Elliptic Genera and q-Series Development in Analysis, String Theory, and N=2 Superconformal Field Theory

L. Bonora [(a) 1], A. A. Bytsenko [(b) 2], M. Chaichian [(c) 3]
and A. E. Gonçalves [(b) 4]

[(a)] *International School for Advanced Studies (SISSA/ISAS)*
Via Bonomea 265, 34136 Trieste and INFN, Sezione di Trieste, Italy

[(b)] *Departamento de Física, Universidade Estadual de Londrina*
Caixa Postal 6001, Londrina-Paraná, Brazil

[(c)] *Department of Physics, University of Helsinki*
P.O. Box 64, FI-00014 Helsinki, Finland

Abstract

In this article we examine the Ruelle type spectral functions $\mathcal{R}(s)$, which define an overall description of the content of the work. We investigate the Gopakumar-Vafa reformulation of the string partition functions, describe the $N = 2$ Landau-Ginzburg model in terms of Ruelle type spectral functions. Furthermore, we discuss the basic properties satisfied by elliptic genera in $N = 2$ theories, construct the functional equations for $\mathcal{R}(s)$, and analyze the modular transformation laws for the elliptic genus of the Landau-Ginzburg model and study their properties in details.

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[1]E-mail: bonora@sissa.it
[2]E-mail: aabyts@gmail.com
[3]E-mail: masud.chaichian@helsinki.fi
[4]E-mail: aedsongoncalves@gmail.com

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1 Introduction

In this article we shall make considerable use of Ruelle type functions $\mathcal{R}(s)$, which should give a well-balanced description of the content of the whole article. These functions are connected to symmetric functions (so-called S-functions $s_\lambda(x)$); the theory of S-functions was developed by Schur [1]. These functions play important role in the representation theory of finite-dimensional classical Lie algebras. Functions $\mathcal{R}(s)$ are an alternating product of more complicate factors, each of which are so-called Patterson-Selberg zeta-functions Z_Γ [2].

Now let us consider three-geometry with an orbifold description H^3/Γ. The complex unimodular group $G = SL(2,\mathbb{C})$ acts on the real hyperbolic three-space H^3 in a standard way, namely for $(x,y,z) \in H^3$ and $g \in G$, one gets $g\cdot(x,y,z) = (u,v,w) \in H^3$. Thus for $r = x+iy,\ g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $u+iv = [(ar+b)\overline{(cr+d)} + a\bar{c}z^2]\cdot[|cr+d|^2+|c|^2z^2]^{-1}$, $w = z\cdot[|cr+d|^2+|c|^2z^2]^{-1}$. Here the bar denotes the complex conjugation. Let $\Gamma \in G$ be the discrete group of G defined as

$$\begin{aligned} \Gamma &= \{\mathrm{diag}(e^{2n\pi(\mathrm{Im}\,\tau+i\mathrm{Re}\,\tau)},\ e^{-2n\pi(\mathrm{Im}\,\tau+i\mathrm{Re}\,\tau)}) : n\in\mathbb{Z}\} = \{\mathfrak{g}^n : n\in\mathbb{Z}\}\,, \\ \mathfrak{g} &= \mathrm{diag}(e^{2\pi(\mathrm{Im}\,\tau+i\mathrm{Re}\,\tau)},\ e^{-2\pi(\mathrm{Im}\,\tau+i\mathrm{Re}\,\tau)})\,. \end{aligned} \tag{1.1}$$

One can define a Selberg-type zeta function for the group $\Gamma = \{\mathfrak{g}^n : n\in\mathbb{Z}\}$ generated by a single hyperbolic element of the form $\mathfrak{g} = \mathrm{diag}(e^z, e^{-z})$, where $z = \alpha+i\beta$ for $\alpha,\beta>0$. In fact, we will take $\alpha = 2\pi \mathrm{Im}\,\tau$, $\beta = 2\pi \mathrm{Re}\,\tau$. For the standard action of $SL(2,\mathbb{C})$ on H^3 one has

$$\mathfrak{g}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} e^\alpha & 0 & 0 \\ 0 & e^\alpha & 0 \\ 0 & 0 & e^\alpha \end{bmatrix} \begin{bmatrix} \cos(\beta) & -\sin(\beta) & 0 \\ \sin(\beta) & \cos(\beta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \tag{1.2}$$

Therefore, $\mathfrak{g}$ is the composition of a rotation in $\mathbb{R}^2$ with complex eigenvalues $\exp(\pm i\beta)$ and a dilatation $\exp(\alpha)$. There exists the Patterson-Selberg spectral function $Z_\Gamma(s)$, meromorphic on $\mathbb{C}$, given for $\mathrm{Re}\, s > 0$ by the formula [3]

$$\log Z_\Gamma(s) = -\frac{1}{4}\sum_{n=1}^{\infty} \frac{e^{-n\alpha(s-1)}}{n[\sinh^2\left(\frac{\alpha n}{2}\right) + \sin^2\left(\frac{\beta n}{2}\right)]}\,. \tag{1.3}$$

The Patterson-Selberg function can be attached to H^3/Γ as follows :

$$Z_\Gamma(s) := \prod_{k_1,k_2 \in \mathbb{Z}_+ \cup \{0\}} [1 - (e^{i\beta})^{k_1}(e^{-i\beta})^{k_2} e^{-(k_1+k_2+s)\alpha}]\,. \tag{1.4}$$

Zeros of $Z_\Gamma(s)$ are the complex numbers

$$\zeta_{n,k_1,k_2} = -\left(k_1 + k_2\right) + i\left(k_1 - k_2\right)\beta/\alpha + 2\pi i n/\alpha \ \ (n \in \mathbb{Z}). \tag{1.5}$$

In our applications we shall consider a compact hyperbolic three-manifold G/Γ with $G = SL(2,\mathbb{C})$. By combining the characteristic class representatives of field theory elliptic genera one can compute quantum partition functions in terms of the spectral functions of hyperbolic three-geometry, we shall demonstrate that in the course of this article.

Let us introduce next Ruelle type spectral functions $\mathcal{R}(s)$ associated with hyperbolic three-geometry [4], which can be continued meromorphically to the entire complex plane $\mathbb{C}$. Let χ be an orthogonal representation of $\pi_1(X)$. Using the Hodge decomposition, the vector space $H(X;\chi)$ of twisted cohomology classes can be embedded into $\Omega(X;\chi)$ as the space of harmonic forms. This embedding induces a norm $|\cdot|^{RS}$ on the determinant line $\det H(M;\chi)$. The Ray-Singer norm $||\cdot||^{RS}$ on $\det H(X;\chi)$ is defined by [5]

$$||\cdot||^{RS} \stackrel{def}{=} |\cdot|^{RS} \prod_{p=0}^{\dim X}\left[\exp\left(-\frac{d}{ds}\zeta(s|L_p)|_{s=0}\right)\right]^{(-1)^p p/2}, \tag{1.6}$$

where the zeta function $\zeta(s|L_p)$ of the Laplacian acting on the space of p-forms orthogonal to the harmonic forms has been used. For a closed connected orientable smooth manifold of odd dimension and for Euler structure $\eta \in \mathrm{Eul}(X)$, the Ray-Singer norm of its cohomological torsion $\tau_{an}(X;\eta) = \tau_{an}(X) \in \det H(X;\chi)$ is equal to the positive square root of the absolute value of the monodromy of χ along the characteristic class $c(\eta) \in H^1(X)$ [6]: $||\tau_{an}(X)||^{RS} = |\mathrm{det}_\chi c(\eta)|^{1/2}$. In the special case where the flat bundle χ is acyclic, we have

$$[\tau_{an}(X)]^2 = |\mathrm{det}_\chi c(\eta)| \prod_{p=0}^{\dim X}\left[\exp\left(-\frac{d}{ds}\zeta(s|L_p)|_{s=0}\right)\right]^{(-1)^{p+1}p}. \tag{1.7}$$

For a closed oriented hyperbolic three-manifolds of the form $X = H^3/\Gamma$, and for acyclic χ, the L^2-analytic torsion has the form [7, 8]: $[\tau_{an}(X)]^2 = \mathcal{R}(0)$, where $\mathcal{R}(s)$ is the Ruelle function (it can be continued meromorphically to the entire complex plane $\mathbb{C}$). The function $\mathcal{R}(s)$ is an alternating product of more complicate factors, each of which is a Selberg type zeta function $Z_\Gamma(s)$ (see Sect. 3.1)

1.1 Structure of the article and our key results

We begin in Sect. 2 with algebra of power series (of commuting variables). Then the polynomial ring of symmetric functions $\Lambda(X)$ over X we analyze in subsect. 2.1. Algebraic properties of $\Lambda(X)$, including plethysms and the Cauchy-Binet formulas, we examine in detail in subsect. 2.2.

