CORRELATION IMAGING IN INVERSE SCATTERING IS TOMOGRAPHY ON PROBABILITY DISTRIBUTIONS

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Abstract. Scattering from a non-smooth random field on the time domain is studied for plane waves that propagate simultaneously through the potential in variable angles. We first derive sufficient conditions for stochastic moments of the field to be recovered from correlations between amplitude measurements of the leading singularities, detected in the exterior of a region where the potential is almost surely supported. The result is then applied to show that if two sufficiently regular random fields yield the same data, they have identical laws as function-valued random variables.

1. Introduction

Randomness is often an inherent part of any computational model for an applied inverse problem. For instance, it can reflect the chaotic evolution of the system or the perspective that the unknown object of interest is rough and vastly complex. Ultimately, the observational noise is most often probabilistic in nature. If the statistics of the system can be described to a good approximation, it can be desirable to transform the problem paradigm by considering correlations or other statistical moments of the data distribution and how that information relates to the relevant system parameters. This approach is often called *correlation based imaging* in literature and it has been recently studied for a variety of inverse problems (see e.g. applications in seismic imaging [20]).

In this paper we consider scattering of waves from a time-independent random potential V supported on a fixed compact set in $\mathbb{R}^n$. We study an inverse problem of recovering the law of V given certain correlation data in the exterior of the potential. The wave propagation is governed by

$$\begin{aligned}(\Box - V(x))\, u(x,t,\tilde{\boldsymbol{\theta}}) &= 0,\\ u(x,t,\tilde{\boldsymbol{\theta}}) &= \sum_{j=1}^{N} \delta\left(t - x\cdot\tilde{\theta}_j\right) + u_{\mathrm{sc}}(x,t,\tilde{\boldsymbol{\theta}}),\\ u_{\mathrm{sc}}(x,t,\tilde{\boldsymbol{\theta}}) &= 0, \quad \text{for } t \ll 0,\end{aligned} \tag{1}$$

where $(x,t) \in \mathbb{R}^{n+1}$, $\tilde{\boldsymbol{\theta}} = \tilde{\boldsymbol{\theta}}^N := (\tilde{\theta}_1, \ldots, \tilde{\theta}_N)$, $\Box := \partial_t^2 - \Delta$ is the wave operator, and the potential

$$V : \mathbb{R}^n \times \Omega \to \mathbb{R},\ V(x) = V(x,\omega)$$

is a random generalized function, that is, a measurable map from the probability space Ω into a linear subspace of generalized functions which, in this paper, will be contained in the Sobolev space $H^2(\mathbb{R}^n) := W^{2,2}(\mathbb{R}^n)$ endowed with the Borel σ-algebra. Above, the incoming wave is given by a superposition of N plane waves. We shall omit the parameter $\omega \in \Omega$ from notation and write $V(x)$ instead of $V(x,\omega)$.

Given a family of directions $\theta_1, \dots, \theta_k \in \mathbb{S}^{n-1}$ which are not necessarily distinct from each other, let us denote the trajectory of the plane wave $\delta(t - \theta_j \cdot x)$, $j \in \{1, \dots, k\}$ by

$$\Sigma_j := \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : x \cdot \theta_j = t\}.$$

In the following $\Sigma_j(t)$ stands for the $(n-1)$-dimensional hyperplane in Σ_j given a fixed $t \in \mathbb{R}$. It is well-known that after suitable time $T > 0$ the potential V produces a discontinuity in the scattered field across the hyperplane $\Sigma_j(t)$ for $t > T$. More importantly, the discontinuities carry information regarding the integral values of V over the bicharacteristic lines. By setting measurement devices outside the region where the potential is almost surely supported one captures "the shadows" of the potential in different angles from the outgoing wave fronts as they collide into the detectors. After repeating the observation one can consider empirical correlations between patterns captured by separate measurement devices.

