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# Linear Time Maximum Segmentation Problems in Column Stream Model\*

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**Abstract.** We study a lossy compression scheme linked to the biological problem of founder reconstruction: The goal in founder reconstruction is to replace a set of strings with a smaller set of founders such that the original connections are maintained as well as possible. A general formulation of this problem is NP-hard, but when limiting to reconstructions that form a segmentation of the input strings, polynomial time solutions exist. We proposed in our earlier work (WABI 2018) a linear time solution to a formulation where minimum segment length was bounded, but it was left open if the same running time can be obtained when the targeted compression level (number of founders) is bounded and lossyness is minimized. This optimization is captured by the Maximum Segmentation problem: Given a threshold  $M$  and a set  $\mathcal{R} = \{\mathcal{R}_1, \dots, \mathcal{R}_m\}$  of strings of the same length  $n$ , find a minimum cost partition  $P$  where for each segment  $[i, j] \in P$ , the *compression level*  $|\{\mathcal{R}_k[i, j] : 1 \leq k \leq m\}|$  is bounded from above by  $M$ . We give linear time algorithms to solve the problem for two different (compression quality) measures on  $P$ : the average length of the intervals of the partition and the length of the minimal interval of the partition. These algorithms make use of positional Burrows–Wheeler transform and the range maximum queue, an extension of range maximum queries to the case where the input string can be operated as a queue. For the latter, we present a new solution that may be of independent interest. The solutions work in a streaming model where one column of the input strings is introduced at a time.

**Keywords:** Pan-genome indexing, founder reconstruction, dynamic programming, positional Burrows–Wheeler transform, range maximum queue

## 1 Introduction

Given a *set of recombinants*  $\mathcal{R} = \{\mathcal{R}_1, \dots, \mathcal{R}_m\}$ , i.e. a set of strings of the same length, a *set of founders* is a set of strings of the same length where all the recombinants can be mapped on the founders for the common positions, i.e. for each position  $k$  all the characters at the position  $k$  of the recombinants are included in all the characters at the position  $k$  of the founders.

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Minimizing the number of *crossovers*, i.e. positions where recombinants need to change between mapped founders, corresponds to the problem *founder sequence reconstruction* [9,8,2]. As this problem is NP-hard [8,2], Ukkonen suggested taking a polynomial variant of this problem through construction of a *segmentation* [9]. Here a segmentation is a decomposition of the set of recombinants into blocks. For a partition  $P$ , the corresponding segmentation corresponds to a set of *segments* where for  $[i, j] \in P$ , the segment of the interval  $[i, j]$  is the set of strings  $\{\mathcal{R}_k[i, j] : 1 \leq k \leq m\}$ .

Ukkonen proposed three different measures for segmentations of recombinants:  $\lambda_{min}$  (the minimum size of the intervals),  $\lambda_{ave}$  (the ratio between the length of a string of  $\mathcal{R}$  and the number of segments) and  $\lambda_{max}$  (the maximum of segment sizes) and gave an optimal solution in  $O(mn^2)$  time where  $\lambda_{min}$  is bounded by a given user-defined value  $M$  and  $\lambda_{max}$  is minimized (*Minimum Segmentation problem*) and an optimal solution in  $O(mn)$  time where  $\lambda_{ave}$  is bounded and  $\lambda_{max}$  is minimized [9]. Norri et al. improved the first result to  $O(mn)$  time by using the *positional Burrows–Wheeler transform* (pBWT) [3]. This problem corresponds to finding the segmentation that minimizes the maximal segment size where each interval of the corresponding partition have length bigger than a threshold  $K$ .

Ukkonen [9] proved that we can find in  $O(n(m + M^3))$  time a set of founders which minimizes the number of crossovers in the case where all the segments of the segmentation have the same size  $M$ . Norri et al. [7] apply the algorithm of Ukkonen to the case where the segments have different sizes; experimental results show that this approach works well in practice, although optimality is not guaranteed in this case.

Instead of minimizing  $\lambda_{max}$  where  $\lambda_{min}$  is bounded as in the Minimum Segmentation problem, in this paper we study the problem where  $\lambda_{max}$  is bounded and we want to maximize  $\lambda_{min}$  or  $\lambda_{ave}$ . In other words, we take the dual problem of Minimum Segmentation problem where we bound the maximum size of segments, i.e. the number of founders, and we want to optimize the partition corresponding to this segmentation (either maximize the size of the minimal interval of the partition or minimize the number of intervals). This formulation is motivated by the ability to control the size of the *pan-genome index* proposed in [10]: Multiple alignment of thousands of human genomes can be replaced with a multiple alignment of founders. We demonstrated in [7] that reduction from 5009 sequences to 130 founders gives average distance 9358 bases between two crossovers. Such preservation of continuity is sufficient for the approach in [10]. With our new formulation, we can directly control the target number of founders without needing to try out different bounds for the segment length. In more general terms, the approach can be seen as a lossy compression scheme, where the targeted compression level (number of founders) is fixed and the *compression quality* (preservation of continuities) is optimized. Such general formulation might find other applications beyond genome research.

We propose two different algorithms to solve maximization of  $\lambda_{min}$  and  $\lambda_{ave}$ , respectively. In Section 3, we give a greedy algorithm which finds an optimal solution in  $O(mn)$  for the first problem and in Section 4, we present a dynamic

programming algorithm using a *Range Maximum Queue* to solve the second problem in linear time. The Range Maximum Queue is an extension of *min-queue* [5]. A min-queue supports minimum value queries on queue in constant time, with the update operations taking also constant time. Our structure supports querying maximum value over a range of items in the queue in constant time, with the update operations taking amortized constant time.

