Master's Thesis

Asymptotics of Randomly Weighted Sums of Heavy-Tailed Random Variables

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11/11/2020

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HELSINGIN YLIOPISTO — HELSINGFORS UNIVERSITET — UNIVERSITY OF HELSINKI

Tiedekunta/Osasto — Fakultet/Sektion — Faculty	Koulutusohjelma — Utbildningsprogram — Degree programme
Faculty of science	MAST

Tekijä — Författare — Author

Atte Pyrylä

Työn nimi — Arbetets titel — Title

Asymptotics of Randomly Weighted Sums of Heavy-Tailed Random Variables

Oppiaine — Läroämne — Subject

Applied Mathematics

Työn laji — Arbetets art — Level	Aika — Datum — Month and year	Sivumäärä — Sidoantal — Number of pages
Master's thesis	Fall 2020	56 p.

Tiivistelmä — Referat — Abstract

In this thesis we will look at the asymptotic approach to modeling randomly weighted heavy-tailed random variables and their sums. The heavy-tailed distributions, named after the defining property of having more probability mass in the tail than any exponential distribution and thereby being heavy, are essentially a way to have a large tail risk present in a model in a realistic manner. The weighted sums of random variables are a versatile basic structure that can be adapted to model anything from claims over time to the returns of a portfolio, while giving the primary random variables heavy-tails is a great way to integrate extremal events into the models. The methodology introduced in this thesis offers an alternative to some of the prevailing and traditional approaches in risk modeling.

Our main result that we will cover in detail, originates from "Randomly weighted sums of subexponential random variables" by Tang and Yuan (2014), it draws an asymptotic connection between the tails of randomly weighted heavy-tailed random variables and the tails of their sums, explicitly stating how the various tail probabilities relate to each other, in effect extending the idea that for the sums of heavy-tailed random variables large total claims originate from a single source instead of being accumulated from a bunch of smaller claims. A great merit of these results is how the random weights are allowed for the most part lack an upper bound, as well as, be arbitrarily dependent on each other.

As for the applications we will first look at an explicit estimation method for computing extreme quantiles of a loss distributions yielding values for a common risk measure known as Value-at-Risk. The methodology used is something that can easily be adapted to a setting with similar preexisting knowledge, thereby demonstrating a straightforward way of applying the results.

We then move on to examine the ruin problem of an insurance company, developing a setting and some conditions that can be imposed on the structures to permit an application of our main results to yield an asymptotic estimate for the ruin probability. Additionally, to be more realistic, we introduce the approach of crude asymptotics that requires little less to be known of the primary random variables, we formulate a result similar in fashion to our main result, and proceed to prove it.

Avainsanat — Nyckelord — Keywords

Asymptotics, Randomly weighted sum, Risk modeling, Subexponential distribution

Säilytyspaikka — Förvaringsställe — Where deposited

Kumpula Campus Library

Muita tietoja — Övriga uppgifter — Additional information

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1 Introduction

When modeling risks that are linked to extremal events, finding out a way to reasonably estimate complicated tail distributions is needed to ensure a good model. In insurance and finance, these difficult to tackle tail distributions often arise from models of weighted sums of heavy-tailed random variables. One of the main ways to approach this problem is establishing asymptotic relation between the tail in question and something that one is able to compute the numerical value for. In this thesis we will mainly look into the asymptotic approach, first introduced by Tang & Yuan in 2014[12], focusing on the randomly weighted sums of the form

$$S_n^\theta = \sum_{i=1}^{n} \theta_i X_i,$$

where we have subexponential primary random variables X_i with the weights θ_i that are independent of the primary random variables, but not necessarily bounded or independent of each other. Subexponential distributions is one of the most well researched subclasses of heavy-tailed distributions, and is known to have many nice properties that make it extremely relevant for practical modeling.

One widely applicable interpretation, and a good starting point for an application, is to have the random weights capture the dependence structure, while the primary random variables represent the magnitude. An alternative and more specific case is found by looking at an insurance company, where two main types of risks contribute to the ruin, financial risks related to investment returns, and liability risks related to size and amount of insurance claims that occur, and it is quite standard to assume that the two are independent. Here the conditions we introduce will match any general type of modeling. The soft bounds and the murky dependence structure for the random weights accommodate most of the models of moderate financial market randomness like inflation or the yields of a conservative investment strategy. The primary random variables being subexponential and close enough to each other in terms of the distribution conveniently describes bundled liabilities on discrete and independent time intervals, where factors like seasonal weather, geolocation etc., can cause slight variation in the distribution.

The main result of this thesis will be three theorems establishing the sufficient conditions for the following asymptotic relation to hold

$$\mathrm{P}\left(\bigvee_{i=1}^{n} S_i^\theta > x\right) \sim \mathrm{P}\left(S_n^\theta > x\right) \sim \mathrm{P}\left(\bigvee_{i=1}^{n} \theta_i X_i > x\right) \sim \sum_{i=1}^{n} \mathrm{P}\left(\theta_i X_i > x\right),$$

where the asymptotic relation $f(x) \sim g(x)$ is understood as $f(x)/g(x) \to 1$, $x \to \infty$. The relation can be depicted as the heavy tails of the primary random variables dissolving any dependencies contained in the random weights, resulting asymptotics very similar to those that define the class of subexponential distributions.

The main result and the proofs behind it originate from Tang and Yuan [12] and their earlier work. Taking this into the account, one of the main objectives of this thesis is to have a more elaborate and in depth presentation of the proofs, while seeking for the wider context of the result, mixing in with the other content various small examples.

Chapter 2 is devoted to providing any necessary preliminaries. In Chapter 4, we go over in detail the proofs for the main theorems, and any auxiliary results. In Chapters 5 and 6, we look at some potential applications. First proposing an application to risk measures that rely on the computation of a tail probability originating from some set of underlying multiline cash flows with tangled dependencies, with the focus on Value-at-Risk,

$$\mathrm{VaR}_q\left(Y\right) = \inf\left\{y \in \mathbb{R} : F_Y(y) \geq q\right\},$$

where $F_Y(z)$ is the distribution function of Y, and $q \in (0,1)$ is the given confidence level. The calculations done and the general methodology in Chapter 5, showcase a straightforward way to adapt the result to practical use.

As another potential application, we will look at ruin probabilities, adapting the ruin problem to fit our setting, and some possible random weight constructions for stochastic inflation coefficients. Estimating the ruin probabilities could be considered one of the more promising applications of the main result, since the random weight structure allows a fair bit of flexibility regarding inflation coefficients and other stochastic discount factors that might prove problematic and require simplifying assumptions in other competing approaches. We further display possible asymptotic approaches to the ruin problem by having random weights present in a setting of crude asymptotics that in general require less detailed assumptions on the distributions of the random variables.

2 Preliminaries

This chapter will provide the preliminaries for notation, some of which is specific to this topic, and some more general prerequisites, like definitions of distribution classes and some basic theorems.

2.1 Notation and Miscellaneous Preliminaries

- Let $f(\cdot)$ and $g(\cdot)$ be positive functions, we will use the following notation:

  1. $f(x) \sim g(x)$, when $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$
  2. $g(x) \gtrsim f(x)$ or $f(x) \lesssim g(x)$, when $\limsup_{x\to\infty} \frac{f(x)}{g(x)} \leq 1$
  3. $f(x) \asymp g(x)$, when the functions are weakly equivalent i.e.

$$0 < \liminf_{x\to\infty} \frac{f(x)}{g(x)} \leq \limsup_{x\to\infty} \frac{f(x)}{g(x)} < \infty.$$

- Let F be a distribution function, we denote its tail by $\overline{F}(x) = 1 - F(x)$, and the tail of its n-fold convolution by $\overline{F^{n*}}(x) = 1 - F^{n*}(x) = \mathrm{P}(X_1 + \cdots + X_n > x)$, where $X_1, \ldots, X_n$ are i.i.d.

- We say that a distribution has an *ultimate right tail (or the tail has right-unbounded support)* if $\overline{F}(x) > 0$ for all $x \geq 0$.

- A random variable θ is said to be *degenerate* at a point x, if $\mathrm{P}(\theta = x) = 1$.

- *Big-O notation*, the notation $f(x) = O(g(x))$, $x \to \infty$ is used, when

$$\limsup_{x\to\infty} \left| \frac{f(x)}{g(x)} \right| < \infty.$$

Remark 2.1. Big-O notation, gives us an alternative expression for the weak equivalence $f(x) \asymp g(x)$, since the original condition is equivalent to both $f(x) = O(g(x))$ and $g(x) = O(f(x))$ holding simultaneously.

Remark 2.2. Big-O is transitive, in the sense that if $f(x) = O(g(x))$ and $g(x) = O(h(x))$, then $f(x) = O(h(x))$.

- *Little-o notation*, the notation $f(x) = o(g(x))$, $x \to \infty$ is used, when

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0.$$

  Throughout the writing, especially when it comes to the big-O and the little-o notations, if not otherwise specified, all the limits hold, as $x \to \infty$.

- For real numbers we use the notation

$$x^+ = x \vee 0 = \max\{x, 0\}$$
$$x^- = -(x \wedge 0) = -\min\{x, 0\}$$
$$x_{(n)} = \max\{x_1, \dots, x_n\}.$$

- The natural logarithm is denoted by both $\log(\cdot)$ and $\ln(\cdot)$, the former is used in the more theoretical context, honoring traditions, while the latter is closer to practical, applications in mind notation, for specifying distribution and density functions.

- We will on several occasions refer to "conditioning on a random variable", the formal meaning is the following.

  Since by the defining notation, for an event A and a random variable X,

$$\mathrm{P}(A \mid X) = \mathrm{P}(A \mid \sigma(X)) = \mathrm{E}[\mathbb{1}_A \mid \sigma(X)] = \mathrm{E}[\mathbb{1}_A \mid X],$$

  we can by applying the double expectation property of conditional expectation obtain that

$$\mathrm{P}(A) = \mathrm{E}[\mathbb{1}_A] = \mathrm{E}\left[\mathrm{E}[\mathbb{1}_A \mid X]\right] = \mathrm{E}\left[\mathrm{P}(A \mid X)\right].$$

  This can be used to temporarily fix values, for instance, of the random weights, to then apply a result on the conditional probability inside the expectation.

Theorem 2.1. Markov's inequality *Let X be a non-negative random variable, and $a > 0$, then*

$$\mathrm{P}(X \geq a) \leq \frac{\mathrm{E}X}{a}.$$

Theorem 2.2. Dominated convergence theorem*(Durrett (2019)[4])*
(i) If a sequence of complex valued functions f_n converges pointwise to f almost everywhere, $|f_n| \leq g$ for all n, and g is integrable, then $\int f_n d\mu \to \int f d\mu$.
(ii) If a sequence of random variables X_n converges to X almost surely, $|X_n| \leq Y$, for all n, and $\mathrm{E}Y < \infty$, then $\mathrm{E}X_n \to \mathrm{E}X$.

Theorem 2.3. Bonferroni's inequalities*(Durrett (2019)[4]) Let $A_1, A_2, \dots, A_n$ be events* *and* $A = \bigcup_{i=1}^{n} A_i$. *Then*

$$\mathrm{P}\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} \mathrm{P}\left(A_i\right)$$
$$\mathrm{P}\left(\bigcup_{i=1}^{n} A_i\right) \geq \sum_{i=1}^{n} \mathrm{P}\left(A_i\right) - \sum_{1\leq i<j\leq n} \mathrm{P}\left(A_i \cap A_j\right)$$
$$\mathrm{P}\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} \mathrm{P}\left(A_i\right) - \sum_{1\leq i<j\leq n} \mathrm{P}\left(A_i \cap A_j\right) + \sum_{1\leq i<j<k\leq n} \mathrm{P}\left(A_i \cap A_j \cap A_k\right).$$

2.2 Heavy-tailed Distributions

Definition 2.1. A random variable X is said to have a *(right) heavy-tailed distribution*, if its moment generating function $M_X(s) = \mathrm{E}(e^{sX})$ is infinite on $(0, \infty)$.

We will now introduce the subclasses of Heavy-tailed distributions that will be of interest throughout this thesis, all of these are fairly well covered in the literature, maybe with the exception of Class $\mathcal{A}$ that has only truly become prominent in the more modern research.

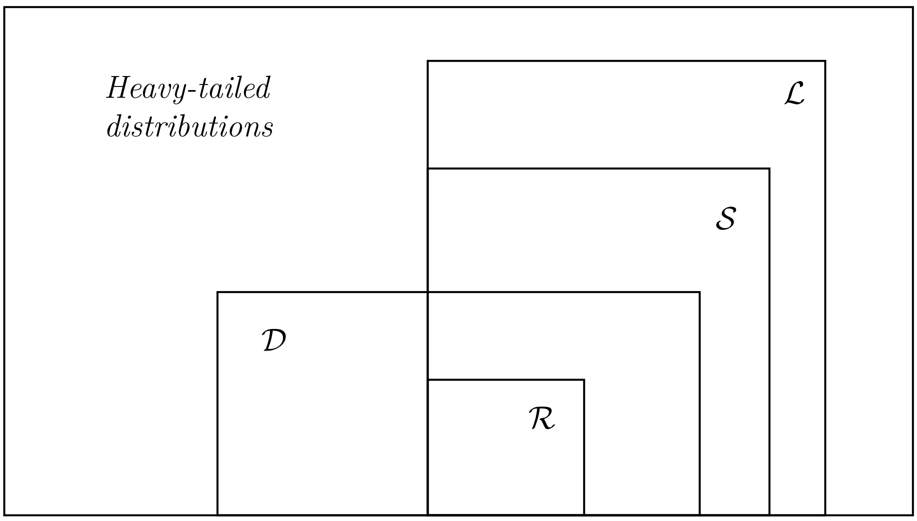

Figure 2.1. Subclasses of Heavy-tailed distributions.(for a more comprehensive picture see Embrechts et al.(1997)[5])

2.2.1 Class $\mathcal{L}$, Long Tailed Distributions

A major subclass of Heavy-tailed distributions defined by an extremely useful asymptotic property of the tail. Often Class $\mathcal{L}$ is used as an auxiliary class providing a major tool for the proofs, since its tail property specified in (2.3) carries to further subclasses, especially to subexponential distributions.

Definition 2.2. A distribution function F on $\mathbb{R}$ is said to be *long tailed*, $F \in \mathcal{L}$, if its (ultimate) right tail satisfies

$$\overline{F}(x+y) \sim \overline{F}(x) \text{ for all } y \in \mathbb{R}. \tag{2.3}$$

Remark 2.4. The relation (2.3) holds uniformly over any compact set of y. As a consequence, there exists a positive function $l(\cdot)$, with $l(x) \leq \frac{x}{2}$ and $l(x) \uparrow \infty$, such that (2.3) holds uniformly for $-l(x) \leq y \leq l(x)$, where the uniformity is understood as

$$\lim_{x\to\infty} \sup_{y\in[-l(x),l(x)]} \left| \frac{\overline{F}(x+y)}{\overline{F}(x)} - 1 \right| = 0.$$

Example 2.1. There are many shortcuts to showing that a distribution is long tailed, such as knowing that it is in one of the many subclasses of Class $\mathcal{L}$, nonetheless, a proof by utilizing the definition is quite often reasonable. Take, for example the Pareto distribution with its tail $\overline{F}(x) = \left(\frac{\kappa}{\kappa+x}\right)^{\alpha}$, where $\alpha, \kappa > 0$, and support on $(0, \infty)$.

Proof. Pareto distribution is long tailed.

$$\frac{\overline{F}(x+y)}{\overline{F}(x)} = \frac{\left(\frac{\kappa}{\kappa+(x+y)}\right)^{\alpha}}{\left(\frac{\kappa}{\kappa+x}\right)^{\alpha}} = \left(1 + \frac{y}{\kappa+x}\right)^{\alpha} \longrightarrow 1, \text{ as } x \to \infty.$$

□

2.2.2 Class $\mathcal{S}$, Subexponential Distributions

A subclass of heavy-tailed distributions, named after the tails that decrease slower than any exponential tail. Most heavy-tailed distributions likely to be encountered in practice are subexponential, including the Pareto, the Burr, the log-normal, and the log-gamma distributions. For a thorough overview of individual distributions in Class $\mathcal{S}$, see Embrechts et al.(1997)[5].

Definition 2.3. A distribution function on $\mathbb{R}_+ = [0, \infty)$ with an ultimate right tail is said to be subexponential, $F \in \mathcal{S}$, if it holds that

$$\lim_{x\to\infty} \frac{\overline{F^{2*}}(x)}{\overline{F}(x)} = 2. \tag{2.5}$$

Lemma 2.1. *A sufficient condition for Relation* (2.5) *to hold is*

$$\limsup_{x\to\infty} \frac{\overline{F^{2*}}(x)}{\overline{F}(x)} \leq 2. \tag{2.6}$$

Proof. (Embrechts et al.(1997)[5], and Amussen & Albrecher (2010)[1]) Since F stands for a positive random variable, it holds for X_1 and X_2 independent and distributed by F that

$$\begin{aligned} &\mathrm{P}(X_1 + X_2 > x) = \overline{F^{2*}}(x) \geq \mathrm{P}(\max(X_1, X_2)) \\ &= \mathrm{P}(X_1 > x) + \mathrm{P}(X_2 > x) - \mathrm{P}(X_1 > x, X_2 > x) = 2\overline{F}(x) - \left(\overline{F}(x)\right)^2. \end{aligned}$$

Thereby it follows that

$$\liminf_{x\to\infty} \frac{\overline{F^{2*}}(x)}{\overline{F}(x)} \geq \liminf_{x\to\infty} \frac{2\overline{F}(x) - \left(\overline{F}(x)\right)^2}{\overline{F}(x)} \geq 2,$$

which given (2.6) yields (2.5). □

Remark 2.7. Definition 2.3 could be equivalently (see Embrechts et al.(1997)[5]) expressed in the following form that happens to reveal one of the main asymptotic properties of subexponential distributions.

For a sequence of i.i.d. $\{X_i, i \geq 1\}$, corresponding distribution function F on $\mathbb{R}_+ = [0, \infty)$ is a *subexponential*, $F \in \mathcal{S}$, if one of the following equivalent conditions holds:

$$\text{(i)} \lim_{x\to\infty} \frac{\overline{F^{n*}}(x)}{\overline{F}(x)} = n \text{ for all } n \geq 2,$$

$$\text{(ii)} \lim_{x\to\infty} \frac{\mathrm{P}(X_1 + \cdots + X_n > x)}{\mathrm{P}(\max(X_1, \ldots, X_n) > x)} = 1 \text{ for all } n \geq 2.$$

By virtue of the equivalence, we can preface the main result by writing

$$\mathrm{P}(\max(X_1, \ldots, X_n) > x) \sim \overline{F^{n*}}(x) \sim n\overline{F}(x),$$

where the first relation is so called principle of a single big jump.

Remark 2.8. In general, a distribution function F on $\mathbb{R}$ is called subexponential if $F^+(x) = F(x)\mathbb{1}\{x \geq 0\}$ is subexponential.

