2F_1$:

$$\Gamma_1 = \frac{l+1}{l-1}(1-C)2F_1(-l,-l; -2l; 2C) - \frac{2}{l-1}2F_1(-l,-l-1; -2l; 2C),$$

(2.20)

$$\Gamma_2 = \frac{l}{l+2}(1-C)2F_1(l+1,l+1; 2l+2; 2C) + \frac{2}{l+2}2F_1(l+1,l; 2l+2; 2C),$$

(2.21)

where $C$ is the compactness parameter. In the case of a black hole, i.e. $C = 1/2$ and $k_{l}^{el} = 0$, the surficial Love numbers are not simply equal to zero — as in the cases of the other introduced Love numbers — but

$$h_l = \frac{l+1}{2l-2}(l!)^2.$$

(2.22)
In this section, we will examine electric-type Love numbers. First, we will briefly introduce these quantities, particularly their dimensionless form, in the next subsection. In Subsection 2.2.2, gravitational-wave physics of colliding neutron stars has been discussed superficially. This is an important topic to cover because gravitational waves allow us to estimate the tidal response of neutron stars. Finally, we will briefly consider how to compute these Love number for different neutron-star configurations (Subsection 2.2.3).

2.2.1 Tidal deformability

The \( l = 2 \) electric-type Love number is a fascinating quantity because it can be used to probe the dense-matter EoS using data from double-neutron-star-merger events (e.g. [4, 34, 138] or articles I–III). Even though the current estimates [6, 34] for this variable are not very precise, future observations will certainly clear the picture. Therefore, it is essential to understand the basic characteristics of the Love numbers.

We cannot talk about the tidal Love numbers without discussing the roles of tidal and multipole moments first. In the Newtonian limit, the tidal moment can be expressed as

\[
\mathcal{E}_L := -\frac{1}{(l-2)!} \frac{\partial^l \Phi_{\text{ext}}}{\partial x^{i_1} \partial x^{i_2} \ldots \partial x^{i_l}},
\]

where \( \Phi_{\text{ext}} \) is the external (Newtonian) potential induced by the companion body given in the center-of-mass frame. Here, \( L \) is a multi-index defined so that \( L := i_1 i_2 \ldots i_l \) where the indices \( i_k \) represent the spatial components. The induced multipole moment can correspondingly be written as

\[
Q^L_{\text{tidal}} := \int \varepsilon x (L) \, d^3 x,
\]

where \( x (L) \) represents the symmetric and traceless part of \( x^L := x^{i_1} x^{i_2} \ldots x^{i_l} \). In the case of the quadrupole moment, this means that \( x (L) = x (i_1 i_2) = x^{i_1} x^{i_2} - \delta^{i_1 i_2} r^2 / 3 \) where \( \delta^{ab} \) is the Kronecker delta. [131, 136]

If we examine linear-order effects — i.e. the metric is almost flat — the tidal response can be shown to be

\[
Q^L_{\text{tidal}} = -\lambda_l \mathcal{E}_L,
\]

where the tidal parameter

\[
\lambda_l := \frac{2}{(2l-1)!!} R^{2l+1} k^l_l
\]

is given in the conventional form [136, 139].* On the other hand, the tidal deformability (also known as the tidal polarization or electric deformability) typically refers to the dimensionless version of the tidal parameter \( \lambda_l \): [139]

\[
\bar{\lambda}_l := \frac{\lambda_l}{M^{2l+1}} = \frac{2}{(2l-1)!!} C^{-2l-1} k^l_l.
\]

*Be aware that some articles call the parameter \( \lambda_l \) as the Love number (e.g. [140]) — instead of \( k^l_l \) — or as the tidal deformability (e.g. [141]).
The above treatment is given in the Newtonian limit but it can be translated into the GR framework even though the definitions given in Eqs. (2.23) and (2.24) do not hold anymore. Incidentally, the metric component $g_{tt}$ and the Newtonian potential are connected if static neutron-star solutions are considered:

$$\Phi_{\text{ext}} = \frac{1 - g_{tt}}{2}. \quad (2.28)$$

Therefore, the definitions of the (electric-type) tidal moment $E_L^\text{tidal}$ and the corresponding multiple moment $Q_L^\text{tidal}$ can be reformulated. The generalization procedure, fortunately, leaves Eqs. (2.25), (2.26), and (2.27) unchanged. [140, 142]

### 2.2.2 Gravitational-wave observations

In the Universe, there are various binary systems where two objects orbit around each other. Because these objects lose energy to gravitational waves, their orbits are not, however, stable as predicted by GR. Therefore, they will inevitably approach each other until they finally merge. Around the collision point, the generated gravitational-wave signal may be strong enough to be detected by terrestrial instruments. At the moment, this means that target objects have to either be neutron stars or black holes. The first successful discovery of such gravitational waves was made in 2015 (event GW150914∗) [143] and numerous similar observations have been made since [6, 144–151]. In the light of this, we will briefly explore the basic properties of gravitational waves in this subsection. We will especially focus on the tidal deformability that is currently the most essential feature to neutron stars.

A weak gravitational field resembles flat Minkowski space so that the spacetime is only weakly curved. In such a case, the metric can be expressed as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad (2.29)$$

where $\eta_{\mu\nu}$ represents the Minkowski metric and $h_{\mu\nu}$ is a small perturbation so that $|h_{\mu\nu}| \ll 1$. One may show that the Einstein field equations can be reformulated so that they can be written as a linear second-order partial differential equations respect to $h_{\mu\nu}$. Furthermore, it has been proven that using a gauge condition

$$\tilde{h}^{\mu\rho,\rho} = 0, \quad (2.30)$$

where $\tilde{h}_{\mu\nu} = h_{\mu\nu} - h_\rho^\rho \eta_{\mu\nu}/2$, the Einstein tensor gets a simple form:

$$G_{\mu\nu} = -\frac{1}{2} \tilde{h}_{\mu\nu,\rho}^\rho. \quad (2.31)$$

If we consider the vacuum solution, then the energy-momentum tensor $T_{\mu\nu}$ is zero and the Einstein field equations can be given as a wave equation:

$$\tilde{h}_{\mu\nu,\rho}^\rho = 0. \quad (2.32)$$

∗Here, the code GW150914 indicates that the gravitational-wave event detected on September 14, 2015.
Here, the small perturbation $\tilde{h}_{\mu\nu}$ represents a gravitational wave. [152, 153]

The simplest solution that fulfills the above conditions is a plane wave:

$$\tilde{h}_{\mu\nu} = \text{Re} [A_{\mu\nu} \exp (ik_{\mu}x^{\nu})] ,$$

(2.33)

where $A_{\mu\nu}$ and $k_{\mu}$ are the amplitude and the wave vector, respectively. These variables have to agree with the following conditions:

$$k_{\mu}k^{\mu} = 0,$$

(2.34)

$$A_{\mu\nu}k^{\mu} = 0.$$

(2.35)

The first equation implies that the wave travels with the speed of light $c$, and the latter one highlights the fact that the wave is transverse. These kinds of gravitational waves have two different types of linear polarization modes which are typically labeled as $+$ and $\times$. The nonvanishing components of the perturbation $h_{\mu\nu}$ — when the transverse traceless gauge is used and the wave propagates in the $z$ direction — are

$$h_{+} := h_{11} = -h_{22} = \text{Re} [A_{+} \exp (-ik_{0}(t-z))] ,$$

(2.36)

$$h_{\times} := h_{12} = h_{21} = \text{Re} [A_{\times} \exp (-ik_{0}(t-z))] ;$$

(2.37)

because now $\tilde{h}_{\mu\nu} = h_{\mu\nu}$. [152, 153]

Assuming that the wavelength of the gravitational-wave signal is much longer than the dimensions of the detector, then the detected waveform is a linear combination of the polarization amplitudes:

$$h(t) = F_{+} h_{+} + F_{\times} h_{\times} ,$$

(2.38)

where $F_{+}$ and $F_{\times}$ are functions that depend on the orientations of the source and the detector. In the case of two point masses, these polarization amplitudes obtain the relatively simple forms:

$$h_{+} = \bar{A}(t) \left( 1 + \cos^{2} \iota \right) \cos \left[ \bar{\phi}(t) \right] ,$$

(2.39)

$$h_{\times} = 2 \bar{A}(t) \cos \iota \sin \left[ \bar{\phi}(t) \right] .$$

(2.40)

Here, $\iota$ is the orbital inclination angle, $\bar{\phi}$ is the phase of the gravitational-wave signal, and $\bar{A}$ is a source-dependent function of the time $t$. Hence, the waveform $h$ is given as

$$h(t) = \bar{A}(t) \sqrt{F_{+}^{2} (1 + \cos^{2} \iota)^{2} + 4 F_{\times}^{2} \cos^{2} \iota \cos [\phi(t)]} .$$

(2.41)

The new phase $\phi$ is obtained after adding the phase offset

$$\Delta \bar{\phi} = - \arctan \left[ \frac{2F_{\times} \cos \iota}{F_{+} (1 + \cos^{2} \iota)} \right]$$

(2.42)

with respect to the old phase $\bar{\phi}$, i.e. $\phi = \bar{\phi} + \Delta \bar{\phi}$. [154, 155]

*Although more complicated solutions also exist, these can always be represented as linear combinations of plane-wave solutions.*
The previous procedure can be generalized so that the observed gravitational-wave signal emitted by a binary system is simply expressed as

\[ h(t) = a(t) \cos [\phi(t)], \]  

(2.43)

where \( a \) is a source-dependent function of the time \( t \) \[139, 156\]. Using the stationary-phase approximation, the Fourier-transformed signal can be given as

\[ h(f) = \mathcal{A}(f) \exp [i\Psi(f)], \]  

(2.44)

where \( \mathcal{A} \) is also another source-dependent function and \( f \) is the gravitational-wave frequency \[156, 157\]. The Fourier-transformed version of the phase shift \( \Psi \) is typically expressed using a post-Newtonian expansion, and it describes many interesting properties of the considered binary.

There are two different effects that have to be taken into account when one is studying an expansion of the gravitational waves close to the Newtonian limit: i) the speed correction* \((v/c)\) and ii) the small deviation from the flat Minkowskian metric \((R_S/d)\) part. Here, \( v \) and \( d \) are the characteristic speed and size of the considered system whereas \( R_S \) is its Schwarzschild radius, i.e. \( R_S = 2GM/c^2 \). Naïvely speaking, one might assume that these two expansion parameters could be treated independently. This is actually true when we are interested in systems where the process is not driven by gravitation, such a charged particle in oscillating electric field. However, self-gravitating bodies, such as binary neutron star mergers, do not fulfill this condition due to the virial theorem. Therefore, the small (post-Newtonian) expansion parameter,

\[ \epsilon \sim \left( \frac{v}{c} \right)^2 \sim \frac{R_S}{d}, \]  

(2.45)

has to contain both effects. \[158\]

At its simplest, the behavior of a compact binary can be described as a system of two point-like masses. In that case, the system radiate energy via gravitational waves according to the Newtonian quadrupole formula, and the inspiral rate of the system can then be given as

\[ \frac{dr_o}{dt} = -\frac{64}{5} \frac{G^3 \mu_M M_2^2}{r_o^4}, \]  

(2.46)

where \( r_o \) is the orbital separation, \( M_{tot} \) is the total mass of the system, and \( \mu_M \) is the reduced mass. These mass parameters are defined as

\[ M_{tot} = M_1 + M_2, \]  

(2.47)

\[ \mu_M = \frac{M_1 M_2}{M_{tot}}, \]  

(2.48)

where \( M_1 \) and \( M_2 \) represent the masses of the individual binary components. Using the separation-of-variables technique, Eq. (2.46) can be solved so that

\[ t = t_c - \frac{5}{256} \frac{G^3 \mu_M M_2^2}{r_o^4}, \]  

(2.49)

*In this paragraph, we are explicitly showing the speed-of-light parameter \( c \) for clarity.
where the integration constant $t_c$ is the coalescence time, i.e. $t_c := t(r \to 0)$. Because the frequency of a quadrupolar gravitational wave

$$f = \frac{(GM_{\text{tot}})^{1/2}}{\pi r_o^{3/2}}, \quad (2.50)$$

the time variable $t$ can also be written as

$$t(f) = t_c - 2(8\pi f)^{-8/3}(GM)^{-5/3}, \quad (2.51)$$

where $M$ is the chirp mass of the binary system defined as

$$M = \frac{(M_1 M_2)^{3/5}}{M_{\text{tot}}^{1/5}}. \quad (2.52)$$

Similarly, one can examine the phase

$$\phi(t) = 2\pi \int_{t_c}^{t} f(t') dt' \quad (2.53)$$

that can be formulated as

$$\phi(f) = \phi_c - 2(8\pi G M f)^{-5/3}, \quad (2.54)$$

where $\phi_c := \phi(t = t_c)$ is the coalescence phase. As introduced in Eq. (2.44), the stationary-phase approximation can be used to calculate the Fourier-transformed waveform. In this case, the Newtonian phase parameter is

$$\Psi(f) = 2\pi ft_c - \phi(f) - \frac{\pi}{4}, \quad (2.55)$$

or equivalently,

$$\Psi(f) = 2\pi ft_c - \phi_c - \frac{\pi}{4} + \frac{3}{4}(8\pi G M f)^{-5/3}, \quad (2.56)$$

where $f > 0$. [156]

Based on this result, the general post-Newtonian expansion can be formulated as

$$\Psi(f) = 2\pi ft_c - \phi_c - \frac{\pi}{4} + \frac{3}{4}(8\pi G M f)^{-5/3} \Delta \Psi_{\text{PN}}(f), \quad (2.57)$$

where $\Delta \Psi_{\text{PN}}$ contains information about the post-Newtonian behavior [157]. By now, the point-like corrections for waveform phase have been calculated up to the third-and-a-half post-Newtonian order* [159] and the other types of adjustment factors have also been computed. For example, the spins of neutron stars and the eccentricity of the binary system change the gravitational-wave signal as well, and hence, the phase $\Psi$ has to be modified accordingly [157].