We consider all of the Maximum Segmentation problems in a specific streaming data model. As every input of the Maximum Segmentation problems consists of a bound and a set of strings of the same length, we define the *Column Stream Model* such that we are given the bound and a stream that yields the set of strings of the same length, column-by-column. In essence, at the  $k^{th}$  step we have the  $k^{th}$  character of each input string. A justification of this model can be found in Appendix.

## 2 Preliminaries

In this section we present the problems of Maximum Segmentation and some terminologies we will need.

To begin we define some notations for strings and sets. Given a *string*  $w = a_1 \dots a_n$ , the *length* of  $w$ , denoted by  $|w|$ , is  $n$ , the  $i^{th}$  element of  $w$ , denoted by  $w[i]$ , is  $a_i$  and the *substring* denoted by  $w[i, j]$  is  $a_i \dots a_j$ . We use an analogous notation for the arrays. Given two integers  $i$  and  $j$  with  $i \leq j$ , we denote by  $[i, j]$  the set of integers between  $i$  and  $j$ , i.e.  $\{k \in \mathbb{Z} : i \leq k \leq j\}$ . Given a finite set  $S$ , a *partition*  $P = \{S_1, \dots, S_k\}$  is a set of subsets of  $S$  such that  $\cup_{S_i \in P} S_i = S$  and  $i \neq j \Rightarrow S_i \cap S_j = \emptyset$ . The *cardinality* of  $S$  is denoted by  $|S|$ .

The input of our problems is a set of *recombinants*  $\mathcal{R} = \{\mathcal{R}_1, \dots, \mathcal{R}_m\}$  which is a set of  $m$  strings of the same length  $n$  ( $|\mathcal{R}_1| = \dots = |\mathcal{R}_m| = n$ ). In what follows, we use  $m$  as the number of strings of  $\mathcal{R}$  and  $n$  as the length of each string of  $\mathcal{R}$ .

For an integer interval  $[i, j]$  with  $1 \leq i \leq j \leq n$ , we denote by  $\mathcal{R}[i, j]$  the set of all the substrings  $\mathcal{R}_k[i, j]$  with  $k \in [1, m]$ , i.e.  $\mathcal{R}[i, j] = \{\mathcal{R}_k[i, j] : k \in [1, m]\}$ . Given a partition  $P$  of  $[1, n]$ , we define the following three measures:  $\lambda_{min}(\mathcal{R}, P) = \min_{[i, j] \in P} |[i, j]|$ ,  $\lambda_{ave}(\mathcal{R}, P) = \frac{n}{|P|}$  and  $\lambda_{max}(\mathcal{R}, P) = \max_{[i, j] \in P} |\mathcal{R}[i, j]|$ . When there is no confusion, we just use the notations  $\lambda_{min}$ ,  $\lambda_{ave}$  and  $\lambda_{max}$ .

*Example 1.* The set  $\mathcal{R}$  of 6 recombinants of size 10:

1	2	3	4	5	6	7	8	9	10
0	1	1	2	2	1	0	2	2	1
0	1	1	2	1	2	0	1	0	1
2	1	0	2	1	2	0	2	1	0
0	2	1	2	2	1	0	2	2	1
2	1	0	2	2	1	0	2	2	1
0	2	1	2	1	2	0	1	0	1

By taking the partition  $P_1 = \{[1, 3], [4, 7], [8, 10]\}$ ,

1 2 3	4 5 6 7	8 9 10
0 1 1	2 2 1 0	2 2 1
0 1 1	2 1 2 0	1 0 1
2 1 0	2 1 2 0	2 1 0
0 2 1	2 2 1 0	2 2 1
2 1 0	2 2 1 0	2 2 1
0 2 1	2 1 2 0	1 0 1

one has  $\lambda_{min}(\mathcal{R}, P_1) = \min\{|[1, 3]|, |[4, 7]|, |[8, 10]|\} = \min\{3, 4, 3\} = 3$ ,  $\lambda_{ave}(\mathcal{R}, P_1) = \frac{10}{3}$  and  $\lambda_{max}(\mathcal{R}, P_1) = \max\{|\mathcal{R}[1, 3]|, |\mathcal{R}[4, 7]|, |\mathcal{R}[8, 10]|\} = \max\{3, 2, 3\} = 3$ .

By taking the partition  $P_2 = \{[1, 2], [3, 6], [7, 8], [9, 10]\}$ ,

1 2	3 4 5 6	7 8	9 10
0 1	1 2 2 1	0 2	2 1
0 1	1 2 1 2	0 1	0 1
2 1	0 2 1 2	0 2	1 0
0 2	1 2 2 1	0 2	2 1
2 1	0 2 2 1	0 2	2 1
0 2	1 2 1 2	0 1	0 1

one has  $\lambda_{min}(\mathcal{R}, P_2) = \min\{|[1, 2]|, |[3, 6]|, |[7, 8]|, |[9, 10]|\} = \min\{2, 4, 2, 2\} = 2$ ,  $\lambda_{ave}(\mathcal{R}, P_2) = \frac{10}{4} = 2.5$  and  $\lambda_{max}(\mathcal{R}, P_2) = \max\{|\mathcal{R}[1, 2]|, |\mathcal{R}[3, 6]|, |\mathcal{R}[7, 8]|, |\mathcal{R}[9, 10]|\} = \max\{3, 4, 2, 2\} = 4$ .