Motivated by this line of thought, we study correlations in the exterior of the potential. We assume that the observations approximate the following generalised function to a good approximation. We will refer to it as the exterior data:

$$(\mathbf{x}, \boldsymbol{\theta}) \mapsto D^k(\mathbf{x}, \boldsymbol{\theta}) = D^k_V(\mathbf{x}, \boldsymbol{\theta}) \quad \text{and for all } (\mathbf{x}, \boldsymbol{\theta}) \in \prod_{j=1}^k (\mathbb{R}^n \times \mathbb{S}^n), \tag{2}$$

where $\mathbf{x} = (x_1, \dots, x_k)$, $\boldsymbol{\theta} = (\theta_1, \dots, \theta_k)$ and

$$D^k(\mathbf{x}, \boldsymbol{\theta}) := \mathbb{E}\left(\lim_{s \to \infty} \prod_{j=1}^k [u_{sc}]_{\Sigma_j}(x_j + s\theta_j, \tilde{\boldsymbol{\theta}}) \right).$$

The notation $[u_{sc}]_{\Sigma_j}(x, \tilde{\boldsymbol{\theta}})$ stands for the jump across Σ_j,

$$[u_{sc}]_{\Sigma_j}(x, \tilde{\boldsymbol{\theta}}) = (\mathrm{Tr}^+_{\Sigma_j} - \mathrm{Tr}^-_{\Sigma_j}) u_{sc}(x, x \cdot \theta_j, \tilde{\boldsymbol{\theta}}),$$

where $\tilde{\boldsymbol{\theta}} \in \prod_{j=1}^N \mathbb{S}^{n-1}$, $N = \# \cup_{j=1}^k \{\theta_j\}$ is a parametrisation of distinct elements in $\theta_1, \dots, \theta_k$, that is, a bijection $j \mapsto \tilde{\boldsymbol{\theta}}_j$ from $\{1, \dots, N\}$ into $\bigcup_{j=1}^k \{\theta_j\}$. The traces $\mathrm{Tr}^{\pm}_{\Sigma_j}$ stand for restrictions to the boundary Σ_j from the upper and lower half-spaces

$$\{(x,t) \in \mathbb{R}^{n+1} : 0 \le \pm(t - x \cdot \theta_j),\ x \in \mathbb{R}^n \setminus \overline{B(0,R)}\}$$

and the radius R is chosen large enough so that the support of the potential is almost surely contained in the ball $B(0,R)$. The angles $\tilde{\theta}_j$, $j = 1, \dots, N$ of the incoming waves $\delta(t - x \cdot \tilde{\theta}_j)$ are considered to be distinct from each other for technical reasons. The vectors θ_j, $j = 1, \dots, k$ are not required to be distinct. We give a precise definition of the the restrictions in the beginning of the next section.

The exterior data is invariant with respect to shifts along the trajectories:

$$D^k(\mathbf{x}, \boldsymbol{\theta}) = D^k(x_1 + s_1\theta_1, \dots, x_k + s_k\theta_k, \boldsymbol{\theta}),\ s_1, \dots, s_k \in \mathbb{R} \tag{3}$$

Therefore, without loss of information, it can alternatively be given on the tangent bundle $\prod_{j=1}^k T\mathbb{S}^{n-1}$ by identifying each fiber $T_\theta \mathbb{S}^{n-1}$, with the orthogonal complement $\{\theta\}^\perp \times \{\theta\} \subset \mathbb{R}^n \times \{\theta\} = T_\theta \mathbb{R}^n$, $\{\theta\}^\perp := \{x \in \mathbb{R}^n : x \cdot \theta = 0\}$ (see Appendix). The data is then expressed as a generalised function

$$\boldsymbol{\theta} \mapsto D^k(\boldsymbol{\theta}) \in \mathcal{D}'\left(T_{\boldsymbol{\theta}} \prod_{j=1}^k \mathbb{S}^{n-1} \right),\ \boldsymbol{\theta} \in \prod_{j=1}^k \mathbb{S}^{n-1},$$

given by

$$D^k(\boldsymbol{\theta})(y_1,\dots,y_k) = D^k\big((y_1,0),\dots,(y_k,0),\boldsymbol{\theta}\big)$$

in coordinates $y_j = (y_j^1,\dots,y_j^{n-1}) \in \mathbb{R}^{n-1}$ of $T_{\theta_j}\mathbb{S}^{n-1}$, $j = 1,\dots,k$. For a smooth potential one obtains more practical form, $D^k \in C^\infty(\prod_{j=1}^k T\mathbb{S}^{n-1})$ which is applied in Proposition 2.1. In comparison with the typical far-field measurement the exterior data contains additional tangential parameters $(y_1,\dots,y_k)$. The data models measurements of statistical correlation between amplitude peaks of leading singularities which scatter from randomly varying objects as a result of interaction between the associated potential and the incident waves. The tangential parameters correspond to points in the surface of a detector plate or a film. The interpretation requires that during a single measurement the random state of the target varies slowly compared to the speed of propagation of waves (e.g. the speed of light or sound).

