University of Helsinki
Faculty of Science
Department of Mathematics and Statistics

Master's Thesis

Geometric proof and applications of the Borsuk-Ulam theorem

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November 2020

HELSINGIN YLIOPISTO — HELSINGFORS UNIVERSITET — UNIVERSITY OF HELSINKI

Tiedekunta/Osasto — Fakultet/Sektion — Faculty	Laitos — Institution — Department
Faculty of Science	Department of Mathematics and Statistics

Tekijä — Författare — Author

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Työn nimi — Arbetets titel — Title

Geometric proof and applications of the Borsuk-Ulam theorem

Oppiaine — Läroämne — Subject

Mathematics

Työn laji — Arbetets art — Level	Aika — Datum — Month and year	Sivumäärä — Sidoantal — Number of pages
Master's Thesis	November 2020	61 pages

Tiivistelmä — Referat — Abstract

This thesis provides a proof and some applications for the famous result in topology called the Borsuk-Ulam theorem. The standard formulation of the Borsuk-Ulam theorem states that for every continuous map from an n-sphere to n-dimensional Euclidean space there are antipodal points that map on top of each other. Even though the claim is quite elementary, the Borsuk-Ulam theorem is surprisingly difficult to prove. There are many different kinds of proofs to the Borsuk-Ulam theorem and nowadays the standard method of proof uses heavy algebraic topology. In this thesis a more elementary, geometric proof is presented.

Some fairly fundamental geometric objects are presented at the start. The basics of affine and convex sets, simplices and simplicial complexes are introduced. After that we construct a specific simplicial complex and present a method, iterated barycentric subdivision, to make it finer. In addition to simplicial complexes, the theory we are using revolves around general positioning and perturbations. Both of these subjects are covered briefly.

A major part in our proof of the Borsuk-Ulam theorem is to show that a certain homotopy function F from a specific $n+1$-manifold to the n-dimensional Euclidean space can be by approximated another map G. Moreover this approximation can be done in a way so that the kernel of G is a symmetric 1-manifold. The foundation for approximating F is laid with iterated barycentric subdivision. The approximation function G is obtained by perturbating F on the vertices of the simplicial complex and by extending it locally affinely. The perturbation is done in a way so that the image of vertices is in a general position.

After proving the Borsuk-Ulam theorem, we present a few applications of it. These examples show quite nicely how versatile the Borsuk-Ulam theorem is. We prove two formulations of the Ham Sandwich theorem. We also deduce the Lusternik-Schnirelmann theorem from the Borsuk-Ulam theorem and with that we calculate the chromatic numbers of the Kneser graphs. The final application we prove is the Topological Radon theorem.

Avainsanat — Nyckelord — Keywords

Borsuk-Ulam theorem, Topology, Geometry, Simplicial complex

Säilytyspaikka — Förvaringsställe — Where deposited

Kumpula Campus Library

Muita tietoja — Övriga uppgifter — Additional information

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1 Introduction

The Borsuk-Ulam theorem is one of the central theorems in topology. The statement of the theorem is easy to understand for anyone with the most basic knowledge of topology. The standard formulation of the Borsuk-Ulam theorem states that for every continuous map $f: \mathbb{S}^n \to \mathbb{R}^n$ there is a point $\boldsymbol{x} \in \mathbb{S}^n$ so that $f(\boldsymbol{x}) = f(-\boldsymbol{x})$. Even though the claim is quite elementary, the Borsuk-Ulam theorem is surprisingly difficult to prove. Nowadays the standard method of proof is to calculate the cohomology rings of real projective spaces $\mathbb{RP}^n$ with $\mathbb{Z}/2\mathbb{Z}$ coefficients. Details for this kind of proof can be found for example in G.E. Bredons book *Topology and Geometry* [1]. Calculating the cohomology ring requires quite a lot of machinery and it is a very abstract process.

In this thesis the reader is provided with a more elementary proof of the Borsuk-Ulam theorem. We follow the geometric proof in Jiří Matoušek's book *Using the Borsuk-Ulam Theorem: Lectures on Topological Methods in Combinatorics and Geometry* [2, page 30]. The proof is originally published in the article *Borsuk's theorem through complementary pivoting* by Imre Bárány [3] and it is presented in quite a similar form in Matoušek's book. There are many more different kinds of proofs to the Borsuk-Ulam theorem. Matoušek presents a few and cites many others.

In addition to having many different kinds of proofs, the Borsuk-Ulam theorem has many formulations. For example Theorems 5.2 and 6.6 are equivalent with the standard form of the Borsuk-Ulam theorem. Once again, many more equivalent forms (and the proofs for the equivalences) can be found in Matoušek's book. Verifying equivalences between different formulations is often a somewhat simple task. On the other hand proving any one of the equivalent forms requires a hefty amount of more work.

Some basic knowledge on topology is expected from the reader. This includes elementary properties of $\mathbb{R}^n$ and concepts like compactness and homeomorphisms. That being said, in Chapter 2 we go through some fairly fundamental things. We introduce affine and convex sets, simplices and simplicial complexes. The chapter is based on Joseph J. Rotman's book *An Introduction to Algebraic Topology* [4].

Chapter 3 is devoted to constructing a sufficient triangulation on the space $\Diamond^n \times [0, 1]$. The notation $\Diamond^n$ is used for the unit sphere of the l_1 norm and $\Diamond^n$ is homeomorphic to $\mathbb{S}^n$ in a natural way. The space $\Diamond^n$ is used, so that we can fit actual simplices on its surface. Thus we avoid the minor technicality of bending the simplices to the round surface $\mathbb{S}^n$.

A major part in our proof of the Borsuk-Ulam theorem is to show that certain continuous map $F: \Diamond^n \times [0, 1] \to \mathbb{R}^n$ can be approximated with another map G so that the kernel of G, that is the preimage $G^{-1}(\mathbf{0})$, is a symmetric 1-manifold. This is the case because $\Diamond^n \times [0, 1]$ is an $n + 1$-manifold. No actual theory on manifolds is used in the thesis. (We do not even define manifolds). Instead we use triangulations and general positioning to prove this. Chapter 3 lays the foundations for the approximation.

In Chapter 4 we define general positioning in a few environments. We prove that with small nudges we can reach a general position in any situation. The idea of general positioning is to avoid certain unwanted behavior that only occurs in some special cases. We add a few more lemmas in Chapter 5 which combined with Chapters 3 and 4 allow us to prove that the kernel of the approximation function G consists of a path and some loops. This enables us to prove the Borsuk-Ulam theorem.

The last chapter is devoted to presenting a few applications of the Borsuk-Ulam theorem. These examples show quite nicely how versatile the Borsuk-Ulam theorem is. It can be used in many fields of mathematics, even in combinatorics which almost exclusively studies finite structures. This is demonstrated in Section 6.3 when we calculate the chromatic numbers of the Kneser graphs.

Finally I want to thank my supervisor Marja Kankaanrinta for the interesting subject, helpful comments and useful sources. The fruitful comments of Pekka Pankka also clarified the text. I am also grateful to my colleagues Jouko Kelomäki, Luukas Hallamaa and Ossi Niemimäki who have provided not only mathematical but also emotional support, which helped me to complete my thesis.

2 Preliminaries

First we establish some notation. Vectors of $\mathbb{R}^n$ are denoted by bold symbols $\boldsymbol{x}, \boldsymbol{y}$, where as the scalars are written with regular symbols c, d. An open ball $\{\boldsymbol{y} \in \mathbb{R}^n : ||\boldsymbol{y} - \boldsymbol{x}|| < \varepsilon\}$ is denoted by $B(\boldsymbol{x}, \varepsilon)$. The notation $\mathbb{S}^n$ refers to the unit sphere of $\mathbb{R}^{n+1}$, that is the set $\{\boldsymbol{x} \in \mathbb{R}^{n+1} : ||\boldsymbol{x}|| = 1\}$. Naturally there is more notation to come, but it will be introduced on the way.

2.1 Affine and convex sets

Definition 2.1. A subset $A \subset \mathbb{R}^n$ is called convex if for every pair $\boldsymbol{x}, \boldsymbol{y} \in A$ the line segment $\{t\boldsymbol{x} + (1-t)\boldsymbol{y} \mid t \in [0,1]\}$ is contained in A. We call $B \subset \mathbb{R}^n$ affine if for every pair $\boldsymbol{x}, \boldsymbol{y} \in B$ the whole line $\{t\boldsymbol{x} + (1-t)\boldsymbol{y} \mid t \in \mathbb{R}\}$ is contained in B.

It follows from the definitions that (possibly infinite) intersections of convex sets are convex and intersections of affine spaces are affine. This leads to our next definitions.

Definition 2.2. Let $A \subset \mathbb{R}^n$. The intersection of all convex sets containing A is called the convex hull of A and denoted as $\operatorname{conv}(A)$. Similarly $\operatorname{aff}(B)$ is the intersection of all affine sets containing B.

Definition 2.3. Let $\boldsymbol{x}_0, \dots, \boldsymbol{x}_k$ be vectors in $\mathbb{R}^n$. The sum

$$\sum_{i=0}^{k} c_i \boldsymbol{x}_i \tag{2.1}$$

where $c_0, \dots, c_k \in \mathbb{R}$ is called a linear combination of $\boldsymbol{x}_0, \dots, \boldsymbol{x}_k$. If in addition $\sum c_i = 1$ we say that 2.1 is an affine combination. In the case where $\sum c_i = 1$ and $c_i \geq 0$ for all i, we call 2.1 a convex combination.

Lemma 2.4. *Let $A \subset \mathbb{R}^n$ be any set. The convex hull of A is the set of all convex combinations of points in A.*

Proof. Denote C as the set of all convex combinations of points in A. The convex hull of A is the smallest convex set containing A. In order to prove that $\operatorname{conv}(A) \subset C$ it is therefore enough to prove that C is convex along with the obvious fact that $A \subset C$.

Let $\boldsymbol{x}, \boldsymbol{y} \in C$. There exist finite sets $M, N \subset A$ and coefficients $c_{\boldsymbol{u}}, d_{\boldsymbol{v}} \geq 0$ such that

$$\boldsymbol{x} = \sum_{\boldsymbol{u} \in M} c_{\boldsymbol{u}} \boldsymbol{u} \quad \text{and} \quad \boldsymbol{y} = \sum_{\boldsymbol{v} \in N} d_{\boldsymbol{v}} \boldsymbol{v}$$

with $\sum_{\boldsymbol{u} \in M} c_{\boldsymbol{u}} = 1 = \sum_{\boldsymbol{v} \in N} d_{\boldsymbol{v}}$. We prove that C is convex by showing that the line segment of $\boldsymbol{x}$ and $\boldsymbol{y}$ is contained in C. For any $t \in [0, 1]$

$$t\boldsymbol{x} + (1-t)\boldsymbol{y} = \sum_{\boldsymbol{u} \in M} t c_{\boldsymbol{u}} \boldsymbol{u} + \sum_{\boldsymbol{v} \in N} (1-t) d_{\boldsymbol{v}} \boldsymbol{v}$$

is a linear combination of vectors in $M \cup N \subset A$. Moreover it is a convex combination as

$$\sum_{\boldsymbol{u} \in M} t c_{\boldsymbol{u}} + \sum_{\boldsymbol{v} \in N} (1-t) d_{\boldsymbol{v}} = 1$$

and the coefficients $tc_{\boldsymbol{u}}$ and $(1-t)d_{\boldsymbol{v}}$ are all non-negative. Hence the line segment of $\boldsymbol{x}$ and $\boldsymbol{y}$ is contained in C. Thus C is convex and $\operatorname{conv}(A) \subset C$.

For all $k \in \mathbb{N} \setminus \{0\}$ denote C_k as the set of all convex combinations of at most k vectors of A. By induction we show that $C_k \subset \operatorname{conv}(A)$ for all $k \in \mathbb{N} \setminus \{0\}$. Let $\boldsymbol{x} \in C_1$. Now

$$\boldsymbol{x} = \sum_{i=1}^{1} c_i \boldsymbol{u}_i$$

for some $c_1 \in \mathbb{R}$ and $\boldsymbol{u}_1 \in A$. Because the combination is convex, $c_1 = 1$. Thus

$$\boldsymbol{x} = \sum_{i=1}^{1} c_i \boldsymbol{u}_i = \boldsymbol{u}_1 \in A \subset \operatorname{conv}(A).$$

proving $C_1 \subset \operatorname{conv}(A)$.

Suppose now that $C_k \subset \operatorname{conv}(A)$. Let $\boldsymbol{x} \in C_{k+1}$. We can write

$$\boldsymbol{x} = \sum_{i=1}^{k+1} c_i \boldsymbol{u}_i$$

for some $\boldsymbol{u}_i \in A$ and $c_i \geq 0$ with $\sum c_i = 1$. Now we consider two cases.

Case 1: $c_{k+1} = 0$. Now $\boldsymbol{x}$ is in fact a convex combination of only k vectors of A, and so $\boldsymbol{x} \in C_k$. Because $C_k \subset \operatorname{conv}(A)$ by assumption, $\boldsymbol{x} \in \operatorname{conv}(A)$.

Case 2: $c_{k+1} > 0$. Now the sum

$$\begin{aligned}\boldsymbol{x} &= \sum_{i=1}^{k} c_i \boldsymbol{u}_i + c_{k+1}\boldsymbol{u}_{k+1} \\ &= (1 - c_{k+1}) \sum_{i=1}^{k} \frac{c_i}{1 - c_{k+1}} \boldsymbol{u}_i + c_{k+1}\boldsymbol{u}_{k+1}\end{aligned}$$

is in the line segment of two points. The first of them is in $\operatorname{conv}(A)$ by induction assumption and we assumed $\boldsymbol{u}_{k+1} \in A \subset \operatorname{conv}(A)$ as well. The convex hull of A is of course convex, so $\boldsymbol{x} \in \operatorname{conv}(A)$.

We have now proven that $C_k \subset \operatorname{conv}(A)$ for all k. Finally

$$C = \bigcup_{k=1}^{\infty} C_k \subset \operatorname{conv}(A)$$

completes the proof for $\operatorname{conv}(A) = C$.

□

Lemma 2.5. *Let $B \subset \mathbb{R}^n$ be any set. The smallest affine set containing B is the set of all affine combinations of points of B.*

Proof. This proof is essentially same as the proof of the previous lemma. The only difference here is that this time the coefficients c_i are allowed to be negative as well. □

Corollary 2.6. *Let $B \subset \mathbb{R}^n$ be any set and $\boldsymbol{y} \in B$. Then*

$$\operatorname{aff}(B) = \boldsymbol{y} + \operatorname{span}\{\boldsymbol{u} - \boldsymbol{y} \mid \boldsymbol{u} \in B\}.$$

Moreover, if $C \subset \mathbb{R}^n$ is affine, then $C = \boldsymbol{x} + L$ for some linear subspace $L \subset \mathbb{R}^n$ and some $\boldsymbol{x} \in \mathbb{R}^n$.

Proof. Any $\boldsymbol{x} \in \operatorname{aff}(B)$ is an affine combination of B, and thus can be written as a finite sum

$$\begin{aligned}\boldsymbol{x} &= \sum_{\boldsymbol{u} \in B} c_{\boldsymbol{u}} \boldsymbol{u} \\ &= \sum_{\boldsymbol{u} \in B} c_{\boldsymbol{u}} \boldsymbol{u} + \boldsymbol{y} - \left(\sum_{\boldsymbol{u} \in B} c_{\boldsymbol{u}} \right) \boldsymbol{y} \\ &= \boldsymbol{y} + \sum_{\boldsymbol{u} \in B} c_{\boldsymbol{u}} (\boldsymbol{u} - \boldsymbol{y}) \\ &\in \boldsymbol{y} + \operatorname{span}\{\boldsymbol{u} - \boldsymbol{y} \mid \boldsymbol{u} \in B\}.\end{aligned}$$

To prove the other direction, any point of $\boldsymbol{y} + \operatorname{span}\{\boldsymbol{u} - \boldsymbol{y} \mid \boldsymbol{u} \in B\}$ can be written as an affine combination of B by the same calculation. For the affine set C, we notice that $C = \operatorname{aff}(C)$. Hence C is of the form $\boldsymbol{x} + L$ by the first claim of this lemma.

□

This corollary states that there are not so many affine sets in $\mathbb{R}^n$. Convex sets are a much richer category of sets. For every subset $U \subset \mathbb{S}^{n-1}$ removing U from the closed unit ball $\overline{B}$ results in a convex set $\overline{B} \setminus U$. It is much harder to classify convex sets as they are in abundance. We shall be mostly looking at convex sets spanned by just finitely many points.

Definition 2.7. A set of vectors $\{\boldsymbol{v}_0, \ldots, \boldsymbol{v}_k\} \subset \mathbb{R}^n$ is called linearly independent, if

$$\sum_{i=0}^{k} c_i \boldsymbol{v}_i = 0$$

and $c_0, \ldots, c_k \in \mathbb{R}$ imply that $c_i = 0$ for all i. We say that $\boldsymbol{u}_0, \ldots, \boldsymbol{u}_k \in \mathbb{R}^n$ are affine independent, if

$$\sum_{i=0}^{k} d_i \boldsymbol{u}_i = 0 \quad \text{and} \quad \sum_{i=0}^{k} d_i = 0$$

with $d_0, \ldots, d_k \in \mathbb{R}$ imply that $d_i = 0$ for all i. We also consider the empty set both affine and linearly independent.

Immediately one can notice that every linearly independent set is also affine independent. Although affine independence at first sight does not seem as that useful, in this thesis it is very much so. Affine independence allows us to define simplices which are at the heart of our proof for the Borsuk-Ulam theorem. In addition to that, affine independence enables us to speak about general positioning, a useful configuration for many proofs to come.

Let us give another characterization for affine independence. Some authors like Rotman [4] actually use this as the definition.

Lemma 2.8. *A set of vectors $\{\boldsymbol{v}_0, \ldots, \boldsymbol{v}_k\} \subset \mathbb{R}^m$ is affine independent if and only if the set $\{\boldsymbol{v}_1 - \boldsymbol{v}_0, \ldots, \boldsymbol{v}_k - \boldsymbol{v}_0\}$ is linearly independent.*

Proof. Suppose that $\boldsymbol{v}_0, \ldots, \boldsymbol{v}_k$ are affine independent. Let $c_1, \ldots, c_k \in \mathbb{R}$ be coefficients with

$$\sum_{i=1}^{k} c_i(\boldsymbol{v}_i - \boldsymbol{v}_0) = \mathbf{0}.$$

Arranging the terms

$$\mathbf{0} = \sum_{i=1}^{k} c_i(\boldsymbol{v}_i - \boldsymbol{v}_0) = \sum_{i=1}^{k} c_i \boldsymbol{v}_i + \left(- \sum_{i=1}^{k} c_i \right) \boldsymbol{v}_0$$

makes it evident that by affine independence $c_i = 0$ for every i. The second part of the proof uses a similar rearranging of the terms and is left out.

□

Lemma 2.9. *Let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \subset \mathbb{R}^m$ be affine independent and denote $A = \operatorname{aff}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)$. Assume also that $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n \subset A$ are affine independent and denote $B = \operatorname{aff}(\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n)$. Then $A = B$.*

Proof. From Corollary 2.6 we get that

$$A = \operatorname{span}(\boldsymbol{v}_2 - \boldsymbol{v}_1, \dots, \boldsymbol{v}_m - \boldsymbol{v}_1) + \boldsymbol{v}_1.$$

and that

$$B = \operatorname{span}(\boldsymbol{w}_2 - \boldsymbol{w}_1, \dots, \boldsymbol{w}_m - \boldsymbol{w}_1) + \boldsymbol{w}_1.$$

Moreover these spanning sets are linearly independent by Lemma 2.8. We denote the linear subspaces $L_1 = \operatorname{span}(\boldsymbol{v}_2 - \boldsymbol{v}_1, \dots, \boldsymbol{v}_m - \boldsymbol{v}_1)$ and $L_2 = \operatorname{span}(\boldsymbol{w}_2 - \boldsymbol{w}_1, \dots, \boldsymbol{w}_m - \boldsymbol{w}_1)$. Writing $\boldsymbol{w}_1$ as an affine combination of $\boldsymbol{v}_1, \dots, \boldsymbol{v}_m$ and arranging the terms shows that $\boldsymbol{v}_1 - \boldsymbol{w}_1 \in L_1$. Since $B \subset A$ we get that

$$L_2 \subset L_1 + \boldsymbol{v}_1 - \boldsymbol{w}_1 = L_1.$$

Also $\dim(L_1) = \dim(L_2)$, so $L_1 = L_2$. Now it follows that

$$\begin{aligned} B &= L_2 + \boldsymbol{w}_1 = L_1 + \boldsymbol{w}_1 = L_1 + \boldsymbol{w}_1 + \boldsymbol{v}_1 - \boldsymbol{w}_1 \\ &= L_1 + \boldsymbol{v}_1 = A, \end{aligned}$$

which proves the claim. □

Lemma 2.9 suggests that dimension could be defined for affine independent sets too. Indeed this is the case, however we do not need dimension for affine independent sets as such, so we do not present it.

Lemma 2.10. *Let $L\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear map and let $A \in \mathbb{R}^n$. Then* $\operatorname{conv}(L(A)) = L(\operatorname{conv}(A))$.

Proof. Let $\boldsymbol{y} \in \operatorname{conv}(L(A))$. By Lemma 2.4 and linearity for some convex coefficients c_i and some $\boldsymbol{x}_i \in A$

$$\boldsymbol{y} = \sum c_i L(\boldsymbol{x}_i) = L\left(\sum c_i \boldsymbol{x}_i\right) \in L(\operatorname{conv}(A)).$$

The proof for the other inclusion is essentially the same.

□

We present two more lemmas, with relatively straight forward proofs that we leave out.

Lemma 2.11. *Let $\boldsymbol{v}_0, \dots, \boldsymbol{v}_k \in \mathbb{R}^m$ be affine independent, $c \neq 0$ and $\boldsymbol{x} \in \mathbb{R}^m$. The sets $\{c\boldsymbol{v}_0, \dots, c\boldsymbol{v}_k\}$ and $\{\boldsymbol{v}_0 + \boldsymbol{x}, \dots, \boldsymbol{v}_k + \boldsymbol{x}\}$ are affine independent*

Lemma 2.12. *Let $\boldsymbol{v}_0, \dots, \boldsymbol{v}_k \in \mathbb{R}^m$ be affine independent and $\boldsymbol{v}_{k+1} \in \mathbb{R}^m$ such that $\boldsymbol{v}_{k+1} \notin \operatorname{aff}(\boldsymbol{v}_0, \dots, \boldsymbol{v}_k)$. Then $\boldsymbol{v}_0, \dots, \boldsymbol{v}_{k+1}$ are affine independent.*

2.2 Simplices and simplicial complexes

Definition 2.13. Let $V = \{\boldsymbol{v}_0, \dots, \boldsymbol{v}_m\} \subset \mathbb{R}^n$ be an affine independent set. We say that $\sigma = \operatorname{conv}(V)$ is an m-simplex. The points $\boldsymbol{v}_0, \dots, \boldsymbol{v}_m$ are called vertices of σ and we denote $\operatorname{vert}(\sigma) = V$. For any subset of vertices $F \subset V$ we call $\operatorname{conv}(F)$ a face of σ. The dimension of a simplex σ is the number of vertices minus one and denoted with $\dim(\sigma)$.

The 0-simplices are points, 1-simplices are line segments, 2-simplices are triangles and so on. We also consider the empty set a simplex to simplify some technicalities along the way. The dimension of the empty set is -1. Naturally a face of a simplex is a simplex itself due to the fact that a subset of an affine independent set is also affine independent.

Definition 2.14. Let $\sigma \subset \mathbb{R}^m$ be an n-simplex. The relative boundary of σ is the union of all its $n-1$-faces. Denote the relative boundary of σ with B. The relative interior of σ is the set $\sigma \setminus B$.

The relative boundary and interior of a simplex $\sigma \subset \mathbb{R}^m$ actually are the topological interior and boundary of σ when we consider σ as a subset of the space $\operatorname{aff}(\sigma)$. However, if $\dim(\sigma) < m$, then the topological interior of σ in the underlying space $\mathbb{R}^m$ is empty.

Lemma 2.15. *Let $\boldsymbol{v}_0, \dots, \boldsymbol{v}_m \in \mathbb{R}^n$ be affine independent. For every $\boldsymbol{x} \in \operatorname{aff}(\boldsymbol{v}_0, \dots, \boldsymbol{v}_m)$ there are unique coefficients $c_0, \dots, c_m \in \mathbb{R}$ such that*

$$\boldsymbol{x} = \sum_{i=0}^{m} c_i \boldsymbol{v}_i \quad \textit{and} \quad \sum_{i=0}^{m} c_i = 1.$$

Proof. We already showed in Lemma 2.5 that there exist some coefficients . Let us now prove the uniqueness. Suppose that

$$\sum_{i=0}^{m} c_i \boldsymbol{v}_i = \boldsymbol{x} = \sum_{i=0}^{m} d_i \boldsymbol{v}_i$$

where $c_i, d_i \in \mathbb{R}$ and $\sum c_i = 1 = \sum d_i$. Subtracting the terms gives us

$$\sum_{i=0}^{m} (c_i - d_i) \boldsymbol{v}_i = 0 \quad \text{and} \quad \sum_{i=0}^{m} (c_i - d_i) = 0.$$

The vectors $\boldsymbol{v}_0, \dots, \boldsymbol{v}_m$ are affine independent, so $c_i - d_i = 0$ for all i. Hence the coefficients $c_0, \dots, c_m$ are unique.

□

Definition 2.16. Let $\boldsymbol{v}_0, \dots, \boldsymbol{v}_m \in \mathbb{R}^n$ be affine independent For every $\boldsymbol{x} \in \operatorname{aff}(\boldsymbol{v}_0, \dots, \boldsymbol{v}_m)$ the unique coefficients found in Lemma 2.15 are called barycentric coordinates of $\boldsymbol{x}$ (with respect to $\boldsymbol{v}_0, \dots, \boldsymbol{v}_m$). The projection maps $\lambda_i\colon \operatorname{aff}(\boldsymbol{v}_0, \dots, \boldsymbol{v}_m) \to \mathbb{R}$ that pick a single barycentric coordinate are called barycentric projections.

The barycentric coordinates are mostly used in a simplex, but at times they can be useful for the whole affine space. For a point $\boldsymbol{x}$ inside a simplex σ the barycentric coordinates are always non-negative.

Lemma 2.17. *Barycentric projections are continuous.*

Proof. Let $\boldsymbol{v}_0, \ldots, \boldsymbol{v}_m \in \mathbb{R}^n$ be affine independent and denote $V = \text{aff}(\boldsymbol{v}_0, \ldots, \boldsymbol{v}_m)$. Denote the linear subspace $W = \text{span}(\boldsymbol{v}_1 - \boldsymbol{v}_0, \ldots, \boldsymbol{v}_m - \boldsymbol{v}_0)$. Corollary 2.6 states that

$$V = \boldsymbol{v}_0 + W$$

and Lemma 2.8 that the vectors $\boldsymbol{v}_1 - \boldsymbol{v}_0, \ldots, \boldsymbol{v}_m - \boldsymbol{v}_0$ form a basis for W.

First we can define a translation map $g \colon V \to W,\ g(\boldsymbol{x}) = \boldsymbol{x} - \boldsymbol{v}_0$. For every $i > 0$ we define the map $h_i \colon W \to \mathbb{R}$ on the basis with

$$\begin{aligned} &h_i(\boldsymbol{v}_i - \boldsymbol{v}_0) = 1 \\ &h_i(\boldsymbol{v}_j - \boldsymbol{v}_0) = 0 &&\text{whenever } i \neq j. \end{aligned}$$

and by extending linearly. Finally we define h_0 with

$$h_0(\boldsymbol{x}) = 1 - \sum_{i=1}^{n} h_i(\boldsymbol{x}).$$

Now all maps $h_i \circ g$ are continuous. More importantly $h_i \circ g$ gives us the ith barycentric coordinates for any $\boldsymbol{x} \in V$. Because the coordinates are unique, it follows that the barycentric projections are continuous.

□

Corollary 2.18. *A simplex is a compact subset of the underlying Euclidean space.*

Proof. Let $\sigma \subset \mathbb{R}^n$ be a simplex with $\text{vert}(\sigma) = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{m+1}\}$. Denote λ_i the barycentric projection to $\boldsymbol{v}_i$ and define a map $\lambda \colon \text{aff}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{m+1}) \to \mathbb{R}^{m+1}$ coordinate-wise by $\lambda = (\lambda_1, \ldots, \lambda_{m+1})$. We can now write the simplex σ as a preimage of a closed set

$$\sigma = \lambda^{-1}\{(x_1, \ldots, x_{m+1}) \in \mathbb{R}^{m+1} \mid x_i \geq 0 \text{ for all } i\}.$$

This means that σ is closed in $\text{aff}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{m+1})$. Since affine spaces are linear subspaces translated with some vector, we know that $\text{aff}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{m+1})$ is closed in $\mathbb{R}^n$. This proves that σ is closed.