**Definition 1.** *The problem of  $\lambda_{min}$ -Maximum Segmentation Partition (or  $\lambda_{min}$ -MSP) is, given a bound  $M$  and a set of recombinants  $\mathcal{R}$ , to find a partition of  $[1, n]$  which maximizes  $\lambda_{min}(\mathcal{R}, P)$  subject to  $\lambda_{max}(\mathcal{R}, P) \leq M$ . The problem of  $\lambda_{min}$ -Maximum Segmentation Length (or  $\lambda_{min}$ -MSL) is, given a bound  $M$  and a set of recombinants  $\mathcal{R}$ , to find the measure  $\lambda_{min}$  for an optimal partition  $P$  of  $\lambda_{min}$ -MSP.*

We denote analogously by  $\lambda_{ave}$ -Maximum Segmentation Partition (or  $\lambda_{ave}$ -MSP) and  $\lambda_{ave}$ -Maximum Segmentation Length (or  $\lambda_{ave}$ -MSL) the similar problems in which  $\lambda_{min}(\mathcal{R}, P)$  is substituted with  $\lambda_{ave}(\mathcal{R}, P)$ .

Hereafter we assume that  $M$  is at least  $\max\{|\mathcal{R}[k, k]| : k \in [1, n]\}$ , otherwise all the Maximum Segmentation problems admit no solution.

*Remark 1.* The notation of  $\lambda_{min}(\mathcal{R}, P)$  and  $\lambda_{ave}(\mathcal{R}, P)$  corresponds to the “ $\lambda_{min}$ ” and “ $\lambda_{ave}$ ” of [9]. The Maximum Segmentation Length problem corresponds to the segmented version of *Maximum fragment length* of [9]. The *Minimum Segmentation problem* of [7] corresponds to finding the measure  $\lambda_{max}$  of a partition  $P$  which minimizes  $\lambda_{max}$  where  $\lambda_{min}$  is bounded from below by an integer in input.

In the two following sections we present different algorithms to find in linear time (in the size of the input) an optimal solution of  $\lambda_{ave}$ -MSP and  $\lambda_{ave}$ -MSL (see Section 3) and  $\lambda_{min}$ -MSP and  $\lambda_{min}$ -MSL (see Section 4).

### 3 $\lambda_{ave}$ -Maximum Segmentation problems

As  $\lambda_{ave} = \frac{n}{|P|}$ , maximizing  $\lambda_{ave}$  corresponds to minimizing  $|P|$ , i.e. the number of intervals of  $P$ . The idea of our algorithm solving the  $\lambda_{ave}$ -Maximum Segmentation problems is to use a greedy algorithm from left to right depending on values given by a special data structure, the *positional Burrows–Wheeler Transform*. We begin by explaining which values we want to compute, what is the positional Burrows–Wheeler Transform and how we can use it to build our values and finally we give a proof of the correctness for our greedy algorithm.

#### 3.1 Optimal solution of $\lambda_{ave}$ -MSP

**Lemma 1.** *Let  $i, j, i'$  and  $j'$  be four integers such that  $i \leq i' \leq j' \leq j$ . We have  $|\mathcal{R}[i', j']| \leq |\mathcal{R}[i, j]|$ .*

*Proof.* The property is due to the fact that each string of  $\mathcal{R}[i', j']$  is a substring of  $\mathcal{R}[i, j]$  on the same interval.  $\square$

Hence, for a fixed  $j$ , the function  $|\mathcal{R}[i, j]|$  is decreasing in  $i$ . For a bound  $M$  and an integer  $k$ , we define  $c_k$  as the value such that  $|\mathcal{R}[c_k, k]| > M$  and  $|\mathcal{R}[c_k + 1, k]| \leq M$ ; in the case  $|\mathcal{R}[1, k]| \leq M$ , we take  $c_k = 0$ .

*Remark 2.* We can equivalently define  $c_k$  as follows:

$$\begin{aligned} c_k &= \min\{j \in [1, k] : |\mathcal{R}[j, k]| \leq M\} - 1 \\ &= \max\{j \in [1, k] : |\mathcal{R}[j, k]| > M\}. \end{aligned}$$

With all the values of  $c_k$  for all  $k \in [1, n]$ , we can build an optimal solution of  $\lambda_{ave}$ -MSP.

**Lemma 2.** *The solution  $P = \{[1, b_p], \dots, [b_3 + 1, b_2], [b_2 + 1, b_1]\}$  is an optimal solution of the  $\lambda_{ave}$ -Maximum Segmentation Partition problem where  $b_1 = n$ ,  $b_{k+1} = c_{b_k}$  for  $k \geq 1$ , and  $c_{b_p} = 0$ .*

*Proof.* The set  $P = \{[1, b_p], \dots, [b_3 + 1, b_2], [b_2 + 1, b_1]\}$  is a partition of  $[1, n]$  because  $b_1 = n$  and  $b_{k+1} = c_{b_k}$ . We are to prove that  $P$  is an optimal solution. Let  $P_{opt} = \{[1, o_{p'}], \dots, [o_2 + 1, o_1]\}$  be an optimal partition with  $o_1 = n$ . We denote  $P_1 = P_{opt}$  and  $P_k = P_{k-1} \setminus ([o_{k+1} + 1, o_k] \cup [o_k + 1, b_{k-1}]) \cup ([o_{k+1} + 1, b_k] \cup [b_k + 1, b_{k-1}])$ . We are going to prove by induction on  $k$  that each  $P_k$  is optimal (for  $k \in [1, p']$ ). The base of induction  $k = 1$  holds by definition ( $P_1 = P_{opt}$ ). We assume that  $P_{k-1}$  is an optimal solution. As  $b_k = c_{b_{k-1}}$ , we have  $o_k \geq b_k$ .