Furthermore, one has to take into account the nonpointlike behavior of individual neutron stars when the distance between the binary components becomes small [157]. The phase-shift

*The determined post-Newtonian expansion terms are typically labeled using half-integer values because the expansion parameter $\epsilon$ contains the squared, dimensionless speed parameter $(v/c)^2$ [158]. Hence, the term related to the $\epsilon^{0.5}$ term corresponds to the first possible post-Newtonian correction called as the 0.5PN order, for instance.
term for the electric-type tidal deformability appears the first time at the fifth order of the post-Newtonian expansion [140]. According to [139], the leading-order result for every electric-type multipole moment can be written in a general form as

$$\Delta \Psi_{\bar{\lambda}_l} = -\sum_{J=1}^{2} \frac{40 (2l - 1)!!(4l + 3)(l + 1)}{3 (4l - 3)(2l - 3)} \left( \frac{M_J}{M_{tot}} \right)^{2l-1} \eta_M \bar{\lambda}_{l,J} v^{4l+2} + 24 \left( \frac{M_J}{M_{tot}} \right)^{4} \delta_{l2} \bar{\lambda}_{2,J} v^{10} \right] + \mathcal{O} \left( v^{4(l+1)} \right).$$

(2.58)

Here, \(v = (\pi G M_{tot} f)^{1/3}\) and \(\eta_M = M_1 M_2 / M_{tot}^2\) are post-Newtonian (orbital) velocity parameter and the symmetric mass ratio, while the subscript index \(J\) refers to the individual binary components. It has also been showed that the magnetic-type Love numbers have a similar relation, whose leading-order term appears at the sixth post-Newtonian order [139].

Especially, the \(l = 2\) electric-type, dimensionless tidal deformability parameter,

$$\Lambda := \bar{\lambda}_2 = \frac{2}{3} C^{-5} k_{2l}^{gl},$$

(2.59)

is frequently used when the latest gravitational-wave results of the first double-neutron-star-merger event GW170817 observed by LIGO\(^\ast\) and Virgo collaborations [6, 34] have been utilized (see e.g [4, 160] or papers I–III). Therefore, we shall call this particular parameter the tidal deformability henceforth. Unfortunately, the individual tidal-deformability values of the binary components cannot be sorted out from the gravitational-wave signal without additional information about the microscopic properties of the binary components as can be seen from Eq. (2.58), for instance. Hence, the dimensionless combined tidal-deformability parameter [6, 157],

$$\bar{\Lambda} = \frac{16}{13} \frac{\left( M_1 + 12 M_2 \right) M_1^3 \lambda_1 + (12 M_1 + M_2) M_2^3 \lambda_2}{M_{tot}^5},$$

(2.60)

has also been used (e.g. [5, 161]) because the leading-order contribution of the tidal deformability for the gravitational-wave-phase parameters can simply be written as [157]

$$\Delta \Psi_{\Lambda} = -\frac{39}{2} \bar{\Lambda} v^{10} + \mathcal{O} \left( v^{12} \right).$$

(2.61)

### 2.2.3 Calculating electric-type tidal Love number

Because one of the main components of Eq. (2.59) is the electric-type tidal Love number \(k_{2l}^{gl}\), it is important to be able to calculate its value for realistic EoS models. This problem was first tackled at the end of 2000s [140, 142, 162]. Later, these works have been extended including higher orders of the multipole moments [136, 137], for example. A simplified formulation to calculate the gravitational Love numbers for nonrotating neutron stars was also introduced in [131].

\(\ast\)The abbreviation LIGO stands for the Laser Interferometer Gravitational-Wave Observatory.
According to [131], the so-called master equation of the electric Love number can be expressed as
\[
\eta'' + \eta(\eta - 1) + A\eta - B = 0,
\]
where
\[
A = \frac{2}{B} \left[ 1 - 3 \frac{Gm}{r} - 2\pi Gr^2(\varepsilon + 3p) \right],
\]
\[
B = \frac{1}{B} \left[ l(l + 1) - 4\pi Gr^2(\varepsilon + p)(c_s^{-2} + 3) \right].
\]

At the center of the star, \(\eta\) has to be equal to \(l\) to fulfill the boundary conditions. After solving the value of \(\eta\) at the surface of the star \((r = R)\), the Love number can be determined:
\[
k_{el}^2 = -\frac{1}{2} \frac{\eta_s - l - 4C/(1 - 2C)}{\eta_s + l + 1 - 4C/(1 - 2C)} B_1 - RB_1',
\]
where \(\eta(R) := \eta_s\) and
\[
B_1(r) := 2F_1 \left( l+1, l+3; 2l+2; \frac{2GM}{r} \right).
\]

Here, \(2F_1\) is the ordinary hypergeometric function.

Calculating the magnetic-type Love number is a very similar process. Further details about the procedure can be found from [131, 136, 137], for instance.

### 2.3 Universal relations

As a rule, various properties of neutron stars are highly dependent on the exact form of the neutron-star-matter EoS. Hence, it was a surprise when [48, 163] discovered that the relationships between certain pairs of neutron-star quantities are almost independent of the underlying EoS when they examined many popular and realistic EoS models. Nowadays, many similar relationships has been found, and they are collectively known as the approximative or EoS-insensitive universal relations.

The best known universal relation is the I-Love-Q one [48, 163] that connects three different variables together: the moment of inertia \(I\), the electric-type Love number \(k_{el}^2\), and the spin-induced quadrupole moment \(Q_S\). To be able to relate these variables, one needs to rewrite them in dimensionless form. As we have shown above, the dimensionless form of the Love number \(k_{el}^2\) is the tidal deformability \(\Lambda\), Eq. (2.59). The other two quantities are normalized as
\[
\tilde{I} = \frac{I}{G^2 M^3},
\]
\[
\tilde{Q} = -\frac{MQ_S}{(I\Omega)^2},
\]
where \(\Omega\) is the angular velocity of the neutron star. Utilizing these formulas, the I-Love-Q relation is given as
\[
\ln y_u = \sum_{i=0}^{4} U_i \ln^i x_u,
\]
Table 2.1: Parameters for the EoS-insensitive I-Love-Q relation given by [49, 50]. These values are obtained after fitting Eq. (2.69) with EoS data from numerous realistic neutron-star-matter models.

<table>
<thead>
<tr>
<th>$x_u$</th>
<th>$y_u$</th>
<th>$U_0$</th>
<th>$U_1$</th>
<th>$U_2$</th>
<th>$U_3$</th>
<th>$U_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Lambda$</td>
<td>$\tilde{I}$</td>
<td>1.496</td>
<td>0.05951</td>
<td>0.02238</td>
<td>$-6.953 \times 10^{-4}$</td>
<td>$8.345 \times 10^{-6}$</td>
</tr>
<tr>
<td>$\Lambda$</td>
<td>$\tilde{Q}$</td>
<td>0.1940</td>
<td>0.09163</td>
<td>0.04812</td>
<td>$-4.283 \times 10^{-3}$</td>
<td>$1.245 \times 10^{-4}$</td>
</tr>
<tr>
<td>$\bar{Q}$</td>
<td>$\bar{I}$</td>
<td>1.393</td>
<td>0.5471</td>
<td>0.03028</td>
<td>0.01926</td>
<td>$4.434 \times 10^{-4}$</td>
</tr>
</tbody>
</table>

where $x_u$ and $y_u$ are the target pair while $U_i$ are the corresponding fitting constants. The up-to-date values for these parameters $U_i$ are given in Table 2.1. Lately, another similar fitting formula has been introduced [50].

As the original study already stressed, numerous realistic EoSs follow the above relations extremely well. To be more specific, fractional errors between the fitted formulas and conventional EoSs are a few per cent at most [48–50, 163]. Therefore, various universal relationships can be used to connect seemingly unrelated neutron-star quantities together very accurately without needing to know the EoS. Nevertheless, some articles (e.g. [III, 164, 165]) have found that certain exotic EoSs can significant violate the universality. That is why it is important to study these relations more closely and to figure out the source of these anomalies.

Beyond the basic I-Love-Q relation, many other similar universal relations have been found. Some of the most studied ones are relations connecting different Love numbers together (e.g. [49, 133, 134, 139]). Especially, the LIGO and Virgo collaborations used those to boost their analysis of the gravitational-wave event GW170817 [34]. Furthermore, it is also possible to extend this method and to define analogous I-Love-Q relations for white dwarfs as well [166]. Nevertheless, we are going to only further examine the original I-Love-Q relation for neutron stars in this thesis.

### 2.3.1 Moment of Inertia

In order to consider the I-Love-Q relations in the case of an arbitrary EoS, we need to be able to calculate the moment of inertia $I$ for various neutron stars. This quantity is one of the characteristic features of a rotating body, and hence, we need to extend the static metric given in Eq. (2.1). In the general axial-symmetric case, this means that the line element can be written as

$$
\text{d}s^2 = \bar{A}(r, \theta)\text{d}t^2 - \bar{B}(r, \theta)\text{d}r^2 - r^2\bar{C}(r, \theta) \left\{ \text{d}\theta^2 + \sin^2\theta \left[ \text{d}\phi - \bar{D}(r, \theta)\text{d}t \right]^2 \right\},
$$

(2.70)
where functions $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, and $\tilde{D}$ are positive [8, 167]. When one considers a slowly rotating setting, the metric is given to the second order in the angular velocity $\Omega$ as

$$
\begin{align*}
\text{d}s^2 &= A \left[1 + 2\Omega^2 (h_0 + h_2 P_2)\right] \text{d}t^2 - \left[1 + 2\Omega^2 \frac{m_0 + m_2 P_2}{rB}\right] \frac{\text{d}r^2}{B} \\
&\quad - r^2 \left[1 + \Omega^2 (v_2 - h_2) P_2\right] \left[\text{d}\theta^2 + \sin^2 \theta (\text{d}\phi - \omega \text{d}t)^2\right], 
\end{align*}
$$

(2.71)

where $\omega$ is the angular velocity of the local inertial frame and $P_2$ is the second-order Legendre polynomial defined so that $P_2 := P_2 (\cos \theta) = (3 \cos^2 \theta - 1) / 2$. The introduced perturbation functions $h_0$, $h_2$, $m_0$, $m_2$, and $v_2$ are radius dependent. [168]

J. B. Hartle [167] has shown that the only nonvanishing terms of the Einstein field equations — expanding the metric to linear order — are the $t-\phi$ ones. Based on these findings, the internal solution can be written in the form of a differential equation:

$$
\frac{d}{dr} \left( r^4 \frac{d\mathfrak{g}}{dr} \right) + 4r^3 \frac{d\mathfrak{g}}{dr} = 0
$$

(2.72)

that specifies the relative angular velocity

$$
\mathfrak{g} := \frac{\tilde{\omega}}{\Omega} = 1 - \frac{\omega}{\Omega},
$$

(2.73)

where $j := \sqrt{AB}$. At the center of the star, the first derivative of the relative angular velocity $\mathfrak{g}$ has to vanish, i.e. $\mathfrak{g}'(r = 0) = 0$. Secondly, the boundary condition at the surface of a star has to be $\mathfrak{g}(R) = 1 - 2I/R^3$ to match with the exterior solution. Now, the moment of inertia of the neutron star,

$$
I = \frac{8\pi}{3} \int_0^R \frac{r^4 \mathfrak{g} (\varepsilon + p)}{B} \text{d}r,
$$

(2.74)

is calculable when the behavior of the variable $\mathfrak{g}$ is known. [III, 8, 167]

### 2.3.2 Spin-induced quadrupole moment

Based on the metric given by Eq. (2.71), the second-order rotational effects can also be solved. As we can see based on the used formulation, the nonvanishing contributions are the monopole and quadrupole modes. In this subsection we will study the quadrupole one only. If one is interested to know more about the monopole mode, see references [167, 169] for details.

The original work of J. B. Hartle [167] gives the equations for the perturbation functions $h_2$, $m_2$, and $v_2$ but here we are going to use formulations based on the ones given in [48]:

$$
\begin{align*}
\frac{dK_2}{dr} &= - \frac{dh_2}{dr} + \left(1 - 3\frac{Gm}{r} - 4\pi Gr^2 p\right) \frac{h_2}{Br} + \left(1 - \frac{Gm}{r} + 4\pi Gr^2 p\right) \frac{m_2}{(Br)^2}, \\
\frac{dh_2}{dr} &= \left(1 - \frac{Gm}{r} + 4\pi Gr^2 p\right) \frac{dK_2}{dr} + \left[3 - 4\pi Gr^2 (\varepsilon + p)\right] \frac{h_2}{Br} + 2 \frac{K_2}{Br} \\
&\quad + \left(1 + 8\pi Gr^2 p\right) \frac{m_2}{(Br)^2} + \frac{ABr}{12} \left(\frac{dg}{dr}\right)^2 - \frac{4\pi Gr^3 A}{3} \frac{\varepsilon + p}{B} g^2, \\
m_2 &= - Br h_2 + \frac{ABr^4}{6} \left[Br \left(\frac{dg}{dr}\right)^2 + 16\pi Gr(\varepsilon + p) g^2\right],
\end{align*}
$$

(2.75-2.77)
where we have introduced a new variable $K_2 := v_2 - h_2$. The boundary conditions demand that both $h_2$ and $K_2$ vanish when $r = 0$ or $r \to \infty$ \cite{167}. Furthermore, it has been shown by \cite{48} that these interior solutions can be expanded as Taylor series around the origin:

\[
\begin{align*}
    h_2(r) &= Br^2 + \mathcal{O}(r^4), \\
    K_2(r) &= -Br^2 + \mathcal{O}(r^4), \\
    m_2(r) &= -Br^3 + \mathcal{O}(r^5),
\end{align*}
\]

where $B$ is a constant depending on the exact value of the quadrupole moment $Q_S$. On the other hand, investigating the exterior metric leads to the following solutions

\[
\begin{align*}
    h_2^{\text{ext}}(r) &= \frac{3A}{C(1-2C)} \left[ 1 - 3C + \frac{4}{3} C^3 + \frac{(1-2C)^2}{2C} \ln(1-2C) \right] + \left( \frac{I}{MR} \right)^2 C(1+C), \\
    K_2^{\text{ext}}(r) &= \frac{3A}{C} \left[ 1 + C - \frac{2}{3} C^2 + \frac{1-2C^2}{2C} \ln(1-2C) \right] - \left( \frac{I}{MR} \right)^2 C(1+2C), \\
    m_2^{\text{ext}}(r) &= \frac{3AR}{C} \left[ 1 - 3C + \frac{4}{3} C^3 + \frac{2}{3} C^3 + \frac{(1-2C)^2}{2C} \ln(1-2C) \right] \\
    &\quad - \left( \frac{I}{MR} \right)^2 C(1 - 7C + 10C^2),
\end{align*}
\]

where the integration constant $A$ will be fixed matching the interior and exterior solutions at the surface of the star \cite{48}.