At present q-series have reappeared in quantum physics and mathematics – Lie algebras, statistics, transcendental number theory, additive number theory and in classical analysis. In this paper we apply q-series approach to several interesting physical models, as in Sections 3 and Sect. 4.

We discuss the infinite specializations and q-series in Sect. 3. Then the theory of Ruelle type functions $\mathcal{R}(s)$, which are an alternating product of more complicate factors, each of them is the Patterson-Selberg zeta function, with its connection to Euler series is developed in subsect. 3.1. Explicit formulas for the refined-shifted topological vertex are deduced in terms of Ruelle type spectral functions in subsect. 3.2. In subsection 3.3 we investigate the Gopakumar-Vafa reformulation for string partition functions in terms of Ruelle type spectral functions.

Finally in Sect. 4 we analyze the $N = 2$ superconformal field theory. First of all we investigate Bailey's transform and infinite hierarchy of Bailey's chain, including Ruelle type functions, subsection 4.1. Then in subsection 4.2 we begin by showing how some Rogers-Ramanujan identities can be rewrite in term of $\mathcal{R}(s)$. The $N = 2$ Landau-Ginzburg model we describe and investigate in subsect. 4.3. The relation between $N = 2$ minimal models and Landau-Ginzburg theories has been proposed by E. Witten [9]. We discuss the basic properties satisfied by elliptic genera in $N = 2$ theories. We construct the functional equations for $\mathcal{R}(s)$ functions and analyze the modular transformation laws for the elliptic genus of the Landau-Ginzburg model.

2 Algebra of power series (of commuting variables)

Let W be a set of all sequences $\{s_n\}_{n=1}^{\infty}$ with integral non-negative terms. Let for each sequence $s \in W$ the correspond number α_s.

Definition 2.1 *By definition such correspondence assigne a formal power series with coef-*

ficients $\sum_{s\in W}\alpha_s\prod_{n=1}^{\infty}x_n^{s_n}$*;* $x_1, x_2, \ldots$*. Define*

$$\left(\sum_{s\in W}\alpha_s\prod_{n=1}^{\infty}x_n^{s_n}\right)+\left(\sum_{s\in W}\beta_s\prod_{n=1}^{\infty}x_n^{s_n}\right)\stackrel{def}{=}\sum_{s\in W}(\alpha_s+\beta_s)\prod_{n=1}^{\infty}x_n^{s_n};$$

$$\beta\sum_{s\in W}\alpha_s\prod_{n=1}^{\infty}x_n^{s_n}\stackrel{def}{=}\sum_{s\in W}\beta\alpha_s\prod_{n=1}^{\infty}x_n^{s_n},\quad \beta\in\mathbb{K},$$

$$\left(\sum_{s\in W}\alpha_s\prod_{n=1}^{\infty}x_n^{s_n}\right)\left(\sum_{s\in W}\beta_s\prod_{n=1}^{\infty}x_n^{s_n}\right)\stackrel{def}{=}\sum_{s\in W}\gamma_s\prod_{n=1}^{\infty}x_n^{s_n}. \tag{2.1}$$

Where $\gamma_s=\sum_{s',s''}\alpha_{s'}\beta_{s''}$*, the summation is realized for pair of sequances which satisfy the condition* $s'_n+s''_n=s_n$ *for any natural* n*.*

2.1 Polynomial ring of symmetric functions

Our aim in this section is to exploit the Hopf algebra of the ring $\Lambda(X)$ of symmetric functions of the independent variables $(x_1, x_2, \ldots)$, finite or contably infinite in number, that constitute the alphabet X.

Let $\mathbb{Z}[x_1,\ldots,x_n]$ be the polynomial ring, or the ring of formal power series, in n commuting variables $x_1,\ldots,x_n$. The symmetric group S_n acts on this ring by permuting the variables. For $\pi\in S_n$ and $f\in\mathbb{Z}[x_1,\ldots,x_n]$ we have $\pi f(x_1,\ldots,x_n)=f(x_{\pi(1)},\ldots,x_{\pi(n)})$. We are interested in the subring of functions invariant under this action, $\pi f=f$, that is to say the ring of symmetric polynomials in n variables: $\Lambda(x_1,\ldots,x_n)=\mathbb{Z}[x_1,\ldots,x_n]^{S_n}$. This ring may be graded by the degree of the polynomials, so that $\Lambda(X)=\oplus_n\Lambda^{(n)}(X)$, where $\Lambda^{(n)}(X)$ consists of homogenous symmetric polynomials in $x_1,\ldots,x_n$ of total degree n.

In order to work with an arbitrary number of variables, following Macdonald [10], we define the ring of symmetric functions $\Lambda=\lim_{n\to\infty}\Lambda(x_1,\ldots,x_n)$ in its stable limit $(n\to\infty)$. There exist various bases of $\Lambda(X)$:

A $\mathbb{Z}$ basis of $\Lambda^{(n)}$ is provided by the monomial symmetric functions $\{m_\lambda\}$, where λ is any partitions of n.

The other (integral and rational) bases for $\Lambda^{(n)}$ are indexed by the partitions λ of n. There are the complete, elementary and power sum symmetric functions bases defined multiplicatively in terms of corresponding one part functions by: $h_\lambda=h_{\lambda_1}h_{\lambda_2}\cdots h_{\lambda_n}$, $e_\lambda=e_{\lambda_1}e_{\lambda_2}\cdots e_{\lambda_n}$ and $p_\lambda=p_{\lambda_1}p_{\lambda_2}\cdots p_{\lambda_n}$ where the one part functions are defined for $\forall n\in\mathbb{Z}_+$ by

$$h_n(X)=\sum_{i_1\le i_2\cdots\le i_n}x_{i_1}x_{i_2}\cdots x_{i_n},\qquad e_n(X)=\sum_{i_1<i_2\cdots<i_n}x_{i_1}x_{i_2}\cdots x_{i_n},\qquad p_n(X)=\sum_i x_i^n, \tag{2.2}$$

with the convention $h_0=e_0=p_0=1$, $h_{-n}=e_{-n}=p_{-n}=0$. Three of these bases are multiplicative, with $h_\lambda=h_{\lambda_1}h_{\lambda_2}\cdots h_{\lambda_n}$, $e_\lambda=e_{\lambda_1}e_{\lambda_2}\cdots e_{\lambda_n}$ and $p_\lambda=p_{\lambda_1}p_{\lambda_2}\cdots p_{\lambda_n}$. The relationships between the various bases we just mention at this stage by the transitions

$$p_\rho(X)=\sum_{\lambda\vdash n}\chi_\rho^\lambda s_\lambda(X)\quad\text{and}\quad s_\lambda(X)=\sum_{\rho\vdash n}\mathfrak{z}_\rho^{-1}\,\chi_\rho^\lambda\,p_\rho(X)\,. \tag{2.3}$$

