DETERMINATION OF A RIEMANNIAN MANIFOLD FROM THE DISTANCE DIFFERENCE FUNCTIONS

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Abstract. Let (N,g) be a Riemannian manifold with the distance function $d(x,y)$ and an open subset $M \subset N$. For $x \in M$ we denote by D_x the distance difference function $D_x : F \times F \to \mathbb{R}$, given by $D_x(z_1,z_2) = d(x,z_1) - d(x,z_2)$, $z_1, z_2 \in F = N \setminus M$. We consider the inverse problem of determining the topological and the differentiable structure of the manifold M and the metric $g|_M$ on it when we are given the distance difference data, that is, the set F, the metric $g|_F$, and the collection $\mathcal{D}(M) = \{D_x;\ x \in M\}$. Moreover, we consider the embedded image $\mathcal{D}(M)$ of the manifold M, in the vector space $C(F \times F)$, as a representation of manifold M. The inverse problem of determining (M,g) from $\mathcal{D}(M)$ arises e.g. in the study of the wave equation on $\mathbb{R} \times N$ when we observe in F the waves produced by spontaneous point sources at unknown points $(t,x) \in \mathbb{R} \times M$. Then $D_x(z_1,z_2)$ is the difference of the times when one observes at points z_1 and z_2 the wave produced by a point source at x that goes off at an unknown time. The problem has applications in hybrid inverse problems and in geophysical imaging.

Keywords: Inverse problems, distance functions, embeddings of manifolds, wave equation.

Contents

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1. Introduction

1.1. **Motivation of the problem.** Let us consider a body in which there spontaneously appear point sources that create propagating waves. In various applications one encounters a geometric inverse problem where we detect such waves either outside or at the boundary of the body and aim to determine the unknown wave speed inside the body. As an example of such situation, one can consider the micro-earthquakes that appear very frequently near active faults. The related inverse problem is whether the surface observations of elastic waves produced by the micro-earthquakes can be used in the geophysical imaging of Earth's subsurface [25, 58], that is, to determine the speed of the elastic waves in the studied volume. In this paper we consider a highly idealized version of the above inverse problem: We consider the problem on an n dimensional manifold N with a Riemannian metric g. The distance function determined by this metric tensor corresponds physically to the travel time of a wave between two points. The Riemannian distance of points $x, y \in N$ is denoted by $d(x, y)$. For simplicity we assume that the manifold N is compact and has no boundary. Instead of considering measurements on boundary, we assume that the manifold contains an unknown part $M \subset N$ and the metric is known outside the set M. When a spontaneous point source produces a wave at some unknown point $x \in M$ at some unknown time $t \in \mathbb{R}$, the produced wave is observed at the point $z \in N \setminus M$ at time $T_{x,t}(z) = d(z, x) + t$. These observation times at two points $z_1, z_2 \in N \setminus M$ determine the *distance difference function*

$$D_x(z_1, z_2) = T_{x,t}(z_1) - T_{x,t}(z_2) = d(z_1, x) - d(z_2, x). \tag{1}$$

Physically, this function corresponds to the difference of times at z_1 and z_2 of the waves produced by the point source at (x, t), see Fig 1. and Section 3. The assumption that there is a large number point sources and that we do measurements over a long time can be modeled by the assumption that we are given the set $N \setminus M$ and the family of functions

$$\{D_x \ ; \ x \in X\} \subset C((N \setminus M) \times (N \setminus M)),$$

where $X \subset M$ is either the whole manifold M or its dense subset, see Remark 2.5 below.

1.2. **Definitions and the main result.** Let (N_1, g_1) and (N_2, g_2) be compact and connected Riemannian manifolds without boundary. Let $d_j(x, y)$ denote the Riemannian distance of points $x, y \in N_j$, $j = 1, 2$. Let $M_j \subset N_j$ be open sets and define closed sets $F_j = N_j \setminus M_j$. Suppose $F_j^{int} \neq \emptyset$. This is a crucial assumption and we provide a counterexample for a case $F_j^{int} = \emptyset$ in Appendix A.

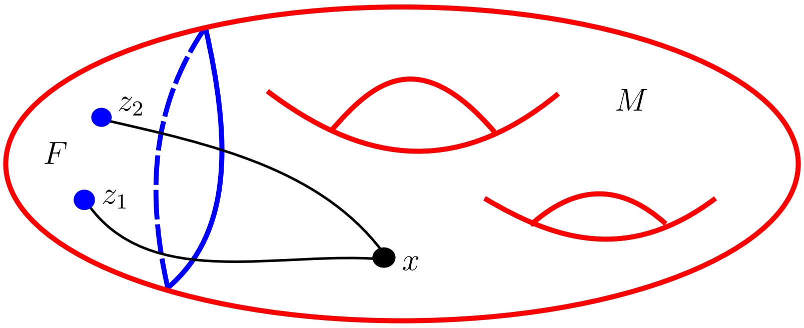

FIGURE 1. *The distance difference functions are related to observation on the closed manifold N that contains an unknown open subset M and its known complement $F = N \setminus M$. The distance difference function D_x associated to a source point $x \in M$ has, at the observation points $z_1, z_2 \in F$, the value $D_x(z_1, z_2) = d(x, z_1) - d(x, z_2)$. Consider the wave equation and a wave that is produced by a point source at x that goes off at an unknown time and that is observed on F. Then the difference of the times when the wave is observed at the points z_1 and z_2 is equal to $D_x(z_1, z_2)$. The time difference inverse problem is to determine the topology and the isometry type of (N, g) from such observations when x runs over a dense subset of M.*

Below, we assume that we know F_j as a differentiable manifold, that is, we know the atlas of C^∞-smooth coordinates on F_j, and the metric tensor $g_j|_{F_j}$ on F_j, but we do not know the manifold $(M_j, g_j|_{M_j})$. We assume F_j to be a smooth manifold with smooth boundary $\partial F_j = \partial M_j$.

Definition 1.1. *For $j = 1, 2$ and all points $x \in N_j$ we define the distance difference function*

$$D^j_x : F_j \times F_j \to \mathbb{R},\ D^j_x(z_1, z_2) := d_j(z_1, x) - d_j(z_2, x)$$

where $F_j = N_j \setminus M_j$. Recall that here d_j is the Riemannian distance function of manifold N_j. We denote by

$$\mathcal{D}_j : N_j \to C(F_j \times F_j), \quad \mathcal{D}_j(x) = D^j_x$$

the map from a point x to the corresponding distance difference function D^j_x. The pair $(F_j, g_j|_{F_j})$ and the collection

$$\mathcal{D}_j(M_j) = \{D^j_x\, ;\ x \in M_j\} \subset C(F_j \times F_j)$$

of the distance difference functions of the points $x \in M_j$ is called the distance difference data for the set M_j.

We emphasize that the above collections $\{D^j_x(\cdot,\cdot);\ x\in M_j\}$ are given as unindexed subsets of $C(F_j\times F_j)$, that is, for a given element $D^j_x(\cdot,\cdot)$ of this set we do not know what is the corresponding "index point" x.

To prove the uniqueness of this inverse problem, we assume the following:

$$\text{(2)}\quad \text{There is a diffeomorphism } \phi: F_1\to F_2 \text{ such that } \phi^*g_2|_{F_2}=g_1|_{F_1},$$

$$\text{(3)}\quad \{D^1_x(\cdot,\cdot)\ ;\ x\in M_1\}=\{D^2_y(\phi(\cdot),\phi(\cdot))\ ;\ y\in M_2\}.$$

The following proposition states that using the small data $\mathcal{D}_j(M_j)$ we can construct the bigger data set $\mathcal{D}_j(N_j)$.

Proposition 1.2. *Assume that* (2)-(3) *are valid. Then:*

- (i) *The map $\phi: F_1\to F_2$, is a metric isometry, that is, $d_1(z,w)=d_2(\phi(z),\phi(w))$ for all $z,w\in F_1$.*
- (ii) *The collections $\mathcal{D}_j(N_j)=\{D^j_x(\cdot,\cdot);\ x\in N_j\}\subset C(F_j\times F_j)$ are equivalent in the following sense*

$$\{D^1_x(\cdot,\cdot)\ ;\ x\in N_1\}=\{D^2_y(\phi(\cdot),\phi(\cdot))\ ;\ y\in N_2\}. \tag{4}$$

We postpone the proof of this proposition and the other results in the introduction and give the proofs later in the paper.

The main theorem of the paper is the following:

Theorem 1.3. *Let (N_1,g_1) and (N_2,g_2) be compact and connected Riemannian manifolds, without boundary, of dimension $n\geq 2$. Let $M_j\subset N_j$ be open sets and define closed sets $F_j=N_j\setminus M_j$. Assume that $F_j^{int}\neq\emptyset$ and that F_j is a smooth manifold with smooth boundary $\partial F_j=\partial M_j$. Also, suppose that assumptions* (2)-(3) *are valid, that is, the distance difference data for sets M_1 and M_2 are equivalent. Then the manifolds (N_1,g_1) and (N_2,g_2) are isometric.*

We prove Theorem 1.3 in Section 2. This proof is divided into 5 subsections. In the first we set notations and consider some basic facts about geodesics. In the second we prove Proposition 1.2. In the third we show that manifolds (N_j,g_j) are homeomorphic. In the fourth subsection we will construct smooth atlases with which we show that manifolds (N_j,g_j) are diffeomorphic. In the fifth subsection we will use techniques developed in papers [46] and [43] to prove that manifolds (N_j,g_j) are isometric.

Finally, in Section 3 we give an example how the main result can be applied for an inverse source problem for a geometric wave equation.

1.3. **Embeddings of a Riemannian manifold.**

A classical distance function representation of a Riemannian manifold is the Kuratowski-Wojdyslawski embedding,

$$\mathcal{K}: x\mapsto \mathrm{dist}_M(x,\cdot),$$

from M to the space of continuous functions $C(M)$ on it. The mapping $\mathcal{K} : M \to C(M)$ is an isometry so that $\mathcal{K}(M)$ is an isometric representation of M in a vector space.

Another important example is the Berard-Besson-Gallot representation [10]

$$\mathcal{G} : M \to C(M \times \mathbb{R}_+), \quad \mathcal{G}(x) = \Phi_M(x, \cdot, \cdot)$$

where $(x, y, t) \mapsto \Phi_M(x, y, t)$ is the heat kernel of the manifold (M, g). The asymptotics of the heat kernel $\Phi_M(x, y, t)$, as $t \to 0$, determines the distance $d(x, y)$, and by endowing $C(M \times \mathbb{R}_+)$ with a suitable topology, the image $\mathcal{G}(M) \subset C(M \times \mathbb{R}_+)$ can be considered as an embedded image of the manifold M.

Theorem 1.3 implies that the set $\mathcal{D}(M) = \{D_x;\ x \in M\}$ can be considered as an embedded image (or a representation) of the manifold (M, g) in the space $C(F \times F)$ in the embedding $x \mapsto D_x$. Moreover, in the proof of Theorem 1.3 we show that $(F, g|_F)$ and the set $\mathcal{D}(M)$ determine uniquely an atlas of differentiable coordinates and a metric tensor on $\mathcal{D}(M)$. These structures make $\mathcal{D}(M)$ a Riemannian manifold that is isometric to the original manifold M. Note that the metric is different than the one inherited from the inclusion $\mathcal{D}(M) \subset C(F \times F)$. Hence, $\mathcal{D}(M)$ can be considered as a representation of the manifold M, given in terms of the distance difference functions, and we call it the *distance difference representation* of the manifold of M in $C(F \times F)$.

The embedding $\mathcal{D}$ is different to the above embeddings $\mathcal{K}$ and $\mathcal{G}$ in the following way that makes it important for inverse problems: With $\mathcal{D}$ one does not need to know a priori the set M to consider the function space $C(F \times F)$ into which the manifold M is embedded. Similar types of embedding have been also considered in the context of boundary distance functions, see Subsection 1.4.1.

In addition to the above tensor g on N, let us consider a sequence of metric tensors g_k, $k \in \mathbb{Z}_+$ on the manifold N and assume that $g_k|_F = g|_F$ on $F \subset N$. We denote the Riemannian manifolds $(N \setminus F, g_k|_{N \setminus F})$, having the boundary ∂F, by (M_k, g_k). Also, we denote by $\mathcal{D}(M_k) \subset C(F \times F)$ the distance difference representations of the manifolds (M_k, g_k) and let $d_H(X_1, X_2)$ denote the Hausdorff distance of sets $X_1, X_2 \subset C(F \times F)$. When $d_H(\mathcal{D}(M_k), \mathcal{D}(M)) \to 0$, as $k \to \infty$, an interesting open question is, if the manifolds (M_k, g_k) converge to (M, g) in the Gromov-Hausdorff topology. This type of questions have been studied for other representations e.g. in [2, 10], but this question is outside the context of this paper.