Proposition 2.1. *Every subexponential distribution is long tailed. (see Embrechts et al.(1997)[5] Lemma 1.3.5(a))*

Proposition 2.2. *Any long tailed distribution with a right tail weakly equivalent to a subexponential tail is subexponential.*

Proof. (*The underlying proof is by Klüppelberg (1988)[6]*) Let $G \in \mathcal{L}$, $F \in \mathcal{S}$, and $\overline{G}(x) \asymp \overline{F}(x)$. Assume that $G(0) = F(0) = 0$. Then since G and F have weakly equivalent tails, we can find $m, M \in (0, \infty)$ such that

$$m \leq \frac{\overline{G}(x)}{\overline{F}(x)} \leq M$$

for all $x \in (0, \infty)$. For fixed $v > 0$ and $x > 2v$, we have

$$\frac{\overline{G^{2*}}(x)}{\overline{G}(x)} = 2\int_0^v \frac{\overline{G}(x-y)}{\overline{G}(x)} dG(y) + \int_v^{x-v} \frac{\overline{G}(x-y)}{\overline{G}(x)} dG(y) + \frac{\overline{G}(x-y)}{\overline{G}(x)} \cdot \overline{G}(v). \tag{2.9}$$

Then by partial integration

$$\begin{aligned}\int_v^{x-v} \frac{\overline{G}(x-y)}{\overline{G}(x)} dG(y) &\leq \frac{M}{m} \int_v^{x-v} \frac{\overline{F}(x-y)}{\overline{F}(x)} dG(y) \\ &\leq \frac{M}{m} \left\{ \overline{G}(v) \cdot \frac{\overline{F}(x-y)}{\overline{F}(x)} + M \int_v^{x-v} \frac{\overline{F}(x-y)}{\overline{F}(x)} dF(y) \right\}.\end{aligned}$$

Now, since $G \in \mathcal{L}$ and $F \in \mathcal{S} \subset \mathcal{L}$ it holds that

$$\begin{aligned}&\lim_{v\to\infty} \limsup_{x\to\infty} 2\int_0^v \frac{\overline{G}(x-y)}{\overline{G}(x)} dG(y) = \lim_{v\to\infty} 2\int_0^v 1 dG(y) = 2, \\ &\lim_{v\to\infty} \limsup_{x\to\infty} \frac{\overline{G}(x-y)}{\overline{G}(x)} \cdot \overline{G}(v) = \lim_{v\to\infty} \overline{G}(v) = 0, \\ &\lim_{v\to\infty} \limsup_{x\to\infty} \overline{G}(v) \cdot \frac{\overline{F}(x-y)}{\overline{F}(x)} = \lim_{v\to\infty} \overline{G}(v) = 0,\end{aligned}$$

and moreover by writing (2.9) for F, and since $F \in \mathcal{S}$, we have

$$\lim_{v\to\infty} \limsup_{x\to\infty} \int_v^{x-v} \frac{\overline{F}(x-y)}{\overline{F}(x)} dF(y) = 0. \tag{2.10}$$

It follows that

$$\limsup_{x\to\infty} \frac{\overline{G^{2*}}(x)}{\overline{G}(x)} \leq 2,$$

and by Lemma 2.1 we are able to conclude that $G \in \mathcal{S}$. □

Proposition 2.3. *The class of subexponential distributions covers the class $\mathcal{R}$ of regularly varying tailed distributions that is distributions with a tail $\overline{F}$ that satisfies for some $0 \leq \alpha < \infty$*

$$\overline{F}(xy) \sim y^{-\alpha}\overline{F}(x), \text{ for all } y > 0,$$

in the case of $\alpha = 0$ the distribution is called slowly varying tailed.

Equivalently, F is regularly varying tailed if

$$\overline{F}(x) = x^{-\alpha}L(x),$$

where $L(x)$ is slowly varying $L(x) \sim L(xy)$.

Proof. (*Regularly varying tailed distributions are subexponential.*)*(Amussen & Albrecher (2010)[1])* Assume $\overline{F}(x) = x^{-\alpha}L(x)$, with $\alpha \geq 0$ and L slowly varying. Let $0 < \delta < 1/2$. If $X_1 + X_2 > x$, then either X_1 or X_2 exceeds $(1-\delta)x$, or they both exceed δx. Thereby

$$\begin{aligned}\limsup_{x\to\infty} \frac{\overline{F^{2*}}(x)}{\overline{F}(x)} &\leq \limsup_{x\to\infty} \frac{2\overline{F}((1-\delta)x) + \left(\overline{F}(\delta x)\right)^2}{\overline{F}(x)} \\ &= \limsup_{x\to\infty} \frac{2L\left((1-\delta)x\right) / \left((1-\delta)x\right)^{\alpha}}{L(x)/x^{\alpha}} + 0 = \frac{2}{(1-\delta)^{\alpha}}\end{aligned}$$

Letting $\delta \to 0$, yields $\limsup_{x\to\infty} \overline{F^{2*}}(x)/\overline{F}(x) \leq 2$, which by Lemma 2.1 gives $F \in \mathcal{S}$. □

Often a distribution is shown to be subexponential via an auxillary result. The following characterization theorem for $\mathcal{S}$ is, for instance, applicable to Weibull and log-normal distributions, neither of which is regularly varying tailed.

Proposition 2.4. *(characterization theorem for $\mathcal{S}$) Suppose F is absolutely continuous with density f and hazard rate (failure rate) $q(x) = f(x)/\overline{F}(x)$ decreasing for $x \geq x_0$ with limit 0 at ∞. Then*

(i) $F \in \mathcal{S}$ if and only if

$$\lim_{x\to\infty} \int_0^x e^{yq(x)} f(y)dy = 1$$

(ii) If

$$\int_0^{\infty} e^{xq(x)} f(x)dx < \infty$$

then $F \in \mathcal{S}$.

For further details and proofs, see Embrechts et al.(1997)[5], and Amussen & Albrecher (2010)[1].

Example 2.2. By applying Proposition 2.4, we can show that the Weibull distribution, originating from the reliability theory, is subexponential.

The Weibull distribution has the tail function

$$\overline{F}(x) = e^{-cx^{\tau}},\ x \geq 0,$$

where $0 < \tau < 1$ and $c > 0$. Then $f(x) = c\tau x^{\tau-1}e^{-cx^{\tau}}$, and $q(x) = c\tau x^{\tau-1}$ decreasing with the limit 0 at ∞, since $\tau < 1$.

Moreover, we have

$$\int_0^{\infty} e^{xq(x)} f(x)dx = \int_0^{\infty} e^{c(\tau-1)x^{\tau}} c\tau x^{\tau-1}dx < \infty,$$

for $0 < \tau < 1$. By applying Proposition 2.4 (ii), we conclude that the Weibull $F \in \mathcal{S}$.

2.2.3 Class $\mathcal{D}$, Dominatedly-varying Tailed Distributions

Definition 2.4. A distribution function F on $\mathbb{R}$ is said to be *dominatedly-varying tailed*, $F \in \mathcal{D}$, if its right tail satisfies

$$\overline{F}(xy) = O(\overline{F}(x)) \text{ for all } 0 < y < 1. \tag{2.11}$$

Remark 2.12. The intersection $\mathcal{L} \cap \mathcal{D}$ is a subclass of subexponential distributions that covers the class of regularly varying-tailed distributions $\mathcal{R}$. In conclusion, we have

$$\mathcal{R} \subset \mathcal{L} \cap \mathcal{D} \subset \mathcal{S} \subset \mathcal{L}, \tag{2.13}$$

which is also visualized in Figure 2.1.

Example 2.3. The aforementioned Pareto distribution is also dominatedly-varying tailed,

$$\frac{\overline{F}(xy)}{\overline{F}(x)} = \frac{\left(\frac{\kappa}{\kappa+xy}\right)^{\alpha}}{\left(\frac{\kappa}{\kappa+x}\right)^{\alpha}} = \frac{\left(\frac{\kappa}{x}+1\right)^{\alpha}}{\left(\frac{\kappa}{x}+y\right)^{\alpha}} \longrightarrow y^{-\alpha},\ \text{as } x \to \infty.$$

Simultaneously, we end up showing that Pareto distribution is regularly varying.

Remark 2.14. Not all popular distributions are in Class $\mathcal{D}$, for example log-normal and Weibull distributions are excluded. (Tang & Tsitsiashvili(2003)[11])

Proof. Weibull distribution is not in Class $\mathcal{D}$.

Let $y = 1/2$, then

$$\limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = \limsup_{x\to\infty} \frac{e^{-c(xy)^{\tau}}}{e^{-cx^{\tau}}} = e^{cx^{\tau}(1-y^{\tau})} = \infty,$$

since $(1 - y^{\tau}) = 1 - 1/2^{\tau} > 0$ for $0 < \tau < 1$. □

2.2.4 Class $\mathcal{A}$, Subversively-varying Tailed Subexponential Distributions

Definition 2.5. A distribution function F is in Class $\mathcal{A}$, $F \in \mathcal{A}$, if its subexponential and its right tail is subversively varying that is

$$\limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} < 1 \text{ for some } y > 1. \tag{2.15}$$

Class $\mathcal{A}$ could be characterized as a subclass of subexponential distributions that does not include exceptionally heavy tails due to the additional condition.

Lemma 2.2. *Let F be a distribution function supported on $[0,\infty)$ with a density function $f(x)$ which is eventually non-increasing. Then the following statements are equivalent:*
(i) Relation (2.15) holds for some $y > 1$;
(ii) Relation (2.15) holds for any $y > 1$;
(iii) the hazard rate of F, $q(x) = f(x)/\overline{F}(x)$, satisfies

$$\liminf_{x\to\infty} xq(x) > 0. \tag{2.16}$$

Proof. The proof is by Konstantinides et.al.[7].
$(i) \Rightarrow (iii)$ For the fixed $y > 1$ in (i) and all sufficiently large $x > 0$, it holds that

$$\frac{\overline{F}(xy)}{\overline{F}(x)} = 1 - \frac{\int_x^{xy} f(t)dt}{\overline{F}(x)} \geq 1 - \frac{(xy-x)f(x)}{\overline{F}(x)} = 1 - (y-1)xq(x),$$

from which it follows that

$$\liminf_{x\to\infty} xq(x) \geq \frac{1}{y-1} \liminf_{x\to\infty} \left(1 - \frac{\overline{F}(xy)}{\overline{F}(x)}\right) > 0.$$

$(iii) \Rightarrow (ii)$ For any fixed $y > 1$ and all large enough $x > 0$,

$$\frac{\overline{F}(xy)}{\overline{F}(x)} = \frac{\overline{F}(xy)}{\int_x^{xy} f(t)dt + \overline{F}(xy)} \leq \frac{\overline{F}(xy)}{(y-1)xf(xy) + \overline{F}(xy)} = \frac{1}{(y-1)xq(xy)+1},$$

which when combined with 2.16, implies that 2.15 holds for any $y > 1$.
$(ii) \Rightarrow (i)$ is trivial. □

Remark 2.17. The subversively varying tailed distributions define their own class, the class $\mathcal{E}$ the extended rapidly varying tailed distributions. Thereby the expression $\mathcal{A} = \mathcal{S} \cap \mathcal{E}$ is quite common in literature.

Remark 2.18. Class $\mathcal{A}$ has practically speaking mild conditions, since Condition 2.15 is fulfilled by almost all distributions with an ultimate right tail and any kind of practical value, including the Pareto, the log-normal, the Weibull, the log-gamma, the Burr, the Benktander I and II distributions (*Konstantinides et.al.*[7]). Class $\mathcal{A}$ also happens to cover Class $\mathcal{S}$ almost completely, notable exclusion being the distributions with slowly varying tail.

Generally speaking, not that many slowly varying tailed distributions are popular enough to have a name, but there are some like the log-Cauchy distribution that is defined by the tail

$$\overline{F}(x) = \frac{1}{2} - \frac{1}{\pi}\arctan(\ln x), \quad x > 0.$$

By applying l'Hôpital's rule, we can verify that the distribution is indeed slowly varying tailed,

$$\begin{aligned}\lim_{x\to\infty}\frac{\overline{F}(xy)}{\overline{F}(x)} &= \lim_{x\to\infty}\frac{\frac{d}{dx}(\frac{1}{2}-\frac{1}{\pi}\arctan(\ln(xy)))}{\frac{d}{dx}(\frac{1}{2}-\frac{1}{\pi}\arctan(\ln x))} = \lim_{x\to\infty}\frac{-\frac{y}{xy}\frac{1}{1+\ln(xy)^2}}{-\frac{1}{x}\frac{1}{1+\ln(x)^2}}\\ &= \lim_{x\to\infty}\frac{1+\ln(x)^2}{1+\ln(xy)^2} = \lim_{x\to\infty}\frac{\ln(x)^2+1}{\ln(x)^2+2\ln(x)\ln(y)+\ln(y)^2+1} = 1.\end{aligned}$$

Slowly varying tailed distributions can also be generated by choosing a long tailed F (by Proposition 2.1 a subexponential F will suffice), noting that

$$\overline{F}(\ln x) \sim \overline{F}(\ln x + \ln y) = \overline{F}(\ln(xy)),$$

and doing the necessary adjustments to the support etc. to obtain a valid distribution tail.

2.3 Matuszewska Indices

We first introduce the original definitions by Matuszewska (see Bingham et al.(1987)[2]), since they are used to establish the language for Definition2.8.

Definition 2.6. Let g be positive. Its *Upper Matuszewska index* $\alpha(g)$ is the infimum of those α for which there exists a constant $C = C(\alpha) > 0$ such that for each $\Lambda > 1$,

$$\frac{g(\lambda x)}{g(x)} \leq C(1+o(1))\lambda^{\alpha},$$

as $x \to \infty$ and uniformly in $\lambda \in [1, \Lambda]$.

Definition 2.7. Let g be positive. Its *Lower Matuszewska index* $\beta(g)$ is the supremum of those β for which there exists a constant $D = D(\beta) > 0$ such that for each $\Lambda > 1$,

$$\frac{g(\lambda x)}{g(x)} \geq D(1 + o(1))\lambda^{\beta},$$

as $x \to \infty$ and uniformly in $\lambda \in [1, \Lambda]$.

Definition 2.8. We say that the positive function g has bounded increase, if $\alpha(g) < \infty$; bounded decrease, if $\beta(g) > -\infty$; positive increase, if $\beta(g) > 0$; positive decrease, if $\alpha(g) < 0$.

Then the definition that we will actually refer to, and which by convention is used to simplify the notation for the study of distribution functions.

Definition 2.9. *Matuszewska indices (for Distribution Functions)*, for the purposes of this thesis, are defined for a distribution with an ultimate right tail as the following

$$M^*(F) = \inf\left\{-\frac{\log \overline{F}_*(y)}{\log y} : y > 1\right\} \tag{2.19}$$

and

$$M_*(F) = \sup\left\{-\frac{\log \overline{F}^*(y)}{\log y} : y > 1\right\}, \tag{2.20}$$

where

$$\overline{F}_*(y) = \liminf_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} \text{ and } \overline{F}^*(y) = \limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)}.$$

If we were to follow the original definition and naming by Matuszewska, $M^*(F)$ and $M_*(F)$ would correspond to the upper and lower Matuszewska indices of $f = 1/\overline{F}$, as can be seen by Corollary 2.1.6 and Theorem 2.1.5 of Bingham et al.(1987)[2], but since it is more convenient, we will refer to them as the upper and lower Matuszewska indices of F.

Let us then introduce a little recollection of some useful results related to Matuszewska indices that we would like to refer to at certain points in the proofs later on. (*All of these can also be found in Tang & Yuan (2014)[12]*)

Two distributions with weakly equivalent tails have the same Matuszewska indices.

$F \in \mathcal{D}$ if and only if $0 \leq M^*(F) < \infty$, or, equivalently, the function $\overline{F}$ has bounded decrease.

Relation (2.15) holds if and only if $0 < M_*(F) \leq \infty$, or, equivalently the function $\overline{F}$ has positive decrease. Thus, $F \in \mathcal{A}$ can be characterized as $F \in \mathcal{S}$ with a tail of positive decrease.

If $0 \leq M^*(F) < \infty$, then for every $\beta > M^*(F)$, there exists some positive constants C_1 and x_1 such that the inequality

$$\frac{\overline{F}(x)}{\overline{F}(xy)} \leq C_1 y^{\beta} \tag{2.21}$$

holds for all $xy \geq x \geq x_1$.

If $0 < M_*(F) \leq \infty$, then for every $0 < \gamma < M_*(F)$, there exists some positive constants C_2 and x_2 such that the inequality

$$\frac{\overline{F}(xy)}{\overline{F}(x)} \leq C_2 y^{-\gamma} \tag{2.22}$$

holds for all $xy \geq x \geq x_2$.

From (2.21) and (2.22), we see that the relations

$$x^{-\beta} = o\left(\overline{F}(x)\right) \text{ and } \overline{F}(x) = o\left(x^{-\gamma}\right) \tag{2.23}$$

hold for every $\beta > M^*(F)$ and $\gamma < M_*(F)$. Therefore, for a random variable X distributed by F, if $M_*(F) > 1$ then $\mathrm{E}[X^+] < \infty$, while if $\mathrm{E}[X^+] < \infty$ then $M^*(F) \geq 1$.

3 Main Results

The main theorems covered in this thesis will be the following three by Tang & Yuan [12], concerning the relations in (3.1), where we have the real valued primary random variables $X_1, \ldots, X_n$, independent and distributed by $F_1, \ldots, F_n$, respectively, and the non-negative random weights $\theta_1, \ldots, \theta_n$, not degenerate at 0, and arbitrarily dependent on each other, but independent of the primary random variables. For the weighted sums we use the notation

$$S_n^{\theta} = \sum_{i=1}^{n} \theta_i X_i.$$

$$\mathrm{P}\left(\bigvee_{i=1}^{n} S_i^{\theta} > x\right) \sim \mathrm{P}\left(S_n^{\theta} > x\right) \sim \mathrm{P}\left(\bigvee_{i=1}^{n} \theta_i X_i > x\right) \sim \sum_{i=1}^{n} \mathrm{P}\left(\theta_i X_i > x\right) \tag{3.1}$$

Given the basic setting, we will now introduce the additional assumptions with small descriptions, and the main theorems.

(A1) $F_i \in \mathcal{L}$ for all $i = 1, \ldots, n$

Used in conjunction with (A2) or (A4), and allows us to use the tail property, $\overline{F}(x+y) \sim \overline{F}(x)$ for all $y \in \mathbb{R}$, as a tool in the proofs.

(A2) There exists a distribution $F \in \mathcal{S}$ such that $\overline{F}_i(x) \asymp \overline{F}(x)$ for all $i = 1, \ldots, n$.

By Proposition 2.2 while also assuming (A1), we know that the weakly equivalent tails connect the distributions in the sense that the tails are similar, and that the distributions are all in Class $\mathcal{S}$.

(A3) $\theta_1, \ldots, \theta_n$ are bounded from above.

Alongside the basic assumption that the random weights $\theta_1, \ldots, \theta_n$ are non-negative, and not degenerate at 0, creates the bounds for the random weights. Remarkably these bounds are not very restrictive, since there is no lower bound $b > 0$.

Theorem 3.1. *Let the assumptions (A1), (A2), and (A3) hold simultaneously, then* (3.1) *holds.*

(A4) There exists a distribution $F \in \mathcal{A}$ such that $\overline{F}_i(x) \asymp \overline{F}(x)$ for all $i = 1, \ldots, n$.

The weakly equivalent tails connect the distributions by their shared Matuszewska indeces, especially $M_*(F) > 0$, and by Proposition 2.2, meaning that all the distributions are in Class $\mathcal{A}$.

(A5) The relation

$$\mathrm{P}\left(\theta_i > ux\right) = o\left(1\right) \mathrm{P}\left(\theta_i X_i > x\right), x \to \infty, u > 0, \tag{3.2}$$

holds for all $i = 1, \ldots, n$.

