FUNCTIONS OF BOUNDED VARIATION AND THE AM–MODULUS IN $\mathbb{R}^n$

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Abstract. Moduli of path families are widely used to study Sobolev functions. Similarly, the recently introduced approximation (AM–) modulus is helpful in the theory of functions of bounded variation (BV) in $\mathbb{R}^n$ [14]. We continue this direction of research. Let Γ_E be the family of all paths which meet $E \subset \mathbb{R}^n$. We introduce the outer measure $E \mapsto AM(\Gamma_E)$ and compare it with other $(n-1)$-dimensional measures. In particular, we show that $AM(\Gamma_E) = 2\,\mathcal{H}^{n-1}(E)$ whenever E lies on a countably $(n-1)$–rectifiable set. Further, we study functions which have bounded variation on AM–a.e. path and we relate these functions to the classical BV functions which have only bounded essential variation on AM–a.e. path. We also characterize sets E of finite perimeter in terms of the AM–modulus of two path families naturally associated with E.

1. Introduction

The approximation modulus, abbreviated as AM–modulus, was introduced in [14] to study functions of bounded variation (BV) in metric measure spaces, see also [11] and [15]. The AM–modulus offers a counterpart to the Fuglede theorem [10] which states that functions in the Sobolev space $W^{1,p}(\mathbb{R}^n)$, $p \geq 1$, are absolutely continuous on every path in $\mathbb{R}^n$ except of a family whose M_p–modulus is zero and in [14] it was shown that BV functions in metric measure spaces have bounded essential variation on AM almost every path. In Section 4 we show that BV functions in $\mathbb{R}^n$ have bounded variation in the classical sense, and not only bounded essential variation, on AM almost every path in $\mathbb{R}^n$. This leads to a new characterization of BV functions in $\mathbb{R}^n$.

Let Γ_E denote the family of all paths which meet a set $E \subset \mathbb{R}^n$. It turns out that the set function $E \mapsto AM(\Gamma_E)$ is a non–trivial measure in $\mathbb{R}^n$ and we compare this to the $(n-1)$–dimensional Hausdorff measure $\mathcal{H}^{n-1}(E)$ and pay special attention to the case when E lies on a "regular" set $A \subset \mathbb{R}^n$, for instance on a countably $(n-1)$–rectifiable set A in Section 5.

In the last section, Section 6, we give two characterizations for sets E of finite perimeter in terms of the AM–modulus. The perimeter of E

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coincides with the AM–modulus of the family of curves connecting the measure theoretic interior to the measure theoretic exterior of E and with the AM–modulus of the family of curves $\gamma : [a,b] \to \mathbb{R}^n$ which meet the measure theoretic boundary of E at some point $\gamma(t)$, $t \in (a,b)$.

Note that there is a parallel approach to measuring curves [4], which is related to modulus [2] and can be also used to characterize BV spaces [1].

Our notation is standard and more specialized symbols and concepts are explained in due course.

2. Preliminaries and AM–modulus

Let $A \subset \mathbb{R}$ and $f\colon A \to [-\infty,\infty]$. The *total variation* of f on A is

$$V(f,A) = \sup\Big\{\sum_{j=1}^{m} |f(x_j) - f(x_{j-1})| : \\ x_0 < x_1 < x_2 < \dots < x_m, \quad x_j \in A\Big\}$$

where $|f(x_j) - f(x_{j-1})| = \infty$ if $|f(x_j) - f(x_{j-1})|$ is undefined, i.e. of the form $|\infty - \infty|$. If A is (Lebesgue) measurable and f is measurable in A, then the *essential variation* of f on A is

$$\tilde V(f,A) = \inf\big\{V(u,A) : \ u = f \text{ a.e. in } A\big\};$$

obviously $\tilde V(f,A) \le V(f,A)$ and $\tilde V(f,[a,b]) = \tilde V(f,(a,b))$. If $f \in L^1_{\text{loc}}((a,b))$, then by [3, Theorem 3.27], see also [8, 5.10.1 Theorem 1],

$$\tilde V(f,(a,b)) = \sup\Big\{\int_A f\varphi'\,dt : \ \varphi \in C^1_0((a,b)),\ |\varphi| \le 1\Big\}. \tag{1}$$

For the properties of $V(f,A)$ and $\tilde V(f,A)$ see e.g. [3, Section 3.2], [8, Section 5.10.1] and [14].

A continuous mapping $\gamma\colon [a,b] \to \mathbb{R}^n$ is called a *curve*. We say that a curve γ is a *path* if it has a finite and non–zero total length; in this case we parametrize γ by its arclength. We consider paths with various lengths but (with some abuse of notation) do not mark the dependence of the total length ℓ on γ. The *locus* of γ is defined as $\gamma([0,\ell])$ and denoted by $\langle\gamma\rangle$.

If γ is a path and $f : \langle\gamma\rangle \to [-\infty,\infty]$, we write $V(f,\gamma) = V(f \circ \gamma,[0,\ell])$ and $\tilde V(f,\gamma) = \tilde V(f\circ\gamma,[0,\ell])$ provided that $f\circ\gamma$ is measurable on $[0,\ell]$.

We refer to [14] and [11] for the properties of the AM–modulus and to [5] and [10] for those of the M_p–modulus. For completeness we recall the definition for the AM–modulus. Let Γ be a family of paths in $\mathbb{R}^n$. A sequence (ρ_i) of Borel functions on $\mathbb{R}^n$ with values in $[0,\infty]$ is said to be *AM–admissible* (or simply *admissible*) for Γ if for every $\gamma \in \Gamma$

$$\liminf_i \int_\gamma \rho_i \, ds \geq 1, \qquad .$$

and we define the *AM-modulus* of Γ as

$$AM(\Gamma) := \inf\Big\{\liminf_i \int_{\mathbb{R}^n} \rho_i \, dx \colon (\rho_i)_i \text{ is admissible for } \Gamma\Big\}.$$

The AM–modulus, as the M_p–modulus, is monotone and countably subadditive and we say that a property holds for AM–a.e. curve if the property fails on the family Γ such that $AM(\Gamma) = 0$.

With an arbitrary set $E \subset \mathbb{R}^n$ we associate the family Γ_E of all paths that meet E. Further, let Γ_E^c be the family of all paths γ in $\mathbb{R}^n$ that *cross* E, i.e. there is an *interior* point $t \in (0,\ell)$ such that $\gamma(t) \in E$. Although Γ_E looks more natural, in the subsequent sections it becomes evident that $AM(\Gamma_E^c)$ is better related to the $(n-1)$–dimensional Hausdorff measure in $\mathbb{R}^n$ than $AM(\Gamma_E)$.

The following lemma clarifies the roles of Γ_E and Γ_E^c.

Lemma 1. $AM(\Gamma_E) = 2\, AM(\Gamma_E^c)$.

Proof. Let (ρ_i) be an AM–admissible sequence for Γ_E and $\gamma \in \Gamma_E^c$. Since $\gamma(t_0) \in E$ for some $t_0 \in (0,\ell))$, the paths $\gamma_1 = \gamma\lfloor_{[0,t_0]}$ and $\gamma_2\lfloor_{[t_0,\ell]}$ belong to Γ_E and hence

$$\begin{aligned}\liminf_i \int_\gamma \rho_i \, ds &= \liminf_i (\int_{\gamma_1} \rho_i \, ds + \int_{\gamma_2} \rho_i \, ds) \\ &\geq \liminf_i \int_{\gamma_1} \rho_i \, ds + \liminf_i \int_{\gamma_2} \rho_i \, ds \geq 2.\end{aligned}$$

Thus the sequence $(\rho_i/2)$ is AM–admissible for Γ_E^c and we obtain $AM(\Gamma_E^c) \leq AM(\Gamma_E)/2$.

For the reverse inequality let (ρ_i^c) be an AM–admissible sequence for Γ_E^c and let $\gamma \in \Gamma_E \setminus \Gamma_E^c$. We can assume that $\gamma(\ell) \in E$. Let $\tilde{\gamma}$ be the path $\tilde{\gamma}(t) = \gamma(t)$ for $t \in [0,\ell]$ and $\tilde{\gamma}(t) = \gamma(2\ell - t)$ for $t \in (\ell, 2\ell]$. Then $\tilde{\gamma} \in \Gamma_E^c$ and

$$1 \leq \liminf_i \int_{\tilde{\gamma}} \rho_i^c \, ds = 2 \, \liminf_i \int_\gamma \rho_i^c \, ds.$$

Thus $(2\rho_i^c)$ is an AM–admissible sequence for Γ_E and, consequently, $AM(\Gamma_E) \leq 2\, AM(\Gamma_E^c]$. □

The following auxiliary lemma is often useful.

Lemma 2. *If* $U \subset \mathbb{R}^n$ *is open and* $E \subset U$, *then* $AM(\Gamma_E) = AM(\Gamma_E(U))$ *where* $\Gamma_E(U) = \{\gamma \in \Gamma_E : \langle\gamma\rangle \subset U\}$.