1.4. **Earlier results and the related inverse problems.** The inverse problem for the distance difference function is closely related to many other inverse problems. We review some results below:

1.4.1. *Boundary distance functions and the inverse problem for a wave equation.* The reconstruction of a compact Riemannian manifold (M, g) with boundary from distance information has been considered e.g. in [27, 30]. There, one defines for $x \in M$ the boundary distance function $r_x : \partial M \to \mathbb{R}$ given by $r_x(z) = d(x, z)$. Assume that one is given the boundary ∂M and the collection of boundary distance functions corresponding to all $x \in M$, that is,

$$\partial M \quad \text{and} \quad \mathcal{R}(M) := \{r_x \in C(\partial M);\ x \in M\}. \tag{5}$$

It is shown in [27, 30] that only knowing the boundary distance data (5) one can reconstruct the topology of M, the differentiable structure of M (i.e., an atlas of C^∞-smooth coordinates), and the Riemannian metric tensor g. Thus $\mathcal{R}(M) \subset C(\partial M)$ can be considered as an isometric copy of M, and the pair $(\partial M, \mathcal{R}(M))$ is called the boundary distance representation of M, see [27, 30]. Similar results for non-compact manifolds are considered in [17]. Constructive solutions to determine the metric from the boundary distance functions have been developed in [14, 15] using a Riccati equation [56] for metric tensor in boundary normal coordinates and in [55] using the properties of the conformal killing tensor.

Physically speaking, functions r_x are determined by the wave fronts of waves produced by the delta-sources $\delta_{x,0}$ that take place at the point x at time $s = 0$. The distance difference functions $D_x^{\partial M}$ are determined by the wave fronts of waves produced by the delta-sources $\delta_{x,s}$ that take place at the point x at an unknown time $s \in \mathbb{R}$.

Many hyperbolic inverse problems with time-independent metric reduce to the problem of reconstructing the isometry type of the manifold from its boundary distance functions. Indeed, in [26, 27, 29, 31, 32, 35, 51, 52] it has been show that the boundary measurements for the scalar wave equation, Dirac equation, and for Maxwell's system (with isotropic scalar impedance) determine the boundary distance functions of the Riemannian metric associated to the wave velocity.

1.4.2. *Hybrid inverse problems.* Hybrid inverse problems are based on coupling two physical models together. In a typical setting of these problems, the first physical system is such that by controlling the boundary values of its solution, one can produce high amplitude waves, that create, e.g. due to energy absorption, a source for the second physical system. Typically, the second physical system corresponds to a hyperbolic equation with the metric

$$ds^2 = c(x)^{-2}((dx^1)^2 + \cdots + (dx^n)^2)$$

corresponding to the wave speed $c(x)$. Examples of such hybrid inverse problems are encountered in thermo-acoustic and photo-acoustic imaging see e.g. [1, 5, 6, 7, 8, 59, 61, 60, 57] and quantitative elastography

[4, 22, 23]. In some cases one can use beam forming in the first physical system to make the source for the second physical system to be strongly localized, that is, to be close to a point-source, see e.g. [4, 23].

To simplify the above hybrid inverse problem, one often can do approximations by assuming that the wave speed in the second physical system is either a constant or precisely known. Usually one also assumes that the time moment when the source for the second physical system is produced is exactly known. However, when these approximations are not made, the wave speed $c(x)$ needs to be determined, too. When the source of the second physical system is produced at the given time in the whole domain M, the problem is studied in [42, 62]. In the cases when the source of the second physical system are close to a point sources, one can try to determine $c(x)$ from the wavefronts that are produced by the point sources and are observed outside the domain M. This problem can be uniquely solved by Theorem 1.3 and we consider it in detail below in Section 3.

1.4.3. *Inverse problems of micro-earthquakes.* The earthquakes are produced by the accumulated elastic strain that at some time suddenly produce an earthquake. As mentioned above, the small magnitude earthquakes (e.g. the micro-earthquakes of magnitude $1 < M < 3$) appear so frequently that the surface observations of the produced elastic waves have been proposed to be used in the imaging of the Earth near active faults [25, 58]. The so-called time-reversal techniques to study the inverse source and medium problems arising from the micro-seismology have been developed in [3, 16, 24].

In geophysical studies, one often approximates the elastic waves with scalar waves satisfying a wave equation. Let us also assume that the sources of such earthquakes are point-like and that one does measurements over so long time that the source-points are sufficiently dense in the studied volume. Then the inverse problem of determining the the speed of the waves in the studied volume from the surface observations of the micro-earthquakes is close to the problem studied in this paper. We note that the above assumptions are highly idealized: For example, considering the system of elastic equations would lead to a problem where travel times are determined by a Finsler metric instead of a Riemannian one. One possible way to continue the line of research conducted in this paper, would be to study, if the result of Theorem 1.3 holds on Finsler manifolds. The authors have not yet addressed this issue.

1.4.4. *Broken scattering relation.* If the sign in the definition of the distance difference functions is changed in (1), we come to distance sum functions

$$(6)\quad D^+_x(z_1, z_2) = d(z_1, x) + d(z_2, x), \quad x \in M,\ z_1, z_2 \in N \setminus M.$$

This function gives the length of the broken geodesic that is the union of the shortest geodesics connecting z_1 to x and the shortest geodesics connecting x to z_2. Also, the gradients of $D_x^+(z_1,z_2)$ with respect to z_1 and z_2 give the velocity vectors of these geodesics. The functions (6) appear in the study of the radiative transfer equation on manifold (N,g), see [13, 47, 48, 49, 54]. Also, the inverse problem of determining the manifold (M,g) from the broken geodesic data, consisting of the initial and the final points and directions, and the total length, of the broken geodesics, has been considered in [33].

2. PROOF OF THE MAIN RESULT

2.0.1. *Notations and basic facts on pre-geodesics.* When we are concerning only one manifold, we use the shorthand notations M, N, F, and g instead of ones with sub-indexes.

Let (N,g) be a compact and connected Riemannian n-manifold without boundary and $n \geq 2$. We assume that $M \subset N$ is an open set of N and the set $F = N \setminus M$ is a compact, F contains an open set and has a smooth boundary. Suppose we know the Riemannian structure of manifold $(F, g|_F)$.

We denote the Riemannian connection of the metric g by ∇. A unit speed geodesic of (N,g) emanating from a point $(p,\xi) \in SN$ is denoted by $\gamma_{p,\xi}(t) = \exp_p(t\xi)$. Here, $SN = \{(p,\xi) \in TN;\ \|\xi\|_g = 1\}$. We use a short hand notation $D_t := \nabla_{\dot\gamma_{p,\xi}(t)}$ for the covariant differentiation in the direction $\dot\gamma_{p,\xi}$ for vector fields along geodesic $\gamma_{p,\xi}$.

Let $p \in N$ and choose some smooth coordinates (U,X) at point p. Denote the Christoffel symbols of connection ∇ by $\Gamma^k_{i,j}$.

We say that a curve $\alpha([t_1,t_2])$ is distance minimizing if the length of this curve is equal to the distance between its end points $\alpha(t_1)$ and $\alpha(t_2)$. Also, a geodesic that is distance minimizing is called a minimizing geodesic.

We say that a curve $\alpha([t_1,t_2])$ is a *pre-geodesic*, if $\alpha(t)$ is a C^1-smooth curve such that $\dot\alpha(t) \neq 0$ on $t \in [t_1,t_2]$, and $\alpha([t_1,t_2])$ can be re-parameterized so that it becomes a geodesic.

Let us next recall some properties of the pre-geodesics. Let us consider a geodesic curve $\gamma : \mathbb{R} \to N$, satisfying in local coordinates the equation

$$(7)\quad D_t\dot\gamma(t) = \frac{d^2\gamma^k}{dt^2}(t) + \Gamma^k_{i,j}(\gamma(t))\frac{d\gamma^i}{dt}(t)\frac{d\gamma^j}{dt}(t) = 0,\ k \in \{1,\dots,n\}.$$

We need the following result, often credited to Levi-Civita [38].

Lemma 2.1. *Let $\kappa : \mathbb{R} \to \mathbb{R}$ be continuous and $\widetilde\gamma : \mathbb{R} \to N$ be a C^2-curve that satisfies the equation*

$$(8)\quad \frac{d^2\widetilde\gamma^k}{ds^2}(s) + \Gamma^k_{i,j}(\widetilde\gamma(s))\frac{d\widetilde\gamma^i}{ds}(s)\frac{d\widetilde\gamma^j}{ds}(s) = \kappa(s)\frac{d\widetilde\gamma^k}{ds}(s),\ k \in \{1,\dots,n\}.$$

Then there exists a change of parameters $t : \mathbb{R} \to \mathbb{R}$ *satisfying*

$$\frac{dt}{ds}(s) = \exp\Big(\int\limits_0^s \kappa(\tau) d\tau \Big). \tag{9}$$

such that curve $\gamma(t) := \widetilde{\gamma}(s(t))$ *solves the geodesic equation (7). Here* $s(t)$ *is the inverse function for* $t(s)$*.*

Proof. The proof is a direct computation. □

Let us now consider a family $\mathcal{C}$ of C^2-smooth curves defined on U. We denote by Ω the subbundle of TU that is deternined by the velocity fields $(c, \dot{c})$, $c \in \mathcal{C}$. More precisely a vector $(p, v) \in TU$ satisfies $(p, v) \in \Omega$ if and only if there exist $a, t \in \mathbb{R}, c \in \mathcal{C}$ such that $(p, v) = (c(t), a\dot{c}(t))$. Let $f : \Omega \to \mathbb{R}$ be a function that satisfies

$$f(av) = af(v), \text{ for all } a \in \mathbb{R} \text{ and } v \in \Omega, \tag{10}$$

i.e., it is homogeneous of degree 1. Moreover we assume that f satisfies the equation

$$\frac{d^2\widetilde{\gamma}^k}{ds^2}(s) + \Gamma^k_{i,j}(\widetilde{\gamma}(s))\frac{d\widetilde{\gamma}^i}{ds}(s)\frac{d\widetilde{\gamma}^j}{ds}(s) = f\Big(\frac{d\widetilde{\gamma}}{ds}(s)\Big)\frac{d\widetilde{\gamma}^k}{ds}(s), \tag{11}$$

for any $\widetilde{\gamma} \in \mathcal{C}$. By Lemma 2.1 each $\widetilde{\gamma} \in \mathcal{C}$ is a pre-geodesic of connection ∇.

Next we will show that also the converse result for the pre-geodesics hold. Let $\widetilde{\gamma}$ a pre-geodesic passing over the point p. We assume that $\widetilde{\gamma}(0) = p$. Let $s(t)$ be such a re-parametrization of $\widetilde{\gamma}$ that $\widetilde{\gamma}(s(t)) =: \gamma(t)$ satisfies the geodesic equation (7), $s(0) = 0$ and $\frac{d}{dt}\widetilde{\gamma}(s(t))|_{t=0} \in S_pN$. Then by the chain rule it holds that

$$\frac{d^2\widetilde{\gamma}^k}{ds^2}(s) + \Gamma^k_{i,j}(\widetilde{\gamma}(s))\frac{d\widetilde{\gamma}^i}{ds}(s)\frac{d\widetilde{\gamma}^j}{dt}(s) = -\frac{\ddot{s}(t)}{\dot{s}(t)^2}\frac{d\widetilde{\gamma}^k}{ds}(s), \; k \in \{1, \dots, n\}.$$

Let Ω be the subbundle of TU that is determined by the velocity vectorfield $(\gamma, \dot{\gamma})$. We define $f : \Omega \to \mathbb{R}$

$$f(q, v) = \mp \|v\|_g \frac{\ddot{s}(t)}{\dot{s}(t)^2}, \text{ if } \frac{v}{\|v\|_g} = \pm\dot{\gamma}(t), \text{ for some } t \in \mathbb{R}.$$

Thus equations (7) and (11) are equivalent in the sense that curves satisfying the latter one, for appropriate f, are also geodesics of metric g, but parametrized in a different way.

The distance function of N is denoted by $d(x, y) = d_N(x, y)$ for $x, y \in N$. The normal vector field of ∂M, pointing inside M, is denoted by ν. The boundary cut locus function is $\tau_{\partial M} : \partial M \to \mathbb{R}_+$,

$$\tau_{\partial M}(z) = \sup\{t > 0;\ d(\gamma_{z,\nu}(t), \partial M) = t\}. \tag{12}$$

Also, we use the cut locus function of N that is $\tau : TN \to \mathbb{R}_+$,

$$\tau(x, \xi) = \sup\{t > 0;\ d(\exp_x(t\xi), x) = t\}. \tag{13}$$

Functions $\tau_{\partial M}(z)$ and $\tau(x,\xi)$ are continuous and satisfy the inequality (see Lemma 2.13 of [27])

$$\tau(z,\nu(z)) > \tau_{\partial M}(z), \quad z \in \partial M. \tag{14}$$

2.1. **Extension of data.** In this subsection we prove Proposition 1.2.