The assumption that replaces the upper bound limitation (A3) in Theorem 3.2. We will take a further look into the alternative ways to express and satisfy this condition with some examples at the beginning of the next section.

Theorem 3.2. *Let the assumptions (A1), (A4), and (A5) hold simultaneously, then* (3.1) *holds.*

(A6) $F_i \in \mathcal{L} \cap \mathcal{D}$ and $E\left[\theta_i^{\beta_i}\right] < \infty$ for some $\beta_i > M^*(F_i)$ and all $i = 1, \ldots, n$.

We now restrict the distributions little differently compared to the assumption (A4) of Theorem 3.2, notably we drop the weak tail equivalence requirement, and gain the slowly varying tailed distributions, but we also lose some popular distributions like the log-normal and the Weibull distributions.

Theorem 3.3. *Let the assumption (A6) hold, then* (3.1) *holds.*

The results lay out the asymptotic relations for the tail behaviors of weighted subexponential random variables and their sums, in essence we have an extension to the single big jump principle in the presence of random weights.

In practice,

$$\sum_{i=1}^{n} \mathrm{P}\left(\theta_i X_i > x\right)$$

will often be the easiest one to compute, thereby the results indicate, what else this computation allows us to estimate. Being a way to approximate

$$\mathrm{P}\left(S_n^{\theta} > x\right),$$

we have an immediate application to the computation of the tail probabilities, a prerequisite for getting a numerical value out of many common risk measures such as the ones defined in Chapter 5, Value-at-Risk (VaR) and Conditional Tail Expectation (CTE). The results could also be used to find a value for a ruin probability, since it often boils down to the computation of the probability

$$\mathrm{P}\left(\bigvee_{i=1}^{n} S_i^{\theta} > x\right),$$

as we'll see in Chapter 6.

To broadly characterize the main theorems, the upper-bound imposed on the random weights slightly limits the applicability of Theorem 3.1, but it offers the base case for Theorem 3.2. Theorem 3.2 reaches a nice level of practical applicability by removing the upper-bound restriction, while keeping almost the complete variety of distributions commonly seen in applications. Theorem 3.3 covers the heavier tails of subexponential distributions, and to its merit it drops the requirement of weakly equivalent tails.

4 The Proofs

First, we take a further look into Condition (A5) concerning the random weights in Theorem 3.2, to better understand, when it holds, and the underlying structure it implies. The following lemma and remark are ways to characterize Relation (3.2) and it will be useful later on.

Lemma 4.1. *Relation* (3.2) *in (A5) is equivalent to the existence of a positive auxiliary function* $a(\cdot) : [0, \infty) \to [0, \infty)$*, with* $a(x) \uparrow \infty$ *and* $a(x) = o(x)$*, as* $x \to \infty$*, such that*

$$\mathrm{P}(\theta_i > a(x)) = o(1)P(\theta_i X_i > x). \tag{4.1}$$

Proof. *(A slightly more elaborate version of the proof of Lemma 3.2 found in Tang [10].)*
" $\Leftarrow$ " If θ_i doesn't have an ultimate right tail, then for any large enough x,

$$\mathrm{P}(\theta_i > ux) = 0.$$

If θ_i has an ultimate right tail, then we note that $ux/x = u$ while $a(x)/x \to 0$ as $x \to \infty$, and thereby for any large enough x, it holds that

$$\frac{\mathrm{P}(\theta_i > ux)}{\mathrm{P}(\theta_i > a(x))} \leq 1,$$

and by writing

$$\frac{\mathrm{P}(\theta_i > ux)}{\mathrm{P}(\theta_i X_i > x)} = \frac{\mathrm{P}(\theta_i > ux)}{\mathrm{P}(\theta_i > a(x))} \frac{\mathrm{P}(\theta_i > a(x))}{\mathrm{P}(\theta_i X_i > x)} \leq \frac{\mathrm{P}(\theta_i > a(x))}{\mathrm{P}(\theta_i X_i > x)} \longrightarrow 0, \text{ as } x \to \infty,$$

we have shown the first half of the proof.
" $\Rightarrow$ " Since for each $n = 1, 2, \dots$,

$$\lim_{x\to\infty} \frac{\mathrm{P}(\theta_i > x/n)}{\mathrm{P}(\theta_i X_i > x)} = 0$$

there exists an increasing sequence of positive real numbers, $\{x_n, n = 1, 2, \dots\}$, with $x_{n+1} > (n + 1)x_n$ for all n, such that the inequality

$$\frac{\mathrm{P}(\theta_i > x/n)}{\mathrm{P}(\theta_i X_i > x)} \leq \frac{1}{n}$$

holds whenever $x \geq x_n$, $n = 1, 2, \dots$. Thereby all the given conditions hold simultaneously with

$$a(x) = \sup_{0\leq y\leq x} b(y), \text{ where } b(x) = \sum_{n=1}^{\infty} \frac{x}{n} \mathbf{1}_{(x_n \leq x < x_{n+1})}.$$

This concludes the proof. □

Lemma 4.2. *Any of the following is sufficient for Relation* (3.2) *in (A5) to hold*

(i) $\mathrm{P}(\theta_i > xy) = o\left(\overline{F}_i(x)\right)$ *for some* $y > 0$;

(ii) $\mathrm{P}(\theta_i > xy) = o(1)\mathrm{P}(\theta_i > x)$ *for some* $y > 1$;

(iii) $0 < M_*(G_i) \leq \infty$ *and* $\mathrm{E}\left[\left(X_i^+\right)^{\beta_i}\right] = \infty$ *for some* $0 < \beta_i < M_*(G_i)$,

where G_i *is the distribution function of* θ_i.

Proof. (The proof is from Tang [10], Corollary 2.1, with the notation adapted.)
Assume that G_i has an unbounded support. Since a bounded support trivially yields $\mathrm{P}(\theta_i > ux) = 0$ for all x large enough.

(i) For each $u > 0$, as $x \to \infty$,

$$\frac{\mathrm{P}(\theta_i > ux)}{\mathrm{P}(\theta_i X_i > x)} \leq \frac{\mathrm{P}(\theta_i > ux)}{\mathrm{P}(X_i > ux/y)\,\mathrm{P}(\theta_i > y/u)} \longrightarrow 0.$$

(ii) For each $u > 0$, as $x \to \infty$,

$$\frac{\mathrm{P}(\theta_i > ux)}{\mathrm{P}(\theta_i X_i > x)} \leq \frac{\mathrm{P}(\theta_i > ux)}{\mathrm{P}(X_i > y/u)\mathrm{P}(\theta_i > ux/y)} \longrightarrow 0.$$

(iii) First noting that applying Relation 2.22 to G_i yields

$$\frac{\overline{G_i}(x/y)}{\overline{G_i}(uy \cdot x/y)} \geq \frac{1}{C_2}(uy)^{\gamma},$$

and then we get that for each $u > 0$, as $x \to \infty$,

$$\frac{\mathrm{P}(\theta_i X_i > x)}{\mathrm{P}(\theta_i > ux)} \geq \int_{1/u}^{x/x_2} \frac{\mathrm{P}(\theta_i > x/y)}{\mathrm{P}(\theta_i > ux)} F(dy) \geq \frac{1}{C_2}\int_{1/u}^{x/x_2} (uy)^{\gamma} F(dy) \longrightarrow \infty.$$

Thereby, $\mathrm{P}(\theta_i > ux) = o(1)\,\mathrm{P}(\theta_i X_i > x)$. □

Remark 4.2. For the distributions with rapidly varying tail $\mathcal{R}_{-\infty}$ that is it an ultimate right tail and

$$\lim_{x\to\infty} \frac{\overline{F}(yx)}{\overline{F}(x)} = \begin{cases} 0, & \text{if } y > 1, \\ \infty, & \text{if } 0 < y < 1, \end{cases}$$

the part (ii) of Lemma 4.2 holds trivially. Allowing many popular distributions that have rapidly varying such as exponential, Weibull and log-normal to be used as the distribution of the random weights.

A particularly useful tool for determining whether or not a distribution is rapidly varying tailed is the *mean excess function*

$$e(u) = \mathrm{E}(X - u \mid X > u) = \frac{\int_u^{\infty} \overline{F}(t)dt}{\overline{F}(u)}, \quad u > 0.$$

Since by combining Embrechts et.al.[5] Example 3.4.8 and Theorem A3.12 (with the remark), we have that for a distribution with an ultimate right tail

$$\overline{F} \in \mathcal{R}_{-\infty} \text{ if and only if } \lim_{u\to\infty} \frac{e(u)}{u} = 0.$$

Name	Tail $\overline{F}$ or density f	Parameters	Mean excess $e(u)$	Tail type
Exponential	$\overline{F}(x) = e^{-\lambda x}$	$\lambda > 0$	λ^{-1}	Light
Gamma	$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$	$\alpha, \beta > 0$	$\beta^{-1}\left(1 + \frac{\alpha-1}{\beta u} + o\left(\frac{1}{u}\right)\right)$	Light
Weibull	$\overline{F}(x) = e^{-cx^\tau}$	$c > 0$, $0 < \tau < 1$	$\frac{u^{1-\tau}}{c\tau}(1 + o(1))$	Heavy
Lognormal	$f(x) = \frac{1}{\sqrt{2\pi}\sigma x} e^{-(\ln x - \mu)^2/(2\sigma^2)}$	$\mu \in \mathbb{R}$, $\sigma > 0$	$\frac{\sigma^2 u}{\ln u - \mu}(1 + o(1))$	Heavy
Benktander-type-I	$\overline{F}(x) = (1 + 2(\beta/\alpha)\ln x)\, e^{-\beta(\ln x)^2 - (\alpha+1)\ln x}$	$\alpha, \beta > 0$	$\frac{u}{\alpha + 2\beta \ln u}$	Heavy
Benktander-type-II	$\overline{F}(x) = e^{\alpha/\beta} x^{-(1-\beta)} e^{-\alpha x^\beta/\beta}$	$\alpha > 0$, $0 < \beta < 1$	$\frac{u^{1-\beta}}{\alpha}$	Heavy

Table 4.1. Some rapidly varying tailed distributions and their mean excess functions.

4.1 General Structure of the Proofs

To establish sufficient conditions for the relations in (3.1) to hold, we will now formulate the basis used in the proofs.

The last relation in (3.1) follows if it holds that

$$\sum_{1 \le j \ne k \le n} \mathrm{P}(\theta_j X_j > x, \theta_k X_k > x) = o(1) \sum_{i=1}^{n} \mathrm{P}(\theta_i X_i > x). \tag{4.3}$$

To see how this is sufficient, we will apply the first two of the Bonferroni's inequalities

$$\mathrm{P}\left(\bigcup_{i=1}^{n} A_i\right) \le \sum_{i=1}^{n} \mathrm{P}\left(A_i\right)$$

and

$$\mathrm{P}\left(\bigcup_{i=1}^{n} A_i\right) \ge \sum_{i=1}^{n} \mathrm{P}\left(A_i\right) - \sum_{1 \le i < j \le n} \mathrm{P}\left(A_i \cap A_j\right).$$

We get

$$\mathrm{P}\left(\bigvee_{i=1}^{n}\theta_i X_i > x\right) \leq \sum_{i=1}^{n}\mathrm{P}\left(\theta_i X_i > x\right),$$

and

$$\begin{aligned}\mathrm{P}\left(\bigvee_{i=1}^{n}\theta_i X_i > x\right) &\geq \sum_{i=1}^{n}\mathrm{P}\left(\theta_i X_i > x\right) - \sum_{1\leq j\neq k\leq n}\mathrm{P}(\theta_j X_j > x, \theta_k X_k > x)\\ &= \sum_{i=1}^{n}\mathrm{P}\left(\theta_i X_i > x\right) - o(1)\sum_{i=1}^{n}\mathrm{P}(\theta_i X_i > x),\end{aligned}$$

where we used (4.3) to write the equality. Then letting $x \uparrow \infty$ yields the result.

The rest of the relations in (3.1) are proven by showing that

$$\mathrm{P}(S_n^{\theta} > x) \gtrsim \sum_{i=1}^{n}\mathrm{P}(\theta_i X_i > x) \tag{4.4}$$

and

$$\mathrm{P}\left(\sum_{i=1}^{n}\theta_i X_i^{+} > x\right) \lesssim \sum_{i=1}^{n}\mathrm{P}(\theta_i X_i > x) \tag{4.5}$$

hold.

Since, for $x \geq 0$ we have

$$\mathrm{P}\left(\sum_{i=1}^{n}\theta_i X_i^{+} > x\right) \geq \mathrm{P}\left(\bigvee_{i=1}^{n} S_i^{\theta} > x\right) \geq \mathrm{P}\left(S_i^{\theta} > x\right),$$

and by (4.4) and (4.5), we get

$$\sum_{i=1}^{n}\mathrm{P}(\theta_i X_i > x) \gtrsim \mathrm{P}\left(\sum_{i=1}^{n}\theta_i X_i^{+} > x\right) \gtrsim \mathrm{P}\left(\bigvee_{i=1}^{n} S_i^{\theta} > x\right) \gtrsim \mathrm{P}\left(S_i^{\theta} > x\right) \gtrsim \sum_{i=1}^{n}\mathrm{P}(\theta_i X_i > x),$$

which yields

$$\mathrm{P}\left(\bigvee_{i=1}^{n} S_i^{\theta} > x\right) \sim \mathrm{P}\left(S_i^{\theta} > x\right) \sim \sum_{i=1}^{n}\mathrm{P}\left(\theta_i X_i > x\right).$$

In terms of presentation, Tang & Yuan[12], where the main results originate from, is a typical research paper, aimed at an advanced audience, it opts for a concise display of the new results, while leaving the parts that are similar to the earlier work as a guided exercise for the reader. For the rest of Chapter 4, we will, for the most part, structure our proofs similarly as in Tang & Yuan[12], while aiming to be more elaborate in presentation, filling in steps left out of the original, solving the parts left as an exercise, and adding the central bits from the earlier work, to overall make the presentation more self-contained and easily approachable.

4.2 Proof of Theorem 3.1, Weights Bounded Above, Primary Random Variables in Class $\mathcal{S}$

Lemma 4.3. *(Lemma 5.1. of Tang & Tsitsiashvili [11])*
Let X_1 and X_2 be two independent random variables distributed by F_1 and F_2, respectively. If $F_1 \in \mathcal{S}$, $F_2 \in \mathcal{L}$, and $\overline{F}_2(x) = O\left(\overline{F}_1(x)\right)$, then for any fixed $0 < a \leq 1$, the relation

$$\mathrm{P}\left(X_1 + cX_2 > x\right) \sim \overline{F}_1(x) + \overline{F}_2(x/c) \tag{4.6}$$

holds uniformly for $c \in [a, 1]$, where the uniformity is understood as

$$\lim_{x\to\infty} \sup_{c\in[a,1]} \left| \frac{\mathrm{P}\left(X_1 + cX_2 > x\right)}{\overline{F}_1(x) + \overline{F}_2(x/c)} - 1 \right| = 0.$$

Corollary 4.1. *Let X_1 and X_2 be two independent random variables distributed by F_1 and F_2, respectively. If $F_1, F_2 \in \mathcal{L}$ and there exists some $F \in \mathcal{S}$ such that $\overline{F}_i(x) \asymp \overline{F}(x)$ for both $i = 1, 2$, then for any fixed $0 < a \leq b < \infty$, Relation* (4.6) *holds uniformly for $c \in [a, b]$.*

Proof. Observe that Remark 2.1 implies $\overline{F}(x)_1 = O(\overline{F}(x))$ and $\overline{F}(x) = O(\overline{F}_2(x))$, then by Remark 2.2, we have that $\overline{F}_2(x) = O\left(\overline{F}_1(x)\right)$, furthermore Proposition 2.2 yields $F_1, F_2 \in \mathcal{S}$.

Now, if $b \leq 1$, then by Lemma 4.3, we immediately obtain that (4.6) holds uniformly for $c \in [a, b]$. Hence, we assume that $a \leq 1 < b$, and derive

$$\sup_{c\in[a,b]} \left| \frac{\mathrm{P}\left(X_1 + cX_2 > x\right)}{\overline{F}_1(x) + \overline{F}_2(x/c)} - 1 \right| = \left(\sup_{c\in[a,1]} + \sup_{c\in(1,b]} \right) \left| \frac{\mathrm{P}\left(X_1 + cX_2 > x\right)}{\overline{F}_1(x) + \overline{F}_2(x/c)} - 1 \right| = K_1 + K_2.$$

By Lemma 4.3, it holds for any $\varepsilon > 0$ and all large enough $x > 0$ that $K_1 \leq \varepsilon$.

We denote $c' = 1/c$ and $x' = x/c$, and rewrite K_2 as

$$K_2 = \sup_{c\in(1,b]} \left| \frac{\mathrm{P}\left(c'X_1 + X_2 > x'\right)}{\overline{F}_1(x'/c') + \overline{F}_2(x')} - 1 \right|.$$

Applying Lemma 4.3 once more, yields that $K_2 \le \varepsilon$ holds for all large enough $x' > 0$, or equivalently for all large enough $x > 0$. By the arbitrariness of $\varepsilon > 0$, we have

$$\lim_{x\to\infty} \sup_{c\in[a,b]} \left| \frac{\mathrm{P}\left(X_1 + cX_2 > x\right)}{\overline{F}_1(x) + \overline{F}_2(x/c)} - 1 \right| = 0,$$

and the result follows. □

Lemma 4.4. *(Lemma 1 of Tang & Yuan[12]) Let $X_1, \ldots, X_n$ be n real-valued independent random variables, distributed by $F_1, \ldots, F_n$, respectively, satisfying the conditions (A1) and (A2). Then for every fixed $0 < a \le b < \infty$, it holds uniformly for all $(c_1, \ldots, c_n) \in [a, b]^n$ that*

$$\mathrm{P}\left(\sum_{i=1}^{n} c_i X_i > x\right) \sim \sum_{i=1}^{n} \mathrm{P}\left(c_i X_i > x\right). \tag{4.7}$$

Proof. (The proof follows the structure given by Tang & Tsitsiasvili [11] for the i.i.d. case, while applying any changes required.)
We give the proof by induction approach. It is trivial that the relation holds for $n = 1$. Now we assume that by induction the relation

$$\mathrm{P}\left(\sum_{i=1}^{n} c_i X_i > x\right) \sim \sum_{i=1}^{n} \mathrm{P}\left(c_i X_i > x\right),$$

holds for $n = m$ for some positive integer m. We aim to prove that the relation

$$\mathrm{P}\left(\sum_{i=1}^{m+1} c_i X_i > x\right) \sim \sum_{i=1}^{m+1} \mathrm{P}\left(c_i X_i > x\right) \tag{4.8}$$

holds uniformly for $(c_1, \ldots, c_{m+1}) \in [a, b]^{m+1}$. We rewrite (4.8) as

$$\mathrm{P}\left(\sum_{i=1}^{m} c_i' X_i + X_{m+1} > x'\right) \sim \sum_{i=1}^{m} \mathrm{P}\left(c_i' X_i > x'\right) + \mathrm{P}\left(X_{m+1} > x'\right), \tag{4.9}$$

where $c_i' = c_i / c_{m+1}$ for $1 \le i \le m$ and $x' = x / c_{m+1}$. Note that $c_{m+1} \in [a, b]$ and that

$$0 < \frac{a}{b} \le c_i' \le \frac{b}{a} < \infty \qquad \text{for all} \quad 1 \le i \le m.$$

Thereby, without loss of generality we can assume $c_{m+1} = 1$ in (4.8). As a consequence, it suffices to prove that the relation

$$\mathrm{P}\left(\sum_{i=1}^{m} c_i X_i + X_{m+1} > x\right) \sim \sum_{i=1}^{m} \mathrm{P}\left(c_i X_i > x\right) + \mathrm{P}\left(X_{m+1} > x\right), \tag{4.10}$$

holds uniformly for $(c_1, \dots, c_m) \in [a,b]^m$.