Because the spin-induced quadrupole moment $Q_S$ corresponds to the $P_2/r^3$ term of the Newtonian potential, or in other words

\[
\Omega^2 h_2 = \frac{Q}{r^3} + \ldots,
\]

one is able to figure out that

\[
Q_S = - \left( \frac{I^2}{M} + \frac{8}{5} AG^2 M^3 \right) \Omega^2
\]

is the form of the quadrupole moment \cite{III,48}. To solve this quantity for a given neutron star, we need to match the perturbation functions $h_2$ and $K_2$ at the surface of the star, which gives us the constants $A$ and $B$. An effective way to do this is to follow the procedure introduced in \cite{48}.

### 2.4 Maximum mass

Thus far, over 30 neutron stars have been found with somewhat precisely measured masses (see review of \cite{170}). However, the theoretical maximum mass $M_{\text{TOV}}$ — also known as the TOV mass — for nonrotating neutron stars is still unknown because it depends on the underlying EoS. Loosely speaking, the mass of a neutron star, however, increases as the density increases.
After the limit density $\varepsilon(M_{\text{TOV}})$, whose value is still unknown, this is no longer the case. Actually, those ultra-dense solutions are then too dense to be supported by the strong interaction, and therefore, these kinds of configurations likely collapse into black holes. Nonetheless, it is hypothesized [171] that a new stable compact-star branch exists after the neutron star one, but this claim has not been verified to date. (Cf. Fig. 2.2a.)

Because $M_{\text{TOV}}$ is one of the most characteristic features of the neutron-star population, it can also be used to probe the underlying EoS. In order to do so, one has to define a set of theoretical conditions that every realistic EoS needs to satisfy. Among the already introduced beta-stability and charge-neutrality requirements, the following conditions are required to be met: [8, 172, 173]

1. The pressure $p$ and the energy density $\varepsilon$ are always nonnegative,
2. microscopic stability holds: $\frac{dp}{d\varepsilon} \geq 0$,
3. the speed of sound $c_s$ cannot exceed the speed of light $c$ (subluminality; see however [1]),
4. neutron-star solutions are stable against radial oscillations and convection (see the next subsection).

We state that the TOV mass is the largest mass of the neutron-star branch so that all above conditions are satisfied. In addition, it can be shown that a test similar to the condition 4 can often be used to locate the stable branch:

$$\frac{\partial M(\varepsilon_c)}{\partial \varepsilon_c} \geq 0,$$

where $\varepsilon_c$ is the energy density at the center of the neutron star. However, this formulation only gives the necessary condition for stability but it is not sufficient. [8] Therefore, it is a useful tool to quickly determine the upper limit of the TOV mass for certain trial EoSs.

By measuring the masses of the most massive neutron stars, we can also specify a lower limit for the TOV mass. So far, three very massive neutron stars have indisputably been observed:

- PSR J1614−2230 ($1.928^{+0.017}_{-0.017} M_\odot$) [174],
- PSR J0348+0432 ($2.01^{+0.04}_{-0.04} M_\odot$) [121], and
- PSR J0740+6620 ($2.14^{+0.10}_{-0.09} M_\odot$) [122].

All the above pulsars are located in binary systems where the companion star is a relatively massive white dwarf. As in most other similar systems, the measurements were carried out using a GR effect — the Shapiro delay. [170] This phenomenon describes how an electromagnetic signal slows down when it passes nearby a massive object [175] — like a white-dwarf companion. Hence, it can be used to precisely determine the masses of the components of the system [170, 176].

*These are 68.3-per-cent confidence or credibility intervals.
In addition to these three objects, some possible neutron-star candidates with high masses (e.g. the Vela X-1 pulsar) have also been observed in x-ray binaries. These results are less reliable due to large observational uncertainties, unfortunately. Overall, the observations, however, support the claim that the maximum mass for nonrotating neutron stars is at least two solar-masses.

From a theoretical point of view, the upper limit of the TOV mass can be assessed by inspecting certain kinds of extreme EoS models together with observational constraints. Maybe the most conservative way to evaluate it is to consider a marginally causal EoS. In this case, the speed of sound $c_s$ is equal to the speed of light $c$ and the EoS takes a straightforward form:

$$p(\varepsilon \geq \varepsilon_0) = p_0 + \varepsilon - \varepsilon_0,$$

where $\varepsilon_0$ and $p_0$ are the energy density and pressure corresponding to the point where the EoS becomes only marginally causal. It has been shown that $M_{\text{TOV}} \lesssim 4M_\odot$ when the matching density is about equal to $n_s$. A robuster estimation for the upper limit of TOV mass can be achieved studying more realistic EoSs. Articles I and II examined three different interpolation methods to approximate the dense-matter EoS between the well-known lower and upper limits, given by ab initio nuclear- and particle-physics calculations, respectively (see Sections 3.2 and 3.3 for details). Forcing the four above mentioned conditions, these studies found out that $M_{\text{TOV}} \lesssim 3.7M_\odot$ what is in line with the above evaluation. Furthermore, using tidal deformability data from the gravitational-wave event GW170817 [34], this estimate can be further refined so that $M_{\text{TOV}} \lesssim 3.0M_\odot$.

It is believed that the event GW170817 has an electromagnetic counterpart produced by the formed kilonova [7]. Several studies [16–19] have used this likely multi-messenger observation to limit the upper bound of the TOV mass. All these articles point to the direction that $M_{\text{TOV}} \lesssim 2.3M_\odot$, and other similar studies [13–15] have come to a comparable conclusion. However, it is important to consider whether the assumptions of the studies are realistic. For example, are the (directly or indirectly implemented) EoSs able to model the collapse well enough to make these kinds of predictions?

Lastly, if rotating neutron stars are examined, one is able to show that their maximum masses are higher than the TOV limit. It has even been estimated that for uniformly rotating neutron stars, the maximum mass is about 1.2 times larger than the TOV mass. Moreover, differentially rotating neutron stars, where the angular velocity $\Omega(r = 0)$ can be much larger than the Keplerian (or mass-shedding) angular frequency $\Omega_K$, can even have greater masses but these configurations are typically highly unstable. [1, 173, 178]

### 2.4.1 Stability against radial oscillations and convection

In addition to static solutions, it is also essential to study the behavior of different neutron-star configurations when small perturbations are present. It is possible that these kinds of oscillations may, e.g., grow exponentially creating unstable neutron-star solutions, even though
2.4 Maximum mass

the corresponding static setups are perfectly stable. In the 1960s, S. Chandrasekhar [179, 180] studied this particular problem. To be specific, he examined infinitesimal, adiabatic radial oscillations of spherically symmetric objects, and he could formulate the corresponding stability condition. Because we are considering spherical stars at zero temperature, adiabatic oscillations are our area of interest.

Let us assume that the small radial displacement of a fluid element from the equilibrium configuration is given as

\[ \Delta r(t, r) = \sqrt{A(r)} \frac{1}{r^2} e^{i\sigma_n t} u_n(r), \]

where \( r \) is the unperturbed radius and the time dependence is sinusoidal, i.e. \( e^{i\sigma_n t} \). Following the formulation given in [8], the eigenvalue equations for the normalized amplitudes of the vibration \( u_n \) take a Sturm–Liouville form:

\[ \frac{d}{dr} \left( \Pi \frac{du_n}{dr} \right) + \left( Q + \sigma_n^2 W \right) u_n = 0, \]

where the functions read

\[ \Pi = \Gamma p \frac{jA}{r^2}, \]

\[ Q = -\frac{4}{r^2} jB \frac{dp}{dr} - \frac{8\pi G}{r^2} p(\varepsilon + p) j^3 + \frac{jA}{r^2 (\varepsilon + p)} \left( \frac{dp}{dr} \right)^2, \]

\[ W = \varepsilon + p \frac{1}{r^2} jB, \]

\[ \Gamma = \varepsilon + p \frac{1}{p} \left( \frac{\partial p}{\partial \varepsilon} \right)_S. \]

Here, \( \sigma_n^2 \) are the eigenvalues of modes \( n \) and \( S \) is the entropy while \( A \) and \( B \) are the metric components given in Eq. (2.7) so that \( j = \sqrt{AB} \).

The boundary conditions of the Sturm–Liouville equation can be expressed as [8, 180]

\[ \begin{cases} u_n = \mathcal{O}(r^3) & \text{at around } r = 0, \\ \frac{du_n}{dr} = 0 & \text{at } r = R. \end{cases} \]

It can be proven that these conditions imply that the eigenvalues \( \sigma_n^2 \) are all real, and they form a monotonically increasing sequence, i.e., \( \sigma_0^2 < \sigma_1^2 < \sigma_2^2 < \ldots \). Moreover, there also exists a unique eigenfunction \( u_n \) for every eigenvalue \( \sigma_n^2 \) so that this function has \( n \) nodes in the range \( r \in ]0, R[ \)
(see Fig. 2.2b). [8, 182] Because the time dependence of the radial perturbation is defined by the term \( e^{i\sigma_n t} \), one can see that the oscillation becomes unstable if any of the eigenvalues \( \sigma_n^2 \) is negative. Due to the above introduced properties, this means that one only needs to examine the fundamental mode \( \sigma_0^2 \) to be able to determine the stability of the configuration against radial oscillations.

In addition to radial oscillations, one also needs to examine possible convective instabilities. In this context, it is reasonable to inspect the Schwarzschild discriminant

\[ S_d(r) := \frac{dp}{dr} - \frac{\Gamma p}{p + \varepsilon} \frac{d\varepsilon}{dr}. \]
Figure 2.2: Stability of compact stars. Panel a contains a schematic mass-radius curve, and it demonstrates the white-dwarf (brown), neutron-star (black), and hypothetical twin-star (blue) sequences. The red sections correspond to unstable configurations, and the purple dots represent various critical points (not shown in the spiral part where \( \varepsilon \to \infty \); for more information about it, see e.g. [181]). In panel b, the behavior of the normalized radial displacement \( \Delta r/r \) has been illustrated as a function of \( r \) for the first three modes (\( n = 0, 1, 2 \)). The equation-of-state model is from article III. For this stable (\( \sigma_0^2 > 0 \)) neutron-star configuration, the mass parameter \( m_0 \) is equal to 320 MeV and the quark chemical potential \( \mu_q \) is 429 MeV at \( r = 0 \). The vertical axis is normalized so that \( \Delta r/r = 1 \) at \( r = 0 \), and the unit of the horizontal axis is the radius of the star \( R \). Note that the vertical axis has been truncated.

If this quantity becomes negative at some depth, it indicates that the target configuration is unstable against convection. This also implies that the temperature gradient is superadiabatic, i.e. denser matter is also colder, at some depth. Respectively, a stable configuration is subadiabatic, or \( S_d(r) > 0 \), for all \( r < R \). \[172, 183, 184\] Using the definition of the polytropic (adiabatic) exponent \( \Gamma \), one can write the discriminant as

\[
S_d = \frac{dp}{dr} - \left( \frac{\partial p}{\partial \varepsilon} \right)_S \frac{d\varepsilon}{dr}.
\] (2.96)

Based on this formulation, it can be seen that the Schwarzschild discriminant is always equal to zero for any isentropic system, i.e.

\[
\left( \frac{\partial p}{\partial \varepsilon} \right)_S = \frac{dp}{d\varepsilon}.
\] (2.97)

This implies that these neutron-star solutions are marginally stable against convection as expected.

In reality, it is laborious to check if a solution is stable against radial oscillations using Eq. (2.89). Hence, a simpler approach is often used to locate possible instability, or critical, points at zero temperature:

\[
\frac{\partial M}{\partial R} = 0.
\] (2.98)
2.5 Radius measurements

Figure 2.3: Mass-radius measurements of various neutron stars. All the different curves represent 68-per-cent probability contours. See the main text for more information.

To be exact, this condition determines the points at which one of the eigenvalues $\sigma_n^2$ changes its sign if it is not a saddle point [183, 185]. At zero temperature, it has been shown [183] that if the mass-radius curve rotates clockwise (counterclockwise) around the zero point of the derivative, one of the unstable (stable) modes becomes stable (unstable; see Fig. 2.2a). Because of this, if the lowest stable mode is known at some point, then the stability of any configuration can be determined using this visual method. Specifically, if a branch — such as the neutron-star one — is known to be stable and the density is increased, the next critical point, which is not a saddle point, is the maximum-mass configuration.

2.5 Radius measurements

Together with the mass, the radius is one of the most important macroscopic neutron-star quantities. As briefly mentioned in the previous section, some precise mass measurements have been obtained but the situation is quite the opposite in the case of the radius. In other words, no neutron stars with precisely known radius have been measured yet. Above all, the measurements have encountered systematical uncertainties in the past [186] and it is not entirely clear that the current approaches are free from such errors [187].

In Fig. 2.3, 68-per-cent mass-radius probability contours for twelve different neutron stars are given. The reader should notice that the dashed elliptic lines correspond to the two independent analyses (blue [188] and red [189]) of the same object PSR J0030+0451 using the data of the NICER (Neutron Star Interior Composition Explorer Mission) telescope. These studies examined how the hot spots — the hot regions on the surface — of the neutron star emit thermal radiation. Using a state-of-the-art theoretical prediction, the groups were able to find suitable models to fit with the observational data, and hence, to estimate the mass and the radius of the pulsar J0030+0451.
In addition to the NICER measurement, three other mass-radius measurements (solid lines in Fig. 2.3) are obtained by observing thermonuclear x-ray bursts from accreting neutron stars in low-mass x-ray binary (LMXB) systems. In the past, various radius estimates of these objects differed enormously; both very small (e.g. [190–192]) and theoretically realistic (e.g. [193]) values were reported. The article [186] has, however, shown that soft (high-accretion-rate) bursts are problematic if one wants to measure masses and radii. It argues that during a soft-state burst the target neutron star is not fully visible leading to an underestimation of the radius. Hence, works with exceptionally small radius estimations, such as soft-state-burst studies, are excluded from this short overview.

In the article [193], two neutron stars from LMXB systems 4U 1724−307 (light brown) and SAX J1810.8−2609 (cyan) are considered. This particular work uses the so-called cooling-tail method* together with realistic atmosphere models to predict the two-dimensional distribution of mass and radius. Furthermore, this paper reports mass-radius measurements for the 4U 1702−429 system as well, but the updated analysis of [197] is used instead (black curve; model D). This more up-to-date approach, also known as the direct atmosphere model, generalizes the cooling-tail method so that the complete evolution of the radiation spectra (the energy dependence) is taken into account [197]. Hence, the newer article [197] was able to produce a more accurate estimation than the previous work [193]. As Fig. 2.3 suggests, this result together with the analysis of NICER data are the most precise mass-radius estimations so far. Because the mass of the pulsar 4U 1702−429 is most probably larger than the mass of the J0030+0451 system, it should have the largest potential to constrain the neutron-star-matter EoS (cf. [198]).