Let $z_1, z_2 \in \partial F = \partial M$. Then using the triangular inequality and that $d(z_1,z_2) = D_{z_2}(z_1,z_2)$ we see easily that

$$d(z_1,z_2) = \sup_{x\in M} D_x(z_1,z_2). \tag{15}$$

Thus $\mathcal{D}(M)$ determines the distances of the boundary points, that is, the function $d|_{\partial M\times\partial M} : \partial M \times \partial M \to \mathbb{R}$.

Lemma 2.2. *Suppose that* (2)-(3) *are valid. Then for every* $w, z \in F_1$ *it holds that* $d_1(w,z) = d_2(\phi(w),\phi(z))$.

The proof of the lemma below is very simple, but as Lemma 2.2 shows how the given data is extended to a larger data set, we present its proof. Notice that Lemma 2.2 proofs (i) of the Proposition 1.2.

Proof. Let $w, z \in F_1$. Let γ be a minimizing unit speed geodesic in N_1 from z to w and denote $S = \gamma([0, d_1(w,z)]) \cap \partial M_1$. When $S = \emptyset$, using $\phi^* g_2 = g_1$ we see that $d_1(w,z) \geq d_2(\phi(w),\phi(z))$.

Next, consider the case when $S \neq \emptyset$. Let $e_1, e_2 \in S$ be such that

$$d_1(w,e_1) = \min\{d_1(w,x) : x \in S\} \text{ and } d_1(z,e_2) = \min\{d_1(z,x) : x \in S\}.$$

As (2)-(3) is valid, the formula (15) implies $d_1(e_1,e_2) = d_2(\phi(e_1),\phi(e_2))$. Since $\phi : F_1 \to F_2$ satisfies $\phi^* g_2 = g_1$,

$$\begin{aligned} d_1(w,z) &= d_1(w,e_1) + d_1(e_1,e_2) + d_1(e_2,z) \\ &\geq d_2(\phi(w),\phi(e_1)) + d_2(\phi(e_1),\phi(e_2)) + d_2(\phi(e_2),\phi(z)) \\ &\geq d_2(\phi(w),\phi(z)) \end{aligned}$$

The opposite inequality follows by changing the roles of N_1 and N_2. ☐

Let us consider the case when $x \in F_1$. Then, Lemma 2.2 implies that for $z_1, z_2 \in F_1$ we have

$$\begin{aligned} D^1_x(z_1,z_2) &= d_1(x,z_1) - d_1(x,z_2) \\ &= d_2(\phi(x),\phi(z_1)) - d_2(\phi(x),\phi(z_2)) \\ &= D^2_{\phi(x)}(\phi(x),\phi(z_2)). \end{aligned} \tag{16}$$

Hence,

$$\{D^1_x(\cdot,\cdot) \ ; \ x \in F_1\} \subset \{D^2_y(\phi(\cdot),\phi(\cdot)) \ ; \ y \in F_2\}. \tag{17}$$

Changing roles of N_1 and N_2 and considering $\phi^{-1} : F_2 \to F_1$ instead of the diffeomorphism $\phi : F_1 \to F_2$, we see that in formula (17) we have the equality. This and formula (3), together with Lemma 2.2, imply Proposition 1.2. q.e.d.

2.2. **Manifolds N_1 and N_2 are homeomorphic.** To simplify the notations, we will next in our considerations omit the sub-indexes of sets M_1, N_1, and F_1 and just consider the sets M, N, and F.

Let $x \in N$ and define a function $D_x : F \times F \to \mathbb{R}$ by a formula

$$D_x(z_1, z_2) = d(x, z_1) - d(x, z_2).$$

Let $\mathcal{D} : N \to C(F \times F)$ be given by $\mathcal{D}(x) = D_x$. Next, we consider the function space $C(F \times F)$ with the norm $\|f\|_\infty = \sup_{x,y \in F} |f(x, y)|$

Theorem 2.3. *The image $\mathcal{D}(N) = \{D_x;\ x \in N\} \subset C(F \times F)$ is a topological manifold homeomorphic to manifold N. Moreover, $\mathcal{D}(M)$ is homeomorphic to M.*

Proof. The proof consists of four short steps.

Step 1 First, we will show that $\mathcal{D}$ is 2-Lipschitz and therefore continuous. Let $x, y \in N$. Using the triangular inequality we see that

$$\begin{aligned} \|D_x - D_y\|_\infty &= \sup_{z_1,z_2 \in F} |D_x(z_1, z_2) - D_y(z_1, z_2)| \\ &\le \sup_{z_1,z_2 \in F} |d(x, z_1) - d(y, z_1)| + |d(x, z_2) - d(y, z_2)| \qquad (18) \\ &\le 2d(x, y). \end{aligned}$$

Step 2. Next we will show that $\mathcal{D}$ is injective. Suppose that $x, y \in N$ are such that $D_x = D_y$ and $x \neq y$. Let $q \in F^{int}$ and denote $\ell_x = d(q, x)$ and $\ell_y = d(q, y)$. Next, without loss of generality, we assume that $\ell_x \le \ell_y$. Also, let $\eta \in S_qN$ be such that $\gamma_{q,\eta}([0, \ell_x])$ is a minimizing geodesic from q to x. Let $s_1 > 0$ be such that $s_1 < \min(\ell_x, \ell_y)$ and $\gamma_{q,\eta}([0, s_1]) \subset F^{int}$. Consider a point $p = \gamma_{q,\eta}(s)$ with $s \in [0, s_1]$. Then we see that

$$\begin{aligned} (d(q, p) + d(p, y)) - d(q, y) &= d(q, p) + D_y(p, q) \\ &= d(q, p) + D_x(p, q) \\ &= (d(q, p) + d(p, x)) - d(q, x) = 0 \end{aligned}$$

and hence p is on a minimizing geodesic from q to y.

Let us consider a minimizing geodesic α from p to y with the length $\ell_y - s$. Then the union of the geodesics $\gamma_{q,\eta}([0, s])$ and α is a distance minimizing curve from q to y and thus this union is a geodesic. This implies that α is a continuation of the geodesics $\gamma_{q,\eta}([0, s])$ and hence $y = \gamma_{q,\eta}(\ell_y)$. Summarizing, $\gamma_{q,\eta}([0, \ell_x])$ and $\gamma_{q,\eta}([0, \ell_y])$ are distance minimizing geodesics from q to x and y, respectively. Since $x \neq y$, we have $\ell_x \neq \ell_y$. Then, as we have assumed that $\ell_x \le \ell_y$, we see that $\ell_x < \ell_y$.

Let $\widehat{q} \in F^{int}$ be a such point that $\widehat{q}$ is not on the curve $\gamma_{q,\eta}(\mathbb{R})$. Clearly, such a point exists as N has the dimension $n \ge 2$. Let $\widehat{\ell}_x = d(\widehat{q}, x)$ and $\widehat{\ell}_y = d(\widehat{q}, y)$. Also, let $\widehat{\eta} \in S_{\widehat{q}}N$ be such that $\gamma_{\widehat{q},\widehat{\eta}}([0, \widehat{\ell}_x])$ is minimizing geodesic from $\widehat{q}$ to x. As above, we see that then $\gamma_{\widehat{q},\widehat{\eta}}([0, \widehat{\ell}_x])$

and $\gamma_{\widehat{q},\widehat{\eta}}([0,\widehat{\ell}_y])$ are distance minimizing geodesics from $\widehat{q}$ to x and y, respectively. However, the geodesics $\gamma_{q,\eta}(\mathbb{R})$ and $\gamma_{\widehat{q},\widehat{\eta}}(\mathbb{R})$ do not coincide as point sets and hence the vectors $\dot\gamma_{q,\eta}(\ell_x) \in T_xN$ and $\dot\gamma_{\widehat{q},\widehat{\eta}}(\widehat{\ell}_x) \in T_xN$ are not parallel. Recall that $\ell_x < \ell_y$. In the case when $\widehat{\ell}_x < \widehat{\ell}_y$, let β be the geodesic segment $\gamma_{\widehat{q},\widehat{\eta}}([\widehat{\ell}_x,\widehat{\ell}_y])$ connecting x to y. In the case when $\widehat{\ell}_x > \widehat{\ell}_y$, let β be the geodesic segment $\gamma_{\widehat{q},\widehat{\eta}}([\widehat{\ell}_y,\widehat{\ell}_x])$ connecting x to y.

Then we see that the union of the paths $\gamma_{q,\eta}([0,\ell_x])$ and β is a distance minimizing path from q to y. As the vectors $\dot\gamma_{q,\eta}(\ell_x)$ and $\dot\gamma_{\widehat{q},\widehat{\eta}}(\widehat{\ell}_x)$ are not parallel, we see that the union of these curves is not a geodesic. This is contradiction and hence there are no $x, y \in N$ such that $D_x = D_y$ and $x \neq y$. Thus, $\mathcal{D} : N \to C(F \times F)$ is an injection.

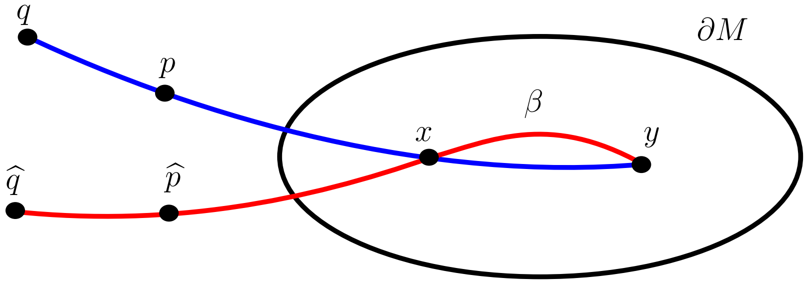

Figure 2. *The setting in Step 2 in the proof of Theorem 2.3. We consider points $x, y \in N$ and points p and q such that p is on a distance minimizing geodesic from q to x. Then this geodesic can be extended to a distance minimizing geodesic from q to y. Similarly, the point $\widehat{p}$ is on a distance minimizing geodesic from $\widehat{q}$ to x and this geodesic can be extended to a distance minimizing geodesic from $\widehat{q}$ to y. Then the union of the (blue) geodesic from q to x and the (red) geodesic β is a length minimizing curve from q to y that is not a geodesic.*

Step 3. So far we have proved the continuity and injectivity of mapping $\mathcal{D}$. Since the domain N of the mapping $\mathcal{D}$ is compact and $(C(F \times F), \|\cdot\|_\infty)$ is a Hausdorff space as a metric space, it holds by basic results of topology that mapping $\mathcal{D} : N \to \mathcal{D}(N)$ is a homeomorphism.

Step 4. By assumption $M \subset N$ is open and therefore mapping $\mathcal{D} : M \to \mathcal{D}(M)$ is open. This proves that the mapping $\mathcal{D} : M \to \mathcal{D}(M)$ is a homeomorphism. $\square$

Define a mapping

$$\Phi : C(F_2 \times F_2) \to C(F_1 \times F_1),\ \Phi(f) = f \circ (\phi \times \phi). \tag{19}$$

Here $f \times h : X \times X \to Y \times Y$ is defined as $(f \times h)(x_1, x_2) = (f(x_1), h(x_2)) \in Y \times Y$ for mappings $f, h : X \to Y$.

Theorem 2.4. *Suppose that Riemannian manifolds* (N_1, g_1) *and* (N_2, g_2) *are as in section 1.2 and the assumptions of the Proposition 1.2 are valid. Let* Φ *be given by (19). Then the mapping*

$$\Psi := \mathcal{D}_1^{-1} \circ \Phi \circ \mathcal{D}_2 : N_2 \to N_1 \tag{20}$$

is a homeomorphism. In addition, it holds that $\Psi^{-1}|_{F_1} = \phi$.

Proof. Due the Theorem 2.3, for the first claim, we only have to prove that mapping Φ is a homeomorphism. Note that mapping Φ has an inverse mapping $h \mapsto h \circ (\phi^{-1} \times \phi^{-1})$. Let $(x, y) \in F_1 \times F_1$ and $f, h \in C(F_2 \times F_2)$ then it follows

$$|(\Phi(f) - \Phi(h))(x, y)| = |f(\phi(x), \phi(y)) - h(\phi(x), \phi(y))| \le \|f - h\|_\infty.$$

This proves the continuity of Φ. A similar argument where ϕ is replaced by ϕ^{-1} proves that mapping Φ is a homeomorphism.