For any $\varepsilon > 0$, by the induction assumption, there exists some constant $B_1 > 0$ such that the inequalities

$$(1-\varepsilon)\sum_{i=1}^{m} \mathrm{P}\left(c_i X_i > x\right) \leq \mathrm{P}\left(\sum_{i=1}^{m} c_i X_i > x\right) \leq (1+\varepsilon)\sum_{i=1}^{m} \mathrm{P}\left(c_i X_i > x\right) \tag{4.11}$$

hold uniformly for $(c_1, \dots, c_m) \in [a,b]^m$ and $x \geq B_1$. Then we divide the probability on the left-hand side of (4.10) into two summands as follows

$$\mathrm{P}\left(\sum_{i=1}^{m} c_i X_i + X_{m+1} > x\right) = L_1 + L_2 = \left(\int_{-\infty}^{x-B_1} + \int_{x-B_1}^{\infty}\right) \mathrm{P}\left(\sum_{i=1}^{m} c_i X_i > x - t\right) dF_{m+1}(t).$$

First we will address L_1. By (4.11) we have

$$\begin{aligned}
L_1 &\leq (1+\varepsilon)\int_{-\infty}^{x-B_1} \mathrm{P}\left(\sum_{i=1}^{m} c_i X_i > x - t\right) dF_{m+1}(t) \\
&= (1+\varepsilon)\sum_{i=1}^{m} \mathrm{P}\left(c_i X_i^{+} + X_{m+1} > x, X_{m+1} \leq x - B_1\right) \\
&= (1+\varepsilon)\sum_{i=1}^{m} \left[\mathrm{P}\left(c_i X_i^{+} + X_{m+1} > x\right) - \mathrm{P}\left(c_i X_i^{+} + X_{m+1} > x, X_{m+1} > x - B_1\right)\right] \\
&\leq (1+\varepsilon)\sum_{i=1}^{m} \left[\mathrm{P}\left(c_i X_i^{+} + X_{m+1} > x\right) - \mathrm{P}\left(c_i X_i^{+} + X_{m+1} > x, X_{m+1} > x\right)\right] \\
&= (1+\varepsilon)\sum_{i=1}^{m} \left[\mathrm{P}\left(c_i X_i^{+} + X_{m+1} > x\right) - \mathrm{P}\left(X_{m+1} > x\right)\right] \\
&= (1+\varepsilon)\sum_{i=1}^{m} \mathrm{P}\left(c_i X_i^{+} + X_{m+1} > x\right) - (1+\varepsilon)m\mathrm{P}\left(X_{m+1} > x\right).
\end{aligned}$$

Symmetrically,

$$
\begin{aligned}
L_1 &\geq (1-\varepsilon) \int_{-\infty}^{x-B_1} \sum_{i=1}^{m} \mathrm{P}\left(c_i X_i > x-t\right) dF_{m+1}(t) \\
&= (1-\varepsilon) \sum_{i=1}^{m} \mathrm{P}\left(c_i X_i^+ + X_{m+1} > x, X_{m+1} \leq x - B_1\right) \\
&= (1-\varepsilon) \sum_{i=1}^{m} \left[\mathrm{P}\left(c_i X_i^+ + X_{m+1} > x\right) - \mathrm{P}\left(c_i X_i^+ + X_{m+1} > x, X_{m+1} > x - B_1\right)\right] \\
&\geq (1-\varepsilon) \sum_{i=1}^{m} \left[\mathrm{P}\left(c_i X_i^+ + X_{m+1} > x\right) - \mathrm{P}\left(X_{m+1} > x - B_1\right)\right] \\
&= (1-\varepsilon) \sum_{i=1}^{m} \mathrm{P}\left(c_i X_i^+ + X_{m+1} > x\right) - (1-\varepsilon) m \mathrm{P}\left(X_{m+1} > x - B_1\right).
\end{aligned}
$$

By Corollary 4.1 and $F_{m+1} \in \mathcal{L}$, for each $i = 1, \ldots, m$, there exists some constant $\beta_i > 0$ such that the inequalities

$$
\begin{aligned}
&(1-\varepsilon)\left(\overline{F}_i(x/c_i) + \overline{F}_{m+1}(x)\right) \\
&\leq \mathrm{P}\left(c_i X_i^+ + X_{m+1} > x\right) \\
&\leq (1+\varepsilon)\left(\overline{F}_i(x/c_i) + \overline{F}_{m+1}(x)\right)
\end{aligned}
$$

and

$$
\overline{F}_{m+1}(x - B_1) \leq (1+\varepsilon)\overline{F}_{m+1}(x)
$$

hold uniformly for $c_i \in [a, b]$ and $x \geq \beta_i$. Then by substituting these into the upper and lower bounds for L_1 that were derived, we obtain that uniformly for $(c_1, \ldots, c_m) \in [a, b]^m$ and $x \geq \max\{B_1, \beta_1, \ldots, \beta_m\}$

(4.12)

$$
(1-\varepsilon)^2 \sum_{i=1}^{m} \overline{F}_i(x/c_i) - 2\left(\varepsilon - \varepsilon^2\right) m\overline{F}_{m+1}(x) \leq L_1 \leq (1+\varepsilon)^2 \sum_{i=1}^{m} \overline{F}_i(x/c_i) + \left(\varepsilon + \varepsilon^2\right) m\overline{F}_{m+1}(x).
$$

Next, we will cover L_2. Deriving an upper bound is basically trivial, since for all $x \geq \max\{\beta_1, \ldots, \beta_m\}$,

$$
\begin{aligned}
L_2 &= \int_{x-B_1}^{\infty} \mathrm{P}\left(\sum_{i=1}^{m} c_i X_i > x - t\right) dF_{m+1}(t) \leq \int_{x-B_1}^{\infty} 1 dF_{m+1}(t) \\
&= \overline{F}_{m+1}\left(x - B_1\right) \leq (1+\varepsilon)\overline{F}_{m+1}(x).
\end{aligned}
$$

To get a lower bound, we can choose some $C > 0$ such that

$$
\mathrm{P}\left(b \sum_{i=1}^{m} \min\{X_i, 0\} > -C\right) \geq (1-\varepsilon),
$$

and then $B_2 > 0$ such that we have uniformly for $(c_1, \dots, c_m) \in [a,b]^m$ and $x \geq B_2$,

$$
\begin{aligned}
L_2 &= \int_{x-B_1}^{\infty} \mathrm{P}\left(\sum_{i=1}^{m} c_i X_i > x - t\right) dF_{m+1}(t)) \\
&\geq \int_{x-B_1}^{\infty} \mathrm{P}\left(b\sum_{i=1}^{m} \min\{X_i, 0\} > x - t\right) dF_{m+1}(t) \\
&\geq \int_{x+C}^{\infty} \mathrm{P}\left(b\sum_{i=1}^{m} \min\{X_i, 0\} > x - t\right) dF_{m+1}(t) \\
&\geq \mathrm{P}\left(b\sum_{i=1}^{m} \min\{X_i, 0\} > -C\right) \overline{F}_{m+1}(x+C) \\
&\geq (1-\varepsilon)^2 \overline{F}_{m+1}(x).
\end{aligned}
$$

Thereby, we have shown that uniformly for $(c_1, \dots, c_m) \in [a,b]^m$ and
$x \geq \max\{B_2, \beta_1, \dots, \beta_m\}$,

$$
(1-\varepsilon)^2 \overline{F}_{m+1}(x) \leq L_2 \leq (1+\varepsilon)\overline{F}_{m+1}(x). \tag{4.13}
$$

By combining (4.12) and (4.13), we see that the inequalities

$$
\mathrm{P}\left(\sum_{i=1}^{m} c_i X_i + X_{m+1} > x\right) \geq (1-\varepsilon)^2 \sum_{i=1}^{m} \overline{F}_i(x/c_i) - \left(2m\varepsilon - 2m\varepsilon^2 - (1-\varepsilon)^2\right) \overline{F}_{m+1}(x)
$$

and

$$
\mathrm{P}\left(\sum_{i=1}^{m} c_i X_i + X_{m+1} > x\right) \leq (1+\varepsilon)^2 \sum_{i=1}^{m} \overline{F}_i(x/c_i) + \left(\varepsilon m - \varepsilon^2 m + \varepsilon + 1\right) \overline{F}_{m+1}(x)
$$

hold uniformly for $(c_1, \dots, c_m) \in [a,b]^m$ and $x \geq \max\{B_1, B_2, \beta_1, \dots, \beta_m\}$. By arbitrariness of $\varepsilon > 0$, we see that Relation (4.10) holds, which concludes the proof. □

Lemma 4.5. *(Lemma 2 of Tang & Yuan[12]) Let X and θ be two non-negative independent random variables. If X is long tailed, θ is not degenerate at 0, and* $\mathrm{P}(\theta > ux) = o(1)\mathrm{P}(\theta X > x), x \to \infty$ *for all $u > 0$, then the product θX is long tailed.*

Proof. Write, given the $a(x)$ as in Lemma 4.1,

$$
\begin{aligned}
&\mathrm{P}(\theta X > x + y) \\
&\quad = \mathrm{P}(\theta X > x+y, \theta > a(x)) + \mathrm{P}(\theta X > x+y, 0 < \theta \leq a(x)) + \mathrm{P}(\theta X > x+y, \theta = 0)
\end{aligned}
$$

and

$$
\mathrm{P}(\theta X > x) = \mathrm{P}(\theta X > x, \theta > a(x)) + \mathrm{P}(\theta X > x, 0 < \theta \leq a(x)) + \mathrm{P}(\theta X > x, \theta = 0).
$$

Then, fix $y \in \mathbb{R}$, and note that, for large enough x,

$$\mathrm{P}(\theta X > x + y, \theta = 0) = \mathrm{P}(\theta X > x, \theta = 0) = 0,$$

and that

$$\mathrm{P}(\theta X > x + y, \theta > a(x)) \leq \mathrm{P}(\theta > a(x)) = o(1)\mathrm{P}(\theta X > x)$$

and

$$\mathrm{P}(\theta X > x, \theta > a(x)) \leq \mathrm{P}(\theta > a(x)) = o(1)\mathrm{P}(\theta X > x).$$

For the remaining parts, fix $\lambda \in (0, 1)$, since X is long tailed, and $a(x) \uparrow \infty$, $a(x) = o(x)$, $x \to \infty$, we have

$$\lim_{x\to\infty} \left| \frac{\mathrm{P}\left(X > \frac{x}{a(x)} + \frac{y}{\lambda}\right)}{\mathrm{P}\left(X > \frac{x}{a(x)}\right)} - 1 \right| = 0.$$

Thereby for any $\varepsilon > 0$, we can find x_ε such that for all $x \geq x_\varepsilon$ it holds that

$$(1-\varepsilon)\mathrm{P}\left(X > \frac{x}{a(x)}\right) \leq \mathrm{P}\left(X > \frac{x}{a(x)} + \frac{y}{\lambda}\right) \leq (1+\varepsilon)\mathrm{P}\left(X > \frac{x}{a(x)}\right).$$

Assuming sufficiently large x so that $\lambda < a(x)$, we note, since for any $t \in (\lambda, a(x)]$,

$$\frac{x}{t} \geq \frac{x}{a(x)} \qquad \text{and} \qquad \frac{|y|}{t} \leq \frac{|y|}{\lambda},$$

we have for the same ε and x_ε that for all $x \geq x_\varepsilon$,

$$(1-\varepsilon)\mathrm{P}\left(X > \frac{x}{t}\right) \leq \mathrm{P}\left(X > \frac{x+y}{t}\right) \leq (1+\varepsilon)\mathrm{P}\left(X > \frac{x}{t}\right),$$

and further that for all $x \geq x_\varepsilon$ it holds that

$$\begin{aligned}
&(1-\varepsilon)\mathrm{P}\left(\theta X > x, \lambda < \theta \leq a(x) \mid \theta = t\right) \\
&\leq \mathrm{P}\left(\theta X > x + y, \lambda < \theta \leq a(x) \mid \theta = t\right) \\
&\leq (1+\varepsilon)\mathrm{P}\left(\theta X > x, \lambda < \theta \leq a(x) \mid \theta = t\right).
\end{aligned}$$

Let $G(t)$ be the distribution function of θ. Now by integrating, we get

$$\begin{aligned}
&(1-\varepsilon)\int_0^\infty \mathrm{P}\left(\theta X > x, \lambda < \theta \leq a(x) \mid \theta = t\right) dG(t) \\
&\leq \int_0^\infty \mathrm{P}\left(\theta X > x + y, \lambda < \theta \leq a(x) \mid \theta = t\right) dG(t) \\
&\leq (1+\varepsilon)\int_0^\infty \mathrm{P}\left(\theta X > x, \lambda < \theta \leq a(x) \mid \theta = t\right) dG(t).
\end{aligned}$$

By the dominated convergence theorem, and the arbitrariness of $\lambda > 0$, we obtain that

$$\begin{aligned}&\int_0^\infty \mathrm{P}\left(\theta X > x, 0 < \theta \le a(x) \mid \theta = t\right) dG(t)\\ &\sim \int_0^\infty \mathrm{P}\left(\theta X > x + y, 0 < \theta \le a(x) \mid \theta = t\right) dG(t),\end{aligned}$$

as $x \to \infty$, which allows us to write,

$$\begin{aligned}&\mathrm{P}(\theta X > x + y, 0 < \theta \le a(x)) = \mathrm{E}\left[\mathrm{P}(\theta X > x + y, 0 < \theta \le a(x) \mid \theta)\right]\\ &\sim \mathrm{E}\left[\mathrm{P}(\theta X > x, 0 < \theta \le a(x) \mid \theta)\right] = \mathrm{P}(\theta X > x, 0 < \theta \le a(x)).\end{aligned}$$

By combining everything, we see that the result follows. □

Proof of Theorem 3.1. (The underlying proof is by Tang & Yuan[12])
Without loss of generality, assume that the random weights $\theta_1, \dots, \theta_n$ are bounded above by 1, and let $x \ge 0$.

First, addressing Relation (4.3).

For Theorem 3.1, Relation (4.3) holds quite trivially, since we have an upper bound for the weights θ_i. Say $\theta_i \le b$ for all $i = 1, \dots, N$, then as $x \to \infty$

$$\begin{aligned}0 &\le \sum_{1 \le j \ne k \le n} \mathrm{P}(\theta_j X_j > x, \theta_k X_k > x)\\ &\le \sum_{1 \le j \ne k \le n} \mathrm{P}(b X_j > x) \mathrm{P}(\theta_k X_k > x)\\ &= o(1) \sum_{i=1}^n \mathrm{P}(\theta_i X_i > x).\end{aligned}$$

Then we move on to show that Relation (4.4) holds.

First assume that the random variables $X_1, \dots, X_n$ are non-negative. It follows that $S_n^\theta \ge \bigvee_{i=1}^n \theta_i X_i$, and further $\mathrm{P}(S_n^\theta > x) \ge \mathrm{P}(\bigvee_{i=1}^n \theta_i X_i > x)$, which in combination with Relation (4.3) yields Relation (4.4).

Now consider the general case where the random variables $X_1, \dots, X_n$ may obtain negative values. For an arbitrary subset $I \subseteq \{1, \dots, n\}$, write $I^c = \{1, \dots, n\} \setminus I$ and

$$\Omega_I(X) = \{\omega : X_i \ge 0 \text{ for } i \in I \text{ and } X_j < 0 \text{ for } j \in I^c\}.$$

Since each random weight is bounded above by 1, and $x \ge 0$, it follows that

$$\begin{aligned}\mathrm{P}(S_n^\theta > x) &= \sum_{\emptyset \ne I \subseteq \{1,\dots,n\}} \mathrm{P}\left(\sum_{i \in I} \theta_i X_i + \sum_{j \in I^c} \theta_j X_j > x, \Omega_I(X)\right)\\ &\ge \sum_{\emptyset \ne I \subseteq \{1,\dots,n\}} \mathrm{P}\left(\sum_{i \in I} \theta_i X_i + \sum_{j \in I^c} X_j > x, \Omega_I(X)\right),\end{aligned}$$

where we follow the convention that the sum over an empty set is equal to 0.

Then conditioning on X_j for $j \in I^c$, we have based on the non-negative case that

$$\begin{aligned}
\mathrm{P}\left(\sum_{i\in I}\theta_i X_i + \sum_{j\in I^c} X_j > x, \Omega_I(X)\right) &= \mathrm{E}\left[\mathrm{P}\left(\sum_{i\in I}\theta_i X_i + \sum_{j\in I^c} X_j > x, \Omega_I(X) \mid \{X_j; j\in I^c\}\right)\right] \\
&\gtrsim \mathrm{E}\left[\sum_{i\in I}\mathrm{P}\left(\theta_i X_i + \sum_{j\in I^c} X_j > x, \Omega_I(X) \mid \{X_j; j\in I^c\}\right)\right] \\
&= \sum_{i\in I}\mathrm{P}\left(\theta_i X_i + \sum_{j\in I^c} X_j > x, \Omega_I(X)\right)
\end{aligned}$$

We then apply the dominated convergence theorem and the fact that based on Lemma 4.5, $\theta_i X_i$ is long tailed, since θ_i is bounded and X_i is long tailed, and obtain

$$\sum_{i\in I}\mathrm{P}\left(\theta_i X_i + \sum_{j\in I^c} X_j > x, \Omega_I(X)\right) \sim \sum_{i\in I}\mathrm{P}(\theta_i X_i > x, \Omega_I(X)),$$

furthermore, asymptotically we have

$$\begin{aligned}
\mathrm{P}(S_n^\theta > x) &\gtrsim \sum_{\emptyset\neq I\subseteq\{1,\ldots,n\}}\sum_{i\in I}\mathrm{P}(\theta_i X_i > x, \Omega_I(X)) \\
&= \sum_{i=1}^{n}\sum_{I: i\in I\subseteq\{1,\ldots,n\}}\mathrm{P}(\theta_i X_i > x, \Omega_I(X)) \\
&= \sum_{i=1}^{n}\mathrm{P}(\theta_i X_i > x),
\end{aligned}$$

which is Relation (4.4).