Proof. If $\gamma \in \Gamma_E \setminus \Gamma_E(U)$, then γ has a subpath $\tilde{\gamma} \in \Gamma_E(U)$ and thus $AM(\Gamma_E(U)) \geq AM(\Gamma_E)$. On the other hand, $\Gamma_E(U) \subset \Gamma_E$ and thus $AM(\Gamma_E(U)) \leq AM(\Gamma_E)$. □

For $E \subset \mathbb{R}^n$ we set $\phi(E) = AM(\Gamma_E^c) = \frac{1}{2}AM(\Gamma_E)$ and call ϕ the *AM–modulus measure*. The following theorem justifies the name.

Theorem 1. *ϕ is a metric outer measure on $\mathbb{R}^n$. Therefore, all Borel sets are ϕ-measurable.*

Proof. Obviously $\phi(\emptyset) = 0$ and ϕ is monotone. For the countable subadditivity let $E = \cup_j E_j$. We may assume that $\phi(E) < \infty$. Choose $\varepsilon > 0$. For each j find an AM–admissible sequence $(\rho_i^j)_i$ for $\Gamma_{E_j}^c$ such that

$$\int_{\mathbb{R}^n} \rho_i^j \, d\mu \le AM(\Gamma_{E_j}^c) + 2^{-j}\varepsilon$$

for each j and i. This is possible by passing to subsequences. Now

$$\rho_i = \sum_j \rho_i^j$$

is AM–admissible for Γ_E^c and we obtain

$$\phi(E) = AM(\Gamma_E^c) \le \liminf_i \int_{\mathbb{R}^n} \rho_i \, dx$$

$$\le \liminf_i \big(\sum_j AM(\Gamma_{E_j}^c) + 2^{-j}\varepsilon\big) \le \sum_j AM(\Gamma_{E_j}^c) + \varepsilon$$

and letting $\varepsilon \to 0$ we obtain

$$\phi(E) = AM(\Gamma_E^c) \le \sum_j \phi(E_j)$$

as required. We have shown that ϕ is an outer measure in $\mathbb{R}^n$.

It remains to show that ϕ is a metric outer measure, i.e. if sets $E_1, E_2 \subset \mathbb{R}^n$ are such that $d(E_1, E_2) > \delta$ for some $\delta > 0$, then

$$\phi(E_1) + \phi(E_2) = \phi(E_1 \cup E_2). \tag{2}$$

Now there are disjoint open sets U_i, $i = 1, 2$, with $E_i \subset U_i$ and then (2) follows from Lemma 2. Therefore ϕ is a metric outer measure and, consequently, all Borel sets are ϕ–measurable. □

For the set theoretic terminology below we refer to [9, 2.2.10] and [13].

Theorem 2. *Let $E \subset \mathbb{R}^n$ be an arbitrary set. Then there exists a ϕ-measurable (co–Suslin) set $F \subset \mathbb{R}^n$ such that $E \subset F$ and $\phi(F) = \phi(E)$. The AM–modulus measure ϕ also has this property.*

Proof. In this proof we use the version $\phi(E) = \frac{1}{2}AM(\Gamma_E)$ of the definition of ϕ. Since closed sets are ϕ-measurable, each co–Suslin set is ϕ-measurable, [9, Theorem 2.2.12]. We follow the proof of Proposition 1.5 in [6]. Let $\mathcal{L}$ be the space of all curves $\gamma\colon [0,1] \to \mathbb{R}^n$ with Lipschitz constant ≤ 1 equipped with the distance $d(\gamma_1, \gamma_2) = \sup_{t\in[0,1]} |\gamma_1(t) - \gamma_2(t)|$.

Then $\mathcal{L}$ is a complete metric space. If (ρ_i) is an admissible sequence for Γ_E, consider the mapping $\Phi_i : \mathcal{L} \to [0, \infty]$,

$$\Phi_i(\gamma) = \int_\gamma \rho_i \, ds, \qquad \gamma \in \mathcal{L}.$$

Then Φ_i are Borel measurable, see [12] (see also [6]), and thus

$$\mathcal{L}_1 = \{\gamma \in \mathcal{L} \colon \gamma \text{ is nonconstant, } \liminf_i \Phi_i(\gamma) < 1\}$$

is also a Borel set in $\mathcal{L}$. Now, the evaluation mapping $\kappa \colon \gamma \mapsto \gamma(0)$ is continuous and thus $G = \kappa(\mathcal{L}_1)$ is a Suslin set. Set $F = \mathbb{R}^n \setminus G$. If $x \in G \cap E$, then there exists $\gamma \in \mathcal{L}_1$ such that $\gamma(0) = x$. Thus $\gamma \in \Gamma_E$, which is a contradiction, as $\mathcal{L}_1 \cap \Gamma_E = \emptyset$. Hence $E \subset F$. The sequence (ρ_i) is admissible for Γ_F and thus

$$\phi(F) \le \tfrac{1}{2} \liminf_i \int_{\mathbb{R}^n} \rho_i \, dx.$$

Taking infimum over all admissible sequences we obtain $\phi(F) \le \phi(E)$, whereas the converse inequality is obvious because $E \subset F$. □

Remark 1. Lemmata 1 and 2 and Theorems 1 and 2 hold in metric measure spaces with similar proofs.

We need estimates between $\phi(E) = AM(\Gamma_E^c)$ and the $(n-1)$–*dimensional Hausdorff measure*

$$\mathcal{H}^{n-1}(E) = \sup_{\delta > 0} \mathcal{H}^{n-1}_\delta(E),$$

where

$$\mathcal{H}^{n-1}_\delta(E) = \inf\Big\{2^{1-n}\boldsymbol{\alpha}_{n-1} \sum_{i=1}^\infty \big(\operatorname{diam} E_i\big)^{n-1} \colon \ E \subset \bigcup_{i=1}^\infty E_i \,, \ \operatorname{diam} E_i < \delta\Big\}$$

are the Hausdorff δ-contents and $\boldsymbol{\alpha}_{n-1} = \mathcal{L}^{n-1}(B^{n-1}(0,1))$.

Let $G(n, n-1)$ be the Grassmannian manifold of $(n-1)$–dimensional linear subspaces of $\mathbb{R}^n$. With each $V \in G(n, n-1)$ we associate the orthogonal projection $\Pi_V : \mathbb{R}^n \to V$. For $E \subset \mathbb{R}^n$ we define

$$\mathcal{G}^{n-1}(E) = \sup_{\delta > 0} \mathcal{G}^{n-1}_\delta(E),$$

where

$$\mathcal{G}^{n-1}_\delta(E) = \inf\Big\{\sum_{i=1}^\infty \sup\{\mathcal{H}^{n-1}(\Pi_V(E_i)) \colon \ V \in G(n, n-1)\} \colon \\ E_i \text{ Borel}, E \subset \bigcup_{i=1}^\infty E_i, \ \operatorname{diam} E_i < \delta\Big\}.$$

This is the $(n-1)$-dimensional *Gross measure*, see [9, 2.10.4].

We let $\mathbb{H}_n$ denote the coordinate plane $\{x \in \mathbb{R}^n : x_n = 0\}$ and identify $\mathbb{R}^{n-1} = \mathbb{H}_n$.

Inequality (3) below (with a different constant) was considered in metric measure spaces in [14, Theorem 3.17].

Theorem 3. *If $E \subset \mathbb{R}^n$, $n > 1$, then*

$$\phi(E) \le c_n \mathcal{H}^{n-1}(E), \tag{3}$$

where

$$c_n = \frac{n^n \boldsymbol{\alpha}_n}{2(n-1)^{n-1}\boldsymbol{\alpha}_{n-1}}.$$

Proof. We may assume that $\mathcal{H}^{n-1}(E) < \infty$. For $j = 1, 2, \dots$ choose a covering E_i^j, $i = 1, 2, \dots$, of E such that $d_i^j := \operatorname{diam} E_i^j \le 2^{-j}$ and

$$2^{1-n}\boldsymbol{\alpha}_{n-1}\sum_{i=1}^{\infty}\bigl(d_i^j\bigr)^{n-1} \le \mathcal{H}^{n-1}_{2^{-j}}(E) + 2^{-j} \le \mathcal{H}^{n-1}(E) + 2^{-j}.$$