If  $b_k \geq o_{k+1} + 1$ , by Lemma 1, as  $o_{k+1} + 1 \leq b_k \leq o_k$ , we have  $|\mathcal{R}[o_{k+1} + 1, b_k]| \leq |\mathcal{R}[o_{k+1} + 1, o_k]| \leq M$  and thus  $P_k$  is a solution and  $|P_k| = |P_{k-1}|$ . Hence by induction,  $P_k$  is an optimal solution.

If  $b_k < o_{k+1} + 1$ ,  $P_k^* = P_{k-1} \setminus ([o_{k+2} + 1, o_{k+1}] \cup [o_{k+1} + 1, o_k] \cup [o_k + 1, b_{k-1}]) \cup ([o_{k+2} + 1, b_k] \cup [b_k + 1, b_{k-1}])$  is a solution and  $|P_k^*| = |P_{k-1}| - 1$  and thus  $P_{k-1}$  is not optimal which is impossible by induction.

As  $P_{p'}$  is optimal and  $P_{p'} = P$ , the partition  $P$  is an optimal solution.  $\square$

### 3.2 pBWT and linear algorithm for $\lambda_{ave}$ -MSP

Given a string  $T[1, m] = t_1 \dots t_m$ , we denote by  $\overleftarrow{T}$  the string corresponding to the reverse of  $T$ , i.e.  $\overleftarrow{T} = t_m \dots t_1$ . The *positional Burrows–Wheeler Transform* [3] (or pBWT) of a set of recombinants  $\mathcal{R}$  is two sets of  $n$  arrays of size  $m$  – an array  $a_k$  and an array  $d_k$  for all  $k \in [1, n]$  – where for  $k \in [1, n]$ ,  $a_k[1, m]$  is a permutation of  $[1, m]$  such that  $\overleftarrow{\mathcal{R}_{a_k[1]}[1, k]} \leq \dots \leq \overleftarrow{\mathcal{R}_{a_k[m]}[1, k]}$  lexicographically and  $d_k[i] = 1 + \max\{j \in [1, k] : \mathcal{R}_{a_k[i]}[j] \neq \mathcal{R}_{a_k[i-1]}[j]\}$ , for  $i \in [2, m]$  and  $d_k[1] = k + 1$ .

Durbin [3] showed that we can compute recursively  $a_k$  and  $d_k$  from  $a_{k-1}$  and  $d_{k-1}$  in  $O(m)$  time for a binary alphabet and Mäkinen and Norri [6] further generalized the construction for integer alphabets of size  $O(m)$ .

**Lemma 3 ([6]).** *The arrays  $a_k$  and  $d_k$  can be computed from  $a_{k-1}$  and  $d_{k-1}$  in  $O(m)$  time, assuming the input alphabet is  $[0, |\Sigma| - 1]$  with  $|\Sigma| = O(m)$ .*

Norri et al. [7] use three different arrays  $s_k$ ,  $t_k$ ,  $e_k$  to store the array  $d_k$  in increasing sorted order where  $s_k$  contains all distinct elements from  $d_k$  in the increasing sorted order (so that the length of  $s_k$  might be less than  $m$ ),  $e_k$  is the normalized array  $d_k$  where  $s_k[e_k[j]] = d_k[j]$  for all  $j \in [1, m]$  and  $t_k$  is an array of the same length as  $s_k$  such that, for any  $j$ ,  $t_k[j]$  indicates the number of times the value  $s_k[j]$  occurs in  $d_k$ .

**Lemma 4 ([7]).** *The arrays  $a_k$ ,  $s_k$ ,  $e_k$  and  $t_k$  can be computed from  $a_{k-1}$ ,  $s_{k-1}$ ,  $e_{k-1}$  and  $t_{k-1}$  in  $O(m)$  time, assuming the input alphabet is  $[0, |\Sigma| - 1]$  with  $|\Sigma| = O(m)$ .*

With  $s_k$ ,  $e_k$  and  $t_k$  we can redefine  $c_k$ . As  $|\mathcal{R}[s_k[j] - 1, k]| = \sum_{i \in [j, |t_k|]} t_k[i]$ , one has

$$c_k = \max\{j \in [1, |s_k|] : \sum_{i \in [j, |t_k|]} t_k[i] > M\}. \quad (1)$$

With this new definition of  $c_k$  we obtain the following theorem.

**Theorem 1.** *Given a bound  $M$  and a set of recombinants  $\mathcal{R}$ , there is an algorithm that computes an optimal solution of the  $\lambda_{ave}$ -Maximum Segmentation Partition problem in a streaming fashion in  $O(mn)$  time and  $O(m + n)$  space.*

*Proof.* By Lemma 4 and Equation 1, we can build  $c_k$  in  $O(m)$  time for each value  $k \in [1, n]$ . Lemma 2 gives us an optimal solution by using the values  $c_k$ . Finally we build and store all the values  $c_k$  and make a backtracking from  $n$  to 0.  $\square$

### 3.3 Right greedy and linear algorithm for $\lambda_{ave}$ -MSL

Lemma 2 gives us a greedy solution from right to left, a “Left greedy” version. Here we present a “Right greedy” version working from left to right.

**Lemma 5.** *The solution  $P = \{[b_1, b_2 - 1], [b_2, b_3 - 1], \dots, [b_p, n]\}$  is an optimal solution of the  $\lambda_{ave}$ -MSP problem where  $b_1 = 1$  and  $b_{k+1} = \min\{j \in [b_k, n] : c_j \geq b_k\}$ .*

*Proof.* By Lemma 1, we know that for all  $k \in [1, n]$  and for all  $j \in [k, n]$ ,  $c_k \leq c_j$ . Hence, we can adapt the proof of Lemma 2 by extending the optimal solution of the right to prove this result.  $\square$

By using the solution of Lemma 5 instead of Lemma 2, we do not need to store all the array of  $c_k$  to build the solutions of  $\lambda_{ave}$ -MSL and of  $\lambda_{ave}$ -MSP and we can extend the result of Theorem 1.