The model (1) can also be interpreted in the frequency domain. Taking Fourier transform with respect to time converts the model into a time-harmonic system,

$$(\Delta + \lambda^2 + V(x))\widehat{u}(x,\lambda,\tilde{\boldsymbol{\theta}}) = 0,$$

associated with angular dependency λ^2, an electric potential V, the incident wave $\widehat{u}_I(x,\lambda,\tilde{\boldsymbol{\theta}}) = \sum_{j=1}^N e^{-i\lambda x\cdot\tilde{\theta}_j}$ and the Sommerfeld radiation condition. In quantum mechanics such a model typically arises from the time-dependent Schrödinger equation,

$$\big(V(x) - \Delta\big)\Psi(x,s,\tilde{\boldsymbol{\theta}}) = i\frac{2m}{\hbar}\partial_s\Psi(x,s,\tilde{\boldsymbol{\theta}}).$$

More precisely, the stationary wave $\widehat{u}(x,\lambda,\tilde{\boldsymbol{\theta}})$ equals the spatial part $\Psi(x,0,\tilde{\boldsymbol{\theta}},\lambda)$ of the solution $\Psi(x,s,\tilde{\boldsymbol{\theta}},\lambda) = \Psi(x,0,\tilde{\boldsymbol{\theta}},\lambda)e^{-i\lambda s/\hbar}$ with energy $E = \lambda^2$ and the general solution is a superposition of these waves. Singularities of a scattered wave in the time domain appear in the frequency domain as slower decay of the Fourier transform in directions that belong to the wave front set. The leading singularity of $u_{sc}(x,t,\tilde{\boldsymbol{\theta}})$ in time corresponds to the high frequency asymptote of $\widehat{u}_{sc}(x,\lambda,\tilde{\boldsymbol{\theta}}) = \Psi(x,0,\tilde{\boldsymbol{\theta}},\lambda) - \widehat{u}_I(x,\lambda,\tilde{\boldsymbol{\theta}})$, that is, the first term of an asymptotic series expansion with respect to the variable λ. The exterior data for a potential in $C_c^\infty(\mathbb{R}^n)$ with the support almost surely contained in a ball $B(0,R)$ is the time domain counterpart of

$$\mathbb{E}\left(c_{k,n}\prod_{j=1}^k \lim_{\lambda\to\infty} e^{-i\lambda x_j\cdot\theta_j}\lambda\Psi_{sc}(x_j,0,\tilde{\boldsymbol{\theta}},\lambda)\right), \quad \mathbf{x} \in \prod_{j=1}^k \mathbb{R}^n \setminus \overline{B(0,R)}, \quad \lambda \in \mathbb{R}\setminus\{0\}$$

which can be obtained directly from the progressive wave expansion [42]. The exterior data therefore describes statistical correlations in scattering patterns produced by multiple high-energy particles interacting with the random potential. Similar interpretation of data is expected to be valid for potentials with less regularity but will not be studied here.

1.1. **The Results.** For a random field

$$V : \mathbb{R}^n \times \Omega \to \mathbb{R} : (x,\omega) \mapsto V(x,\omega)$$

we introduce the following three conditions:

(C1) $V \in H^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ almost surely
(C2) There is a compact set $K \subset \mathbb{R}^n$ such that V is supported in K almost surely
(C3) There is a constant $a > 0$ such that $\mathbb{E}e^{a\|V\|_{H^2}} < \infty$,

The compact set K in (C2) can be replaced by a ball. A field with almost surely bounded H^2-norm, $\|V\|_{H^2} \in L^\infty(\Omega)$, satisfies (C3).

Our main results are given by the following theorems.