The next step is establishing Relation (4.5). Begin by assuming first that the random weights are positive. Let I and I^c be as before and write

$$\Omega_I^\varepsilon(\theta) = \{\omega : \theta_i > \varepsilon \text{ for } i\in I \text{ and } \theta_j \leq \varepsilon \text{ for } j \in I^c\},\ 0<\varepsilon<1.$$

We have,

$$\begin{aligned}
\mathrm{P}\left(\sum_{i=1}^{n}\theta_i X_i^+ > x\right) &= \sum_{I\subseteq\{1,\ldots,n\}}\mathrm{P}\left(\sum_{i\in I}\theta_i X_i^+ + \sum_{j\in I^c}\theta_j X_j^+ > x, \Omega_I^\varepsilon(\theta)\right) \\
&\leq \sum_{I\subseteq\{1,\ldots,n\}}\mathrm{P}\left(\sum_{i\in I}\theta_i X_i^+ + \sum_{j\in I^c}\varepsilon X_j^+ > x, \Omega_I^\varepsilon(\theta)\right).
\end{aligned}$$

Now, based on Lemma 4.4,

$$
\begin{aligned}
&\mathrm{P}\left(\sum_{i\in I}\theta_i X_i^+ + \sum_{j\in I^c}\varepsilon X_j^+ > x, \Omega_I^\varepsilon(\theta)\right)\\
&\sim \sum_{i\in I}\mathrm{P}\left(\theta_i X_i > x, \Omega_I^\varepsilon(\theta)\right) + \sum_{j\in I^c}\mathrm{P}\left(\varepsilon X_j > x, \Omega_I^\varepsilon(\theta)\right)\\
&= \sum_{i\in I}\mathrm{P}\left(\theta_i X_i > x, \Omega_I^\varepsilon(\theta)\right) + \sum_{j\in I^c}\mathrm{P}\left(\varepsilon X_j > x\right)\mathrm{P}\left(\Omega_I^\varepsilon(\theta)\right)\\
&= \sum_{i\in I}\mathrm{P}\left(\theta_i X_i > x, \Omega_I^\varepsilon(\theta)\right) + \sum_{j\in I^c}\mathrm{P}\left(\varepsilon X_j > x, \theta_j > \varepsilon\right)\frac{\mathrm{P}\left(\Omega_I^\varepsilon(\theta)\right)}{\mathrm{P}\left(\theta_j > \varepsilon\right)}.
\end{aligned}
$$

Yielding,

$$
\begin{aligned}
&\mathrm{P}\left(\sum_{i=1}^n \theta_i X_i^+ > x\right)\\
&\lesssim \sum_{I\subseteq\{1,\ldots,n\}}\left(\sum_{i\in I}\mathrm{P}\left(\theta_i X_i > x, \Omega_I^\varepsilon(\theta)\right) + \sum_{j\in I^c}\mathrm{P}\left(\varepsilon X_j > x, \theta_j > \varepsilon\right)\frac{\mathrm{P}\left(\Omega_I^\varepsilon(\theta)\right)}{\mathrm{P}\left(\theta_j > \varepsilon\right)}\right)\\
&\lesssim \sum_{i=1}^n \sum_{I: i\in I\subseteq\{1,\ldots,n\}}\mathrm{P}\left(\theta_i X_i > x, \Omega_I^\varepsilon(\theta)\right) + \sum_{j=1}^n \sum_{I: j\notin I\subseteq\{1,\ldots,n\}}\mathrm{P}\left(\theta_j X_j > x\right)\frac{\mathrm{P}\left(\Omega_I^\varepsilon(\theta)\right)}{\mathrm{P}\left(\theta_j > \varepsilon\right)}\\
&= \sum_{i=1}^n \mathrm{P}\left(\theta_i X_i > x, \theta_i > \varepsilon)\right) + \sum_{j=1}^n \mathrm{P}\left(\theta_j X_j > x\right)\frac{\mathrm{P}\left(\theta_j \le \varepsilon)\right)}{\mathrm{P}\left(\theta_j > \varepsilon\right)}\\
&\le \left(1 + \max_{1\le j\le n}\frac{\mathrm{P}\left(\theta_j \le \varepsilon\right)}{\mathrm{P}\left(\theta_j > \varepsilon\right)}\right)\sum_{i=1}^n \mathrm{P}\left(\theta_i X_i > x\right).
\end{aligned}
$$

Since each θ_j is strictly positive, it follows by letting $\varepsilon \downarrow 0$ that Relation (4.5) holds.

Finally, consider the general case, where weights can get value 0 with a positive probability. Let I and I^c be as before and write

$$\Omega_I^0(\theta) = \{\omega : \theta_i > 0 \text{ for } i \in I \text{ and } \theta_j = 0 \text{ for } j \in I^c\}$$

Then the previous case, where the weights were strictly positive, and some creative manipulation with the set $\Omega_I^0(\theta)$, permit us to use of Relation (4.5), to write

$$\begin{aligned}
\mathrm{P}\left(\sum_{i=1}^{n}\theta_i X_i^+ > x\right) &= \sum_{\emptyset\neq I\subseteq\{1,\dots,n\}} \mathrm{P}\left(\sum_{i\in I}\theta_i X_i^+ > x, \Omega_i^0(\theta)\right) \\
&\lesssim \sum_{\emptyset\neq I\subseteq\{1,\dots,n\}} \sum_{i\in I} \mathrm{P}\left(\theta_i X_i > x, \Omega_i^0(\theta)\right) \\
&= \sum_{i=1}^{n} \mathrm{P}\left(\theta_i X_i > x\right).
\end{aligned}$$

This concludes the proof of Theorem 3.1.

□

4.3 Proof of Theorem 3.2, Unbounded Weights, Primary Random Variables in Class $\mathcal{A}$

Lemma 4.6. *(Lemma 3 of Tang & Yuan[12]) In addition to the conditions of Lemma 4.4, assume that $M_*(F) > 0$. Then it holds uniformly for all $(c_1, \dots, c_n) \in (0, 1]^n$ that*

$$\mathrm{P}\left(\sum_{i=1}^{n} c_i X_i > x\right) \lesssim \sum_{i=1}^{n} \mathrm{P}(c_i X_i > x).$$

Proof. (Tang & Yuan[12]) We may assume without loss of generality that $c_1 = 1$, since by rewriting

$$\mathrm{P}\left(\sum_{i=1}^{n} c_i X_i > x\right) = \mathrm{P}\left(\sum_{i=1}^{n} \tilde{c}_i X_i > \tilde{x}\right),$$

where $\tilde{c}_i = c_i/c_{(n)}$ and $\tilde{x} = x/c_{(n)}$, we have that each $\tilde{c}_i$ lies in $(0, 1]$ with at least one them equaling 1, and $\tilde{x} \to \infty$ as $x \to \infty$.

For every $0 < \varepsilon < 1$ and $I \subseteq \{2, \dots, n\}$, write $I^c = \{2, \dots, n\} \setminus I$ and

$$A_I = \{(c_2, \dots, c_n) : 0 < c_i \le \varepsilon \text{ for } i \in I \text{ and } \varepsilon < c_j \le 1 \text{ for } j \in I^c\}.$$

Based on Lemma 4.4 and Inequality (2.22), uniformly over each A_I it holds for arbitrarily fixed $0 < \gamma < M_*(F)$ and some $C > 0$ that

$$\begin{aligned}
\frac{\mathrm{P}\left(\sum_{i=1}^n c_i X_i > x\right)}{\sum_{i=1}^n \mathrm{P}\left(c_i X_i > x\right)} &\lesssim \frac{\mathrm{P}\left(X_1^+ + \sum_{i\in I} \varepsilon X_i^+ + \sum_{j\in I^c} c_j X_j^+ > x\right)}{\mathrm{P}\left(X_1 > x\right) + \sum_{j\in I^c} \mathrm{P}\left(c_j X_j > x\right)} \\
&\sim \frac{\mathrm{P}(X_1 > x) + \sum_{i\in I} \mathrm{P}(\varepsilon X_i > x) + \sum_{j\in I^c} \mathrm{P}(c_j X_j > x)}{\mathrm{P}\left(X_1 > x\right) + \sum_{j\in I^c} \mathrm{P}\left(c_j X_j > x\right)} \\
&= 1 + \frac{\sum_{i\in I} \mathrm{P}(\varepsilon X_i > x)}{\mathrm{P}\left(X_1 > x\right) + \sum_{j\in I^c} \mathrm{P}\left(c_j X_j > x\right)} \\
&\leq 1 + \frac{\sum_{i\in I} \mathrm{P}(\varepsilon X_i > x)}{\mathrm{P}\left(X_1 > x\right)} \\
&\lesssim 1 + C\varepsilon^\gamma n.
\end{aligned}$$

The result follows, since $\{A_I : I \subseteq \{2, \dots, n\}\}$ forms a finite partition of $(0, 1]^{n-1}$ and ε can be chosen to be arbitrarily close to 0. □

Lemma 4.7. *(Lemma 4 of Tang & Yuan[12]) Let $X_1, \dots, X_n$ be n non-negative independent random variables, each with an ultimate right tail, then it holds uniformly for all $(c_1, \dots, c_n) \in (0, 1]^n$ that*

$$\mathrm{P}\left(\sum_{i=1}^n c_i X_i > x\right) \gtrsim \sum_{i=1}^n \mathrm{P}(c_i X_i > x).$$

Proof. First note that

$$\mathrm{P}\left(\sum_{i=1}^n c_i X_i > x\right) \geq \mathrm{P}\left(\bigvee_{i=1}^n c_i X_i > x\right),$$

and then by applying Bonferroni's inequality, we get

$$\begin{aligned}
\mathrm{P}\left(\bigvee_{i=1}^n c_i X_i > x\right) &\geq \sum_{i=1}^n \mathrm{P}(c_i X_i > x) - \sum_{1\leq i,j\leq n} \mathrm{P}\left(c_i X_i > x, c_j X_j > x\right) \\
&\geq \sum_{i=1}^n \mathrm{P}(c_i X_i > x) - \sum_{1\leq i,j\leq n} \mathrm{P}\left(c_i X_i > x\right) \mathrm{P}\left(c_j X_j > x\right) \\
&\sim \sum_{i=1}^n \mathrm{P}(c_i X_i > x).
\end{aligned}$$

□

By combining Lemmas 4.6 and 4.7, we obtain the following extension to Lemma 4.4.

Lemma 4.8. *(Lemma 5 of Tang & Yuan[12]) Let $X_1, \ldots, X_n$ be n non-negative independent random variables, distributed by $F_1, \ldots, F_n$, respectively, satisfying the conditions (A1) and (A4). Then relation*

$$\mathrm{P}\left(\sum_{i=1}^{n} c_i X_i > x\right) \sim \sum_{i=1}^{n} \mathrm{P}\left(c_i X_i > x\right).$$

holds uniformly for all $(c_1, \ldots, c_n) \in (0, 1]^n$.

Lemma 4.9. *(Lemma 6 of Tang & Yuan[12]) Let X, $\{Y_1, \ldots, Y_n\}$ and $\{\theta_0, \theta_1, \ldots, \theta_n\}$ be three independent groups of non-negative random variables. If X is distributed by $F \in \mathcal{L}$ with $M_*(F) > 0$, and $\mathrm{P}(\theta_i > ux) = o(1)\mathrm{P}(\theta_i X > x), x \to \infty$ for all $u > 0$ and $i = 0, 1, \ldots, n$, then*

$$\mathrm{P}\left(\theta_0 X - \sum_{i=1}^{n} \theta_i Y_i > x\right) = \mathrm{P}(\theta_0 X > x) - o(1) \sum_{i=1}^{n} \mathrm{P}(\theta_i X > x).$$

Proof. (Tang & Yuan[12]) Since for non-negative random variables, it obviously holds that $\mathrm{P}\left(\theta_0 X - \sum_{i=1}^{n} \theta_i Y_i > x\right) \leq \mathrm{P}\left(\theta_0 X > x\right)$, we can focus on the opposite inequality.

Based on Lemma 4.1, we know that there is some positive auxiliary function $a(\cdot)$, with $a(x) \uparrow \infty$ and $a(x) = o(x)$, such that $\mathrm{P}\left(\theta_i > a(x)\right) = o(1)\mathrm{P}\left(\theta_i X > x\right)$ holds for all $i = 0, 1, \ldots, n$. Let $\theta_{(n)} = \bigvee_{i=1}^{n} \theta_i$.

For arbitrarily fixed $0 < \varepsilon < 1$, we derive

$$\begin{aligned}
\mathrm{P}\left(\theta_0 X - \sum_{i=1}^{n} \theta_i Y_i > x\right) &\geq \mathrm{P}\left(\theta_0 X - \sum_{i=1}^{n} \theta_i Y_i > x, \varepsilon\theta_{(n)} < \theta_0 \leq a(x)\right) \\
&\geq \mathrm{P}\left(\theta_0 X - \frac{\theta_0}{\varepsilon} \sum_{i=1}^{n} Y_i > x, \varepsilon\theta_{(n)} < \theta_0 \leq a(x)\right) \\
&= \mathrm{E}\left[\mathrm{P}\left(\theta_0 X - \frac{\theta_0}{\varepsilon} \sum_{i=1}^{n} Y_i > x, \varepsilon\theta_{(n)} < \theta_0 \leq a(x) \mid \theta_0\right)\right] \\
&\sim \mathrm{E}\left[\mathrm{P}\left(\theta_0 X > x, \varepsilon\theta_{(n)} < \theta_0 \leq a(x) \mid \theta_0\right)\right] \\
&= \mathrm{P}\left(\theta_0 X > x, \varepsilon\theta_{(n)} < \theta_0 \leq a(x)\right) \\
&\geq \mathrm{P}\left(\theta_0 X > x\right) - \mathrm{P}\left(\theta_0 X > x, \varepsilon\theta_{(n)} \geq \theta_0\right) - \mathrm{P}\left(\theta_0 > a(x)\right),
\end{aligned}$$

where in the fourth step we condition on θ_0, then apply the dominated convergence theorem and the fact that $F \in \mathcal{L}$. It holds for all $0 < \gamma < M_*(F)$, some $C > 0$ and all large enough x that

$$\begin{aligned}
\mathrm{P}\left(\theta_0 X > x, \varepsilon\theta_{(n)} \geq \theta_0\right) &\leq \mathrm{P}\left(\varepsilon\theta_{(n)} X > x\right) \\
&\leq \mathrm{P}\left(\varepsilon\theta_{(n)} X > x, \theta_{(n)} \leq a(x)\right) + \mathrm{P}\left(\theta_{(n)} > a(x)\right) \\
&\leq C\varepsilon^{\gamma}\mathrm{P}\left(\theta_{(n)} X > x\right) + \sum_{i=1}^{n} \mathrm{P}\left(\theta_i > a(x)\right) \\
&\lesssim C\varepsilon^{\gamma} \sum_{i=1}^{n} \mathrm{P}\left(\theta_i X > x\right),
\end{aligned}$$

where the third step follows from Relation (2.22). Furthermore, we can now write

$$\begin{aligned}
\mathrm{P}\left(\theta_0 X - \sum_{i=1}^{n} \theta_i Y_i > x\right) &\gtrsim \mathrm{P}\left(\theta_0 X > x\right) - C\varepsilon^{\gamma} \sum_{i=1}^{n} \mathrm{P}\left(\theta_i X > x\right) - \mathrm{P}\left(\theta_0 > a(x)\right) \\
&= \mathrm{P}\left(\theta_0 X > x\right) - C\varepsilon^{\gamma} \sum_{i=1}^{n} \mathrm{P}\left(\theta_i X > x\right) - o(1)\mathrm{P}\left(\theta_i X > x\right),
\end{aligned}$$

and since ε was arbitrary, we conclude the proof. □

Proof of Theorem 3.2. (The underlying proof is by Tang & Yuan[12])
First addressing Relation (4.3). From Relation (4.1), it follows that

$$\begin{aligned}
&\mathrm{P}\left(\theta_i X_i > x\right) \\
&= \mathrm{P}\left(\theta_i X_i > x, \theta_i > a(x)\right) + \mathrm{P}\left(\theta_i X_i > x, \theta_i \leq a(x)\right) \leq \mathrm{P}\left(\theta_i > a(x)\right) + \mathrm{P}\left(a(x) X_i > x\right) \\
&\leq o(1)\mathrm{P}\left(\theta_i X_i > x\right) + \mathrm{P}\left(a(x) X_i > x\right),
\end{aligned}$$

and by conditioning on θ_j and θ_k for each term, we can then write

$$\begin{aligned}
&\sum_{1\leq j\neq k\leq n} \mathrm{P}(\theta_j X_j > x, \theta_k X_k > x) \\
&= \sum_{1\leq j\neq k\leq n} \mathrm{E}\left[\mathrm{P}(\theta_j X_j > x, \theta_k X_k > x \mid \theta_j, \theta_k)\right] \\
&= \sum_{1\leq j\neq k\leq n} \mathrm{E}\left[\mathrm{P}(\theta_j X_j > x \mid \theta_j, \theta_k)\right] \mathrm{E}\left[\mathrm{P}(\theta_k X_k > x \mid \theta_j, \theta_k)\right] \\
&= \sum_{1\leq j\neq k\leq n} \mathrm{P}(\theta_j X_j > x)\mathrm{P}(\theta_k X_k > x) \\
&\leq \sum_{1\leq j\neq k\leq n} \left(o(1)\mathrm{P}(\theta_j X_j > x) + \mathrm{P}(a(x) X_j > x)\right) \mathrm{P}(\theta_k X_k > x) \\
&= o(1) \sum_{i=1}^{n} \mathrm{P}(\theta_i X_i > x).
\end{aligned}$$

Assume $x \geq 0$, and make a temporary assumption that the random weights are strictly positive.