**Theorem 2.** *Given a bound  $M$  and a set of recombinants  $\mathcal{R}$ , there is an algorithm that computes an optimal solution of the  $\lambda_{ave}$ -Maximum Segmentation Length problem in a streaming fashion in  $O(mn)$  time and  $O(m)$  space. One can also find in  $O(mn)$  time and  $O(m + |P|)$  space the corresponding partition  $P$ , thus solving the  $\lambda_{ave}$ -Maximum Segmentation Partition problem.*

## 4 $\lambda_{min}$ -Maximum Segmentation problems

In this section, we give an  $O(mn)$  time algorithm building an optimal solution of  $\lambda_{min}$ -MSP and  $\lambda_{min}$ -MSL. We focus on solving the problem  $\lambda_{min}$ -MSL by using dynamic programming algorithm; the corresponding partition (solution of  $\lambda_{min}$ -MSP) can be reconstructed by “backtracking” in a standard way (see [9]).

### 4.1 Dynamic programming algorithm

Given an integer  $M$  and a set of recombinants  $\mathcal{R}$ , the  $\lambda_{min}$ -Maximum Segmentation Length problem seeks to maximize the smallest cardinality of the intervals of a partition  $P$  subject to  $\lambda_{max} \leq M$ . In other words, this problem is to compute

$$\max_{P \in \mathcal{P}_{M, \mathcal{R}}} \min\{j - i + 1 : [i, j] \in P\} \quad (2)$$

where  $\mathcal{P}_{M, \mathcal{R}}$  is the set of all partitions  $P$  of  $[1, n]$  such that for all  $[i, j] \in P$ ,  $|\mathcal{R}[i, j]| \leq M$ .

To solve  $\lambda_{min}$ -MSL, we define the following recursion which solves (2):

$$N(k) = \begin{cases} \infty & \text{If } k = 0, \\ \max_{c_k \leq j < k} \min\{N(j), k - j\} & \text{Otherwise.} \end{cases} \quad (3)$$



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**Algorithm 1** The algorithm  $Next(x, k)$ .