Let $x \in F_1$ and denote $y = \phi(x)$. Then

$$\begin{aligned} \Psi^{-1}(x) &= (\mathcal{D}_2^{-1} \circ \Phi^{-1} \circ \mathcal{D}_1)(x) = \mathcal{D}_2^{-1}(D_x^1(\phi^{-1}(\cdot) \times \phi^{-1}(\cdot))) \\ &\overset{(16)}{=} \mathcal{D}_2^{-1}(D_y^2) = y. \end{aligned}$$

□

Remark 2.5. *As the map* $\mathcal{D} : M \to \mathcal{D}(M)$, $x \mapsto D_x$, *is a homeomorphism, we see that for a dense set* $X \subset M$ *we have*

$$\mathcal{D}(M) = cl(\mathcal{D}(X)) = cl\{D_x \ ; \ x \in X\} \subset C(F \times F)\},$$

where the closure cl *is taken with respect to the topology of* $C(F \times F)$. *This means that the distance difference functions corresponding to* x *in a dense set* X *determine the distance difference functions corresponding to the points in the whole set* M.

2.3. Manifolds N_1 and N_2 are diffeomorphic.

Our next goal is to construct such smooth atlases for manifolds N_i that homeomorphism $\Psi : N_2 \to N_1$ of Theorem 2.4 is a diffeomorphism.

Lemma 2.6. *Let* (N, g) *be a compact Riemannian manifold of dimension* n, $x \in N$ *and* $\xi \in T_xN$, $\|\xi\|_g = 1$. *Let* $\gamma_{x,\xi} : [0, \ell] \to N$ *be a distance minimizing geodesic. Let* $0 < h < \ell$, $z = \gamma_{x,\xi}(h)$. *Then there exists a neighborhood* V *of* z *such that the set*

$$U = \{(z_i)_{i=1}^n \in V^n : dim\, span((F(z_i) - \xi)_{i=1}^n) = n\} \tag{21}$$

is open and dense in $V^n := V \times V \times \ldots \times V$. *Here* $F(q) := \frac{(\exp_x)^{-1}(q)}{\|(\exp_x)^{-1}(q)\|_g}$, $q \in V$.

Moreover for every $(z_i)_{i=1}^n \in U$ *there exists an open neighborhood* W *of* x *such that*

$$H : W \to \mathbb{R}^n, \ H(y) = (d(y, z_i) - d(y, z))_{i=1}^n \tag{22}$$

is a smooth coordinate mapping.

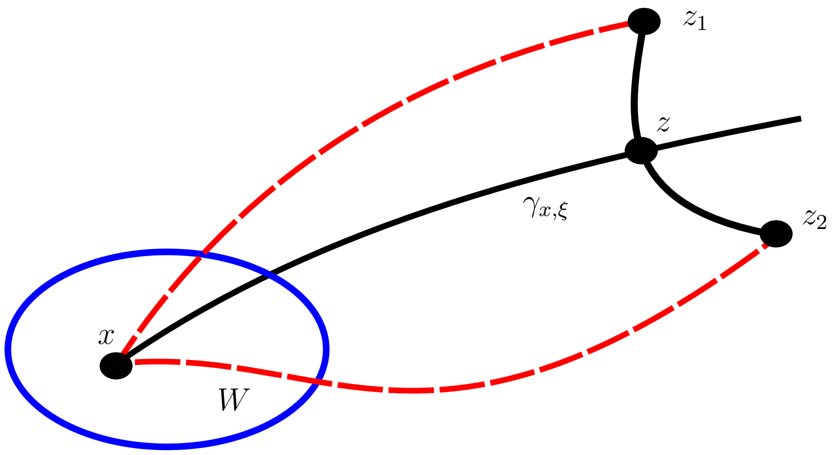

FIGURE 3. *A schematic picture of the coordinate system H.*

Proof. Since the geodesic $\gamma_{x,\xi}([0,\ell])$ is distance minimizing, the geodesic segment $\gamma_{x,\xi}([0,h])$ from x to z has no cut points. Moreover, there exist neighborhoods V_x and V of x and z such that the mapping $(p,q) \mapsto d(p,q)$ is smooth on $V_x \times V$. As the geodesic $\gamma_{x,\xi}([0,h])$ has no cut points, the differential of $\exp_x$ at $v := h\xi \in T_xN$ is invertible. In particularly the map $F : V \to S_xN$ is well defined and smooth.

Now we study the properties of the set U, given in (21). Consider the function

$$T : (S_xN)^n \to \mathbb{R},\ T((v_i)_{i=1}^n) = \det(v_1 - \xi, \dots, v_n - \xi).$$

Then it holds that $(z_i)_{i=1}^n \in U$ if and only if $T((F(z_i))_{i=1}^n) \neq 0$. Therefore the set U is open.

We define a set

$$O := T^{-1}(\mathbb{R} \setminus \{0\}) \subset (S_xN)^n.$$

Then O is open. Our aim is to prove that the set O is also dense. We note that $(S_xN)^n$ is a real analytic manifold and the map T is real analytic since, it is a polynomial. Also the constant map $0 =: (v_i)_{i=1}^n \mapsto 0$ is real analytic. By Lemma 4.3 of [20] the functions T and 0 coincide in $(S_xN)^n$ if and only if they coincide in some open subset of $(S_xN)^n$. Thus to prove that O is dense, it suffices to prove that there exists $(v_i)_{i=1}^n \in (S_xN)^n$ such that $T((v_i)_{i=1}^n \neq 0$.

To simplify the notations we assume $S_xN = S^{n-1} \subset \mathbb{R}^n$ and $\xi = e_n$, where $e_1, \dots, e_n$ is the standard orthonormal basis of $\mathbb{R}^n$. Denote $v_i = e_i, i \in \{1, \dots, n-1\}$ and $v_n = \frac{v_1+v_2}{\sqrt{2}}$. Then $T((v_i)_{i=1}^n) \neq 0$, since

$$\text{span}(v_1 - \xi, \dots, v_{n-1} - \xi, \frac{v_1 + v_2}{\sqrt{2}} - \xi) = \text{span}(e_1, \dots, e_{n-1}, e_n) = \mathbb{R}^n.$$

We conclude that the set U is dense in V^n, since $O \subset (S_xN)^n$ is dense and F is an open map.

Finally we will show that the mapping H, defined in (22), is a smooth coordinate map at some neighborhood W of x. Choose $(z_j)_{j=1}^n \in U$. By the preparations made above, it holds that the gradients

$$-\nabla(d(\cdot, z_i) - d(\cdot, z))|_x = F(z_i) - \xi$$

are linearly independent. Then due to the Inverse function theorem, there exists such a neighborhood W of x that the function

$$H : W \to \mathbb{R}^n,\ H(y) = (d(y, z_i) - d(y, z))_{i=1}^n$$

is a smooth coordinate mapping. □

Next we consider the homeomorphism $\Psi : N_2 \to N_1$ of Theorem 2.4.

Theorem 2.7. *Suppose that Riemannian manifolds* (N_1, g_1) *and* (N_2, g_2) *are as in section 1.2 and Proposition 1.2 is valid. Then mapping* $\Psi : N_2 \to N_1$*, given in formula* (20)*, is a diffeomorphism.*

Proof. Note that for any $p \in N_2$ and all $q, r \in F_2$ it holds that

$$D_p^2(q, r) = D^1_{\Psi(p)}(\Psi(q), \Psi(r)). \tag{23}$$

Let $x \in N_2$, $y \in F_2^{int}$ and denote $\widetilde{x} = \Psi(x)$ and $\widetilde{y} = \Psi(y)$. Let $h \in (0, d_2(x, y))$ be such that $z := \gamma_{x,\xi_2}(h) \in F_2^{int}$ and $\gamma_{x,\xi_2}([h, d_2(x, y)]) \subset F_2^{int}$, where γ_{x,ξ_2} is a minimizing unit speed geodesic from x to y and $\widetilde{z} = \Psi(z) \in F_1^{int}$. Note that by the choice of z it holds that it is not a cut point of x on curve γ_{x,ξ_2}. Therefore mapping $p \mapsto D_p^2(r, q)$ is C^∞-smooth, when p is sufficiently close to x and r, q are sufficiently close to z. Since

$$D_x^2(y, z) = D^1_{\widetilde{x}}(\widetilde{y}, \widetilde{z}), d_2(z, y) \geq d_1(\widetilde{z}, \widetilde{y}) \text{ and } d_2(x, y) = d_2(x, z) + d_2(z, y),$$

we deduce using the triangle inequality that

$$d_1(\widetilde{x}, \widetilde{y}) = d_1(\widetilde{x}, \widetilde{z}) + d_1(\widetilde{z}, \widetilde{y}).$$

Therefore, there exists a unit speed distance minimizing geodesic $\gamma_{\widetilde{x},\xi_1}$ from $\widetilde{x}$ to $\widetilde{y}$ that contains the point $\widetilde{z}$. Hence, the mapping $\widetilde{p} \mapsto D^1_{\widetilde{p}}(\widetilde{r}, \widetilde{q})$ is smooth, when $\widetilde{p}$ is sufficiently close to $\widetilde{x}$ and $\widetilde{r}, \widetilde{q}$ are sufficiently close to $\widetilde{z}$.

Choose a neighborhood V_2 of z such that the map $F_2 : V_2 \to S_x N_2, F_2(q) := \frac{(\exp_x)^{-1}(q)}{\|(\exp_x)^{-1}(q)\|_g}, q \in V_2$ is well defined. Since Ψ is homeomorphism we may assume that $\Psi(V_2) = V_1$, which is a neighborhood of $\widetilde{z}$ such that the map $F_1 : V_1 \to S_x N_1, F_1(q) := \frac{(\exp_x)^{-1}(q)}{\|(\exp_x)^{-1}(q)\|_g}, q \in V_1$ is well defined.

We want to show that there exist points $(z_i)_{i=1}^n \in V_2$ for which the collections

$$((F_2(z_i) - \xi_2))_{i=1}^n \in T_x N_2 \text{ and } ((F_1(\Psi(z_i)) - \xi_1))_{i=1}^n \in T_{\widetilde{x}} N_1$$

are linearly independent. Let us define

$$U_i := \{(z_j)_{j=1}^n \in V_i^n : \dim \operatorname{span}((F_i(z_j) - \xi_i)_{j=1}^n) = n\},\ i \in \{1, 2\}.$$

By Lemma 2.6 the sets U_i are open and dense. Since $\Psi : N_2 \to N_1$ is a homeomorphism, also the map $\Psi^n : N_2^n \to N_1^n$ defined by

$$\Psi^n((q_i)_{i=1}^n) = (\Psi(q_i))_{i=1}^n$$

is a homeomorphism. Therefore $U_1 \cap \Psi^n(U_2)$ is open and dense in V_1^n. Due to the choice of vector $\xi_1 \in S_{\widetilde{x}}N_1$, there exist points $(z_i)_{i=i}^n \in U_2$ that satisfy $(\Psi(z_1))_{i=1}^n \in U_1$.

By Lemma 2.6 there exists a neighborhood W_2 of z such that the map

$$H : W_2 \to \mathbb{R}^n,\ H(y) = (d_2(y, z_i) - d_2(y, z))_{i=1}^n$$

is a smooth coordinate map, $W_1 := \Psi(W_2)$ is a neighborhood of $\widetilde{x}$ and moreover the map

$$\widetilde{H} : W_1 \to \mathbb{R}^n,\ \widetilde{H}(y) = (d_1(y, \Psi(z_i)) - d_2(y, \widetilde{z}))_{i=1}^n$$

is also a smooth coordinate map. We conclude that by equation (23) we have shown $H(W_2) = \widetilde{H}(W_1)$ and more importantly

$$\widetilde{H} \circ \Psi \circ H^{-1} = Id$$

Since the point $x \in N_2$ above is arbitrary and H and $\widetilde{H}$ are smooth coordinate mappings for x and $\widetilde{x}$, respectively, the above implies that Ψ is a diffeomorphism. □

2.4. **Riemannian metrics g_1 and $\Psi_* g_2$ coincide in N_1.**

In this section we will show that manifolds (N_1, g_1) and (N_2, g_2) that satisfy (2)-(3) are isometric.