We begin by showing that Relation (4.4) holds. Let $I \subseteq \{1, \dots, n\}$ be an arbitrary subset, write $I^c = \{1, \dots, n\} \setminus I$, and

$$\Omega_I(X) = \{\omega : X_i \geq 0 \text{ for } i \in I, X_j < 0 \text{ for } j \in I^c\}.$$

Recall the equivalence between Relations (3.2) and (4.1), with the positive auxiliary function $a(\cdot)$ specified in Lemma 4.1 for all $i = 1, \dots, n$, it holds that

$$\begin{aligned}
\mathrm{P}\left(S_n^\theta > x\right) &= \sum_{\emptyset \neq I \subset \{1,\dots,n\}} \mathrm{P}\left(\sum_{i \in I} \frac{\theta_i}{\theta_{(n)}} X_i + \sum_{j \in I^c} \frac{\theta_j}{\theta_{(n)}} X_j > \frac{x}{\theta_{(n)}}, \Omega_I(X)\right) \\
&\geq \sum_{\emptyset \neq I \subset \{1,\dots,n\}} \mathrm{P}\left(\sum_{i \in I} \frac{\theta_i}{\theta_{(n)}} X_i + \sum_{j \in I^c} \frac{\theta_j}{\theta_{(n)}} X_j > \frac{x}{\theta_{(n)}}, \theta_{(n)} \leq a(x), \Omega_I(X)\right)
\end{aligned}$$

Noting that $\frac{\theta_i}{\theta_{(n)}} \in (0, 1]$, conditioning on $\theta_1, \dots, \theta_n$ (which we denote by Θ) and X_j for $j \in I^c$, and applying Lemma 4.7, yields

$$\begin{aligned}
&\mathrm{P}\left(S_n^\theta > x\right) \\
&\geq \sum_{\emptyset \neq I \subset \{1,\dots,n\}} \mathrm{P}\left(\sum_{i \in I} \frac{\theta_i}{\theta_{(n)}} X_i + \sum_{j \in I^c} \frac{\theta_j}{\theta_{(n)}} X_j > \frac{x}{\theta_{(n)}}, \theta_{(n)} \leq a(x), \Omega_I(X)\right) \\
&= \sum_{\emptyset \neq I \subset \{1,\dots,n\}} \mathrm{E}\left[\mathrm{P}\left(\sum_{i \in I} \frac{\theta_i}{\theta_{(n)}} X_i + \sum_{j \in I^c} \frac{\theta_j}{\theta_{(n)}} X_j > \frac{x}{\theta_{(n)}}, \theta_{(n)} \leq a(x), \Omega_I(X) \mid \Theta, \{X_j; j \in I^c\}\right)\right] \\
&\gtrsim \sum_{\emptyset \neq I \subset \{1,\dots,n\}} \mathrm{E}\left[\sum_{i \in I} \mathrm{P}\left(\frac{\theta_i}{\theta_{(n)}} X_i + \sum_{j \in I^c} \frac{\theta_j}{\theta_{(n)}} X_j > \frac{x}{\theta_{(n)}}, \theta_{(n)} \leq a(x), \Omega_I(X) \mid \Theta, \{X_j; j \in I^c\}\right)\right] \\
&= \sum_{\emptyset \neq I \subset \{1,\dots,n\}} \sum_{i \in I} \mathrm{P}\left(\frac{\theta_i}{\theta_{(n)}} X_i + \sum_{j \in I^c} \frac{\theta_j}{\theta_{(n)}} X_j > \frac{x}{\theta_{(n)}}, \theta_{(n)} \leq a(x), \Omega_I(X)\right) \\
&\geq \sum_{\emptyset \neq I \subset \{1,\dots,n\}} \sum_{i \in I} \mathrm{P}\left(\frac{\theta_i}{\theta_{(n)}} X_i + \sum_{j \in I^c} \frac{\theta_j}{\theta_{(n)}} X_j > \frac{x}{\theta_{(n)}}, \Omega_I(X)\right) - \mathrm{P}\left(\theta_{(n)} > a(x)\right) \\
&= \sum_{\emptyset \neq I \subset \{1,\dots,n\}} \sum_{i \in I} \mathrm{P}\left(\theta_i X_i + \sum_{j \in I^c} \theta_j X_j > x, \Omega_I(X)\right) - \mathrm{P}\left(\theta_{(n)} > a(x)\right).
\end{aligned}$$

Then applying Lemma 4.9, and further interchanging the order of summation, yields

$$
\begin{aligned}
&\sum_{\emptyset \neq I \subset \{1,\dots,n\}} \sum_{i \in I} \mathrm{P}\left(\theta_i X_i + \sum_{j \in I^c} \theta_j X_j > x, \Omega_I(X)\right) - \mathrm{P}\left(\theta_{(n)} > a(x)\right) \\
&= \sum_{\emptyset \neq I \subset \{1,\dots,n\}} \sum_{i \in I} \left(\mathrm{P}\left(\theta_i X_i > x, \Omega_I(X)\right) - o(1) \sum_{j \in I^c} \mathrm{P}\left(\theta_j X_i > x, \Omega_I(X)\right)\right) - \mathrm{P}\left(\theta_{(n)} > a(x)\right) \\
&\geq \sum_{\emptyset \neq I \subset \{1,\dots,n\}} \sum_{i \in I} \left(\mathrm{P}\left(\theta_i X_i > x, \Omega_I(X)\right) - o(1) \sum_{j \in I^c} \mathrm{P}\left(\theta_j X_i > x\right)\right) - \mathrm{P}\left(\theta_{(n)} > a(x)\right) \\
&= \sum_{i=1}^{n} \mathrm{P}\left(\theta_i X_i > x\right) - o(1) \sum_{\emptyset \neq I \subset \{1,\dots,n\}} \sum_{i \in I, j \in I^c} \mathrm{P}\left(\theta_j X_i > x\right) - \mathrm{P}\left(\theta_{(n)} > a(x)\right) \\
&\geq \sum_{i=1}^{n} \mathrm{P}\left(\theta_i X_i > x\right) - o(1) \sum_{\emptyset \neq I \subset \{1,\dots,n\}} \sum_{i \in I, j \in I^c} \mathrm{P}\left(\theta_j X_i > x\right) - \sum_{i=1}^{n} \mathrm{P}\left(\theta_i > a(x)\right) \\
&= \sum_{i=1}^{n} \mathrm{P}\left(\theta_i X_i > x\right) - o(1) \sum_{\emptyset \neq I \subset \{1,\dots,n\}} \sum_{i \in I, j \in I^c} \mathrm{P}\left(\theta_j X_i > x\right) - o(1) \sum_{i=1}^{n} \mathrm{P}\left(\theta_i X_i > x\right).
\end{aligned}
$$

Where we would like to show that $o(1) \sum_{\emptyset \neq I \subset \{1,\dots,n\}} \sum_{i \in I, j \in I^c} \mathrm{P}\left(\theta_j X_i > x\right) = o(1), x \to \infty$, we do this by showing that $\mathrm{P}(\theta_i X_i > x) \asymp \mathrm{P}(\theta_j X_i > x)$. By writing

$$
\begin{aligned}
\frac{\mathrm{P}(\theta_i X_i > x)}{\mathrm{P}(\theta_j X_i > x)} &= \frac{\mathrm{P}(\theta_i X_i > x, \theta_i > ux) + \mathrm{P}(\theta_i X_i > x, \theta_i \leq ux)}{\mathrm{P}(\theta_j X_i > x, \theta_j > ux) + \mathrm{P}(\theta_j X_i > x, \theta_j \leq ux)} \\
&\leq \frac{\mathrm{P}(\theta_i > ux) + \mathrm{P}(ux X_i > x)}{\mathrm{P}(ux X_i > x)} = 1 + o(1) \frac{\mathrm{P}(\theta_i X_i > x)}{\mathrm{P}\left(X_i > 1/u\right)},
\end{aligned}
$$

we see that $\mathrm{P}(\theta_i X_i > x) \asymp \mathrm{P}(\theta_j X_i > x)$ holds for all $1 \leq i, j \leq n$.

By combining the results so far, we have that

$$
\begin{aligned}
\mathrm{P}\left(S_n^\theta > x\right) &\gtrsim \sum_{i=1}^{n} \mathrm{P}\left(\theta_i X_i > x\right) - o(1) \sum_{\emptyset \neq I \subset \{1,\dots,n\}} \sum_{i \in I, j \in I^c} \mathrm{P}\left(\theta_j X_i > x\right) - o(1) \sum_{i=1}^{n} \mathrm{P}\left(\theta_i X_i > x\right) \\
&\sim \sum_{i=1}^{n} \mathrm{P}\left(\theta_i X_i > x\right).
\end{aligned}
$$

Next the proof of Relation (4.5). We write, applying Lemma 4.8 at the fourth step,

$$\begin{aligned}
\mathrm{P}\left(\sum_{i=1}^{n}\theta_i X_i^+ > x\right) &= \mathrm{P}\left(\sum_{i=1}^{n}\frac{\theta_i}{\theta_{(n)}}X_i^+ > \frac{x}{\theta_{(n)}}, \theta(n) \le a(x)\right) + \mathrm{P}\left(\sum_{i=1}^{n}\frac{\theta_i}{\theta_{(n)}}X_i^+ > \frac{x}{\theta_{(n)}}, \theta(n) > a(x)\right) \\
&\le \mathrm{P}\left(\sum_{i=1}^{n}\frac{\theta_i}{\theta_{(n)}}X_i^+ > \frac{x}{\theta_{(n)}}, \theta(n) \le a(x)\right) + \mathrm{P}\left(\theta(n) > a(x)\right) \\
&\le \mathrm{P}\left(\sum_{i=1}^{n}\frac{\theta_i}{\theta_{(n)}}X_i^+ > \frac{x}{\theta_{(n)}}, \theta(n) \le a(x)\right) + \sum_{i=1}^{n}\mathrm{P}(\theta_i > a(x)) \\
&\lesssim \sum_{i=1}^{n}\mathrm{P}\left(\frac{\theta_i}{\theta_{(n)}}X_i > \frac{x}{\theta_{(n)}}, \theta(n) \le a(x)\right) + \sum_{i=1}^{n}\mathrm{P}(\theta_i > a(x)) \\
&= \sum_{i=1}^{n}\mathrm{P}\left(\frac{\theta_i}{\theta_{(n)}}X_i > \frac{x}{\theta_{(n)}}, \theta(n) \le a(x)\right) + o(1)\sum_{i=1}^{n}\mathrm{P}(\theta_i X_i > x) \\
&\le \sum_{i=1}^{n}\mathrm{P}\left(\theta_i X_i > x\right) + o(1)\sum_{i=1}^{n}\mathrm{P}(\theta_i X_i > x) \\
&\sim \sum_{i=1}^{n}\mathrm{P}(\theta_i X_i > x).
\end{aligned}$$

So far we have only done the part, where the random weights are strictly positive, to extend the result to the non-negative random weights, we will in essence copy the last part of the proof of Theorem 3.1, simply justifying the use of Relation (4.4) by what we have proven here instead.

Consider the general case, where weights can get value 0 with a positive probability. Let I and I^c be as before and write

$$\Omega_I^0(\theta) = \{\omega : \theta_i > 0 \text{ for } i \in I \text{ and } \theta_j = 0 \text{ for } j \in I^c\}$$

Then the previous case, where the weights were strictly positive, and some creative manipulation with the set $\Omega_I^0(\theta)$, permit us to use of Relation (4.5), to write

$$\begin{aligned}
\mathrm{P}\left(\sum_{i=1}^{n}\theta_i X_i^+ > x\right) &= \sum_{\emptyset \ne I \subseteq \{1,\dots,n\}} \mathrm{P}\left(\sum_{i\in I}\theta_i X_i^+ > x, \Omega_i^0(\theta)\right) \\
&\lesssim \sum_{\emptyset \ne I \subseteq \{1,\dots,n\}} \sum_{i\in I}\mathrm{P}\left(\theta_i X_i > x, \Omega_i^0(\theta)\right) \\
&= \sum_{i=1}^{n}\mathrm{P}\left(\theta_i X_i > x\right).
\end{aligned}$$

This concludes the proof of Theorem 3.2.

□

4.4 Proof of Theorem 3.3, Unbounded Weights, Primary Random Variables in $\mathcal{L} \cap \mathcal{D}$

Lemma 4.10. *(Lemma 7 of Tang & Yuan[12]) Let X be a random variable with a dominatedly-varying right tail and upper Matuszewska index M^*, let θ be a non-negative random variable with $\mathrm{E}\left[\theta^\beta\right] < \infty$ for some $\beta > M^*$, let $\{\Delta_t, t \in \mathcal{T}\}$ be a set of random events satisfying $\lim_{t\to t_0} P(\Delta_t) = 0$ for some t_0 in the closure of the index set $\mathcal{T}$, and let $\{\theta, \{\Delta_t, t \in \mathcal{T}\}\}$ be independent of X. Then*

$$\lim_{t\to t_0} \limsup_{x\to\infty} \frac{\mathrm{P}(\theta X > x, \Delta_t)}{\mathrm{P}(\theta X > x)} = \lim_{t\to t_0} \limsup_{x\to\infty} \frac{\mathrm{P}(\theta X > x, \Delta_t)}{\mathrm{P}(X > x)} = 0.$$

Proof. (Tang & Yuan[12] with additional detail)
Theorem 3.3(iv) of Cline & Samorodnitsky (1994)[3], can be simplified to say that if $F \in \mathcal{D}$ and $\mathrm{E}\left[\theta^\beta\right] < \infty$ for some $\beta > 0$, then

$$0 < \liminf_{x\to\infty} \frac{\mathrm{P}(\theta X > x)}{\mathrm{P}(X > x)} \leq \limsup_{x\to\infty} \frac{\mathrm{P}(\theta X > x)}{\mathrm{P}(X > x)} < \infty.$$

That is $\mathrm{P}(\theta X > x) \asymp \mathrm{P}(X > x)$, and since we can write

$$\frac{\mathrm{P}(\theta X > x, \Delta_t)}{\mathrm{P}(\theta X > x)} = \frac{\mathrm{P}(\theta X > x, \Delta_t)}{\mathrm{P}(X > x)} \frac{\mathrm{P}(X > x)}{\mathrm{P}(\theta X > x)},$$

it follows that we only need to prove the second relation in Lemma 4.10. Choose c to be such that $M^* < c\beta < \beta$, and do the split

$$\mathrm{P}(\theta X > x, \Delta_t) = \mathrm{P}(\theta X > x, \Delta_t, \theta \leq x^c) + \mathrm{P}(\theta X > x, \Delta_t, \theta > x^c).$$

By Inequality (2.21), there exists a constant $C > 0$ such that for any large enough x,

$$\begin{aligned}
\mathrm{P}(\theta X > x, \Delta_t, \theta \leq x^c) &\leq \mathrm{P}((\theta \vee 1)X > x, \Delta_t, \theta \leq x^c) \\
&= \mathrm{E}\left[\mathrm{P}((\theta \vee 1)X > x, \Delta_t, \theta \leq x^c \mid \theta)\right] \\
&= \mathrm{E}\left[\mathrm{P}((\theta \vee 1)X > x, \theta \leq x^c \mid \theta)\mathrm{P}(\Delta_t, \theta \leq x^c \mid \theta)\right] \\
&\leq \mathrm{E}\left[C\mathrm{P}(X > x)(\theta \vee 1)^\beta \mathrm{E}\left[\mathbb{1}_{\Delta_t} \mid \theta\right]\right] \\
&= C\mathrm{P}(X > x)\mathrm{E}\left[(\theta \vee 1)^\beta \mathbb{1}_{\Delta_t}\right],
\end{aligned}$$

where we have $\mathrm{E}\left[(\theta \vee 1)^\beta \mathbb{1}_{\Delta_t}\right] \to 0$, as $t \to t_0$, since by $\mathrm{P}(\Delta_t) = \mathrm{E}\left[\mathbb{1}_{\Delta_t}\right] \to 0$, and $\mathrm{E}\left[\theta^\beta\right] < \infty$, we have that $(\theta \vee 1)^\beta \mathbb{1}_{\Delta_t} \to 0$ almost surely, and then we simply apply the dominated convergence theorem.

For the second part, by Markov's inequality and the first relation in (2.23),

$$\mathrm{P}(\theta X > x, \Delta_t, \theta > x^c) \le \mathrm{P}(\theta > x^c) = \mathrm{P}\left(\theta^\beta > x^{c\beta}\right) \le x^{-c\beta}\mathrm{E}\left[\theta^\beta\right] = o\left(\overline{F}(x)\right)\mathrm{E}\left[\theta^\beta\right]$$

By applying these upper bounds, we see that

$$\begin{aligned} 0 \le \lim_{t\to t_0}\limsup_{x\to\infty}\frac{\mathrm{P}(\theta X > x, \Delta_t)}{\mathrm{P}(X > x)} &\le \lim_{t\to t_0}\limsup_{x\to\infty}\frac{C\mathrm{P}(X > x)\mathrm{E}\left[(\theta\vee 1)^\beta \mathbb{1}_{\Delta_t}\right] + x^{-c\beta}\mathrm{E}\left[\theta^\beta\right]}{\mathrm{P}(X > x)} \\ &= \lim_{t\to t_0} C\mathrm{E}\left[(\theta\vee 1)^\beta \mathbb{1}_{\Delta_t}\right] \\ &= 0, \end{aligned}$$

and the result follows. □

Proof of Theorem 3.3. (The underlying proof by Tang & Yuan[12])
Addressing Relation (4.3), by Lemma 4.10 with the choice $\mathcal{T} = \{t > 0; t \in \mathbb{R}\}$, $\mathrm{P}(\Delta_t) = \mathrm{P}(\theta_i X_i > x_t)$, and $x_t = 1/t$, with $t_0 = 0$ so that $x_t \to \infty$ as $t \to t_0$, we obtain that

$$\mathrm{P}(\theta_j X_j > x, \theta_k X_k > x) = o(1)\mathrm{P}(\theta_k X_k > x)$$

for any pair $j, k \in \{1, \dots n\}$, $j \ne k$, noting that by assumtion $\theta_j X_j$ and X_k are independent. Since the sum in Relation (4.3) is finite, it follows that Relation (4.3) holds in the setting of Theorem 3.3.

Then we show that Relation (4.4) holds.

Since by Lemma 4.5 each $\theta_i X_i$ is long tailed, we can find a positive function $l(\cdot)$, with $l(x) \uparrow \infty$ and $l(x) \le x/2$, such that the relation $\mathrm{P}(\theta_i X_i > x + y) \sim \mathrm{P}(\theta_i X_i > x)$ holds uniformly for $-l(x) \le y \le l(x)$ and $i = 1, \dots, n$.

By applying Bonferroni's inequality we can write

$$
\begin{aligned}
&\mathrm{P}\left(S_n^{\theta} > x\right) \\
&\geq \mathrm{P}\left(S_n^{\theta} > x, \bigvee_{i=1}^{n} \theta_i X_i > x + l(x)\right) \\
&\geq \sum_{i=1}^{n} \mathrm{P}\left(S_n^{\theta} > x, \theta_i X_i > x + l(x)\right) - \sum_{1\leq j<k\leq n} \mathrm{P}\left(S_n^{\theta} > x, \theta_j X_j > x + l(x), \theta_k X_k > x + l(x)\right) \\
&\geq \sum_{i=1}^{n} \mathrm{P}\left(S_n^{\theta} > x, \theta_i X_i > x + l(x)\right) - \sum_{1\leq j<k\leq n} \mathrm{P}\left(\theta_j X_j > x + l(x), \theta_k X_k > x + l(x)\right) \\
&= \sum_{i=1}^{n} \left(\mathrm{P}\left(\theta_i X_i > x + l(x)\right) - \mathrm{P}\left(S_n^{\theta} \leq x, \theta_i X_i > x + l(x)\right)\right) + o(1) \sum_{i=1}^{n} \mathrm{P}\left(\theta_i X_i > x + l(x)\right) \\
&= (1 + o(1)) \sum_{i=1}^{n} \mathrm{P}\left(\theta_i X_i > x + l(x)\right) - \sum_{i=1}^{n} \mathrm{P}\left(S_n^{\theta} \leq x, \theta_i X_i > x + l(x)\right) \\
&\geq (1 + o(1)) \sum_{i=1}^{n} \mathrm{P}\left(\theta_i X_i > x + l(x)\right) - \sum_{i=1}^{n} \mathrm{P}\left(\theta_i X_i > x + l(x), \sum_{j=1, j\neq i}^{n} \theta_j X_j < -l(x)\right) \\
&\sim (1 + o(1)) \sum_{i=1}^{n} \mathrm{P}\left(\theta_i X_i > x\right) - \sum_{i=1}^{n} \mathrm{P}\left(\theta_i X_i > x + l(x), \sum_{j=1, j\neq i}^{n} \theta_j X_j < -l(x)\right) \\
&\sim \sum_{i=1}^{n} \mathrm{P}\left(\theta_i X_i > x\right),
\end{aligned}
$$

where in the fourth step we apply Lemma 4.10, then in the second last step we use the long tail property, and finally in the last step we once again apply Lemma 4.10 with the note that, since $l(x) \uparrow \infty$, it holds that

$$
\begin{aligned}
\mathrm{P}\left(\sum_{j=1, j\neq 1}^{n} \theta_j X_j < -l(x)\right) &= 1 - \mathrm{P}\left(\sum_{j=1, j\neq i}^{n} \theta_j X_j \geq -l(x)\right) \\
\longrightarrow 1 - \mathrm{P}\left(\sum_{j=1, j\neq i}^{n} \theta_j X_j \geq -\infty\right) &= 1 - 1 = 0.
\end{aligned}
$$

To show that Relation (4.5) holds,first we note that

$$
\mathrm{P}\left(\sum_{i=1}^{n} \theta_i X_i^{+} > x, \bigvee_{k=1}^{n} \theta_k X_k \leq \frac{x}{n}\right) = 0,
$$

and then we observe that

$$
\begin{aligned}
&\mathrm{P}\left(\sum_{i=1}^{n}\theta_i X_i^+ > x\right)\\
&\leq \mathrm{P}\left(\bigvee_{i=1}^{n}\theta_i X_i > x - l(x)\right) + \mathrm{P}\left(\sum_{i=1}^{n}\theta_i X_i^+ > x,\ \bigvee_{j=1}^{n}\theta_j X_j \leq x - l(x),\ \bigvee_{k=1}^{n}\theta_k X_k > \frac{x}{n}\right)\\
&\leq \sum_{i=1}^{n}\mathrm{P}\left(\theta_i X_i > x - l(x)\right) + \sum_{k=1}^{n}\mathrm{P}\left(\theta_k X_k > \frac{x}{n},\ \sum_{i=1, i\neq k}^{n}\theta_i X_i^+ > l(x)\right)\\
&\sim \sum_{i=1}^{n}\mathrm{P}\left(\theta_i X_i > x - l(x)\right) + o(1)\sum_{k=1}^{n}\mathrm{P}\left(\theta_k X_k > \frac{x}{n}\right)\\
&\sim \sum_{i=1}^{n}\mathrm{P}\left(\theta_i X_i > x\right),
\end{aligned}
$$

where in the second last step we used Lemma 4.10, and in the last step we applied the long tail property.