Let $\boldsymbol{x} \in \sigma$. Writing $\boldsymbol{x}$ as a convex combination, and using the triangle inequality gives us that

$$||\boldsymbol{x}|| = ||\sum_i c_i \boldsymbol{v}_i|| \leq \sum_i c_i ||\boldsymbol{v}_i|| \leq \max_i ||\boldsymbol{v}_i||.$$

Hence σ is bounded.

□

Lemma 2.19. *Let σ be a simplex in $\mathbb{R}^n$ and let A, B be faces of σ. Then $A \cap B$ is a simplex and $A \cap B$ is a face of both A and B.*

Proof. We prove that

$$\operatorname{conv}(A \cap B) = \operatorname{conv}(A) \cap \operatorname{conv}(B),$$

from which the statements in the lemma follow. Because $\operatorname{conv}(A) \cap \operatorname{conv}(B)$ is convex and it contains $A \cap B$, we get that $\operatorname{conv}(A \cap B) \subset \operatorname{conv}(A) \cap \operatorname{conv}(B)$. To prove the other inclusion we need the fact that $A \cup B \subset V$ is an affine independent set.

Let $\boldsymbol{y} \in \operatorname{conv}(A) \cap \operatorname{conv}(B)$. We can now write

$$\sum_{\boldsymbol{u} \in A} c_{\boldsymbol{u}} \boldsymbol{u} = \boldsymbol{y} = \sum_{\boldsymbol{v} \in B} d_{\boldsymbol{v}} \boldsymbol{v}$$

where $\sum_{\boldsymbol{u} \in A} c_{\boldsymbol{u}} = 1 = \sum_{\boldsymbol{v} \in B} d_{\boldsymbol{v}}$ and $c_{\boldsymbol{u}}, d_{\boldsymbol{v}} \geq 0$. Arranging the terms we get to the form

$$\boldsymbol{0} = \sum_{\boldsymbol{u} \in A} c_{\boldsymbol{u}} \boldsymbol{u} - \sum_{\boldsymbol{v} \in B} d_{\boldsymbol{v}} \boldsymbol{v} = \sum_{\boldsymbol{u} \in A \setminus B} c_{\boldsymbol{u}} \boldsymbol{u} + \sum_{\boldsymbol{v} \in B \setminus A} -d_{\boldsymbol{v}} \boldsymbol{v} + \sum_{\boldsymbol{t} \in A \cap B} (c_{\boldsymbol{t}} - d_{\boldsymbol{t}}) \boldsymbol{t},$$

where all the coefficients sum up to 0. The set $A \cup B$ is affine independent, so $c_{\boldsymbol{u}} = 0$ for all $\boldsymbol{u} \in A \setminus B$ and $d_{\boldsymbol{v}} = 0$ for all $\boldsymbol{v} \in B \setminus A$. Hence the only coefficients that can be non-zero are the ones corresponding to points in $A \cap B$. Thus $\boldsymbol{y} \in \operatorname{conv}(A \cap B)$ and so $\operatorname{conv}(A \cap B) \subset \operatorname{conv}(A) \cap \operatorname{conv}(B)$.

□

Definition 2.20. Let $\mathcal{F}$ be a finite collection of simplices in $\mathbb{R}^n$. We say that $\mathcal{F}$ is a simplicial complex if two conditions hold.

1. For every $\sigma \in \mathcal{F}$ all the faces of σ are contained in $\mathcal{F}$.

2. If $A, B \in \mathcal{F}$ then the intersection $A \cap B$ is a face of both A and B and it is contained in $\mathcal{F}$ as well.

The union of all simplices of $\mathcal{F}$ is called the polyhedron of $\mathcal{F}$ and denoted by $|\mathcal{F}|$. For a subspace X of $\mathbb{R}^n$ we say that $\mathcal{F}$ is a triangulation of X if $|\mathcal{F}| = X$. The dimension of a simplicial complex $\mathcal{F}$ is defined by $\dim(\mathcal{F}) = \max\{\dim(\sigma) \mid \sigma \in \mathcal{F}\}$. For a simplicial complex $\mathcal{F}$ we use notation $\operatorname{vert}(\mathcal{F})$ to refer to the union $\bigcup_{\sigma \in \mathcal{F}} \operatorname{vert}(\sigma)$.

With this definition there are not so many subspaces of $\mathbb{R}^n$ that we can triangulate. Because we defined simplicial complexes to only have finitely many simplices, our polyhedra are always compact. Moreover we cannot make smooth round surfaces like the sphere $\mathbb{S}^n$ or a closed ball with simplices.

Both of these limitations can be overcome by extending the definitions of a simplicial complex and a triangulation. For example in [1] a simplicial complex is allowed to have more than a finite amount of simplices. Also a polyhedron is there defined to be any space homeomorphic to the union of simplices of a complex. That definition allows the triangulations of many more spaces including the common $\mathbb{R}^n$ and $\mathbb{S}^n$. Even so, we stick to simpler definitions as they are sufficient for our purposes.

Next we prove a lemma that helps us to show that a collection of simplices is indeed a simplicial complex.

Lemma 2.21. *Let $\mathcal{M}$ be a set of simplices in $\mathbb{R}^n$ such that for all $\sigma_1, \sigma_2 \in \mathcal{M}$ the intersection $\sigma_1 \cap \sigma_2$ is a face of both σ_1 and σ_2. Denote $\mathcal{F}$ as the set of simplices of $\mathcal{M}$ and all of their faces. Then $\mathcal{F}$ is a simplicial complex.*

Proof. Let A be a simplex in $\mathcal{F}$. Now A is a face of some $\sigma \in \mathcal{M}$. The collection $\mathcal{F}$ contains all of the faces of σ. Because the faces of A are a subset of the faces of σ, the collection $\mathcal{F}$ contains all of the faces of σ as well.

Now let A and B be simplices in $\mathcal{F}$. There exists $\sigma_1, \sigma_2 \in \mathcal{M}$ such that A is a face of σ_1 and B is a face of σ_2. Because A and $\sigma_1 \cap \sigma_2$ are faces of σ_1, their intersection $A \cap \sigma_1 \cap \sigma_2$ is a face of $\sigma_1 \cap \sigma_2$ by Lemma 2.19. Moreover we assumed that $\sigma_1 \cap \sigma_2$ is a face of σ_2.

"Being a face of" is a transitive relation among simplices so $A \cap \sigma_1 \cap \sigma_2$ is a face of σ_2. Once again we can use Lemma 2.19 to state that the intersection $(A \cap \sigma_1 \cap \sigma_2) \cap B = A \cap B$ is a face of B. By switching the roles of A and B we can also prove that $A \cap B$ is a face of A as well. Thus $\mathcal{F}$ is a simplicial complex. □

3 Triangulating the space $\Diamond^n \times [0, 1]$

Let us immediately clear up our notation.

Definition 3.1. Denote $|| \cdot ||_1$ as the l_1-norm in $\mathbb{R}^{n+1}$ defined by $||\boldsymbol{x}||_1 = \sum_{i=1}^{n+1} |x_i|$. We use the symbol $\Diamond^n = \{\boldsymbol{x} \in \mathbb{R}^{n+1} : ||\boldsymbol{x}||_1 = 1\}$ for the unit sphere of the l_1-norm.

Lemma 3.2. *There exists a homeomorphism $h \colon \Diamond^n \to \mathbb{S}^n$ such that $h(-\boldsymbol{x}) = -h(\boldsymbol{x})$.*

Proof. Define $h(\boldsymbol{x}) = \boldsymbol{x}/||\boldsymbol{x}||$. This map is bijective and continuous. Because $\Diamond^n$ is compact, it is a homeomorphism. □

The spaces $\Diamond^n$ and $\mathbb{S}^n$ are essentially the same, as the previous lemma points out. Despite of that we use $\Diamond^n$ instead of $\mathbb{S}^n$ in proving the Borsuk-Ulam theorem. The reason behind this is that $\Diamond^n$ has flat surfaces which are easy to triangulate. If we wanted to use $\mathbb{S}^n$ we would have to bend the simplices. It would also be doable, but our method is arguably simpler. Without having to bend simplices with a homeomorphism, we can use the narrow Definition 2.20 for a simplicial complex.

3.1 Constructing a symmetric triangulation

In this section we have a simple but a relatively lengthy goal. We want to construct a simplicial complex $\mathcal{G}$, such that $|\mathcal{G}| = \Diamond^n \times [0,1]$. After constructing $\mathcal{G}$, we shall prove that it is also symmetric with respect to an antipodality on $\Diamond^n \times [0,1]$. More on that later.

Definition 3.3. Denote $\boldsymbol{e}_1, \dots, \boldsymbol{e}_{n+1}$ as the canonical base vectors of $\mathbb{R}^{n+1}$ with $\boldsymbol{e}_i$ having 1 at coordinate i and 0 at other coordinates. Let $S = \{-1, 1\}^{n+1}$ be the set of all binary $(n+1)$-tuples of signs. For every $\boldsymbol{s} = (s_1, \dots, s_{n+1}) \in S$ we define the n-simplex $\sigma^{\boldsymbol{s}} = \operatorname{conv}\{s_1\boldsymbol{e}_1, \dots, s_{n+1}\boldsymbol{e}_{n+1}\}$. The collection of all these simplices is denoted by $\mathcal{M}^n = \{\sigma^{\boldsymbol{s}} \mid \boldsymbol{s} \in S\}$.

Lemma 3.4. *If $\sigma^{\boldsymbol{s}}, \sigma^{\boldsymbol{t}} \in \mathcal{M}^n$ then $\sigma^{\boldsymbol{s}} \cap \sigma^{\boldsymbol{t}}$ is a face of both $\sigma^{\boldsymbol{s}}$ and $\sigma^{\boldsymbol{t}}$.*

Proof. Let $\sigma^{\boldsymbol{s}}$ and $\sigma^{\boldsymbol{t}}$ be simplices of $\mathcal{M}^n$ where $\boldsymbol{s} = (s_1, \dots, s_{n+1})$ and $\boldsymbol{t} = (t_1, \dots, t_{n+1})$ are binary tuples. Define

$$A = \operatorname{conv}\{s_i\boldsymbol{e}_i \mid i \in \{1, \dots, n+1\},\ s_i = t_i\}.$$

We want to prove that $\sigma^{\boldsymbol{s}} \cap \sigma^{\boldsymbol{t}} = A$, from which we can deduce that $\sigma^{\boldsymbol{s}} \cap \sigma^{\boldsymbol{t}}$ is a face of both $\sigma^{\boldsymbol{s}}$ and $\sigma^{\boldsymbol{t}}$. Clearly A is a subset of both $\sigma^{\boldsymbol{s}}$ and $\sigma^{\boldsymbol{t}}$, so $A \subset \sigma^{\boldsymbol{s}} \cap \sigma^{\boldsymbol{t}}$.

Let $\boldsymbol{x} \in \sigma^{\boldsymbol{s}} \cap \sigma^{\boldsymbol{t}}$. We can write

$$\sum_{i=1}^{n+1} c_i s_i \boldsymbol{e}_i = \boldsymbol{x} = \sum_{i=1}^{n+1} d_i t_i \boldsymbol{e}_i$$

where $c_i, d_i \geq 0$ and $\sum c_i = 1 = \sum d_i$. From this we get coordinate-wise that $c_i s_i = d_i t_i$. This means that $c_i = 0 = d_i$ whenever s_i and t_i are different signs. Thus $\boldsymbol{x}$ is a convex combination of the terms where $s_i = t_i$ which means that $\boldsymbol{x} \in A$. $\square$

Lemma 3.5. *The union of all simplices of $\mathcal{M}^n$ is $\Diamond^n$.*

Proof. Let $\boldsymbol{x} \in \bigcup \mathcal{M}^n$, so $\boldsymbol{x} \in \sigma^{\boldsymbol{s}}$ for some binary tuple $\boldsymbol{s}$. Now we can write

$$\boldsymbol{x} = \sum_{i=1}^{n+1} c_i s_i \boldsymbol{e}_i$$

where $c_i \geq 0$ and $\sum c_i = 1$. Calculating the l_1-norm

$$||\boldsymbol{x}||_1 = \left\| \sum_{i=1}^{n+1} c_i s_i \boldsymbol{e}_i \right\|_1 = \sum_{i=1}^{n+1} |c_i s_i| = \sum_{i=1}^{n+1} c_i = 1$$

proves that $\boldsymbol{x} \in \Diamond^n$, and so $\mathcal{M}^n \subset \Diamond^n$.

Now we assume that $\boldsymbol{x} = (x_1, \dots, x_{n+1}) \in \Diamond^n$. Define the binary tuple $\boldsymbol{s} = (s_1, \dots, s_{n+1})$ where $s_i = 1$ if $x_i \geq 0$ and $s_i = -1$ if $x_i < 0$. We can write

$$\boldsymbol{x} = \sum_{i=1}^{n+1} x_i \boldsymbol{e}_i = \sum_{i=1}^{n+1} |x_i| s_i \boldsymbol{e}_i$$

which proves that $\boldsymbol{x} \in \sigma^{\boldsymbol{s}}$. Moreover $\sigma^{\boldsymbol{s}} \in \mathcal{M}^n$ which concludes the proof that $\bigcup \mathcal{M}^n = \Diamond^n$. □

Lemmas 3.4 and 3.5 accompanied with Lemma 2.21 actually form a triangulation $\mathcal{T}$ for $\Diamond^n$. This $\mathcal{T}$ will also be a simplicial subcomplex of the triangulation $\mathcal{G}$ for $\Diamond^n \times [0,1]$ when we construct it. In $\mathcal{G}$ the simplicial complex $\mathcal{T}$ lies both in the top $\Diamond^n \times \{1\}$ and the bottom $\Diamond^n \times \{0\}$ parts of the the cylinder-like shape $\Diamond^n \times [0,1]$. These are interesting facts that help one to picture the situation.

Now we move on to the next step: increasing the dimension of simplices of $\mathcal{M}$ by one to get a triangulation for $\Diamond^n \times [0,1]$. We proved that the simplices of $\mathcal{M}$ cover $\Diamond^n$ so the Cartesian products $\sigma \times [0,1]$ where $\sigma \in \mathcal{M}$ cover $\Diamond^n \times [0,1]$. We will refer to these Cartesian products as shards. The triangulation of $\Diamond^n \times [0,1]$ will be done separately for each shard. Next we define a simple tool to help us with our indexing.

Definition 3.6. Let $(X, \preceq)$ be a finite totally ordered set of m elements. Define an order bijection $\rho_{X,\preceq}\colon \{1,\dots,m\} \to X$ where $\rho_{X,\preceq}(n)$ is the nth smallest element of X with respect to $\preceq$. When the order and the set is obvious from the context, we write simply ρ instead of $\rho_{X,\preceq}$.

Definition 3.7. Let $\sigma \subset \mathbb{R}^n$ be an $(m-1)$-simplex and $\preceq$ a total order for $\operatorname{vert}(\sigma)$. For $i \in \{1,\dots,m\}$ we define

$$\sigma_i = \operatorname{conv}\{(\rho(1),0),\dots,(\rho(i),0),(\rho(i),1),\dots,(\rho(m),1)\} \subset \mathbb{R}^{n+1},$$

and denote $\mathcal{N}(\sigma, \preceq) = \{\sigma_i \mid i \in \{1,\dots,m\}\}$.

Regarding the last dimension, a simplex σ_i has the first i vertices on the bottom. After that come the rest $m-i+1$ of the vertices on the top. Let us now check that these in fact are simplices.

Lemma 3.8. *Every $\sigma_k \in \mathcal{N}(\sigma, \preceq)$ is a simplex.*

Proof. Suppose that $c_1,\dots,c_k,c'_k,\dots,c'_m \in \mathbb{R}$ with

$$\sum_{i=1}^{k} c_j(\rho(i),0) + \sum_{i=k}^{m} c'_i(\rho(i),1) = (\mathbf{0},0)$$

and $\sum c_i + \sum c'_i = 0$. The points $\rho(1),\dots,\rho(m)$ are affine independent, so $c_i, c'_i = 0$ for all $i \neq k$ and $c_k + c'_k = 0$. The last coordinate must also be 0 so $c'_k = 0$ and thus $c_k = 0$ also. This shows that the set $\{(\rho(1),0),\dots,(\rho(k),0),$ $(\rho(k),1),\dots,(\rho(m),1)\}$ is affine independent, so σ_k is a simplex. □

Definition 3.9. Denote $V = \{\boldsymbol{e}_1, -\boldsymbol{e}_1, \dots, \boldsymbol{e}_{n+1}, -\boldsymbol{e}_{n+1}\}$ and index it with

$$\begin{aligned}
\boldsymbol{v}_1 &= \boldsymbol{e}_1\\
\boldsymbol{v}_2 &= -\boldsymbol{e}_1\\
&\vdots\\
\boldsymbol{v}_{2n+1} &= \boldsymbol{e}_{n+1}\\
\boldsymbol{v}_{2n+2} &= -\boldsymbol{e}_{n+1}.
\end{aligned}$$

Notice that this is the set of all vertices of simplices of $\mathcal{M}$. From this we get a total order $\preceq$ for V where $\boldsymbol{v}_i \preceq \boldsymbol{v}_k$ if $i \leq k$. Moreover a total order is inherited to each subset of V. Denote $\preceq_{\boldsymbol{s}}$ as the order inherited from $\preceq$ to a $\operatorname{vert}(\sigma^{\boldsymbol{s}}) \subset V$ and $\rho_{\boldsymbol{s}}$ as the order bijection corresponding to it. We shall define

$$\mathcal{N} = \bigcup_{\boldsymbol{s}} \mathcal{N}(\sigma^{\boldsymbol{s}}, \preceq_{\boldsymbol{s}}),$$

where $\boldsymbol{s}$ goes through all of the binary tuples $\{-1,1\}^{n+1}$. Finally we denote $\mathcal{G}$ to be the collection of all simplices of $\mathcal{N}$ plus all of faces of each simplex of $\mathcal{N}$.

Notice that because two orders $\preceq_{\boldsymbol{s}}$ and $\preceq_{\boldsymbol{t}}$ are inherited from the same order, they agree on $\operatorname{vert}(\sigma^{\boldsymbol{s}}) \cap \operatorname{vert}(\sigma^{\boldsymbol{t}})$. Moreover if $\boldsymbol{v}_i \preceq \boldsymbol{v}_k$ and $\boldsymbol{v}_i \neq \pm \boldsymbol{v}_k$ then $-\boldsymbol{v}_i \preceq -\boldsymbol{v}_k$. The latter will come in handy when proving the antipodality of $\mathcal{G}$.

Lemma 3.10. *The collection $\mathcal{G}$ is a simplicial complex.*

The idea of the proof is to show that an intersection of two simplices of $\mathcal{N}$ is a face of both of the simplices. Once we do that we can apply Lemma 2.21 which then proves the claim.

Proof. Let $\sigma_k^{\boldsymbol{s}}$ and $\sigma_l^{\boldsymbol{t}}$ be simplices of $\mathcal{N}$, so $\boldsymbol{s}, \boldsymbol{t} \in S$ and $k, l \in \{1, \ldots, n+1\}$. First we want to write out what we mean with $\sigma_k^{\boldsymbol{s}}$ and $\sigma_l^{\boldsymbol{t}}$. Define $p \in \{1, \ldots, 2n+2\}$ as the unique index with $\rho_{\boldsymbol{s}}(k) = \boldsymbol{v}_p$. A definition after another gives us that

$$\begin{aligned}
\operatorname{vert}(\sigma_k^{\boldsymbol{s}}) &= \{(\rho_{\boldsymbol{s}}(1), 0), \ldots, (\rho_{\boldsymbol{s}}(k), 0), (\rho_{\boldsymbol{s}}(k), 1), \ldots, (\rho_{\boldsymbol{s}}(n+1), 1)\} \\
&= \{(\rho_{\boldsymbol{s}}(i), 0) \mid i \leq k\} \cup \{(\rho_{\boldsymbol{s}}(i), 1) \mid k \leq i\} \\
&= \{(\rho_{\boldsymbol{s}}(i), 0) \mid \rho_{\boldsymbol{s}}(i) \preceq \rho_{\boldsymbol{s}}(k)\} \cup \{(\rho_{\boldsymbol{s}}(i), 1) \mid \rho_{\boldsymbol{s}}(k) \preceq \rho_{\boldsymbol{s}}(i)\} \\
&= \{(\boldsymbol{v}_i, 0) \mid \boldsymbol{v}_i \preceq \boldsymbol{v}_p, \quad \boldsymbol{v}_i \in \operatorname{vert}(\sigma^{\boldsymbol{s}})\} \cup \{(\boldsymbol{v}_i, 1) \mid \boldsymbol{v}_p \preceq \boldsymbol{v}_i, \quad \boldsymbol{v}_i \in \operatorname{vert}(\sigma^{\boldsymbol{s}})\} \\
&= \{(\boldsymbol{v}_i, 0) \mid i \leq p, \quad \boldsymbol{v}_i \in \operatorname{vert}(\sigma^{\boldsymbol{s}})\} \cup \{(\boldsymbol{v}_i, 1) \mid i \geq p, \quad \boldsymbol{v}_i \in \operatorname{vert}(\sigma^{\boldsymbol{s}})\},
\end{aligned}$$

which means that

$$\sigma_k^{\boldsymbol{s}} = \operatorname{conv}\big(\{(\boldsymbol{v}_i, 0) \mid i \leq p, \quad \boldsymbol{v}_i \in \operatorname{vert}(\sigma^{\boldsymbol{s}})\} \cup \{(\boldsymbol{v}_i, 1) \mid i \geq p, \quad \boldsymbol{v}_i \in \operatorname{vert}(\sigma^{\boldsymbol{s}})\}\big).$$

Similarly when we define $q \in \{1, \ldots, 2n+2\}$ as the unique index with $\rho_{\boldsymbol{t}}(l) = \boldsymbol{v}_q$, we get that

$$\sigma_l^{\boldsymbol{t}} = \operatorname{conv}\big(\{(\boldsymbol{v}_i, 0) \mid i \leq q, \quad \boldsymbol{v}_i \in \operatorname{vert}(\sigma^{\boldsymbol{t}})\} \cup \{(\boldsymbol{v}_i, 1) \mid i \geq q, \quad \boldsymbol{v}_i \in \operatorname{vert}(\sigma^{\boldsymbol{t}})\}\big).$$

By symmetry we can assume that $p \leq q$. Define

$$\begin{aligned}
B = \operatorname{conv}\big(&\{(\boldsymbol{v}_i, 0) \mid i \leq p, \quad \boldsymbol{v}_i \in \operatorname{vert}(\sigma^{\boldsymbol{s}} \cap \sigma^{\boldsymbol{t}})\} \\
&\cup \{(\boldsymbol{v}_i, 1) \mid i \geq q, \quad \boldsymbol{v}_i \in \operatorname{vert}(\sigma^{\boldsymbol{s}} \cap \sigma^{\boldsymbol{t}})\}\big).
\end{aligned}$$

From the definition we can see that B is a face of both $\sigma_k^{\boldsymbol{s}}$ and $\sigma_l^{\boldsymbol{t}}$. Next we want to prove that $\sigma_k^{\boldsymbol{s}} \cap \sigma_l^{\boldsymbol{t}} = B$ where $\sigma_k^{\boldsymbol{s}} \cap \sigma_l^{\boldsymbol{t}} \subset B$ is the non-trivial part.

Let $(\boldsymbol{x}, r) \in \sigma_k^{\boldsymbol{s}} \cap \sigma_l^{\boldsymbol{t}}$. Because $(\boldsymbol{x}, r) \in \sigma_k^{\boldsymbol{s}}$, we have

$$(\boldsymbol{x}, r) = \sum_{i=1}^{p} a_i(\boldsymbol{v}_i, 0) + \sum_{i=p}^{2n+2} b_i(\boldsymbol{v}_i, 1)$$

where $a_i, b_i \geq 0$ and $\sum a_i + \sum b_i = 1$ and $a_i, b_i = 0$ if $\boldsymbol{v}_i \notin \mathrm{vert}(\sigma^{\boldsymbol{s}})$. Thus

$$\boldsymbol{x} = \sum_{i=1}^{p} a_i \boldsymbol{v}_i + \sum_{i=p}^{2n+2} b_i \boldsymbol{v}_i.$$

However, since $(\boldsymbol{x}, r) \in \sigma_k^{\boldsymbol{s}} \cap \sigma_l^{\boldsymbol{t}}$, it follows that $\boldsymbol{x} \in \sigma^{\boldsymbol{s}} \cap \sigma^{\boldsymbol{t}}$ and

$$\boldsymbol{x} = \sum_{i=1}^{2n+2} y_i \boldsymbol{v}_i$$

for some $y_i \geq 0$ and $\sum y_i = 1$ and $y_i = 0$ if $\boldsymbol{v}_i \notin \mathrm{vert}(\sigma^{\boldsymbol{s}} \cap \sigma^{\boldsymbol{t}})$.

The set
$$\{a_i \mid i < p\} \cup \{a_p + b_p\} \cup \{b_i \mid i > p\}$$
forms barycentric coordinates for $\boldsymbol{x}$ with respect to $\sigma^{\boldsymbol{s}}$ (along with some unimpactful zeroes). The coefficients y_i also form barycentric coordinates for $\boldsymbol{x}$ with respect to $\sigma^{\boldsymbol{s}} \cap \sigma^{\boldsymbol{t}}$. Lemma 3.4 gives us that $\sigma^{\boldsymbol{s}} \cap \sigma^{\boldsymbol{t}}$ is a face of $\sigma^{\boldsymbol{s}}$. Therefore the coefficients y_i form barycentric coordinates for $\boldsymbol{x}$ with respect to $\sigma^{\boldsymbol{s}}$ also. Because barycentric coordinates are unique, we get that

$$\begin{aligned} a_i &= y_i && \text{for } i < p \\ a_p + b_p &= y_p && \\ b_i &= y_i && \text{for } i > p. \end{aligned}$$

Thus $a_i, b_i = 0$ whenever $\boldsymbol{v}_i \notin \mathrm{vert}(\sigma^{\boldsymbol{s}} \cap \sigma^{\boldsymbol{t}})$. Similarly because $(\boldsymbol{x}, r) \in \sigma_l^{\boldsymbol{t}}$ we can make the same argument to prove that for some $c_i, d_i \geq 0$

$$(\boldsymbol{x}, r) = \sum_{i=1}^{q} c_i(\boldsymbol{v}_i, 0) + \sum_{i=q}^{2n+2} d_i(\boldsymbol{v}_i, 1)$$

where $\sum c_i + \sum d_i = 1$ and $c_i, d_i = 0$ if $\boldsymbol{v}_i \notin \mathrm{vert}(\sigma^{\boldsymbol{s}} \cap \sigma^{\boldsymbol{t}})$. For the case $p = q$ this is actually sufficient to show that $(\boldsymbol{x}, r) \in B$. Thus we can assume that $p < q$ for the rest of the proof.

Next we compare the barycentric coordinates a_i, b_i to c_i, d_i in the simplex $\sigma^{\boldsymbol{s}} \cap \sigma^{\boldsymbol{t}}$ for $\boldsymbol{x}$. From that we get

$$\begin{aligned} b_q &= c_q + d_q && \\ b_i &= d_i && \text{for all } i \in \{q+1, \ldots, 2n+2\} \end{aligned}$$

among with some other identities which we do not need. The last coordinate r yields an equation

$$\sum_{i=p}^{2n+2} b_i = r = \sum_{i=q}^{2n+2} d_i,$$

which accompanied with the identities leads to

$$\sum_{i=p}^{q} b_i = d_q = b_q - c_q \leq b_q.$$

Since all of the coefficients are non-negative, $b_i = 0$ for all $i \in \{p, \dots, q-1\}$. This shows that $(\boldsymbol{x}, r) \in B$ and so $\sigma_k^{\boldsymbol{s}} \cap \sigma_l^{\boldsymbol{t}} = B$, which proves that $\sigma_k^{\boldsymbol{s}} \cap \sigma_l^{\boldsymbol{t}}$ is a face of both $\sigma_k^{\boldsymbol{s}}$ and $\sigma_l^{\boldsymbol{t}}$. Finally Lemma 2.21 states that $\mathcal{G}$ is a simplicial complex.

□

Now we shall prove a very technical lemma that aids us in proving Lemma 3.12.

Lemma 3.11. *Let $\boldsymbol{u}_1, \dots, \boldsymbol{u}_n$ be affine independent in $\mathbb{R}^m$. For natural numbers $n \geq b \geq c \geq 1$ denote*

$$C_{b,c} = \operatorname{conv}\{(\boldsymbol{u}_1, 0), \dots, (\boldsymbol{u}_b, 0), (\boldsymbol{u}_c, 1), \dots, (\boldsymbol{u}_n, 1)\}.$$

If $b > c$ and $\boldsymbol{x} \in C_{b,c}$ then $\boldsymbol{x} \in C_{b-1,c}$ or $\boldsymbol{x} \in C_{b,c+1}$.

Proof. Suppose that $b > c$ and $\boldsymbol{x} \in C_{b,c}$. We can write

$$\boldsymbol{x} = \sum_{i=1}^{b} d_i(\boldsymbol{u}_i, 0) + \sum_{i=c}^{n} d_i'(\boldsymbol{u}_i, 1)$$

where

$$\sum_{i=1}^{b} d_i + \sum_{i=c}^{n} d_i' = 1$$

and $d_i, d_i' \geq 0$ for all i. We look at two cases.