Theorem 1.1. *Let $V : \mathbb{R}^n \times \Omega \to \mathbb{R}$ be a random field that satisfies the conditions* (C1) *and* (C2) *above. Then the k^{th} moment map $M^k \in \mathcal{E}'\left(\prod_{j=1}^k \mathbb{R}^n\right)$, given by*

$$M^k(\mathbf{x}) = \mathbb{E}\left(\prod_{j=1}^k V(x_j)\right), \tag{4}$$

is uniquely determined by the exterior data (2) *for any $k \in \mathbb{N}$.*

In particular, the proof of Theorem 1.1 provides a reconstruction strategy to recover an explicitly defined sequence M^k_ϵ, $\epsilon > 0$ of smooth functions that converges to M^k. Moreover, as Gaussian random fields are determined by their mean field and covariance function, the Theorem 1.1 yields an immediate corollary for this important class of random models.

Corollary 1.1. *If the potential V in Theorem 1.1 is a Gaussian random field, then the probability distribution of V is uniquely determined given the exterior data*

$$(\mathbf{x}, \boldsymbol{\theta}) \mapsto D^j(\mathbf{x}, \boldsymbol{\theta}) \quad \textit{for } j = 1, 2, \textit{and for all } (\mathbf{x}, \boldsymbol{\theta}) \in \prod_{j=1}^k (\mathbb{R}^n \times \mathbb{S}^n).$$

Theorem 1.1 can be applied to derive sufficient conditions for the associated laws $V_*\mathbb{P} : A \mapsto \mathbb{P}\{\omega \in \Omega : V(\cdot, \omega) \in A\}$ in $H^2(\mathbb{R}^n)$ to be unique, i.e., the only positive Borel measure related to the data:

Theorem 1.2. *Let V and W be two random fields that satisfy the conditions* (C1), (C2), (C3) *and yield the same exterior data for every $k \in \mathbb{N}$. Then the potentials have the same laws (i.e. probability distributions):*

$$V_*\mathbb{P} = W_*\mathbb{P}.$$

The proof of Theorem 1.2 relies partly on a result from [14] considering determinateness for Euclidean multivariate moment problem. This is the main reason to consider fields of the form (C3).

1.2. **Previous literature.**

A natural path for acquiring the spatial correlation data in practise is averaging a large number of independent observations of the scattered field in time. Similar problem setup often appears in wave and particle propagation in heterogeneous media. Typically, heterogeneous medium is modelled as a realization of random field with a priori known statistics. In literature, multi-scale analysis or homogenization is often utilized with the aim of capturing the effective properties of the propagation. Notice that our work does not assume any scale separation. We refer to the articles [6, 1, 27, 17, 11] for various perspectives on wave propagation (whether classical or quantum) in random media.

The origin of the randomness can also be a specific source in the considered system, see e.g. the early work [12] on inverse random source problems. Since then correlation based imaging in random source problems have been considered widely in the framework of different PDE models by Li, Bao and others [35, 3, 33, 2, 4, 34]. Other applications include telescope imaging [23] and seismic imaging [19, 20, 21, 22, 24]. Imaging in random media has also been studied by Borcea and others [5, 7, 8, 44], and for backscattering by Shevtsov [46].

Our paper provides continuation to previous work by the authors in [32, 25, 9], where the averaging procedure to estimate correlations is based on a single realization of the observed data, i.e., the random potential or boundary condition is sampled only once. Such an approach can reveal valuable information of the leading order statistics of the unknown field. However, the full probability distribution of the unknown is not recovered unlike here.

The literature on inverse scattering for deterministic potentials is rather wide and we cite here only a few works in the field. In [10], Colton and Kirsch introduced the linear sampling method to determine the support of an imperfect conductor given the far-filed of the scattered wave. Uniqueness for the inverse acoustic medium problem was proved by Nachman [38], Novikov [39], and Ramm [43]. Uniqueness for the inverse backscattering problem in a generic class of potentials was proved by Eskin and Ralston [15, 16]. Uniqueness for angularly controlled potentials has been proved by Rakesh and Uhlmann [42]. Single measurement inverse problems for the wave equation is explored by Rakesh [41] and by Liu and others [26, 36]. Use of moments in inverse problems for partial differential equations has previously been studied by Kurylev and others in [31, 30, 29].

Finally, we want to point out that there exists a variety of criteria for the moment problem to be determinate. These might provide potential alternatives for applications, where the exponential moment is not bounded. This aspect is not the focus of our work and we refer to [45] and references therein regarding such generalizations.