In addition to the previous measurements, eight neutron stars in quiescent — only slight or not at all accreting — LMXB systems have also been considered and illustrated in Fig. 2.3 (dotted lines). In general, it is nontrivial to determine the distance to a celestial body. Hence, the analyses of [193, 197] examine neutron stars without precise prior knowledge about distance to a target object. The spectral-fit approach, where the desirable atmosphere model is directly fitted to the x-ray burst spectrum, is used to study quiescent LMXB targets, and it is a powerful tool if distance data are at hand. In this inspection, seven mass-radius distributions derived using this approach from [199] [M28 (orange), M30 (black), NGC 6304 (citrus), NGC 6397 (green), ωCen (magenta), 47 Tuc X5 (blue), and 47 Tuc X7 (red)] and one from [200] [M13 (purple)] are used assuming uniform emission from the star, i.e., without hot spots. This assumption has been made because of the simplistic nature of our review. As one can notice, some credibility limits of these measurements are enormous compared to the results of the other analysis techniques.

All in all, the take-home message from Fig. 2.3 is that no neutron stars with precisely estimated radius have yet been identified reliably. In recent years, some promising studies — such as [188, 189, 197] — have however appeared indicating that the anticipated, precise mass-radius measurements will be available in the near future.

*A more advanced approach to measure the radius of a neutron star than the one introduced in Section 2.1.2. In this method, the evolution of the blackbody temperature and normalization are compared to numerically calculated atmosphere-model results during the cooling stage. (See [194–196] for details.)
Chapter 3

Quantum Chromodynamics

At the moment, the Standard model (SM) of particle physics is the state-of-the-art theory to describe the Universe at the smallest scales. The beauty of the SM is its ability to combine three out of the four fundamental interactions — the electromagnetic, the weak and the strong one — and all observed elementary particles into one single entity. Likewise, its predictive power is remarkable.

One of the key parts of the SM is quantum chromodynamics (QCD) which describes the strong interaction. More specifically, QCD describes the interactions between quarks and gluons which are the matter particles and force carriers of the theory, respectively. Practically, one is not able to encounter free, individual quarks due to color confinement [201]. This property states that strongly-interacting particles can only appear in color-charge-neutral states, and these composite particles — hadrons — contain several valence quarks that define their quantum properties. The most famous manifestation of hadrons are nucleons — i.e. the proton and neutron — which are the basic building blocks of atomic nuclei. On the other hand, it has been hypothesized (e.g. [202]) that gluons can exist on their own as glueballs due to the complex nature of the underlying theory, but these configurations have never been observed to this day.

In this chapter, the basic properties of QCD will be first introduced — including a discussion of the phase diagram of the theory. It is followed by two sections where the current state of low- and high-density perturbative approaches have been explained in short. Lastly, the usage of holographic theories to model the strong interaction will be briefly examined.

3.1 Basic properties

So far, six different quark flavors — or species — have been observed in three different generations. The heaviest known quark flavor — the top quark — was the last to be discovered, and it was found in 1995 [203, 204]. For everyday phenomena, the two lightest quarks — the up and

†Even thought the experimental results and lattice QCD calculations support the color-confinement hypothesis, the condition has never been proven analytically.

‡The color charge is the QCD counterpart of the electric charge of electromagnetism.
down ones — have the greatest direct impact because they serve as the valence quarks of the nucleons. With increasing density, for example, the significance of the third lightest quark flavor — the strange one — increases as well. In general, baryons — composite particles consisting of an odd number of valence quarks — with nonzero strangeness are known as hyperons, and they may play an important role inside neutron stars (see the previous chapter). Furthermore, the other two heavier quark flavors — the charm and bottom ones — were also discovered in collider experiments as early as the 1970s [205–207].

Gluons — the gauge bosons of QCD — are massless particles as are the photons — the force carrier of quantum electrodynamics (QED). Nonetheless, photons and gluons differ in many ways. First, the SU(3) gauge symmetry of QCD demands that eight different gluon types should exist whereas there is only one gauge boson type in QED. Moreover, the gluons also carry color charges\(^*\) but photons are electrically neutral particles. It is also worthwhile to mention that the gluons can, therefore, self interact unlike the photons. This feature will be introduced in the next subsection.

### 3.1.1 Lagrangian density

The most fundamental quantity describing any physical system is its Lagrangian which contains information about the dynamics and symmetries of the system. The Lagrangian (density) for QCD can be given as

\[
L_{QCD} = \sum_f \bar{\psi}_f (i\not{D} - m_f) \psi_f - \frac{1}{2} \text{Tr} [F_{\mu\nu} F^{\mu\nu}],
\]

where \(f\) denotes the quark flavor and the Feynman slash notation has been utilized — i.e. \(\not{D} := \gamma^\mu D_\mu\). Here, \(\gamma^\mu\) are the Dirac gamma matrices (see Appendix A for details). One should also notice that the Einstein summation convention has been used with repeated indices excluding \(f\).

The first term represents quarks where \(\psi_f\) corresponds to quark field of flavor \(f\), and it is a Dirac four-spinor. For QCD, whose (color-)gauge symmetry group is SU(3), every \(\psi_f\) has three components, in color space, corresponding to each color charge. The first term of Eq. (3.1) can be further split into two subterms: the kinetic and mass one, respectively. Here, \(m_f\) is the mass of the quark flavor \(f\). In the kinetic term, the (gauge-)covariant derivative reads

\[
D_\mu := \partial_\mu + ig t_a A_\mu^a,
\]

where \(g\) is the strong coupling constant, the \(t_a\)’s are the generators of the Lie algebra \(su(3)\) of the SU(3) group, and the \(A_\mu^a\)’s are the color gauge fields of the gluons. Here, the index \(a\) is running from 1 to 8 representing the eight different gluons. In this case, the generators can be given using the Gell-Mann matrices \(\lambda_a\) (see Appendix A) so that

\[
t_a = \frac{\lambda_a}{2}.
\]

\(^*\)To be more specific, a gluon carries a superpositional combination of one color and one anticolor charges.
It can be shown that the Lagrangian is invariant under a local SU(3) gauge transformation:

\[ \psi_f(x) \mapsto U(x)\psi_f(x), \]

\[ A^a_\mu(x) \mapsto U(x)A^a_\mu(x)U^\dagger(x) - \frac{1}{ig}U(x)\partial_\mu U^\dagger(x), \]

where the unitary transformation function is

\[ U(x) = \exp[-it_a\theta^a(x)] \]

so that \( \theta^a(x) \) is a function that depends on \( x \). On the one hand, one can use the SU(3) symmetry as a starting point and to construct the QCD Lagrangian demanding that it contains all possible gauge-invariant terms up to mass dimension four. Following this procedure, one is able to create Eq. (3.1) with two additional modifications. Firstly, the Lagrangian contains another gluonic term — namely one proportional to \( \epsilon^{\mu\nu\alpha\beta} \text{Tr}(F_{\mu\nu}F_{\alpha\beta}) \), where \( \epsilon^{\mu\nu\alpha\beta} \) is the Levi-Civita symbol. Secondly, the quark-mass term should also contain a chiral phase parameter \( \theta' \) so that \( m_f \mapsto m_f \exp(i\gamma_5\theta') \). On the other hand, experimental evidence [208, 209] suggests that these additional terms are very small or even nonexisting.

The second term of the QCD Lagrangian is the gluonic part, where the field strength tensor \( F_{\mu\nu} \) can be given as

\[ F_{\mu\nu} := t_a F^a_{\mu\nu} = \frac{i}{g} [D_\mu, D_\nu]. \]

Its components can be written as

\[ F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s f_{abc} A^b_\mu A^c_\nu, \]

where \( f_{abc} \) is a totally antisymmetric structure constant (see Appendix A).

The feature that set QCD — or any non-Abelian field theory in general — apart from the simple QED framework with Abelian U(1) symmetry is the existence of the last term in Eq. (3.8). Due to this contribution, the gauge bosons of non-Abelian theories — e.g. the gluons in QCD — self interact. These interactions among three (3g) and four (4g) gluons can be presented as

\[ L_{3g} = -\frac{g}{2} f^a_{bc} (\partial^\mu A^c_\nu - \partial^\nu A^c_\mu) A^b_\mu A^c_\nu, \]

\[ L_{4g} = -\frac{g^2}{4} f_{abc} f^{ade} A^b_\mu A^c_\nu A^d_\mu A^e_\nu. \]

### 3.1.2 Asymptotic freedom and the scale parameter

To examine the running of the coupling \( g \), the so-called beta function can be defined as

\[ \beta(\alpha_s) := Q \frac{d\alpha_s}{dQ}, \]

where \( Q \) is the reference scale and \( \alpha_s := g^2/(4\pi) \). To date, the beta function of QCD has been calculated up to the five-loop order in the minimal subtraction (MS) scheme [210, 211]. The one-loop order result is given as

\[ \beta(\alpha_s) = \frac{2N_f - 11N_c}{6\pi} \alpha_s^2 + \mathcal{O}(\alpha_s^3), \]
where $N_f$ and $N_c$ are the number of quark flavors and colors, respectively [212, 213]. So far, six different flavors and three color charges have been detected, and notable signs of extra quark flavors (or colors) have not been found (see e.g. [80, 214–216]). If $N_f/N_c < 11/2$, the one-loop beta function is negative indicating that the coupling decreases with increasing energy — or with decreasing distance — asymptotically. This ultraviolet behavior of QCD is known as asymptotic freedom and it was discovered in 1973 [212, 213]. The beta function also suggests that high-density QCD phenomena can only be probed using perturbation methods because the coupling constant $g$ is then small. QED, on the other hand, behaves differently — its one-loop beta function is strictly positive

$$\beta(\alpha) = \frac{2\alpha^2}{3\pi} + \mathcal{O}(\alpha^3),$$

(3.13)

where $\alpha$ is the fine-structure constant [217].

One can also reformulate the beta function using a separation of variables:

$$\int \frac{dQ}{Q} = \int \frac{d\alpha_s}{\beta(\alpha_s)}.$$ 

(3.14)

When solving the differential equation, then the dimensionful constant of integration can be expressed as

$$\Lambda_{QCD} = Q \exp \left[- \int \frac{d\alpha_s}{\beta(\alpha_s)} \right].$$

(3.15)

In the case of the one-loop-level result [Eq. (3.12)], this simply implies that

$$\alpha_s^{-1}(Q) = \frac{11N_c - 2N_f}{6\pi} \ln \left( \frac{Q}{\Lambda_{QCD}} \right).$$

(3.16)

Based on this formulation, it is easy to see that $\Lambda_{QCD}$ specifies the scale of the theory, and it is, therefore, known as the QCD scale parameter. If one wants to evaluate the value of $\Lambda_{QCD}$, one can use the world average of the strong coupling constant

$$\alpha_s(m_Z) = 0.1179 \pm 0.0010,$$

(3.17)

where the reference energy is equal to the mass of the Z boson, $m_Z = 91.1876 \pm 0.0021$ GeV [80]. This rough, one-loop-level estimation suggests that $\Lambda_{QCD} \approx 200$ MeV.

### 3.1.3 Phase diagram

One of the most interesting open problems in the study of QCD is to figure out the structure of the phase diagram. This diagram is usually illustrated as a function of the temperature $T$ and baryon chemical potential $\mu_B$ (see Fig. 3.1). Even though the starting point — the Lagrangian of QCD — is well-known, any suitable first-principle method to probe the diagram at arbitrary temperatures and chemical potentials has not been developed yet. However, some useful approaches to tackle this problem already exist but their scope is very limited.
Figure 3.1: Schematic phase diagram of quantum chromodynamics. The three major phases — the hadronic (blue), quark-gluon-plasma (yellow), and quark-matter (red) ones — are only shown. The solid lines correspond to the first-order phase transitions whereas the dashed lines represent crossovers. Likewise, the dots illustrate the critical points (A–C) and the triple point (D).

One of such method is perturbation theory. With this tool, it is possible to investigate the behavior of matter when the coupling of QCD is weak. Unfortunately, this implies that perturbative methods are only applicable when the temperature or density is large. Another widely used method is lattice field theory — an \textit{ab initio} approach to study QCD nonperturbatively. Due to the notorious sign problem [218], this method is only appropriate when the baryon chemical potential $\mu_B$ is close to zero. Nevertheless, this region is one of the best-known part of the phase diagram at the moment (see details below). It is also possible to investigate the problem from a more phenomenological perspective using various model calculations (e.g. [219–224]). Although this is an effective, qualitative way to scan the phase diagram, there is no guarantee that a model in hand captures the essential features of the underlying theory. So, the predictions of these models should always be treated with caution.

A schematic phase diagram of Fig. 3.1 illustrates the three main phases — the hadronic, quark-gluon-plasma, and quark-matter ones. Here, we will focus on these most notable phases because the phase diagram is still largely unknown. Nevertheless, it has been suggested that other possible phases — such the quarkyonic-matter one — may also exist, but we will not consider them in this thesis (see e.g. the review of [225] for further information). We will, however, present some interesting but speculative features of the diagram to stimulate discussion.

As hinted above, the behavior of the phase diagram is well understood at small chemical potentials. In this region, it has been observed that hadronic matter transforms into quark-gluon plasma via a smooth crossover (see Fig. 3.1) — a continuous change from one phase to another without distinguishable transition point — by increasing the temperature as lattice studies [226, 227] suggested. Even though the exact location of the crossover point cannot be pinpointed,
lattice studies [228–231] have been able to estimate the pseudo-critical temperature $T_c$ based on the behaviors of several observables. These analyses propose that the value of $T_c$ is about 155 MeV. In addition to these theoretical studies, collider experiments at CERN (European Organization for Nuclear Research) and BNL (Brookhaven National Laboratory) have managed to create quark-gluon plasma — plasma-like state consisting of deconfined quarks and gluons (see the detailed review of [232]).

Another well-studied part of the phase diagram is highlighted with purple color within the hadronic phase in Fig. 3.1. This corresponds to the first-order transition between the gaseous (low $\mu_B$) and liquid-like (high $\mu_B$) hadronic phases. The corresponding critical point (marked as A) is located at $n_c/n_s \approx 0.3–0.4$ and $T_c \approx 16–18$ MeV. (See the review of [233].)