For each partition λ, the Schur function is defined by

$$s_\lambda(X) \equiv s_\lambda(x_1, x_2, \dots, x_n) = \frac{\sum_{\sigma\in S_n} \mathrm{sgn}(\sigma) X^{\sigma(\lambda+\delta)}}{\prod_{i<j}(x_i - x_j)}\,, \tag{2.4}$$

where $\delta = (n-1, n-2, \dots, 1, 0)$. In fact both h_n and e_n are special Schur functions, $h_n = s_{(n)}$, $e_n = s_{(1^n)}$, and their generating functions are expressed in terms of the power-sum p_n:

$$\sum_{n\geq 0} h_n z^n = \exp(\sum_{n=1}^{\infty}(p_n/n)z^n), \qquad \sum_{n\geq 0} e_n z^n = \exp(-\sum_{n=1}^{\infty}(p_n/n)(-z)^n)\,. \tag{2.5}$$

The Jacobi-Trudi formula [10] express the Schur functions in terms of h_n or e_n: $s_\lambda = \det(h_{\lambda_i - i+j}) = \det(e_{\lambda' - i+j})$, where λ' is the conjugate of λ. An involution $\iota : \Lambda \to \Lambda$ can be defined by $\iota(p_n) = (-1)^{n-1}p_n$. Then it follows that $\iota(h_n) = e_n$. Also we have $\iota(s_\lambda) = s_{\lambda'}$. χ^λ_ρ is the character of the irreducible representation of the symmetric groups S_n specified by λ in the conjugacy class specified by ρ. These characters satisfy the orthogonality conditions

$$\sum_{\rho\vdash n} \mathfrak{z}_\rho^{-1}\ \chi^\lambda_\rho\ \chi^\mu_\rho = \delta_{\lambda,\mu} \quad \text{and} \quad \sum_{\lambda\vdash n} \mathfrak{z}_\rho^{-1} \chi^\lambda_\rho \chi^\lambda_\sigma\ = \delta_{\rho,\sigma}\,. \tag{2.6}$$

The significance of the Schur function basis lies in the fact that with respect to the usual Schur-Hall scalar product $\langle \cdot \,|\, \cdot \rangle_{\Lambda(X)}$ on $\Lambda(X)$ we have

$$\langle s_\mu(X) \,|\, s_\nu(X)\rangle_{\Lambda(X)} = \delta_{\mu,\nu} \quad \text{and} \quad \text{therefore} \quad \langle p_\rho(X) \,|\, p_\sigma(X)\rangle_{\Lambda(X)} = \mathfrak{z}_\rho \delta_{\rho,\sigma}\,, \tag{2.7}$$

where $\mathfrak{z}_\lambda = \prod_i i^{m_i} m_i!$ for $\lambda = (1^{m_1}, 2^{m_2}, \cdots)$.

2.2 Algebraic properties of $\Lambda(X)$

The ring $\Lambda(X)$ of symmetric functions over X has a Hopf algebra structure, and there are two further algebraic and two coalgebraic operations. For notation and basic properties we refer the reader to [11, 12, 13] and references therein.

Plethysms (compositions) are denoted by $\circ$ or by means of square brackets [ ]; plethysm coproduct is denoted by ∇. The corresponding coproduct maps are specified by Δ for the outer coproduct. Notation δ we use for the inner coproduct.

The coproduct coefficients themselves are obtained from the products by duality using the Schur-Hall scalar product and the self-duality of $\Lambda(X)$. For example, for all $A, B \in \Lambda(X)$:

$$\begin{aligned} A\circ B = A[B]; \quad \Delta(A) \;&=\; A_{(1)} \otimes A_{(2)}; \\ \delta(A) = A_{[1]} \otimes A_{[2]}; \quad \nabla(A) \;&=\; A_{\langle 1\rangle} \otimes A_{\langle 2\rangle}\,. \end{aligned} \tag{2.8}$$

In terms of the Schur function basis $\{s_\lambda\}_{\lambda\vdash n, n\in\mathbb{Z}_+}$ the product and coproduct maps give rise to the particular sets of coefficients specified as follows:

$$\begin{aligned} s_\mu s_\nu &= \sum_\lambda c^\lambda_{\mu,\nu} s_\lambda\,; \qquad \Delta(s_\lambda) = s_{\lambda_{(1)}} \otimes s_{\lambda_{(2)}} = \sum_{\mu,\nu} c^\lambda_{\mu,\nu} s_\mu \otimes s_\nu\,; \\ s_\mu * s_\nu &= \sum_\lambda g^\lambda_{\mu,\nu} s_\lambda\,; \qquad \delta(s_\lambda) = s_{\lambda_{[1]}} \otimes s_{\lambda_{[2]}} = \sum_{\mu,\nu} g^\lambda_{\mu,\nu} s_\mu \otimes s_\nu\,; \\ s_\mu[s_\nu] &= \sum_\lambda p^\lambda_{\mu,\nu} s_\lambda\,; \qquad \nabla(s_\lambda) = s_{\lambda_{\langle 1\rangle}} \otimes s_{\lambda_{\langle 2\rangle}} = \sum_{\mu,\nu} p^\lambda_{\mu,\nu} s_\mu \otimes s_\nu\,. \end{aligned} \tag{2.9}$$

Here the $c^\lambda_{\mu,\nu}$ are Littlewood-Richardson coefficients, the $g^\lambda_{\mu,\nu}$ are Kronecker coefficients and the $p^\lambda_{\mu,\nu}$ are plethysm coefficients. All these coefficients are non-negative integers. The Littlewood-Richardson coefficients can be obtained, for example, by means of the Littlewood-Richardson rule [14, 15] or the hive model [16]. The Kronecker coefficients may determined directly from characters of the symmetric group or by exploiting the Jacobi-Trudi identity and the Littlewood-Richardson rule, while plethysm coefficients have been the subject of a variety methods of calculation [17, 18]. Note that the above sums are finite, since $c^\lambda_{\mu,\nu} \geq 0$ iff $|\lambda| = |\mu| + |\nu|$; $g^\lambda_{\mu,\nu} \geq 0$ iff $|\lambda| = |\mu| = |\nu|$; $p^\lambda_{\mu,\nu} \geq 0$ iff $|\lambda| = |\mu|\,|\nu|$.

The Schur-Hall scalar product may be used to define skew Schur functions $s_{\lambda/\mu}$ through the identities $c^\lambda_{\mu,\nu} = \langle s_\mu\, s_\nu | s_\lambda\rangle = \langle s_\nu\, | s^\perp_\mu(s_\lambda)\rangle = \langle s_\nu | s_{\lambda/\mu}\rangle$, so that $s_{\lambda/\mu} = \sum_\nu\ c^\lambda_{\mu,\nu}\ s_\nu$.

In what follows we shall make considerable use of several infinite series of Schur functions. The most important of these are the mutually inverse pair defined by

$$\mathcal{F}(t;X) = \prod_{i\geq 1}(1 - t\,x_i)^{-1} = \sum_{m\geq 0} h_m(X)t^m\,, \tag{2.10}$$

$$\mathcal{G}(t;X) = \prod_{i\geq 1}(1 - t\,x_i) = \sum_{m\geq 0}(-1)^m e_m(X)t^m\,, \tag{2.11}$$

where as Schur functions $h_m(X) = s_{(m)}(X)$ and $e_m(X) = s_{(1^m)}(X)$. For convenience, in the case $t = 1$ we write $\mathcal{F}(1;X) = \mathcal{F}(X)$ and $\mathcal{G}(1;X) = \mathcal{G}(X)$.

Note on plethysms. Plethysms are defined as compositions whereby for any $A, B \in \Lambda(X)$; the plethysm $A[B]$ is A evaluated over an alphabet Y whose letters are the monomials of $B(X)$, with each letter repeated as many times as the multiplicity of the corresponding monomial. For example, the Schur function plethysm is defined by $s_\lambda[s_\mu](X) = s_\lambda(Y)$, where $Y = s_\mu(X)$.