Definition 2.8. *Let $z_1 \in F$ and $\xi \in S_{z_1}N$. Define a set $\sigma(z_1, \xi)$ by*

$$\sigma_N(z_1, \xi) := \{x \in N \quad ; \quad D_x(\cdot, z_1) \text{ is } C^1\text{-smooth in a neighborhood of } z_1 \text{ and } \nabla D_x(\cdot, z_1)|_{z_1} = \xi\}. \tag{24}$$

Lemma 2.9. *Let $z_1 \in F$ and $\xi \in S_{z_1}N$. Then it follows*

$$\sigma_N(z_1, \xi) = \gamma_{z_1, -\xi}(\{s \ ; \ 0 < s < \tau(z_1, -\xi)\}), \tag{25}$$

Roughly speaking, Lemma 2.9 means that sets $\sigma_N(z_1, \xi)$, that can be determined using data (4), are unparameterized geodesics on N.

Proof. First we recall that for all $x \in N$ the distance function $d(\cdot, x)$ is not smooth near $y \in N \setminus \{x\}$ if and only if point y is in a cut locus of x. See for instance Section 9 of Chapter 5 of [56].

First, consider the case when $x \in \sigma_N(z_1, \xi)$. Then by the definition of $\sigma_N(z_1, \xi)$ the distance function $d(\cdot, x)$ is C^∞-smooth in a neighbourhood z_1 so that z_1 is not in a cut locus of x, or equivalently, x is not in a cut locus of z_1. Also, have that $x \neq z_1$. Hence, there exists an unique distance minimizing unit speed geodesic from x to z_1. Since this geodesic has the velocity

$$\nabla d(\cdot, x)|_{z_1} = \nabla D_x(\cdot, z_2)|_{z_1} = \xi$$

at z_1, it follows that $x \in \gamma_{z_1,-\xi}(\{s\ ;\ 0 < s < \tau(z_1,-\xi)\})$.

Second, consider the case when $x \in \gamma_{z_1,-\xi}(\{s\ ;\ 0 < s < \tau(z_1,-\xi)\})$. Then function $D_x(\cdot, z_1)$ is smooth near z_1 and

$$\nabla D_x(\cdot, z_1)|_{z_1} = \dot\gamma(d(x, z_1)) = -\dot\gamma_{z_1,-\xi}(0) = \xi.$$

Thus, $x \in \sigma_N(z_1, \xi)$. □

The Lemma 2.9 will be the key element to prove that the mapping Ψ is an isometry.

Definition 2.10. *Let N be a smooth manifold and let g and $\widetilde{g}$ be metric tensors on N. We say that metric tensors g and $\widetilde{g}$ are geodesically equivalent, if for all geodesics $\gamma : I_1 \to N$ of metric g and $\widetilde{\gamma} : \widetilde{I}_1 \to N$ of metric $\widetilde{g}$ there exist changes of parameters $\alpha : I_2 \to I_1$ and $\widetilde{\alpha} : \widetilde{I}_2 \to \widetilde{I}_1$ such that*

$$\gamma \circ \alpha \textit{ is a geodesic of metric } \widetilde{g}$$

and

$$\widetilde{\gamma} \circ \widetilde{\alpha} \textit{ is a geodesic of metric } g.$$

A trivial example of two geodesically equivalent Riemannian metrics are g and cg, where $c > 0$ is a constant. Other, more interesting examples of geodesically equivalent Riemannian metrics are

(1) The Southern hemisphere of the sphere S^2 and the plane $\mathbb{R}^2$ and that are mapped to each other in a gnomonic projection, i.e. the great circles are mapped to straight line.
(2) Unit disc in $\mathbb{R}^2$ and the Beltrami-Klein model of a hyperbolic plane.

Our first goal is to show that when the distance difference data on N_1 and N_2 satisfy (2)-(3), we have that metric tensors g_1 and $(\Psi^{-1})^* g_2$ are geodesically equivalent. By Lemma 2.9 we know all the geodesics of N_1 that exit unknown region M_1, as point sets. Next we will show that this information is enough to deduce the geodesic equivalence of g_1 and $(\Psi^{-1})^* g_2$.

Since the mapping Ψ is a diffeomorphism, it holds that each geodesic of (N_2, g_2) is mapped to some smooth curve of (N_1, g_1). By formula (4) and Lemma 2.9, it holds that sets $\sigma_N(z, \xi) \subset N_1$, with $z \in F_1$ and $\xi \in S_z N_1$, are images of geodesics of (N_2, g_2) in the mapping Ψ. Note that the geodesic segments $\sigma_N(z, \xi) \subset N_1$ are not self-intersecting, since a cut point occurs before a geodesic stops being one-to-one.

Let $z \in F_2$, $\xi \in S_z N_2$ and $t_2 = \tau_2(z, -\xi)$. Then $t \mapsto \Psi(\gamma^2_{z,-\xi}(t))$, $t \in [0, t_2)$ is smooth and not self-intersecting curve on N_1. By Proposition 1.2 and Theorem 2.4 we have

$$\Psi(\gamma^2_{z,-\xi}((0, t_2))) = \sigma_{N_1}(\Psi(z), \Psi_* \xi) = \sigma_{N_1}(\phi^{-1}(z), (\phi^{-1})_* \xi). \tag{26}$$

Let $w = \phi^{-1}(z)$ and $\eta = (\phi^{-1})_*\xi$. Then by Lemma 2.9 we have $\sigma_{N_1}(w,\eta) = \gamma^1_{w,-\eta}(\{s;\ 0<s<t_1\})$, where $t_1 = \tau_1(w,-\eta)$. Furthermore, it is easy to see that there is a re-parametrization

$$s:(0,t_1)\to(0,t_2) \text{ such that } \gamma^1_{w,-\eta}(t) = \Psi(\gamma^2_{z,-\xi}(s(t))),\ t\in(0,t_1). \tag{27}$$

For $p \in N_1$, we define a collection $\mathcal{C}(p)$ of geodesics γ of (N_1,g_1) and real numbers $t_0 \in \mathbb{R}$, given by

$$\begin{aligned}\mathcal{C}(p) \ = \ &\{(\gamma,t_0)\,;\ \gamma:(a,b)\to N_1 \text{ is a geodesic of } (N_1,g_1),\\ &\gamma(t_0)=p, \text{ and there are } z\in F_1^{int},\ \xi\in S_zN_1\\ &\text{such that } \gamma((a,b)) = \sigma_{N_1}(z,\xi)\}.\end{aligned}$$

Here γ is given as a pair of the set $\mathrm{dom}(\gamma) = (a,b)\subset\mathbb{R}$, $-\infty\le a<b\le\infty$, where the mapping γ is defined and the function $\gamma:\mathrm{dom}(\gamma)\to N_1$. Also, $t_0\in(a,b)$. Moreover, above $\gamma((a,b)) = \sigma_{N_1}(z,\xi)$ means that the sets $\gamma((a,b))\subset N_1$ and $\sigma_{N_1}(z,\xi)\subset N_1$ are the same, or equivalently, that $\gamma((a,b))$ and $\sigma_{N_1}(z,\xi)$ are the same as unparameterized curves.

For a moment we consider only metric g_1. Assume that p is a point in N_1 and q is point of F_1^{int} such that $q = \gamma_{p,\xi}(\ell)$, $\ell>0$ and the geodesic $\gamma_{p,\xi}([0,\ell])$ has no cut points. Then there is a neighborhood $U\subset F_1^{int}$ of q and a neighborhood $V\subset T_pN_1$ of $\ell\xi$ such that $\exp_p : V\to U$ is a diffeomorphism. Assuming that the neighborhood V is small enough, we see that for any $\ell v\in V$, $\|v\|_{g_1}=1$, there is $s>0$ such that the geodesic $\gamma_{p,v}([-s,\ell])$ has no cut points. Let $s_0(p,v)\in(0,\infty]$ be the supremum of such s. Then, for the geodesic $\gamma_{p,v}:(-s_0(p,v),\ell)\to N_1$ we have $(\gamma_{p,v},0)\in\mathcal{C}(p)$. This proves that set

$$\begin{aligned}\Omega_p := \{v\in T_pN_1 \quad ; \quad &\text{there are } (c,t_p)\in\mathcal{C}(p),\ c(t_p)=p\\ &\text{and } \dot c(t_p) \text{ is proportional to } \pm v\}\end{aligned}$$

contains a non-empty open double cone Σ_p, that is, an open set that satisfies $rv\in\Sigma_p$ for all $v\in\Sigma_p$ and $r\in\mathbb{R}\setminus\{0\}$. Note that the complement of Ω_p in T_pN_1 is non-empty, if in manifold M_1 there are closed geodesics, or geodesics that are trapping in both directions in M_1 and go through the point p.

Let point $p\in N_1$ and (U,X) be coordinates near p, $X:U\to\mathbb{R}^n$, and denote $X(q) = (x^j(q))_{j=1}^n$. Recall that a pre-geodesic $\widetilde\gamma$ on (N_1,g_1) satisfies the formula (11), that is,

$$\Big[\frac{d^2\widetilde\gamma^k}{ds^2}(s) + \Gamma^k_{i,j}(\widetilde\gamma(s))\frac{d\widetilde\gamma^i}{ds}(s)\frac{d\widetilde\gamma^j}{ds}(s)\Big]\Big|_{s=s_p} = f\Big(\frac{d\widetilde\gamma}{ds}\Big)\frac{d\widetilde\gamma^k}{ds}(s)\Big|_{s=s_p}, \tag{28}$$

$k\in\{1,2,\dots,n\}$. Here $\gamma(s_p)=p$ and f is some function that is homogeneous of degree 1 on the subbundle of TU that is determined by the velocity vectorfield of $\widetilde\gamma$.

Next, we change the point of view and consider the equation (28) as a system of equations for the "unknown" (Γ,f) with the given coefficients

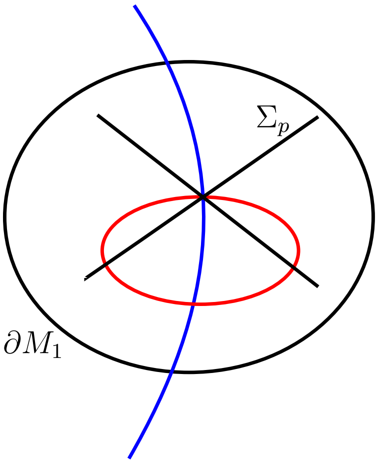

FIGURE 4. *For all $p \in M_1$ there exists an open conic set $\Sigma_p \subset T_pN_1$ such that for every $\xi \in \Sigma_p$ the geodesic $\gamma_{p,\xi}$ of (N_1, g_1) can be extended to a distance minimizing geodesic (blue curve in the figure) that enters the set $F = N \setminus M$. When the distance difference data for g_1 and g_2 coincide, these geodesics have to be pre-geodesic also with respect to the metric $\Psi^* g_2$. Note that there may be g_1-geodesics emanating from p to directions $\xi \notin \Sigma_p$ that do not intersect the set F. Such geodesics can be e.g. closed loops in M_1 (red curve).*

$\frac{d\widetilde{\gamma}}{ds}(s)|_{s=s_p} \in \Omega_p$ and $\frac{d^2\widetilde{\gamma}}{ds^2}(s)|_{s=s_p}$ where $(\widetilde{\gamma}, s_p) \in \mathcal{C}(p)$. Here Γ stands for a collection of Christoffel symbols $\Gamma^k_{i,j}$ and $f : \Omega \to \mathbb{R}$ is a function that satisfies equation (10) on the subbundle

$$\Omega := \bigcup_{p \in U} \Omega_p \subset TU.$$

Suppose that we also have another Riemannian connection which Christoffel symbols $\widetilde{\Gamma}^k_{i,j}$ in the (U, X)-coordinates have the form

$$\widetilde{\Gamma}^k_{i,j} = \Gamma^k_{i,j} + \delta^k_i \varphi_j + \delta^k_j \varphi_i, \tag{29}$$

for some smooth functions $\varphi_i : U \to \mathbb{R}$, $i = 1, 2, \ldots, n$. Here, δ^k_i is one when $k = i$ and zero otherwise. Let $\varphi(x) = \varphi_i(x) dx^i$ be a smooth 1-form that has functions $(\varphi_i)^n_{i=1}$ as the coefficients. We need the following consequence of Lemma 2.1:

Lemma 2.11. *Let (U, X) a smooth coordinate chart. If the Christoffel symbols $\widetilde{\Gamma}$ and Γ satisfy the equation* (29) *for some 1-form φ and pair (f, Γ), $f : \Omega \to \mathbb{R}$ is homogenous of degree 1, is a solution of* (11)*, with $s = s_p$, for all $(\widetilde{\gamma}, s_p) \in \mathcal{C}(p)$, then pair $(\widetilde{\Gamma}, \widetilde{f})$ where*

$$\widetilde{f}(v) = f(v) + 2\varphi(v), \ v \in \Omega. \tag{30}$$

is also a solution of (11)*, with $s = s_p$, for all $(\widetilde{\gamma}, s_p) \in \mathcal{C}(p)$.*