Set

$$\rho_j(x) = \sum_i \frac{1}{2r_i^j}\chi_{E_i^j + B_i^j}(x)$$

where $B_i^j = B(0, r_i^j)$ and

$$r_i^j = \frac{1}{2(n-1)}\, d_i^j.$$

Then ρ_j are Borel functions and the sequence (ρ_j) is AM–admissible for Γ_E^c. Indeed, if $\gamma \in \Gamma_E^c$, then there is $t_0 \in (0, \ell)$ such that $\gamma(t_0) \in E$. The path $\gamma\lfloor_{[0,t_0]}$ is non–constant and hence there is j_0 such that for $j \ge j_0$, $\gamma\lfloor_{[0,t_0]}$ meets both E_i^j and the complement of $E_i^j + B_i^j$ for some $i = i(j)$. Now the path $\gamma\lfloor_{[0,t_0]}$ travels in $E_i^j + B_i^j$ at least distance r_i^j. The same applies to the path $\gamma\lfloor_{[t_0,\ell]}$, possibly with a different j_0, and hence γ travels in $E_i^j + B_i^j$ at least distance $2r_i^j$. Thus

$$\liminf_j \int_\gamma \rho_j\, ds \ge 1$$

as required. Using the isodiametric inequality we estimate

$$\begin{aligned}
AM(\Gamma_E^c) &\le \liminf_j \int_{\mathbb{R}^n} \rho_j\, dx \le 2^{-n}\boldsymbol{\alpha}_n \liminf_j \sum_i \frac{1}{2r_i^j}\operatorname{diam}(E_i^j + B_i^j)^n \\
&= 2^{-n}\boldsymbol{\alpha}_n \liminf_j \sum_i \frac{1}{2r_i^j}(d_i^j + 2r_i^j)^n. \\
&= 2^{-n}\Bigl(\frac{n}{n-1}\Bigr)^n \boldsymbol{\alpha}_n \liminf_j \sum_i \frac{n-1}{d_i^j}\,(d_i^j)^n \\
&= \frac{n^n\boldsymbol{\alpha}_n}{2(n-1)^{n-1}\boldsymbol{\alpha}_{n-1}} \liminf_j 2^{1-n}\boldsymbol{\alpha}_{n-1}\sum_i (d_i^j)^{n-1} \\
&\le \frac{n^n\boldsymbol{\alpha}_n}{2(n-1)^{n-1}\boldsymbol{\alpha}_{n-1}} \liminf_j (\mathcal{H}^{n-1}(E) + 2^{-j})
\end{aligned}$$

and (3) follows. □

Theorem 4. *For every* $V \in G(n, n-1)$ *and* $E \subset \mathbb{R}^n$

$$\mathcal{H}^{n-1}(\Pi_V(E)) \le \phi(E) \tag{4}$$

and if E *is a Borel set, then*

$$\mathcal{G}^{n-1}(E) \le \phi(E). \tag{5}$$

Proof. We first prove (4). Let $V \in G(n, n-1)$. We may assume that $V = \mathbb{H}_n = \mathbb{R}^{n-1}$. Let (ρ_i) be an AM–admissible sequence for Γ_E^c. Since each ρ_i is a Borel function, the set

$$A = \Big\{ y \in \mathbb{R}^{n-1} : \liminf_i \int_{-\infty}^{\infty} \rho_i(y,t)\, dt \ge 1 \Big\}$$

is a measurable set in $\mathbb{R}^{n-1}$. For each $x' \in \Pi_V(E)$ choose $x \in \Pi_V^{-1}(x') \cap E$ and define

$$\gamma_{x'}(t) = x + (t-1)e_n, \quad t \in [0,2].$$

Now $\gamma_{x'}$ belongs to Γ_E^c. By the Fubini theorem and the Fatou lemma

$$\liminf_i \int_{\mathbb{R}^n} \rho_i \, dx \ge \liminf_i \int_A \Big(\int_{-\infty}^{\infty} \rho_i((y,t))\, dt \Big)\, dy$$

$$\ge \int_A \Big(\liminf_i \int_{-\infty}^{\infty} \rho_i((y,t))\, dt \Big)\, dy \ge \mathcal{H}^{n-1}(A) \ge \mathcal{H}^{n-1}(\Pi_V(E))$$

and since this holds for every AM–admissible sequence, the estimate (4) follows.

If E is a Borel set and $\delta > 0$, we write E as a pairwise disjoint union of Borel sets E_i with $\operatorname{diam} E_i < \delta$. Using countable subadditivity of $\mathcal{G}_\delta^{n-1}$, countable additivity of ϕ on Borel sets and (4), we obtain

$$\mathcal{G}_\delta^{n-1}(E) \le \sum_i \mathcal{G}_\delta^{n-1}(E_i) \le \sum_i \phi(E_i) \le \phi(E).$$

Passing to the supremum over δ we obtain (5). □

Remark 2. It is easy to see that $\mathcal{H}^{n-1} = \phi$ for $n = 1$.

Remark 3. Note that for the estimate (4) we do not need to suppose that E is a Borel set. We do not know if (5) holds for all sets $E \subset \mathbb{R}^n$.

3. *BVC* functions in $\mathbb{R}^n$

Let Ω be an open set in $\mathbb{R}^n$ and $u \colon \Omega \to [-\infty, \infty]$. We say that u is a *BVC function*, $u \in \mathcal{BVC}(\Omega)$, if there exists a sequence (g_i) of non–negative Borel functions in Ω such that

$$|u(\gamma(\ell) - u(\gamma(0))| \le \liminf_i \int_\gamma g_i \, ds \tag{6}$$

for each path γ in Ω and

$$\liminf_i \int_\Omega g_i \, dx < \infty. \tag{7}$$

Equation (6) is understood in the sense that if the left hand side is undefined, i.e. of the form $|\infty - \infty|$, then its value is ∞. The sequence (g_i) is called a *BV upper bound* for u in Ω.

Note that we assume no a priori regularity, besides (6), for $u \in \mathcal{BVC}(\Omega)$. However, in Theorem 5 below we show that $u \in L^1_{\rm loc}(\Omega)$.

If $u \in \mathcal{BVC}(\Omega)$, then [14, Theorem 4.1] shows that for AM–a.e. path in Ω and for every BV upper bound (g_i)

$$V(u,\gamma) \le \liminf_i \int_\gamma g_i\,ds < \infty. \tag{8}$$

The condition (6) can be relaxed and we say that a sequence (g_i) of non–negative Borel functions in Ω is a *weak BV upper bound* for u in Ω if (7) is satisfied and (6) holds for AM–a.e. path in Ω. The next lemma shows that weak upper bounds can be used instead of ordinary BV upper bounds.

Lemma 3. *If (g_i) is a weak BV upper bound for u and $\varepsilon > 0$, then there exists a BV upper bound $(\tilde g_i)$ for u and*

$$\liminf_i \int_\Omega \tilde g_i\,dx \le \liminf_i \int_\Omega g_i\,dx + \varepsilon.$$

Proof. Let Γ_0 be the family of all paths in Ω such that

$$V(u,\gamma) > \liminf_i \int_\gamma g_i\,ds.$$

Since $AM(\Gamma_0) = 0$, for each $\varepsilon > 0$ there exists a sequence (v_i) of non-negative Borel functions such that

$$\liminf_i \int_\gamma v_i\,ds = \infty$$

for every $\gamma \in \Gamma_0$ and

$$\int_\Omega v_i\,dx < \varepsilon$$

for every i, see [11, Theorem 7 and Remark 6]. Let $\tilde g_i = g_i + v_i$. Now $(\tilde g_i)$ satisfies 6 and

$$\liminf_i \int_\Omega \tilde g_i\,dx = \liminf_i \int_\Omega (g_i + v_i)\,dx \le \liminf_i \int_\Omega g_i\,dx + \varepsilon$$

and so $(\tilde g_i)$ is the required BV upper bound for u. □

Lemma 3 shows that (8) holds for each weak upper bound for $u \in \mathcal{BVC}(\Omega)$ and consequently u needs to be defined only in $\Omega \setminus E$ where E satisfies $AM(\Gamma_E) = 0$ and so the values of u on E are immaterial. For example, precisely represented functions u in the Sobolev class $W^{1,p}(\Omega)$, $p \ge 1$, are BVC functions since they are absolutely continuous on M_p–a.e. path and hence on AM–a.e. path in Ω. The constant sequence $(|\nabla u|)$ is a weak BV upper bound for u.

For an open set $U \subset \Omega$ we define

$$|D_{BVC}u|(U) = \inf\{\liminf_j \int_\Omega g_i\,dx : \\ (g_i) \text{ is a } BV \text{ upper bound for } u \text{ in } U\}.$$

By Lemma 3 weak BV upper bounds can be used as well.

Next we show that a BVC function u in Ω belongs to $L^1_{\mathrm{loc}}(\Omega)$. Since a BVC function has bounded variation on a.e. line parallel to coordinate axis one might think that this property already implies measurability. However, W. Sierpiński [17] constructed a non–measurable set $A \subset \mathbb{R}^2$ such every line (not only parallel to coordinate axis) meets A at most in two points. The function χ_A has thus bounded variation on every line segment but it is not measurable. Hence the role of the weak BV upper bound (g_j) is essential.

We employ a weak type estimate for a maximal function associated with a weak BV upper bound (g_j) of a BVC function u in Ω. We set $g_j = 0$ in $\mathbb{R}^n \setminus \Omega$. To simplify the notation, we use the symbol $\mathbf{g}$ for the sequence (g_j) and introduce the maximal function

$$M\mathbf{g}(x) = \liminf_j Mg_j(x)$$

where Mg_j denotes the ordinary Hardy–Littlewood maximal function of g_j.