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```

1:  $z \leftarrow k - N(x)$ ;
2:  $w \leftarrow \text{Argmax}\{N(u) : x \leq u < z\}$ ;
3: if  $x < z$  and  $N(w) > N(x)$  then
4:   return  $Next(x + 1, k)$ ;
5: else
6:   return  $x$ ;

```

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Given  $k$  between 1 and  $n$ , we denote by  $\text{Previous}(k)$  the set of previous values of  $k$  by (3), i.e.  $\text{Previous}(k) = \text{Argmax}_{c_k \leq j < k} \min\{N(j), k - j\} = \{j \in [c_k, k - 1] : N(k) = \min\{N(j), k - j\}\}$ .

We can exhibit two recursive properties, one on  $c_k$  and one on  $\text{Previous}(k)$  (with Algorithm 1).

**Lemma 6.** *Given  $k \in [1, n - 1]$ , we have*

1.  $c_k \leq c_{k+1}$ ,
2. For all  $j \in \text{Previous}(k)$ ,  $Next(\max\{j, c_{k+1}\}, k + 1) \in \text{Previous}(k + 1)$ .

*Proof.* For 1, we straightforwardly obtain  $c_k \leq c_{k+1}$  due to Lemma 1.

For 2, we begin by proving that for all  $j_k \in \text{Previous}(k)$ , there exists  $j_{k+1} \in \text{Previous}(k + 1)$  with  $j_k \leq j_{k+1}$ . Assume that it is not the case. Let be  $j_k \in \text{Previous}(k)$  such that  $\forall j_{k+1} \in \text{Previous}(k + 1)$ ,  $j_{k+1} < j_k$ . In this case, we have  $c_{k+1} < j_k$  and for all  $j' \in [c_{k+1}, j_k - 1]$ ,  $N(j') \leq N(k) = \min\{N(j_k), k - j_k\}$ . If  $N(j_k) \leq k - j_k$ , we have that  $N(j_k) < k - j_k + 1$  and thus  $N(k + 1) \leq N(k)$ . As  $j_k \in [c_{k+1}, k + 1]$ ,  $j_k$  is an element of  $\text{Previous}(k + 1)$  which is impossible. Otherwise, we have  $N(j_k) > k - j_k$ ,  $N(j_k) \geq k - j_k + 1$  and thus  $j_k \in \text{Previous}(k + 1)$  which is also impossible.

Now, we know that we can search  $j_{k+1}$  in  $[\max\{j_k, c_{k+1}\}, k + 1]$ . In Algorithm 1, we decrease the size of the interval by the left until finding an element of  $\text{Previous}(k + 1)$ . Indeed for  $Next(x, k)$ , if  $N(w) > N(x)$ , there exists  $u \in [x, k - N(x) - 1]$  such that  $N(u) > N(x)$  and thus  $\min\{N(x), k - x\} < \min\{N(u), k - u\}$  and  $x \notin \text{Previous}(k)$ . Otherwise, we know that for all  $u \in [x, k - N(x) - 1]$ ,  $\min\{N(x), k - x\} \geq N(u) \geq \min\{N(u), k - u\}$  and for all  $u \in [k - N(x), k]$ ,  $\min\{N(x), k - x\} \geq k - u \geq \min\{N(u), k - u\}$ . Hence, we have for  $x = Next(\max\{j_k, c_{k+1}\}, k + 1)$ , for all  $u \in [\max\{j_k, c_{k+1}\}, k + 1]$ ,  $\min\{N(x), k + 1 - x\} \geq \min\{N(u), k + 1 - u\}$  and thus  $x \in \text{Previous}(k + 1)$ .  $\square$

**Theorem 3.** *Given a bound  $M$  and a set of recombinants  $\mathcal{R}$ , there is an algorithm that computes an optimal solution of the  $\lambda_{\min}$ -Maximum Segmentation Length problem in a streaming fashion in  $O(nm)$  time and  $O(m + \mathcal{C})$  space where  $\mathcal{C} = \max\{k - c_k + 1 : k \in [1, n]\}$ . Using an additional array of length  $n$ , one can also find in  $O(n)$  time the corresponding partition, thus solving the  $\lambda_{\min}$ -Maximum Segmentation Partition problem.*

*Proof.* By using a Range Maximum Queue (see Lemma 8) on the table of  $N(\cdot)$  initialized in size  $\mathcal{C}$ , we can build one recursive step of  $Next$  (Algorithm 1) in

$O(1)$ . By using the pBWT, we can precompute to find  $\mathcal{C}$  and build all the  $c_k$  in  $O(nm)$  time (see Lemma 4 and Equation 1).

By Lemma 6, we can call  $O(k)$  times the algorithm *Next* to build  $N(k)$ . Hence, we can solve the  $\lambda_{\min}$ -Maximum Segmentation Length in the optimal  $O(nm)$  time.  $\square$

## 4.2 Range Maximum Queue

Our algorithm for solving  $\lambda_{\min}$ -MSL (Theorem 3) requires a Range Maximum Queue data structure. We begin by presenting a semi-dynamic RMQ data structure that can answer RMQ queries on the array  $Q$  in constant time and can “extend”  $Q$  to the right (see Lemma 7).

**Lemma 7.** *There exists a data structure that maintains an integer array  $Q[1, n]$  and supports the append query, which adds a new element to the end of  $Q$  and increments  $n$ , in  $O(1)$  amortized time and the Range Maximum Query, which, for given  $i \in [1, n]$  and  $j \in [i, n]$ , computes a position  $h \in [i, j]$  such that  $Q[h] = \max\{Q[\ell] : i \leq \ell \leq j\}$ , in  $O(1)$  time.*

*Proof.* Our solution is a straightforward modification of the classical static Range Minimum Query approach used in, for instance, [4] and [1].

Let  $n$  be the current length of  $Q$ . Denote  $b = \lceil \frac{\log n}{4} \rceil$ . We split  $Q[1, n]$  into blocks of length  $b$ . As is standard, a Range Maximum Query on  $Q[i, j]$  is reduced to two queries inside blocks and one “block-aligned” query: provided  $i$  and  $j$  belong to different blocks (i.e.,  $\lfloor (i-1)/b \rfloor < \lfloor (j-1)/b \rfloor$ ), the new three query ranges are  $Q[i, i']$ ,  $Q[i'+1, i'']$ ,  $Q[i''+1, j]$  (each might be empty) such that  $i'$  and  $i''$  are multiples of  $b$ ,  $i' - i < b$ , and  $j - i'' < b$ .