For all $A, B, C \in \Lambda(X)$ we have the following rules, due to Littlewood [15], for manipulating plethysms:

$$\begin{aligned} (A+B)[C] &= A[C] + B[C]\,; \qquad A[B+C] = A_{(1)}[B]A_{(2)}[C]\,; \\ (AB)[C] &= A[C]B[C]\,; \qquad A[BC] = A_{[1]}[B]A_{[2]}[C]\,; \\ A[B[C]] &= (A[B])[C]\,. \end{aligned} \tag{2.12}$$

These rules enable us to evaluate plethysms not only of outer and inner products but also of outer and inner coproducts.

The Cauchy-Binet formulas. It is often convenient to represent an alphabet in an additive manner X, as itself an element of the ring $\Lambda(X)$ in the sense that $x_1 + x_2 + \cdots = h_1(X) = e_1(X) = p_1(X) = s_{(1)}(X)$. As elements of $\Lambda(X) \otimes \Lambda(Y)$ we have $X + Y = \sum_{j=1} x_j + \sum_{j=1} y_j$, $XY = \sum_j x_j \sum_j y_j$. With this notation, the outer coproduct gives

$$\begin{aligned} \Delta(\mathcal{F}) &= \mathcal{F}_{(1)} \otimes \mathcal{F}_{(2)} = \mathcal{F} \otimes \mathcal{F}; \qquad \mathcal{F}(X{+}Y) = \prod_i (1-x_i)^{-1} \prod_j (1-y_j)^{-1}\,; \\ \Delta(\mathcal{G}) &= \mathcal{G}_{(1)} \otimes \mathcal{G}_{(2)} = \mathcal{G} \otimes \mathcal{G}; \qquad \mathcal{G}(X{+}Y) = \prod_i (1-x_i) \prod_j (1-y_j)\,, \end{aligned}$$

so that $\mathcal{F}(X{+}Y) = \mathcal{F}(X)\,\mathcal{F}(Y)$ and $\mathcal{G}(X{+}Y) = \mathcal{G}(X)\,\mathcal{G}(Y)$. For the inner coproduct:

$$\begin{aligned} \delta(\mathcal{F}) &= \mathcal{F}_{[1]} \otimes \mathcal{F}_{[2]}; \qquad \mathcal{F}(XY) = \prod_{i,j} (1-x_i y_j)^{-1}\,; \\ \delta(\mathcal{G}) &= \mathcal{G}_{[1]} \otimes \mathcal{G}_{[2]}; \qquad \mathcal{G}(XY) = \prod_{i,j} (1-x_i y_j)\,. \end{aligned}$$

The expansions of the products on the right hand sides of these expressions is effected remarkably easily by evaluating the inner coproducts on the left:

$$\begin{aligned} \delta(\mathcal{F}) &= \sum_{k\geq 0} \delta(h_k) = \sum_{k\geq 0} \sum_{\lambda\vdash k} s_\lambda \otimes s_\lambda\,; \\ \delta(\mathcal{G}) &= \sum_{k\geq 0} (-1)^k \,\delta(e_k) = \sum_{k\geq 0} (-1)^k \sum_{\lambda\vdash k} s_\lambda \otimes s_{\lambda'}\,. \end{aligned}$$

This gives immediately the well known Cauchy and Cauchy-Binet formulas:

$$\mathcal{F}(XY) = \prod_{i,j} (1 - x_i\, y_j)^{-1} = \sum_\lambda s_\lambda(X)\, s_\lambda(Y)\,; \tag{2.13}$$

$$\mathcal{G}(XY) = \prod_{i,j} (1 - x_i\, y_j) = \sum_\lambda (-1)^{|\lambda|} s_\lambda(X)\, s_{\lambda'}(Y)\,. \tag{2.14}$$

Generally speaking, for any $F(X) \in \Lambda(X)$ with dual $F^\perp(X)$, by linearly extending the above result we have $F(X)\ \mathcal{F}(XY) = F^\perp(Y)\,(\mathcal{F}(XY))$.

3 Infinite specializations and q-series

The study of q-series has been extensively enriched due to success in investigation of q-analogs of the classical special functions. The large amount of activity of q-series has been appeared in Lie algebras, statistical and quantum mechanics, trancendental number theory, and computer algebra.

By considering infinite specializations, i.e. setting $X = (x_1, x_2, \ldots, x_r, \ldots) = (q, q^2, \ldots q^r, \ldots)$, $q = e^{2\pi i\tau}$, we present the following useful identities:

$$\begin{aligned} \mathcal{F}(q; XY) &= \prod_{i,j} (1 - q x_i y_j)^{-1} = \sum_\alpha q^\alpha s_\alpha(X) s_\alpha(Y), \\ \mathcal{G}(q; XY) &= \prod_{i,j} (1 - q x_i y_j) = \sum_\alpha (-q)^{|\alpha|} s_\alpha(X) s_{\alpha'}(Y). \end{aligned} \tag{3.1}$$

3.1 Spectral functions of hyperbolic three-geometry

The most important Euler series can be represent in the form of the Ruelle type (Patterson-Selberg) spectral function $\mathcal{R}(s)$ of hyperbolic three-geometry, in fact:

$$
\begin{aligned}
\prod_{n=\ell}^{\infty}(1-q^{an+\varepsilon}) &= \prod_{p=0,1} Z_\Gamma(\underbrace{(a\ell+\varepsilon)(1-i\varrho(\tau))+1-a}_{s}+a(1+i\varrho(\tau)p)^{(-1)^p} \\
&= \mathcal{R}(s=(a\ell+\varepsilon)(1-i\varrho(\tau))+1-a), \qquad (3.2)\\
\prod_{n=\ell}^{\infty}(1+q^{an+\varepsilon}) &= \prod_{p=0,1} Z_\Gamma(\underbrace{(a\ell+\varepsilon)(1-i\varrho(\tau))+1-a+i\sigma(\tau)}_{s}+a(1+i\varrho(\tau)p)^{(-1)^p} \\
&= \mathcal{R}(s=(a\ell+\varepsilon)(1-i\varrho(\tau))+1-a+i\sigma(\tau))\,. \qquad (3.3)
\end{aligned}
$$

Here $q = \exp(2\pi i\tau)$, $\varrho(\tau) = \mathrm{Re}\,\tau/\mathrm{Im}\,\tau$, $\sigma(\tau) = (2\,\mathrm{Im}\,\tau)^{-1}$, a is a real number, $\varepsilon, b \in \mathbb{C}$, $\ell \in \mathbb{Z}_+$.

Various expansions different from power series expansion. Let us consider

$$
\mathcal{Q}(q) = \prod_{n=1}^{\infty}(1-q^n)^{-C_n} = 1+\sum_{n=1}^{\infty}\mathcal{B}_n q^n. \qquad (3.4)
$$

Usualy $\mathcal{Q}(q)$ associated wth some generating function, and $\mathcal{B}_n$ is related sequence that will be study.

Theorem 3.1 *([19], Theorem 10.3) Let in Eq. (3.4) C_n and $\mathcal{B}_n$are integers, then*

$$
\begin{aligned}
n\mathcal{B}_n &= \sum_{k=1}^{n} D_k\mathcal{B}_{n-k}, \qquad (3.5)\\
D_k &= \sum_{d|k} dC_d. \qquad (3.6)
\end{aligned}
$$

Note that if either sequence C_n or $\mathcal{B}_n$ is given, then the other is uniquely determined by (3.5) and (3.6).