Case 1: $d_b \geq d_c'$. Now

$$\begin{aligned}
\boldsymbol{x} = & \sum_{i=1}^{b} d_i(\boldsymbol{u}_i, 0) + \sum_{i=c}^{n} d_i'(\boldsymbol{u}_i, 1) - d_c'((\boldsymbol{u}_c, 1) + (\boldsymbol{u}_b, 0)) + d_c'((\boldsymbol{u}_c, 0) + (\boldsymbol{u}_b, 1)) \\
= & \sum_{i=1,\ i\neq c}^{b-1} d_i(\boldsymbol{u}_i, 0) + \sum_{i=c+1,\ i\neq b}^{n} d_i'(\boldsymbol{u}_i, 1) \\
& + 0(\boldsymbol{u}_c, 1) + (d_b - d_c')(\boldsymbol{u}_b, 0) + (d_c + d_c')(\boldsymbol{u}_c, 0) + (d_b' + d_c')(\boldsymbol{u}_b, 1)
\end{aligned}$$

which means that $\boldsymbol{x}$ is a convex combination of vectors $(\boldsymbol{u}_1, 0), \dots, (\boldsymbol{u}_b, 0)$, $(\boldsymbol{u}_{c+1}, 1), \dots, (\boldsymbol{u}_n, 1)$. Thus $\boldsymbol{x} \in C_{b,c+1}$.

Case 2: $d_b < d'_c$. The calculation in this case is very similar:

$$\begin{aligned}\boldsymbol{x} &= \sum_{i=1}^{b} d_i(\boldsymbol{u}_i, 0) + \sum_{i=c}^{n} d'_i(\boldsymbol{u}_i, 1) - d_b((\boldsymbol{u}_c, 1) + (\boldsymbol{u}_b, 0)) + d_b((\boldsymbol{u}_c, 0) + (\boldsymbol{u}_b, 1)) \\ &= \sum_{i=1,\ i\neq c}^{b-1} d_i(\boldsymbol{u}_i, 0) + \sum_{i=c+1,\ i\neq b}^{n} d'_i(\boldsymbol{u}_i, 1) \\ &\quad + (d'_c - d_b)(\boldsymbol{u}_c, 1) + 0(\boldsymbol{u}_b, 0) + (d_c + d_b)(\boldsymbol{u}_c, 0) + (d'_b + d_b)(\boldsymbol{u}_b, 1).\end{aligned}$$

Hence $\boldsymbol{x} \in C_{b-1,c}$.

□

Lemma 3.12. *The simplicial complex $\mathcal{G}$ is a triangulation for $\lozenge^n \times [0,1]$. In other words $|\mathcal{G}| = \lozenge^n \times [0,1]$.*

Proof. Every simplex of $\mathcal{N}$ is contained in $\lozenge^n \times [0,1]$. Thus $|\mathcal{G}| \subset \lozenge^n \times [0,1]$. Let $\boldsymbol{x} \in \lozenge^n \times [0,1]$. Lemma 3.5 shows that $\boldsymbol{x} \in \sigma^{\boldsymbol{s}} \times [0,1]$ for some $\boldsymbol{s} \in S$. Recall that $\operatorname{vert}(\sigma^{\boldsymbol{s}})$ has an order $\preceq_{\boldsymbol{s}}$ and an order bijection ρ such that

$$\operatorname{vert}(\sigma^{\boldsymbol{s}}) = \{\rho(1), \ldots, \rho(n+1)\}.$$

We use notation from the previous Lemma 3.11 so that

$$C_{b,c} = \operatorname{conv}\{(\rho(1), 0), \ldots, (\rho(b), 0), (\rho(c), 1), \ldots, (\rho(n+1), 1)\}.$$

Notice that $\sigma^{\boldsymbol{s}} \times [0,1] = C_{n+1,1}$. Also for every i the simplex $\sigma^{\boldsymbol{s}}_i = C_{i,i}$. Because $\boldsymbol{x} \in C_{n+1,1}$, we can use Lemma 3.11 iteratively n times to get that $\boldsymbol{x} \in C_{i,i}$ for some i. Hence $\boldsymbol{x} \in \sigma^{\boldsymbol{s}}_i$ which means that $\lozenge^n \times [0,1] \subset |\mathcal{G}|$.

□

Definition 3.13. Let X be a topological space. We say that a continuous map $v \colon X \to X$ is an antipodality on X, if $v \circ v = \mathrm{id}_X$. A function $f \colon X \to \mathbb{R}^n$ is called antipodal, if $(f \circ v)(\boldsymbol{x}) = -f(\boldsymbol{x})$. A triangulation $\mathcal{S}$ on the space X is called symmetric with respect to antipodality v, if $\sigma \in \mathcal{S}$ implies that $v(\sigma) \in \mathcal{S}$ as well.

It follows immediately from the definition that an antipodality must be a bijection. The most common antipodality is on $\mathbb{R}^n$ or $\mathbb{S}^n$ where $\boldsymbol{x} \mapsto -\boldsymbol{x}$. This map is so familiar that it is rarely useful to call it an antipodality. To use more algebraic terms, we can say that an antipodality is a continuous $\mathbb{Z}_2$ group action on a topological space. For now we leave antipodal functions aside and focus on antipodal simplicial complexes. We shall provide a connection between these terms in Chapter 4.2.

Lemma 3.14. *Let v be an antipodality on $\lozenge^n \times [0,1]$ defined by $v(\boldsymbol{x}, t) = (-\boldsymbol{x}, t)$. The simplicial complex $\mathcal{G}$ is symmetric with respect to v.*

We first prove that $\mathcal{N}$ is symmetric with respect to v. Then showing that the symmetry extends to $\mathcal{G}$ is a simple task.

Proof. Let $\sigma_k^{\boldsymbol{s}} \in \mathcal{N}$ where $\boldsymbol{s} = (s_1, \ldots, s_{n+1})$ is a binary tuple and $k \in \{1, \ldots, n+1\}$. Recall that

$$\mathrm{vert}(\sigma^{\boldsymbol{s}}) = \{s_1\boldsymbol{e}_1, \ldots, s_{n+1}\boldsymbol{e}_{n+1}\}$$

where

$$s_1\boldsymbol{e}_1 \preceq_{\boldsymbol{s}} s_2\boldsymbol{e}_2 \preceq_{\boldsymbol{s}}, \ldots, \preceq_{\boldsymbol{s}} s_{n+1}\boldsymbol{e}_{n+1}.$$

Thus $\rho_{\boldsymbol{s}}(i) = s_i\boldsymbol{e}_i$ for all i. Repeating the same argument for $-\boldsymbol{s}$ yields to $-\rho_{\boldsymbol{s}}(i) = -s_i\boldsymbol{e}_i = \rho_{-\boldsymbol{s}}(i)$. for all i.

The map v is linear. In Lemma 2.10 we showed that the convex hull operation commutes with linear maps. Therefore

$$\begin{aligned} v(\sigma_k^{\boldsymbol{s}}) &= v\big(\mathrm{conv}\{(\rho_{\boldsymbol{s}}(1), 0), \ldots, (\rho_{\boldsymbol{s}}(k), 0), (\rho_{\boldsymbol{s}}(k), 1), \ldots, (\rho_{\boldsymbol{s}}(n+1), 1)\}\big) \\ &= \mathrm{conv}\{v(\rho_{\boldsymbol{s}}(1), 0), \ldots, v(\rho_{\boldsymbol{s}}(k), 0), v(\rho_{\boldsymbol{s}}(k), 1), \ldots, v(\rho_{\boldsymbol{s}}(n+1), 1)\} \\ &= \mathrm{conv}\{(\rho_{-\boldsymbol{s}}(1), 0), \ldots, (\rho_{-\boldsymbol{s}}(k), 0), (\rho_{-\boldsymbol{s}}(k), 1), \ldots, (\rho_{-\boldsymbol{s}}(n+1), 1)\} \\ &= \sigma_k^{-\boldsymbol{s}}, \end{aligned}$$

which proves that $v(\sigma_k^{\boldsymbol{s}}) \in \mathcal{N}$.

We have now shown that $\mathcal{N}$ is symmetric with respect to v. Now let $\sigma \in \mathcal{G}$. Then σ is a face of some $\sigma_k^{\boldsymbol{s}}$ and so $v(\sigma)$ is a face of $v(\sigma_k^{\boldsymbol{s}}) = \sigma_k^{-\boldsymbol{s}} \in \mathcal{N}$ by earlier calculations. This means that $v(\sigma) \in \mathcal{G}$ and thus $\mathcal{G}$ is antipodal.

□

3.2 Barycentric subdivision

The geometrical object that one encounters first when studying topology is a ball. A ball has several good properties. Balls form a basis for the standard topology of a Euclidean space. They are also easy to define and the definition works well regardless of the Euclidean dimension, or even regardless of the metric. Simplices are not so well-rounded objects as balls, but they have one advantage. A simplex can be split in a way so that the parts themselves are simplices as well. In this section we introduce a process called barycentric subdivision, which does just that. This process cannot be mimicked with a ball. One cannot split a ball and end up with smaller balls.

We will continue to tinker with the simplicial complex $\mathcal{G}$ we just constructed. The goal is to split the simplices of $\mathcal{G}$ into smaller simplices with iterated barycentric subdivision. Once all of the simplices are small enough, we can use the fact that a continuous map f from the compact space $\Diamond^n \times [0, 1]$ is uniformly continuous. Therefore f behaves quite nicely inside the small simplices and f can be approximated with a piecewise affine map. This section is based on Ryszard Engelking's book *General Topology* [5].

Definition 3.15. Let $\sigma \subset \mathbb{R}^n$ be a simplex with vertices $\boldsymbol{v}_0, \ldots, \boldsymbol{v}_m$. The barycenter of σ, denoted often with $b(\sigma)$, is the point

$$\sum_{i=0}^{n} \frac{1}{m+1}\boldsymbol{v}_i.$$

Definition 3.16. Let $\mathcal{S}$ be a simplicial complex. Consider sequences of the form

$$\mathcal{L} = (F_{-1}, F_0, F_1, \dots, F_n)$$

where every F_i is an i-simplex of S and $F_{-1} = \emptyset$. We also require that F_i is a face of F_{i+1} for all i. Denote $C(\mathcal{S})$ as the set of all these sequences. For any $\mathcal{L} \in C(\mathcal{S})$ we define

$$B(\mathcal{L}) = \text{conv}\{b(F_0), \dots, b(F_n)\}$$

where $b(F_i)$ is the barycenter of F_i. Finally we will define $M(\mathcal{S}) = \{B(\mathcal{L}) \mid L \in C(\mathcal{S})\}$.

Lemma 3.17. *Every $B(\mathcal{L})$ is a simplex.*

Proof. Let $\mathcal{L} = (F_{-1}, F_0, F_1, \dots, F_n) \in C(\mathcal{S})$, where $F_i = \text{conv}\{\boldsymbol{v}_0, \dots, \boldsymbol{v}_i\}$. We want to show that $b(F_0), \dots, b(F_n)$ are affine independent. Suppose that $c_0, \dots, c_n \in \mathbb{R}$ are coefficients such that

$$\sum_{i=0}^{n} c_i b(F_i) = \mathbf{0}$$

and $\sum c_i = 0$. The calculation

$$\begin{aligned}
\mathbf{0} &= \sum_{i=0}^{n} c_i b(F_i) \\
&= \sum_{i=0}^{n} \left(\frac{c_i}{i+1} \sum_{j=0}^{i} \boldsymbol{v}_j \right) \\
&= c_0 \boldsymbol{v}_0 + \frac{c_1}{2}(\boldsymbol{v}_0 + \boldsymbol{v}_1) + \dots + \frac{c_n}{n+1}(\boldsymbol{v}_0 + \dots + \boldsymbol{v}_n) \\
&= \left(c_0 + \frac{c_1}{2} + \dots + \frac{c_n}{n+1} \right) \boldsymbol{v}_0 + \dots + \frac{c_n}{n+1} \boldsymbol{v}_n
\end{aligned}$$

motivates us to write

$$p_i = \sum_{j=i}^{n} \frac{c_j}{j+1}.$$

Arranging the summands gives us that $\sum p_i = \sum c_i = 0$. Thus we can use the fact that $\boldsymbol{v}_0, \dots, \boldsymbol{v}_n$ are affine independent to get that $p_i = 0$ for all i. Writing the link between the coefficients c_i and p_i in a matrix form

$$\begin{bmatrix} 1 & \frac{1}{2} & \dots & \frac{1}{n+1} \\ 0 & \frac{1}{2} & \dots & \frac{1}{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{n+1} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_n \end{bmatrix}$$

makes it evident that $c_i = 0$ for all i also. This proves that $b(F_0), \dots, b(F_n)$ are affine independent and so $B(\mathcal{L})$ is an n-simplex.

□

Lemma 3.18. *The intersection of two simplices of $M(\mathcal{S})$ is a face of both of them.*

Proof. Let $(F_{-1}, \ldots, F_{n_1})$ and $(G_{-1}, \ldots, G_{n_2})$ be sequences in $C(\mathcal{S})$. The proof here is split into two parts. First we show that

$$B(F_{-1}, \ldots, F_{n_1}) \cap B(G_{-1}, \ldots, G_{n_2}) = B(F_{-1}, \ldots, F_m) \cap B(G_{-1}, \ldots, G_m)$$

for some $m \leq \min(n_1, n_2)$ with $F_m = G_m$. In the second part we show that

$$B(F_{-1}, \ldots, F_m) \cap B(G_{-1}, \ldots, G_m)$$

is a face of both $B(F_{-1}, \ldots, F_m)$ and $B(G_{-1}, \ldots, G_m)$.

Denote the vertices so that $F_i = \operatorname{conv}\{\boldsymbol{v}_0, \ldots, \boldsymbol{v}_i\}$ and let

$$\boldsymbol{x} \in B(F_{-1}, \ldots, F_{n_1}) \cap B(G_{-1}, \ldots, G_{n_2}).$$

For every $i \geq 0$ the barycenter $b(F_i)$ is contained in F_{n_1} and thus

$$\boldsymbol{x} \in B(F_{-1}, \ldots, F_{n_1}) = \operatorname{conv}\{b(F_0), \ldots, b(F_{n_1})\} \subset F_{n_1}.$$

Similarly we argue $\boldsymbol{x} \in G_{n_2}$. Since F_{n_1} and G_{n_2} are simplices from the simplicial complex $\mathcal{S}$, their intersection is also a face of both. Moreover we can define $m_1 = \dim(F_{n_1} \cap G_{n_2})$.

Because $\boldsymbol{x} \in B(F_{-1}, \ldots, F_{n_1})$ we have that

$$\boldsymbol{x} = \sum_{i=0}^{n_1} c_i b(F_i)$$

for some coefficients $c_i \geq 0$ such that $\sum c_i = 1$. Suppose that $c_k > 0$ for some $k > m_1$. Then

$$\boldsymbol{x} = \sum_{i=0,\, i \neq k}^{n_1} c_i b(F_i) + \frac{c_k}{k+1} \sum_{i=0}^{k} \boldsymbol{v}_i$$

which means that the barycentric coordinates for $\boldsymbol{v}_0, \ldots, \boldsymbol{v}_k$ are positive. But $\boldsymbol{x}$ lies on $F_{n_1} \cap G_{n_2}$ which is an m_1-dimensional face of F_{n_1}. Hence no more than $m_1 + 1$ barycentric coordinates of $\boldsymbol{x}$ can be non-zero. This is a contradiction. It must be that $c_k = 0$ for all $k > m_1$, which means that $\boldsymbol{x} \in B(F_{-1}, \ldots, F_{m_1})$. Similarly we deduce that $\boldsymbol{x} \in B(G_{-1}, \ldots, G_{m_1})$ and so

$$B(F_{-1}, \ldots, F_{n_1}) \cap B(G_{-1}, \ldots, G_{n_2}) \subset B(F_{-1}, \ldots, F_{m_1}) \cap B(G_{-1}, \ldots, G_{m_1}).$$

Now we have formed a process which we can inductively repeat. Next we denote $m_2 = \dim(F_{m_1} \cap G_{m_1})$, generally $m_{i+1} = \dim(F_{m_i} \cap G_{m_i})$. Iterating the previous argument shows that

$$B(F_{-1}, \ldots, F_{m_i}) \cap B(G_{-1}, \ldots, G_{m_i}) \subset B(F_{-1}, \ldots, F_{m_{i+1}}) \cap B(G_{-1}, \ldots, G_{m_{i+1}})$$

for all i. The generated sequence of integers $(m_i)_{i\in\mathbb{N}}$ is descending as

$$m_i = \dim(F_{m_i}) \geq \dim(F_{m_i} \cap G_{m_i}) = m_{i+1}.$$

However it cannot be strictly descending as all dimensions of simplices are integers and bounded below by -1. Hence $m_i = m_{i+1}$ for some i. This means that

$$\dim(F_{m_i}) = \dim(G_{m_i}) = m_i = m_{i+1} = \dim(F_{m_i} \cap G_{m_i}).$$

Because $F_{m_i} \cap G_{m_i}$ is a face of both F_{m_i} and G_{m_i}, we get that

$$F_{m_i} = F_{m_i} \cap G_{m_i} = G_{m_i}.$$

From this point on denote simply $m = m_i$. The inclusions from the iterated argument show that

$$B(F_{-1}, \dots, F_{n_1}) \cap B(G_{-1}, \dots, G_{n_2}) \subset B(F_{-1}, \dots, F_m) \cap B(G_{-1}, \dots, G_m)$$

which means that the sets coincide. This completes the first part of the proof.

To prove the second claim we denote

$$D = \operatorname{conv}\{b(F_i) \mid 0 \leq i \leq m,\ \operatorname{vert}(F_i) = \operatorname{vert}(G_i)\}.$$

Clearly D is a face of both $B(F_{-1}, \dots, F_m)$ and $B(G_{-1}, \dots, G_m)$. Because of that, the task turns into showing that $B(F_{-1}, \dots, F_m) \cap B(G_{-1}, \dots, G_m) = D$, where $B(F_{-1}, \dots, F_m) \cap B(G_{-1}, \dots, G_m) \subset D$ is the nontrivial part.

The vertices of $F_m = G_m$ are indexed in such a way that $F_i = \operatorname{conv}\{\boldsymbol{v}_0, \dots, \boldsymbol{v}_i\}$. For some permutation $\alpha\colon \{0, \dots, m\} \to \{0, \dots, m\}$ we have that

$$G_i = \operatorname{conv}\{\boldsymbol{v}_{\alpha(0)}, \dots, \boldsymbol{v}_{\alpha(i)}\}.$$

Let $\boldsymbol{x} \in B(F_{-1}, \dots, F_m) \cap B(G_{-1}, \dots, G_m)$. Because $\boldsymbol{x} \in B(F_{-1}, \dots, F_m)$ we get that

$$\boldsymbol{x} = \sum_{i=0}^{m} c_i b(F_i)$$

for some coefficients $c_i \geq 0$ with $\sum c_i = 1$. We write p_i for the barycentric coordinate of vertex $\boldsymbol{v}_i$ for $\boldsymbol{x}$ in the simplex F_m. By a similar calculation as in Lemma 3.17 we get that

$$p_i = \sum_{j=i}^{m} \frac{c_j}{j+1}.$$

Especially it means that

$$0 \leq p_m \leq \dots \leq p_0 \leq 1.$$

Correspondingly, when writing

$$\boldsymbol{x} = \sum_{i=0}^{m} d_i b(G_i),$$

for convex coefficients d_i we can deduce that

$$0 \leq p_{\alpha(m)} \leq \cdots \leq p_{\alpha(0)} \leq 1.$$

Suppose now that $\mathrm{vert}(F_i) \neq \mathrm{vert}(G_i)$ for some fixed i. The sets $\mathrm{vert}(F_i)$ and $\mathrm{vert}(G_i)$ are of the same size, so there exists $\boldsymbol{v}_j \in \mathrm{vert}(F_i) \setminus \mathrm{vert}(G_i)$. This means that $j \leq i$ and that $j \notin \{\alpha(0), \ldots, \alpha(i)\}$. Hence $j = \alpha(j')$ for some $j' > i$.

Now we denote $A = \{\boldsymbol{v}_0, \ldots, \boldsymbol{v}_m\}$ and repeat the argument for $A \setminus \mathrm{vert}(F_i)$ and $A \setminus \mathrm{vert}(G_i)$. Because $\mathrm{vert}(F_i)$ and $\mathrm{vert}(G_i)$ are distinct and of the same size, the sets $A \setminus \mathrm{vert}(F_i)$ and $A \setminus \mathrm{vert}(G_i)$ are also distinct and of the same size. Thus there exists

$$\boldsymbol{v}_k \in (A \setminus \mathrm{vert}(F_i)) \setminus (A \setminus \mathrm{vert}(G_i)).$$

This means that $k > i$ and that $k \in \{\alpha(0), \ldots, \alpha(i)\}$. Hence $k = \alpha(k')$ for some $k' \leq i < j'$. Using the deduced orders of the barycentric coordinates gives us now that

$$p_i \leq p_j = p_{\alpha(j')} \leq p_{\alpha(k')} = p_k \leq p_i.$$

and thus $p_i = p_k$. Hence

$$\sum_{j=i}^{m} \frac{c_j}{j+1} = p_i = p_k = \sum_{j=k}^{m} \frac{c_j}{j+1}$$

and so $c_i = 0$. We have now shown that if $F_i \neq G_i$, then $c_i = 0$. This concludes that $\boldsymbol{x} \in D$.

All in all we showed that

$$B(F_{-1}, \ldots, F_{n_1}) \cap B(G_{-1}, \ldots, G_{n_2}) = D$$

and that D is a face of both $B(F_{-1}, \ldots, F_m)$ and $B(G_{-1}, \ldots, G_m)$. Since $B(F_{-1}, \ldots, F_m)$ and $B(G_{-1}, \ldots, G_m)$ are faces of the simplices $B(F_{-1}, \ldots, F_{n_1})$ and $B(G_{-1}, \ldots, G_{n_2})$, respectively, it follows that the intersection of two simplices of $M(\mathcal{S})$ is a face of both of them.

□

Now Lemmas 2.21, 3.17 and 3.18 lead to the definition of one of our main tools.

Definition 3.19. Let $\mathcal{S}$ be a simplicial complex. The barycentric subdivision $\mathrm{Sd}(\mathcal{S})$ is the simplicial complex of all simplices of $M(\mathcal{S})$ and all of their faces.

Lemma 3.20. *Let $\mathcal{S}$ be a simplicial complex. The polyhedron of $\mathcal{S}$ is the polyhedron of the barycentric subdivision of $\mathcal{S}$, that is $|\mathcal{S}| = |\mathrm{Sd}(\mathcal{S})|$.*

Proof. In the proof of Lemma 3.18 we stated that for every $\mathcal{L} \in C(\mathcal{S})$, the simplex $B(\mathcal{L})$ is contained is some $\sigma \in \mathcal{S}$. Hence every simplex of $\mathrm{Sd}(\mathcal{S})$ is also contained in $|\mathcal{S}|$. To prove that $|\mathcal{S}| \subset |\mathrm{Sd}(\mathcal{S})|$ we need to do a bit more work.

Let $\boldsymbol{x} \in \mathcal{S}$, so $\boldsymbol{x} \in \sigma$ for some $\sigma \in \mathcal{S}$ with $\operatorname{vert}(\sigma) = \{\boldsymbol{v}_0, \ldots, \boldsymbol{v}_m\}$. Denote p_i for the barycentric coordinate of $\boldsymbol{v}_i$ for $\boldsymbol{x}$. For some permutation α of $\{0, \ldots, m\}$ it holds that

$$0 \leq p_{\alpha(m)} \leq \cdots \leq p_{\alpha(0)} \leq 1.$$

For simplicity we denote $\boldsymbol{u}_i = \boldsymbol{v}_{\alpha(i)}$ and $t_i = p_{\alpha(i)}$. Consider the sequence $(F_{-1}, \ldots, F_m)$, where $F_i = \operatorname{conv}\{\boldsymbol{u}_0, \ldots, \boldsymbol{u}_i\}$ and $F_{-1} = \emptyset$. Our goal is to show that $\boldsymbol{x} \in B(F_{-1}, \ldots, F_m)$.

We define $c_m = (m+1)t_m$ and for $i \in \{0, \ldots, m-1\}$ we define inductively

$$c_i = (i+1)(t_i - t_{i+1}).$$

Now clearly every c_i is non-negative. The calculation

$$\begin{aligned}
\sum_{i=0}^{m} c_i b(F_i) &= \sum_{i=0}^{m} \left(\frac{c_i}{i+1} \sum_{j=0}^{i} \boldsymbol{u}_j \right) \\
&= \sum_{i=0}^{m-1} \left((t_i - t_{i+1}) \sum_{j=0}^{i} \boldsymbol{u}_j \right) + t_m \sum_{j=0}^{m} \boldsymbol{u}_j \\
&= (t_0 - t_1)\boldsymbol{u}_0 + \cdots + (t_{m-1} - t_m) \left(\sum_{j=0}^{m-1} \boldsymbol{u}_j \right) + t_m \sum_{j=0}^{m} \boldsymbol{u}_j \\
&= t_0 \boldsymbol{u}_0 + t_1(-\boldsymbol{u}_0 + (\boldsymbol{u}_0 + \boldsymbol{u}_1)) + \cdots + t_m \left(-\sum_{j=0}^{m-1} \boldsymbol{u}_j + \sum_{j=0}^{m} \boldsymbol{u}_j \right) \\
&= t_0 \boldsymbol{u}_0 + t_1 \boldsymbol{u}_1 + \cdots + t_m \boldsymbol{u}_m = \boldsymbol{x}
\end{aligned}$$

shows that $\boldsymbol{x}$ is a linear combination of the barycenters $b(F_i)$. A similar calculation shows that $\sum c_i = \sum t_i = 1$. Therefore the combination of the barycenters is a convex one and so $\boldsymbol{x} \in B(F_{-1}, \ldots, F_m)$. Hence $\boldsymbol{x} \in |\operatorname{Sd}(\mathcal{S})|$ and thus $|\mathcal{S}| = |\operatorname{Sd}(\mathcal{S})|$.

□

Lemma 3.21. *Let σ be an n-simplex with vertices $\boldsymbol{v}_0, \ldots, \boldsymbol{v}_n$ and denote $I = \{0, \ldots, n\}$.*

(i) If $\boldsymbol{x} \in \sigma$, then $\sup_{\boldsymbol{y} \in \sigma} ||\boldsymbol{x} - \boldsymbol{y}|| = \max_{j \in I} ||\boldsymbol{x} - \boldsymbol{v}_j||$.

(ii) $\operatorname{diam}(\sigma) = \max_{i,j \in I} ||\boldsymbol{v}_i - \boldsymbol{v}_j||$.

Proof. Fix $\boldsymbol{x} \in \sigma$ and let $\boldsymbol{y} \in \sigma$. Then

$$\boldsymbol{y} = \sum_{i=0}^{n} c_i \boldsymbol{v}_i$$

where c_i are non-negative coefficients and $\sum c_i = 1$. We can calculate

$$\begin{aligned}
||\boldsymbol{x} - \boldsymbol{y}|| &= ||\left(\sum_{i=0}^{n} c_i\right)\boldsymbol{x} - \sum_{i=0}^{n} c_i \boldsymbol{v}_i|| \\
&= ||\sum_{i=0}^{n} c_i(\boldsymbol{x} - \boldsymbol{v}_i)|| \\
&\leq \sum_{i=0}^{n} c_i ||\boldsymbol{x} - \boldsymbol{v}_i|| \\
&\leq \sum_{i=0}^{n} c_i (\max_{j\in I} ||\boldsymbol{x} - \boldsymbol{v}_j||) \\
&= \max_{j\in I} ||\boldsymbol{x} - \boldsymbol{v}_j||
\end{aligned}$$

to prove that $\sup_{\boldsymbol{y}\in\sigma} ||\boldsymbol{x} - \boldsymbol{y}|| \leq \max_{j\in I} ||\boldsymbol{x} - \boldsymbol{v}_j||$. This is sufficient to prove *(i)*.

Now let $\boldsymbol{x}, \boldsymbol{y} \in \sigma$. By *(i)* we get that $||\boldsymbol{x} - \boldsymbol{y}|| \leq \max_{j\in I} ||\boldsymbol{x} - \boldsymbol{v}_j||$. Also for all $\boldsymbol{v}_j$ by *(i)* we have $||\boldsymbol{x} - \boldsymbol{v}_j|| \leq \max_{i\in I} ||\boldsymbol{v}_i - \boldsymbol{v}_j||$. Thus

$$||\boldsymbol{x} - \boldsymbol{y}|| \leq \max_{j\in I} ||\boldsymbol{x} - \boldsymbol{v}_j|| \leq \max_{i,j\in I} ||\boldsymbol{v}_i - \boldsymbol{v}_j||$$

and so $\sup_{\boldsymbol{x},\boldsymbol{y}\in\sigma} ||\boldsymbol{x} - \boldsymbol{y}|| \leq \max_{i,j\in I} ||\boldsymbol{v}_i - \boldsymbol{v}_j||$. This proves the second claim *(ii)*.