Proof. Let $(\widetilde{\gamma}, s_p) \in \mathcal{C}(p)$. A direct computation shows that

$$\begin{aligned}(\delta^k_i \varphi_j + \delta^k_j \varphi_i)\tfrac{d\widetilde{\gamma}^i}{ds}(s)\tfrac{d\widetilde{\gamma}^j}{ds}(s) &= \varphi_j \tfrac{d\widetilde{\gamma}^k}{ds}(s)\tfrac{d\widetilde{\gamma}^j}{ds}(s) + \varphi_i \tfrac{d\widetilde{\gamma}^i}{ds}(s)\tfrac{d\widetilde{\gamma}^k}{ds}(s)\\ &= 2\tfrac{d\widetilde{\gamma}^k}{ds}(s)\Big(\varphi_i \tfrac{d\widetilde{\gamma}^i}{ds}(s)\Big) = 2\tfrac{d\widetilde{\gamma}^k}{ds}(s)\varphi\Big(\tfrac{d\widetilde{\gamma}}{ds}(s)\Big)\end{aligned} \tag{31}$$

Use this and substitute equation (29) into equation (11) to obtain

$$\frac{d^2\widetilde{\gamma}^k}{ds^2}(s) + \widetilde{\Gamma}^k_{i,j}(p)\frac{d\widetilde{\gamma}^i}{ds}(s)\frac{d\widetilde{\gamma}^j}{ds}(s)\Big|_{s=s_p} = \frac{d\widetilde{\gamma}^k}{ds}(s)\Big[f\Big(\frac{d\widetilde{\gamma}}{ds}(s)\Big) + 2\varphi\Big(\frac{d\widetilde{\gamma}}{ds}(s)\Big)\Big]$$

that proves the claim. ☐

The following lemma gives a converse result for Lemma 2.11. It is obtained by using, in a quite straightforward way, results of V. Matveev [46, Sec. 2] for general affine connections on pseudo-Riemannian manifolds. However, for the convenience of the reader, we give a detailed proof for the lemma and analyze at the same time the smoothness of the 1-form $x \mapsto \varphi(x)$ in a local coordinate neighborhood $U \subset M$.

Lemma 2.12. *Let (U, X) a smooth coordinate chart. Let functions $f : \Omega \to \mathbb{R}$ and $\widetilde{f} : \Omega \to \mathbb{R}$ be homogeneous of degree 1. Suppose that pairs (f, Γ) and $(\widetilde{\Gamma}, \widetilde{f})$ both solve at all points $p \in U$ the system* (11) *for all such coefficients $\frac{d\gamma}{ds}(s)|_{s=s_p} \in \Omega_p$ and $\frac{d^2\gamma}{ds^2}(s)|_{s=s_p}$ that $(\gamma, s_p) \in \mathcal{C}(p)$. Then the Christoffel symbols Γ and $\widetilde{\Gamma}$ satisfy equation* (29) *in U with a C^∞-smooth 1-form φ in U.*

Proof. Define a pair $(\overline{f}, \overline{\Gamma})$ as

$$\overline{f} = f - \widetilde{f} \text{ and } \overline{\Gamma}^k_{i,j} = \Gamma^k_{i,j} - \widetilde{\Gamma}^k_{i,j}.$$

As a difference of two connection coefficients, $\overline{\Gamma}$ is a tensor. By substitution of pairs (f, Γ) and $(\widetilde{\Gamma}, \widetilde{f})$ into equation (11) and by subtracting the obtained equations, we obtain at $p \in U$

$$\overline{\Gamma}^k_{i,j} v^i v^j = \overline{f}(v) v^k, \text{ for every } v \in \Omega_p. \tag{32}$$

Note that (32) defines a smooth extension of $\overline{f} : \Omega \to \mathbb{R}$ to $TU \setminus \{0\}$, given by

$$\overline{f}(v) = \frac{\overline{f}(v) v^k g_{k\ell} v^\ell}{g(v,v)} = \frac{\overline{\Gamma}^k_{i,j}(p) v^i v^j g_{k\ell}(p) v^\ell}{g_{ab}(p) v^a v^b}, \ (p, v) \in TU \setminus \{0\}. \tag{33}$$

Here, the rightmost term is smooth in $TU \setminus \{0\}$.

Recall that Ω_p contains an open double cone $\Sigma_p \subset \Omega_p$. Our next goal is to show that there exist a linear function $\varphi : T_pN \to \mathbb{R}$ such that the restriction of function $\overline{f}$, to $\Sigma_p \subset \Omega_p$, is equal to $2\varphi|_{\Sigma_p}$. Define a family of symmetric bi-linear mappings

$$\sigma^k : T_pN \times T_pN \to \mathbb{R}, \ \sigma^k(u, v) = \overline{\Gamma}^k_{i,j} v^i u^j, \ k \in \{1, \dots, n\}.$$

Since mappings σ^k are symmetric, the parallelogram equation

$$0=\sigma^k(u+v,u+v)+\sigma^k(u-v,u-v)-2\sigma^k(u,u)-2\sigma^k(v,v)$$

holds.

Next, let $u\in\Sigma_p$, $u\neq 0$. Then there is $\varepsilon=\varepsilon(u)>0$ such that, if $v\in T_pN$ satisfies $\|v\|<\varepsilon$, then $u-v\in\Sigma_p$.

Let us next consider $v\in\Sigma_p$ with $\|v\|<\varepsilon$. Then $u-v,u+v\in\Sigma_p\subset\Omega_p$. By the parallelogram equality for the function σ^k and (32) we have (34)

$$\begin{aligned}0&=\overline{f}(u+v)(u+v)+\overline{f}(u-v)(u-v)-2\overline{f}(u)u-2\overline{f}(v)v\\&=(\overline{f}(u+v)+\overline{f}(u-v)-2\overline{f}(u))u+(\overline{f}(u+v)-\overline{f}(u-v)-2\overline{f}(v))v.\end{aligned}$$

If vectors u and v are linearly independent, we get a system

$$\left\{\begin{array}{l}\overline{f}(u+v)+\overline{f}(u-v)-2\overline{f}(u)=0\\ \overline{f}(u+v)-\overline{f}(u-v)-2\overline{f}(v)=0.\end{array}\right.\tag{35}$$

By summing up these two equations, we get

$$\overline{f}(u+v)=\overline{f}(u)+\overline{f}(v).\tag{36}$$

Observe that the system (35) is valid also when $v=\lambda u$, $\lambda\in\mathbb{R}$. Recall that the mappings f and $\widetilde{f}$ are solutions of (11) and therefore, they satisfy the equation (10), i.e., they commute with scalar multiplication in Ω_p.

So far we have proved that $\overline{f}(u+\cdot)$ and $\overline{f}(u)+\overline{f}(\cdot)$ coincide in set $B_p(0,\epsilon)\cap\Sigma_p$. Since $\overline{f}$ is homogeneous of degree 1 it holds by (36) that

$$\overline{f}(u+av)=\overline{f}(u)+a\overline{f}(v),\ v\in B_p(0,\epsilon)\cap\Sigma_p,\ -1<a<1.\tag{37}$$

We define a linear function

$$2\varphi:T_pN\to\mathbb{R},\ 2\varphi(v)=\lim_{r\to 0}\frac{\overline{f}(u+rv)-\overline{f}(u)}{r}=\nabla_u\overline{f}(u)\cdot v.\tag{38}$$

If $v\in\Sigma_p$ and r is small enough, then $rv\in B_p(0,\varepsilon)\cap\Sigma_p$ and therefore by formula (37) it holds that

$$2\varphi(v)=\overline{f}(v)\quad\text{for every } v\in\Sigma_p.\tag{39}$$

As Σ_p is open, and φ and $\overline{f}$ are linear, this holds for all $v\in T_pN$ and thus $\varphi(v)$ given by the formula (38) is independent on the choice of used $u\in\Sigma_p$. In local coordinates (U,X) we have by (33) and (39) that

$$\varphi(\frac{\partial}{\partial x^\ell}):=\frac{1}{2}\sum_{i,k,j=1}^{n}\frac{1}{g_{\ell\ell}(x)}\overline{\Gamma}^k_{i,j}(x)\delta^i_\ell\delta^j_\ell g_{k\ell}(x)$$

defines a C^∞-smooth 1-form $x\mapsto\varphi(x)$ in U, that is an extension of $\overline{f}:\Omega\to\mathbb{R}$.

Define a connection

$$\widehat{\Gamma}^k_{i,j}:=\widetilde{\Gamma}^k_{i,j}+\delta^k_i\varphi_j+\delta^k_j\varphi_i,$$

and choose $v = \frac{d}{ds}\gamma(s)|_{s=s_p} \in \Sigma_p$. Since pairs (f, Γ) and $(\widetilde{\Gamma}, \widetilde{f})$ are both solutions of (11) the above considerations yield

$$\begin{aligned}
&\Big[\frac{d^2\gamma^k}{ds^2}(s) + \Gamma^k_{i,j}(p)\frac{d\gamma^i}{ds}(s)\frac{d\gamma^j}{ds}(s)\Big]\Big|_{s=s_p} = \Big[f\Big(\frac{d\gamma}{ds}(s)\Big)\frac{d\gamma^k}{ds}(s)\Big]\Big|_{s=s_p} \\
= \;& \frac{d\gamma^k}{ds}(s)\Big[2\varphi\Big(\frac{d\gamma}{ds}(s)\Big) + \widetilde{f}\Big(\frac{d\gamma}{ds}(s)\Big)\Big]\Big|_{s=s_p} \\
= \;& \Big[\frac{d^2\gamma^k}{ds^2}(s) + \widetilde{\Gamma}^k_{i,j}(p)\frac{d\gamma^i}{ds}(s)\frac{d\gamma^j}{ds}(s)\Big]\Big|_{s=s_p} + \frac{d\gamma^k}{ds}(s)\Big[2\varphi\Big(\frac{d\gamma}{ds}(s)\Big)\Big]\Big|_{s=s_p} \\
\overset{(31)}{=} \;& \Big[\frac{d^2\gamma^k}{ds^2}(s) + \widehat{\Gamma}^k_{i,j}(p)\frac{d\gamma^i}{ds}(s)\frac{d\gamma^j}{ds}(s)\Big]\Big|_{s=s_p}.
\end{aligned}$$

Therefore we have

$$\Gamma^k_{i,j}(p)\frac{d\gamma^i}{ds}(s)\frac{d\gamma^j}{ds}(s)\Big|_{s=s_p} = \widehat{\Gamma}^k_{i,j}(p)\frac{d\gamma^i}{ds}(s)\frac{d\gamma^j}{ds}(s)\Big|_{s=s_p}. \tag{40}$$

Thus we have shown that for all $v \in \Sigma_p$ the equation

$$\Gamma^k_{i,j}(p)v^iv^j = \widehat{\Gamma}^k_{i,j}(p)v^iv^j \tag{41}$$

is valid. Since set Σ_p is open, it holds that

$$\Gamma^k_{\ell,m}(p) = \partial_{v^\ell v^m}\Gamma^k_{i,j}(p)v^iv^j = \partial_{v^\ell v^m}\widehat{\Gamma}^k_{i,j}(p)v^iv^j = \widehat{\Gamma}^k_{\ell,m}(p).$$

As above $p \in U$ is arbitrary, this proves the claim. □

Proposition 2.13. *Suppose that Riemannian manifolds* (N_1, g_1) *and* (N_2, g_2) *are as in Section 1.2 and* (2)-(3) *are valid. Let* $p \in N_1$ *and* (U, X) *be coordinates in a neighborhood of* p. *Then it holds that the Christoffel symbols* Γ *and* $\widetilde{\Gamma}$ *of metrics* g_1 *and* $(\Psi^{-1})^*g_2$, *respectively, satisfy equation* (29) *in* U *with some 1-form* φ, *where* Ψ *is as in* (20).

Proof. Let (U, X) be a local coordinate system in N_1. Our aim is to use the Lemma 2.12 to prove the claim of this Lemma. To do so we need to construct a function $\widetilde{f} : \Omega \to \mathbb{R}$ that satisfies (10) and moreover for any $q \in U$ the pair $(\widetilde{\Gamma}, \widetilde{f})$ solves the system (11) for all such coefficients $\frac{d\gamma}{ds}(s)|_{s=s_q} \in \Omega_q$ and $\frac{d^2\gamma}{ds^2}(s)|_{s=s_q}$ that $(\gamma, s_q) \in \mathcal{C}(q)$.