□

5 Measuring Heavy-tailed Risk

The light-tailed Gaussian distributions have been historically prominent in risk management, but by assuming that the tails are light, it can be argued, causes severe underestimation of the true risks present in the real world, and leads to insufficient cash reservoirs. The clear alternative, is applying the heavy-tailed distributions, which eliminates the problem of underestimation, but at the same time complicates the numerical risk estimation, since not every tool that works for the light-tailed is available for the heavy-tailed distributions, the most obvious being variance, requiring moments that commonly don't exist for the heavy-tailed distributions. This can be fixed by introducing risk metrics and measures that focus on the tail, for example Value-at-Risk (VaR) and conditional tail expectation (CTE).

Our main result offers a way to estimate the tails of sums, total claims of a portfolio, via an alternative that can possibly be much easier to compute. Thereby, we propose an application to computation of tail focused risk measures. The approach isn't perfect, but serves as a starting point, and a reason to demonstrate the calculus that one could encounter, when applying the main results to something more explicit.

5.1 VaR - Value at Risk

One of the most prominent risk measures is so called Value-at-Risk measure that given a confidence level q, which is typically 0.95 or 0.99, Solvency II Accord designed by the European Comission sets $q = 0.995$ over a one-year time horizon, gives the corresponding boundary value for the magnitude of the loss that can then be used for risk assessment. In effect VaR replaces the value of the maximum loss, often infinite, with a maximum loss given a high probability.

The main criticism at VaR is that it basically ignores the magnitude of extreme tail risks, and other weaknesses include the lack of subadditivity, making it a non-coherent risk measure. All this has lead to VaR in many parts of the world being currently phased out as a risk measure in favor of alternatives like CTE and the spectral risk measures that have nicer mathematical properties, but it still remains relevant as a computational tool.

Definition 5.1. *Value-at-Risk* for a random variable Y, at a confidence level $q \in (0, 1)$, is defined as

$$\mathrm{VaR}_q\left(Y\right) = \inf\left\{y \in \mathbb{R} : F_Y(y) \geq q\right\},$$

where $F_Y(z)$ is the distribution function of Y.

In the case that Y is continuous, we have

$$\mathrm{P}\left(Y \leq \mathrm{VaR}_q(Y)\right) = q.$$

Remark 5.1. When looking for rigorous mathematical properties, it is useful to know that Value-at-Risk is in essence the *generalized inverse* of a distribution function.

Being able to compute VaR is still at the core of many modern risk measures such as the Conditional Tail Expectation CTE.

Definition 5.2. *Conditional Tail Expectation* for a random variable Y, given a risk level $q \in (0, 1)$, is defined as

$$\mathrm{CTE}_q(Y) = \mathrm{E}[Y \mid Y > \mathrm{VaR}_q(Y)].$$

We now derive a straightforward, but possibly a little rough, method for estimating VaR, in the case that one of the main theorems assures us that the main asymptotic relations hold.

By the main results

$$\mathrm{P}\left(S_n^\theta > x\right) \sim \sum_{i=1}^{n} \mathrm{P}\left(\theta_i X_i > x\right),$$

$$\mathrm{P}\left(S_n^\theta > x\right) \approx \sum_{i=1}^{n} \mathrm{P}\left(\theta_i X_i > x\right) \text{ for large } x.$$

Based on the ultimate right tail,

$$\mathrm{VaR}_q\left(S_n^\theta\right) \to \infty, \quad a \to 1.$$

This leads us to estimate in the continuous case,

$$\begin{aligned}
&\mathrm{P}\left(S_n^\theta > \mathrm{VaR}_q\left(S_n^\theta\right)\right) = 1 - q = \sum_{i=1}^{n} \mathrm{P}\left(\theta_i X_i > x\right) \\
&\Rightarrow \mathrm{VaR}_q\left(S_n^\theta\right) \approx x, \quad q \uparrow 1.
\end{aligned}$$

Or if we would like to conform to the definition a bit more,

$$\begin{aligned}
\mathrm{VaR}_q\left[S_n^\theta\right] &= \inf\left\{y \in \mathbb{R} : \mathrm{P}\left(S_n^\theta \le y\right) \ge q\right\}, \quad 0 < q < 1, \\
&= \inf\left\{x \in \mathbb{R} : \mathrm{P}\left(S_n^\theta > x\right) \le 1 - q\right\} \\
&\approx \inf\left\{x \in \mathbb{R} : \sum_{i=1}^{n} \mathrm{P}\left(\theta_i X_i > x\right) \le 1 - q\right\}, \quad q \uparrow 1,
\end{aligned}$$

What we, of course, don't know here is the rate of convergence, which we could estimate by simulation, this would also clarify which distributions behave nicely in this regard. Considering the general roughness that characterizes practical VaR estimation, this methodology might be viable as is, in some cases, but in others it might produce misleading estimates, especially when $q = 0.95$.

5.2 An Application to Estimating VaR

As it remarkably quickly turns out, computing the tail of a product distribution is not as trivial as one might first think, and has in fact occasionally papers written on it. Anyways, computing the tail is by no means by and large impossible, and once someone has found a way to do the calculations, the result can be then quite effortlessly implemented in the applications.

Example 5.1. Let $\theta_i \sim \mathrm{Exp}(\lambda_i)$ and $X_i \sim \mathrm{Pareto}(\alpha, \kappa_i)$.
In order to check the preliminaries, justifying the application of Theorem 3.3, we first observe that based on the earlier examples, we know that Pareto distribution is in $\mathcal{L} \cap \mathcal{D}$, and we can compute that for a Pareto distribution $M^*(F) = M_*(F) = \alpha$.

Then we want to show that $\mathrm{E}\left[\theta_i^{\beta_i}\right] < \infty$, holds for some $\beta_i > \alpha$. Write,

$$\begin{aligned}
\mathrm{E}\left[\theta_i^{\beta_i}\right] &= \int_0^\infty x^{\beta_i} \lambda_i e^{-\lambda_i x} dx = \frac{1}{\lambda_i^{\beta_i}} \int_0^\infty (\lambda_i x)^{\beta_i} e^{-\lambda_i x} \lambda_i dx \\
&= \frac{1}{\lambda_i^{\beta_i}} \int_0^\infty u^{\beta_i} e^{-u} du = \frac{\Gamma(\beta_i + 1)}{\lambda_i^{\beta_i}} < \infty.
\end{aligned}$$

To make the calculations a little easier, we will note that by shifting $x \in (0, \infty)$ to $x \in (\kappa, \infty)$, and then writing the Pareto tail as

$$\overline{F}(x) = \left(\frac{\kappa}{x}\right)^{\alpha},$$

we will maintain an equivalence with the earlier.

Now to the actual computation of the tail. *(The computation method is heavily inspired by Obeid & Kadry(2020)[9].)* We have,

$$\begin{aligned}
\mathrm{P}\left(\theta_i X_i > x\right) &= \int_0^{\infty} \overline{F}_{X_i}(x/y) f_{\theta_i}(y) dy \\
&= \int_0^{x/\alpha} \left(\frac{\kappa_i y}{x}\right)^{\alpha} \lambda_i e^{-\lambda_i y} dy + \int_{x/\alpha}^{\infty} 1 \cdot \lambda_i e^{-\lambda_i y} dy \\
&= \left(\int_0^{\infty} - \int_{x/\alpha}^{\infty}\right) \left(\frac{\kappa_i y}{x}\right)^{\alpha} \lambda_i e^{-\lambda_i y} dy + e^{-\frac{\lambda_i x}{\alpha}} \\
&=: I_1 - I_2 + e^{-\frac{\lambda_i x}{\alpha}}.
\end{aligned}$$

Calculation of I_1

$$\begin{aligned}
I_1 &= \int_0^{\infty} \left(\frac{\kappa_i y}{x}\right)^{\alpha} \lambda_i e^{-\lambda_i y} dy \\
&= \left(\frac{\kappa_i}{\lambda_i x}\right)^{\alpha} \int_0^{\infty} \left(\lambda_i y\right)^{\alpha} e^{-\lambda_i y} \lambda_i dy \\
&= \left(\frac{\kappa_i}{\lambda_i x}\right)^{\alpha} \Gamma(\alpha + 1).
\end{aligned}$$

Calculation of I_2

$$\begin{aligned}
I_2 &= \int_{x/\alpha}^{\infty} \left(\frac{\kappa_i y}{x}\right)^{\alpha} \lambda_i e^{-\lambda_i y} dy \\
&= \left(\frac{\kappa_i}{\lambda_i x}\right)^{\alpha} \int_{x/\alpha}^{\infty} \left(\lambda_i y\right)^{\alpha} e^{-\lambda_i y} \lambda_i dy \\
&= \left(\frac{\kappa_i}{\lambda_i x}\right)^{\alpha} \int_{\frac{\lambda_i x}{\alpha}}^{\infty} u^{\alpha} e^{-u} du \\
&= \left(\frac{\kappa_i}{\lambda_i x}\right)^{\alpha} \Gamma\left(\alpha + 1, \frac{\lambda_i x}{\alpha}\right),
\end{aligned}$$

where $\Gamma(a, x) = \int_x^{\infty} t^{a-1} e^{-t} dt$ is the upper incomplete gamma function.

So that we obtain,

$$\mathrm{P}\left(\theta_i X_i > x\right) = \left(\frac{\kappa_i}{\lambda_i x}\right)^{\alpha} \left(\Gamma(\alpha + 1) - \Gamma\left(\alpha + 1, \frac{\lambda_i x}{\alpha}\right)\right) + e^{-\frac{\lambda_i x}{\alpha}},$$

which leads us to estimate

$$\mathrm{P}\left(S_n^\theta > x\right) \sim \sum_{i=1}^n \mathrm{P}\left(\theta_i X_i > x\right) = \sum_{i=1}^n \left(\left(\frac{\kappa_i}{\lambda_i x}\right)^\alpha \left(\Gamma(\alpha+1) - \Gamma\left(\alpha+1, \frac{\lambda_i x}{\alpha}\right)\right) + e^{-\frac{\lambda_i x}{\alpha}}\right).$$

and further

$$\mathrm{VaR}_q\left[S_n^\theta\right] \approx \inf_{x\in\mathbb{R}} \left\{\sum_{i=1}^n \left(\left(\frac{\kappa_i}{\lambda_i x}\right)^\alpha \left(\Gamma(\alpha+1) - \Gamma\left(\alpha+1, \frac{\lambda_i x}{\alpha}\right)\right) + e^{-\frac{\lambda_i x}{\alpha}}\right) \le 1-q\right\}.$$

Now one can just plug in values for the parameters, and use a computer to solve the numerical value.

6 Application to Ruin Theory

6.1 The Basic Setting

We consider, in the setting by Tang & Tsitsiashvili (2003) [11], the surplus process of the insurance company in discrete time, characterized by the recursive equation,

$$S_0 = x, \qquad S_n = \xi_n S_{n-1} + (\eta_n - Z_n), n \ge 1.$$

Here $S_0 = x > 0$ represents the initial surplus of the company, η_n is the total premium income of the year n, and Z_n is the total paid claims of the year n. ξ_n is the stochastic inflation coefficient for the period from $n-1$ to n. We introduce random variables $X_n = \eta_n - Z_n$, $n \ge 1$ distributed by $F_1, \ldots, F_n$, respectively, and write $Y_n = \xi_n^{-1}$, $n \ge 1$ and then we suppose that $X_1, \ldots, X_n$ are independent of each other and $Y_1, \ldots, Y_n$.

Then iterating the original recursive equation yields,

$$S_0 = x, \qquad S_n = x\prod_{i=1}^n \xi_i - \sum_{k=1}^n X_k \prod_{i=k+1}^n \xi_i, \qquad n \ge 1.$$

And by utilizing the discount factors, we can then write

$$\tilde{S}_0 = x, \qquad \tilde{S}_n = S_n \prod_{i=1}^n Y_i = x - \sum_{k=1}^n X_k \prod_{i=1}^k Y_i, \qquad n \ge 1.$$

Denote the ruin probability within a finite time horizon $n \ge 1$ by

$$\psi(x, n) = \mathrm{P}\left(\min_{0\le m\le n} S_m < 0 \mid S_0 = x\right),$$

which we can alternatively write as

$$\psi(x,n) = \mathrm{P}\left(\min_{0\le m\le n} \tilde{S} < 0 \mid \tilde{S}_0 = x\right) = \mathrm{P}\left(\max_{1\le m\le n} \sum_{k=1}^{n} X_k \prod_{i=1}^{k} Y_i > x\right).$$

Now to conform to our earlier notation, denote

$$\theta_k = \prod_{i=1}^{k} Y_i, \qquad 1 \le k \le n.$$

Finally, we have

$$\psi(x,n) = \mathrm{P}\left(\max_{1\le m\le n} \sum_{k=1}^{n} \theta_k X_k > x\right).$$

From here it is fairly obvious, how to apply the main results, assuming that the distributions of ξ_n's project a favorable structure for θ_k's.

Example 6.1. Clearly, for $n \ge 1$, if the random variables ξ_i are bounded from below in the sense that $\mathrm{P}\left(a \le \xi_i < \infty\right) = 1$ for some $0 < a < \infty$ and all $i = 1,\dots,n$, then it holds for the random variables $\theta_k = \prod_{i=1}^{k} \xi_i^{-1}$ that there exists $0 < b < \infty$ such that $\mathrm{P}\left(0 \le \theta_i \le b\right) = 1$ for all $i = 1,\dots,n$, and we see that the θ_k's meet the condition (A3).

Example 6.2. Let $G_1,\dots,G_n$ be a series of independent random variables distributed by $G_i \sim \mathcal{N}(\mu_i,\sigma_i^2)$, where $\mu_i \in (-\infty,\infty)$ and $\sigma_i^2 > 0$, for each i respectively. Then it holds that

$$\sum_{i=1}^{k} G_i \sim \mathcal{N}\left(\sum_{i=1}^{k}\mu_i, \sum_{i=1}^{k}\sigma_i^2\right),$$

for any $k = 1,\dots,n$.

We set $Y_i = e^{G_i}$, and it follows that $Y_i \sim \mathrm{Lognormal}\left(\mu_i,\sigma_i^2\right)$. Then

$$\theta_k = \prod_{i=1}^{k} Y_i = e^{\sum_{i=1}^{k} G_i},$$

and we further observe

$$\ln\left(\theta_k\right) = \ln\left(\prod_{i=1}^{k} Y_i\right) = \ln\left(e^{\sum_{i=1}^{k} G_i}\right) = \sum_{i=1}^{k} G_i \sim \mathcal{N}\left(\sum_{i=1}^{k}\mu_i, \sum_{i=1}^{k}\sigma_i^2\right)$$

that is $\theta_k \sim \mathrm{Lognormal}\left(\sum_{i=1}^{k}\mu_i, \sum_{i=1}^{k}\sigma_i^2\right)$.

Denote $\tilde{\mu}_k = \sum_{i=1}^{k}\mu_i$ and $\tilde{\sigma}_k^2 = \sum_{i=1}^{k}\sigma_i^2$. It is widely know that for a log-normal distributed random variable all the origin moments exist, i.e. for every $m \in \mathbb{N}^+$, we have

$$\mathrm{E}\left(\theta_k^m\right) = e^{m\tilde{\mu}_k + m^2\tilde{\sigma}_k^2/2} < \infty.$$

We can see that in this case θ_i's satisfy the requirements of Theorem 3.3, as well as, by Remark 4.2 the requirements of Theorem 3.2, and then assuming that X_i's satisfy their own respective conditions, we obtain that

$$\psi(x,n) \sim \sum_{i=1}^{n} \mathrm{P}\left(X_i \prod_{j=1}^{i} Y_j \right).$$

6.2 Crude Asymptotics of the Ruin Probability

Another way to approach the ruin problem is via crude (rough) asymptotics that is the asymptotic relation holds for the logarithms of the things related $\log(f(x)) \sim \log(g(x))$.

We will now look at a slightly different setting, similar to one represented in the lecture notes and the exercises of the course Advanced Risk Theory by Lehtomaa (2020)[8], where the loss of the year n is represented by $Z_n = \theta_n \eta_n$.

We assume that $\theta_1, \dots, \theta_n$ are independent of $\eta_1, \dots, \eta_n$, for all θ_i it holds that $\mathrm{P}(a \le \theta_i \le b) = 1$, where $0 < a \le b < \infty$, $\eta, \eta_1, \dots, \eta_n$ are independent and identically distributed by F_η, in addition $\mathrm{E}(\eta) \in (-\infty, 0)$, and

$$\lim_{x\to\infty} \frac{\log \mathrm{P}(\eta > x)}{\log x} = -\alpha,$$

where $\alpha \in (1, \infty)$.

We denote $Y_n = Z_1 + \dots + Z_n$, $M_n = \max\{Y_1, \dots, Y_n\}$.

We are then interested of the ruin time $T = \inf\{n \ge 0 \mid Y_n > U_0\}$ given the initial capital $U_0 > 0$.

Theorem 6.1. *Let $\beta > 0$. In the given setting it holds that*

$$(i) \quad \limsup_{U_0\to\infty} \frac{\log \mathrm{P}(T \le \beta U_0)}{\log U_0} \le 1 - \alpha$$

and if we assume in addition that $\mathrm{P}(\eta > -c) = 1$ for some $c > 0$, then

$$(ii) \quad \lim_{U_0\to\infty} \frac{\log \mathrm{P}(T \le \beta U_0)}{\log U_0} = 1 - \alpha.$$

In essence, we are justifying a crude estimate $\mathrm{P}\left(T \le \beta U_0\right) \approx U_0^{1-\alpha}$ for some large U_0.

Remark 6.1. The limits in Theorem 6.1 do not depend on the value of $\beta > 0$. Let $\mu > 0$,

$$\begin{aligned} \limsup_{U_0\to\infty} \frac{\log \mathrm{P}(T \le \beta U_0)}{\log U_0} &= \limsup_{U_0\to\infty} \frac{\log \mathrm{P}(T \le \beta\mu U_0)}{\log(\mu U_0)} \\ &= \limsup_{U_0\to\infty} \frac{\log(U_0)}{\log(\mu U_0)} \frac{\log \mathrm{P}(T \le \beta\mu U_0)}{\log(U_0)} = \limsup_{U_0\to\infty} \frac{\log \mathrm{P}(T \le \beta\mu U_0)}{\log(U_0)}. \end{aligned}$$

For the proof we need some additional lemmas, first one from the lecture notes by Lehtomaa [8].