For $C_n = c\cdot n, c = const.$ the following relation holds:

$$
\begin{aligned}
\prod_{n=\ell}^{\infty}(1-q^{an+\varepsilon})^{cn} &= \mathcal{R}(s=(a\ell+\varepsilon)(1-i\varrho(\tau))+1-a)^{c\ell}\\
&\times\prod_{n=\ell+1}^{\infty}\mathcal{R}(s=(an+\varepsilon)(1-i\varrho(\tau))+1-a)^{c}. \qquad (3.7)
\end{aligned}
$$

3.2 The refined-shifted topological vertex

The standard topological vertex $\mathfrak{C}_{\lambda\mu\nu}(q)$. For the particular case (there is no boundary condition), i.e. $\lambda = \mu = \nu = \emptyset$, the vertex $\mathfrak{C}_{\emptyset\emptyset\emptyset}(q) = \mathfrak{C}_3$ coincides with 3d– MacMahon function $\mathfrak{C}_3(q) = \prod_{n=1}^{\infty}(1-q^n)^{-n}$ [20].

$$\prod_{n=1}^{\infty}(1-q^n)^{-n} = \mathcal{R}(s=1-i\varrho(\tau))^{-1}\prod_{n=2}^{\infty}\mathcal{R}(s=n(1-i\varrho(\tau)))^{-1} \tag{3.8}$$

The refined topological vertex $\mathfrak{R}_{\lambda\mu\nu}(q,t)$. $\mathfrak{R}_{\lambda\mu\nu}(q,t)$ can be constructed as a two parameter extension of $\mathfrak{C}_{\lambda\mu\nu}(q)$. In the particular case $\lambda = \mu = \nu = \emptyset$, the vertex $\mathfrak{R}_{\lambda\mu\nu}(q,t) = \mathfrak{R}_{\emptyset\emptyset\emptyset}(q,t)$ coincide with the refined 3d-MacMahon function.

$$\mathfrak{R}_{\emptyset\emptyset\emptyset}(q,t) = \prod_{n,k=1}^{\infty}(1-q^{n-1}t^k) = \prod_{k=1}^{\infty}\mathcal{R}(s=\varepsilon(1-i\varrho(\tau))), \tag{3.9}$$

where $\varepsilon = k\mathrm{log}t/2\pi i\tau$. Finally in the case $t=q$ in Eq. (3.9) we obtain the standard $\mathfrak{C}_3(q)$ relation.

Shifted topological vertex $\mathfrak{S}_{\lambda\mu\nu}(q)$. With generic boundary conditions the shifted topological vertex $\mathfrak{S}_{\lambda\mu\nu}(q)$ is the shifted 3d MacMahon $\mathfrak{S}_{\emptyset\emptyset\emptyset}(q) = \mathfrak{S}_3(q)$, the generating functional of strict plane partitions $\mathfrak{S}_3(q) = \prod_{n=1}^{\infty}\left(\frac{1+q^n}{1-q^n}\right)^n$ [20].

$$\mathfrak{S}_3(q) = \frac{\mathcal{R}(s=1-i\varrho(\tau)+i\sigma(\tau))}{\mathcal{R}(s=1-i\varrho(\tau))}\cdot\frac{\prod_{n=2}^{\infty}\mathcal{R}(s=n(1-i\varrho(\tau)+i\sigma(\tau)))}{\prod_{n=2}^{\infty}\mathcal{R}(s=n(1-i\varrho(\tau)))}. \tag{3.10}$$

Refining the shifted vertex $\mathfrak{T}_{\lambda\mu\nu}(q,t)$. The refining version $\mathfrak{T}_{\lambda\mu\nu}(q,t)$ of the shifted topological vertex $\mathfrak{S}_{\lambda\mu\nu}(q)$ is a two parameters q and t with boundary conditions given by strict 2d partitions λ,μ and ν [20]. In addition, $\mathfrak{T}_{\lambda\mu\nu}(q,t)$ is non cyclic with respect to the permutations of the strict 2d partitions λ,μ,ν, i.e. $\mathfrak{T}_{\lambda\mu\nu} \neq \mathfrak{T}_{\mu\nu\lambda} \neq \mathfrak{T}_{\nu\lambda\mu}$, and what is more

$$\mathfrak{T}_{\lambda\mu\nu}(q,q) = \mathfrak{S}_{\lambda\mu\nu}(q), \quad \mathfrak{T}_{\emptyset\emptyset\emptyset}(q,t) = \mathfrak{T}_3(q,t). \tag{3.11}$$

$$\mathfrak{T}_3(q,t) = \prod_{j=1}^{\infty}\prod_{k=1}^{\infty}\left(\frac{1+q^{j-1}t^k}{1-q^{j-1}t^k}\right) = \prod_{k=1}^{\infty}\frac{\mathcal{R}(s=\varepsilon(1-i\varrho(\tau))+i\sigma(\tau))}{\mathcal{R}(s=\varepsilon(1-i\varrho(\tau)))}, \tag{3.12}$$

recall that $\varepsilon = k\mathrm{log}t/2\pi i\tau$. In this section we have analyzed the refining and the shifting properties of the standard topological vertex $\mathfrak{C}_{\lambda\mu\nu}(q)$, the refined topological vertex $\mathfrak{R}_{\lambda\mu\nu}(q,t)$, the shifted topological vertex $\mathfrak{S}_{\lambda\mu\nu}(q)$, and the refining the shifted vertex $\mathfrak{T}_{\lambda\mu\nu}(q,t)$, in terms of spectral functions $\mathcal{R}(s)$.

3.3 Gopakumar-Vafa reformulation of string partition functions

In order to prove the Gopakumar-Vafa conjecture [21, 22, 23], we need to rewrite the sums over partition by means of infinite products. In this section we propose a spectral function reformulation for such an infinite products.

Denote by F the generating series of Gromov-Witten invariants of a Calabi-Yau three-fold X. Intuitively we can counts the number of stable maps with *connected* domain curves to X

in any given nonzero homology classes. However, because of the existence of automorphisms, one has to perform the weighted count by dividing by the order of the automorphism groups (hence Gromov-Witten invariants are in general rational numbers). Based on M-theory considerations, Gopakumar and Vafa [22] made a remarkable conjecture on the structure of F, in particular, on its integral properties. More precisely, integers n_Σ^g are conjectured to exists such that

$$F = \sum_{\Sigma\in H_2(X)-\{0\}} \sum_{g\geq 0} \sum_{k\in\mathbb{Z}_+} k^{-1} n_\Sigma^g (2\sin(k\lambda/2))^{2g-2} Q^{k\Sigma}\,. \tag{3.13}$$

For given Σ, there are only finitely many nonzero n_Σ^g, $Q^\Sigma = \exp(-\int_\Sigma \omega)$, the holomorphic curve $\Sigma \in H_2(X,\mathbb{Z}) := H_2(X)$ is given by $\int_\Sigma \omega$, where ω is the Kählerian form on X. Let us regard $q = \exp(i\lambda)$, for some real $\lambda = 2\pi\tau$, as an element of $SU(2)$ represented by the diagonal matrix diag (q, q^{-1}). The generating series of *disconnected* Gromov-Witten invariants is given by the string partition function: $Z = \exp F$.

Topological string amplitudes. We shall present now the interpretation of topological string amplitudes in the form of the generating functions of the BPS degeneracies of wrapped M2-branes, which give rise to particles in the five dimensional theory [24]. First consider IIA strings compactified on a CY3-fold X. In this case the theory on the transverse four dimensions has *N=2* supersynnetry. In four dimensions this theory has F-terms which can be calculated exactly [25, 26].