□

Definition 3.22. The mesh of a simplicial complex $\mathcal{S}$ is the largest diameter of the simplices of $\mathcal{S}$ and denoted by $\mathrm{mesh}(\mathcal{S})$.

Lemma 3.23. *Let $\mathcal{S}$ be a simplicial complex of dimension n. Then* $\mathrm{mesh}(\mathrm{Sd}(\mathcal{S})) \leq (n/(n+1))\,\mathrm{mesh}(\mathcal{S})$.

Proof. Let $(F_{-1}, \ldots, F_m) \in C(S)$ with $F_i = \mathrm{conv}\{\boldsymbol{v}_0, \ldots, \boldsymbol{v}_i\}$ and $\dim(\mathcal{S}) = n$. In order to find out the diameter of the simplex $B(F_{-1}, \ldots, F_m)$, by Lemma 3.21 we need only to look at $\max_{i,j} ||b(F_i) - b(F_j)||$.

Let $i, j \in \{0, \ldots, m\}$. By the first part of Lemma 3.21, for some $0 \leq k \leq n$

we have that $||b(F_i) - b(F_j)|| \leq ||b(F_i) - \boldsymbol{v}_k||$. For these i and j it holds that

$$\begin{aligned}
||b(F_i) - b(F_j)|| &\leq ||b(F_i) - \boldsymbol{v}_k|| \\
&= ||\frac{1}{i+1}\sum_{l=0}^{i}\boldsymbol{v}_l - \frac{1}{i+1}\sum_{l=0}^{i}\boldsymbol{v}_k|| \\
&= \frac{1}{i+1}||\sum_{l=0}^{i}(\boldsymbol{v}_l - \boldsymbol{v}_k)|| \\
&\leq \frac{1}{i+1}\sum_{l=0}^{i}||\boldsymbol{v}_l - \boldsymbol{v}_k|| \\
&\leq \frac{i}{i+1}\operatorname{diam}(F_m) \\
&\leq \frac{n}{n+1}\operatorname{mesh}(\mathcal{S}).
\end{aligned}$$

Therefore the diameter of $B(F_{-1}, \dots, B_m)$ is bounded above by $(n/(n+1)) \operatorname{mesh}(\mathcal{S})$. Hence every diameter of a simplex of $M(\mathcal{S})$ is uniformly bounded from above. It follows that the faces of those simplices, simplices of $\operatorname{Sd}(\mathcal{S})$, are also uniformly bounded. This proves that $\operatorname{mesh}(\operatorname{Sd}(\mathcal{S})) \leq (n/(n+1)) \operatorname{mesh}(\mathcal{S})$.

□

Lemma 3.24. *Suppose $\mathcal{S}$ is a triangulation on $\Diamond^n \times [0,1]$ that is symmetric with respect to the map $v(\boldsymbol{x}, t) = (-\boldsymbol{x}, t)$. Then $\operatorname{Sd}(\mathcal{S})$ is also symmetric with respect to v.*

Proof. Let $\sigma \in \operatorname{Sd}(\mathcal{S})$, so σ is a face of some $B(\mathcal{L}) = \operatorname{conv}(b(F_0), \dots, b(F_m))$, where $F_{-1}, \dots, F_n$ is an increasing sequence of simplices of $\mathcal{S}$. As the triangulation $\mathcal{S}$ is symmetric with v, the sequence $v(F_{-1}), \dots, v(F_m)$ is also an increasing sequence of simplices of $\mathcal{S}$. Moreover

$$v(B(\mathcal{L})) = \operatorname{conv}(b(v(F_0)), \dots, b(v(F_m)))$$

which means that $v(B(\mathcal{L}))$ is contained in $\operatorname{Sd}(\mathcal{S})$. Because σ is a face of $B(\mathcal{L})$, then $v(\sigma)$ is a face of $v(B(\mathcal{L}))$. A simplicial complex contains all of its faces, thus $v(\sigma) \in \operatorname{Sd}(\mathcal{S})$.

□

Corollary 3.25. *For every $\varepsilon > 0$ there is a triangulation $\mathcal{S}$ on $\Diamond^n \times [0,1]$ such that, $\mathcal{S}$ is symmetric with respect to the map $v(\boldsymbol{x}, t) = (-\boldsymbol{x}, t)$ and $\operatorname{mesh}(\mathcal{S}) < \varepsilon$.*

Proof. Let $\varepsilon > 0$. We take the triangulation $\mathcal{G}$ of $\Diamond^n \times [0,1]$ that was formed in Chapter 3.1. Choose a large enough $m \in \mathbb{N}$ so that

$$\left(\frac{n}{n+1}\right)^m \operatorname{mesh}(\mathcal{G}) < \varepsilon.$$

We form $\mathcal{S}$ by applying barycentric subdivision on $\mathcal{G}$ over and over again m times. Now Lemma 3.23 ensures that $\text{mesh}(\mathcal{S}) < \varepsilon$ while Lemmas 3.14 and 3.24 state that $\mathcal{S}$ is symmetric with v.

□

4 General positioning in $\mathbb{R}^n$

In this chapter we continue with affine sets and simplices and introduce a concept called general positioning. As the name implies, "usually" finite sets are in a general position. For example the set of m-tuples in $\mathbb{R}^n$ that are not in general position can be considered as set of measure zero in $\mathbb{R}^{mn}$. The point in general positioning is to avoid irregular behavior that happens in certain special cases.

4.1 Hyperplanes and finite sets

Definition 4.1. An affine set spanned by n affine independent points of $\mathbb{R}^n$ is called a hyperplane.

We shall use notation $H(\boldsymbol{u}, b)$ for the set $\{\boldsymbol{x} \in \mathbb{R}^n \mid \langle \boldsymbol{x}, \boldsymbol{u} \rangle = b\}$ where $\boldsymbol{u}$ and b are parameters with $\boldsymbol{u} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}$ and $b \in \mathbb{R}$. In the Lemma 4.3 we characterize the hyperplanes of $\mathbb{R}^n$ using this notation and the dot-product. For that we need a small proposition.

Lemma 4.2. *Let $\boldsymbol{x}_1, \dots, \boldsymbol{x}_n \in \mathbb{R}^n$ be affine independent, $\boldsymbol{u} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}$ and $b \in \mathbb{R}$. Suppose also that $\langle \boldsymbol{x}_i, \boldsymbol{u} \rangle = b$ for all i. Then $\text{aff}(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n) = H(\boldsymbol{u}, b)$.*

Proof. First we show that $\text{aff}(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n) \subset H(\boldsymbol{u}, b)$. Since $\text{aff}(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n)$ is the smallest affine set containing $\boldsymbol{x}_1, \dots, \boldsymbol{x}_n$, it suffices to show that $H(\boldsymbol{u}, b)$ is affine. Let $\boldsymbol{s}, \boldsymbol{t} \in H(\boldsymbol{u}, b)$, so $\langle \boldsymbol{s}, \boldsymbol{u} \rangle = b = \langle \boldsymbol{t}, \boldsymbol{u} \rangle$. For all $a \in \mathbb{R}$ we have

$$\langle a\boldsymbol{s} + (1-a)\boldsymbol{t}, \boldsymbol{u} \rangle = a\langle \boldsymbol{s}, \boldsymbol{u} \rangle + (1-a)\langle \boldsymbol{t}, \boldsymbol{u} \rangle = b$$

which means that $H(\boldsymbol{u}, b)$ is affine. Thus $\text{aff}(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n) \subset H(\boldsymbol{u}, b)$.

Now let $\boldsymbol{s} \in H(\boldsymbol{u}, b)$. Since $\langle \boldsymbol{s} - \boldsymbol{x}_1, u \rangle = 0$, we have that $\boldsymbol{s} - \boldsymbol{x}_1 \in \text{span}(\boldsymbol{u})^\perp$. The linear subspace $\text{span}(\boldsymbol{u})^\perp$ has a dimension $n-1$, and the vectors $\boldsymbol{x}_2 - \boldsymbol{x}_1, \dots, \boldsymbol{x}_n - \boldsymbol{x}_1$ are linearly independent by Lemma 2.8. Thus $\boldsymbol{x}_2 - \boldsymbol{x}_1, \dots, \boldsymbol{x}_n - \boldsymbol{x}_1$ form a basis for $\text{span}(\boldsymbol{u})^\perp$, so there exist coefficients $c_2, \dots, c_n \in \mathbb{R}$ with

$$\boldsymbol{s} - \boldsymbol{x}_1 = \sum_{i=2}^{n} c_i(\boldsymbol{x}_i - \boldsymbol{x}_1).$$

By rearranging the terms we see that $\boldsymbol{s}$ is an affine combination

$$\boldsymbol{s} = (1 - \sum_{i=2}^{n} c_i)\boldsymbol{x}_1 + \sum_{i=2}^{n} c_i \boldsymbol{x}_i,$$

which is contained in $\text{aff}(\boldsymbol{x}_1, \dots, \boldsymbol{x}_n)$. □

Lemma 4.3. *Every hyperplane is of the form* $H(\boldsymbol{u}, b) = \{\boldsymbol{x} \in \mathbb{R}^n \mid \langle \boldsymbol{x}, \boldsymbol{u} \rangle = b\}$ *for some* $\boldsymbol{u} \in \mathbb{S}^{n-1}$ *and* $b \in \mathbb{R}$ *. On the other hand, for every* $\boldsymbol{u} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}$ *and* $b \in \mathbb{R}$*, the set* $H(\boldsymbol{u}, b)$ *is a hyperplane.*

Proof. Let H be a hyperplane spanned by affine independent $\boldsymbol{x}_1, \dots, \boldsymbol{x}_n \in \mathbb{R}^n$. We want to find suitable $\boldsymbol{u} \in \mathbb{S}^{n-1}$ and $b \in \mathbb{R}$. The vectors $\boldsymbol{x}_1, \dots, \boldsymbol{x}_n$ are affine independent, so by Lemma 2.8 the vectors $\boldsymbol{x}_2 - \boldsymbol{x}_1, \dots, \boldsymbol{x}_n - \boldsymbol{x}_1$ are linearly independent. Denote with A the $n-1$-dimensional linear subspace $\mathrm{span}(\boldsymbol{x}_2 - \boldsymbol{x}_1, \dots, \boldsymbol{x}_n - \boldsymbol{x}_1)$. The orthogonal complement $A^\perp$ has dimension 1 and is therefore spanned by a single vector.

Define $b = ||pr_{A^\perp}(\boldsymbol{x}_1)||$. If $b > 0$, then define $\boldsymbol{u} = b^{-1} pr_{A^\perp}(\boldsymbol{x}_1)$. In the case where $b = 0$ we choose $\boldsymbol{u}$ from the set $A^\perp \cap \mathbb{S}^{n-1}$. This set consists of two antipodal points so the choice of $\boldsymbol{u}$ is not unique here. The calculation

$$\begin{aligned} \langle \boldsymbol{x}_i, \boldsymbol{u} \rangle &= \langle \boldsymbol{x}_1, \boldsymbol{u} \rangle + \langle \boldsymbol{x}_i - \boldsymbol{x}_1, \boldsymbol{u} \rangle \\ &= \langle \boldsymbol{x}_1, \boldsymbol{u} \rangle \\ &= \langle \boldsymbol{x}_1 - pr_{A^\perp} \boldsymbol{x}_1 + pr_{A^\perp} \boldsymbol{x}_1, \boldsymbol{u} \rangle \\ &= b^{-1} \langle pr_{A^\perp} \boldsymbol{x}_1, pr_{A^\perp} \boldsymbol{x}_1 \rangle \\ &= b \end{aligned}$$

shows that $\langle \boldsymbol{x}_i, \boldsymbol{u} \rangle = b$ for all i. Hence by Lemma 4.2 we have that $H = H(\boldsymbol{u}, b)$.

Let $\boldsymbol{u} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}$ and $b \in \mathbb{R}$. The orthogonal complement $\mathrm{span}(\boldsymbol{u})^\perp$ has dimension $n-1$. Hence there are some linearly independent vectors $\boldsymbol{x}_1, \dots, \boldsymbol{x}_{n-1}$ that span it. Lemma 2.8 states that the set $\boldsymbol{x}_1, \dots, \boldsymbol{x}_{n-1}, \boldsymbol{0}$ is affine independent. If $b = 0$ then we already get from Lemma 4.2 that

$$H(\boldsymbol{u}, b) = \mathrm{aff}(\boldsymbol{x}_1, \dots, \boldsymbol{x}_{n-1}, \boldsymbol{0}).$$

Because of that we can assume $b \neq 0$ and denote $m = b \langle \boldsymbol{u}, \boldsymbol{u} \rangle^{-1}$. By Lemma 2.11 the set $\{m(\boldsymbol{x}_1 + \boldsymbol{u}), \dots, m(\boldsymbol{x}_{n-1} + \boldsymbol{u}), m\boldsymbol{u}\}$ is affine independent. For all i we can calculate the dot-product

$$\langle m(\boldsymbol{x}_i + \boldsymbol{u}), \boldsymbol{u} \rangle = m \langle \boldsymbol{u}, \boldsymbol{u} \rangle = b$$

and we also know that $\langle m\boldsymbol{u}, \boldsymbol{u} \rangle = b$. Hence by Lemma 4.2, the set $H(\boldsymbol{u}, b)$ coincides with the hyperplane spanned by $\{m(\boldsymbol{x}_1 + \boldsymbol{u}), \dots, m(\boldsymbol{x}_{n-1} + \boldsymbol{u}), m\boldsymbol{u}\}$. □

With this characterization it is easy to state a few properties of hyperplanes. Every hyperplane $H(\boldsymbol{u}, b)$ is a closed set. The complement $\mathbb{R}^n \setminus H(\boldsymbol{u}, b)$ is a union of two path components. These path-components are called half-spaces and in chapter 6.1 we shall prove an interesting result related to them called the ham sandwich theorem.

Lemma 4.4. *Every hyperplane has a measure of zero.*

Proof. The idea is to assume that a hyperplane would have a positive measure. Then we can fit an infinite amount of disjoint copies of a positive measured part of the hyperplane in a bounded set. This leads to a contradiction.

Let $H(\boldsymbol{u}, b) = \{\boldsymbol{x} \in \mathbb{S}^n \mid \langle \boldsymbol{x}, \boldsymbol{u} \rangle = b\}$ be a hyperplane with parameters $\boldsymbol{u} \in \mathbb{S}^n$ and $b \in \mathbb{R}$. Because $H(\boldsymbol{u}, b)$ is closed, it is measurable. Suppose $\mu(H(\boldsymbol{u}, b)) > 0$. By the convergence of the Lebesgue measure

$$0 < \mu\left(H(\boldsymbol{u}, b)\right) = \mu\left(\bigcup_{r=1}^{\infty}(B(\boldsymbol{0}, r) \cap H(\boldsymbol{u}, b))\right) = \lim_{r\to\infty} \mu\left(B(\boldsymbol{0}, r) \cap H(\boldsymbol{u}, b)\right),$$

which implies that there exists a radius $r \in \mathbb{N}$ and $\varepsilon > 0$ such that $\mu(H(\boldsymbol{u}, b) \cap B(\boldsymbol{0}, r)) \geq \varepsilon$. Simple calculation proves that $H(\boldsymbol{u}, b) + c\boldsymbol{u} = H(\boldsymbol{u}, b + c)$ where $c \in \mathbb{R}$ and the summation means translation of the set. Moreover by translation invariance of the Lebesgue measure we get that for all $c \in \mathbb{R}$

$$\mu(H(\boldsymbol{u}, b + c) \cap B(c\boldsymbol{u}, r)) = \mu(H(\boldsymbol{u}, b) \cap B(\boldsymbol{0}, r)) \geq \varepsilon.$$

Additionally, for all $c \in [0, 1] \cap \mathbb{Q}$ the sets $H(\boldsymbol{u}, b+c) \cap B(c\boldsymbol{u}, r)$ are disjoint and

$$H(\boldsymbol{u}, b + c) = H(\boldsymbol{u}, b) + c\boldsymbol{u} \subset B(\boldsymbol{0}, r) + B(\boldsymbol{0}, 2) = B(\boldsymbol{0}, r + 2),$$

which means that they are contained in the ball $B(\boldsymbol{0}, r + 2)$. Denote

$$A = \bigcup_{c\in[0,1]\cap\mathbb{Q}} (H(\boldsymbol{u}, b + c) \cap B(c\boldsymbol{u}, r)).$$

Now we can calculate the measure of A in two ways. Because $A \subset B(\boldsymbol{0}, r + 2)$ by monotonicity we get $\mu(A) < \infty$. However, because A is a disjoint union of sets

$$\mu(A) = \sum_{c\in[0,1]\cap\mathbb{Q}} \mu(H(\boldsymbol{u}, b + c) \cap B(c\boldsymbol{u}, r)) = \infty,$$

as each of the summands are bounded below by ε. This is a contradiction and therefore $\mu(H(\boldsymbol{u}, b)) = 0$.

□

Definition 4.5. Let $A \subset \mathbb{R}^n$ be a finite set. We say that A is in general position, if every $K \subset A$ with $\operatorname{card}(K) \leq n + 1$ is an affine independent subset. If $B \subset \mathbb{R}^n$ is a finite set not containing $\boldsymbol{0}$ and $B \cup \{\boldsymbol{0}\}$ is in a general position, we say that B is in a general position with respect to $\boldsymbol{0}$.

If $A \subset \mathbb{R}^n$ is in a general position, then no $n+1$ points of A lie on a common hyperplane.

Lemma 4.6. *Let $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m$ be a sequence of points in $\mathbb{R}^n$. For every $\varepsilon > 0$ there exists an injective shifting function $s\colon \{1, \ldots, m\} \to \mathbb{R}^n$ such that the image of s is in a general position and $||\boldsymbol{x}_i - s(i)|| < \varepsilon$ for all i.*

Proof. We construct s by induction. The idea here is that hyperplanes, their subsets and unions are null sets. Therefore there is space in between the hyperplanes where we can shift our current point.

Let $\varepsilon > 0$. We keep the first point in place by defining $s(1) = \boldsymbol{x}_1$. Clearly the singleton $\{s(1)\}$ is in a general position. Suppose now that we have acquired $U_k = \{s(1), \ldots, s(k)\}$, which is in general position. Denote

$$\mathcal{F} = \{\text{aff}(A) \mid A \subset U_k,\ \text{card}(A) \leq n\}$$

and G as the union of all sets of $\mathcal{F}$. Because G is a finite union of hyperplanes and subsets of hyperplanes it is a null set by Lemma 4.4. Since G is a null set, there exists a point $\boldsymbol{y} \in \mathbb{R}^n \setminus G$ such that $||\boldsymbol{x}_{k+1} - \boldsymbol{y}|| < \varepsilon$. We define $s(k+1) = \boldsymbol{y}$.

The set $\{s(1), \ldots, s(k)\}$ was already assumed to be in a general position. So the only way that $\{s(1), \ldots, s(k+1)\}$ would not be in a general position is that $s(k+1)$ would not be affine independent with some n or less points of $\{s(1), \ldots, s(k)\}$. However, this is not the case either, as Lemma 2.12 proves. Therefore $\{s(1), \ldots, s(k+1)\}$ is in a general position. The existence of s is proven via induction. □

We add three more slightly different versions of this shifting function. The different versions are not necessarily interesting in themselves. They are tailor-made for the different proofs to come.

Lemma 4.7. *Let $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m$ be a sequence of points in $\mathbb{R}^n$. For every $\varepsilon > 0$ there exists an injective shifting function $s'\colon \{1, \ldots, m\} \to \mathbb{R}^n$ such that the image of s' is in a general position with respect to $\mathbf{0}$ and $||\boldsymbol{x}_i - s'(i)|| < \varepsilon$ for all i.*

Proof. We apply the inductive process of the proof of Lemma 4.6 to the sequence $\mathbf{0}, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_m$ and get a shifting function $s\colon \{1, \ldots, m+1\} \to \mathbb{R}^n$. The first point $\mathbf{0}$ is kept in place, so $s(1) = \mathbf{0}$, and no other point can land on $\mathbf{0}$. We get the sufficient s' by shifting the indices and defining $s'(i) = s(i+1)$. □

Lemma 4.8. *Let $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m$ be a sequence of points in $\mathbb{S}^n$. For every $\varepsilon > 0$ there exists an injective shifting function $s'\colon \{1, \ldots, m\} \to \mathbb{S}^n$ such that the image of s' is in a general position with respect to $\mathbf{0}$ and $||\boldsymbol{x}_i - s'(i)|| < \varepsilon$ for all i.*

Proof. This is a minor variation to the proofs given above. Instead of using measure, we use the fact that a hyperplane is nowhere dense[1] in $\mathbb{S}^n$. □

Lemma 4.9. *Let X be a finite set and v be an antipodality on X such that $v(\boldsymbol{x}) \neq \boldsymbol{x}$ and $v(v(\boldsymbol{x})) = \boldsymbol{x}$ for all $\boldsymbol{x} \in X$. Suppose $f\colon X \to \mathbb{R}^n$ is an antipodal function, meaning that $f \circ v = -f$. For all $\varepsilon > 0$ there exists a map $g\colon X \to \mathbb{R}^n$ with the following properties:*

[1]A subset $A \subset X$ is nowhere dense if the interior of the closure of A is empty. As with null sets, a finite union of nowhere dense sets is nowhere dense. Also if A is nowhere dense and $\boldsymbol{a} \in A$, then for all $\varepsilon > 0$ we can find $\boldsymbol{b} \in X \setminus A$ with $||\boldsymbol{a} - \boldsymbol{b}|| < \varepsilon$.

1. *The map g is injective.*

2. *The map g approximates f, so $||f(\boldsymbol{x}) - g(\boldsymbol{x})|| < \varepsilon$ for all $\boldsymbol{x} \in X$.*

3. *The map g is antipodal.*

4. *For any $A \subset X$ that does not contain a pair of antipodal points, the set $g(A)$ is in a general position with respect to $\mathbf{0}$.*

Proof. Since $v(\boldsymbol{x}) \neq \boldsymbol{x}$ for all $\boldsymbol{x}$, we can index the set X in some way with

$$X = \{\boldsymbol{x}_1, v(\boldsymbol{x}_1), \boldsymbol{x}_2, v(\boldsymbol{x}_2), \ldots, \boldsymbol{x}_m, v(\boldsymbol{x}_m)\}.$$

We define X_0 by adding two exterior elements $\boldsymbol{x}_0, \boldsymbol{x}_0'$ to X and extend v to X_0 with $v(\boldsymbol{x}_0) = \boldsymbol{x}_0'$ and $v(\boldsymbol{x}_0') = \boldsymbol{x}_0$.

Let $\varepsilon > 0$. We define a function $h\colon X_0 \to \mathbb{R}^n$ inductively on the sets $V_k = \{\boldsymbol{x}_0, v(\boldsymbol{x}_0), \ldots, \boldsymbol{x}_k, v(\boldsymbol{x}_k)\}$. In this proof we say that a subset $Y \subset X_0$ is *symmetrically generic*, if any $A \subset Y$ not containing antipodal points has its image $h(A)$ in a general position.

We start defining h on V_0 with $h(\boldsymbol{x}_0) = \mathbf{0} = h(v(\boldsymbol{x}_0))$. So far h is antipodal and the set V_0 is symmetrically generic. Suppose now that we have defined function h antipodally on the set $V_k = \{\boldsymbol{x}_0, v(\boldsymbol{x}_0), \ldots, \boldsymbol{x}_k, v(\boldsymbol{x}_k)\}$ and so that V_k is symmetrically generic. We denote

$$\mathcal{F} = \{A \subset V_k \mid \operatorname{card}(A) \leq n,\ \{\boldsymbol{x}, v(\boldsymbol{x})\} \not\subset A \text{ for any } \boldsymbol{x}\}$$

and

$$G = \bigcup_{A \in \mathcal{F}} \operatorname{aff}(h(A)).$$

Similarly as in the proof of the Lemma 4.6 we can find $\boldsymbol{y} \notin G$ with $||f(\boldsymbol{x}_{k+1}) - \boldsymbol{y}|| < \varepsilon$. We define $h(\boldsymbol{x}_{k+1}) = \boldsymbol{y}$ and $h(v(\boldsymbol{x}_{k+1})) = -\boldsymbol{y}$. First we show that $V_k \cup \{\boldsymbol{x}_{k+1}\}$ is symmetrically generic. Let $A \subset V_k \cup \{\boldsymbol{x}_{k+1}\}$ be a subset not containing antipodal points. We already know that V_k is symmetrically generic, so the only interesting case is where $\boldsymbol{x}_{k+1} \in A$. But $h(\boldsymbol{x}_{k+1}) = \boldsymbol{y} \notin \operatorname{aff}(h(A \setminus \boldsymbol{x}_{k+1}))$, which means that $h(A)$ is in a general position.

Next we show that V_{k+1} is symmetrically generic. Let $A \subset V_{k+1}$ be a set not containing antipodal points. The only new case is where $v(\boldsymbol{x}_{k+1}) \in A$. Now the set $v(A) \subset V_k \cup \{\boldsymbol{x}_{k+1}\}$ does not contain any antipodal points. Since $V_k \cup \{\boldsymbol{x}_{k+1}\}$ is symmetrically generic, it follows that $h(v(A))$ is in a general position. Therefore $-h(v(A)) = h(A)$ is in a general position and V_{k+1} is symmetrically generic.

By induction we define the function h on the whole set X_0. It also follows that h is antipodal and X_0 is symmetrically generic. The sufficient g of this lemma is achieved with $g = h|X$.

□

4.2 Affine and generic maps on simplices and complexes

Definition 4.10. Let σ be a simplex with vertices $\boldsymbol{v}_0, \ldots, \boldsymbol{v}_n$. The map $f\colon \sigma \to \mathbb{R}^m$ is called affine, if

$$f\left(\sum_{i=0}^{n} c_i \boldsymbol{v}_i\right) = \sum_{i=0}^{n} c_i f(\boldsymbol{v}_i)$$

whenever $\sum c_i = 1$.

Lemma 4.11. *Let σ be a simplex and let $g\colon \operatorname{vert}(\sigma) \to \mathbb{R}^m$ be any map. There exists a unique continuous affine map $f\colon \sigma \to \mathbb{R}^m$ with $f|\operatorname{vert}(\sigma) = g$.*

Proof. Let σ be a simplex with vertices $\boldsymbol{v}_0, \ldots, \boldsymbol{v}_n$. Denote λ_i as the barycentric projection for coordinate i. We define $f(\boldsymbol{x}) = \sum_i \lambda_i(\boldsymbol{x}) g(\boldsymbol{v}_i)$. It follows from the definition that f agrees with g on the vertices, and that f is affine. Because barycentric projections are continuous, f is continuous as well.

Suppose now h is another affine map agreeing on the vertices with g. The calculation

$$h(\boldsymbol{x}) = h\left(\sum_i \lambda_i(\boldsymbol{x})\boldsymbol{v}_i\right) = \sum_i \lambda_i(\boldsymbol{x}) h(\boldsymbol{v}_i) = \sum_i \lambda_i(\boldsymbol{x}) g(\boldsymbol{v}_i) = f(\boldsymbol{x})$$

proves that f is unique. □

Lemma 4.12 (Gluing lemma)**.** *Let X and Y be topological spaces and let $X_1, \ldots, X_n$ be closed subsets of X that cover X. Suppose that $f_i\colon X_i \to Y$ is a continuous function and that $f_i|X_i \cap X_j = f_j|X_i \cap X_j$ for all $i, j \in \{1, \ldots, n\}$. There exists a unique continuous function $f\colon X \to Y$, such that $f|X_i = f_i$.*

Proof. Denote $Z_k = \bigcup_{i=1}^{k} X_i$ and $g_1 = f_1$. Clearly g_1 is continuous and well-defined. Suppose now that we have acquired a continuous and well-defined $g_k\colon Z_k \to Y$. Define $g_{k+1}\colon Z_{k+1} \to Y$ with

$$g_{k+1}(x) = \begin{cases} g_k(x) & \text{when } x \in Z_k \\ f_{k+1}(x) & \text{when } x \in X_{k+1}. \end{cases}$$

This is a well-defined map, as g_k agrees with f_{k+1} whenever their domains intersect. Suppose now that U is a closed subset in Y. The pre-image $g_{k+1}^{-1}(U) = g_k^{-1}(U) \cup f_{k+1}^{-1}(U)$ is closed and therefore g_{k+1} is continuous. By the induction principle the map $g_n\colon X \to Y$ is continuous, well-defined and $g_n|X_i = f_i$ for all i. This is the map f and it is uniquely defined by the functions f_i.

□

Definition 4.13. Let X be a polyhedron with triangulation $\mathcal{S}$. We say that a function $f\colon X \to \mathbb{R}^m$ is locally affine, if the restriction of f to any simplex $\sigma \in \mathcal{S}$ is affine.

Lemma 4.14. *Let $\mathcal{S}$ be a simplicial complex. Suppose that $g\colon \operatorname{vert}(\mathcal{S}) \to \mathbb{R}^m$ is any map. There exists a unique continuous and locally affine map $f\colon |\mathcal{S}| \to \mathbb{R}^m$ with $f|\operatorname{vert}(\mathcal{S}) = g$.*

Proof. For each simplex σ in the simplicial complex $\mathcal{S}$ we define a function $f_\sigma \colon \sigma \to \mathbb{R}^m$ with $f(\boldsymbol{x}) = \sum_{\boldsymbol{v} \in \mathrm{vert}(\sigma)} \lambda_{\boldsymbol{v}}(\boldsymbol{x}) g(\boldsymbol{v})$. The proof of Lemma 4.11 states that this is the unique affine map from $\sigma \to \mathbb{R}^m$.