Lemma 4. *Let $\mathbf{g} = (g_j)$ be a sequence of nonnegative measurable functions on $\mathbb{R}^n$ and $t > 0$. Then*

$$\mathcal{L}^n(\{x \in \mathbb{R}^n : M\mathbf{g}(x) > t\}) \le \frac{C_n}{t} \liminf_j \int_{\mathbb{R}^n} g_j(z)\,dz,$$

where C_n depends only on n. Consequently, $M\mathbf{g}(x) < \infty$ for a.e. $x \in \mathbb{R}^n$.

Proof. Set

$$I = \liminf_j \int_{\mathbb{R}^n} g_j(z)\,dz.$$

Choose $\varepsilon > 0$. We may assume that

$$\sup_j \int_{\mathbb{R}^n} g_j(z)\,dz \le I + \varepsilon.$$

Fix $t > 0$ and let

$$E_k = \{x \in \mathbb{R}^n : \inf_{j \ge k} Mg_j(x) > t\}.$$

Then $E_1 \subset E_2 \subset \dots$ and

$$E := \{x \in \mathbb{R}^n : M\mathbf{g}(x) > t\} = \bigcup_k E_k.$$

The sets E_k are measurable. By the Hardy-Littlewood maximal theorem,

$$\mathcal{L}^n(E_k) \le \mathcal{L}^n(\{x \in \mathbb{R}^n \colon Mg_k(x) > t\}) \le \frac{C}{t}\int_{\mathbb{R}^n} g_k(z)\,dz \le \frac{C(I+\varepsilon)}{t}.$$

Letting $E_k \nearrow E$ and $\varepsilon \searrow 0$ we conclude

$$\mathcal{L}^n(E) \le \frac{CI}{t}.$$

□

Theorem 5. *$\mathcal{BVC}(\Omega) \subset L^1_{\mathrm{loc}}(\Omega)$.*

Proof. We first show that $u \in \mathcal{BVC}(\Omega)$ is measurable. It suffices to show that if u is a BVC function in $B_0' = B(x_0, 2r_0) \subset \Omega$, then u is measurable in $B_0 = B(x_0, r_0/2)$. Let (g_j) be a BV upper bound for u in Ω and set $g_j = 0$ in $\mathbb{R}^n \setminus \Omega$.

If $r \in (0, 3r_0/2)$, $x \in B_0$ and $y \in B(x,r)$, then

$$|u(y) - u(x)| \le \liminf_j \int_{\gamma_{x,y}} g_j\,ds = \liminf_j \int_0^1 g_j(x + t(y-x))|y-x|\,dt$$

$$\le \liminf_j \int_0^1 r g_j(x + t(y-x))\,dt =: v^x(y) \tag{9}$$

where $\gamma_{x,y}(t) = x + t(y-x)$, $t \in [0,1]$, is the line segment from x to y. For a fixed x the function v^x is measurable since the functions g_j are Borel functions. Now, the Fubini theorem and the Fatou lemma yield

$$\begin{aligned}
\int_{B(x,r)} v^x(y)\,dy &= \int_{B(x,r)} \Bigl(\liminf_j \int_0^1 r g_j(x + t(y-x))\,dt\Bigr)\,dy \\
&\le \liminf_j \int_{B(x,r)} \Bigl(\int_0^1 r g_j(x + t(y-x))\,dt\Bigr)\,dy \\
&= \liminf_j \int_0^1 r t^{-n} \Bigl(\int_{B(x,tr)} g_j(z)\,dz\Bigr)\,dt \\
&\le \boldsymbol{\alpha}_n r^{n+1} \liminf_j \int_0^1 Mg_j(x)\,dt = \boldsymbol{\alpha}_n r^{n+1} M\mathbf{g}(x),
\end{aligned} \tag{10}$$

where the change of variable from y to $z = x + t(y-x)$ is also used.

To complete the proof let x_1, $x_2 \in B_0$ with $r = |x_2 - x_1| > 0$. Write $B = B(a, r/2)$, where $a = \frac{1}{2}(x_1 + x_2)$. Then $B \subset B(x_1, r) \cap B(x_2, r) \subset B_0'$ and for $y \in B$ we have

$$|u(x_2) - u(x_1)| \le |u(x_2) - u(y)| + |u(x_1) - u(y)| \le v^{x_2}(y) + v^{x_1}(y)$$

and thus integrating the inequality

$$|u(x_2) - u(x_1)| \le v^{x_2}(y) + v^{x_1}(y)$$

over $y \in B$ we obtain from (10)

$$\boldsymbol{\alpha}_n (r/2)^n |u(x_2) - u(x_1)| \le \boldsymbol{\alpha}_n r^{n+1} (M\mathbf{g}(x_1) + M\mathbf{g}(x_2)).$$

This shows that

$$|u(x_2) - u(x_1)| \le 2^n \, |x_2 - x_1| (M\mathbf{g}(x_1) + M\mathbf{g}(x_2)).$$

Let $A_i = \{x \in B_0 : M\mathbf{g}(x) \le i\}$, $i = 1, 2, \dots$. Then $\mathcal{L}^n(B_0 \setminus \cup_i A_i) = 0$ as $\mathcal{L}^n(\{x \in B_0 : M\mathbf{g}(x) = \infty\}) = 0$ by Lemma 4. In the set A_i the function u is Lipschitz and hence measurable and from this it follows that u is measurable in B_0.

Now it easily follows that $u \in L^1_{\mathrm{loc}}(\Omega)$. First note that $|u(x)| < \infty$ for a.e. $x \in \Omega$ by the Fubini theorem, as the family Γ of line segments parallel to a coordinate axis on which u fails to be of finite variation has AM–modulus zero by Theorem 3. Then we find $x \in B_0$ such that $|u(x)| < \infty$ and $M\mathbf{g}(x) < \infty$. An integration of (9) over $y \in B_0$ and (10) show that $u \in L^1(B_0)$. □

Remark 4. In fact, we can obtain better integrability $\mathcal{BVC}(\Omega) \subset L^{n/(n-1)}_{loc}(\Omega)$. This can be obtained following the standard argument by Gagliardo and Nirenberg, or as a consequence of the inclusion $\mathcal{BVC}(\Omega) \subset BV_{\mathrm{loc}}(\Omega)$ presented later (Theorem 7).

4. *BVC* versus *BV* functions

The M_1-modulus cooperates well with the Sobolev spaces $W^{1,1}$, but not so well with the BV spaces. In this respect the AM-modulus is a more efficient tool. Next we show that a precise representative of a BV function in Ω is a BVC function.

We recall some concepts from the theory of BV functions in $\mathbb{R}^n$, see [3] and [8].

A function $u \in L^1_{\mathrm{loc}}(\Omega)$ has *bounded variation* (BV) in Ω if its distributional gradient Du can be represented as a finite vector-valued signed Radon measure Du in Ω, i.e.

$$\int_\Omega u \operatorname{div} \varphi \, dx = -\int_\Omega \varphi \cdot Du$$

for every $\varphi \in C^1_0(\Omega, \mathbb{R}^n)$. Equivalently there are signed Radon measures $D_i u$, $i = 1, \dots, n$, such that

$$\int_\Omega u \, \partial_i \varphi \, dx = -\int_\Omega \varphi D_i u$$

for every $\varphi \in C^1_0(\Omega)$. We let $|Du|$ and $|D_i u|$ denote the total variation of the measures Du and $D_i u$, $i = 1, \dots, n$, respectively. By the Riesz representation theorem, see e.g. [8], we have

$$|Du|(\Omega) = \sup \Big\{ \int_\Omega u \operatorname{div} \varphi \, dx : \varphi \in C^1_0(\Omega, \mathbb{R}^n), \, |\varphi| \le 1 \Big\} \tag{11}$$

and for each $i = 1, \dots, n$

$$|D_i u|(\Omega) = \sup\Big\{ \int_\Omega u\, \partial_i \varphi\, dx : \varphi \in C_0^1(\Omega),\ |\varphi| \le 1 \Big\}. \tag{12}$$

A function $u \in C^1(\Omega)$ with $\int_\Omega |\nabla u|\, dx < \infty$ has bounded variation and

$$|Du|(\Omega) = \int_\Omega |\nabla u|\, dx, \qquad |D_i u|(\Omega) = \int_\Omega |\partial_i u|\, dx.$$

We denote the space of all BV functions in Ω by $\mathcal{BV}(\Omega)$. This is a Dirichlet type space; we leave the symbol $BV(\Omega)$ for the usual BV-space, namely

$$BV(\Omega) = \mathcal{BV}(\Omega) \cap L^1(\Omega).$$

The space $BV(\Omega)$ is equipped with the BV norm

$$\|u\|_{BV} = |Du|(\Omega) + \|u\|_1.$$

Similarly, we define

$$BVC(\Omega) = \mathcal{BVC}(\Omega) \cap L^1(\Omega)$$

and consider the BVC norm

$$\|u\|_{BVC} = |Du|_{BVC}(\Omega) + \|u\|_1.$$

Of course, the BV norm does not satisty the axioms of norm on the family of BV functions. For a precise manipulation with the BV space in the framework on normed linear spaces we identify functions which coincide a.e. and mark the underlying linear space

$$\mathbf{BV}(\Omega) := \big\{[u]_{\mathcal{L}^n} \colon u \in BV(\Omega)\big\},$$

where

$$[u]_{\mathcal{L}^n} = \big\{v \in BV(\Omega) \colon v = u \text{ a.e.}\big\}$$

Concerning the BVC norm, we use a "finer' equivalence relation, so that the underlying linear space is

$$\mathbf{BVC}(\Omega) := \big\{[u]_\phi \colon u \in BVC(\Omega)\big\},$$

where

$$[u]_\phi = \big\{v \in BVC(\Omega) \colon v = u\ \phi\text{-a.e.}\big\}$$

This formalism helps us to explain the diference between the BV and BVC spaces in Remark 5 below.