The third phase, shown in Fig. 3.1 as red, is the (cool) quark-matter one. Many model calculations have proposed that this part of the phase diagram consist of a collection of various color-superconducting phases (see e.g. the review of [234]; see also Fig. 2 of [220] or Figs. 1 and 2 of [235]). This kind of superconducting behavior is similar to the one with electrons described by the Bardeen–Cooper–Schrieffer (BCS) theory [236]. The more complex nature of the strong interaction, however, gives rise to a richer class of physical phenomena, i.e. the number of possible color-superconducting phases is huge. This does not necessarily have to be the case even though model calculations usually favor super-conducting phases. Although the (cool) quark-matter and quark-gluon-plasma phases are explicitly separated by a first-order transition (red solid line) in Fig. 3.1, it is certainly possible that these two phases form a united quark-matter phase together.

Beyond the above points, the phase structure given in Fig. 3.1 is even more speculative. For example, it is not known if the transition between the hadronic and quark-matter phases (blue solid line) is really a first-order one with a critical point B as the figure indicates (see e.g. [224]). Nonetheless, multiple phenomenological studies, such as [219, 220, 222], support this claim. Secondly, the existence of the third critical point (C) has also been hypothesized (e.g. [221, 223]) and some authors have constructed models with even more complex phase structure [237]. In a more typical scenario, the first-order-transition line meets the $\mu_B$-axis without forming this so-called quark-hadron-continuity structure.

3.2 Nuclear physics at low densities

As pointed out in the previous chapter (see Fig. 3.2, as well), the nuclear-physics side of the EoS is well-known up to the crust-core interface — around 0.1 to 0.5 times the saturation density. For example, the works of [98, 99] are often used to describe this part of the EoS. The main degrees of freedom of nuclear matter are not quarks and gluons even though they are its fundamental building blocks. On the contrary, the effective degrees of freedom are confined configurations — such as the nuclei and pions. Hence, it is practical to use an effective-field-theory approach to study nuclear matter — especially near saturation. The most common approach is to utilize the
3.2 Nuclear physics at low densities

Figure 3.2: Known behavior of the equation of state of QCD at zero temperature. The relatively well-known outer- and inner-crust equations of state [98, 99] are illustrated using orange and blue colors, respectively. The low-density chiral-effective-field-theory results of [32] are highlighted with red color (Nucl.) while the high-density perturbative-QCD (pQCD) calculations [11, 238] are given in black.

so-called chiral effective field theory (cEFT) which makes use of the almost chiral nature of the QCD Lagrangian. This can be seen from the chiral formulation of Eq. (3.1) with two lightest quark flavors

$$\mathcal{L}_{\text{QCD}} = \bar{q}_L i \not{D} q_L + \bar{q}_R i \not{D} q_R - \bar{q}_L \tilde{m} q_R - \bar{q}_R \tilde{m} q_L - \frac{1}{2} \text{Tr} [G_{\mu\nu} G^{\mu\nu}], \quad (3.18)$$

where \( \tilde{m} \) is the mass matrix defined so that \( \tilde{m} = \text{diag}(m_u, m_d) \). Using the quark field \( q = (\psi_u \psi_d)^T \), the left- and right-handed component fields are given as \( q_L = (1 - \gamma_5)q/2 \) and \( q_R = (1 + \gamma_5)q/2 \), respectively. In the massless limit, the Lagrangian is invariant under the chiral transformation:

$$q_i \mapsto \exp(-i\theta_{ij}\sigma^j/2)q_i, \quad (3.19)$$

where \( \sigma^j \) are the Pauli matrices (see Appendix A) and \( \theta_{ij} \) are angles while \( i = L, R \). Compared to the nucleonic masses of \( \mathcal{O}(\text{GeV}) \), the light-quark masses \( \mathcal{O}(\text{MeV}) \) are insignificant, and therefore, the light-QCD Lagrangian is approximately chirally symmetric. [239]

In the cEFT framework, the calculations for pure neutron matter can be carried out using, e.g., perturbative methods, and currently, some contributions are even studied up to N\(^4\)LO\(^*\) (see review [240] and reference therein). Here, the small perturbation parameters are \( P/\Lambda_c \) and \( m_\pi/\Lambda_c \), where \( P \) is the typical momentum scale and \( m_\pi \approx 135 \text{ MeV} \) is the pion mass, while \( \Lambda_c \approx 500 \text{ MeV} \) is the breaking scale of the effective theory [35, 240]. The modern cEFT studies of neutron-star matter — such as [5, 35, 138] — often employ two- and three-nucleon interactions up to \( 2n_s \) at most because the uncertainties quickly increase with density. However, some more

\(^*\)Here, N\(^4\)LO refers to the next-to-next-to-next-to-leading order.
ambitious calculations with distinctly higher densities (see e.g. [241]) have been performed as well. In this thesis, the conservative cEFT results of [32] for neutron-star matter up to 1.1\(n_s\) are used (cf. Fig. 3.2).

### 3.3 Perturbative QCD at high densities

In the weak coupling regime, where the temperature or the density are extremely large, perturbative techniques are available. Unfortunately, many interesting regions of the phase diagram — such as the cores of neutron stars — cannot be directly probed using these methods. In addition, some phenomena — such as the instantons — are not obtainable without a proper nonperturbative description. For an in-depth review of the current state of the EoS of perturbative QCD, refer to [242].

As has been pointed out, an old neutron star can be treated as an object at zero temperature. In this limit with massless quarks, the full three-loop (\(\alpha_s^2\)) results of the EoS have been known for a long time [243] (see [244] as well), but recently, the first part of the four-loop contribution — namely the \(\alpha_s^3 \ln \alpha_s\) order — has, besides, been solved [12]. It is anticipated that the four-loop result could be available — i.e. the remaining \(\alpha_s^3 \ln \alpha_s\) and \(\alpha_s^3\) terms will be computed — in the near future. On the other hand, a three-loop-level calculation for realistic strange-quark matter — i.e. including the contribution of the strange-quark mass — has been carried out in [11]. A simple approximative formulation of these results ([238]; see the following subsection as well) has been used to model the high-density behavior of quark matter in the calculations described in this thesis (see Fig. 3.2).

Besides the zero-temperature limit, other perturbative regions have also been considered in the literature. Recently, calculations for dense matter with small, nonvanishing temperature have been carried out up to \(O(g^5)\) [245]. With vanishing chemical potential and large temperature, the last fully perturbative (\(\alpha_s^3 \ln \alpha_s\)) order has already been derived [246] and the corresponding small-chemical-potential extension is also known [244].

#### 3.3.1 Pocket formula

The three-loop zero-temperature pQCD results of [11] with nonzero strange-quark mass can be written in a more compact form as given in [238]. The pressure of beta-stable and electrically neutral QCD matter can be represented using only the baryon chemical potential \(\mu_B\) and the dimensionless scale parameter \(X := 3\bar{\Lambda}/\mu_B\):

\[
p_{pQCD} (\mu_B, X) = p_{FD} (\mu_B) \left( c_1 - \frac{a(X)}{(\mu_B/\text{GeV}) - b(X)} \right),
\]

(3.20)

where \(a(X) = d_1 X^{-\nu_1}\) and \(b(X) = d_2 X^{-\nu_2}\). Here, \(\bar{\Lambda}\) is the renormalization scale — the arbitrary energy scale where the desirable renormalization conditions are assigned to remove unfavourable ultraviolet divergences — in the modified minimal subtraction (\(\overline{\text{MS}}\)) scheme. Likewise, the
function

\[ p_{FD}(\mu_B) = \frac{3}{4\pi^2} \left( \frac{\mu_B}{3} \right)^4 \]  

(3.21)
is the Fermi–Dirac pressure which corresponds to the massless, noninteracting limit of a system of three quark flavors. The fitting parameters \( c_1, d_1, d_2, \nu_1 \) and \( \nu_2 \) are selected so that the model faithfully represents the pressure, quark number density and squared speed of sound (see the original study for details). When \( \mu_B > 2 \text{ GeV}^* \), \( p(\mu_B) > 0 \), and \( 1 \leq X \leq 4 \); the authors of [238] argue that the optimal values for these parameters are

\[ c_1 = 0.9008, \quad d_1 = 0.5034, \quad d_2 = 1.452, \quad \nu_1 = 0.3553, \quad \nu_2 = 0.9101. \]  

(3.22)

### 3.4 Holographic duality

String theory is the leading candidate for a theory of quantum gravity. Although it has not been able to produce any observationally verifiable predictions, one of its side products — the gauge/gravity duality — has. Its main realization is the conjectured AdS/CFT correspondence — proposed by J. Maldacena in 1997 [37] — which connects two very different kind of physical systems — (classical) gravity and quantum (field) theory. Here, AdS and CFT stand for anti-de Sitter space and conformal field theory, respectively, and these terms will be defined more closely in the following subsection.

In this thesis, the AdS/CFT approach has been exploited because it is a first-principle method to study various strongly coupled systems, such as QCD. As stated previously, there are no \textit{ab initio} methods to probe the QCD directly at zero temperature. However, this kind of indirect way can possibly reveal important information about the generic behavior of strongly coupled systems. That is why it is significant to study this approach more closely.

#### 3.4.1 AdS/CFT correspondence

Maybe the most famous and highly used version of the AdS/CFT correspondence is the setup where the gravity side is described by a ten-dimensional type IIB string theory on \( \text{AdS}_5 \times S^5 \) space whereas the corresponding dual field theory is the \( \mathcal{N} = 4 \) supersymmetric Yang–Mills theory living in four-dimensional Minkowski space (see [37]). Here, \( S^5 \) is the 5-sphere. This approach has been used in this thesis, but some other implementations of the correspondence have also been considered in the literature [37, 247].

The gravity side is mainly described by \((d + 1)\)-dimensional anti-de Sitter space (AdS\(_{d+1}\)). It is the vacuum solution of the Einstein field equations with negative cosmological constant \( \Lambda_C \):

\[ G_{\mu\nu} = -\Lambda_C g_{\mu\nu}. \]  

(3.23)

\*Notice that the original research article [238] contains a typographical error. To fix it, the inequality symbol has been reversed.
A common way to represent the line element of AdS space is to use Poincaré coordinates:

$$ds^2 = \frac{L^2}{z^2} \left( \eta_{\mu\nu} dx^\mu dx^\nu - dz^2 \right), \quad (3.24)$$

where $z$ is the extra dimension and $L$ is the curvature radius while indices $\mu, \nu$ range from 0 to $d - 1$. Variables $z$ and $L$ are defined so that $z > 0$ and

$$L^2 = -\frac{d(d-1)}{2\Lambda_C}. \quad (3.25)$$

Nevertheless, this parametrization does not cover the whole space, and hence, another suitable coordinate system has to be exploited at times, such as the so-called global coordinates. [248–250]

The other main component of the correspondence is a CFT. It is a quantum field theory that is invariant under conformal transformations that leave all angles intact but allow, e.g., rotations and translations to happen. Often, the $\mathcal{N} = 4$ supersymmetric Yang–Mills (SYM) theory is used to describe the quantum side of the duality because it resembles QCD and it is a (relatively) well understood theory. Although the particle content is, e.g., rather similar, these two theories differ at the level of details; unlike QCD, the SYM theory is a CFT and supersymmetric, for instance. Put it more precisely, the first feature basically means that the SYM theory cannot have massive particles because it would break the needed scale invariance. The second condition suggests that there are some kinds of (yet experimentally unobserved) symmetries that connect the degrees of freedom of bosons and fermions. Actually, the $\mathcal{N} = 4$ condition underlines the fact that the given SYM theory is as supersymmetric as theoretically possible. [249–251]

So to speak, the CFT lives on the surface of AdS space meaning that the dimension of the AdS space is one larger than that of the corresponding CFT. This property follows the holographic principle which suggests that the surface of a system contains encoded information about its content [252, 253]. One of the most famous examples of the holographic principle is the fact that the entropy of a black hole is proportional to the surface area $A_{BH}$ — and not the volume — of the event horizon: [254–256]

$$S = \frac{A_{BH}}{4G}. \quad (3.26)$$

This statement also implies that a black hole should radiate as S. W. Hawking famously suggested [255, 256].

In this thesis, the AdS/CFT correspondence has been used because it has been conjectured that the dual of a strongly coupled SYM theory is a weakly coupled system described by classical gravity (and vice versa). Often, calculations are much easier to do in the weak limit, and these results can then be converted to describe the strongly coupled system using the duality together with a suitable dictionary. In this matter, one can get information about a QCD-like EoS by studying a simpler gravity-based system. [248, 249]

It is good to be aware that there exists several versions of the AdS/CFT conjecture. The weakest one states that the correspondence is only valid if the ’t Hooft coupling,

$$\lambda_{YM} := g_{YM}^2 N_c = g_s N_c, \quad (3.27)$$
is large. Here, $g_{\text{YM}}$ and $g_s$ are the Yang-Mills and string couplings, respectively. A stronger version, in contrast, allows finite 't Hooft coupling but the string coupling has to vanish when $N_c \rightarrow \infty$. Moreover, the strongest formulation of the conjecture guarantees that the duality stands regardless of the values of $g_s$ and $N_c$ if the dual system is solvable. [249]

It should be noted that the above introduction of the AdS/CFT correspondence is only a brief coverage of the topic. For a thorough introduction of the issue, see [248–250].

### 3.4.2 Applications

So far, the gauge/gravity duality has had many interesting applications in, e.g., condensed matter physics. Particularly, superconductivity and superfluidity are theoretically studied using this correspondence among other methods (see e.g. [39, 40]). In hot QCD, holographic techniques have been used to conjecture that the lower bound of the shear viscosity per the entropy density is $1/(4\pi)$ in quark gluon plasma [38]. This result is in agreement with other estimations (see e.g. [257–260]), and it is therefore often stated to be a manifestation of the power of the correspondence. Unfortunately, the gauge/gravity dual of QCD has not been found yet, and in fact, the AdS/CFT correspondence cannot directly describe QCD which is not a CFT. Consequently, the holographic results should always been taken with a healthy amount of skepticism. One should, nonetheless, remember that the approach might still teach us something useful about some characteristic features of QCD, otherwise not accessible.

In neutron-star physics, the holographic approach is a relatively new tool — the first holographic studies of neutron stars were introduced at around 2010 [261–266]. Recently, several other studies have been published (e.g. [41–47]) — including paper III. In conclusion, one may expect that the pace will only increase in the near future bringing even more realistic holographic setups, and hopefully, interesting and useful outcomes as well.
Chapter 4

Interpolation

The main aim of this thesis is to probe the EoS of cold and dense QCD matter. As pointed out, no \textit{ab initio} method is available to solve this relationship starting from the QCD Lagrangian at zero temperature. Nevertheless, the EoS is known when the density is either very low or very high as illustrated in Fig. 3.2. Hence, many studies (e.g. \cite{25–30}) have used various model calculations to probe the poorly known part of the EoS at $\epsilon \approx 200$–10,000 MeV/fm$^3$. Even though these model calculations are able to produce precise predictions, one cannot be sure about their validity without sufficient observational and experimental input. And currently, we are lacking this kind of information.