2. Proofs of the Results

2.1. **Preliminary Definitions.** Let us introduce some relevant notations and definitions. Given a distribution $v \in \mathcal{D}'(X)$ on a smooth manifold X let $WF(v)$ be the wave front set of v, i.e., the complement of the collection of co-vectors $(z_0, \xi_0) \in X \times (\mathbb{R}^{\dim(X)} \setminus \{0\})$ such that in some neighbourhoods $U \ni z_0$ and $V \ni \xi_0$ the decay estimate

$$\widehat{\varphi v}(\tau\xi) = O(\tau^{-m}), \text{ for } \tau \to \infty, \text{ uniformly in } \xi \in V,$$

holds for every $\varphi \in C_c^\infty(U)$ and $m \in \mathbb{N}$. Let Γ be a closed cone in $T^*(X)$ and define $\mathcal{D}'_\Gamma(X)$ as the collection of distributions $v \in \mathcal{D}'(X)$ such that $WF(v) \subset \Gamma$. Similarly, we define

$$H^s_{\Gamma,loc}(X) := H^s_{loc}(X) \cap \mathcal{D}'_\Gamma(X)$$

for $s \in \mathbb{R}$. Given a submanifold $Y \subset X$ we denote by N^*Y the conormal bundle of Y, that is, the collection of vectors $\xi \in T^*X$ such that $\langle \xi, v\rangle = 0$ for every $v \in T_{\pi(\xi)}Y$, where $\pi : T^*X \to X$ stands for the bundle projection. Given a distribution v on $\mathbb{R}^{n+1}$ with $WF(v) \cap N^*\Sigma_i = \emptyset$, the trace $\mathrm{Tr}_{\Sigma_i}(v) \in \mathcal{D}'(\Sigma_i)$ is well defined and depends continuously on v with respect to the topology of distributions (see [13]).

It is shown in the next section that u_{sc} can be split into two parts $u_{sc} = \widetilde{u}_{sc} + u_R$ where

$$\widetilde{u}_{sc}(\cdot,\cdot,\tilde{\boldsymbol{\theta}}) \in H^{-1}_{\Gamma,loc}\big((\mathbb{R}^n \setminus \overline{B(0,R)}) \times \mathbb{R}\big),$$

with

$$\Gamma := \bigcup_{j=1}^{N} N^*\Sigma_j \cup \{(x,t\ ;\ \xi,k) \in T^*\mathbb{R}^{n+1} : (x,t) \in \mathbb{R}^{n+1},\ \langle \xi, \tilde{\theta}_j\rangle = 0,\ k = 0\}, \tag{5}$$

and $u_R \in H^1_{loc}(\mathbb{R}^{n+1})$ for compactly supported $V \in H^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Given $\theta_i \in \mathbb{S}^{n-1}$, we are interested in discontinuity across Σ_i which appears in the first term of the decomposition.

Amplitude of the peak is defined by the difference

$$[u_{sc}]_{\Sigma_i}(x,\tilde{\boldsymbol{\theta}}) :=(\mathrm{Tr}^+_{\Sigma_i} - \mathrm{Tr}^-_{\Sigma_i})\widetilde{u}_{sc}(x, x\cdot\theta_i, \tilde{\boldsymbol{\theta}}) \tag{6}$$

between the two limits $\mathrm{Tr}^{\pm}_{\Sigma_i}\widetilde{u}_{sc} := \lim_{\epsilon\to 0\pm}\mathrm{Tr}_{\Sigma_i}\circ S^*_\epsilon$, where

$$S^*_\epsilon : H^{-1}_{\Gamma,loc}\big((\mathbb{R}^n\setminus\overline{B(0,R)})\times\mathbb{R}\big) \to H^{-1}_{\Gamma_\epsilon,loc}\big((\mathbb{R}^n\setminus\overline{B(0,R)})\times\mathbb{R}\big),$$

$$\Gamma_\epsilon := \{(x,t-\epsilon;v)\in T^*\mathbb{R}^{n+1} : (x,t;v)\in\Gamma\}$$

is the pull-back generated by $S_\epsilon : (x,t)\mapsto(x,t+\epsilon)$. As shown in Section 2.3, the limits exist for compactly supported potentials in $H^2(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$. It is a consequence of the trace theorem that the amplitude $[u_{sc}]$ does not depend on the choice of the decomposition

$$u_{sc} = \widetilde{u}_{sc} + u_R \in H^{-1}_{\Gamma,loc}\big((\mathbb{R}^n\setminus\overline{B(0,R)})\times\mathbb{R}\big) + H^1_{loc}(\mathbb{R}^{n+1}).$$