Let us make the convention that we will still say e.g. "v is a representative of $u \in BV(\Omega)$" instead of more precise, but clumsy "v is a representative of $[u]_{\mathcal{L}^n} \in \mathbf{BV}(\Omega)$".

Let $u \in L^1_{\text{loc}}(\Omega)$. A point $z \in \Omega$ is said to be a *Lebesgue point* for u if

$$\lim_{r\to 0+} \fint_{B(z,r)} |u(x) - u(z)|\, dx = 0.$$

Given $z \in \mathbb{R}^n$, $r > 0$ and $\mathbf{n} \in \partial B(0,1)$, we write

$$B_{\mathbf{n}}(z,r) = \{x \in B(z,r) \colon (x - z)\cdot \mathbf{n} > 0\}$$

and say that $z \in \Omega$ is an L^1-*approximate jump point* for u if there exist $a, b \in \mathbb{R}$ and $\mathbf{n} \in \partial B(0,1)$ such that $a \neq b$ and

$$\lim_{r\to 0+} \fint_{B_{\mathbf{n}}(z,r)} |u(x) - b|\, dx = \lim_{r\to 0+} \fint_{B_{-\mathbf{n}}(z,r)} |u(x) - a|\, dx = 0.$$

Define the *normal vector* of u at z by $\mathbf{n}_u(z) = (b-a)\mathbf{n}$ where $\mathbf{n}$ is given by the preceding formula; the vector $\mathbf{n}_u(z)$ is unique at L^1-approximate jump points.

We define the L^1-*approximate discontinuity set* S_u as a set of non-Lebesgue points for u and the L^1-*approximate jump set* of J_u as the set of all L^1-approximate jump points for u; now $J_u \subset S_u$.

We call $\bar{u} \colon \Omega \to \mathbb{R}$ a *precise representative* of u if

$$\bar{u}(z) = \lim_{r\to 0} \fint_{B(z,r)} u(x)\, dx$$

whenever for z this limit exists and is finite. At a jump point z of u we set $\bar{u}(z) = (a+b)/2$. Now we have $\bar{u} = u$ a.e. in Ω.

If u is a BV function in Ω and $\bar{u}$ the precise representative of u, then by [3, Theorem 3.78] we have $\mathcal{H}^{n-1}(S_u \setminus J_u) = 0$ and hence by Theorem 3, $AM(\Gamma_{S_u\setminus J_u}) = 0$ and thus $\bar{u}$ is well defined on AM–a.e. path in Ω. The set $S_u \setminus J_u$ contains all points at which the value of a precise representative remains undetermined. The values of $\bar{u}$ at these points x are immaterial, even can be undefined.

Theorem 6. *Let u be a precise representative of a function from $\mathcal{BV}(\Omega)$. Then $u \in \mathcal{BVC}(\Omega)$ and u has bounded variation on AM–a.e. path in Ω. Moreover, there is a sequence of functions $v_j \in C^1(\Omega)$ such that $v_j \to u$ in $L^1_{\mathrm{loc}}(\Omega)$ and $|\nabla v_j|$ is a weak BV upper bound for u with*

$$\begin{aligned} |Du|(\Omega) &= \lim_{j\to\infty} \int_\Omega |\nabla v_j|\, dx, \\ |D_i u|(\Omega) &= \lim_{j\to\infty} \int_\Omega |\partial_i v_j|\, dx, \qquad i = 1, \dots, n. \end{aligned} \tag{13}$$

In particular,

$$|D_{BVC} u|(\Omega) \leq |Du|(\Omega).$$

Proof. As in the proof of [3, Theorem 3.9] or of [8, 5.2.2 Theorem 2] there is a sequence (v_j) of functions $v_j \in C^1(U)$ such that $v_j \to u$ in $L^1_{\mathrm{loc}}(\Omega)$ and (13) holds. The functions v_j are constructed by the standard convolution process with a radial mollifier η, see [8, 4.2.1.], with the aid of a partition of unity which helps to define the approximating function near the boundary. Now at each Lebesgue point $x \in \Omega$ of u, i.e. $x \notin S_u$, $v_j(x) \to u(x)$, see [3, Proposition 3.64] or [8, 4.2.1 Theorem 1], and also at each jump point x of u, $v_j(x) \to u(x) = (a+b)/2$, see [3, Proposition 3.69] or [8, 5.9 Corollary 1].

Let $E = S_u \setminus J_u$. Now $AM(\Gamma_E) = 0$ and for each path γ in Ω but not in Γ_E, $v_i \circ \gamma(t) \to u \circ \gamma(t)$ for every $t \in [0, \ell]$ and hence by the lower semicontinuity of the total variation

$$V(u, \gamma) \le \liminf_j V(v_j, \gamma) \le \liminf_j \int_\gamma |\nabla v_j| \, ds.$$

Thus the sequence $(|\nabla v_j|)$ is a weak BV upper bound for u in Ω and so $u \in \mathcal{BVC}(\Omega)$. □

A converse for Theorem 6 was established in [15, Theorem 5.4]. We will improve the bound obtained there.

Lemma 5. *Let $u \in \mathcal{BVC}(\Omega)$. Let (g_j) a weak BV upper bound for u in Ω. Then $u \in \mathcal{BV}(\Omega)$ and*

$$|Du(U)| \le \liminf_j \int_U g_j \, dx \tag{14}$$

for each convex open set $U \subset \Omega$ with compact closure in Ω.

Proof. We first fix a convex open set $U \subset \Omega$ with compact closure in Ω and show that $u \in \mathcal{BV}(U)$ and, for $i = 1, \dots, n$,

$$\begin{aligned} |D_i u|(U) &= \sup \Big\{ \int_U u \, \partial_i \varphi \, dx \colon \varphi \in C_0^1(U), \, |\varphi| \le 1 \Big\} \\ &\le \liminf_j \int_U g_j \, dx < \infty. \end{aligned} \tag{15}$$

We may assume $i = n$ which simplifies notation. Since U is convex and has compact closure in Ω, for a fixed $y \in \mathbb{R}^{n-1}$, the set $\{t \in \mathbb{R} \colon (y, t) \in \overline{U}\}$ is a closed interval L_y (possibly empty) and from (8) it follows that

$$V(u(y, \cdot), L_y) \le \liminf_j \int_{L_y} g_j(y, t) \, dt \tag{16}$$

for a.e. $y \in \mathbb{R}^{n-1}$. By Theorem 5, $u \in L^1(U)$. Let $\varphi \in C_0^1(U)$, $|\varphi| \le 1$. Applying the Fubini theorem and the Fatou lemma we obtain from (1) and (16)

$$\begin{aligned} \int_U u \, \partial_n \varphi \, dx &= \int_{\mathbb{R}^{n-1}} \Big(\int_{L_y} u(y, t) \, \partial_n \varphi(y, t) \, dt \Big) \, dy \\ &\le \int_{\mathbb{R}^{n-1}} \tilde{V}(u(y, \cdot), L_y) \, dy \le \int_{\mathbb{R}^{n-1}} V(u(y, \cdot), L_y) \, dy \\ &\le \liminf_j \int_U g_j \, dx, \end{aligned}$$

which verifies (15). To prove (14), pick a unit vector $\mathbf{n}$ such that $|Du(U)| = Du(U) \cdot \mathbf{n}$. We may assume that $\mathbf{n} = \mathbf{e}_n$; then

$$|Du(U)| = D_n u(U) \le \liminf_j \int_U g_j \, dx$$

from (15).

The estimate

$$|D_i u|(U) \le \liminf_j \int_U g_j \, dx, \quad i = 1, \dots, n$$

from (15) implies

$$|Du|(B) \le n \liminf_j \int_B g_j \, dx \tag{17}$$

for each ball B with closure in Ω. Using a finite covering with bounded multiplicity of Ω by such balls and an associated partition of unity we obtain

$$|Du|(\Omega) \le C \liminf_j \int_\Omega g_j \, dx \tag{18}$$

where C depends only on n. Hence $u \in \mathcal{BV}(\Omega)$. □

If $u \in \mathcal{BV}(\Omega)$ is a precisely represented BV function, Theorem 6 shows that $|D_{BVC} u|(U) \le |Du|(U)$ for open sets $U \subset \Omega$. In the next theorem we present the converse sharp inequality.