That is why it is good to explore other kinds of approaches as well, such as extrapolation or interpolation tools. In this context, different extrapolation methods have been used to extend the known low-density nuclear EoS to higher density using various techniques. This allows to inspect the properties of matter inside neutron stars, and thus, to describe their macroscopic properties. On the contrary, the interpolation methods do not just take into account the low-density knowledge but also the high-density behavior of the EoS. Simply put, one can then delimit the EoS from two different directions.

Although it is much more common to use the extrapolation methods in the literature (e.g. \cite{32–35, 193, 267}), we will utilize the interpolation approach in this thesis. In our framework, the low-density limit (or constraint) is the cEFT result of \cite{32} up to $n_{cEFT} := 1.1n_s$. Respectively, we will use the pQCD calculations of \cite{11} (or its simplified pocket formulation \cite{238}, to be specific) to describe the high-density quark-matter EoS above $\mu_{pQCD} := 2.6$ GeV. The actual interpolation procedure is as follows. The first matching happens at $n_{cEFT}$ so that

1. $p_{cEFT}(n_{cEFT}) = p_1(1.1n_s)$ and $\varepsilon_{cEFT}(n_{cEFT}) = \varepsilon_1(1.1n_s)$,

where the subscripts cEFT and 1 corresponds to the cEFT model and the (first) interpolation section, respectively. Secondly, the (possible) matching conditions between two piecewise parts of the interpolation function are

2. $p_i(\mu_{t,i}) = p_{i+1}(\mu_{t,i})$ and $\varepsilon_i(\mu_{t,i}) = \varepsilon_{i+1}(\mu_{t,i})$,
where $\mu_{t,i}$ is the transition point. Here, $i = 1, 2, \ldots, N_p$, where $N_p$ is the number of piecewise segments. Lastly, the interpolated part has to be matched with the pQCD EoS:

$$p_{N_p}(\mu_{p\text{QCD}}) = p_{\text{pQCD}}(\mu_{p\text{QCD}}) \quad \text{and} \quad \varepsilon_{N_p}(\mu_{p\text{QCD}}) = \varepsilon_{\text{pQCD}}(\mu_{p\text{QCD}}).$$

It is noteworthy to mention that the chemical potential always behaves like $\mu_B(n_{c\text{EFT}}) < \mu_{t,1} < \mu_{t,2} < \ldots < \mu_{t,N_p} < \mu_{p\text{QCD}}$.

The interpolation function alone is not enough because an EoS candidate has to fulfill certain physical conditions. In Section 2.4, we introduced four conditions that every realistic neutron-star EoS should fulfill. Here, we are going to use a simplified set of these conditions:

A) $0 \leq c_s^2 \leq 1$ \quad and \quad B) $\frac{\partial M}{\partial \varepsilon_c} \leq 0$.

In other words, the first item means that the EoS has to be subluminal ($c_s^2 \leq 1$) and microscopically stable ($c_s^2 \geq 0$). However, we also allow first-order phase transitions where the speed of sound is exactly zero, and then, the EoS is only marginally stable. Secondly, because we know that the low-density EoS satisfies all original conditions, we just need to use at the simpler stability condition B). In practice, this means that all neutron-star solutions are stable up to $M = M_{\text{TOV}}$.

It is also reasonable to use astronomical observations to indirectly restrict the underlying EoS. In this thesis, we will use two main constraints: maximum-mass and tidal-deformability measurements. Put simply, we can express these as

a) $M_{\text{TOV}} \geq 1.97M_\odot$ \quad and \quad b) $70 < \Lambda_{1.4} < 580$,

where $\Lambda_{1.4} := \Lambda(1.4M_\odot)$. We will often refer to the first condition as the two-solar-mass constraint, and it corresponds to the lower limit of the 68-per-cent confidence interval of the mass estimation of PSR J0348+0432 [121]. The second item is the updated version of the tidal-deformability estimation (90-per-cent credibility interval) for a $1.4M_\odot$ neutron star based on the gravitational-wave event GW170817 [34]. It should be pointed out that we also used the old estimation [6] in paper I:

b*) $\Lambda_{1.4} < 800$.

This chapter is structured as followed. In the first three sections, we will introduce three different interpolation functions: piecewise-defined polytropes, the spectral representation, and the speed-of-sound method. In addition, we will compare these functions as well as examine the basic properties of the interpolation methods.

### 4.1 Piecewise polytropes

A common and simple way to interpolate the EoS is to use polytropes. It is typical that the actual interpolation function is defined in a piecewise manner so that the expression consists
of multiple polytropic segments, also known as monotropes. This method became popular in neutron-star research after the study of [72]. They found out that already a few monotropes are enough to efficiently extrapolate several realistic model EoSs. Since then, this method has also been used for interpolation purposes (e.g. [I, 4, 36]).

One of the possible formulations of a polytrope is

\[ p = \kappa \varepsilon^\gamma, \]  

(4.1)

where \( \kappa \) is a (polytropic) constant. Here, the defining variable is known as the polytropic index

\[ \gamma = \frac{\partial \ln p}{\partial \ln \varepsilon} = \frac{\varepsilon \partial p}{p \partial \varepsilon}, \]  

(4.2)

and it defines the overall behavior of the approximated EoS in the density interval studied.\(^*\)

Using the (adiabatic) speed of sound

\[ c_s^2 := \left(\frac{\partial p}{\partial \varepsilon}\right)_S, \]  

(4.3)

where \( S \) is the entropy, we can further simplify the polytropic index at zero temperature:

\[ \gamma = \frac{\varepsilon}{p} c_s^2. \]  

(4.4)

It should be pointed out that this EoS model can be used to easily describe a first-order phase transition by setting \( \gamma = 0 \). In that case, the pressure stays constant with increasing energy density.

It is also possible to define another polytrope so that the pressure is directly proportional to the baryon number density \( n_B \):\(^†\)

\[ p(n_B) = K n_B^\Gamma, \]  

(4.5)

where \( K \) and \( \Gamma \) are the related polytropic coefficients. Accordingly, the adiabatic index is given as

\[ \Gamma(n_B) = \frac{\partial \ln p}{\partial \ln n_B} = \frac{n_B}{p} \frac{\partial p}{\partial n_B} = \frac{\varepsilon + p}{p} c_s^2. \]  

(4.6)

Although it might not seem obvious, these two polytropic setups are related via

\[ \Gamma = \gamma + c_s^2. \]  

(4.7)

Furthermore, the energy density \( \varepsilon \) can be calculated starting from [268]

\[ d\varepsilon = n_B T ds + \frac{\varepsilon + p}{n_B} dn_B, \]  

(4.8)

\(^*\)It is good to note that another constant \( N \) is often used instead of the polytropic index \( \gamma \) so that the conversion is defined as \( \gamma = 1 + 1/N \).

\(^†\)It is worthwhile to mention that one can replace the baryon number density \( n_B \) by the baryon mass density \( \rho_B = n_B m_B \), where \( m_B \) is the corresponding mass of the baryon. However, one needs to notice that the adiabatic coefficient \( K \) have to be transformed accordingly: \( K \rightarrow K/m_B^\Gamma \).
where $s$ is the entropy per baryon. In the zero-temperature limit, the energy density is then given by

$$
\epsilon(n_B) = \frac{p(n_B)}{\Gamma - 1} + \left( \epsilon_0 - \frac{p_0}{\Gamma - 1} \right) \frac{n_B}{n_0},
$$

(4.9)
or if the adiabatic index $\Gamma = 1$, then

$$
\epsilon(n_B) = p \ln \left( \frac{n_B}{n_0} \right) + \epsilon_0 \frac{n_B}{n_0}.
$$

(4.10)

Here, $n_0$ is a reference density so that $\epsilon_0 := \epsilon(n_0)$ and $p_0 := p(n_0)$. This quantity is needed when the TOV equations are solved.

When one examines an isentropic system, the baryon chemical potential is given by

$$
\mu_B = \left( \frac{\partial \epsilon}{\partial n_B} \right)_{S,V} = \frac{\epsilon + p}{n_B}.
$$

(4.11)

Using this simple relation, one can reformulate Eq. (4.5) so that

$$
p(\mu_B) = K \left[ n_0^{\Gamma - 1} + \frac{\Gamma - 1}{K \Gamma} (\mu_B - \mu_0) \right]^{\frac{1}{\Gamma - 1}},
$$

(4.12)
or if $\Gamma = 1$, then

$$
p(\mu_B) = p_0 \exp (\mu_B - \mu_0).
$$

(4.13)

Here, $\mu_0$ is a reference chemical potential defined as $\mu_0 := (\epsilon_0 + p_0)/n_0$.

Fig. 4.1 contains two example EoSs illustrating $\Gamma$-polytropes. The red dotted line corresponds to a quadrutrope — a piecewise polytrope with four $\Gamma$-monotropic segments — that satisfies the conditions A), B) and b) but is unable to meet the two-solar-mass limit. In contrast, the blue dotted line represents a quadrutrope that does not fulfill the tidal-deformability constraint b) but supports massive stars.

### 4.2 Spectral representation

Even though it is easy and relatively efficient to interpolate the EoS using polytropes, this method has its own weaknesses as well. For instance, one of its prominent flaws is the speed-of-sound behavior; although the zero-temperature EoS $\epsilon(p)$ is continuous, its first derivative — i.e. the squared speed of sound — is not. This is particularly problematic because the speed of sound is often needed — for example when one wants to determine the tidal deformability of a neutron star (see Section 2.2). It is also troublesome to impose a realistic subluminality analysis when the speed of sound is defined in a piecewise manner and is very spiky.

There exist multiple possible solutions to this problem but a promising one is the spectral method introduced by L. Lindblom [269]. This technique assumes that the adiabatic index $\Gamma$ is a continuous function with respect to the pressure. Hence, we can formulate an ordinary differential equation from Eq. (4.6) at zero temperature [269, 270]:

$$
\frac{d\epsilon}{dp} = \frac{\epsilon(p) + p}{p \Gamma(p)}.
$$

(4.14)
4.2 Spectral representation

Figure 4.1: Example interpolated equations of state (a) and the corresponding mass-radius curves (b) obtained from the Tolman–Oppenheimer–Volkoff equations. For completeness, the behaviors of the squared speeds of sound $c_s^2$ as functions of the energy density $\varepsilon$ (c) and the pressures $p$ normalized by the Fermi–Dirac limit $p_{FD}$ against the baryon chemical potential $\mu_B$ (d) are also shown. The polytropic (spectral) [speed-of-sound-squared] equations of state are illustrated by dotted (dashed) [solid] curves and the dots represent the maximum-mass configurations. In panel c, the black horizontal line corresponds to the conformal limit, $c_s^2 = 1/3$. Moreover, the chiral-effective-field-theory [32] and perturbative-QCD [238] equations of states are highlighted using the light blue color in panels a and d.
This equation can be solved utilizing an integrating factor [269, 270]

$$\tilde{\mu}(p) = \exp\left[-\int_{p_0}^{p} \frac{dp'}{p'\Gamma(p')}\right], \quad (4.15)$$

where the energy density is

$$\varepsilon(p) = \frac{1}{\tilde{\mu}(p)} \left[\varepsilon_0 + \int_{p_0}^{p} \frac{\tilde{\mu}(p')}{\Gamma(p')} dp'\right]. \quad (4.16)$$

Similarly, the corresponding baryon number density can be written as

$$n_B(p) = n_0 \exp\left[\int_{p_0}^{p} \frac{p'\Gamma(p')}{\tilde{\gamma}(p')} dp'\right], \quad (4.17)$$

using Eqs. (4.8) and (4.14).

The easiest way to create a spectral representation of $\Gamma$ would be using a linear combination of basis functions $\Phi(p)$. However, the EoS has to be thermodynamically stable which means that the adiabatic index has to always be nonnegative, $\Gamma \geq 0$. Therefore, it is convenient to define

$$\Gamma = \exp\left[\sum_k \tilde{\gamma}_k \Phi_k(p)\right], \quad (4.18)$$

where $\tilde{\gamma}_k$ are the spectral coefficients [269, 270]. Typically, the basis functions are chosen to be $\Phi_k(p) = [\log(p/p_0)]^k$ (see e.g. the original study [269]) but the Chebyshev polynomials of the first kind $\Phi_k(p) = T_k(\tilde{p})$, where the pressure $p$ is scaled so that $\tilde{p} \in [-1, 1]$, have also been used [II].

In addition, other similar spectral methods have been developed. For instance, the spectral expansion can be written using the relativistic enthalpy,

$$h(p) = \int_{p_0}^{p} \frac{dp'}{p'/\varepsilon(p')}, \quad (4.19)$$

as the base variable instead of the pressure [269, 271, 272]. This formulation has many advantages compared to the original one when performing numerical studies — e.g. the model has better behavior near the surface and the center of the star [273, 274]. In particular, the radius of a star can then be solved accurately.

It is also possible to define a completely new base function. An interesting option to replace the adiabatic index $\Gamma$ is [270]

$$\Upsilon = \frac{1 - c_s^2}{\varepsilon_s^2}. \quad (4.20)$$

At zero temperature, Eq. (4.3) can now be reformulated as

$$\frac{d\varepsilon}{dp} = 1 + \Upsilon, \quad (4.21)$$

from where the needed thermodynamical variables can be calculated [270]. As can be easily noted, the advantage of this base function is the automatically enforced subluminality — i.e.
\( c_s^2 \leq 1 \). In other words, this condition does not have to be separately implemented as the case of the polytropes or the original spectral method.

For illustrative purposes, Fig. 4.1 represents one example spectral EoS (black dashed line). This particular EoS uses the Chebyshev polynomials up to degree five, i.e.

\[
\Gamma = \exp \left[ \sum_{k=0}^{5} \tilde{\gamma}_k T_k(p) \right],
\]

(4.22)
to model the intermediate density range, and it satisfies all four conditions given at the beginning of the chapter.

### 4.3 Speed-of-sound method

Even though the different spectral methods are capable of modeling the EoS with high precision [270–272], they are often too resource intensive for practical usage. Furthermore, the spectral-interpolation techniques are not ideal to describe possible first-order phase transitions due to their smooth nature. It is, therefore, essential to consider more cost effective solutions to model the EoS and its derivative in continuous manner. One of the simplest ways to achieve this goal is to interpolate the speed of sound — or its square — and several suitable implementations have been proposed in the literature (e.g. [35, 129, 198, 275]). In this section, we have, however, considered a method that has been introduced in article II.