Let $p \in U$ and $(c_1, t_1) \in \mathcal{C}(p)$. With out loss of generality we may assume that $t_1 = 0$ and $\dot{c}_1(0) = \xi \in S_pN \cap \Omega_p$. By definition of $\mathcal{C}(p)$, it holds that there is a unique reparametrization $t \mapsto s_\xi(t) =: s(t)$ of c_1 such that for curves c_1 and $c_2 = c_1 \circ s$ we have $s(0) = p$, $\dot{c}_2(0) = \xi$ and

$$\begin{cases} \ddot{c}^k_1(t) + \dot{c}^i_1(t)\dot{c}^j_1(t)\Gamma^k_{i,j}(c_1(t)) = 0, \\ \ddot{c}^k_2(t) + \dot{c}^i_2(t)\dot{c}^j_2(t)\widetilde{\Gamma}^k_{i,j}(c_2(t)) = 0. \end{cases} \tag{42}$$

Using the chain rule we can write the latter equation as

$$\ddot{c}^k_1(s(t)) + \dot{c}^i_1(s(t))\dot{c}^j_1(s(t))\widetilde{\Gamma}^k_{i,j}(c_1(s(t))) = -\frac{\ddot{s}(t)}{\dot{s}(t)^2}\dot{c}^k_1(s(t)).$$

We define $f : \Omega \to \mathbb{R}$

$$f(q, v) = \frac{\ddot{s}_v(t)}{\dot{s}_v(t)^2}, \text{ if } v = \dot{\gamma}(0) \text{ for some } (\gamma, 0) \in \mathcal{C}(q).$$

Above s_v is such a reparametrization of γ that, $s_v(0) = 0$, $\frac{d}{dt}\gamma(s(t))|_{t=0} = v$ and (42) is valid, when c_1 is replaced with γ and c_2 is replaced with $\gamma \circ s_v$. Note that function f is well defined and satisfies the equation (10), since geodesic equation (7) is preserved under affine re-parametrizations. Therefore it holds that for any $q \in U$ the pairs $(\Gamma, 0)$ and $(\widetilde{\Gamma}, f)$ both solve the system (11) for all such coefficients $\frac{d\gamma}{ds}(s)|_{s=s_q} \in \Omega_q$ and $\frac{d^2\gamma}{ds^2}(s)|_{s=s_q}$ that $(\gamma, s_q) \in \mathcal{C}(q)$. The claim follows then from Lemma 2.12. □

Lemma 2.14. *Suppose that the connections Γ and $\widetilde{\Gamma}$ corresponding to metric tensors g and $\widetilde{g}$, respectively, satisfy the equation* (29) *with a 1-form φ. Then the metric tensors g and $\widetilde{g}$ are geodesically equivalent.*

Proof. Let $\gamma(t)$ be a geodesic with respect to the metric g. Then γ satisfies the geodesic equation (7). Substitute Γ with $\widetilde{\Gamma}$ into (7) to get the equation

$$\frac{d^2\gamma^k}{dt^2}(t) + \widetilde{\Gamma}^k_{i,j}(\gamma(t))\frac{d\gamma^i}{dt}(t)\frac{d\gamma^j}{dt}(t) = 2\frac{d\gamma^k}{dt}(t)\varphi\Big(\frac{d\gamma}{dt}(t)\Big).$$

Write $\kappa(t) = 2\varphi(\dot{\gamma}(t))$ and use Lemma 2.1 to show that there exists a change of parameters $s \mapsto t(s)$ such that $s \mapsto \gamma(t(s))$ is a geodesic with respect to the metric $\widetilde{\Gamma}$. As the roles of g and $\widetilde{g}$ can be exchanged, the claim follows. □

By the Lemma 2.14, the equivalence of the distance difference data (2)-(3) implies the geodesic equivalence of metric tensors g and $\Psi_* g_2$ on N_1. In the following theorem, that shows that metric tensors g and $\Psi_* g_2$ coincide also in N_1, we will use the implications of the Matveev-Topalov theorem [43]. Their result is also concerned in the appendix of the extended preprint version of this paper [37] and its generalizations have been considered in [11, 63].

Lemma 2.15. *Suppose that manifold N satisfies assumptions of Section 1.2 and g and $\widetilde{g}$ are two metric tensors on N. Suppose that these metrics are geodesically equivalent on manifold N and coincide in set $F^{int} \neq \emptyset$. Then $g = \widetilde{g}$ in whole N.*

Proof. Define a smooth mapping $I_0 : TN \to \mathbb{R}$ as

$$I_0((x, v)) = \Big(\frac{\det(g_x)}{\det(\widetilde{g}_x)}\Big)^{\frac{2}{n+1}} \widetilde{g}_x(v, v), \tag{43}$$

where $\widetilde{g}_x(v, v) = \widetilde{g}_{jk}(x)v^j v^k$. Note that the function $x \mapsto \frac{\det(g_x)}{\det(\widetilde{g}_x)}$ is coordinate invariant.

Let γ_g be a geodesic of metric g. Define a smooth path β in TN as $\beta(t) = (\gamma_g(t), \dot\gamma_g(t))$, i.e., β is an integral curve of the geodesic flow of metric g. The Matveev-Topalov theorem [43] states that if g and $\widetilde{g}$ are geodesically equivalent, then there are several invariants related to the $(1,1)$-tensor $G = g^{-1}\widetilde{g}$, given in local coordinates by $G^j_k(x) = g^{ji}(x)\widetilde{g}_{ik}(x)$, that are constants along integral curves $\beta(t)$. In particular, the function $t \mapsto I_0(\beta(t))$ is a constant.

A corollary of Matveev-Topalov theorem, [43, Cor. 2] (see also [44, Cor. 2] and [11, Thm. 3]), is that the number $n(x)$ of the different eigenvalues of the map $G(x) : T_xN \to T_xN$ is constant at almost every point $x \in N$. Since $G(x) = I$ for $x \in F^{int}$, we have that $n(x) = 0$ in the set F^{int} having a positive measure. This implies that $n(x) = 0$ for almost all $x \in N$. Hence for almost all $x \in N$ there is $c(x) \in \mathbb{R}_+$ such that we have $G(x) = c(x)I$, so that $\widetilde{g}_{ik}(x) = c(x)g_{ik}(x)$. As G is continuous, this holds for all $x \in N$. Summarizing, the first implication of the Matveev-Topalov theorem is that g and $\widetilde{g}$ are conformal on the whole manifold N.

Let x_0 be a point of N. Since we assumed that metrics g and $\widetilde{g}$ coincide in set F, we have for any point $z \in F$ and vector $v \in T_zN$ that formula (43) has form

$$I_0(z, v) = \widetilde{g}_z(v, v) = g_z(v, v). \tag{44}$$

Let $\gamma(t) := \gamma^g_{z,\xi}(t)$, $\xi \in S_zN$, $z \in F$ be a g-geodesic passing through x_0 such that $x_0 = \gamma(t_0)$ for some $t_0 \geq 0$. The $I_0((z, \xi)) = 1$ and by the Matveev-Topalov theorem, I_0 is constant along the integral curves of geodesic flow of g. Thus, we have

$$I_0(x_0, \dot\gamma(t_0)) = I_0(z, \xi) = 1. \tag{45}$$

Define W_{x_0} to be the set of all g-unit vectors of $T_{x_0}N$ with respect to metric g, such that every vector in W_{x_0} is a velocity vector of some g-geodesic starting from F and passing trough x_0. Recall that set $W^{int}_{x_0} \subset S_{x_0}N$ is not empty.

Let $X = (x^1, \dots, x^n)$ be any coordinate chart at x_0. Formula (45) shows that for every $\xi \in W_{x_0}$ we have

$$g_{ij}(x_0)\xi^i\xi^j = 1 = I_0(x_0, \xi) = \Big(\frac{\det(g_{x_0})}{\det(\widetilde{g}_{x_0})}\Big)^{\frac{2}{n+1}} \widetilde{g}_{ij}(x_0)\xi^i\xi^j. \tag{46}$$

Consider an open cone

$$W^{int}_{x_0} \cdot \mathbb{R}_+ := \{tw \in T_{x_0}N : t > 0, w \in W^{int}_{x_0}\}.$$

Then the equation (46) holds for all $\xi \in W^{int}_{x_0} \cdot \mathbb{R}_+$ and since the set $W^{int}_{x_0} \cdot \mathbb{R}_+$ is open and both sides of equation (46) are smooth in ξ, we obtain the equation

$$g_{ij}(x_0) = \Big(\frac{\det(g_{x_0})}{\det(\widetilde{g}_{x_0})}\Big)^{\frac{2}{n+1}} \widetilde{g}_{ij}(x_0), \text{ for all } i, j \in \{1, \dots, n\}, \tag{47}$$

as a second order derivative with respect to ξ of equation (46).

Denote $f(p) := \frac{\det(g(p))}{\det(\widetilde{g}(p))}$. Then the above yields

$$(f(x_0))^{\frac{2}{n+1}}\, \widetilde{g}_{jk}(x_0) = g_{jk}(x_0), \text{ for all } j,k \in \{1,\dots,n\}. \tag{48}$$

Taking determinants of both sides of (48) we see that

$$(f(x_0))^{\frac{2n}{n+1}-1} = 1. \tag{49}$$

Since we we have assumed the dimension of manifold N is at least 2, we see from equation (49) that $f(x_0) = 1$. By formula (48) this implies $g = \widetilde{g}$ on M. □

Theorem 1.3 follows now from Theorems 2.4 and 2.7 and Lemmas 2.14 and 2.15. □

3. Application for an inverse problem for a wave equation

Here we consider an application of Theorem 1.3 for an inverse problem for a wave equation with spontaneous point sources.

3.0.1. *Support sets of waves produced by point sources.* Let (N,g) be a closed Riemannian manifold. Denote the Laplace-Beltrami operator of metric g by Δ_g. We consider a wave equation

$$\begin{cases} (\partial_t^2 - \Delta_g)G(\cdot,\cdot,y,s) = \kappa(y,s)\delta_{y,s}(\cdot,\cdot), & \text{in } \mathcal{N} \\ G(x,t,y,s) = 0, \quad \text{for } t < s,\ x \in N. \end{cases} \tag{50}$$

where $\mathcal{N} = N \times \mathbb{R}$ is the space-time. The solution $G(x,t,y,s)$ is the wave produced by a point source located at the point $y \in M$ and time $s \in \mathbb{R}$ having the magnitude $\kappa(y,s) \in \mathbb{R} \setminus \{0\}$. Above, we have $\delta_{y,s}(x,t) = \delta_y(x)\delta_s(t)$ corresponds to a point source at $(y,s) \in \mathcal{N}$.

3.0.2. *Inverse coefficient problem with spontaneous point source data.* Assume that there are two manifolds (N_1,g_1) and (N_2,g_2) satisfying the assumptions given in Section 1.2 and

$$\text{There exists an isometry } \phi : F_1 \to F_2 \tag{51}$$

$$W_1 = W_2 \tag{52}$$

where W_1 and W_2 are collections of supports of waves produced by point sources taking place at unknown points at unknown time, that is,

$$W_1 = \{\operatorname{supp}(G^1(\cdot,\cdot,y_1,s_1)) \cap (F_1 \times \mathbb{R})\,;\ y_1 \in M_1,\ s_1 \in \mathbb{R}\} \subset 2^{F_1 \times \mathbb{R}}$$

and

$$W_2 = \{\operatorname{supp}(G^2(\phi(\cdot),\cdot,y_2,s_2)) \cap (F_1 \times \mathbb{R})\,;\ y_2 \in M_2,\ s_2 \in \mathbb{R}\} \subset 2^{F_1 \times \mathbb{R}}$$

where functions G^j, $j = \{1,2\}$ solve equation (50) on manifold N_j. Here $2^{F_j \times \mathbb{R}} = \{V;\ V \subset F_j \times \mathbb{R}\}$ is the power set of $F_j \times \mathbb{R}$. Roughly speaking, W_j corresponds to the data that one makes by observing, in the set F_j, the waves that are produced by spontaneous point sources that that go off, at an unknown time and at an unknown location, in the set M_j.