Theorem 7. *If $u \in \mathcal{BVC}(\Omega)$, then $u \in \mathcal{BV}(\Omega)$ and*

$$|Du|(\Omega) \le |D_{BVC} u|(\Omega).$$

Proof. Let (g_j) be a BV upper bound for u in Ω. By Lemma 5, u is a BV function in Ω. By the Besicovitch differentiation theorem [3, Theorem 2.22], there exists a set $N \subset \Omega$ such that $\Omega \setminus N$ is contained in the support of $|Du|$,

$$|Du|(\Omega \cap N) = 0 \tag{19}$$

and for each $y \in \Omega \setminus N$ there exists

$$\lim_{r \to 0} \frac{Du(B(y,r))}{|Du|(B(y,r)} \in \partial B(0,1). \tag{20}$$

Given $\varepsilon > 0$, with each $y \in \Omega$ we associate the family $R(y)$ of all radii $r > 0$ with the properties $\overline{B}(y,r) \subset \Omega$,

$$|Du|(B(y,r)) \le (1+\varepsilon)|Du(B(y,r))| \tag{21}$$

and

$$|Du|(\partial B(y,r)) = 0. \tag{22}$$

This defines a fine centered covering of $\Omega \setminus N$. By the Vitali-Besicovitch covering theorem [3, Theorem 2.19], there exists a countable (possibly finite) pairwise disjoint collection $\{B(y_i, r_i)\}$ of balls such that $y_i \in \Omega \setminus N$, $r_i \in R(y_i)$ and

$$|Du|((\Omega \setminus N) \setminus \bigcup_i \overline{B}(y_i, r_i)) = 0.$$

However, since $|Du|(N) = 0$ and in view of (22), we can improve to

$$|Du|(\Omega \setminus \bigcup_i B(y_i, r_i)) = 0.$$

Now Lemma 5 and (21) yield

$$
\begin{aligned}
(23) \qquad |Du|(\Omega) &\le \sum_i |Du|(B(y_i, r_i)) \le (1+\varepsilon) \sum_i |Du(B(y_i, r_i))| \\
&\le (1+\varepsilon) \sum_i \liminf_j \int_{B(y_i,r_i))} g_j \, dx \le (1+\varepsilon) \liminf_j \int_\Omega g_j \, dx.
\end{aligned}
$$

Since (g_j) was arbitrary, letting $\varepsilon \to 0$ we obtain the desired inequality. □

The above results lead to the following characterization of BV functions.

Theorem 8. *Let u be a function on Ω. Then u is a BV function in Ω if and only if there exists a BVC function v on Ω such that $u = v$ a.e.*

Proof. This is immediate from Theorem 6 and Theorem 7. □

Remark 5. We have proved inclusions

$$\big\{u \in BV(\Omega) \colon u \text{ is precisely represented}\big\} \subset BVC(\Omega) \subset BV(\Omega).$$

These inclusions are both strict: The characteristic function of a singleton is in $BVC(\Omega)$ but it is not precisely represented. The characteristic function of the set of all $x \in \Omega$ which have at least one rational coordinate is in $BV(\Omega)$ but not in $BVC(\Omega)$. Let us associate a precise representative $\bar{u}$ with any $u \in BV(\Omega)$. Consider the induced mappings

$$
\begin{aligned}
\epsilon \colon [u]_{\mathcal{L}^n} &\mapsto [\bar{u}]_\phi \colon \mathbf{BV}(\Omega) \to \mathbf{BVC}(\Omega), , \\
\varkappa \colon [u]_\phi &\mapsto [u]_{\mathcal{L}^n} \colon \mathbf{BVC}(\Omega) \to \mathbf{BV}(\Omega).
\end{aligned}
$$

If v_1, v_2 are precise representatives of $u \in BV(\Omega)$, then $v_1 = v_2$ $\mathcal{H}^{n-1}$-a.e. and the more $v_1 = v_2$ ϕ-a.e., so that ϵ is well defined. Both ϵ and $\varkappa$ are linear. We will show that ϵ is injective, but not surjective and $\varkappa$ is surjective, but not injective. Therefore we cannot identify the spaces $\mathbf{BVC}(\Omega)$ and $\mathbf{BV}(\Omega)$. Indeed, the injectivity of ϵ is obvious. For any $\mathbf{u} \in \mathbf{BV}(\Omega)$ we have $\mathbf{u} = \varkappa(\epsilon(\mathbf{u}))$, so that $\varkappa$ is surjective. Now, consider the characteristic function u of the singleton $\{0\}$ in $(-1, 1)$. Then $\phi(\{0\}) = 1$, hence $[u]_\phi \ne [0]_\phi$, but

$$\varkappa([u]_\phi) = [0]_{\mathcal{L}^n} = \varkappa([0]_\phi).$$

Therefore $\varkappa$ is not injective. The same u disproves surjectivity of ϵ, as $[u]_\phi$ does not belong to its range. Note that $|D_{BVC}u|((-1, 1)) = 2$.

The representatives u of $u \in BV(\Omega)$ can be classified as non-BVC representatives, "bad" BVC representatives and "good" BVC representatives. Here, we mean that u is a good representative if $[u]_\phi =$

$\epsilon([u]_{\mathcal{L}^n})$. If u is a good representative, then $|Du|(\Omega) = |D_{BVC}u|(\Omega)$ (but not conversely, consider the characteristic function of $[0,1)$ in $(-1,1)$).

5. AM–modulus and $(n-1)$–rectifiable sets

A set $A \subset \mathbb{R}^n$ is called *countably $\mathcal{H}^{n-1}$–rectifiable*, or (following [16]) simply $(n-1)$*–rectifiable*, if there are Lipschitz maps $h_i : \mathbb{R}^{n-1} \to \mathbb{R}^n$, $i = 1, 2, \dots$, such that $\mathcal{H}^{n-1}(A \setminus \cup_{i=1}^{\infty} h_i(\mathbb{R}^{n-1})) = 0$.

For the theory of rectifiable sets see [9], [16] and [3].

We first state the lower estimate as a separate lemma.

Lemma 6. *Suppose that $A \subset \mathbb{R}^n$ is $(n-1)$–rectifiable. If $E \subset A$, then*

$$\mathcal{H}^{n-1}(E) \leq AM(\Gamma_E^c) \tag{24}$$

Proof. Assume first that E is $\mathcal{H}^{n-1}$-measurable. The result of [3, Proposition 2.66] states that (due to $(n-1)$–rectifiability)

$$\mathcal{H}^{n-1}(E) = \sup\Big\{ \sum_{i=1}^{\infty} \mathcal{H}^{n-1}(\Pi_{V_i}(K_i)) \colon V_i \in G(n, n-1), \\ K_i \subset E \text{ compact and pairwise disjoint} \Big\}.$$

If K_i and V_i are as above, by (4) we have

$$\sum_{i=1}^{\infty} \mathcal{H}^{n-1}(\Pi_{V_i}(K_i)) \leq \sum_{i=1}^{\infty} \phi(K_i) \leq \phi(E).$$

Hence $\mathcal{H}^{n-1}(E) \leq \phi(E)$ and in the general case we use Theorem 2 to find a ϕ-measurable set $F \supset E$ such that $\phi(F) = \phi(E)$ and obtain

$$\mathcal{H}^{n-1}(E) \leq \mathcal{H}^{n-1}(F) \leq \phi(F) = \phi(E).$$

□

Comparing the previous lemma with Theorem 3 we already know that $\mathcal{H}^{n-1}$ and ϕ are comparable on $(n-1)$–rectifiable sets. However, we can profit from rectifiability also in the sharp constant of the upper estimate and prove that $\mathcal{H}^{n-1}$ and ϕ in fact agree on $(n-1)$–rectifiable sets.

We start with a simple estimate of Minkowski content. If $E \subset \mathbb{R}^n$ is an arbitrary set, we define

$$E_r = \{x \in \mathbb{R}^n \colon \operatorname{dist}(x, E) < r\}$$

and

$$\underline{\mathcal{M}}^{n-1}(E) = \liminf_{r \to 0+} \frac{\mathcal{L}^n(E_r)}{2r}.$$

This is the *lower Minkowski content* of E.