The derivate of Eq. (4.11) can be expressed as

\[
\frac{d\mu_B}{dn_B} = \frac{\mu_B}{n_B} c_s^2.
\]

(4.23)

This ordinary differential equation can be solved using a separation of variables:

\[
n_B(\mu_B) = n_0 \exp \left[ \int_{\mu_B}^{\mu_0} \frac{d\mu'}{\mu' c_s^2(\mu')} \right],
\]

(4.24)

where \( \mu_0 \) is the reference chemical potential so that \( n_0 := n_B(\mu_0) \). Moreover, it is easy to see that the baryon number density is

\[
n_B(\mu_B) = \frac{dp}{d\mu_B},
\]

(4.25)

and by separating the variables the corresponding pressure can be found out to be

\[
p(\mu_B) = p_0 + \int_{\mu_0}^{\mu_B} d\mu' n_B(\mu').
\]

(4.26)

One of the simplest speed-of-sound-squared models is the piecewise-linear one: [II]

\[
c_s^2(\mu) = \frac{(\mu_{i+1} - \mu_B)c_{s,i+1}^2 + (\mu_B - \mu_i)c_{s,i+1}^2}{\mu_{i+1} - \mu_i},
\]

(4.27)

where \( \mu_i < \mu_B < \mu_{i+1} \) and \( c_{s,i}^2 := c_s^2(\mu_i) \). According to Eq. (4.24), the number density can be expressed as a product:

\[
n_B(\mu_B) = n_0 \prod_{i=1}^{N} n_i(\mu_B),
\]

(4.28)
where the factors are defined as

\[
\pi_i(\mu_B) := \left[ \frac{\mu_i - \hat{\mu}}{(\mu_i - \mu_{i-1})c_{s,i-1}^2} \right]^{a_{i,0}}. \tag{4.29}
\]

Here, \( \hat{\mu} := \max(\hat{\mu}_i, \mu_{i-1}) \), \( \bar{\mu}_i := \min(\mu_B, \mu_i) \), and

\[
a_{i,j} := \frac{\mu_i - \mu_{i-1}}{(c_{s,i}^2 + j)\mu_{i-1} - (c_{s,i-1}^2 + j)\mu_i}. \tag{4.30}
\]

On the other hand, the corresponding pressure is a simple sum,

\[
p(\mu_B) = p_0 + \sum_{i=1}^{N} \left[ \bar{p}_i(\mu_i) - \bar{p}_i(\mu_{i-1}) \right], \tag{4.31}
\]

as specified by Eq. (4.26). Here, the summands can be written as

\[
\bar{p}_i(\mu_B) = \mu_B n_B(\mu_B) \left[ 1 + a_{i,1} \left( 1 + \frac{c_{s,i-1}^2 - c_{s,i}^2}{\mu_i c_{s,i-1}^2 - \mu_{i-1} c_{s,i}^2} - \mu_B \right) \right], \tag{4.32}
\]

where \( {}_2F_1 \) is the ordinary hypergeometric function.

In Fig. 4.1, we have illustrated an example speed-of-sound EoS (magenta line). This is the first EoS of Supplementary Table of paper II, and it consists of five independent segments. In addition to meeting all the conditions given at the beginning of this chapter, we also demand that the speed of sound is continuous at the matching points. Notably, the maximum speed of sound of this particular EoS is remarkably low as well.

### 4.4 Properties and differences

All above interpolation methods have their distinct advantages and downsides. Therefore, we will briefly overview the two most distinctive features of these approaches in this section. First, we will investigate the impact of the degrees of freedom using polytropes. In this context, we use the term degree of freedom to denote the number of free parameters of an interpolation model. Although we will only consider a polytropic approach, it is presumable that the analyses can be extended to apply with the other related techniques as well. Second, the different interpolation methods introduced in this work will be compared.

#### 4.4.1 Degrees of freedom

In this subsection, we will shortly investigate the effects of the degrees of freedom in our interpolation systems. The calculations will be carried out using piecewise polytropes with two to four segments. The behavior of polytropic EoSs with two and three pieces, i.e. the bitropic and tritropic EoSs, was first studied in [36]. In article I, this study has been extended by adding fourth polytropic segment — i.e. studying quadrutropes.

For this analysis, approximately 160,000 subluminal \( \Gamma \)-bitropic EoSs have been produced. We randomly generated these EoSs using uniform distributions \( \mu_t \in [\mu_B(1.1n_s), \mu_{pQCD}] \) and
Figure 4.2: Effect of increasing the number of the degrees of freedom. The panel a shows the possible limits of the equation-of-state configurations given by three different subluminal interpolation setups: bitrope (black dotted), tritrope (orange dashed), and quadrutrope (background). In panel b, the corresponding mass-radius clouds are presented. In addition, the equations of state have been divided into three subcategories. The green band represents the allowed region whereas the cyan and purple regions break the two-solar-mass and tidal-deformability constraints, respectively.

As one is able to see from Fig. 4.2a, the additional polytropic segments (new degrees of freedom) expand the EoS cloud, i.e. the area where all interpolated EoSs are located. It is however disputable if these extensions contain any realistic EoS solutions. Nonetheless, we cannot ignore them without a proper investigation and further observational inputs, and therefore, articles I and II still include them.

In the case of the mass-radius cloud, one is able to notice that the allowed region also increases (see Fig. 4.2b). Especially, the existence of extremely massive neutron stars with large radii is allowed by the presence of a large number of degrees of freedom. As shown in papers I and II, the current tidal-deformability measurements [condition b)] heavily disfavor these solutions (purple region). If requirements a) and b) are taken into account, the behaviors of different mass-radius clouds are relatively similar regardless of the number of polytropic segments. However, some differences arise when extreme borderline solutions are considered.
Figure 4.3: Comparison of the three interpolation methods: the speed-of-sound (background regions), quadrutropic (dotted black lines), and spectral (dashed orange lines) ones. In panel a, the subluminal equation-of-state clouds are illustrated whereas the corresponding mass-radius regions are shown in panel b. The color coding of the background is given in Fig. 4.2. This graph is Extended Data Fig. 1 from article II.

4.4.2 Comparing different methods

It is essential to compare different interpolation methods. In this way, we can gain valuable information about their behavior, advantages, and limitations. Here, we are going to briefly summarize the main points given in article II where the above-introduced interpolation methods have been investigated.

Fig. 4.3 suggests that the speed-of-sound interpolation is the most general tool to model the EoS — i.e. it covers the largest areas in both the mass-radius and $\varepsilon$-$p$ planes. The enveloped regions of the polytropic EoSs are rather similar to the speed-of-sound ones, but the method is unable to produce solutions with large radii and masses. It is likely that this is caused by its speed-of-sound behavior. The spectral method, on the other hand, is able to create these kinds of continuous solutions. Nevertheless, the extensively smooth behavior of these EoSs prevents it from covering the mass-radius and EoS clouds as efficiently as the two other methods. In other words, it cannot mimic EoSs with strong first-order phase transitions, for example, as shown in [269].
Chapter 5

Summary of the Publications

This thesis consists of three scientific articles published in peer-review journals, and the main findings of the publications are summarized and discussed in this chapter. In the first section, the unknown part of the dense-matter EoS at zero temperature will be investigated using interpolative approaches. This choice is an unusual one because extrapolative techniques are often used in the literature instead, as pointed out in the previous chapter. In paper I, we focused on probing the EoS and the mass-radius behavior of neutron stars using data from the gravitational-wave event GW170817. Article II then considered the possibility that deconfined quark matter could exist in the cores of neutron stars quantitatively. In the latter section, we shall examine a situation where the high-density limit is given by a holographic model. This setup is presented in paper III, and it relies on rather unorthodox machinery. Notwithstanding, we were able to show that certain exotic EoSs break the I-Love-Q relations.

5.1 Neutron-star matter

The object of this section is to probe the neutron-star-matter EoS using interpolative methods explained in the previous chapter. Throughout the section, we demand that the theoretical conditions A) and B) as well as a) and b) [or occasionally b*]] given at the beginning of the previous chapter hold. At first, we will investigate how theoretical knowledge of the EoS and the latest observational — especially, the tidal deformability — data constrain the behavior of the unknown part of the EoS and the mass-radius relation. After this, we will examine the possibility that neutron stars contain quark matter in their cores.

5.1.1 Pinning down the equation of state and the mass-radius curve

As has been highlighted previously, the dense-matter EoS at zero temperature is poorly known from approximately $n_s$ to $40n_s$. This, on the other hand, means that the mass-radius relationship has also remained vague for the most part because most of the radius and mass of a neutron star are explained by the unknown part of the EoS (see Fig. 2.1). However, the improvement
Figure 5.1: Tidal deformability $\Lambda$ of a $1.4M_\odot$ neutron star as a function of its radius $R$. Quadrutrope equation-of-state models that do not fulfill the two-solar-mass constrain are shown in cyan. The remaining area is then divided into three subsections using the tidal deformability data from the GW170817 event so that $\Lambda(1.4M_\odot) < 400$ (green), $400 < \Lambda(1.4M_\odot) < 800$ (purple), and $\Lambda(1.4M_\odot) > 800$ (red). The empirical fit Eq. (5.1) is illustrated using the orange dashed line. This is Fig. 2 from paper I.

of theoretical calculations — such as the pQCD and cEFT ones — will reduce the size of the unknown region in the future. It is also possible to take advantage of recent observational findings and to use them to rule out unphysical configurations.

In article I, we formulated an extension of a previous study of [36] (see also [276]) where the piecewise-defined polytropic approximation is used to interpolate the EoS utilizing the two-solar-mass neutron-star observations [120, 121] as a constraint. To be specific, the unknown section was modeled using bitropic and tritropic EoSs. In [36], it was showed that this mass limit together with the standard physical limitations, such as subluminality, eliminate stars with very small radii, or correspondingly, soft EoSs in the low-density regime.

The original setup of [36] was, at first, adjusted including a fourth monotropic segment. Furthermore, we made use of the LIGO/Virgo estimate for the tidal deformability of a $1.4M_\odot$ star based on the gravitational wave event GW170817. This is the main improvement of our study compared to [36]. As Figs. 1 and 3 of paper I suggest, this tidal-deformability constraint [$\Lambda_{1.4} < 800$; condition b*)] has an effect opposite to the two-solar-mass limit. Firstly, this new restriction excludes the stiffest EoSs in the low-density region. Secondly, it disfavors the most massive neutron-star models bringing the TOV mass down slightly. This, on the other hand, means that our analysis prefers neutron-star solutions with moderate radii. For example, the radius of a $1.4M_\odot$ neutron star is likely to be between 9.9 and 13.6 kilometers according to this study.

Secondly, we also considered the relationship between the radius $R_{1.4}$ and tidal deformability $\Lambda_{1.4}$ of a $1.4M_\odot$ neutron star (Fig. 5.1). We found out that the relation can be approximated
5.1 Neutron-star matter

Figure 5.2: Interpolated equation-of-state cloud. The green (yellow) area represents the allowed $c_s^2$-interpolation band from paper II whereas the light blue regions correspond to the known chiral-effective-field-theory (Nucl.) [32] and perturbative-QCD (pQCD) [238] equations of state. **Left panel:** The dashed, orange lines are the approximative polytropic extensions with $\gamma = 1$ (high) and 2.5 (low density) while the black curves illustrate the extrapolated results of [12, 112], respectively. **Right panel:** The allowed cloud is divided into subregions using the maximum value of the squared speed of sound $c_s^2$ (color coding). In addition, the locations of $1.44M_\odot$ (blue diamonds), $2M_\odot$ (orange squares), and maximally massive (red dots) configurations are also shown. Note that the right plot is Extended Data Fig. 3 from article II.

with a simple function

$$A_{1.4}(R_{1.4}) = 2.88 \times 10^{-6}(R_{1.4}/\text{km})^{7.5}.$$  \hfill (5.1)

In the literature, similar empirical behavior has been observed (e.g. [277–279]) with slightly differing exponents. This is, however, not a surprise because the fit depends on the data set, i.e. the constraints used to limit the EoS. Therefore, the numerical values of our result should only be treated as a guideline.

5.1.2 Existence of quark-matter cores

While paper I focused on probing the general behavior of the EoS and the corresponding mass-radius curve, the goal of article II is a more specific. As introduced in Section 3.1.3, hadronic matter will undergo a phase transition (or a smooth crossover) from the hadronic phase to the quark-matter one at some density. However, it is unclear where this transition (or crossover) exactly happens at zero temperature, in particular, whether this density is ever reached inside neutron stars. Hence, it is an open question whether a neutron star can contain quark matter in its core.

The quark-core hypothesis has been addressed in several studies (e.g. [28, 74, 108, 128, 129, 282]) using different model calculations. In paper II, we, however, chose a novel approach where we examined a collection of $c_s^2$-interpolated EoSs. By considering a family of these solutions
Figure 5.3: Behaviors of three characteristic thermodynamic variables: the polytropic index $\gamma$, the squared speed of sound $c_s^2$, and the normalized pressure by the Fermi-Dirac limit $p/p_{FD}$. The black curves represent realistic hadronic equations of state from [112, 280, 281] whereas the thin gray lines form a representative subset of $c_s^2$-interpolated equations of state. In panel a, the full three-dimensional graph is shown while panel b (c) illustrates its two-dimensional projection to the $p/p_{FD}-\gamma$ ($c_s^2-\gamma$) plane. The solid blue diamonds and red dots (open cyan diamonds and magenta circles) denote $1.4M_\odot$ and maximally massive interpolated (hadronic) neutron-star solutions, respectively. This graph is Fig. 2 from paper II.
Figure 5.4: Size of the quark core of a maximally massive (*left panel*) and a two-solar-mass (*right panel*) neutron star. The inset of the *left panel* illustrates a two-solar-mass neutron star with radius of 12 km and a 6.5 km quark core. Color coding follows Fig. 5.2 (right panel). These plots are Fig. 3 (*left*) and Extended Data Fig. 4 (*right*) from article II.

(Fig. 5.2, left panel), one may notice that the EoS cloud basically consists of two straight lines in the log-log graph with a bend around 400 to 700 MeV/fm$^3$ of the energy density. This means that the relationship between the pressure and the energy density is approximately polytropic: $p \propto \varepsilon^\gamma$ — or bitropic to be more specific. By investigating individual EoS models, we found out that a typical pulsar with mass around $1.4 M_\odot$ lies to the left of the kink of the EoS cloud but maximally-massive stars to the right as the right panel of Fig. 5.2 indicates. Unfortunately, the two-solar-mass neutron-star solutions are at around the bending point, meaning that their faith is unsettled.