Next we show that for two simplices A and B the maps f_A and f_B agree on the intersection $A \cap B$. Let $\boldsymbol{x} \in A \cap B$. The intersection $A \cap B$ is a face of A and B. Therefore the barycentric coordinates of $\boldsymbol{x}$ in A are 0 for the vertices that are not contained in $A \cap B$. The same is true for the barycentric coordinates of $\boldsymbol{x}$ in B. Hence

$$f_A(\boldsymbol{x}) = \sum_{\boldsymbol{v} \in \mathrm{vert}(A)} \lambda_{\boldsymbol{v}}(\boldsymbol{x}) g(\boldsymbol{v}) = \sum_{\boldsymbol{u} \in \mathrm{vert}(B)} \lambda_{\boldsymbol{u}}(\boldsymbol{x}) g(\boldsymbol{u}) = f_B(\boldsymbol{x}).$$

We proved in Lemma 2.18 that every simplex in compact. Therefore the simplices of $\mathcal{S}$ are closed in $|\mathcal{S}|$. Now we can apply the gluing lemma to the collection of maps $(f_\sigma)_{\sigma \in \mathcal{S}}$ to get the unique continuous and locally affine map f.

□

It actually follows from Lemma 4.14 that if X is a polyhedron and $f \colon X \to \mathbb{R}^n$ is a continuous map, then f can be approximated with a locally affine map g. The space X is compact, so f is uniformly continuous. Once we iterate the barycentric subdivision enough and set $g(\boldsymbol{x}) = f(\boldsymbol{x})$ on the vertices, we get a good approximation of f by extending g locally affinely.

We will make this idea even stronger by perturbating g so that it is generic in every simplex. First we need a small lemma on affine sets.

Lemma 4.15. *Let $H \subset \mathbb{R}^n$ be a hyperplane and $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m \subset \mathbb{R}^n$ be affine independent. Denote $A = \mathrm{aff}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m)$. Suppose that A intersects with H, but not fully. That is $\emptyset \neq A \cap H \neq H$. Then $A \cap H$ is an affine hull of some $m-1$ affine independent points of $\mathbb{R}^n$.*

Proof. Because A and H are both affine sets, their intersection is also affine. Hence $A \cap H$ is spanned by some k amount of affine independent points. We show that $k = m-1$ by showing that every other choice for k leads to a contradiction.

Case: $k \geq m$. There are affine independent $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_k$, such that $A \cap H = \mathrm{aff}(\boldsymbol{w}_1, \ldots, \boldsymbol{w}_k)$. Since $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_k \in A$ it follows from Lemma 2.9 that $A = A \cap H$. This contradicts our assumptions.

Case: $1 \leq k \leq m-2$. There are affine independent $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_k$ so that $A \cap H = \mathrm{aff}(\boldsymbol{w}_1, \ldots, \boldsymbol{w}_k)$. Since $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$ are affine independent and $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_k \in A$, we have that at least two of the vectors $\boldsymbol{v}_i$ and $\boldsymbol{v}_j$ are such that $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_k$ and $\boldsymbol{v}_i, \boldsymbol{v}_j$ are affine independent together.

As a hyperplane, $H = \{\boldsymbol{x} \in \mathbb{R}^n \mid \langle \boldsymbol{x}, \boldsymbol{u} \rangle = b\}$ for some $\boldsymbol{u} \in \mathbb{S}^{n-1}$ and $b \in \mathbb{R}$. Because $\boldsymbol{v}_i, \boldsymbol{v}_j \notin \mathrm{aff}(\boldsymbol{w}_1, \ldots, \boldsymbol{w}_k) = A \cap H$, we have that $\langle \boldsymbol{v}_i, \boldsymbol{u} \rangle \neq b \neq \langle \boldsymbol{v}_j, \boldsymbol{u} \rangle$. We can assume that $\langle \boldsymbol{v}_i, \boldsymbol{u} \rangle \neq \langle \boldsymbol{v}_j, \boldsymbol{u} \rangle$. If this is not the case, then $\langle 2^{-1}(\boldsymbol{v}_i + \boldsymbol{w}_1), \boldsymbol{u} \rangle \neq \langle \boldsymbol{v}_j, \boldsymbol{u} \rangle$ and we can can follow though the same argument using the point $2^{-1}(\boldsymbol{v}_i + \boldsymbol{w}_1)$ instead of $\boldsymbol{v}_i$.

Because $\langle \boldsymbol{v}_i, \boldsymbol{u} \rangle \neq \langle \boldsymbol{v}_j, \boldsymbol{u} \rangle$ there is a constant $a \in \mathbb{R}$ so that

$$\langle a\boldsymbol{v}_i + (1-a)\boldsymbol{v}_j, \boldsymbol{u} \rangle = b.$$

Denote $\boldsymbol{x} = a\boldsymbol{v}_i + (1-a)\boldsymbol{v}_j$. The point $\boldsymbol{x}$ is an affine combination of $\boldsymbol{v}_i$ and $\boldsymbol{v}_j$. However, we also know that $\boldsymbol{x} \in A \cap H = \operatorname{aff}(\boldsymbol{w}_1, \ldots, \boldsymbol{w}_k)$ so $\boldsymbol{x}$ is an affine combination of $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_k$. It follows that $\boldsymbol{x}$ can be written in two different ways as an affine combination of $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_k, \boldsymbol{v}_i, \boldsymbol{v}_j$. This is a contradiction, as we know that $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_k, \boldsymbol{v}_i, \boldsymbol{v}_j$ are affine independent.

Case: $k = 0$. This is not possible, as we assumed $A \cap H \neq \emptyset$.

□

Definition 4.16. Let X be a finite set. We say that a function $f\colon X \to \mathbb{R}^m$ is generic, if it is injective and the image $f(X)$ is in a general position with respect to $\mathbf{0}$. Let $\sigma \subset \mathbb{R}^n$ be a simplex and $h\colon \sigma \to \mathbb{R}^k$ an affine map. We say that h is generic if $h|\operatorname{vert}(\sigma)$ is generic in the finite sense.

Lemma 4.17. *Let $\sigma \subset \mathbb{R}^n$ be an $m+1$-simplex and $h\colon \sigma \to \mathbb{R}^m$ an affine and generic map. Then $h(\boldsymbol{x}) \neq \mathbf{0}$ whenever $\boldsymbol{x}$ is contained in an $m-1$ face of σ.*

Proof. We use contraposition to prove the claim. Suppose $h(\boldsymbol{x}) = \mathbf{0}$ for some $\boldsymbol{x}$ that is contained in an $m-1$-face of σ. Then for some vertices $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$ and coefficients $c_1, \ldots, c_m \in \mathbb{R}$ we have that

$$\boldsymbol{x} = \sum_{i=1}^{m} c_i \boldsymbol{v}_i$$

and $\sum c_i = 1$. The map h is affine, so

$$\mathbf{0} = h(\boldsymbol{x}) = \sum_{i=1}^{m} c_i h(\boldsymbol{v}_i) = \sum_{i=1}^{m} c_i h(\boldsymbol{v}_i) + (-1)\mathbf{0}.$$

This means that the set $\{\mathbf{0}, h(\boldsymbol{v}_0), \ldots, h(\boldsymbol{v}_m)\}$ is not affine independent and therefore h is not generic.

□

Lemma 4.18. *Let $\triangle \subset \mathbb{R}^{n+2}$ be a standard $n+1$-simplex, the vertices of which are the canonical base vectors $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{n+2}$. Suppose $h\colon \triangle \to \mathbb{R}^n$ is a generic affine map.*

(i) The kernel of h is either an empty set, or a line segment $\operatorname{conv}(\boldsymbol{z}, \boldsymbol{w})$ *for some $\boldsymbol{z}, \boldsymbol{w} \in \triangle$.*

In case $\ker(h) = \operatorname{conv}(\boldsymbol{z}, \boldsymbol{w})$ *(so the kernel is not empty), the following is true for the points $\boldsymbol{z}, \boldsymbol{w} \in \triangle$.*

(ii) The kernel of h cannot be a singleton, so $\boldsymbol{z} \neq \boldsymbol{w}$.

(iii) The points $\boldsymbol{z}$ and $\boldsymbol{w}$ are not contained in any $n-1$-face of $\triangle$.

(iv) The points $\boldsymbol{z}$ *and* $\boldsymbol{w}$ *are contained in n-faces of* $\triangle$.

(v) The points $\boldsymbol{z}$ *and* $\boldsymbol{w}$ *do not lie on a common n-face.*

(vi) The line segment $\operatorname{conv}(\boldsymbol{z}, \boldsymbol{w})$ *meets the relative boundary of* $\triangle$ *at exactly two points,* $\boldsymbol{z}$ *and* $\boldsymbol{w}$.

Proof. (i) The images of the base vectors $h(\boldsymbol{e}_1), \ldots, h(\boldsymbol{e}_{n+2})$ define a linear map $L\colon \mathbb{R}^{n+2} \to \mathbb{R}^n$. This L is a linear extension of h, so $L|\triangle = h$ and $\ker(L) \cap \triangle = \ker(h)$. Since the images of the vertices are in a general position, we have that the image of L is the whole $\mathbb{R}^n$ and so the dimension of the kernel of L is 2. Thus $\ker(L)$ is the affine hull of three affine independent points. Using Lemma 4.15 gives us that $\ker(L) \cap \operatorname{aff}(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{n+2})$ is empty or $\ker(L) \cap \operatorname{aff}(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{n+2}) = \operatorname{aff}(\boldsymbol{x}, \boldsymbol{y})$ for some $\boldsymbol{x} \neq \boldsymbol{y}$. Supposing it is non-empty yields that

$$\ker(h) = \operatorname{aff}(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{n+2}) \cap \ker(L) \cap \triangle = \operatorname{aff}(\boldsymbol{x}, \boldsymbol{y}) \cap \triangle.$$

An intersection of a compact convex set $\triangle$ and a closed affine set $\operatorname{aff}(\boldsymbol{x}, \boldsymbol{y})$ is a compact and convex subset of $\operatorname{aff}(\boldsymbol{x}, \boldsymbol{y})$. Therefore $\ker(h)$ is empty, or $\ker(h) = \operatorname{conv}(\boldsymbol{z}, \boldsymbol{w})$ for some $\boldsymbol{z}, \boldsymbol{w} \in \triangle$.

(ii) We show that $\ker(h)$ cannot be a singleton by showing that besides $\boldsymbol{z}$ there is another point contained in $\ker(h)$. This also proves that $\boldsymbol{z} \neq \boldsymbol{w}$. By Lemma 4.17 the point $\boldsymbol{z}$ cannot lie on an $n-1$-face of $\triangle$. In case $\boldsymbol{z}$ is in the relative interior of $\triangle$, the intersection of $\operatorname{aff}(\boldsymbol{x}, \boldsymbol{y})$ and some small enough ball $B(\boldsymbol{z}, \varepsilon)$ is a line segment contained in $\triangle$. We can find our second point in there. The case where $\boldsymbol{z}$ lies on an n-face of $\triangle$, we can only find points of $\ker(h)$ on one side of $\boldsymbol{z}$. The latter situation is a bit more complicated and we provide a rigorous argument for it.

Without loss of generality, we can assume that $\boldsymbol{z}$ is contained in the face $\operatorname{conv}(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{n+1})$. Thus

$$\boldsymbol{z} = \sum_{i=1}^{n+1} c_i \boldsymbol{e}_i$$

for strictly positive coefficients c_i with $\sum c_i = 1$. Denote $\varepsilon = \min_i(c_i)$. Now we can choose $\boldsymbol{z}' \in \operatorname{aff}(\boldsymbol{x}, \boldsymbol{y}) \subset \operatorname{aff}(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{n+2})$, such that $||\boldsymbol{z} - \boldsymbol{z}'|| < \varepsilon$ and $\boldsymbol{z} \neq \boldsymbol{z}'$. We take a look at two cases.

Case 1: The projection of the last coordinate is non-negative, $pr_{n+2}(\boldsymbol{z}') \geq 0$. Because $\boldsymbol{z}' \in \operatorname{aff}(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{n+2})$, we know that

$$\boldsymbol{z}' = \sum_{i=1}^{n+2} d_i \boldsymbol{e}_i$$

for some real coefficients d_i with $\sum d_i = 1$. Moreover each d_i is non-negative, as we chose $\boldsymbol{z}'$ close enough to $\boldsymbol{z}$ and assumed also $d_{n+2} = pr_{n+2}(\boldsymbol{z}') \geq 0$. This means that $\boldsymbol{z}' \in \triangle \cap \operatorname{aff}(\boldsymbol{x}, \boldsymbol{y}) = \ker(h)$ proving the claim.

Case 2: Now assume $pr_{n+2}(\boldsymbol{z}') < 0$. This is essentially same as the previous case, but we use $2\boldsymbol{z} - \boldsymbol{z}'$ instead of $\boldsymbol{z}'$ to force the last coordinate positive.

(iii) This is a straight consequence of Lemma 4.17.

(iv) Suppose that $\boldsymbol{z}$ is contained in the relative interior of $\triangle$. Then for small enough $\varepsilon > 0$ we have that

$$(1+\varepsilon)\boldsymbol{z} - \varepsilon\boldsymbol{w} \in \triangle \cap \operatorname{aff}(\boldsymbol{x}, \boldsymbol{y}) = \operatorname{conv}(\boldsymbol{z}, \boldsymbol{w}).$$

However $(1+\varepsilon)\boldsymbol{z} - \varepsilon\boldsymbol{w}$ is not contained in $\operatorname{conv}(\boldsymbol{z}, \boldsymbol{w})$ as $\boldsymbol{z} \neq \boldsymbol{w}$ by *(ii)*. This contradiction proves that neither $\boldsymbol{z}$ nor $\boldsymbol{w}$ can lie in the relative interior of $\triangle$. Hence they are contained in some n-faces of $\triangle$.

(v) This argument is only a minor variation to the proof of *(iv)*. If $\boldsymbol{z}$ and $\boldsymbol{w}$ would lie on an n-face F_n of $\triangle$, then by *(iii)* they would be lying on the relative interior of F_n. Once more, for a small $\varepsilon > 0$, the point $(1+\varepsilon)\boldsymbol{z} - \varepsilon\boldsymbol{w}$ would, and would not be contained in the set $\operatorname{conv}(\boldsymbol{z}, \boldsymbol{w}) = \triangle \cap \operatorname{aff}(\boldsymbol{x}, \boldsymbol{y})$.

(vi) We have now proven in *(iii)-(v)* that $\boldsymbol{z}$ and $\boldsymbol{w}$ lie on disjoint n-faces of $\triangle$ and that they are not contained in faces smaller than that. Thus

$$\boldsymbol{z} = \sum_{i=1}^{n+2} c_i \boldsymbol{e}_i \quad \text{and} \quad \boldsymbol{w} = \sum_{i=1}^{n+2} d_i \boldsymbol{e}_i$$

for some non-negative coefficients c_i and d_i with $\sum c_i = 1 = \sum d_i$. Also $c_j = 0$ for exactly one index j and similarly $d_k = 0$ for exactly one k. Importantly $j \neq k$ as $\boldsymbol{z}$ and $\boldsymbol{w}$ are not contained in the same n-face.

Every point in $\operatorname{conv}(\boldsymbol{z}, \boldsymbol{w})$ other than $\boldsymbol{w}$ and $\boldsymbol{z}$ is of the form $a\boldsymbol{w} + (1-a)\boldsymbol{z}$ for some $0 < a < 1$. Now

$$a\boldsymbol{w} + (1-a)\boldsymbol{z} = \sum_{i=1}^{n+2} (ac_i + (1-a)d_i)\boldsymbol{e}_i$$

which means that points of $\operatorname{conv}(\boldsymbol{z}, \boldsymbol{w})$ other than $\boldsymbol{w}$ and $\boldsymbol{z}$ are contained in the relative interior of $\triangle$.

□

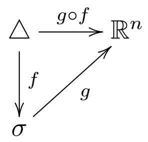
Lemma 4.19. *Let $\sigma \subset \mathbb{R}^m$ be any $n+1$-simplex and $g\colon \sigma \to \mathbb{R}^n$ a generic affine map. Then all properties of Lemma 4.18 hold for the kernel of g.*

Proof. Let $\sigma \subset \mathbb{R}^m$ be simplex with vertices $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{n+2}$ and let $\triangle \subset \mathbb{R}^{n+2}$ be standard $n+1$-simplex with canonical vertices $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{n+2}$. Suppose also that $g\colon \sigma \to \mathbb{R}^n$ is a generic affine map. Define the function $f\colon \triangle \to \sigma$,

$$f(\boldsymbol{x}) = \sum_{i=1}^{m} \lambda_i(\boldsymbol{x})\boldsymbol{v}_i$$

where λ_i is the standard projection for coordinate i. This f is a continuous bijection. Moreover in Corollary 2.18 we stated that $\triangle$ is compact. Thus f is a homeomorphism. In addition f maps the faces of $\triangle$ to the faces of σ and convex sets to convex sets.

The following diagram makes it easier to see how g inherits the properties of Lemma 4.18.



The map $g \circ f$ is affine and generic. Therefore $\ker(g \circ f)$ has all the properties of Lemma 4.18. Mapping with f preserves all these properties and

$$f(\ker(g \circ f)) = f f^{-1}(\ker(g)) = \ker(g).$$

Hence all properties of Lemma 4.18 hold for the kernel of g.

□

5 Proof of the Borsuk-Ulam theorem

We recall the statement of the Borsuk-Ulam theorem and provide an equivalent formulation to it. For any map $f \colon X \to \mathbb{R}^n$ we say that $\boldsymbol{x} \in X$ is a zero of f if $f(\boldsymbol{x}) = \mathbf{0}$.

Theorem 5.1 (Borsuk-Ulam theorem, standard form)**.** *Let $f \colon \mathbb{S}^n \to \mathbb{R}^n$ be a continuous map. There exists $\boldsymbol{x} \in \mathbb{S}^n$ with $f(\boldsymbol{x}) = f(-\boldsymbol{x})$.*

Theorem 5.2 (Borsuk-Ulam theorem, antipodal form)**.** *Let $f \colon \mathbb{S}^n \to \mathbb{R}^n$ be a continuous antipodal map. The antipodality here is the standard antipodality of $\mathbb{S}^n$, so in other words we require that $f(-\boldsymbol{x}) = -f(\boldsymbol{x})$ for all $\boldsymbol{x}$. Then f has a zero.*

Lemma 5.3. *The standard and antipodal forms of the Borsuk-Ulam theorem are equivalent.*

Proof. Suppose the standard form of the Borsuk-Ulam theorem holds. Let $f \colon \mathbb{S}^n \to \mathbb{R}^n$ be a continuous antipodal map. It immediately follows that there exists $\boldsymbol{x} \in \mathbb{S}^n$ with $f(\boldsymbol{x}) = f(-\boldsymbol{x}) = -f(\boldsymbol{x})$. Thus $f(\boldsymbol{x}) = \mathbf{0}$.

Now we suppose that the antipodal form of the Borsuk-Ulam theorem holds. Let $f \colon \mathbb{S}^n \to \mathbb{R}^n$ be a continuous function. We define a continuous antipodal function $g \colon \mathbb{S}^n \to \mathbb{R}^n$ by $g(\boldsymbol{x}) = f(\boldsymbol{x}) - f(-\boldsymbol{x})$. By the antipodal form, there exists $\boldsymbol{x} \in \mathbb{S}^n$ with $g(\boldsymbol{x}) = \mathbf{0}$. This means that $f(\boldsymbol{x}) = f(-\boldsymbol{x})$. □

In this chapter we present a proof for the Borsuk-Ulam theorem. Jiří Matoušek in his book [2] calls this method of proof the homotopy extension argument although it does not really require any theory of homotopy. The proof of Theorem 5.2 will be an immediate consequence of Lemmas 5.6 and 5.12. For both of these lemmas we will need a few more results. We shall tackle Lemma 5.6 first.

Lemma 5.4. *Let σ be an n-simplex. Denote $\mathcal{F}$ as the simplicial complex consisting of σ and all faces of σ. There exists a point $\boldsymbol{y} \in \sigma$, so that $\boldsymbol{y}$ is not contained in any $n-1$-simplex of any iteration of barycentric subdivision of $\mathcal{S}$.*

Proof. We first prove this for an n-simplex $\triangle \in \mathbb{R}^n$ with vertices $\mathbf{0}, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$. After that we show that it holds for the arbitrary n-simplex σ.

Denote by $\mathcal{T}$ the simplicial complex of $\triangle$ and its faces. We also denote by $F(\mathcal{T}^i)$ as the set of all $n-1$-simplices in the ith barycentric subdivision of $\mathcal{T}$. Every $n-1$-simplex in $\mathbb{R}^n$ is contained in a hyperplane spanned by its vertices. From Lemma 4.4 we get that $n-1$-simplices are null sets. It follows that the countable union

$$A = \bigcup_{i=0}^{\infty} \bigcup F(\mathcal{T}^i)$$

is a null set. On the other hand it is clear that $\triangle$ is a set of positive measure. This proves that there exists a point $\boldsymbol{z} \in \triangle \setminus A$ that in not contained in any $n-1$-simplex of any barycentric subdivision of $\triangle$.

Mapping vertices of $\triangle$ to vertices of σ bijectively and extending the map affinely generates a homeomorphism $h \colon \triangle \to \sigma$. More importantly, the barycentric coordinates of $\boldsymbol{x}$ match the barycentric coordinates of $h(\boldsymbol{x})$ for every $\boldsymbol{x}$. Barycentric subdivision depends only on the barycentric coordinates which means that $\mathrm{Sd}^i(\mathcal{F}) = h(\mathrm{Sd}^i(\mathcal{T}))$. It follows that $h(\boldsymbol{z})$ is not contained in any $n-1$-simplex of any iteration of the barycentric subdivision of $\mathcal{S}$. □

Lemma 5.5. *Let $\sigma \subset \mathbb{R}^n$ be an n-simplex with vertices $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{n+1}$. Suppose also that $\mathbf{0}$ is contained in the interior of σ. There exists $\varepsilon > 0$ so that whenever $f \colon \sigma \to \mathbb{R}^n$ is an affine map satisfying:*

1. *$||f(\boldsymbol{v}_i) - \boldsymbol{v}_i|| < \varepsilon$ for all vertices $\boldsymbol{v}_i$.*
2. *f is a generic affine map.*

Then f is injective and $\mathbf{0}$ is contained in the interior of the image $f(\sigma)$.

Proof. Since $\mathbf{0}$ is contained in the interior of σ, we know that there exists some $\varepsilon > 0$ so that the distance between $\mathbf{0}$ and any proper face of σ is at least 2ε. Let f be an affine map satisfying 1 and 2. Since f is an affine map that satisfies 2, it follows that the image $f(\sigma)$ is also an n-simplex and f is injective. From 1 and the fact that f is affine we obtain $||\boldsymbol{x} - f(\boldsymbol{x})|| < \varepsilon$ for all $\boldsymbol{x} \in \sigma$. Thus the distance $\mathrm{dist}(\mathbf{0}, f(A)) > \varepsilon$ for all proper faces A of σ. Hence $\mathbf{0}$ is not contained on the boundary of $f(\sigma)$.

We make a counter assumption that $\mathbf{0}$ is not contained in the interior of $f(\sigma)$. We stated that $\mathbf{0}$ is not on the boundary of $f(\sigma)$, so $\mathbf{0}$ must lie outside of $f(\sigma)$. Since $f(\sigma)$ is compact, there is a point $\boldsymbol{y} \in f(\sigma)$ with $||\boldsymbol{y}|| = \inf_{\boldsymbol{z} \in f(\sigma)} ||\boldsymbol{z}||$. Moreover $\boldsymbol{y}$ is contained in the boundary of $f(\sigma)$, so $\boldsymbol{y}$ is contained in some proper face $f(A)$ of $f(\sigma)$. Now the calculation

$$\varepsilon < \mathrm{dist}(0, f(A)) = ||\boldsymbol{y}|| \leq ||f(\mathbf{0})|| < \varepsilon$$

leads to a contradiction, which means that $\mathbf{0}$ is contained in the interior of $f(\sigma)$. □

Convention. For the rest of this chapter we refer to the sets $\Diamond^n \times \{0\}$ and $\Diamond^n \times \{1\}$ as the bottom and the top, respectively.

Lemma 5.6. *Suppose there exists a continuous antipodal map $f: \mathbb{S}^n \to \mathbb{R}^n$ that has no zeros. Then we can construct a triangulation $\mathcal{S}$ on $\Diamond^n \times [0,1]$ and a map $G: \Diamond^n \times [0,1] \to \mathbb{R}^n$ with the following properties:*

(i) G is antipodal with respect to v, where v is an antipodality on $\Diamond^n \times [0,1]$ defined by $v(\boldsymbol{x}, t) = (-\boldsymbol{x}, t)$.

(ii) When restricted to any simplex $\sigma \in \mathcal{S}$, the map G is a generic affine map.

(iii) G contains no zeros on the top.

(iv) G contains two zeros on the bottom. Both of them are contained in some n-simplices but not contained in any $n-1$-simplices of $\mathcal{S}$.

(v) G contains no other zeros on the bottom.

Proof. Let $f: \mathbb{S}^n \to \mathbb{R}^n$ be a continuous antipodal map without zeros. Lemma 3.2 gives us a homeomorphism $h: \Diamond^n \to \mathbb{S}^n$, which preserves the antipodal structure. Hence the map $f \circ h: \Diamond^n \to \mathbb{R}^n$ is continuous, antipodal and without zeros. Since $\Diamond^n$ is compact, it follows that there exists $\varepsilon > 0$ with $||(f \circ h)(\boldsymbol{x})|| \geq \varepsilon$ for all $\boldsymbol{x}$.

We take the original triangulation $\mathcal{G}$ of $\Diamond^n \times [0,1]$ that was presented in Chapter 3.1. Let A be some n-simplex of $\mathcal{G}$ that is contained in the bottom part $\Diamond^n \times \{0\}$. Using Lemma 5.4 we get a point $\boldsymbol{a} \in A$ that is not contained in any barycentric subdivision of A.

The last coordinate of $\boldsymbol{a}$ is 0, so we can consider it as an element of $\mathbb{R}^{n+1}$. We take a basis of $\mathbb{R}^{n+1}$ consisting of $\boldsymbol{a}$ and some $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n$. Let $p: \mathbb{R}^{n+1} \to \mathbb{R}^n$ be a linear map sending $\boldsymbol{a}$ to $\mathbf{0}$ and each $\boldsymbol{b}_i$ to the canonical base vector $\boldsymbol{e}_i$. It is easy to see that $\ker(p) = \operatorname{span}(\boldsymbol{a})$ and so $\boldsymbol{a}$ and $-\boldsymbol{a}$ are the only points in $\Diamond^n \subset \mathbb{R}^{n+1}$ that p maps to $\mathbf{0}$. Because p is linear, it has some nice properties. It is continuous and the restriction of p to $\Diamond^n$ is antipodal. Moreover p is affine, when restricted to any simplex in $\mathbb{R}^{n+1}$.

Let F be a homotopy from p to $f \circ h$. More precisely $F: \Diamond^n \times [0,1] \to \mathbb{R}^n$ is a continuous function defined by $F(\boldsymbol{x}, t) = (1-t)p(\boldsymbol{x}) + t(f \circ h)(\boldsymbol{x})$. Consider the antipodality v on $\Diamond^n \times [0,1]$ defined by $v(\boldsymbol{x}, t) = (-\boldsymbol{x}, t)$. The maps p and $f \circ h$ are antipodal with respect to the standard antipodality on $\Diamond^n$. It follows that the homotopy F is antipodal with respect to v, namely $F(v(\boldsymbol{x})) = -F(\boldsymbol{x})$. We will be constructing G as an approximation for F.

Because $\Diamond^n \times [0,1]$ is compact, the function F is uniformly continuous. Hence there exists $\delta > 0$ so that $||F(\boldsymbol{x}) - F(\boldsymbol{y})|| < \varepsilon/2$ whenever $||\boldsymbol{x} - \boldsymbol{y}|| < \delta$. With iterated barycentric subdivision on $\mathcal{G}$ we get an antipodal triangulation $\mathcal{S}$ on $\Diamond^n \times [0,1]$ with $\operatorname{mesh}(\mathcal{S}) < \delta$ as in Corollary 3.25.

On the bottom the map F agrees with p. The points $\boldsymbol{a}$ and $-\boldsymbol{a}$, which are zeros of F were defined in a way so that neither of them is contained in any $n-1$-simplex of $\mathcal{S}$. Denote $\triangle$ as the unique n-simplex containing $\boldsymbol{a}$. We want to show that p is injective on $\triangle$. (Once again we consider $\triangle$ as a subset of $\mathbb{R}^{n+1}$.)