Lemma 7. $\phi(E) \leq \underline{\mathcal{M}}^{n-1}(E)$ *for every set* $E \subset \mathbb{R}^n$

Proof. We find a sequence (r_i), $r_i \searrow 0$, such that

$$\frac{\mathcal{L}^n(E_{r_i})}{2r_i} \to \underline{\mathcal{M}}^{n-1}(E)$$

and set $\rho_i = \chi_{E_{r_i}}/(2r_i)$. Let $\gamma \in \Gamma_E^c$. Then there exists $i_0 \in \mathbb{N}$ and $t \in (0,\ell)$ such that $\gamma(t) \in E$ and $[t-r_i, t+r_i] \subset [0,\ell]$ for each $i \geq i_0$. Since (by our convention) γ is parametrized by its arclength, we have $\gamma([t-r_i, t+r_i]) \subset E_{r_i}$ and thus

$$\int_\gamma \rho_i \geq 1, \qquad i \geq i_0.$$

It follows that (ρ_i) is admissible for Γ_E^c and thus

$$\phi(E) \leq \liminf_j \int_{\mathbb{R}^n} \rho_i \, dx = \liminf_j \frac{\mathcal{L}^n(E_{r_i})}{2r_i} = \underline{\mathcal{M}}^{n-1}(E).$$

□

Theorem 9. *If $A \subset \mathbb{R}^n$ is $(n-1)$–rectifiable and $E \subset A$, then*

$$\mathcal{H}^{n-1}(E) = AM(\Gamma_E^c).$$

Proof. By Theorem 3 it suffices to show

$$AM(\Gamma_E^c) \leq \mathcal{H}^{n-1}(E) \tag{25}$$

and we may assume that $\mathcal{H}^{n-1}(E) < \infty$. Also assume first that E is $\mathcal{H}^{n-1}$-measurable. Then as a straightforward consequence of the definition, there exist compact sets $K_j \subset \mathbb{R}^{n-1}$ and Lipschitz maps $f_j \colon K_j \to \mathbb{R}^n$ such that $f_j(K_j)$ are pairwise disjoint subsets of E and

$$\mathcal{H}^{n-1}\Big(E \setminus \bigcup_j f_j(K_j)\Big) = 0.$$

Theorem [9, 3.2.39] states that $\underline{\mathcal{M}}^{n-1}(F) = \mathcal{H}^{n-1}(F)$ for each Lipschitz image of a compact subset of $\mathbb{R}^{n-1}$. Hence by Lemma 7,

$$\begin{aligned}\phi(E) &= \sum_j \phi(f_j(K_j)) \leq \sum_j \underline{\mathcal{M}}^{n-1}(f_j(K_j)) \\ &= \sum_j \mathcal{H}^{n-1}(f_j(K_j)) \leq \mathcal{H}^{n-1}(E).\end{aligned}$$

Here we have used that $\phi(N) = 0$ for each $N \subset \mathbb{R}^n$ with $\mathcal{H}^{n-1}(N) = 0$, this follows from Theorem 3.

Now we may remove the additional assumption of $\mathcal{H}^{n-1}$–measurability. Indeed, since $\mathcal{H}^{n-1}$ is a Borel regular measure, for an arbitrary set $E \subset A$ there exists a Borel set $F \supset E$ such that $\mathcal{H}^{n-1}(F) = \mathcal{H}^{n-1}(E)$ and we can use the estimate above for $F \cap A$. □

6. Perimeter and AM–modulus

We first recall the basic concepts on sets of finite perimeter, see [3] and [8].

Let $E \subset \mathbb{R}^n$ be a Lebesgue measurable set. The *measure theoretic boundary* $\partial_* E$ of E consists of points $x \in \mathbb{R}^n$ such that $\Theta^n(x, E) > 0$ and $\Theta^n(x, \mathbb{R}^n \setminus E) > 0$ where

$$\Theta^n(x, A) = \limsup_{r \to 0} \frac{\mathcal{L}^n(B(x,r) \cap A)}{\mathcal{L}^n(B(x,r))}$$

is the upper Lebesgue density of A at x. The *measure theoretic interior* $\mathrm{int}_* E$ of E and the *measure theoretic exterior* $\mathrm{ext}_* E$ are the sets of points $x \in \mathbb{R}^n$ where $\Theta^n(x, \mathbb{R}^n \setminus E) = 0$ and $\Theta^n(x, E) = 0$, respectively. The sets $\partial_* E$, $\mathrm{int}_* E$ and $\mathrm{ext}_* E$ are Borel sets.

A set $E \subset \mathbb{R}^n$ has *finite perimeter* if $\chi_E \in BV(\mathbb{R}^n)$, i.e. E has finite Lebesgue measure and $P(E) = |D\chi_E|(\mathbb{R}^n) < \infty$.

It is also possible to consider sets E which have finite perimeter in an open set $\Omega \subset \mathbb{R}^n$, i.e. χ_E is a BV function in Ω, but, for simplicity, we only consider $\Omega = \mathbb{R}^n$.

If E has finite perimeter we let $\partial^* E$ denote the *reduced boundary* of E, which consist of all points x from the support of $|D\chi_E|$ for which there exists a unit vector $\mathbf{n}(x)$ such that

$$\lim_{r \to 0} \frac{D\chi_E(B(x,r))}{|D\chi_E(B(x,r))|} = \mathbf{n}(x),$$

see [3, Definition 3.54] [8, 5.7.1]. Then we have

$$|D\chi_E|(A) = \mathcal{H}^{n-1}(\partial^* E \cap A) = \mathcal{H}^{n-1}(\partial_* E \cap A) \tag{26}$$

for each Borel set $A \subset \mathbb{R}^n$, see [3, Theorem 3.59 and Theorem 3.61] or [8, 5.7.3 and 5.8]. From Theorems 6 and 7 we obtain

Corollary 1. *If E has finite perimeter, then the precise representative $\bar{\chi}_E$ of χ_E belongs to $BVC(\mathbb{R}^n)$ and, conversely, if $\chi_E \in BVC(\mathbb{R}^n)$, then E has finite perimeter.*

Next we use the results from the previous sections to express $|D\chi_E|(A)$ for sets $A \subset \mathbb{R}^n$ in terms of the AM–modulus.

Theorem 10. *Suppose that E has finite perimeter and $A \subset \mathbb{R}^n$ a Borel set. Then*

$$|D\chi_E|(A) = AM(\Gamma^c_{A \cap \partial_* E}) \tag{27}$$

and, in particular, $P(E) = AM(\Gamma^c_{\partial_* E})$.

Proof. By the structure theorem for sets of finite perimeter, see [3, Theorem 3.59] or [8, 5.7.3 Theorem 2], the reduced boundary $\partial^* E$ is $(n-1)$–rectifiable and $|D\chi_E|(A) = \mathcal{H}^{n-1}(A \cap \partial^* E)$. Since $\partial^* E \subset \partial_* E$ and $\mathcal{H}^{n-1}(\partial_* E \setminus \partial^* E) = 0$, by [3, Theorem 3.61] or [8, 5.8 Lemma 1],

$|D\chi_E|(A) = \mathcal{H}^{n-1}(A\cap\partial_* E)$. Now the proof follows from Theorem 9. $\square$

Next we characterize the sets E of finite perimeter in terms of the path families $\Gamma^c_{\partial_* E}$ and $\Gamma_{\text{cross}}(E)$, where the latter family consists of all paths in $\mathbb{R}^n$ connecting $\operatorname{int}_* E$ to $\operatorname{ext}_* E$, i.e. each path in $\Gamma_{\text{cross}}(E)$ meets both $\operatorname{int}_* E$ and $\operatorname{ext}_* E$.

Let $E\subset\mathbb{R}^n$ be a set of finite measure. We start by showing that the condition $AM(\Gamma_{\text{cross}}(E))<\infty$ implies that E has finite perimeter. Then we proceed to prove a sharp result in Theorem 11.

Lemma 8. *If $AM(\Gamma_{\text{cross}}(E))<\infty$, then E has finite perimeter. Moreover, if (ρ_j) is an admissible sequence for $\Gamma_{\text{cross}}(E)$, then for each ball $B(z,r)$ and every $i\in\{1,\dots,n\}$*

$$|D_i\chi_E|(B(z,r) \le \liminf_j \int_{B(z,r)} \rho_j\,dx \tag{28}$$

and

$$|D\chi_E(B(z,r))| \le \liminf_j \int_{B(z,r)} \rho_j\,dx. \tag{29}$$

Proof. Assume that χ_E is precisely represented. Fix a ball $B=B(z,r)$ and $i\in\{1,\dots,n\}$ and let Π_i denote the orthogonal projection of $\mathbb{R}^n$ onto the i^{th}–coordinate plane. Given $y\in\Pi_i(B)$, we let $\gamma_y\colon[0,\ell]\to\overline{B}$ denote the path which parametrizes the line segment L_y parallel to the x_i–axis, joins the opposite faces of ∂B and whose continuation passes through y. Since the set $\partial_* E$ has Lebesgue measure zero, the Fubini theorem yields that for a.e. $y\in\Pi_i(B)$, the set $L_y\cap\partial_* E$ has one-dimensional measure zero. Therefore, for any $\lambda<\tilde V(\chi_E,\gamma_y)$ there exists a partition $0<t_0<t_1<\dots<t_m<\ell$ such that $m\ge\lambda$, all the points $\gamma_y(t_\alpha)$ belong to $\operatorname{int}_* E\cup\operatorname{ext}_* E$, and

$$|\chi_E(\gamma_y(t_\alpha))-\chi_E(\gamma_y(t_{\alpha-1}))|\ge 1,\qquad \alpha=1,\dots,m.$$