Let F_g be the topological string amplitudes (Gopakumar). In the A-twisted topologicasl theory F_g arise as integrals over the genus g moduli space of Riemann surfaces and related to the generating functions of the genus g Gromov-Witten invariants. The topological string amplitudes can be compactly organized into the generating function $F(\lambda_s) = \sum_{g=0}^\infty \lambda_s^{2g-2} F_g$, where λ_s is the constant self-dual graviphoton strength. Interesting physical interpretation of the generating function $F(\lambda_s)$ the reader can find in [21, 22]

For second quantized strings and A-model result (the Gopakumar-Vafa conjecture) can be reformulated as follows (see [24]):

$$Z = \prod_{\Sigma\in H_2(X)} \prod_j \prod_{k=-j}^{j} \prod_{m\in\mathbb{Z}_+\cup\{0\}} (1 - q^{2k+m+1} Q^\Sigma)^{(-1)^{2j+1}(m+1)N_\Sigma^g}. \tag{3.14}$$

In Eq. (3.14) $j = g/2$, $k = -j, -j+1, \ldots, j-1, j$, $Q^\Sigma = e^{-T_\Sigma}$ and $q = e^{-i\lambda_s}$. Expressions (3.13) and (3.14) look very much like *counting* the states in a Hilbert space for the case of $j_L = 0$ BPS states [21]. The partition function counts M2-branes [24]. In addition the integrality of Z has to be directly related to the same integrality in F. One can interpret Z as the partition function of a second quantized theory which is built purely out of fields creating M2-branes.

In terms of the Ruelle type spectral functions the string partition function can be reformulated as follows

$$
\begin{aligned}
Z &= \prod_{\Sigma\in H_2(X)} \prod_{j} \prod_{k=-j}^{j} \prod_{m\in\mathbb{Z}_+\cup\{0\}} (1-q^{2k+m+1}Q^{\Sigma})^{(-1)^{2j+1}N^g_{\Sigma}m} \cdot (1-q^{2k+m+1}Q^{\Sigma})^{(-1)^{2j+1}N^g_{\Sigma}} \\
&= \prod_{\Sigma\in H_2(X)} \prod_{j} \prod_{k=-j}^{j} \mathcal{R}(s=(2k+\varphi+1)(1-i\varrho(\tau))) \\
&\times \prod_{n=\ell+1}^{\infty} \mathcal{R}(s=(n+2k+\varphi+1)(1-i\varrho(\tau)))^{(2j+1)N^g_{\Sigma}},
\end{aligned} \tag{3.15}
$$

where $\varphi = \log Q^{\Sigma}/2\pi i\tau = -\int_{\Sigma}\omega/2\pi i\tau$.

4 The *N=2* superconformal field theory

4.1 Bailey's transform and infinite hierarchy of Bailey's chain

Let us begin with notations: $(a;q)_\infty := \prod_{m=0}^{\infty}(1-aq^m)$, and $(a;q)_n := (a;q)_\infty/(aq^n;q)_\infty$. W.N. Bailey made the following observation [27] (see also [19]), which is known as *Bailey's transform.*

Theorem 4.1 *(lectures [19], Theorem 3.1) Let for a suitable convergence conditions,*

$$\beta_n = \sum_{r=0}^{n} \alpha_r u_{n-r} v_{n+r}, \tag{4.1}$$

$$\gamma_n = \sum_{r=n}^{\infty} \delta_r u_{r-n} v_{r+n}, \tag{4.2}$$

then

$$\sum_{n=0}^{\infty} \alpha_n\gamma_n = \sum_{n=0}^{\infty} \beta_n\delta_n. \tag{4.3}$$

Proof.

$$\sum_{n=0}^{\infty} \alpha_n\gamma_n = \sum_{n=0}^{\infty}\sum_{r=n}^{\infty} \alpha_n\delta_r u_{r-n}v_{r+n} = \sum_{r=0}^{\infty}\sum_{n=0}^{r} \alpha_n\delta_r u_{r-n}v_{r+n} = \sum_{r=0}^{\infty}\delta_r\beta_r. \tag{4.4}$$

Important application of the Bailey's transform and Bailey's Lemma the reader can found in [27], § 4

Let for $n \geq 0$

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q;q)_{n-r}(aq;q)_{n+r}}, \tag{4.5}$$

then also

$$\beta'_n = \sum_{r=0}^{n} \frac{\alpha'_r}{(q;q)_{n-r}(aq;q)_{n+r}}. \tag{4.6}$$

In addition

$$\alpha'_r = \alpha_r \frac{(\rho_1;q)_r(\rho_2;q)_r(aq/\rho_1\rho_2)^r}{(aq/\rho_1;q)_r(aq/\rho_2;q)_r}, \tag{4.7}$$

$$\beta'_n = \sum_{r\geq 0} \beta_r \frac{(\rho_1;q)_r(\rho_2;q)_r(aq/\rho_1\rho_2;q)_{n-r}(aq/\rho_1\rho_2)^j}{(q;q)_{n-j}(aq/\rho_1;q)_n(aq/\rho_2;q)_n}. \tag{4.8}$$

Restricted products. We apply Bailey's transform in the case of restricted products.

$$u_{n-r} := \prod_{m=0}^{n-r}(1-q^{m+1}) \equiv (q;q)_{n-r} = \frac{(q;q)_\infty}{(qq^{n-r};q)_\infty} = \prod_{m=0}^{\infty}\frac{(1-q^{m+1})}{(1-q^{m+n-r+1})} \tag{4.9}$$

$$= \frac{\mathcal{R}(s=1-i\varrho(\tau))}{\mathcal{R}(s=(n-r+1)(1-i\varrho(\tau)))}, \tag{4.10}$$

$$v_{n+r} := \prod_{m=0}^{n+r}(1-aq^{m+1}) \equiv (aq;q)_{n+r} = \frac{(q;q)_\infty}{(aq^{n+r};q)_\infty} = \prod_{m=0}^{\infty}\frac{(1-q^{m+1})}{(1-aq^{m+n+r+1})} \tag{4.11}$$

$$= \frac{\mathcal{R}(s=1-i\varrho(\tau))}{\mathcal{R}(s=(n+r+\xi+1)(1-i\varrho(\tau)))}, \tag{4.12}$$

where $\xi = \log a/2\pi i\tau$.

Using Eqs. (4.10) and (4.12) we have:

$$\beta_n = \sum_{r=0}^{n} \alpha_r \frac{\mathcal{R}(s=(n-r+1)(1-\varrho(\tau)))\cdot\mathcal{R}(s=(n-r+1)(1-\varrho(\tau)))}{\mathcal{R}(s=1-i\varrho(\tau))\cdot\mathcal{R}(s=1-i\varrho(\tau))}, \tag{4.13}$$

$$\beta'_n = \sum_{r=0}^{n} \alpha'_r \frac{\mathcal{R}(s=(n-r+1)(1-\varrho(\tau)))\cdot\mathcal{R}(s=(n-r+1)(1-\varrho(\tau)))}{\mathcal{R}(s=1-i\varrho(\tau))\cdot\mathcal{R}(s=1-i\varrho(\tau))}. \tag{4.14}$$

In addition (see [19]),

$$\alpha'_r = \alpha_r \frac{(\rho_1;q)_r(\rho_2;q)_r(aq/\rho_1\rho_2)^r}{(aq/\rho_1;q)_r(aq/\rho_2;q)_r}, \tag{4.15}$$

$$\beta'_n = \sum_{r\geq 0} \beta_r \frac{(\rho_1;q)_r(\rho_2;q)_r(aq/\rho_1\rho_2;q)_{n-r}(aq/\rho_1\rho_2)^r}{(q;q)_{n-r}(aq/\rho_1;q)_n(aq/\rho_2;q)_n}.. \tag{4.16}$$

Important special cases of Bailey's Lemma and iterated of these results where found in [28]. Note that a pair of sequences (α_n, β_n) is called a *Bailey pair*. Thus if (α_n, β_n) is a Bailey pair, then (α'_n, β'_n) is new pair which is given by (4.15) and (4.16).