Let $p(\boldsymbol{z}) = p(\boldsymbol{w})$ for some $\boldsymbol{z}, \boldsymbol{w} \in \triangle$. Since p is linear, it follows that $\boldsymbol{z} - \boldsymbol{w} \in \ker(p) = \operatorname{span}(\boldsymbol{a})$. Thus $\boldsymbol{z} - \boldsymbol{w} = c\boldsymbol{a}$ for some $c \in \mathbb{R}$. Suppose now that $c \neq 0$. This means that $c^{-1}(\boldsymbol{z} - \boldsymbol{w}) = \boldsymbol{a} \in \triangle$. However the sum of barycentric coordinates of $c^{-1}(\boldsymbol{z} - \boldsymbol{w})$ with respect to vertices of $\triangle$ is 0. This implies $c^{-1}(\boldsymbol{z} - \boldsymbol{w}) \notin \triangle$ which is a contradiction. Hence $c = 0$, which means that $\boldsymbol{z} = \boldsymbol{w}$ and p is injective on $\triangle$.

The map F is affine and injective on the n-simplex $\triangle$. Thus the image $F(\triangle) \subset \mathbb{R}^n$ is also an n-simplex. We know that $\mathbf{0}$ is contained in the relative interior of $F(\triangle)$. From Lemma 5.5 we get a threshold $\varepsilon_1 > 0$, so that any smaller generic shift of vertices of $F(\triangle)$ keeps the shifted simplex containing $\mathbf{0}$ in its interior.

Denote $\mathcal{B}$ as the collection of simplices of $\mathcal{S}$ that are contained in the bottom and $\mathcal{B}' = \mathcal{B} \setminus \{\triangle, v(\triangle)\}$. The union

$$C = \bigcup_{\sigma \in \mathcal{B}'} \sigma$$

is a compact set containing no zeros for F. We get a second threshold by setting $\varepsilon_2 = \min_{\boldsymbol{x} \in C} ||F(\boldsymbol{x})|| > 0$. The third threshold is $\varepsilon/2$ and we define $\alpha = \min\{\varepsilon_1, \varepsilon_2, \varepsilon/2\}$.

The function F is antipodal, so in particular it is antipodal on $\operatorname{vert}(\mathcal{S})$. We use Lemma 4.9 to get a function $g\colon \operatorname{vert}(\mathcal{S}) \to \mathbb{R}^n$ with the following properties:

1. The map g is injective.
2. $||F(\boldsymbol{x}) - g(\boldsymbol{x})|| < \alpha$ for all $\boldsymbol{x} \in \operatorname{vert}(\mathcal{S})$.
3. The map g is antipodal with respect to v.
4. For any $A \subset \operatorname{vert}(\mathcal{S})$ that does not contain a pair of antipodal points, the set $g(A)$ is in a general position with respect to $\mathbf{0}$.

Finally we define $G\colon \Diamond^n \times [0,1] \to \mathbb{R}^n$ as the locally affine extension of g that is introduced in Lemma 4.14. Next we check that G satisfies each of the Properties *(i)-(v)*.

(i) Let $\boldsymbol{x} \in \Diamond^n \times [0,1]$. Then $\boldsymbol{x}$ is contained in some $\sigma \in \mathcal{S}$ with vertices $\boldsymbol{u}_1, \dots \boldsymbol{u}_m$ and $\boldsymbol{x} = \sum_i c_i \boldsymbol{u}_i$. Because $\mathcal{S}$ is symmetric, it follows that $v(\sigma)$ is a simplex in $\mathcal{S}$ with vertices $v(\boldsymbol{u}_1), \dots, v(\boldsymbol{u}_m)$. The facts that g is antipodal and

that G is affine both in σ and $v(\sigma)$ enable us to calculate

$$\begin{aligned}-G(\boldsymbol{x}) &= \sum_{i=1}^{m} c_i(-g(\boldsymbol{u}_i)) = \sum_{i=1}^{m} c_i(g(v(\boldsymbol{u}_i)) = G\left(\sum_{i=1}^{m} c_i v(\boldsymbol{u}_i)\right) \\ &= (G \circ v)\left(\sum_{i=1}^{m} c_i \boldsymbol{u}_i\right) = (G \circ v)(\boldsymbol{x}).\end{aligned}$$

(ii) Let σ be a simplex in $\mathcal{S}$. We constructed $\mathcal{S}$ in a way so that no vertices of σ are antipodal to each other. From the Properties 1 and 4 of g it follows that $g|\operatorname{vert}(\sigma)$ is a generic map. The map G is affine on σ, so G is generic when restricted to σ.

(iii) Let $\boldsymbol{x} \in \lozenge^n \times [0,1]$. As previously $\boldsymbol{x}$ is contained in some $\sigma \in \mathcal{S}$ with vertices $\boldsymbol{u}_1, \dots \boldsymbol{u}_m$ and $\boldsymbol{x} = \sum_i c_i \boldsymbol{u}_i$ for some convex coefficients $c_i \geq 0$. We chose the mesh of our simplicial complex to be small enough, so that $||F(\boldsymbol{x}) - F(\boldsymbol{u}_i)|| < \varepsilon/2$ for all vertices $\boldsymbol{u}_i$. From Property 2 of g we get that $||F(\boldsymbol{u}_i) - g(\boldsymbol{u}_i)|| < \varepsilon/2$ for all i. Hence we can calculate

$$\begin{aligned}||F(\boldsymbol{x}) - G(\boldsymbol{x})|| &= ||\sum_{i=1}^{m} c_i F(\boldsymbol{x}) - \sum_{i=1}^{m} c_i g(\boldsymbol{u}_i)|| \\ &= ||\sum_{i=1}^{m} c_i(F(\boldsymbol{x}) - g(\boldsymbol{u}_i))|| \\ &\leq \sum_{i=1}^{m} c_i\big(||F(\boldsymbol{x}) - F(\boldsymbol{u}_i)|| + ||F(\boldsymbol{u}_i) - g(\boldsymbol{u}_i)||\big) \\ &< \varepsilon\end{aligned}$$

which proves that $||F - G||_{\sup} < \varepsilon$. It follows that G has no zeros on the top.

(iv) We want to show that the two zeros of G at the bottom are contained in the same simplices that contain $\boldsymbol{a}, -\boldsymbol{a}$, the two zeros of F at the bottom. We defined $\triangle$ to be the n-simplex containing $\boldsymbol{a}$. Denote the vertices of $\triangle$ with $\boldsymbol{u}_1, \dots, \boldsymbol{u}_{n+1}$. Previously we showed that F is injective and affine when restricted to $\triangle$. Hence the image $F(\triangle)$ is an n-simplex with vertices $F(\boldsymbol{u}_1), \dots F(\boldsymbol{u}_{n+1})$.

Let us define a shifting map $s\colon F(\triangle) \to \mathbb{R}^n$ on the vertices with $s(F(\boldsymbol{u}_i)) = g(\boldsymbol{u}_i)$. Elsewhere on $F(\triangle)$ we define s by extending it affinely. It follows from the Properties 1 and 4 of g that s is a generic affine map. From the Property 2, we get that $||F(\boldsymbol{u}_i) - s(F(\boldsymbol{u}_i))|| < \varepsilon_2$ which was the threshold for Lemma 5.5. Now using the Lemma 5.5 states that $\mathbf{0}$ is contained in the relative interior of the n-simplex $s(F(\triangle))$ and that the function $s \circ F$ is injective when restricted to $\triangle$. Combining this with the fact that $s \circ F$ is affine on $\triangle$, we get that

$$\mathbf{0} = \sum_{i=1}^{n+1} c_i s(F(\boldsymbol{u}_i)) = (s \circ F)\left(\sum_{i=1}^{n+1} c_i \boldsymbol{u}_i\right)$$

for some coefficients $c_i > 0$. We denote $\boldsymbol{b} = \sum c_i \boldsymbol{u}_i$. The point $\boldsymbol{b}$ is contained in the relative interior of $\triangle$. Because the function $s \circ F$ is injective when restricted to $\triangle$, we also know that $\boldsymbol{b}$ is the unique zero of $s \circ F$ in $\triangle$.

When restricted to the simplex $\triangle$ the maps $s \circ F$ and G are both affine and agree on the vertices of $\triangle$. Therefore $s \circ F = G$ on the simplex $\triangle$. It follows that in the simplex $\triangle$ the map G has exactly one zero $\boldsymbol{b}$. Because $\boldsymbol{b}$ is contained in the relative interior of the n-simplex $\triangle$, it is not contained in any $n-1$-simplex of $\mathcal{S}$.

The function G is antipodal as we proved in *(i)*, so $G(v(\boldsymbol{b})) = \mathbf{0}$ also. This means that $v(\boldsymbol{b})$ is the second zero for G on the bottom. Since $\mathcal{S}$ is antipodal, the point $v(\boldsymbol{b})$ is contained in the n-simplex $v(\triangle)$ but not in any $n-1$-simplex of $\mathcal{S}$.

(v) In *(iv)* we showed that G contains exactly two zeros in the simplices $\triangle$ and $v(\triangle)$. We now want to show that there are no other zeros on the bottom for G.

Let $\boldsymbol{x}$ be a point in the bottom that is not contained in $\triangle$ or $v(\triangle)$. Thus $\boldsymbol{x}$ is contained in some simplex $\sigma \in \mathcal{B}'$ with vertices $\boldsymbol{u}_1, \ldots \boldsymbol{u}_k$. We can write $\boldsymbol{x}$ as a convex combination $\sum_i c_i \boldsymbol{u}_i$. From 2 we get that $||g(\boldsymbol{u}_i) - F(\boldsymbol{u}_i)|| < \varepsilon_2$ for every i. Since G and F are affine on σ, we can calculate

$$\begin{aligned} ||G(\boldsymbol{x}) - F(\boldsymbol{x})|| &= ||\sum_{i=1}^{k} c_i(g(\boldsymbol{u}_i) - F(\boldsymbol{u}_i))|| \\ &\leq \sum_{i=1}^{k} c_i||(g(\boldsymbol{u}_i) - F(\boldsymbol{u}_i))|| \\ &< \varepsilon_2. \end{aligned}$$

The threshold ε_2 was defined in a way so that $||F(\boldsymbol{x})|| \geq \varepsilon_2$. Combining these facts with the reverse triangle inequality gives us

$$||G(\boldsymbol{x})|| \geq ||F(\boldsymbol{x})|| - ||G(\boldsymbol{x}) - F(\boldsymbol{x})|| > 0.$$

□

Next we want to show that such a map G that we constructed in the previous lemma cannot exist. We will do so in Lemma 5.12, but a few results are needed before that. We investigate in the technical Lemmas 5.8 and 5.11 how n-simplices are faces of $n+1$-simplices in triangulations of $\lozenge^n \times [0,1]$. Arguably a more elegant proof to these lemmas could be found in J.R. Munkres' book *Elements of Algebraic Topology* [6, corollary 63.3]. But Munkres uses homology in his proofs and the point of this thesis is to provide an elementary proof to the Borsuk-Ulam theorem. Hence we avoid using algebraic topology and include somewhat crude proofs for Lemmas 5.8 and 5.11.

We shall first further investigate some properties of shards of $\lozenge^n \times [0,1]$. Recall that the set $\lozenge^n \times [0,1]$ is a union of shards, i.e., sets of the form

$$\operatorname{conv}(s_1\boldsymbol{e}_1, \ldots, s_{n+1}\boldsymbol{e}_{n+1}) \times [0,1]$$

where $s_i = \pm 1$ and $\boldsymbol{e}_i$ is the ith canonical base vector of $\mathbb{R}^{n+1}$ for each i. Shards are convex sets as products of two convex sets. A shard R contains an affine independent set of points

$$(s_1\boldsymbol{e}_1, 0), \ldots, (s_{n+1}\boldsymbol{e}_{n+1}, 0), (s_{n+1}\boldsymbol{e}_{n+1}, 1).$$

Moreover every $\boldsymbol{r} \in R$ can be uniquely written as an affine combination of these points. This means that R is contained in some hyperplane. A point $(x_1, \ldots, x_{n+2})$ in the intersection of two distinct shards must have $x_i = 0$ for at least one $i \leq n+1$.

Lemma 5.7. *Let $\mathcal{S}$ be a triangulation of $\Diamond^n \times [0,1]$. Then every $\sigma \in \mathcal{S}$ is contained in some shard of $\Diamond^n \times [0,1]$. In addition* $\dim(\mathcal{S}) \leq n+1$.

Proof. Suppose $\sigma \in \mathcal{S}$ is not contained fully in any shard. Then σ contains points $(\boldsymbol{x}, t_1) = (x_1, \ldots, x_{n+1}, t_1)$ and $(\boldsymbol{y}, t_2) = (y_1, \ldots, y_{n+1}, t_2)$ with $x_k > 0 > y_k$ for some k. Without loss of generality we can assume $k = 1$. Setting

$$a = \frac{-y_1}{x_1 - y_1}$$

we get that the first coordinate of $a\boldsymbol{x} + (1-a)\boldsymbol{y}$ is 0. We can now calculate

$$\begin{aligned}
||a\boldsymbol{x} + (1-a)\boldsymbol{y}||_1 &= \sum_{i=1}^{n+1} |ax_i + (1-a)y_i| \\
&= \sum_{i=2}^{n+1} |ax_i + (1-a)y_i| \\
&\leq a\sum_{i=2}^{n+1} |x_i| + (1-a)\sum_{i=2}^{n+1} |y_i| \\
&< a\sum_{i=1}^{n+1} |x_i| + (1-a)\sum_{i=1}^{n+1} |y_i| \\
&= a||\boldsymbol{x}||_1 + (1-a)||\boldsymbol{y}||_1 \\
&= 1.
\end{aligned}$$

This means that the convex combination $a(\boldsymbol{x}, t_1) + (1-a)(\boldsymbol{y}, t_2)$ is not contained in the space $\Diamond^n \times [0,1]$, which contradicts $\sigma \in \mathcal{S}$.

Suppose now that $\triangle \in \mathcal{S}$ is an $n+2$-simplex. We showed that $\triangle$ is contained in some shard R of $\Diamond^n \times [0,1]$. But R is contained in some hyperplane of $\mathbb{R}^{n+2}$ which means that $\triangle$ cannot lie on R. This proves $\dim(\mathcal{S}) \leq n+1$.

□

Lemma 5.8. *Let $\mathcal{S}$ be a triangulation of $\Diamond^n \times [0,1]$. Suppose $\sigma \in \mathcal{S}$ is an n-simplex which is not fully contained in the top or the bottom of $\Diamond^n \times [0,1]$. Then σ is a face of exactly two $n+1$-simplices.*

Before we prove this lemma we shall first prove a smaller proposition to help with this proof.

Lemma 5.9. *Let $\mathcal{S}$ be a triangulation of $\Diamond^n \times [0,1]$. Suppose $\triangle \in \mathcal{S}$ is an n-simplex, $H = \{\boldsymbol{x} \in \mathbb{R}^{n+2} \mid \langle \boldsymbol{x}, \boldsymbol{u} \rangle = 0\}$ is a hyperplane with $\boldsymbol{u} \in \mathbb{S}^{n+1}$ and $\triangle \subset H$. Let A and B be $n+1$-simplices of $\mathcal{S}$ with* $\operatorname{vert}(A) = \operatorname{vert}(\triangle) \cup \{\boldsymbol{a}\}$ *and* $\operatorname{vert}(B) = \operatorname{vert}(\triangle) \cup \{\boldsymbol{b}\}$. *We assume that A and B are contained in a common shard R and that $\langle \boldsymbol{a}, \boldsymbol{u} \rangle > 0$ and $\langle \boldsymbol{b}, \boldsymbol{u} \rangle > 0$. Then $A = B$.*

Proof. The shard R is contained in some hyperplane P. By Lemma 2.9 the hyperplane P is the affine hull of any $n+2$ affine independent vectors contained in P. Hence $P = \operatorname{aff}(\operatorname{vert}(A))$, and we can write any point of R as an affine combination of the vertices of A.

Consider the sequence $(\boldsymbol{x}_j)_{j\in\mathbb{N}}$ in B with $\boldsymbol{x}_j = (1 - j^{-1})b(\triangle) + j^{-1}\boldsymbol{b}$, where $b(\triangle)$ is the barycenter of $\triangle$. Since this sequence is contained in R we can look at the barycentric coordinates of the sequence with respect to $\operatorname{vert}(A)$. The barycentric projections are continuous, so the tuple of coordinates for $\lim \boldsymbol{x}_j$ is

$$\left(\frac{1}{n+1}, \dots, \frac{1}{n+1}, 0\right).$$

The last coordinate of every $\boldsymbol{x}_j$ is positive, because $\langle \boldsymbol{x}_j, \boldsymbol{u} \rangle > 0$ for every j. Thus for some big enough k the point $\boldsymbol{x}_k$ has positive barycentric coordinates with respect to $\operatorname{vert}(A)$. This means that $\boldsymbol{x}_k$ is contained in the relative interior of A.

We now showed that $A \cap B$ contains a point in the relative interior of A. Because $\mathcal{S}$ is a simplicial complex, $A \cap B$ is a face of A. The only face of A containing points of the relative interior of A is the simplex A itself. Thus $A = A \cap B$. Now again $A \cap B$ is an $n+1$-face of B which means that $A = B$. □

Proof of Theorem 5.8. Let $\sigma \in \mathcal{S}$ be an n-simplex that is not fully contained in the top or the bottom. This implies that the barycenter $b(\sigma)$ is not contained in the top or the bottom either. We split into two cases depending on the positioning of σ with regards to the shards.

Case 1: σ not contained in the intersection of two distinct shards. We first show that the barycenter $b(\sigma)$ is not contained in the intersection of any two shards either. Denote by R a shard with $\sigma \subset R$ and denote the kth projection map of $\mathbb{R}^{n+2}$ by p_k. Assume that $b(\sigma)$ is also contained in another shard Q. This implies that $p_k(b(\sigma)) = 0$ for some $k \leq n+1$. It follows that $p_k(\boldsymbol{v}_i) = 0$ for all vertices $\boldsymbol{v}_i$ of σ. Hence σ is contained in the intersection of two distinct shards, contradicting the assumption of this case. Thus $b(\sigma)$ is not contained in the intersection of two distinct shards.

The set $\operatorname{vert}(\sigma) \cup \{\boldsymbol{0}\}$ is affine independent and so it determines a hyperplane

$$H = \operatorname{aff}(\operatorname{vert}(\sigma), \boldsymbol{0}) = \{\boldsymbol{x} \in \mathbb{R}^{n+2} \mid \langle \boldsymbol{x}, \boldsymbol{u} \rangle = 0\}$$

with some $\boldsymbol{u} \in \mathbb{S}^{n+1}$. The shard R contains an affine independent set of points

$$(s_1\boldsymbol{e}_1, 0), \dots, (s_{n+1}\boldsymbol{e}_{n+1}, 0), (s_{n+1}\boldsymbol{e}_{n+1}, 1)$$

with some $s_i \in \{0, 1\}$. On the other hand the hyperplane H contains $\mathbf{0}$ which is affine independent from all of the above. Thus we have a point $\boldsymbol{r} \in R$ that is not contained in H. If such point would not exist, then H would contain $n+3$ affine independent points leading to the contradiction that $H = \mathbb{R}^{n+2}$. Hence $\langle \boldsymbol{r}, \boldsymbol{u} \rangle \neq 0$ and by symmetry we assume $\langle \boldsymbol{r}, \boldsymbol{u} \rangle > 0$

Consider a sequence $(\boldsymbol{x}_j)_{j \in \mathbb{N}}$ in R, where $\boldsymbol{x}_j = (1 - j^{-1})b(\sigma) + j^{-1}\boldsymbol{r}$. Every point $\boldsymbol{x}_j$ is contained in some simplex of $\mathcal{S}$. In particular some $\triangle_1$ of the finitely many simplices of $\mathcal{S}$ contains infinitely many points $\boldsymbol{x}_i$. By Lemma 2.18 the simplex $\triangle_1$ is closed, so $b(\sigma) = \lim \boldsymbol{x}_i \in \triangle_1$. For every j we obtain that $\boldsymbol{x}_j \notin H$ and especially $\boldsymbol{x}_j \notin \sigma$, because $\langle \boldsymbol{x}_j, \boldsymbol{u} \rangle > 0$.

We know that the barycenter $b(\sigma)$ is contained only in the simplex σ and simplices that have σ as their face. Therefore σ is a face of $\triangle_1$. The simplex $\triangle_1$ contains points $\boldsymbol{x}_j$ with $\langle \boldsymbol{x}_j, \boldsymbol{u} \rangle > 0$, but for every $\boldsymbol{s} \in \sigma$ we have that $\langle \boldsymbol{s}, \boldsymbol{u} \rangle = 0$. This means that $\triangle_1 \not\subset \sigma$, so $\dim(\triangle_1) \geq n + 1$. It follows that $\triangle_1$ has a vertex $\boldsymbol{t}_1$ with $\langle \boldsymbol{t}_1, \boldsymbol{u}, \rangle > 0$. However by Lemma 5.7 we know that $\dim(\mathcal{S}) \leq n + 1$, so $\dim(\triangle_1) \leq n + 1$. Thus we have found the first $n + 1$-simplex $\triangle_1$ that has σ as its face.

From the previous sequence $(\boldsymbol{x}_j)_{j \in \mathbb{N}}$ we can make a new sequence $(\boldsymbol{y}_j)_{j \in \mathbb{N}}$ with $\boldsymbol{y}_j = 2b(\sigma) - \boldsymbol{x}_j$. The barycenter $b(\sigma)$ is neither contained in an intersection of any two shards nor at the top or the bottom. This means that every coordinate of $b(\sigma)$ is not 0 and the last coordinate is not 1 as well. With large enough j the points $\boldsymbol{y}_j$ share coordinate-wise the same signs with $b(\sigma)$. Thus $\boldsymbol{y}_j \in R \subset \Diamond^n \times [0, 1]$ with large indices j. This time we have that $\langle \boldsymbol{y}_j, \boldsymbol{u} \rangle < 0$. Repeating a similar argument as previously we can find an $n + 1$-simplex $\triangle_2$ that has σ as its face. In addition $\triangle_2$ has a vertex $\boldsymbol{t}_2$ with $\langle \boldsymbol{t}_2, \boldsymbol{u} \rangle < 0$ and so $\triangle_2$ is distinct from $\triangle_1$.

Now we have found two $n + 1$-simplices $\triangle_1$ and $\triangle_2$ that have σ as their face. We suppose that there is a third $n + 1$-simplex $\triangle_3$ that has σ as its face. The simplices $\sigma, \triangle_1, \triangle_2$ and $\triangle_3$ contain $b(\sigma)$, which is only contained in a single shard R. Therefore all of these simplices are fully contained in a common shard by Lemma 5.7. Denote the vertex of $\triangle_3$ that is not a vertex of σ with $\boldsymbol{t}_3$. If $\langle \boldsymbol{t}_3, \boldsymbol{u} \rangle > 0$, then Lemma 5.9 states that $\triangle_1 = \triangle_3$. Symmetrically: if $\langle \boldsymbol{t}_3, \boldsymbol{u} \rangle < 0$, then $\triangle_2 = \triangle_3$.

Finally we check that $\langle \boldsymbol{t}_3, \boldsymbol{u} \rangle = 0$ is not possible. Assume $\langle \boldsymbol{t}_3, \boldsymbol{u} \rangle = 0$. Then $\triangle_3$ is contained in the intersection of two hyperplanes $\operatorname{aff}(R)$ and H. By Lemma 4.15 this intersection is an affine hull of $n+1$ points. Thus $\triangle_3$ cannot have $n+2$ affine independent vertices, which is a contradiction. This concludes the proof of *Case 1.*

Case 2: σ is contained in the intersection of some two distinct shards Q and R. Suppose σ is contained in some third distinct shard P, which in particular means that

$$\sigma \subset \operatorname{aff}(Q) \cap \operatorname{aff}(R) \cap \operatorname{aff}(P).$$

Using Lemma 4.15 twice states that the set $\operatorname{aff}(Q) \cap \operatorname{aff}(R) \cap \operatorname{aff}(P)$ is empty, or the affine hull of only n points. Therefore σ, an n-simplex with $n + 1$ affine independent vertices cannot be contained in there. This contradiction proves us that σ is only contained in the intersection of two shards.

For some $q_i, r_i \in \{1, -1\}$ we have that

$$Q = \text{conv}(q_1 \boldsymbol{e}_1, \ldots, q_{n+1} \boldsymbol{e}_{n+1}) \times [0, 1]$$
$$R = \text{conv}(r_1 \boldsymbol{e}_1, \ldots, r_{n+1} \boldsymbol{e}_{n+1}) \times [0, 1].$$

Since Q and R are distinct, $q_k = -r_k$ for some k. We define a hyperplane $H = \{\boldsymbol{x} \in \mathbb{R}^{n+2} \mid \langle \boldsymbol{x}, q_k \boldsymbol{e}_k \rangle = 0\}$ and see that $\sigma \subset H$.

Similarly as in *Case 1*, a sequence $(\boldsymbol{x}_j)_{j \in \mathbb{N}}$ where $\boldsymbol{x}_j = (1-j^{-1})b(\sigma)+j^{-1}q_k \boldsymbol{e}_k$ yields us an $n+1$-simplex $\triangle_1$ that contains σ as its face. This simplex $\triangle_1$ contains σ, so it must be fully contained in Q or R. As $\langle \boldsymbol{x}_j, q_k \boldsymbol{e}_k \rangle = j^{-1} > 0$ for all j, we have that $\triangle_1 \subset Q$ and $\triangle_1$ has a vertex $\boldsymbol{t}_1$ such that $\langle \boldsymbol{t}_1, q_k \boldsymbol{e}_k \rangle > 0$. Correspondingly with the sequence $(\boldsymbol{y}_j)_{j \in \mathbb{N}}$ where $\boldsymbol{y}_j = (1-j^{-1})b(\sigma)+j^{-1}r_k \boldsymbol{e}_k$ we find a simplex $\triangle_2 \subset R$. The simplex $\triangle_2$ has σ as its face and a vertex $\boldsymbol{t}_2$ with $\langle \boldsymbol{t}_2, q_k \boldsymbol{e}_k \rangle < 0$.

Now we show that no other simplices that have σ as their face can be found. Suppose $\triangle_3$ is an $n+1$-simplex containing σ as its face. Denote by $\boldsymbol{t}_3$ the vertex of $\triangle_3$ that is not a vertex of σ. If $\langle \boldsymbol{t}_3, q_k \boldsymbol{e}_k \rangle > 0$, then $\triangle_3$ is contained in Q. Then Lemma 5.9 states that $\triangle_3 = \triangle_1$. Similarly if $\langle \boldsymbol{t}_3, q_k \boldsymbol{e}_k \rangle < 0$, we have that $\triangle_3 = \triangle_2$. The case $\langle \boldsymbol{t}_3, q_k \boldsymbol{e}_k \rangle = 0$ is not possible as we can see by repeating the argument in *Case 1*. This concludes *Case 2* completing the whole proof. □

We move on to proving that an n-simplex in the top and bottom parts of $\Diamond^n \times [0, 1]$ is the face of exactly one $n+1$-simplex. First we prove a small lemma.

Lemma 5.10. *Let $\mathcal{S}$ be a triangulation on $\Diamond^n \times [0, 1]$ and $\sigma \in \mathcal{S}$. Then the intersection of σ and the bottom is a simplex of $\mathcal{S}$. Similarly $\sigma \cap (\Diamond^n \times \{1\}) \in \mathcal{S}$ as well.*

Proof. Let $\sigma \in \mathcal{S}$ be a simplex with vertices $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$. We assume that σ is contained in the bottom part $\Diamond^n \times \{0\}$. The case $\sigma \subset \Diamond^n \times \{1\}$ is almost identical. We denote p_{n+2} as the last standard projective map on $\mathbb{R}^{n+2}$ and define

$$B = \text{conv}\{\boldsymbol{v}_i \mid p_{n+2}(\boldsymbol{v}_i) = 0\}.$$

Clearly B is a face of σ, which means that the task turns into proving $\sigma \cap (\Diamond^n \times \{0\}) \subset B$.

Let $\boldsymbol{x} \in \sigma \cap (\Diamond^n \times \{0\})$. This means that $\boldsymbol{x}$ can be written as a convex combination

$$\boldsymbol{x} = \sum_{i=1}^{m} c_i \boldsymbol{v}_i$$

with some non-negative coefficients c_i. Since $p_{n+2}(\boldsymbol{x}) = 0$ and $p_{n+2}(\boldsymbol{v}_i) \geq 0$ for all i, it follows that $c_i = 0$ whenever $p_{n+2}(\boldsymbol{v}_i) > 0$. Hence $\boldsymbol{x} \in B$ and $B = \sigma \cap (\Diamond^n \times \{0\})$. The simplicial complex $\mathcal{S}$ contains all faces of its simplices, which means that $\sigma \cap (\Diamond^n \times \{0\}) \in \mathcal{S}$.