Lemma 6.1. *Let* X *be random variable such that* $\mathrm{P}(X \in [0,a]) = 1$ *for some* $a > 0$ *and* $\mathrm{P}(X > 0) > 0$. *Suppose* $h > 0$ *is given. Then*

$$\mathrm{E}\left(e^{hX}\right) \leq \frac{e^{ha} - 1}{a}\mathrm{E}(X) + 1,$$

and

$$\mathrm{E}\left(e^{hX}\right) \leq \frac{e^{ha} - 1 - ha}{a^2}\mathrm{E}\left(X^2\right) + 1 + h\mathrm{E}(X).$$

Then a lemma that acts as an intermediary result in the proof of the theorem.

Lemma 6.2. *Let* $\delta \in (0,1)$ *and* $\beta > 0$ *be given in our setting. Then*

$$\lim_{x\to\infty} \frac{\log \mathrm{P}(Y_n > 0, M_n \leq n^{1-\delta})}{\log n} = -\infty \tag{6.2}$$

and

$$\lim_{U_0\to\infty} \frac{\log \mathrm{P}\left(T \leq \beta U_0, M_{\lfloor \beta U_0 \rfloor} \leq {U_0}^{1-\delta}\right)}{\log U_0} = -\infty. \tag{6.3}$$

Proof. We will first show Lemma 6.2, and then proceed to prove Theorem 6.1.

Let $\delta \in (0,1)$ be given and denote $h = h_n = n^{-1+\delta/2}$. For $j \in \mathbb{N}$ set

$$Z'_j = Z_j \mathbb{1}\left(Z_j \leq n^{1-\delta}\right).$$

First we show that it is possible to determine constants $\varepsilon' > 0$ and n_0 that do not depend on j, so that

$$\mathrm{E}\left(e^{hZ'_j}\right) \leq e^{-h\varepsilon'} \tag{6.4}$$

holds for all $j \in \mathbb{N}$ and all $n \geq n_0$.

Suppose $\varepsilon > 0$ and $\varepsilon'' > 0$ are given and denote $\mu = \mathrm{E}(\eta)$. Let $c > 0$ be such that

$$\mathrm{E}(\max\{\eta, -c\}) < \mu + \varepsilon''.$$

Now, we can write

$$\begin{aligned}
\mathrm{E}\left(\max\{Z_j, -bc\}\right) &= \mathrm{E}\left(\mathrm{E}\left(\max\{Z_j, -bc\} \mid \theta_j\right)\right) \\
&= \mathrm{E}\left(\theta_j \mathrm{E}\left(\max\left\{\eta_j, \frac{-bc}{\theta_j}\right\} \mid \theta_j\right)\right) \\
&\leq \mathrm{E}\left(\theta_j \mathrm{E}\left(\max\{\eta_j, -c\}\right)\right) \\
&\leq \mathrm{E}\left(\theta_j\right) \cdot (\mu + \varepsilon'') \\
&\leq \mu \mathrm{E}\left(\theta_j\right) + b\varepsilon''.
\end{aligned}$$

If we denote

$$Z''_j = \max\left\{Z'_j, -bc\right\} + bc,$$

and $\mu''_j = \mathrm{E}\left(Z''_j\right)$, it holds that $\mathrm{P}\left(Z''_j \in [0, n^{1-\delta} + bc]\right) = 1$.

We can then apply Lemma 6.1 to get that

$$\mathrm{E}\left(e^{hZ''_j}\right) \leq \frac{e^{h(n^{1-\delta}+bc)} - 1}{h(n^{1-\delta} + bc)} h\mu''_j + 1 \leq (1+\varepsilon)h\mu''_j + 1 \leq e^{(1+\varepsilon)h\mu''_j}$$

for all $n \geq n_\varepsilon$, where n_ε is chosen to be such that

$$\left|\frac{e^{h(n^{1-\delta}+bc)} - 1}{h(n^{1-\delta} + bc)} - 1\right| < \varepsilon.$$

Then

$$\begin{aligned}
\mathrm{E}\left(e^{hZ'_j}\right) &\leq \mathrm{E}\left(e^{hZ''_j}\right) \cdot e^{-hbc} \\
&\leq e^{(1+\varepsilon)h\mu''_j - hbc} \\
&\leq e^{(1+\varepsilon)h(\mu\mathrm{E}(\theta_j) + b\varepsilon'' + bc) - hbc} \\
&= e^{h((1+\varepsilon)\mu\mathrm{E}(\theta_j) + (1+\varepsilon)b\varepsilon'' + \varepsilon bc)} \\
&\leq e^{h((1+\varepsilon)\mu a + (1+\varepsilon)b\varepsilon'' + \varepsilon bc)}
\end{aligned}$$

Recall that $\mu < 0$ by assumption, and we can repeat the argument with any $\varepsilon > 0$ and $\varepsilon'' > 0$. Since

$$(1+\varepsilon)a\mu + (1+\varepsilon)a\varepsilon'' + \varepsilon bc \leq b\mu + 2b\varepsilon'' + \varepsilon bc,$$

when $\varepsilon \leq 1$, it follows that the right hand side in this inequality must be negative, when ε'' and ε are sufficiently small. Thereby, we are able to find a suitable value for $\varepsilon' > 0$ so that (6.4) holds.

Set $Y'_n = Z'_1 + \cdots + Z'_n$. Then

$$\mathrm{E}\left(e^{hY'_n}\right) \geq \mathrm{E}\left(e^{hY'_n}\mathbb{1}\left(Y'_n > 0\right)\right) \geq \mathrm{P}\left(Y'_n > 0\right).$$

Based on (6.4)

$$\begin{aligned}
\mathrm{E}\left(e^{hY'_n}\right) &\leq \mathrm{E}\left(e^{hY'_n}\right) = \mathrm{E}\left(e^{hZ'_1 + \cdots + hZ'_n}\right) = \mathrm{E}\left(e^{hZ'_1}\right) \cdot \ldots \cdot \mathrm{E}\left(e^{hZ'_n}\right) \\
&\leq \left(e^{-h\varepsilon'}\right)^n = e^{-n^{\delta/2}\varepsilon'},
\end{aligned}$$

so

$$\limsup_{n\to\infty} \frac{\log \mathrm{P}\left(Y_n' > 0\right)}{\log n} \le \limsup_{n\to\infty} \frac{\log\left(e^{-n^{\delta/2}\varepsilon'}\right)}{\log n} = -\infty,$$

and it follows that

$$\lim_{n\to\infty} \frac{\log \mathrm{P}\left(Y_n' > 0\right)}{\log n} = -\infty.$$

Since $\mathrm{P}\left(Y_n > 0, M_n \le n^{1-\delta}\right) \le \mathrm{P}\left(Y_n' > 0\right)$, we have that

$$\lim_{n\to\infty} \frac{\log \mathrm{P}\left(Y_n > 0, M_n \le n^{1-\delta}\right)}{\log n} = -\infty, \tag{6.5}$$

which is the first half of Lemma 6.2.

When $U_0 > 0$ is large, given $h' = h'(U_0) = \lceil U_0 \rceil^{1-\delta}$, this implies that for all i

$$\mathrm{E}\left(e^{h' Z_i \mathbb{1}\left(Z_i \le \lceil U_0\rceil^{1-\delta}\right)}\right) \le e^{-h'\varepsilon'}.$$

So by,

$$\begin{aligned}
&e^{-h'U_0}\mathrm{E}\left(e^{h'\sum_{i=1}^n Z_i \mathbb{1}\left(Z_i \le \lceil U_0\rceil^{1-\delta}\right)}\right) \\
&\ge \mathrm{E}\left(e^{h'\left(-U_0 + \sum_{i=1}^n Z_i \mathbb{1}\left(Z_i \le \lceil U_0\rceil^{1-\delta}\right)\right)} \mathbb{1}\left(\sum_{i=1}^n Z_i \mathbb{1}\left(Z_i \le \lceil U_0\rceil^{1-\delta}\right) > U_0\right)\right) \\
&\ge \mathrm{P}\left(\sum_{i=1}^n Z_i \mathbb{1}\left(Z_i \le \lceil U_0\rceil^{1-\delta}\right) > U_0\right),
\end{aligned}$$

we have

$$\mathrm{P}\left(\sum_{i=1}^n Z_i \mathbb{1}\left(Z_i \le \lceil U_0\rceil^{1-\delta}\right) > U_0\right) \le e^{-h'U_0}\left(e^{-h'\varepsilon'}\right)^n \le e^{-h'U_0} \le e^{-U_0^\delta/2}.$$

Let $\beta > 0$ be given, then the union bound yields,

$$\begin{aligned}
\mathrm{P}\left(T \le \beta U_0, M_{\lceil \beta U_0\rceil} \le U_0^{1-\delta}\right) &\le \sum_{n \le \beta U_0} \mathrm{P}\left(Y_n > U_0, M_n \le U_0^{1-\delta}\right) \\
&\le \sum_{n \le \beta U_0} \mathrm{P}\left(\sum_{i=1}^n Z_i \mathbb{1}\left(Z_i \le U_0^{1-\delta}\right) > U_0\right)
\end{aligned}$$

so that

$$\mathrm{P}\left(T \le \beta U_0, M_{\lceil \beta U_0\rceil} \le U_0^{1-\delta}\right) \le \beta U_0 e^{-U_0^\delta/2}.$$

Once we take the limit with respect to $U_0 \to \infty$, we obtain what equals to the second half of Lemma 6.2,

$$\lim_{U_0\to\infty} \frac{\log \mathrm{P}\left(T \leq \beta U_0, M_{\lfloor \beta U_0 \rfloor} \leq {U_0}^{1-\delta}\right)}{\log U_0} \leq \lim_{U_0\to\infty} \frac{\log\left(\beta U_0 e^{-U_0^\delta/2}\right)}{\log U_0} = -\infty.$$

Now knowing this, and writing

$$\mathrm{P}\left(T \leq \beta U_0\right) \leq \mathrm{P}\left(T \leq \beta U_0, M_{\lfloor \beta U_0 \rfloor} \leq {U_0}^{1-\delta}\right) + \mathrm{P}\left(M_{\lfloor \beta U_0 \rfloor} > {U_0}^{1-\delta}\right), \tag{6.6}$$

we see that the first term is very small causes no problems, making it sufficient to solely focus on the latter term, when deriving an upper bound.

We have for all large enough $U_0 > 0$ that

$$\begin{aligned}
\mathrm{P}\left(M_{\lfloor \beta U_0 \rfloor} > {U_0}^{1-\delta}\right) &\leq \mathrm{P}\left(\max\left\{\eta_1, \ldots, \eta_{\lfloor \beta U_0 \rfloor}\right\} > \frac{U_0^{1-\delta}}{b}\right) \\
&\leq \mathrm{P}\left(\max\left\{\eta_1, \ldots, \eta_{\lfloor \beta U_0 \rfloor}\right\} > U_0^{1-2\delta}\right) \\
&= 1 - \left(1 - \overline{F}_\eta\left(U_0^{1-\delta}\right)\right)^{\lfloor \beta U_0 \rfloor} \\
&= 1 - e^{\lfloor \beta U_0 \rfloor \log\left(1-\overline{F}_\eta\left(U_0^{1-\delta}\right)\right)} \\
&\leq 1 - e^{\beta U_0 \log\left(1-\overline{F}_\eta\left(U_0^{1-\delta}\right)\right)}
\end{aligned}$$

Since

$$\lim_{x\to 0} \frac{\log(1-x)}{x} = -1,$$

it holds for all large enough $U_0 > 0$ that

$$\log\left(1 - \overline{F}_\eta\left(U_0^{1-\delta}\right)\right) \geq -(1+2\delta)\overline{F}_\eta\left(U_0^{1-\delta}\right).$$

Choosing $\delta > 0$ so small that $U_0\overline{F}_\eta\left(U_0^{1-2\delta}\right) \to 0$, as $U_0 \to \infty$, thus gives us the estimate

$$\begin{aligned}
\mathrm{P}\left(M_{\lfloor \beta U_0 \rfloor} > {U_0}^{1-\delta}\right) &\leq 1 - e^{\beta U_0 \log\left(1-\overline{F}_\eta\left(U_0^{1-\delta}\right)\right)} \\
&\leq 1 - e^{-\beta U_0 (1+2\delta)\overline{F}_\eta\left(U_0^{1-\delta}\right)} \\
&= 1 - \left(1 - \beta U_0(1+2\delta)\overline{F}_\eta\left(U_0^{1-\delta}\right)(1+o(1))\right) \\
&= (1+o(1))\beta U_0(1+2\delta)\overline{F}_\eta\left(U_0^{1-\delta}\right).
\end{aligned}$$

We derive from this estimate alongside the assumption

$$\lim_{x\to\infty} \frac{\log \overline{F}_\eta(x)}{\log x} = -\alpha$$

that

$$\limsup_{U_0\to\infty} \frac{\log \mathrm{P}\left(M_{\lfloor \beta U_0 \rfloor} > U_0^{1-\delta}\right)}{\log U_0} \leq 1 + \limsup_{U_0\to\infty} (1-2\delta) \frac{\log \overline{F}_\eta\left(U_0^{1-2\delta}\right)}{\log\left(U_0^{1-2\delta}\right)}$$
$$= 1 + (1-2\delta)(-\alpha) = 1-(1-2\delta)\alpha.$$

Since $\delta > 0$ can be chosen to be arbitrarily small , and the first term in 6.6 does not affect the limsup, we have that

$$\limsup_{U_0\to\infty} \frac{\log \mathrm{P}\left(T \leq \beta U_0\right)}{\log U_0} \leq 1-\alpha.$$

Concluding the proof for the first part of Theorem 6.1.

Next we cover the second part.

Let $\beta \in (0,1)$ be fixed. We assume $\mathrm{P}(\eta > -c) = 1$ for some $c > 0$. When $U_0 > 0$ is large, it holds for given $\varepsilon > 0$ that

$$\mathrm{P}\left(T \leq \beta U_0\right) \geq \mathrm{P}\left(Y_{\lfloor \beta U_0 \rfloor} > U_0\right)$$
$$\geq \sum_{i=1}^{\lfloor \beta U_0 \rfloor} \mathrm{P}\left(Z_i > U_0^{1+\varepsilon}, Z_j \leq U_0 \text{ for all } j \leq \lfloor \beta U_0 \rfloor \text{ such that } i \neq j\right),$$

since it follows from $\mathrm{P}(\eta > -c) = 1$ that $\mathrm{P}\left(Y_{\lfloor \beta U_0 \rfloor} - Z_i > -\lfloor \beta U_0 \rfloor bc\right) = 1$.

Now

$$\mathrm{P}\left(Z_i > U_0^{1+\varepsilon}, Z_j \leq U_0 \text{ for all } j \leq \lfloor \beta U_0 \rfloor \text{ such that } i \neq j\right)$$
$$\geq \mathrm{P}\left(Z_i > U_0^{1+\varepsilon}\right) \mathrm{P}\left(\eta \leq \frac{U_0}{b}\right)^{\lfloor \beta U_0 \rfloor}$$
$$\geq \mathrm{P}\left(\eta > \frac{U_0}{a}\right)\left(1-\overline{F}_\eta\left(\frac{U_0}{b}\right)\right)^{\lfloor \beta U_0 \rfloor}.$$

Like earlier

$$\left(1-\overline{F}_\eta\left(\frac{U_0}{b}\right)\right)^{\lfloor \beta U_0 \rfloor} = e^{\beta U_0 \overline{F}_\eta\left(\frac{U_0}{b}\right)(1+o(1))} = 1 + o(1), \quad U_0 \to \infty.$$

So that we have,

$$\mathrm{P}\left(T \leq \beta U_0\right) \geq \lfloor \beta U_0 \rfloor \mathrm{P}\left(\eta > \frac{U_0}{a}\right)(1+o(1)), \quad U_0 \to \infty,$$

and moreover,

$$\liminf_{U_0 \to \infty} \frac{\log \mathrm{P}\left(T \leq \beta U_0\right)}{\log U_0} \geq 1-\alpha,$$

which, combined with the first part, implies

$$\lim_{U_0 \to \infty} \frac{\log \mathrm{P}\left(T \leq \beta U_0\right)}{\log U_0} = 1-\alpha,$$

concluding the proof of the second part of Theorem 6.1. □

7 Conclusion

First we introduced the preliminaries, which can also be useful in case the reader wants to delve further into the topic and similar results, as the research papers rarely have anything as comprehensive as what we had here. Then we moved on to our main result the three theorems that establish sufficient conditions for the asymptotic relations to hold, in effect extending the principle of a single big jump. The asymptotic relations can be used for sufficiently large x to give an estimate of one probability by computing some other probability in the relation, often the easiest to compute is the sum of single line tail probabilities, and later on we had this as an underlying assumption, but by no means should one ignore the possibility for more creative applications. Also worth noting is that what counts as a sufficiently large x, is highly dependent on the application in question, since some application are fine with a bit rougher estimate, while the other demand higher level of precision.

In Theorem 3.1, we had the upper bound for the random weights, which limits the practical use a bit, but as we saw later on in the proofs section, the results that we developed to show Theorem 3.1, could be extended to show other theorems. For practical applications, Theorem 3.2 and 3.3 are more intriguing, given the flexibility from the lack of upper bound and the possibility of an ultimate right tail for the random weights.

In the proofs section, we went over in further detail the proofs from Tang & Yuan (2014)[12], also adding parts that Tang & Yuan[12] skipped over. While the section serves the purpose of providing a comprehensive and clear look into the individual steps in the proofs, it also serves as a point of comparison to see what a research paper tends to leave out, this includes quite a surprising amount of smaller deductions as well as extensions and adaptations of older results.

As for the applications, we first proposed a straightforward application to computing an explicit estimate of Value-at-Risk. Notably, in a similar setting with different distributions, if we had the same kind of knowledge that is a paper with the calculus concerning the individual tail, then we could use essentially same approach to yield same type of an estimate. Of course, what one should probably also do is to use simulation to establish the quality of the obtained estimate.

Our second application was to ruin theory, we established a suitable setting for applying our main result to determine the ruin probability. The setting is in line with the existing framework, so it should not introduce any new restrictions. Examples in regards to the random weight structure are by no means conclusive, one only needs to look at Theorem 3.2 and the related lemmas to see that starting from a model based on data might actually be easier than trying theorize what would for sure work. Similar logarithm tricks may be useful especially if discounting involves investment returns rather than just pure inflation, in which case logarithmic distributions might be more realistic.

Crude asymptotics case that we covered is, in terms of the random weight structure, something akin to Theorem 3.1, and while the applications may differ, one is hard-pressed not to theorize that similar extensions could be possible with further research. Allowing the random weights be unbounded, like in Theorems 3.2 and 3.3, would certainly yield extremely applicable result. Nonetheless, having random weights at all is significant, since it means that not every bit of detail needs to be boiled down to the compromise that is a series of i.i.d. random variables.

As a closing remark, one should note that while we focused on insurance and finance, the scope of extremal events and their modeling is far greater. More and possibly even more exciting applications could be found by looking at heavy-tailed modeling in other fields, after all, randomly weighted sums of random variables are more than just a tool for evaluating a potential monetary loss.

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