Bailey's chain. Using Bailey initial pair (α'_n, β'_n) we can create new pair (α''_n, β''_n) by applying Bailey's Lemma. Continuing this process we create a sequence of Baley pairs – Bailey's chain:

$$(\alpha_n, \beta_n) \longrightarrow (\alpha'_n, \beta'_n) \longrightarrow (\alpha''_n, \beta''_n) \longrightarrow (\alpha'''_n, \beta'''_n) \longrightarrow \cdots \tag{4.17}$$

Let us assume $(\alpha_n, \beta_n) = (\alpha_n^0, \beta_n^0)$, then this allows to extend the Bailey chain (4.17) to the left [19]:

$$\cdots \longrightarrow (\alpha_n^{(-2)}, \beta_n^{(-2)}) \longrightarrow (\alpha_n^{(-1)}, \beta_n^{(-1)}) \longrightarrow (\alpha_n^0, \beta_n^0) \longrightarrow (\alpha_n', \beta_n') \longrightarrow \cdots \tag{4.18}$$

A Bailey pair is uniquely determined given either sequence α_n or β_n. Let α_n is given, then β_n sequence we can find. Eq. (4.5) and Eq. (4.6) can be invert, and result is [29]:

$$\begin{aligned}
\alpha_n &= (1-aq^{2n})\sum_{k=0}^{n}\beta_k\frac{(aq;q)_{n+k-1}(-1)^{n-k}q^{C_2^{n-k}}}{(q;q)_{n-k}} \\
&= (1-aq^{2n})\sum_{k=0}^{n}\beta_k\frac{\prod_{m=0}^{\infty}(1-aq^{m+n+k})(-1)^{n-k}q^{C_2^{n-k}}}{\prod_{m=0}^{\infty}(1-q^{m+n-k+1})}.
\end{aligned} \tag{4.19}$$

Finally we obtain

$$\alpha_n = (1-aq^{2n})\sum_{k=0}^{n}\beta_k\frac{\mathcal{R}(s=(n+k)(1-i\varrho(\tau)))(-1)^{n-k}q^{C_2^{n-k}}}{\mathcal{R}(s=(n-k+1)(1-\varrho(\tau)))}. \tag{4.20}$$

4.2 Rogers-Ramanujan type identities

$$\begin{aligned}
&\prod_{n=0}^{\infty}(1-2kq^n\cos\alpha+k^2q^{2n}) = (ke^{i\alpha};q)_\infty(ke^{-i\alpha};q)_\infty = \prod_{n=0}^{\infty}(1-ke^{i\alpha}q^n)\cdot(1-ke^{-i\alpha}q^n) \\
&= \mathcal{R}(s=\xi_+(1-i\varrho(\tau)))\cdot\mathcal{R}(s=\xi_-(1-i\varrho(\tau))),
\end{aligned} \tag{4.21}$$

$$\begin{aligned}
&\prod_{n=0}^{\infty}(1+2kq^n\cos\alpha+k^2q^{2n}) = (-ke^{i\alpha}q;q)_\infty(-ke^{-i\alpha}q;q)_\infty = \prod_{n=0}^{\infty}(1+ke^{i\alpha}q^n)\cdot(1+ke^{-i\alpha}q^n) \\
&= \mathcal{R}(s=\xi_+(1-i\varrho(\tau))+i\sigma(\tau))\cdot\mathcal{R}(s=\xi_-(1-i\varrho(\tau))+i\sigma(\tau)).
\end{aligned} \tag{4.22}$$

where $\xi_\pm = (\log k \pm i\alpha)/2\pi i\tau$.

$$\begin{aligned}
\theta_1(x) &= 2\sum_{n=0}^{\infty}(-1)^n q^{(n+1/2)^2}\sin(2n+1)x \\
&= 2q^{1/4}\sin x\prod_{n=1}^{\infty}(1-q^{2n})(1-2q^{2n}\cos 2x+q^{4n}),
\end{aligned} \tag{4.23}$$

The Jacobi theta function $\theta_1(\tau, z)$ is

$$\theta_1(\tau,z) = i\sum_{n\in\mathbb{Z}}(-1)^n q^{\frac{1}{2}(n-\frac{1}{2})^2}y^{n-\frac{1}{2}} = e^{2\pi i(1+\tau/8)}y^{-\frac{1}{2}}\prod_{n=1}^{\infty}(1-q^n)(1-yq^{n-1})(1-y^{-1}q^n), \tag{4.24}$$

where $q = e^{2\pi i\tau}$, $y = e^{2\pi iz}$.

4.3 The N=2 Landau-Ginzburg model

Elliptic genus in N=2 superconformal field theory has been proposed by E. Witten [9] in order to understand the relation between N=2 minimal models and Landau-Ginzburg theories. In this section we discuss the basic properties satisfied by elliptic genera in N=2 theories. In the case of the N=2 Landau-Ginzburg model an action takes the form [30]

$$\int d^2zd^2\theta d^2\overline{\theta}X\overline{X}+\left(\int d^2zd^2\theta W(X)+c.c.\ term\right), \tag{4.25}$$

where the superpotential W is a weighted homogeneous polynomial of N chiral superfields $X_1,\ldots,X_N$ with weights $\omega_1,\ldots,\omega_N$,

$$\lambda W(X_1,\ldots,X_N)=W(\lambda^{\omega_1}X_1,\ldots,\lambda^{\omega_N}X_N) \tag{4.26}$$

Assume that W has an isolated critical point at the origin and ω_i*'s* are strictly positive rational numbers such that $\omega_1,\ldots,\omega_N\leq 1/2$. then the elliptic genus of the landau-Ginzburg model can be computed as

$$Z(\tau,z)=\prod_{j=1}^{N}Z_{\omega_j}(\tau,z) \tag{4.27}$$

where

$$Z_\omega(\tau,z)=\frac{\theta_1(\tau,(1-\omega)z)}{\theta_1(\tau,\omega z)}. \tag{4.28}$$

Finally using Eqs. (4.24) we obtain:

$$Z_\omega(\tau,z)=e^{\omega+\tau/8+1/2}\prod_{n=1}^{\infty}\frac{[(1-yq^{n-1})(1-y^{-1}q^n)]|_{y=e^{2\pi i(1-\omega)z}}}{[(1-yq^{n-1})(1-y^{-1}q^n)]|_{y=e^{2\pi i\omega z}}}=e^{\omega+\tau/8+1/2}$$
$$\times\frac{\mathcal{R}(s=(1+z/\tau(1-\omega))(1-i\varrho(\tau)))\mathcal{R}(s=(1-z/\tau)(1-\omega)(1-\varrho(\tau)))}{\mathcal{R}(s=(1+z\omega/\tau)(1-\varrho(\tau)))\mathcal{R}(s=1-z\omega/\tau)(1-i\varrho(\tau))}. \tag{4.29}$$

The modular transformation laws.

Obviously $\prod_{n=1}^{\infty}(1+q^n)=\prod_{n=1}^{\infty}(1-q^{2n})/(1-q^n)=\prod_{n=1}^{\infty}(1-q^{2n-1})^{-1}$, thus $\mathcal{R}(s=1-i\varrho(\tau)+i\sigma(\tau))\mathcal{R}(s=1-\varrho(\tau))=\mathcal{R}(s=1-2i\varrho(\tau))=1$.