To process the query on  $Q[i'+1, i'']$ , we maintain, for each  $k \in [0, \log n]$ , an array  $P_k[1, \lfloor \frac{n}{b} \rfloor]$  storing positions of maximums in ranges of  $2^k$  blocks; more precisely, for  $h \in [1, \lfloor \frac{n}{b} \rfloor]$ , we have  $(h - 2^k)b < P_k[h] \leq hb$  and  $Q[P_k[h]] = \max\{Q[\ell] : (h - 2^k)b < \ell \leq hb\}$ , assuming  $Q[\ell] = +\infty$  for  $\ell \leq 0$ . Then, putting  $k = \lfloor \log((i'' - i')/b) \rfloor$ , the maximum in  $Q[i'+1, i'']$  obviously is  $\max\{Q[P_k[i''/b]], Q[P_k[i'/b + 2^k]]\}$  and we return either  $P_k[i''/b]$  or  $P_k[i'/b + 2^k]$  accordingly. To calculate  $k$  in  $O(1)$  time, we use either a special processor instruction or a precomputed table  $L[1, 2^{\lceil \frac{\log n}{2} \rceil}]$  such that  $L[x] = \lfloor \log x \rfloor$  for  $x \in [1, 2^{\lceil \frac{\log n}{2} \rceil}]$  (hence,  $\lfloor \log x \rfloor = L[x/2^{\lceil \frac{\log n}{2} \rceil}] + \lceil \frac{\log n}{2} \rceil$  for  $x \in [2^{\lceil \frac{\log n}{2} \rceil} + 1, n]$ ). Note that the length of  $L$  is  $O(\sqrt{n})$ .

For “in-block” queries, we maintain an array  $C[1, \lceil \frac{n}{b} \rceil]$  succinctly encoding Cartesian trees for all blocks (see below). The *Cartesian tree*<sup>3</sup> for an array  $A[h, h']$  is a binary tree with vertices  $[h, h']$  whose root is the smallest  $r \in [h, h']$  such that  $A[r] = \max\{A[\ell] : h \leq \ell \leq h'\}$ , and the left (resp., right) child of  $r$  (if any) is the root of the Cartesian tree for  $A[h, r-1]$  (resp.,  $A[r+1, h']$ ). For each  $h \in [1, \lceil \frac{n}{b} \rceil - 1]$ , we encode the Cartesian tree for the block  $Q[(h-1)b+1, hb]$  as a sequence of  $2b$  balanced parentheses and store it as a  $2b$ -bit integer in  $C[h]$  (zero/one bits correspond to opening/closing parentheses);  $C[\lceil \frac{n}{b} \rceil]$  stores the

<sup>3</sup> The original of cartesian tree is for Range Minimum Query.

Cartesian tree for  $Q[(\lceil \frac{n}{b} \rceil - 1)b + 1, n]$ . It is well known that if  $a$  is the lowest common ancestor of two vertices  $p$  and  $q$  ( $p \leq q$ ) in the Cartesian tree for  $A[h, h']$ , then  $A[a] = \max\{A[\ell] : p \leq \ell \leq q\}$ . We precalculate a table  $T[0, 2^{2b} - 1][1, b][1, b]$  such that, given numbers  $p, q \in [1, b]$  and a  $2b$ -bit integer  $x$  encoding the Cartesian tree for an integer array  $A[1, b]$ ,  $T[x][p][q]$  stores the lowest common ancestor of  $p$  and  $q$  in the tree (if  $x$  does not encode any such tree, the value of  $T[x][p][q]$  is undefined). Now, using the table  $T$  and the array  $C$ , one can straightforwardly answer in-block queries in  $O(1)$  time. Note that the size of  $T$  is  $O(2^{2b}b^2) = O(\sqrt{n} \log^{O(1)} n)$ .

It remains to describe how the defined structures are modified. Suppose that a new value is appended to the end of  $Q$  and  $n$  is incremented. If the new  $n$  is a multiple of  $b$ , a new element  $P_k[\lceil \frac{n}{b} \rceil]$  is added to each array  $P_k$ :  $P_0[\lceil \frac{n}{b} \rceil]$  is computed naively in  $O(b)$  time and, for  $k > 0$ ,  $P_k[\lceil \frac{n}{b} \rceil]$  is set to either  $\ell = P_{k-1}[\lceil \frac{n}{b} \rceil]$  or  $\ell' = P_{k-1}[\lceil \frac{n}{b} \rceil - 2^{k-1}]$ , depending on whether  $Q[\ell] < Q[\ell']$ . Thus, we spend  $\Theta(b + \log n) = \Theta(b)$  time to update all  $P_k$ , which is amortized among the previous  $b - 1$  appends to  $Q$  in which  $n$  was not a multiple of  $b$ .

To maintain  $C$ , we utilize a well-known fact that the Cartesian tree for any array  $A[h, h']$  can be constructed in  $O(h' - h)$  time online, i.e., we read  $A[h, h']$  from left to right and, after processing each prefix  $A[h, h'']$ , have the Cartesian tree for  $A[h, h'']$  (e.g., see [4]). The Cartesian tree for the last block  $Q[(\lceil \frac{n}{b} \rceil - 1)b + 1, n]$  of  $Q$  is maintained using this online algorithm and, thus, when  $n$  is incremented, we have a new tree and have to update  $C[\lceil \frac{n}{b} \rceil]$ . To this end, we construct  $b - 1$  tables  $T_j[0, 2^{2j} - 1][1, 2j]$ , for  $j \in [1, b - 1]$ , such that, given  $h \in [1, 2j]$  and a  $2j$ -bit number  $x$  that encodes a tree  $F$  with  $j$  vertices  $[1, j]$  in a balanced parentheses form,  $T_j[x][h]$  contains a  $2(j+1)$ -bit integer encoding a tree obtained from  $F$  by attaching the new vertex  $j + 1$  as a leaf to  $h$  if  $h \leq j$ , or by making  $j + 1$  the new parent of  $h - j$  (so that the old parent of  $h - j$ , if any, is the parent of  $j + 1$ ) if  $h > j$ ;  $T_j[x][h]$  is undefined if  $x$  does not encode any tree. Using the tables  $T_j$  and the online algorithm, one can maintain  $C$  in  $O(n)$  total time. Note that the size of all  $T_j$  is  $O(2^{2b}b^2) = O(\sqrt{n} \log^{O(1)} n)$ .

Finally, if the new value of  $\lceil \log n \rceil$  differs from the old one, we rebuild all structures in a straightforward way: the tables  $T$ ,  $T_j$ , and  $L$  are precomputed in  $O(\sqrt{n} \log^{O(1)} n) = o(n)$  time,  $P_0$  is constructed from  $Q$  in one pass, each array  $P_k$  with  $k \in [1, \log n]$  is computed using  $P_{k-1}$  in  $\Theta(\frac{n}{b}) = \Theta(\frac{n}{\log n})$  time, and the Cartesian trees for all blocks are built in  $O(n)$  total time and encoded in the array  $C$ . Overall, the rebuilding takes  $O(n)$  time. Since this process is initiated only when  $n$  is a power of two, the total time is  $O(n)$  in the end.  $\square$