Set

$$\gamma_{y,\alpha}(t)=\gamma_y(t-t_{\alpha-1}),\qquad t\in[0,t_\alpha-t_{\alpha-1}],\quad \alpha=1,\dots,m.$$

Then each $\gamma_{y,\alpha}$ belongs to $\Gamma_{\text{cross}}(E)$ and thus

$$1\le\liminf_j\int_{\gamma_{y,\alpha}}\rho_j\,ds,\qquad \alpha=1,\dots,m,$$
$$\lambda\le\sum_{\alpha=1}^m\liminf_j\int_{\gamma_{y,\alpha}}\rho_j\,ds\le\liminf_j\int_{\gamma_y}\rho_j\,ds.$$

Then letting $\lambda\to\tilde V(\chi_E,\gamma_y)$ we obtain

$$\tilde V(\chi_E,\gamma_y)\le\liminf_j\int_{\gamma_y}\rho_j\,ds$$

and integrating with respect to y and using the Fatou lemma and the Fubini theorem we conclude that

$$\int_{\Pi_i(B)} \tilde{V}(\chi_E, \gamma_y)\, dy \le \int_{\Pi_i(B)} \Big(\liminf_j \int_{\gamma_y} \rho_j\, ds \Big) dy \le \liminf_j \int_B \rho_j\, dx.$$

Now from (12), (1) and the Fubini theorem we obtain (28). Note that $y \mapsto \tilde{V}(\chi_E, \gamma_y)$ is $\mathcal{L}^{n-1}$–measurable, see [8, 5.10.2 Lemma 1].

Since (28) is true in every ball, we have

$$|D_i\chi_E|(\mathbb{R}^n) \le \liminf_j \int_{\mathbb{R}^n} \rho_j\, dx$$

for every $i \in \{1, \ldots, n\}$ and this implies that $P(E) = |D\chi_E|(\mathbb{R}^n) < \infty$.

To prove (29) note that there exists a unit vector $\mathbf{n}$ such that

$$|D\chi_E(B)| = D\chi_E(B) \cdot \mathbf{n}.$$

We may assume that $\mathbf{n} = \mathbf{e}_n$ and then $D\chi_E(B) \cdot \mathbf{n} = D_n\chi_E(B)$. Since $|D_n\chi_E(B)| \le |D_n\chi_E|(B)$ the result follows from (29). □

Theorem 11. *If $E \subset \mathbb{R}^n$ is a set of finite (Lebesgue) measure, then*

$$AM(\Gamma^c_{\partial_* E}) = P(E) = AM(\Gamma_{\text{cross}}(E)). \tag{30}$$

Proof. If $P(E) < \infty$, then Theorem 10 gives $P(E) = AM(\Gamma^c_{\partial_* E})$. Suppose that $AM(\Gamma^c_{\partial_* E}) < \infty$. By Theorem 4, $\mathcal{G}_{n-1}(\partial_* E) < \infty$ and since the $(n-1)$–dimensional integral geometric measure (with exponent 1) $\mathcal{I}_1^{n-1}$ satisfies $\mathcal{I}_1^{n-1} \le \mathcal{G}_{n-1}$, see [9, 2.10.5–6], $\mathcal{I}_1^{n-1}(\partial_* E) < \infty$. Now, by [9, 4.5.11] it follows that E has finite perimeter (for a more elementary proof of this deep result see [7, Theorem 4.9]). Then Theorem 10 gives $AM(\Gamma^c_{\partial_* E}) = P(E)$.

It remains to consider the second equality in (30). We first show that

$$P(E) \le AM(\Gamma_{\text{cross}}(E)). \tag{31}$$

We may assume that $AM(\Gamma_{\text{cross}}(E)) < \infty$. Then Lemma 8 yields that $\chi_E \in BV(\mathbb{R}^n)$ and we replace χ_E by its precise representative. Let (ρ_j) be an admissible sequence for $\Gamma_{\text{cross}}(E)$. Now we can proceed as in the proof of Theorem 7. Given $\varepsilon > 0$, using the Vitali-Besicovitch covering theorem [3, Theorem 2.19], the definition of reduced boundary, (26) and (29) we find a countable (possibly finite) pairwise disjoint collection $\{B(y_i, r_i)\}$ of balls such that

$$|D\chi_E|(\partial^* E \setminus \bigcup_i B(y_i, r_i)) = 0,$$

$$|D\chi_E|(B(y_i, r_i)) \le (1+\varepsilon)|D\chi_E(B(y_i, r_i))|,$$

and

$$|D\chi_E(B(y_i, r_i))| \le \liminf_j \int_{B(y_i, r_i)} \rho_j\, dx.$$

Summing over i we obtain

$$P(E) \le (1+\varepsilon) \liminf_j \int_{\mathbb{R}^n} \rho_j \, dx$$

and since this holds for every $\varepsilon > 0$ and every admissible sequence (ρ_j), (31) follows.

To prove the reverse inequality to (31) we may assume that $P(E) < \infty$. Now we can use the fact that χ_E is a BV function. By the approximation theorem for BV functions ([3, Theorem 3.9] or [8, 5.2.2 Theorem 2]) a sequence of BV functions $f_i \in C^1(\mathbb{R}^n, \mathbb{R})$ such that

$$\int_{\mathbb{R}^n} |\nabla f_i| \, dx \to P(E)$$

and $f_i(x) \to \chi_E(x)$ at every density point x of E and at every density point x of $\mathbb{R}^n \setminus E$ as $i \to \infty$. Now the sequence $(|\nabla f_i|)$ is an AM–admissible sequence for $\Gamma_{\text{cross}}(E)$ because for $\gamma \in \Gamma_{\text{cross}}(E)$ we find t_1 and t_2 such that $\gamma(t_1) \in \text{ext}_* E$ and $\gamma(t_2) \in \text{int}_* E$ and thus

$$1 = \lim_i |f_i(\gamma(t_2)) - f_i(\gamma(t_1))| \le \liminf_i \int_\gamma |\nabla f_i| \, ds$$

and we obtain

$$AM(\Gamma_{\text{cross}}(E)) \le \liminf_i \int_{\mathbb{R}^n} |\nabla f_i| \, dx = P(E).$$

□

References

[1] L. Ambrosio and S. Di Marino, Equivalent definitions of BV space and of total variation on metric measure spaces. J. Funct. Anal. 266 (2014), no. 7, 4150–4188.

[2] L. Ambrosio, S. Di Marino and G. Savaré, On the duality between p-modulus and probability measures. J. Eur. Math. Soc. (JEMS) 17 (2015), no. 8, 1817–1853.

[3] L. Ambrosio, N. Fusco and D. Pallara, *Functions of bounded variation and free discontinuity problems.* Oxford Mathematical Monographs, Calendron Press, New York 2000.

[4] L. Ambrosio, N. Gigli and G. Savaré, Density of Lipschitz functions and equivalence of weak gradients in metric measure spaces. Rev. Mat. Iberoam. 29 (2013), no. 3, 969–996.

[5] A. Björn, J. Björn, *Nonlinear Potential Theory on Metric Spaces.* EMS Tracts in Mathematics, 17. European Mathematical Society (EMS), Zurich, 2011.

[6] A. Björn, J. Björn and J. Malý, Quasiopen and p-Path Open Sets, and Characterizations of Quasicontinuity. Potential Anal. 46 (2017), no. 1, 181–199.

[7] M. Chlebík, *Going beyond variation of sets.* arXiv:1611.05931v1 [math.MG] 17 Nov 2016.

[8] L. C. Evans and R. Gariepy, *Measure Theory and Fine Properties of Functions.* Studies in Advanced Mathematics, CRC Press, Boca Raton, 1992.

[9] H. Federer, *Geometric Measure Theory.* Springer–Verlag, 1969.

[10] B. Fuglede, Extremal length and functional completion. Acta Math. 98 (1957), 171–219.

[11] V. Honzlová Exnerová, J. Malý and O. Martio, Modulus in Banach Function Spaces. Ark. Mat. 55.1 (2017), 105-130.

[12] E. Järvenpää, M. Järvenpää, K. Rogovin, S. Rogovin and N. Shanmugalingam, Measurability of equivalence classes and MECp-property in metric spaces. Rev. Mat. Iberoam. 23 (2007), no. 3, 811–830.

[13] A. S. Kechris, *Classical Descriptive Set Theory.* Graduate Texts in Mathematics, Springer, 1995.

[14] O. Martio, Functions of bounded variation and curves in metric measure spaces. Adv. Calc. Var., Vol. 9, Issue 4 (2016), 305-322.

[15] O. Martio, The space of functions of bounded variation on curves in metric measure spaces. Conform. Geom. Dyn. 20 (2016), 81-96.

[16] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces: Fractals and Rectifiability.* Cambridge University Press, 1995.

[17] W. Sierpiński, Sur un problème concernant les ensembles mesurables superficiellement, Fundamenta Mathematicae, Vol. 1, Issue 1 (1920), 112-115.

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