However, the above analysis is not sufficient because one needs to examine the bending point of each individual EoS and not to consider the general point given by the overall EoS cloud. This task is, nevertheless, challenging because the kink is not, most of the time, a well-defined point. That is why we introduced a complementary approach where we considered the behavior of certain thermodynamical variables, inspecting the change in the degrees of freedom of the system via them. In article II, three such variables are considered: the polytropic index $\gamma$, the squared speed of sound $c_s^2$, and the pressure normalized by the Fermi-Dirac limit $p/p_{FD}$. A representative sample of the $c_s^2$-interpolated EoSs together with several realistic hadronic EoSs are given in Fig. 5.3. On the one hand, the figure indicates that there is no difference between interpolated and nuclear EoSs in the case of $1.4 M_\odot$ neutron-star solutions meaning that the stars should be purely hadronic. On the other hand, the behaviors of these two EoS classes differ greatly if maximum-mass configurations are considered. In that case, the interpolated EoSs approach the pQCD EoS much more closely.

Even though the analysis given in article II favors the quark-core hypothesis, it does not identify the details of the transition (or the crossover) between the hadronic and quark phases. It is, however, convenient to define a simple, approximative condition for the (pseudo)transition point. This helps us to estimate the prominent properties of the quark core and the suitable
EoSs. In paper II, we defined that matter can be described in terms of quark degrees of freedom if \( \gamma < 1.75 \forall \varepsilon > \varepsilon_0 \), where \( \varepsilon_0 \) is an estimated energy density for the (pseudo)transition point. As Fig. 5.3 indicates, this choice should be a reasonable one, but we have to further emphasize that this particular limit is only an educated guess. This straightforward model, nevertheless, suggests that an EoS that is able to generate hybrid-star configurations should not be too extreme. Alternatively, if the squared speed of sound exceeds 0.7 at some point — or the possible latent heat is over 130 MeV/fm\(^3\) — the maximally-massive neutron-star solutions may not contain any quark matter.

If the neutron-star-matter EoS supports the hypothesis, the produced quark cores can be remarkably large as the left panel of Fig. 5.4 states. For example, a certain kind of EoS model can create a core that contains even 40 per cent of the mass and half of the radius of a maximally-massive neutron star (see Fig. 5.4, left panel). It is worth highlighting that subconformal — i.e. \( \max(c_s^2) < 1/3 \) [283] — EoSs sustain such hybrid-star configurations with especially large quark cores. Furthermore, the latest mass-radius measurements — especially the most precise ones — also support the existence of these types of configurations (see Extended Data Fig. 5 of [II]). Even so, it is still possible that the biggest existing quark-matter core is noticeably small.

In addition, we can also consider two-solar-mass neutron stars that correspond to the most massive observed neutron stars (see Section 2.4 for details). As mentioned earlier, even if a maximally-massive neutron star contains quark matter, lighter stars may not. The quark cores are then essentially absent in two-solar-mass neutron stars if the maximum mass is above 2.25\( M_\odot \) (see Fig. 5.4, right panel). This finding is in good agreement with the approximately 2.3\( M_\odot \) upper limit estimation for the TOV mass given by multiple studies [16–19] using multimessenger observations related to the GW170817 event. Regardless, any EoS solution with \( \max(c_s^2) < 0.4 \) should have a quark core if \( M \gtrsim 2M_\odot \).

As a secondary objective, we also updated the analysis of article I using the upgraded tidal-deformability estimation, \( 70 < \Lambda_{1.4} < 580 \) (see Fig. 4.3). In addition, the same inspection was not only carried out just utilizing the quadrutropes but also the spectral representation and the \( c_s^2 \)-method (see Section 4.4.2).

### 5.2 Holographic compact stars

In article III, a simple holographic setup has been used to mimic the behavior of the quark-matter EoS. This approach has been chosen because the \textit{ab initio} nature of the AdS/CFT correspondence allows to define the EoS rigorously, even in relatively small densities. This is the biggest advantage of the holographic methods because traditional perturbative frameworks cannot model the EoS of QCD if the coupling is too strong. At the same time, the uncertainties of pQCD calculations also dramatically increase as the density decreases.

The holographic model discussed in this publication is from [41]. This zero-temperature,
holographic EoS takes a simple analytical form

\[ \varepsilon = 3p + \frac{m_0^2}{2\pi} \sqrt{3p}, \]  

or equivalently

\[ \varepsilon = \frac{3}{(8\pi)^2} \left( \mu_q^2 - m_0^2 \right) \left( 3\mu_q^2 + m_0^2 \right), \]  

where \( m_0 \) and \( \mu_q \) are a parameter of the model and the quark chemical potential, respectively. This plain model assumes that the constituent masses of the lightest quarks (up, down, and strange) are all equal and given by the parameter \( m_0 \). \(^{\text{III, 41}}\) The biggest difference between paper \text{III} and \cite{41} is the value of \( m_0 \); our study assumes that it is a free parameter whereas it is fixed using a rather arbitrary condition in \cite{41}. It is worth mentioning that the above model does not, however, fully describe all the essences of the strong interaction, and generally speaking, none of the known holographic setups are completely satisfactory in this regard. For example, our calculations are carried out in the large-\( N_c \) limit although \( N_c \) is ultimately set to three. Nevertheless, it is likely that different holographic models can catch many important features of QCD (cf. Section 3.4), and therefore, it is essential to study them as well.

Where the study of \cite{41} only discovered hadronic neutron-star solutions, article \text{III} illustrates that three other kinds of compact-star solutions can be found by decreasing the value of the parameter \( m_0 \). The most conventional of these are quark stars, but these solutions are not compatible with the tidal-deformability measurements of the LIGO/Virgo collaboration. The other two classes are new types of hybrid-star-like solutions. The first of these novel solutions is a reversed hybrid star (HS2) — i.e. the star has a quark-matter crust and a nuclear-matter core (see Fig. 5.5). Although these solutions are able to support two-solar-mass stars, the tidal-deformability conditions are not met when the radius is remarkably large. The other new class
is HS3 that consists of stars with three layers — a large hadronic core as well as a thin quark-matter mantle and nuclear-matter crust (again, see Fig. 5.5). These stars are fully compatible with both the two-solar-mass and tidal-deformability constraints.

The most important discovery in paper III is the deviation from the I-Love-Q relations. As pointed out in Section 2.3, it is believed that these relations are, to large extent, independent of the dense-matter EoS model. Although the purely hadronic neutron-star solutions match well with the fit given in [49], the other models have larger discrepancy from the baseline (Fig. 5.6). Nonetheless, quark-stars and some new hybrid-star solutions could be argued to be almost compatible \([\mathcal{O}(5\%)\)]\). These results highlight the fact that if a discrepancy between the fitted model of [50] and observations is found, it may imply that atypical compact-star configurations have to exist in Nature or the underlying theory of gravity needs to be modified [163]. It is also possible that magnetic or rotational effects have to be taken into account [49]. However, any sign of such inconsistencies has not been detected to date.

In conclusion, the results of article III are qualitatively significant because very few anomalous EoSs deviating from the I-Love-Q relations have been found. In addition to our findings, neutron stars with crystalline quark-matter cores [164, 165] are examples of such unusual configurations. Consequently, one should be aware of this possibility when using these relations.
Figure 5.6: Fractional errors between the universal I-Love-Q relationships of [49] and five holographic equations of state (cf. Fig. 5.5). The subplots represent $\Lambda - \bar{I}$ (top, left), $\bar{Q} - I$ (top, right), and $\Lambda - \bar{Q}$ (bottom) relations (here, $\bar{\lambda} := \Lambda$). The subscript $U$ indicates that the quantity is given by the corresponding universal relation, and the dot-dashed black line is an analytical approximation (see the original study for details). This is a modified version of Fig. 6 from article III.
Chapter 6

Conclusions and Outlook

Neutron-star observations can be used to constrain the EoS of cold and dense QCD matter. The decade-old detection of a two-solar-mass neutron star [120] has ruled out the softest neutron-star-matter EoSs. On the other hand, the revolutionary gravitational-wave observation of two colliding neutron stars [6] has excluded the stiffest EoS candidates at the low-density regime — as demonstrated in this thesis using various interpolation techniques. These results underline the importance of different astronomical observations. Thus, it is more than likely that the prospective new mass and gravitational-wave detections will further limit the behavior of the EoS in the coming years. Simultaneous mass-radius measurements will also help to probe the EoS, and consequently, we are currently investigating this particular issue [284]. Another interesting observable is the moment of inertia. After the upcoming measurement of this quantity for a neutron star with known mass, one is able to examine the equation of state in an altogether new fashion (see e.g. [285, 286]).

Various neutron-star observations are not the only ways to study the macroscopic properties of these objects. Gratifyingly, improving theoretical knowledge of the dense-matter EoS will also delimit the mass-radius relationship of neutron stars. Already, some research groups (e.g. [35, 241]) have investigated the possibility to extend the cEFT results to even higher densities than around one saturation density $n_s$. In turn, particle physicists are working to improve their calculations, such as higher-order pQCD loop corrections and their convergence.

In the future, it may even be feasible that the exploratory holographic approach could overcome the predictive power of pQCD but this is not likely going to happen in the foreseeable future. However, this method has produced some promising results already in the high-temperature regime although the techniques are still under development. In this work, we have, hopefully, taken some important steps to improve the holographic description for low-temperature and high-density systems by studying holographic neutron-star setups. Although article III contains many interesting results, the underlying holographic approach should be treated as a phenomenological toy model — and not as a fully polished tool. Nonetheless, the discovered EoSs that violate the I-Love-Q relationship constitute a significant finding although it is not a completely new result (see e.g. [164, 165]). After all, we are at the beginning of the
journey in terms of the development of holographic tools. Such progress could, nonetheless, lead to a situation where the quark-matter EoS could be known down to relatively small values of the density at (around) zero temperature. By achieving this goal, one could remarkably restrict the fundamental features of neutron stars, and therefore, various studies [42–46] have explored ways to improve the current AdS/CFT methods.

It is remarkable that the matter content of neutron-star cores is not, largely, known even though the core comprises most of the matter inside a heavy neutron star. In this thesis, it has been shown that there exists encouraging evidence supporting the quark-core hypothesis. This implies that the part of the EoS corresponding to the cores of the most massive neutron stars has properties similar to the state-of-the-art pQCD one. It is important to highlight that a light neutron star does not necessarily have to have a quark-matter core — a typical $1.4M_\odot$ neutron star seems to lack one, for instance. Our analysis, however, showed that the viable quark cores could be notably large — particularly if the EoS is strictly subconformal, or nonextreme in general. Nevertheless, it is still possible that quark-matter cores are tiny or entirely absent even in the case of maximally-massive neutron stars. Hence, additional theoretical and observational work needs to be carried out before this mystery is fully solved. We hope that our upcoming probabilistic work [284] will help with this issue because this approach allows us to take full advantage of the latest (mass-radius) measurements (see also [4, 15, 34, 192, 193, 199, 287–290]). In the end, either we will be able to learn a lot about the fundamental properties of the cold and dense, deconfined quark matter directly; or the hypothesis will be ruled out altogether. All the same, both of these scenarios are interesting, per se.

In addition to the points addressed above, the EoS of dense matter should also be studied at low — but nonzero — temperatures. This is important because supernovae, proto-neutron stars, and neutron-star-merger systems, among other things, contain matter with temperatures of the order of 10 to 100 MeV, or $10^{11}$ to $10^{12}$ K (see e.g. [291–294]). Some — mainly nuclear — models have already been created to address the issue (e.g. [295–297]), and a few cEFT-based studies have been published as well (e.g. [298–300]). As briefly pointed out in Section 3.3, the pQCD calculations have also been carried out in this part of the phase diagram [245], but these two theoretical — the pQCD and cEFT — density regions have not yet been connected at low, nonzero temperatures. Nonetheless, the gauge/gravity duality has already been utilized to scrutinize the problem to some degree [43]. Consequently, it can be expected that the characteristics of cool neutron-star matter will be under intense study over the coming years as well.

In view of these considerations, it is very likely that more light will be cast on the EoS of dense matter at zero temperature within a decade or two. This, however, requires seamless cooperation between numerous scientists — from theoretical physicists to observational astronomers.
Appendix A

Generators of the group SU(3)

The eight generators of the group SU(3), \( t_a \), can be defined using the Gell-Mann matrices, \( \lambda_a \):

\[
\begin{align*}
\lambda_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda_2 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & \lambda_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, & \lambda_5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, & \lambda_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\
\lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, & \lambda_8 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.
\end{align*}
\] (A.1)

Here, the connection is given as \( t_a = \lambda_a / 2 \). These matrices fulfill the commutation relation

\[
[t_a, t_b] = i f_{abc} t_c,
\] (A.2)

where \( f_{abc} \) are the completely antisymmetric structure constants and

\[
[t_a, t_b] := t_a t_b - t_b t_a.
\] (A.3)

On the other hand, the structure constants can be written as follows:

\[
f_{abc} = -2i \text{Tr} ([t_a, t_b] t_c).
\] (A.4)

In particular, the nonzero components of \( f_{abc} \) can be expressed as

\[
\begin{align*}
f_{123} &= 1, \\
f_{458} &= f_{678} = \frac{\sqrt{3}}{2}, \\
f_{145} &= f_{245} = f_{257} = f_{345} = -f_{156} = -f_{367} = \frac{1}{2}.
\end{align*}
\] (A.5)
Generators of the group SU(2)

The Hermitian and unitary Pauli matrices, $\sigma_a$, are the generators of the group SU(2):

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

(A.6)

Here, the corresponding commutation relations is

$$[\sigma_a, \sigma_b] = 2i\epsilon_{abc}\sigma_c,$$

(A.7)

where $\epsilon_{abc}$ is the Levi-Civita symbol.

Dirac gamma matrices

In the Dirac basis, the Dirac gamma matrices are given as

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix},$$

(A.8)

where $\mathbf{1}$ and $\mathbf{0}$ are the $2\times2$ identity and null matrices, respectively. Here, the index $k$ takes values from 1 to 3. The most characteristic feature of the Dirac gamma matrices is the anticommutation relation

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu},$$

(A.9)

where $\eta^{\mu\nu}$ is the Minkowski metric and

$$\{\gamma^\mu, \gamma^\nu\} := \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu.$$

(A.10)

In addition, one can define the fifth Dirac gamma matrix so that

$$\gamma^5 := i\gamma^0\gamma^1\gamma^2\gamma^3.$$

(A.11)

In the Dirac basis, it takes a simple form:

$$\gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(A.12)
Bibliography