□

Lemma 5.11. *Let $\mathcal{S}$ be a triangulation of $\Diamond^n \times [0,1]$. Let σ be an n-simplex contained in the top or the bottom. Then σ is a face of exactly one $n+1$-simplex.*

Proof. We can assume that σ is contained in the bottom $\Diamond^n \times \{0\}$ which we denote by B. We also denote $X = \Diamond^n \times [0,1]$ and define maps

$$f\colon X \to X,\ f(x_1, \ldots, x_{n+2}) = \left(x_1, \ldots, x_{n+1}, \frac{1 + x_{n+2}}{2}\right)$$
$$g\colon X \to X,\ g(x_1, \ldots, x_{n+2}) = \left(x_1, \ldots, x_{n+1}, \frac{1 - x_{n+2}}{2}\right).$$

These maps are injective linear maps with an added translation. Therefore the set

$$\mathcal{T} = \{f(\triangle), g(\triangle) \mid \triangle \in \mathcal{S}\}$$

is a collection of simplices. We show that $\mathcal{T}$ is a simplicial complex. Certainly for every $\triangle \in \mathcal{T}$, all the faces of $\triangle$ are contained in $\mathcal{T}$ as well.

Let $\triangle_1$ and $\triangle_2$ be simplices of $\mathcal{T}$. Suppose we have that $\triangle_1 = f(\sigma_1)$ and $\triangle_2 = f(\sigma_2)$ for some $\sigma_1, \sigma_2 \in \mathcal{S}$. Then by injectivity of f, we obtain

$$\triangle_1 \cap \triangle_2 = f(\sigma_1 \cap \sigma_2) \in \mathcal{T}.$$

Similarly, if both $\triangle_1$ and $\triangle_2$ are images of simplices with the map g, their intersection is in $\mathcal{T}$.

Suppose now $\triangle_1 = f(\sigma_1)$ and $\triangle_2 = g(\sigma_2)$ for some $\sigma_1, \sigma_2 \in \mathcal{S}$. We know that $f(\sigma_1)$ and $g(\sigma_2)$ can only intersect in the middle part of X, that is $\Diamond^n \times \{1/2\}$. We also know that the maps f and g agree on the bottom B which enables us to calculate

$$\begin{aligned}
\triangle_1 \cap \triangle_2 &= f(\sigma_1) \cap g(\sigma_2) \cap \left(\Diamond^n \times \left\{\frac{1}{2}\right\}\right) \\
&= f(\sigma_1) \cap f(B) \cap g(\sigma_2) \cap g(B) \\
&= f(\sigma_1 \cap B) \cap g(\sigma_2 \cap B) \\
&= f(\sigma_1 \cap B) \cap f(\sigma_2 \cap B) \\
&= f((\sigma_1 \cap B) \cap (\sigma_2 \cap B)).
\end{aligned}$$

Using Lemma 5.10 states that the intersection $\triangle_1 \cap \triangle_2$ is a simplex in $\mathcal{T}$. Hence $\mathcal{T}$ is a simplicial complex.

It is easy to see that $\mathcal{T}$ is a triangulation of X. Hence σ is a face of exactly two simplices in $\mathcal{T}$ by Lemma 5.8. If one of them is of the form $f(\triangle)$, then the other must be of the form $g(\triangle)$ with the same $n+1$-simplex $\triangle$. This means that in $\mathcal{S}$ the simplex σ is a face of only one $n+1$-simplex, proving the claim. □

Lemma 5.12. *There cannot exist a triangulation $\mathcal{S}$ of $\Diamond^n \times [0,1]$ and a map $G\colon \Diamond^n \times [0,1] \to \mathbb{R}^n$ that fulfill the properties of Lemma 5.6.*

Proof. Suppose towards a contradiction that such a triangulation $\mathcal{S}$ and a map G exist. By Properties *(i)* and *(iv)* of Lemma 5.6 the map G has two zeros on the bottom, $\boldsymbol{b}_0$ and $v(\boldsymbol{b}_0)$. Neither of them are contained in any $n-1$-simplices of $\mathcal{S}$. We show that inside the kernel of G there is a path from $\boldsymbol{b}_0$ to $v(\boldsymbol{b}_0)$. We inductively define a sequence of points $\boldsymbol{b}_i$, n-simplices $\triangle_i$ and $n+1$-simplices σ_i.

We already defined $\boldsymbol{b}_0$ and we will define $\triangle_0$ as the unique n-simplex containing $\boldsymbol{b}_0$ at the bottom. Since $\triangle_0$ is contained in the bottom, by Lemma 5.11 the n-simplex is contained in exactly one $n+1$-simplex of $\mathcal{S}$ which we denote by σ_0. By the Property *(ii)* the map G is generic affine in σ_0. It follows from Lemma 4.19, that $\ker(G) \cap \sigma_0$ is a line segment. One of the end points of this line segment is $\boldsymbol{b}_0$ and the other we denote with $\boldsymbol{b}_1$. Lemma 4.19 further restricts that $\boldsymbol{b}_1$ is contained in another n-face of σ_0, which we denote by $\triangle_1$. Since $\triangle_1$ is distinct from $\triangle_0$ it cannot be contained in the bottom or the top.

Suppose now that we have defined points $\boldsymbol{b}_0, \ldots, \boldsymbol{b}_j$, n-simplices $\triangle_0, \ldots, \triangle_j$ and $n+1$-simplices $\sigma_0, \ldots, \sigma_{j-1}$. We assume that all of these points and simplices are distinct from each other and that $\triangle_j$ is not contained in the top or the bottom. Now by Lemma 5.8 the simplex $\triangle_j$ is a face of exactly two $n+1$-simplices. One of them is σ_{j-1} and the other we denote by σ_j.

We show that σ_j is distinct from $\sigma_0, \ldots, \sigma_{j-1}$. We assume the contrary that $\sigma_j = \sigma_i$ for some $i \leq j-2$. This would mean that σ_i would contain three distinct zeros $\boldsymbol{b}_i$, $\boldsymbol{b}_{i+1}$ and $\boldsymbol{b}_j$ from three distinct n-faces $\triangle_i$, $\triangle_{i+1}$ and $\triangle_j$. It would cause a contradiction between Lemma 4.19 and Property *(ii)*. Hence we have found a new $n+1$-simplex, σ_j.

Inside σ_j the kernel of G is a line segment of two points, $\boldsymbol{b}_j$ and another point, which we denote by $\boldsymbol{b}_{j+1}$. The point $\boldsymbol{b}_{j+1}$ is contained in some n-face of σ_j, which we denote by $\triangle_{j+1}$. The end points $\boldsymbol{b}_j$ and $\boldsymbol{b}_{j+1}$ lie on distinct n-faces, so $\sigma_j \neq \sigma_{j+1}$. The simplex σ_{j+1} is also distinct from $\sigma_0, \ldots, \sigma_{j-1}$ as none of them are faces of σ_j. It also follows that $\boldsymbol{b}_{j+1}$ is distinct from $\boldsymbol{b}_0, \ldots, \boldsymbol{b}_j$.

If $\triangle_{j+1}$ is not contained in the top or the bottom we can continue on with the next step in the induction. We look at the case where $\triangle_{j+1}$ and $\boldsymbol{b}_{j+1}$ are contained in the top or the bottom. By Properties *(iii)*-*(v)* the only simplices containing zeros of G, that lie on the top or the bottom, are $\triangle_0$ and $v(\triangle_0)$. Since $\triangle_{j+1}$ has a zero $\boldsymbol{b}_{j+1}$ and is distinct from $\triangle_0$ it follows that $\triangle_{j+1} = v(\triangle_0)$. Here we terminate the inductive sequence.

There are only finitely many simplices in $\mathcal{S}$. Hence the sequence of points and simplices has to terminate eventually and we obtain a finite sequence of points $\boldsymbol{b}_0, \ldots, \boldsymbol{b}_k$, n-simplices $\triangle_0, \ldots, \triangle_k$ and $n+1$-simplices $\sigma_0, \ldots, \sigma_{k-1}$. The set

$$\bigcup_{i=0}^{k-1} \operatorname{conv}(\boldsymbol{b}_i, \boldsymbol{b}_{i+1})$$

is a path from $\boldsymbol{b}_0$ to $\boldsymbol{b}_k$ contained in the kernel of G.

We show by induction that $v(\triangle_i) = \triangle_{k-i}$ and $v(\sigma_i) = \sigma_{k-i-1}$ for every i, so that the expressions are well-defined. It is already known that $v(\triangle_0) = \triangle_k$. The simplices were constructed in a way so that $\triangle_0$ is a face of σ_0 and $\triangle_k$ is a face of σ_{k-1}. It follows that $v(\triangle_0)$ is a face of $v(\sigma_0)$. By Lemma 5.11 we know that $\triangle_k$ is a face of only one $n+1$-simplex, so we obtain $v(\sigma_0) = \sigma_{k-1}$.

Assume now that $v(\sigma_i) = \sigma_{k-i-1}$ and $v(\triangle_i) = \triangle_{k-i}$ for some $0 \leq i < k$. The map G is antipodal, so if $\tau \in \mathcal{S}$ is a simplex containing a zero, then $v(\tau)$ is also a simplex containing a zero. Thus the n-simplices $v(\triangle_i)$, $v(\triangle_{i+1})$, $\triangle_{k-i-1}$ and $\triangle_{k-i}$ are faces of $v(\sigma_i) = \sigma_{k-i-1}$ containing zeros. Moreover we know that

$$v(\triangle_{i+1}) \neq v(\triangle_i) = \triangle_{k-i} \neq \triangle_{k-i-1}.$$

The map G is generic on σ_{k-i-1}, so there are only two n-faces that contain zeros. Therefore $v(\triangle_{i+1}) = \triangle_{k-i-1}$.

There is one less of $n+1$-simplices σ_j than there are n-simplices $\triangle_j$. Thus if $i = k-1$ we have already proven enough steps and can terminate the induction. Otherwise we know that $v(\sigma_i)$, $v(\sigma_{i+1})$, σ_{k-i-1} and σ_{k-i} are $n+1$-simplices containing $v(\triangle_{i+1}) = \triangle_{k-i-1}$ as their face. In addition

$$v(\sigma_{i+1}) \neq v(\sigma_i) = \sigma_{k-i} \neq \sigma_{k-i-1}.$$

The simplex $\triangle_{k-i-1}$ contains a zero and is distinct from $\triangle_0$ and $\triangle_k$. Thus $\triangle_{k-i-1}$ is not fully contained in the top or the bottom a is therefore a face of exactly two $n+1$-simplices. It follows that $v(\sigma_{i+1}) = \sigma_{k-i-1}$, which proves the claim by induction.

For every i the points $v(\boldsymbol{b}_i)$ and $\boldsymbol{b}_{k-i}$ are both zeros of G. Because every $\triangle_{k-i}$ contains only one zero we know that $v(\boldsymbol{b}_i) = \boldsymbol{b}_{k-i}$ for every i. We are now ready to arrive at a contradiction. We split it into two cases.

Case 1: k is even. Thus $k = 2m$ for some $m \in \mathbb{N}$ and $v(\boldsymbol{b}_m) = \boldsymbol{b}_m$. This is a contradiction, as v has no fixed points on $\Diamond^n \times [0,1]$.

Case 2: k is odd. Now $k = 2m+1$ for some $m \in \mathbb{N}$. We can find a fixed point with the calculation

$$\frac{1}{2}\left(\boldsymbol{b}_m + \boldsymbol{b}_{m+1}\right) = \frac{1}{2}(v(\boldsymbol{b}_{m+1}) + v(\boldsymbol{b}_m)) = v\left(\frac{1}{2}\left(\boldsymbol{b}_m + \boldsymbol{b}_{m+1}\right)\right),$$

which leads to a contradiction.

□

This lemma was the last piece of the puzzle. We have now proven the Borsuk-Ulam theorem.

6 Applications of the Borsuk-Ulam theorem

In this chapter we take a quick look at some applications of the Borsuk-Ulam theorem. The proofs of the Ham-Sandwich theorems, the Lusternik-Schnirelmann theorem and the Kneser graph colorings are based on Matoušek's book [2] and the proof for the topological Radon theorem is from an article called *An Elementary Deduction of the Topological Radon Theorem from Borsuk-Ulam* by Craig R. Guilbault [7]. The applications of the Borsuk-Ulam theorem are of course not limited to just to these four. A lot more can be found in [2] and it seems reasonable to assume that are many applications of the Borsuk-Ulam theorem still to be discovered.

6.1 Ham sandwich theorems

The ham sandwich theorem says that if two people need to split a ham sandwich (consisting of bread, cheese and ham), they can do so in with a straight planar cut that leaves both of them with an equal amount of bread, cheese and ham. Or at least that is where the name of the theorem comes from. The formal statement is a slightly different as we will see in a bit. Since the Ham Sandwich Theorem 6.2 is a statement about measure, we need to recall a basic result from analysis.

Theorem 6.1 (Dominated convergence theorem). *Let $(f_i)_{i\in\mathbb{N}} \colon \mathbb{R}^n \to \mathbb{R}$ be measurable functions converging pointwise to $f \colon \mathbb{R}^n \to \mathbb{R}$ almost everywhere. Suppose also that every f_i is dominated by some integrable function $g \colon \mathbb{R}^n \to \mathbb{R}$, meaning that $|f_i(\boldsymbol{x})| \leq g(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^n$. Then*

$$\lim_i \int f_i \, d\mu = \int f \, d\mu.$$

Proof. See [8] theorem 1.34. □

Theorem 6.2 (Ham sandwich theorem). *Let $A_1, \dots A_n \subset \mathbb{R}^n$ be measurable sets with finite measure. There exists a hyperplane that simultaneously bisects each of the sets A_i. This means that half of the measure lies on each side of the hyperplane for every set A_i.*

Proof. Let $A_1, \dots, A_n \subset \mathbb{R}^n$ be measurable sets with finite measure. For each $\boldsymbol{u} = (u_1, \dots, u_{n+1}) \in \mathbb{S}^n$ we define the sets

$$H(\boldsymbol{u}) = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum_{k=1}^{n} x_k u_k = u_{n+1}\}$$
$$H^+(\boldsymbol{u}) = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid \sum_{k=1}^{n} x_k u_k < u_{n+1}\}.$$

We can see that every $H(\boldsymbol{u})$ is a hyperplane except for $\boldsymbol{u} \pm (0, \dots, 0, 1)$. This special case will be checked later. For antipodal points $\boldsymbol{u}$ and $-\boldsymbol{u}$ the hyperplanes $H(\boldsymbol{u})$ and $H(-\boldsymbol{u})$ are the same, while the sets $H^+(\boldsymbol{u})$ and $H^+(-\boldsymbol{u})$ correspond to opposite half-spaces. The idea of the proof is to find out with the Borsuk-Ulam theorem a sufficient $\boldsymbol{x}_0 \in \mathbb{S}^n$ such that the corresponding hyperplane $H(\boldsymbol{x}_0)$ cuts each of the sets A_i in half. This means that half of the measure for each A_i lies in $H^+(\boldsymbol{x}_0)$ while the other half is in $H^+(-\boldsymbol{x}_0)$.

For all $i \in \{1 \dots n\}$ we define a function $f_i \colon \mathbb{S}^n \to \mathbb{R},\ f_i(\boldsymbol{u}) = \mu(H^+(\boldsymbol{u}) \cap A_i)$. We fix i and show that f_i is continuous. Let $(\boldsymbol{y}_j)_{j\in\mathbb{N}}$ converge to $\boldsymbol{y}$ in $\mathbb{S}^n$. Define $g_j \colon \mathbb{R}^n \to \mathbb{R}$ to be the characteristic function of $H^+(\boldsymbol{y}_j) \cap A_i$ and g

as the characteristic function of $H^+(\boldsymbol{y}) \cap A_i$. First we want to show that g_j converges to g pointwise almost everywhere. After that, we use the dominated convergence theorem to prove that $f_i(\boldsymbol{y}_j) \to f_i(\boldsymbol{y})$ when $j \to \infty$.

Let $\boldsymbol{x} \in H^+(\boldsymbol{y})$. We use upper indices to express the coordinates of our vectors: $\boldsymbol{x} = (x^1, \dots, x^n)$, $\boldsymbol{y}_j = (y_j^1, \dots, y_j^{n+1})$ and so on. To further simplify the notation we say that all of the following sums in this proof are indexed by k and range from 1 to n. Because $\boldsymbol{x} \in H^+(\boldsymbol{y})$ it follows that $\sum x^k y^k < y^{n+1}$. We can choose j_0 such that

$$\left|\sum x^k y_j^k - \sum x^k y^k\right| < \frac{y^{n+1} - \sum x^k y^k}{2} \tag{6.1}$$

and

$$\left|y_j^{n+1} - y^{n+1}\right| < \frac{y^{n+1} - \sum x^k y^k}{2} \tag{6.2}$$

whenever $j \geq j_0$. From 6.1 we get that

$$\sum x^k y_j^k < \frac{y^{n+1} + \sum x^k y^k}{2}$$

and from 6.2 we get

$$\frac{y^{n+1} + \sum x^k y^k}{2} < y_j^{n+1}.$$

This means that $\boldsymbol{x} \in H^+(\boldsymbol{y}_j)$ for all $j \geq j_0$. Moreover $g_j(\boldsymbol{x}) = g(\boldsymbol{x})$ for all $j \geq j_0$ meaning that g_j converges to g in $H^+(\boldsymbol{y})$.

A similar argument proves that g_j converges to g in $H^+(-\boldsymbol{y})$. By Lemma 4.4 the hyperplane $H(\boldsymbol{y})$ is a null set so we do not need to worry about convergence there. It is enough that g_j converges to g almost everywhere and we have now shown this. Because every g_j is dominated by the characteristic function of A_i we can use the dominated convergence theorem to calculate

$$\lim_j f_i(\boldsymbol{y}_j) = \lim_j \int g_j \, d\mu = \int g \, d\mu = f_i(\boldsymbol{y}).$$

Therefore f_i is continuous for all $i \in \{1 \dots n\}$.

Now we can define a continuous map $f \colon \mathbb{S}^n \to \mathbb{R}^n, f = (f_1, \dots, f_n)$. The Borsuk-Ulam theorem states that there exists $\boldsymbol{x}_0 \in \mathbb{S}^n$ such that $f(\boldsymbol{x}_0) = f(-\boldsymbol{x}_0)$. Because $f(\boldsymbol{x}_0) = f(-\boldsymbol{x}_0)$, it follows that all of the coordinate functions of f agree on $\boldsymbol{x}_0$. Thus for all i

$$\mu(H^+(\boldsymbol{x}_0) \cap A_i) = f_i(\boldsymbol{x}_0) = f_i(-\boldsymbol{x}_0) = \mu(H^+(-\boldsymbol{x}_0) \cap A_i),$$

which means that the hyperplane $H(\boldsymbol{x}_0)$ bisects simultaneously all off the sets $A_1, \dots A_n$. This proves the claim except for a small detail.

In the case where the Borsuk-Ulam theorem would give us $\boldsymbol{x}_0 = \pm(0, \dots, 0, 1)$, then $H(\boldsymbol{x}_0)$ would not be a hyperplane but rather the empty set. In this case $H^+(\boldsymbol{x}_0)$ would be either the whole $\mathbb{R}^n$ or the empty set. This is not a problem as here the sets A_i would need to be of measure zero. Hence any proper hyperplane would do the trick.

□

Theorem 6.3 (Ham sandwich theorem for pointed sets)**.** *Let $A_1, \dots A_n$ be finite sets in $\mathbb{R}^n$. There exists an hyperplane that bisects each of the sets A_i, $i \in \{1, \dots, n\}$.*

Here bisection means that each open half-space contains no more than half of the points from each of the sets A_i. If some A_i has an odd number $2k+1$ of points, then each of the half-spaces is allowed to contain at most k points. Therefore at least one of the points must lie on the hyperplane. This way of defining a bisection might be a somewhat unintuitive as it can occur that one of the half-spaces contains more points of A_i than the other. In that case some points of A_i must lie on the hyperplane. However, a stronger result would require more assumptions as we can see in the example 6.4.

Proof. The idea of this proof is to replace the finite sets of points with a finite set of small enough balls. Then we can use the previous ham sandwich theorem to cut these ball sets in half. We use the notation A^ε for the set $\bigcup_{\boldsymbol{a} \in A} B(\boldsymbol{a}, \varepsilon)$ and $H(\boldsymbol{u}, b)$ for the hyperplane $\{\boldsymbol{x} \in \mathbb{R}^n \mid \langle \boldsymbol{x}, \boldsymbol{u} \rangle = b\}$ where $\boldsymbol{u} \in \mathbb{S}^{n-1}$ and $b \in \mathbb{R}$. We first prove the theorem in a simplified case. After that we expand it to any configuration in *Cases 2* and *3*.

Case 1: Each of the sets A_i has an odd number of points. Additionally $U = \bigsqcup_i A_i$ is in a general position. This means that no $n+1$ points of U lie on a same hyperplane. Also any $\boldsymbol{x} \in \mathbb{R}^n$ is allowed to be included in only one A_i at most. We want to prove that there exists such a small $\varepsilon > 0$ that no $n+1$ balls of U^ε are intersected by a common hyperplane.

Let $(j^{-1})_{j \in \mathbb{N}}$ be a sequence of radiuses. We make a counter assumption that for every $j \in \mathbb{N}$ there exists a hyperplane $H(\boldsymbol{u}_j, b_j)$ that intersects $n+1$ balls of $U^{j^{-1}}$. The set U is finite and therefore bounded. Hence there exists $M > 0$ such that the ball sets $U^{j^{-1}}$ are contained in the giant ball $B(\boldsymbol{0}, M)$ for every $j \in \mathbb{N}$. By assumption every $H(\boldsymbol{u}_j, b_j)$ contains some point $\boldsymbol{x}_j \in U^{j^{-1}} \subset B(\boldsymbol{0}, M)$. The Cauchy-Schwarz inequality

$$|b_j| = |\langle \boldsymbol{x}_j, \boldsymbol{u}_j \rangle| \leq ||\boldsymbol{x}_j|| \leq M$$

proves that every b_j lies on the interval $[-M, M]$. Moreover every pair $(\boldsymbol{u}_j, b_j)$ is contained in the compact space $\mathbb{S}^{n-1} \times [-M, M]$. It follows that the sequence $(\boldsymbol{u}_j, b_j)_{j \in \mathbb{N}}$ has a cluster point $(\boldsymbol{u}, b) \in \mathbb{S}^{n-1} \times [-M, M]$. Hence there exists a subset of indices $J_1 \subset \mathbb{N}$ such that the subsequence $(\boldsymbol{u}_j, b_j)_{j \in J_1}$ converges to $(\boldsymbol{u}, b)$ as $j \to \infty$.

For every $j \in J_1$ the hyperplane $H(\boldsymbol{u}_j, b_j)$ intersects some $n+1$ balls of $U^{j^{-1}}$. There are only finitely many ways to choose $n+1$ points from the finite set U. Therefore there must exist an infinite subset $J_2 \subset J_1$ so that the hyperplanes $H(\boldsymbol{u}_j, b_j)$ intersect $n+1$ balls with the same center points for every $j \in J_2$. We call this set of $n+1$ points A and prove that $A \subset H(\boldsymbol{u}, b)$.

Let $\boldsymbol{a} \in A$. Define $\boldsymbol{a}_j$ as the closest point on the hyperplane $H(\boldsymbol{u}_j, b_j)$ to $\boldsymbol{a}$. For every $j \in J_2$ the hyperplane $H(\boldsymbol{u}_j, b_j)$ intersects $B(\boldsymbol{a}, j^{-1})$. Hence $\boldsymbol{a}_j \in B(\boldsymbol{a}, j^{-1})$ and thus $\boldsymbol{a}_j \to \boldsymbol{a}$ as $j \to \infty$ and $j \in J_2$. Now using the continuity of the dot product we get

$$\langle \boldsymbol{a}, \boldsymbol{u} \rangle - b = \lim_{j \to \infty \ j \in J_2} (\langle \boldsymbol{a}_j, \boldsymbol{u}_j \rangle - b_j) = 0,$$

which implies that $\boldsymbol{a} \in H(\boldsymbol{u}, b)$ and thus $A \subset H(\boldsymbol{u}, b)$. This is a contradiction as we assumed that no $n+1$ amount of points would lie on a common hyperplane.

We showed that there exists $\varepsilon > 0$ such that no $n+1$ amount of balls in U^ε are intersected by a common hyperplane. The ham sandwich theorem gives us a hyperplane H that bisects every A_i. Because every A_i^ε has an odd number of balls, H must intersect at least one ball from each A_i so that the measure of each A_i^ε is divided evenly. Because H cannot intersect $n+1$ balls, it therefore intersects exactly n balls. Hence H cuts one ball exactly in half from each A_i and passes through their center points. Therefore H is a bisection also in the discrete sense for the sets $A_1, \ldots, A_n$ which proves the claim in *Case 1.*

Case 2: Every A_i contains odd amount of points but this time their position can be arbitrary. We first move the points of $U = \bigsqcup_i A_i$ slightly to reach a general position so that we can make use of *Case 1.* After that we use a similar compactness argument as previously to prove that the theorem holds also in *Case 2.*

Let $(j^{-1})_{j \in \mathbb{N}}$ be the sequence of distances of shifts. By Lemma 4.6 for every $j \in \mathbb{N}$ there is an injective shifting function $s_j \colon U \to \mathbb{R}^n$ with two properties. Firstly the image $s_j(U)$ is in general position. Secondly $||\boldsymbol{u} - s_j(\boldsymbol{u})|| < j^{-1}$ for all $\boldsymbol{u} \in U$. *Case 1* shows that for all $s_j(U)$ there exists a ham sandwich cut $H(\boldsymbol{u}_j, b_j)$. Once again $(\boldsymbol{u}_j, b_j)_{j \in \mathbb{N}}$ is a sequence in some compact space $\mathbb{S}^{n-1} \times [-M', M']$ where $M' > 0$. Thus there exists a cluster point $(\boldsymbol{u}, b) \in \mathbb{S}^{n-1} \times [-M', M']$ and an infinite $K_1 \subset \mathbb{N}$ with $(\boldsymbol{u}_j, b_j) \to (\boldsymbol{u}, b)$ when $j \to \infty$ and $j \in K_1$. For all $j \in K_1$ we define

$$\begin{aligned} V_j^0 &= \{\boldsymbol{a} \in U \mid \langle s_j(\boldsymbol{a}), \boldsymbol{u}_j \rangle - b_j = 0\} \\ V_j^+ &= \{\boldsymbol{a} \in U \mid \langle s_j(\boldsymbol{a}), \boldsymbol{u}_j \rangle - b_j > 0\} \\ V_j^- &= \{\boldsymbol{a} \in U \mid \langle s_j(\boldsymbol{a}), \boldsymbol{u}_j \rangle - b_j < 0\}. \end{aligned}$$

In addition we define

$$\begin{aligned} V^0 &= \{\boldsymbol{a} \in U \mid \langle \boldsymbol{a}, \boldsymbol{u} \rangle - b = 0\} \\ V^+ &= \{\boldsymbol{a} \in U \mid \langle \boldsymbol{a}, \boldsymbol{u} \rangle - b > 0\} \\ V^- &= \{\boldsymbol{a} \in U \mid \langle \boldsymbol{a}, \boldsymbol{u} \rangle - b < 0\}. \end{aligned}$$

Now all of the triples (V_j^0, V_j^+, V_j^-) and (V^0, V^+, V^-) are partitions of U.

Because U is finite, there are only a finite number of ways to divide it into triples (V_j^0, V_j^+, V_j^-). Therefore at least one partition must occur infinitely many times. Hence we can move on to an infinite subset $K_2 \subset K_1$ where every (V_j^0, V_j^+, V_j^-) is the same for all $j \in K_2$. Furthermore we do not need to specify $j \in K_2$ when we write V_j^0, V_j^+ or V_j^-.

We can now use the fact that every $H(\boldsymbol{u}_j, b_j)$ is a bisection of $s_j(U)$ to show that $H(\boldsymbol{u}, b)$ must be a bisection of U. The idea is that with a small enough shift, the points in the half-spaces cannot move past the cutting hyperplane. They can move to the boundary but not to the other side. This happens, because the hyperplane is a closed set.

Let $\boldsymbol{a} \in V_j^0$ and $j \in K_2$. By taking the limit of j we get that

$$\langle \boldsymbol{a}, \boldsymbol{u} \rangle - b = \lim_{j \to \infty \ j \in K_2} (\langle s_j(\boldsymbol{a}), \boldsymbol{u}_j \rangle - b_j) = 0.$$

Therefore $\boldsymbol{a} \in V^0$ and $V_j^0 \subset V^0$. Now let $\boldsymbol{a} \in V_j^+$ and $j \in K_2$. Similarly we get

$$\langle \boldsymbol{a}, \boldsymbol{u} \rangle - b = \lim_{j \to \infty \ j \in K_2} (\langle s_j(\boldsymbol{a}), \boldsymbol{u}_j \rangle - b_j) \geq 0.$$

which means that $V_j^+ \subset V^0 \cup V^+$. The same argument proves that $V_j^- \subset V^0 \cup V^-$. Combining these subset relations we get

$$U \setminus V_j^+ = V_j^0 \cup V_j^- \subset V^0 \cup V^- = U \setminus V^+$$

which implies $V^+ \subset V_j^+$. Correspondingly we argue that $V^- \subset V_j^-$.

The hyperplane $H(\boldsymbol{u}_j, b_j)$ is a bisection for sets $s_j(A_1), \ldots, s_j(A_n)$. Therefore a half-space of it contains at most half of the points of each set $s_j(A_i)$. These points correspond bijectively to points in V_j^+. Because $V^+ \subset V_j^+$, the set V^+ contains at most half of the points of each A_i. This holds for V^- as well. Now we have proven that $H(\boldsymbol{u}, b)$ is a bisection for the sets $A_1, \ldots, A_n$ completing *Case 2*.