Earlier, the inverse problem for the sources that are delta-distributions in time and localized also in the space has been studied in [16] in the case when the metric g is known. Theorem 1.3 yields the following result telling that the metric g can be determined when a large number of waves produced by the point sources are observed:

Proposition 3.1. *Let* (N_j, g_j), $j = 1, 2$ *be a closed compact Riemannian* n*-manifolds,* $n \geq 2$ *and* $M_j \subset N_j$ *be an open set such that* $F_j = N_j \setminus M_j$ *have non-empty interior. If the spontaneous point source data of these manifolds coincide, that is, we have* (51)-(52)*, then* (N_1, g_1) *and* (N_2, g_2) *are isometric.*

Proof. Let us again omit the sub-indexes of N, M, and F. For $y \in M$, $s \in \mathbb{R}$, and $z \in F$ we define a number

$$\begin{aligned} \mathcal{T}_{y,s}(z) \;=\; & \sup\{t \in \mathbb{R};\ \text{the point } (z,t) \text{ has a neighborhood} \\ & \qquad U \subset \mathcal{N} \text{ such that } G(\cdot,\cdot,y,s)|_U = 0\} \end{aligned}$$

which tells us, what is the first time when the wave $G(\cdot,\cdot,y,s)$ is observed near the point z. Using the finite velocity of the wave propagation for the wave equation, see [21], we see that the support of $G(\cdot,\cdot,y,s)$ is contained in the future light cone of the point $q = (y,s) \in \mathcal{N}$ given by

$$J^+(q) = \{(y', s') \in \mathcal{N};\ s' \geq d(y', y) + s\}.$$

Next, for $\xi = \xi^j \frac{\partial}{\partial x^j} \in T_y N$ we denote the corresponding co-vector by $\xi^\flat = g_{jk}(y)\xi^j dx^j$. Then the results of [18] and [19] on the propagation of singularities for the real principal type operators, in particular for the wave operator, imply that in the set $\mathcal{N}\setminus\{q\}$ Green's function $G(\cdot,\cdot,y,s)$ is a Lagrangian distribution associated to the Lagranian sub-manifold

$$\Sigma_0 = \{(\gamma_{y,\eta}(t), s+t; \dot\gamma_{y,\eta}(t)^\flat, dt) \in T^*\mathcal{N};\ \eta \in S_y N,\ t > 0\}$$

and its principal symbol on Σ_0 is non-zero. In particular, [19, Prop. 2.1] implies that $\Sigma = \Sigma_0 \cup (T_q^*\mathcal{N}, \setminus\{0\})$ coincides with the wave front set $\mathrm{WF}(u)$ of the solution $u = G(\cdot,\cdot,y,s)$. This means that a wave emanating from a point source (y,s) propagates along the geodesics of manifold (N,g). The image of $\mathrm{WF}(u)$ in the projection $\pi : T^*\mathcal{N} \to \mathcal{N}$ coincides the singular support of u. Hence, we see that

$$\begin{aligned} &\mathrm{singsupp}(G(\cdot,\cdot,y,s)) = S(q), \quad \text{where} \\ &S(q) = \{(\exp_y(t\eta), s+t) \in \mathcal{N}; \eta \in S_y N,\ t \geq 0\}. \end{aligned} \tag{53}$$

Since the Riemannian manifold N is complete, the space-time $\mathcal{N}$ is a globally hyperbolic Lorentzian manifold and we have $\partial J^+(q) = S(q)$, see [50]. Summarizing, the above implies that the function $G(\cdot,\cdot,y,s)$ vanishes outside $J^+(q)$ and is non-smooth, and thus it is non-vanishing in a neighborhood of arbitrary point of $\partial J^+(q)$. Thus, for $z \in F$ we have $\mathcal{T}_{y,s}(z) = d(z,y) - s$. Hence the distance difference functions satisfy equation

$$D_y(z_1,z_2) = \mathcal{T}_{y,s}(z_1) - \mathcal{T}_{y,s}(z_2). \tag{54}$$

Thus, when formulas (51)-(52) are valid, we see using equation (54) that the distance difference data of the manifolds N_1 and N_2 coincide, that is, we have (2)-(3). Hence, the claim follows from Theorem 1.3. □

Finally, we note that sets W_j are closely related to the light-observation sets studied in [34] in the study of the inverse problems for non-linear hyperbolic problems with a time-dependent metric. The light-observation set $P_U(q)$ corresponding to a source point $q = (y,s)$ and the observation set U is the intersection of U and the future light cone emanating from q. In fact, the formula (53) implies that in the space time $\mathcal{N} = N \times \mathbb{R}$ the sets W_j coincide with the light-observation sets $P_U(q)$ corresponding to a source point $q = (y,s)$ and the observation set $U = F \times \mathbb{R}$.

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Appendix A: Extensions of data

Assume that we are given the set $F = N \setminus M$ and the metric $g|_F$, but instead of the function $D_x : F \times F \to \mathbb{R}$ we know only its restriction on the boundary $\partial F = \partial M$, that is, the map

$$D_x|_{\partial F \times \partial F} : \partial F \times \partial F \to \mathbb{R},\ D_x|_{\partial F \times \partial F}(z_1,z_2) := d_N(z_1,x) - d_N(z_2,x).$$

Lemma 3.2. *The manifold $F = N \setminus M$, the metric $g|_F$, and the restriction $D_x|_{\partial F \times \partial F}$ of the distance difference function corresponding to $x \in M$ determine the distance difference function $D_x : F \times F \to \mathbb{R}$.*

Proof. We can determine the map $D_x : F \times F \to \mathbb{R}$ by the formula

$$D_x(z_1,z_2) = \inf_\alpha \sup_\beta \Big(\mathcal{L}(\alpha) + D_x|_{\partial F \times \partial F}(\alpha(1),\beta(1)) - \mathcal{L}(\beta) \Big),$$

where the infimum is taken over the smooth curves $\alpha : [0,1] \to F$ from z_1 to $\alpha(1) \in \partial F$ and the supremum is taken over the smooth curves $\beta : [0,1] \to F$ from z_2 to $\beta(1) \in \partial F$. □

This raises the question, if the manifold (N,g) can be reconstructed when we are given a submanifold of codimension 1, e.g. the boundary of the open set M considered above, and the distance difference functions on this submanifold. To consider this, assume that we are given a submanifold $\widetilde{F} \subset N$ of dimension $(n-1)$, the metric $g|_{\widetilde{F}}$ on $\widetilde{F}$, and the collection

$$\{D^x_{\widetilde{F},N};\ x \in N\} \subset C(\widetilde{F} \times \widetilde{F}),$$

where $D^x_{\widetilde{F},N}(z_1, z_2) = d_N(x, z_1) - d_N(x, z_2)$ for $z_1, z_2 \in \widetilde{F}$. The following counterexample shows that such data do not uniquely determine the isometry type of (N, g).

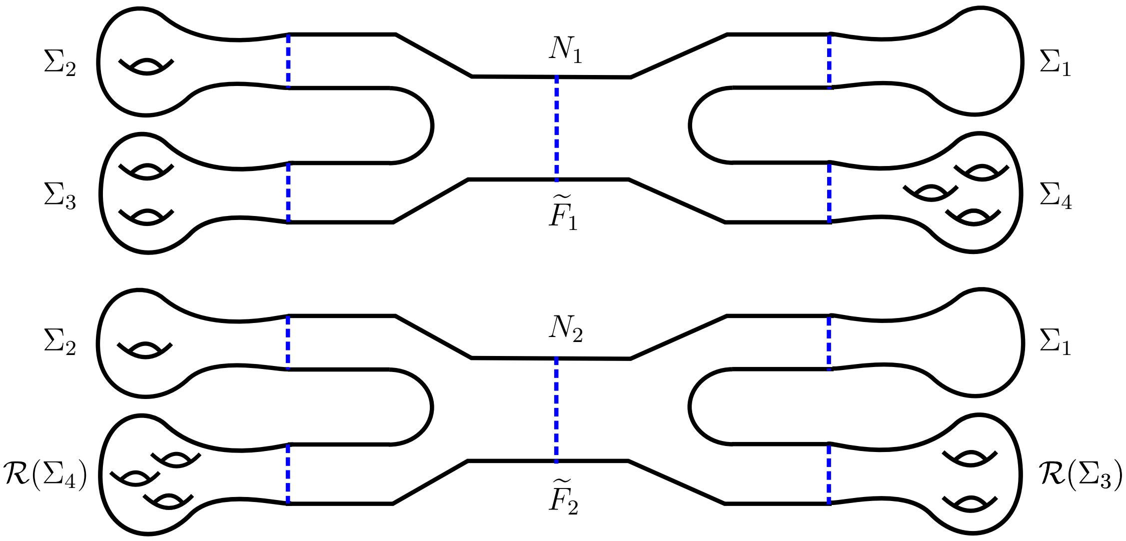

Figure 5. An illustration of manifolds N_1 and N_2 in Example $A1$. When $(n-1)$-dimensional submanifolds $\widetilde{F}_1 = \widetilde{F}_2 = \widetilde{F}$ are identified, the distance difference functions $\{D^x_{\widetilde{F},N_1};\ x \in N_1\}$ and $\{D^x_{\widetilde{F},N_2};\ x \in N_2\}$ coincide.

Example A1. Let $C_r(y) = \{(x_1, x_2) \in \mathbb{R}^2;\ |x_1 - y_1|^2 + |x_2 - y_2|^2 = r^2\}$ be a circle of radius r centered at $y = (y_1, y_2)$. Let $p_1 = (2, 0)$, $p_2 = (-2, 0)$, $L > 3$, and

$$S_0 = C_1(0) \times [-1, 1],$$
$$S_1 = C_1(p_1) \times [2, L],$$
$$S_2 = C_1(p_2) \times [2, L],$$

and $K \subset \mathbb{R}^2 \times [1, 2]$ be a 2-dimensional surface which boundary has three components, $C_1(0) \times \{1\}$, $C_1(p_1) \times \{2\}$, and $C_1(p_2) \times \{2\}$, such that the union $S_0 \cup K \cup S_1 \cup S_2$ is a smooth surface in $\mathbb{R}^3$. Moreover, let $\mathcal{R} : (x_1, x_2, x_3) \mapsto (x_1, x_2, -x_3)$ denote the reflection in the x_3-variable. Observe that then $\mathcal{R}(S_0) = S_0$. We define a smooth surface

$$\Sigma_0 = S_0 \cup K \cup S_1 \cup S_2 \cup \mathcal{R}(K) \cup \mathcal{R}(S_1) \cup \mathcal{R}(S_2).$$

The boundary of Σ_0 consists of 4 circles, namely $\Gamma_1 = C_1(p_1) \times \{L\}$, $\Gamma_2 = C_1(p_1) \times \{-L\}$, $\Gamma_3 = C_1(p_2) \times \{L\}$, and $\Gamma_4 = C_1(p_2) \times \{-L\}$. Let us consider four embedded Riemannian surfaces $\Sigma_j \subset \mathbb{R}^3$, $j = 1, 2, 3, 4$, with boundaries $\partial\Sigma_j$ are equal to Γ_j. Assume that near $\partial\Sigma_j$ the surfaces Σ_j are isometric to the Cartesian product of Γ_j and an interval $[0, \varepsilon]$ with $\varepsilon > 0$, and that the genus of Σ_j is equal to $(j - 1)$. Also, assume that $\Sigma_j \cap \Sigma_k = \emptyset$ for $j, k = 1, 2, 3, 4$ and $\Sigma_0 \cap \Sigma_j = \Gamma_j$ for $j = 1, 2, 3, 4$.

First, let us construct a manifold N_1 by gluing surfaces Σ_0 with $\Sigma_1, \Sigma_2, \Sigma_3$, and Σ_4 such that the boundaries Γ_j are glued with $\partial\Sigma_j$, $j \in \{1, 2, 3, 4\}$.

Second, we construct a manifold N_2 by gluing surfaces Σ_0 with $\Sigma_1, \Sigma_2, \mathcal{R}(\Sigma_3)$, and $\mathcal{R}(\Sigma_4)$ such that the boundaries Γ_j are glued with $\partial\Sigma_j$ with $j \in \{1, 2\}$ but Γ_3 is glued with $\mathcal{R}(\partial\Sigma_4)$ and Γ_4 is glued with $\mathcal{R}(\partial\Sigma_3)$, see Fig. 5. For both manifolds N_1 and N_2 we give the induced Riemannian metric from $\mathbb{R}^3$. Let $\widetilde{F} = \widetilde{F}_1 = \widetilde{F}_2 = S_0 \cap (\mathbb{R}^2 \times \{0\})$.

Let us assume that L above is larger than diam $(K) + 10$. Then on N_ℓ, $\ell = 1, 2$ a minimizing geodesic from $x \in \Sigma_j$, $j \geq 1$ to $z \in \widetilde{F}$ does not intersect the other sets Σ_k with $k \in \{1, 2, 3, 4\} \setminus \{j\}$. Using this we see that the sets $\{D^x_{\widetilde{F}, N_\ell};\ x \in N_\ell\} \subset C(\widetilde{F} \times \widetilde{F})$ are the same for $\ell = 1, 2$. As the manifolds N_1 and N_2 are not isometric, this implies that the data $(\widetilde{F}, g|_{\widetilde{F}})$ and $\{D^x_{\widetilde{F}, N};\ x \in N\}$ do not determine uniquely the manifold (N, g).

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