Next let us introduce some well-known functions and their modular properties under the action of $SL(2,\mathbb{Z})$. The special cases associated with (4.10), (4.12) are:

$$\begin{aligned}
\varphi_1(q) &= q^{-\frac{1}{48}}\prod_{m=1}^{\infty}(1-q^{m+\frac{1}{2}}) \ = q^{-\frac{1}{48}}\mathcal{R}(s=3/2(1-i\varrho(\tau))) \\
&= \frac{\eta_D(q^{\frac{1}{2}})}{\eta_D(q)}\,, && (4.30)\\
\varphi_2(q) &= q^{-\frac{1}{48}}\prod_{m=1}^{\infty}(1+q^{m+\frac{1}{2}}) \ = q^{-\frac{1}{48}}\mathcal{R}(s=3/2(1-i\varrho(\tau))+i\sigma(\tau)) \\
&= \frac{\eta_D(q)^2}{\eta_D(q^{\frac{1}{2}})\eta_D(q^2)}\,, && (4.31)\\
\varphi_3(q) &= q^{\frac{1}{24}}\prod_{m=1}^{\infty}(1+q^{m+1}) \ = q^{\frac{1}{24}}\mathcal{R}(s=2(1-i\varrho(\tau))+i\sigma(\tau)) \\
&= \frac{\eta_D(q^2)}{\eta_D(q)}\,, && (4.32)
\end{aligned}$$

where $\eta_D(q)\equiv q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ is the Dedekind η-function. The linear span of $\varphi_1(q),\varphi_2(q)$ and $\varphi_3(q)$ is $SL(2,\mathbb{Z})$-invariant [31].

Functional equations for the spectral Ruelle functions are

$$\begin{aligned}
&\mathcal{R}(s=(z+b)(1-i\varrho(\tau))+i\sigma(\tau))\cdot\mathcal{R}(s=-(1+z+b)(1-i\varrho(\tau))+i\sigma(\tau))\\
&=q^{-zb-b(b+1)/2}\mathcal{R}(s=-z(1-i\varrho(\tau))+i\sigma(\tau))\cdot\mathcal{R}(s=(1+z)(1-i\varrho(\tau))+i\sigma(\tau))\\
&=q^{-z(b-1)-b(b+1)/2}\mathcal{R}(s=(1-z)(1-i\varrho(\tau)+i\sigma(\tau))\cdot\mathcal{R}(s=z(1-i\varrho(\tau)+i\sigma(\tau))). \quad (4.33)
\end{aligned}$$

The simple case $b=0$ in Eq. (4.33) leads to the symmetry $\tau\to-\tau$, i.e. the symmetry $q\to q^{-1}$. The modular transformation laws are:

$\theta_1(\tau+1,z)=e^{2\pi i(1/8)}\theta_1(\tau,z)$; $\theta_1(-1/\tau,z/\tau)=(-i\tau)^{1/2}e^{2\pi i(1/2)(z^2/\tau)}\theta_1(\tau,z)$, $\theta_1(\tau,-z)=-\theta_1(\tau,z))$.

The double quasi-periodicity law is: $\theta_1(\tau,z+\alpha\tau+\beta)=(-1)^{\alpha+\beta}e^{-2\pi i(1/2)(\alpha^2+2\alpha z)}\theta_1(\tau,z)$, $\alpha,\beta\in\mathbb{Z}$. Note that $\theta_1(\tau,z)$ has no poles but this function (as a function of z) has simple zeros: $\theta_1(\tau,\alpha\tau+\beta)=0$, $\alpha,\beta\in\mathbb{Z}$.

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References

[1] I. Schur, *Über eine Klasse von Matrizen die sich einer gegebenen Matrix zuordnen lassen*. Ges. Abhandlung **1** (1901) 1.

[2] P. Perry and F. Williams, *Selberg zeta function and trace formula for the BTZ black hole*, Int. J. Pure Appl. Math. **9** (2003) 1-21.

[3] A. A. Bytsenko, M. E. X. Guimãraes and F. L. Williams, *Remarks on the spectrum and truncated heat kernel of the BTZ black hole*, Lett. Math. Phys. **79** (2007) 203-211.

[4] L. Bonora and A.A. Bytsenko, *Partition functions for quantum gravity, black holes, elliptic genera and Lie algebra homologies*, Nucl. Phys. B **852** (2011) 508-537.

[5] D. Ray and I. Singer, *R-Torsion and the Laplacian on Riemannian Manifolds*, Adv. Math. **7** (1971) 145-210.

[6] M. Farber and V. Turaev, *Poincaré-Reidemaister Metric, Euler Structures, and Torsion*, J. reine angew. Math. **520** (2000) 195-225.

[7] D. Fried, *Analytic torsion and closed geodesics on hyperbolic manifolds*, Invent. Math. **84** (1986) 523-540.

[8] A. A. Bytsenko, L. Vanzo and S. Zerbini, *Ray-Singer torsion for a hyperbolic 3-manifold and asymptotics of Chern-Simons-Witten invariant*, Nucl. Phys. B **505** (1997) 641-659.

[9] E. Witten *On the Landau-Ginzburg description of N=2 minimal models*, Int. J. Mod. Phys. A **9** (1994) 4783-4800.

[10] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, 2nd Ed. Clarendon Press, Oxford, 1995.

[11] B. Fauser and P. D. Jarvis, *A Hopf laboratory for symmetric functions*, J. Phys. A: Math. Gen. **37** (2004) 1633-1663.

[12] B. Fauser, P. D. Jarvis, R. C. King and B. G. Wybourne, *New branching rules induced by plethysm*, J. Phys A: Math. Gen. **39** (2006) 2611-2655.

[13] B. Fauser, P. D. Jarvis and R. C. King, *Plethysms, replicated Schur functions and series, with applications to vertex operators*, J. Phys. A: Math. Theor. **43** (2010) 405202 (30 pp).

[14] D. E. Littlewood and A. Richardson, *Group characters and algebra*, Phil. Trans. Roy. Soc. London A **233** (1934) 99-141.

[15] D. E. Littlewood, *The Theory of Group Characters*, 2nd Ed. Oxford University Press, Oxford, 1950.

[16] A. S. Buch, *The saturation conjecture (after A. Knutson and T. Tao). With an appendix by W. Fulton*, Enseign. Math. **46** (2000) 43-60.

[17] D. E. Littlewood, *A University Algebra*, Review, Bull. Amer. Math. Soc. **59** (1953) 97-98.

[18] Y. Chen, A. M. Garsia and J. Remmel, *Algorithms for plethysm*, Contemporary Math. **34** (1984) 109-153.

[19] G. E. Andrews, *q-Series: Their Development and Application in Analysis, Number Theory, Combinatorics, Physics, and Computer Algebra*, Expository Lectures from the CBMS Regional Conference No. 66, Providence, Rh. I.: AMS, 1986.

[20] L. B. Drissi, J. Houda and E. H. Saidi, *Refining the Shifted Topological Vertex*, J. Math. Phys. **50** (2009) 0135 ; arXiv:0812.0513 [hep-th].

[21] R. Gopakumar and C. Vafa, *M-theory and topological strings-I*, hep-th/9809187.

[22] R. Gopakumar and C. Vafa, *M-theory and topological strings-II*, hep-th/9812127.

[23] R. Gopakumar and C. Vafa, *On the gauge theory/geometry correspondence*, Adv. Theor. Math. Phys. **3** (1999) 4115-1443.

[24] T. J. Hollowood, A. Iqbal and C. Vafa, *Matrix Models, Geometric Engineering and Elliptic Genera*, JHEP 0803 (2008) 069; [arXiv:hep-th/0310272v4].

[25] I. Antoniadis, E. Gava, K. S. Narain and T. R. Taylor, *Topological amplitudes in string theory*, Nucl. Phys. B 413 (1994) 162

[26] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, *Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes*, Commun. Math. Phys. **165** (1994) 311.

[27] W. N. Bailey *Identities of the Rogers-Ramanujan type*, Proc. London Math. Soc. **50** (1949) 1-10.

[28] P. Paule *On identities of the Rogers-Ramanujan type*, J. Math. Anal. and Appl. **107** (1985) 255-284.

[29] G. E. Andrews, *Multiple series Rogers-Ramanujan type identities*, Pacific J. Math. **114** (1984) 267-283.

[30] T. Kawai, Y. Yamada and S.-K. Yang, *Elliptic Genera and N=2 Superconformal Field Theory*, Nucl. Phys. B **414** (1994) 191-212.

[31] V. G. Kac, *Infinite dimensional Lie algebras*, Cambridge University Press, 1990.