In our algorithm, all the RMQ queries of the semi-dynamic RMQ data structure are made on a window from left to right. We can see this data structure as a *queue*, i.e. a data structure where we can make insertion at the end and deletion at the beginning in constant time. A *Range Maximum Queue* (or RMQe) is the data structure  $Q$  that supports queue operations and range maximum query, which, for given  $i \in [1, n]$  and  $j \in [i, n]$ , computes a position  $h \in [i, j]$  such that  $Q[h] = \max\{Q[\ell] : i \leq \ell \leq j\}$ , in  $O(1)$  time. If we know the maximum number of elements of the RMQe data structure, we can improve the space



$Q[i, i']$  and  $Q[i''+1, j]$  we need to compute  $T[C[i'/b + \lceil \frac{N}{b} \rceil \text{begin}]] [i + \text{start}] [i']$  and  $T[C[i''/b + \lceil \frac{N}{b} \rceil \text{begin}]] [i''] [j + \text{start}]$ .

To remove an element at the beginning of our RMQe we only need to increase  $\text{start}$  by 1:  $\text{start}$  becomes  $\text{start} + 1$  if  $\text{start} < b - 1$  and otherwise  $\text{start}$  becomes 0 and we update  $\text{begin}$  to  $\text{begin} + \lceil \frac{N}{b} \rceil 1$ .

To add an element at the end, we use the same online algorithm of Lemma 7 (see [4]) to maintain the  $P_k$  arrays and  $C$  by updating the elements of index  $\text{end}$  except the fact that if  $n$  is a multiple of  $b$ , we do not create a new element, we just update  $\text{end}$  to  $\text{end} + \lceil \frac{N}{b} \rceil 1$ . □

The lemma can be further strengthened by removing the requirement of knowing the bound  $N$ : In that case, one needs to consider the case when  $\log n$  changes. Unlike in Lemma 7, series of alternating insertions and deletions can now cause  $\lceil \log n \rceil$  to change at each operation, so the amortization argument cannot be used directly. However, this case can be handled by maintaining all structures for two consecutive  $\lceil \log n \rceil$  values: Consider  $x = \lceil \log n \rceil$  to change into  $x+1$  due to insertion. We build all structures for  $x+1$  as in the proof of Lemma 7, but also keep the structures for  $x$ . All insertions and deletions are applied on both structures until  $\lceil \log n \rceil$  becomes  $x-1$  or  $x+2$ . We then build structures for  $x-1$  and keep structures for  $x-1$  and  $x$ , or build structures for  $x+2$  and keep structures for  $x+1$  and  $x+2$ . Now the  $O(n)$  rebuilding cost can be amortized to the  $O(n)$  work done before it takes place.

## 5 Conclusion

In this article, we described linear algorithms for Maximum Segmentation problems (see Table 1). In our Column Stream Model, we assume that we see our data column by column and thus the time complexity is  $\Omega(mn)$  and the space complexity is  $\Omega(m + \mathcal{X})$  where  $\mathcal{X}$  is the size of the output ( $\mathcal{X}$  is equal to 1 for  $\lambda_{\text{ave-MSL}}$  and  $\lambda_{\text{min-MSL}}$  and the cardinality of the optimal partition for  $\lambda_{\text{ave-MSP}}$  and  $\lambda_{\text{min-MSP}}$ ). All of these algorithms can be applied in the random access data model (without data streams): they give exactly the same time complexities.

Problems	Time complexity	Space complexity	Source
$\lambda_{\text{ave-MSP}}$	$O(mn)$	$O(m +  P )$ where $P$ is an optimal partition	Theorem 1
$\lambda_{\text{ave-MSL}}$	$O(mn)$	$O(m)$	Theorem 2
$\lambda_{\text{min-MSP}}$	$O(mn)$	$O(m + n)$	Theorem 3
$\lambda_{\text{min-MSL}}$	$O(mn)$	$O(m + \max\{k - c_k : k \in [1, n]\})$	Theorem 3

**Table 1.** Summary of Maximum Segmentation problems in the column stream model.

As future work, we plan to implement the algorithms and offer them as new features of our founder reconstruction toolbox.<sup>4</sup>

<sup>4</sup> <https://github.com/tsnorri/founder-sequences>

## References

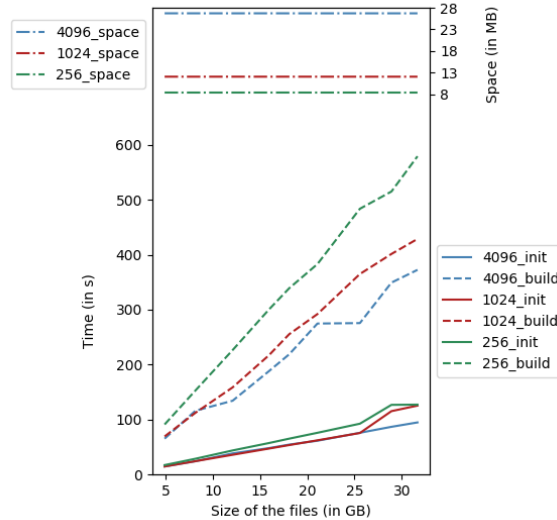
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## Appendix

### About the Column Stream Model

Given an algorithm for a problem with an input  $\mathcal{I}$  and an output  $\mathcal{O}$ , the space complexity of this algorithm corresponds to the space used by  $\mathcal{I}$  and by  $\mathcal{O}$  and the *auxiliary space* which is the temporary space used by this algorithm. Therefore the space complexity is in  $\Omega(|\mathcal{I}| + |\mathcal{O}|)$ . In the case of problems of Maximum Segmentation, all algorithms have a space complexity of  $\Omega(nm)$  where the input is a set of  $m$  strings of size  $n$ . As we want to avoid an auxiliary space of  $\Theta(nm)$  (this could be too big for a computer), we cannot use the random access model. Indeed the random access model corresponds to open all the file in input in the temporary memory. We suggest a specific streaming data model where the set of strings of the same length is seen column by column: the *Column Stream Model*. In this model, the size of the input is in  $\Theta(m)$  which is acceptable.

To prove the realism of this model, we implemented a streaming way to read a file and we tested this implementation with files of different sizes (see Figure 2). The experiments were run on a machine with an Intel Xeon E5-2680 v4 2.4GHz CPU, which has a 35 MB Intel SmartCache. The machine has 256 gigabytes of memory at a speed of 2400MT/s. The code was compiled with `g++` using the `-Ofast` optimization flag.



**Fig. 2.** Time and space complexity to read a set of recombinants depending of the buffer size (256, 1024 and 4096).