Case 3: The sets $A_1, \ldots, A_n$ can be arbitrarily positioned and their cardinality can be arbitrary as well. From each A_i with even cardinality we remove one point. Now we can bisect these odd sets as proven in *Case 2*. We defined bisection here in a way so that the even sets we are allowed to have one more point in each of the open half-spaces than in the odd sets. Hence we can put back the removed points without ruining the bisection.

□

Example 6.4. Let $n = 2$, $A_1 = \{(0, 0)\}$ and $A_2 = \{(1, 0),\ (2, 0),\ (0, 1)\}$. In $\mathbb{R}^2$ hyperplanes are straight lines. Because A_1 is a singleton, any line bisecting it must pass through the origin. Now the only way to simultaneously bisect A_2 is to cut through the x-axis. This way the upper half-space contains one point of A_2 while the lower half-space contains nothing. This is still a legal bisection with our definition. In this case it would be impossible to cut both A_1 and A_2 to exactly two parts requiring both half-spaces to contain the same amount of points for both sets.

6.2 The Lusternik-Schnirelmann theorem

Although we are listing the Lusternik-Schnirelmann theorem as an application of the Borsuk-Ulam theorem, it is often considered as a different formulation to it. This is due to the fact that the equivalence between the Lusternik-Schnirelmann theorem and the Borsuk-Ulam theorem is not hard to prove. Whether we consider it is as a different formulation, or an application is not really important.

Regardless, we shall only prove the Lusternik-Schnirelmann theorem from the Borsuk-Ulam theorem. First we recall a famous result from topology.

Lemma 6.5 (Urysohn lemma)**.** *Let X be a normal space, $A, B \subset X$ closed and $A \cap B = \emptyset$. Then exists a continuous function $f\colon X \to [0,1]$ with $f(A) = \{0\}$ and $f(B) = \{1\}$.*

Proof. See [9] theorem 19.2. ☐

Theorem 6.6 (Lusternik-Schnirelmann theorem)**.** *Let $A_1, \dots, A_{n+1}$ be closed sets covering $\mathbb{S}^n$. Then at least one of the A_i contains a pair of antipodal points.*

Proof. Let $\{A_1, \dots, A_{n+1}\}$ be a closed cover for $\mathbb{S}^n$. Now suppose that for all $i \leq n$ the sets A_i do not contain any pairs of antipodal points. This means that $A_i \cap -A_i = \emptyset$ for all $i \leq n$. Our goal now is to show that the last set A_{n+1} must contain an antipodal pair of points.

The set $-A_i$ is closed as an image of a closed set in automorphism $\boldsymbol{x} \mapsto -\boldsymbol{x}$. For all $i \leq n$ we can use the Urysohn lemma to give us a continuous map $f_i\colon \mathbb{S}^n \to [0,1]$ with $f_i(\boldsymbol{x}) = 0$ for all $\boldsymbol{x} \in A_i$ and $f_i(\boldsymbol{x}) = 1$ for all $\boldsymbol{x} \in -A_i$. Now $f = (f_1, \dots, f_n)\colon \mathbb{S}^n \to \mathbb{R}^n$ is a continuous map. The Borsuk-Ulam theorem states that there exists $\boldsymbol{x}_0$ such that $f(\boldsymbol{x}_0) = f(-\boldsymbol{x}_0)$.

Suppose $\boldsymbol{x}_0 \in A_i$ for some $1 \leq i \leq n$. Now $-\boldsymbol{x}_0 \in -A_i$ and $0 = f_i(\boldsymbol{x}_0) = f_i(-\boldsymbol{x}_0) = 1$, which is a contradiction. Thus $\boldsymbol{x}_0 \notin A_i$ for any $1 \leq i \leq n$. Similarly we can argue that $-\boldsymbol{x}_0 \notin A_i$ for any $1 \leq i \leq n$. Because $\boldsymbol{x}_0$, $-\boldsymbol{x}_0 \in \mathbb{S}^n$ and $\{A_1, \dots, A_{n+1}\}$ is a cover for $\mathbb{S}^n$, the only possibility is that $\boldsymbol{x}_0$, $-\boldsymbol{x}_0 \in A_{n+1}$.

☐

Next we prove a slightly more general version of the Lusternik-Schnirelmann theorem.

Theorem 6.7. *Let $\mathbb{S}^n = A_1 \cup \cdots \cup A_{n+1}$ and assume each A_i is either open or closed. To be clear, the collection of sets A_i can be any mix of open and closed sets. Then at least one A_i contains a pair of antipodal points.*

We first need a small proposition to prove Theorem 6.7. Lemma 6.8 and it proof is from [10] where it is referred as the shrinking lemma.

Lemma 6.8. *Let X be a normal space and $\{A_1, \dots, A_n\}$ a finite open cover for X. Then there exists a closed cover $\{B_1, \dots, B_n\}$ for X where $B_i \subset A_i$ for all i.*

We prove this by proving an equivalent statement: *Suppose $F_1, \dots, F_n$ are closed and $\bigcap_i F_i = \emptyset$. Then exists open sets $U_1, \dots, U_n$ with $\bigcap_i U_i = \emptyset$ and $F_i \subset U_i$ for all i.*

Proof. Let X be a normal space, $F_1, \dots, F_n$ closed and $\bigcap_i F_i = \emptyset$. Because F_1 and $\bigcap_{i=2}^n F_i$ are disjoint, there exist disjoint open sets U_1 and U_2^n with $F_1 \subset U_1$ and $\bigcap_{i=2}^n F_i \subset U_2^n$. Because U_2^n is open, it follows that $\overline{U_1} \cap \bigcap_{i=2}^n F_i = \emptyset$.

Now suppose we have chosen $U_1, \dots, U_k$. We can repeat the previous argument by considering closed disjoint sets F_{k+1} and

$$\bigcap_{i=1}^{k} \overline{U_i} \cap \bigcap_{i=k+2}^{n} F_i.$$

This gives us sufficient open sets $U_1, \dots, U_n$ in a finite number of iterations.

□

Proof of Theorem 6.7. Let $\mathbb{S}^n = A_1 \cup \cdots \cup A_{n+1}$. At first we assume that every A_i is open. By Lemma 6.8 we get a closed cover $\{B_1, \dots, B_{n+1}\}$ where $B_i \subset A_i$ for all i. The Lusternik-Schnirelmann Theorem 6.6 states that one of the closed sets B_i contains a pair of antipodal points. Therefore the corresponding open set A_i contains a pair of antipodal points.

Next assume that $\mathbb{S}^n = A_1 \cup \cdots \cup A_k \cup B_{k+1} \cup \cdots \cup B_{n+1}$ and that $A_1, \dots, A_k$ are open and $B_{k+1}, \dots, B_{n+1}$ are closed. For all $\varepsilon > 0$ we denote the open sets

$$B_i^{\varepsilon} = \{\boldsymbol{x} \in \mathbb{S}^n : ||\boldsymbol{x} - \boldsymbol{b}|| < \varepsilon \text{ for some } \boldsymbol{b} \in B_i\}.$$

Let $(\varepsilon_j)_{j \in \mathbb{N}}$ be a sequence converging to 0. Previously we showed that for all ε_j there is a pair of antipodal points $(\boldsymbol{x}_j, -\boldsymbol{x}_j)$ in some A_i or $B_i^{\varepsilon_j}$. If any pair of those antipodal points would be included in some A_i the claim would follow immediately. Therefore we can assume that this is not the case. Because $\mathbb{S}^n$ is compact, there is a cluster point $\boldsymbol{x}$ and an infinite subset of indices $J_1 \subset \mathbb{N}$ with $\boldsymbol{x}_j \to \boldsymbol{x}$ whenever $j \to \infty$ and $j \in J_1$. There are only a finite amount of sets B_i, so we can move on to another infinite subset of indices $J_2 \subset J_1$ so that $\boldsymbol{x}_j \in B_i^{\varepsilon_j}$ with the same i for every $j \in J_2$. We keep i fixed for the rest of the proof.

For each $j \in J_2$ we can pick $\boldsymbol{a}_j \in B_i$ with $||\boldsymbol{x}_j - \boldsymbol{a}_j|| < \varepsilon_j$. Now we can calculate

$$\begin{aligned}\inf_{\boldsymbol{y} \in B_i} ||\boldsymbol{x} - \boldsymbol{y}|| &\le ||\boldsymbol{x} - \boldsymbol{a}_j|| \\ &\le ||\boldsymbol{x} - \boldsymbol{x}_j|| + ||\boldsymbol{x}_j - \boldsymbol{a}_j|| \\ &\to 0\end{aligned}$$

which proves that $\boldsymbol{x} \in \overline{B_i} = B_i$. Similarly can be proven that $-\boldsymbol{x} \in B_i$. Hence B_i contains a pair of antipodal points $(\boldsymbol{x}, -\boldsymbol{x})$.

□

6.3 Kneser graph colorings

Calculating the chromatic numbers of the Kneser graphs is a problem about finite sets. It is therefore quite surprising how powerfully we can use the Borsuk-Ulam theorem in doing so.

Definition 6.9. A graph is a pair $G = (V, E)$ where V is any set and E is a set of pairs $\{a, b\}$ where $a, b \in V$.

The elements of V are called vertices and the elements of E are edges. Vertices a and b are called adjacent if there is an edge connecting them. We denote $[k] = \{1, \dots, k\}$ for $k \in \mathbb{N}$.

Definition 6.10. Let $G = (V, E)$ be a graph. A k-coloring of G is a map $f\colon V \to [k]$ where $f(x) \neq f(y)$ if x and y are adjacent. The chromatic number of G is the smallest number k for which there exists a k-coloring of G.

Definition 6.11. A Kneser graph $KG_{n,k}$, where $n, k \in \mathbb{N}$ and $k \leq n$, is a graph whose vertices are k-sized subsets of $[n]$. Two vertices of a Kneser graph are connected, if their sets are disjoint.

Theorem 6.12. *Let $n, k \in \mathbb{N}$, $k > 0$ and $2k \leq n$. The chromatic number of the Kneser graph $KG_{n,k}$ is $n - 2k + 2$.*

The proof is done in two parts. First we construct an explicit $n - 2k + 2$-coloring for $KG_{n,k}$. In the second part we prove that no $n - 2k + 1$-coloring exists by using Theorem 6.7.

Proof. A vertex v of $KG_{n,k}$ is a set of k amount of natural numbers between 1 and n. This v we paint with color

$$f(v) = \min\{\, \min(v),\ n - 2k + 2\,\}.$$

Now we show that any to adjacent vertices are not painted with the same color. Let v and v' be adjacent vertices. If $f(v) \neq n - 2k + 1$ then $f(v) = \min(v)$. Because v and v' are disjoint sets then $\min(v) \neq \min(v')$ and thus $f(v) \neq f(v')$. Also if $f(v') \neq n - 2k + 2$ then $f(v) \neq f(v')$ by symmetry.

Suppose now that $f(v) = f(v') = n - 2k + 2$. Then all of the elements of v and v' are bigger than $n - 2k + 1$. Both of the sets contain k elements and they are disjoint. Calculating the elements

$$n = \operatorname{card}[n] \geq \operatorname{card}[n - 2k + 1] + \operatorname{card} v + \operatorname{card} v' = n + 1$$

leads to contradiction. Hence f does not paint adjacent vertices with the same color and is therefore an $n - 2k + 2$-coloring of $KG_{n,k}$.

We now show that no $n - 2k + 1$-coloring is possible. Denote $d = n + 2k - 1$ and suppose that f is a d-coloring for $KG_{n,k}$. Let $s\colon [n] \to \mathbb{S}^d$ be an injective map such that no $d + 1$ points of $s([n])$ lie on the same hyperplane with the origin. The existence of such s can be proven by Lemma 4.8. First we map $[n]$ anywhere on the sphere. Then we shift them however much we want to reach a general position with respect to $\mathbf{0}$.

For all $\boldsymbol{x} \in \mathbb{S}^d$ denote the half-space by $H^+(\boldsymbol{x}) = \{\boldsymbol{y} \in \mathbb{S}^d \mid \langle \boldsymbol{y}, \boldsymbol{x} \rangle > 0\}$ and for all $i \in \{1, \dots d\}$ we define

$$A_i = \{\boldsymbol{x} \in \mathbb{S}^d \mid H^+(\boldsymbol{x}) \text{ contains a } k\text{-tuple of } s([n]) \text{ with color } i\ \}.$$

By referring to the color of the k-tuple of $s([n])$ we refer to the color of the k-tuple in $[n]$. Tuples here are not allowed to contain the same point twice. We can induce the coloring f to $s([n])$, because s is injective.

We prove that every A_i is open. Fix i and let $\boldsymbol{x}_0 \in A_i$. Now a k-tuple $(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_k)$ of color i is contained in $H^+(\boldsymbol{x}_0)$. Define

$$\varepsilon = \min_{j \in \{1,\ldots,k\}} \langle \boldsymbol{a}_j, \boldsymbol{x}_0 \rangle.$$

We fix $j \in \{1, \ldots, k\}$. Since the dot product is continuous, there exists a neighbourhood U_j of $\boldsymbol{x}_0$, so that $|\langle \boldsymbol{a}_j, \boldsymbol{x} \rangle - \langle \boldsymbol{a}_j, \boldsymbol{x}_0 \rangle| < \varepsilon$, whenever $\boldsymbol{x} \in U_j$. This means that for all $\boldsymbol{x} \in U_j$

$$\langle \boldsymbol{a}_j, \boldsymbol{x} \rangle > -\varepsilon + \langle \boldsymbol{a}_j, \boldsymbol{x}_0 \rangle \geq 0$$

and thus $\boldsymbol{a}_j \in H^+(\boldsymbol{x})$. Similarly choosing U_j for every j we get a neighbourhood $U = \bigcap_{j=1}^{k} U_j$ for $\boldsymbol{x}_0$. Now for all $\boldsymbol{x} \in U$ the k-tuple $(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_k)$ of color i is contained in $H^+(\boldsymbol{x})$. Therefore $U \subset A_i$ and thus A_i is open.

Define $A_{d+1} = \mathbb{S}^d \setminus \bigcup_{i=1}^{d} A_i$. Because every other A_i is open, the set A_{d+1} is closed. Moreover the sets $A_1, \ldots, A_{d+1}$ cover $\mathbb{S}^d$. Theorem 6.7 states that some A_i contains a pair of antipodal points $\{\boldsymbol{x}, -\boldsymbol{x}\}$. Let us find a contradiction.

Suppose $\{\boldsymbol{x}, -\boldsymbol{x}\} \subset A_i$ for some $i \leq d$. Because $\boldsymbol{x} \in A_i$, there is an i-colored k-tuple $K \subset H^+(\boldsymbol{x})$. Similarly there exists an i-colored k-tuple $K' \subset H^+(-\boldsymbol{x})$. But the sets $H^+(\boldsymbol{x})$ and $H^+(-\boldsymbol{x})$ are disjoint, so the tuples K and K' have to be disjoint too. Therefore there is an edge between vertices K and K'. Thus K and K' cannot both be of the same color i, which is a contradiction.

Now suppose that $\{\boldsymbol{x}, -\boldsymbol{x}\} \subset A_{d+1}$. For any $i \in \{1, \ldots, d\}$ the point $\boldsymbol{x}$ is not contained in A_i and thus the hyperplane $H(\boldsymbol{x})$ does not contain an i-colored k-tuple. Every k-tuple of $s([\boldsymbol{x}])$ is colored with some color $i \in \{1, \ldots, d\}$. Therefore $H^+(\boldsymbol{x}) \cap s([n])$ contains at most $k-1$ points. Similarly $\mathrm{card}(H^+(-\boldsymbol{x}) \cap s([n])) \leq k - 1$.

The set $H = \mathbb{S}^d \setminus (H^+(\boldsymbol{x}) \cup H^+(-\boldsymbol{x}))$ is an intersection of a hyperplane through the origin and the sphere $\mathbb{S}^d$. By construction of s the set H cannot contain more than d points of $s([n])$. Once again calculating the elements

$$\begin{aligned}
n &= \mathrm{card}([n]) \\
&= \mathrm{card}(s([n])) \\
&= \mathrm{card}(H \cap s([n])) + \mathrm{card}(H^+(\boldsymbol{x}) \cap s([n])) + \mathrm{card}(H^+(-\boldsymbol{x}) \cap s([n])) \\
&\leq (n - 2k + 1) + (k - 1) + (k - 1) \\
&= n - 1
\end{aligned}$$

leads to a contradiction. Hence there does not exist an $n - 2k + 1$ coloring for $KG_{n,k}$ and the chromatic number of $KG_{n,k}$ is $n - 2k + 2$.

□

6.4 The Topological Radon theorem

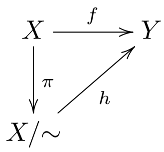
To prove the topological Radon theorem, we first need a some more topological notions. The proofs and diagram of these preliminaries are from [4] while the proof for the topological Radon theorem is from [7].

Definition 6.13. A continuous surjection $f\colon X \to Y$ is called an identification, if a subset $U \subset Y$ is open if and only if $f^{-1}(U)$ is open.

Lemma 6.14. *Let X be compact, Y Hausdorff and $f\colon X \to Y$ continuous surjection. Then f is an identification.*

Proof. If $U \subset Y$ is open, then $f^{-1}(U)$ is open by continuity. Suppose that $f^{-1}(U)$ is open. Then $X \setminus f^{-1}(U)$ is closed and therefore compact. Because f is a surjection, $f(X \setminus f^{-1}(U)) = Y \setminus U$ and it is compact as an image of a compact set. Finally, Y is Hausdorff, so $Y \setminus U$ is closed and thus Y is open. This proves that f is an identification. □

Lemma 6.15. *Let $f\colon X \to Y$ be an identification and let $\sim$ be an equivalence relation on X defined by f. That is $\boldsymbol{x} \sim \boldsymbol{y}$, if $f(\boldsymbol{x}) = f(\boldsymbol{y})$. Then $h\colon X/\!\sim\ \to Y$, $h([\boldsymbol{x}]) = f(\boldsymbol{x})$ is a homeomorphism, that makes the following diagram commute.*



Proof. The definition of $\sim$ makes h a well defined map and the definition of h makes the diagram commute. Because f is a surjection and $f = h \circ \pi$, the map h is surjection. Suppose $h([\boldsymbol{x}]) = h([\boldsymbol{y}])$. Then $f(\boldsymbol{x}) = f(\boldsymbol{y})$ which means that $[\boldsymbol{x}] = [\boldsymbol{y}]$. Therefore h is injective.

Suppose $U \subset Y$ is open. Then $f^{-1}(U)$ is open which means that $\pi^{-1}(h^{-1}(U))$ is open. Thus $h^{-1}(U)$ is open in $X/\!\sim$ and so h in continuous. Suppose now that $V \subset X/\!\sim$ is open. The preimage $\pi^{-1}(V)$ is open and since h is bijective we get

$$\pi^{-1}(V) = \pi^{-1}(h^{-1}(h(V))) = f^{-1}(h(V)).$$

Because the map f is an identification, it follows that $h(V)$ is open. Hence h is an open map.

□

Now we prove a key lemma, which combined with the Borsuk-Ulam theorem gives the Topological Radon theorem quite easily. In the following we denote $\triangle^n$ as the standard n-simplex with vertices $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{n+1}$. That being said, the topological Radon theorem holds for any $n+1$-simplex as simplices of the same dimension are homeomorphic.

Lemma 6.16. *For every $n \in \mathbb{N}$ there is a continuous map $\psi_n\colon \mathbb{S}^n \to \triangle^{n+1}$ such that the antipodal points $\boldsymbol{x}$ and $-\boldsymbol{x}$ are mapped to disjoint faces of $\triangle^{n+1}$ for every $\boldsymbol{x} \in \mathbb{S}^n$.*

Proof. We construct the maps ψ_n inductively. To construct ψ_{n+1} from ψ_n we actually first use spaces homeomorphic to $\mathbb{S}^n$. The idea of the homeomorphisms is to simplify the inductive process.

For every $n \in \mathbb{N}$ define a continuous function $f_{n+1} \colon \mathbb{S}^n \times [-\frac{\pi}{2}, \frac{\pi}{2}] \to \mathbb{S}^{n+1}$, $f_{n+1}(\boldsymbol{x}, t) = (\cos t\ \boldsymbol{x}, \sin t)$. We want to show that every f_{n+1} is a surjection. Let $\boldsymbol{y} = (y_1, \dots, y_{n+2}) \in \mathbb{S}^{n+1}$. Suppose that $y_{n+2} \neq \pm 1$. Then the point $(\boldsymbol{x}, \arcsin(y_{n+2}))$ is mapped to $\boldsymbol{y}$, where

$$\boldsymbol{x} = \frac{1}{\sqrt{1 - y_{n+2}^2}}(y_1, \dots, y_{n+1}) \in \mathbb{S}^n.$$

In the case where $y_{n+2} = \pm 1$, we know that $f(1, 0, \dots, 0, \pm\frac{\pi}{2}) = \boldsymbol{y}$. Hence f_{n+1} is a surjection and by Lemma 6.14 an identification.

Consider the diagram

$$\begin{array}{ccc} \mathbb{S}^n \times [-\frac{\pi}{2}, \frac{\pi}{2}] & \xrightarrow{f_{n+1}} & \mathbb{S}^{n+1} \\ \Big\downarrow \pi & \nearrow h_{n+1} & \\ (\mathbb{S}^n \times [-\frac{\pi}{2}, \frac{\pi}{2}])/\!\sim & & \end{array}$$

where $\sim$ is the equivalence relation defined by f_{n+1}. Lemma 6.15 gives us homeomorphisms h_n for all $n \geq 1$. In addition we define h_0 as the identity map on $\mathbb{S}^0 = \{-1, 1\}$. For all $n \geq 1$ we denote $X^n = (\mathbb{S}^{n-1} \times [-\frac{\pi}{2}, \frac{\pi}{2}])/\!\sim$ and $X^0 = \mathbb{S}^0$.

Denote $[\boldsymbol{x}, t]$ as the equivalence class of $(\boldsymbol{x}, t)$ and $-[\boldsymbol{x}, t] = [-\boldsymbol{x}, -t]$. In addition to preserving the topology from the spaces $\mathbb{S}^n$ to spaces X^n we want the maps h_n to also preserve this antipodality. The calculation

$$\begin{aligned} h_n(-[\boldsymbol{x}, t]) &= (\cos(-t)(-\boldsymbol{x}), \sin(-t)) \\ &= -(\cos t\boldsymbol{x}, \sin t) \\ &= -h_n([\boldsymbol{x}, t]) \end{aligned}$$

shows this to be the case for all h_n. This implies that $h_n^{-1}(-\boldsymbol{x}) = -h_n^{-1}(\boldsymbol{x})$ holds for all n and $\boldsymbol{x}$ also.

The simplices $\triangle^n$ here are the standard simplices with edges as the canonical base vectors. We can naturally consider $\triangle^n$ as a subset of $\triangle^{n+1}$. To do so, we say that $\triangle^n$ is the convex hull of all the edge points of $\triangle^{n+1}$ except for the last one $(0, \dots, 0, 1)$. Doing so makes $\triangle^n$ a face of $\triangle^{n+1}$. Denote b_n as the barycenter of $\triangle^n$. With our natural inclusion we can also consider it as an element of $\triangle^{n+1}$.

Now we have the necessary tools to start inductively defining maps. Because $\mathbb{S}^0 = \{-1, 1\}$ we can define $\lambda_0 \colon \mathbb{S}^0 \to \triangle^1$ with $\lambda_0(1) = (1, 0)$ and $\lambda_0(-1) = (0, 1)$. Clearly λ_0 is continuous and the antipodal points are mapped to disjoint faces of $\triangle^1$. Suppose now, that we have acquired a continuous $\lambda_n \colon X^n \to \triangle^{n+1}$ that maps antipodal points to disjoint faces. We define $\lambda_{n+1} \colon X^{n+1} \to \triangle^{n+2}$

$$\lambda_{n+1}([\boldsymbol{x}, t]) = \begin{cases} (0, \dots, 0, 1) & : t \in [\frac{\pi}{4}, \frac{\pi}{2}] \\ (1 - \frac{4t}{\pi})\lambda_n(h_n^{-1}(\boldsymbol{x})) + \frac{4t}{\pi}(0, \dots, 0, 1) & : t \in [0, \frac{\pi}{4}] \\ \lambda_n(h_n^{-1}(\boldsymbol{x})) & : t \in [-\frac{\pi}{4}, 0] \\ -(1 + \frac{4t}{\pi})\boldsymbol{b}_n + (2 + \frac{4t}{\pi})\lambda_n(h_n^{-1}(\boldsymbol{x})) & : t \in [-\frac{\pi}{2}, -\frac{\pi}{4}]. \end{cases}$$

First we have to see that this gives a well defined function. The piecewise definitions clearly agree whenever $t = \{-\frac{\pi}{4}, 0, \frac{\pi}{4}\}$. The equivalence relation $\sim$ only glues together points whenever $t = \frac{\pi}{2}$ or $t = -\frac{\pi}{2}$. However in that case λ_{n+1} does not depend on $\boldsymbol{x}$. Hence λ_{k+1} is well defined.

All of the four piecewise restrictions of λ_{n+1} are continuous as elementary combinations of continuous functions. Moreover their domains are are closed sets. Hence by the gluing lemma the map λ_{n+1} is continuous.

To check that λ_{n+1} maps antipodal points to disjoint faces of $\triangle^{n+2}$, we have to consider a few cases. Let $[\boldsymbol{x}, t] \in X^{n+1}$.

Case 1: $t \in [\frac{\pi}{4}, \frac{\pi}{2}]$. Now $\lambda_{n+1}([\boldsymbol{x}, t]) = (0, \ldots, 0, 1)$ which is a face of $\triangle^{n+2}$ in itself. Because both $\boldsymbol{b}_n$ and $\lambda_n(h_n^{-1}(-\boldsymbol{x}))$ are contained in $\triangle^{n+1}$, also their convex combinations are contained in $\triangle^{n+1}$ as well. Thus $\lambda_{n+1}([\boldsymbol{x}, t])$ and $\lambda_{n+1}(-[\boldsymbol{x}, t])$ are mapped to disjoint faces, $(0, \ldots, 0, 1)$ and $\triangle^{n+1}$ respectively.

Case 2: $t \in [0, \frac{\pi}{4}]$. Earlier we showed that $\lambda_n(h_n^{-1}(-\boldsymbol{x})) = \lambda_n(-h_n^{-1}(\boldsymbol{x}))$. By induction assumption, the points $\lambda_n(h_n^{-1}(\boldsymbol{x}))$ and $\lambda_n(-h_n^{-1}(\boldsymbol{x}))$ are mapped to disjoint faces of the simplex $\triangle^{n+1}$. We call these faces A and B and consider them now as faces of $\triangle^{n+2}$. The point $\lambda_{n+1}([\boldsymbol{x}, t])$ now lies in face A' whose edges are the edges of A and the point $(0, \ldots, 0, 1)$. Because the point $\lambda_{n+1}(-[\boldsymbol{x}, t])$ is still contained in B, the function λ_{n+1} maps antipodal points $[\boldsymbol{x}, t]$ and $-[\boldsymbol{x}, t]$ to disjoint faces A' and B.

Case 3: $t \in [-\frac{\pi}{2}, 0]$. Now the antipodal point $[-\boldsymbol{x}, -t]$ is contained in either in the *Case 1* or *2*. This means that the antipodal points $[-\boldsymbol{x}, -t]$ and $-[-\boldsymbol{x}, -t] = [\boldsymbol{x}, t]$ are mapped to disjoint faces of $\triangle^{n+2}$.

We have now proven that for all $n \in \mathbb{N}$ there exists a continuous function $\lambda_n \colon X^n \to \triangle^{n+1}$ that maps every pair of antipodal points to disjoint faces of $\triangle^{n+1}$. Defining $\psi_n = \lambda_n \circ h_n^{-1}$ completes the proof.

□

Theorem 6.17 (Topological Radon theorem)**.** *Let* $f \colon \triangle^{n+1} \to \mathbb{R}^n$ *be a continuous map. Then there exist two disjoint faces* A *and* B *of* $\triangle^{n+1}$ *such that the images* $f(A)$ *and* $f(B)$ *are not disjoint.*

Proof. Let $f_n \colon \triangle^{n+1} \to \mathbb{R}^n$ be continuous and let $\psi_n \colon \mathbb{S}^n \to \triangle^{n+1}$ be the map we constructed in Lemma 6.16. The Borsuk-Ulam theorem states that there is a pair of antipodal points $\boldsymbol{x}$ and $-\boldsymbol{x}$ with $(f_n \circ \psi_n)(\boldsymbol{x}) = (f_n \circ \psi_n)(-\boldsymbol{x})$. We constructed ψ_n in a way that the points $\psi_n(\boldsymbol{x})$ and $\psi_n(-\boldsymbol{x})$ lie on some disjoint faces A and B respectively. However the images $f(A)$ and $f(B)$ are not disjoint, as $(f_n \circ \psi_n)(\boldsymbol{x})$ is contained in both of them.

□

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