In particular, the formal notation $(\mathrm{Tr}^+_{\Sigma_i} - \mathrm{Tr}^-_{\Sigma_i})u_{sc}(x, x\cdot\theta_i, \tilde{\boldsymbol{\theta}})$ in the introduction makes sense.

2.2. **Unique recovery for smooth potentials.** Here we prove that arbitrary moments of the random potential are uniquely recovered by the data if we know a priori that the potential is smooth. This partial result is the basis for the full proof of Theorem 1.1 in Section 2.3.

Proposition 2.1. *Let $V:\mathbb{R}^n\times\Omega\to\mathbb{R}$ be a random field that such that the condition* (C2) *holds and $V\in C^\infty(\mathbb{R}^n)$ almost surely. Then $M^k\in\mathcal{E}'\left(\prod_{j=1}^k\mathbb{R}^n\right)$, given by* (4)*, is uniquely determined by the exterior data* (2) *for any $k\in\mathbb{N}$.*

Proof. By linearity and uniqueness of the solution the scattered wave is of the form

$$u_{sc}(x,t,\tilde{\boldsymbol{\theta}}) = \sum_{j=1}^N v_{sc}(x,t,\tilde{\theta}_j),$$

where $v_{sc}(x,t,\theta)$ is the scattered part of $v(x,t,\theta)$, defined by

$$\begin{aligned}(\Box - V(x))v(x,t,\theta) &= 0,\\ v(x,t,\theta) &= \delta(t-x\cdot\theta) + v_{sc}(x,t,\theta),\\ v_{sc}(x,t,\theta) &= 0, \quad \text{for } t=-\mathrm{diam}(K),\end{aligned}$$

for $(x,t,\theta)\in\mathbb{R}^{n+1}\times\mathbb{S}^{n-1}$. Let $a_\alpha(x,\theta)\in C^\infty(\mathbb{R}^n\times\mathbb{S}^{n-1})$, $\alpha\in\mathbb{N}\setminus\{0\}$, be the coefficients of the following asymptotic expansion:

$$v_{sc}(x,t,\theta) = \frac{1}{2}a_1(x,\theta)H(t-x\cdot\theta) + \frac{1}{2}\sum_{\alpha=1}^s a_{\alpha+1}(x,\theta)(t-x\cdot\theta)_+^\alpha \mod C^{s+2}, \tag{7}$$

where the identity is up to functions in $C^{s+2}(\mathbb{R}^{n+1})$ and H is the Heaviside step function,

$$H(x) = \begin{cases}1, & \text{for } x\geq 0\\ 0, & \text{for } x<0.\end{cases}$$

We recall from [42, 40, 47] that for a smooth compactly supported potential the expansion exists and is given recursively by

$$\alpha\theta\cdot\nabla a_{\alpha+1}(x,\theta) = \frac{1}{2}\big(\Delta + V(x)\big)a_\alpha(x,\theta),\ \alpha = 1,2,3,\dots$$

and

$$\theta\cdot\nabla a_1(x,\theta) = V(x).$$

Due to the zero initial value of the scattered wave, the first coefficient equals the line integral:

$$a_1(x,\theta) = \int_{-\infty}^{0} V(x+s\theta)ds.$$

Therefore, we have the identity

$$\lim_{s\to\infty} a_1(x+s\eta,\theta) = \begin{cases} 0, & \eta \neq \theta \\ \int_{\mathbb{R}} V(x+s\theta)ds, & \eta = \theta. \end{cases},$$

for $\eta \in \mathbb{S}^{n-1}$. A restriction of $v_{sc}(x,t,\theta)$ to the boundary $t = x\cdot\theta$ at any distant point along the ray $x+s\theta$, $s\in\mathbb{R}$ is simply the associated ray transform of the potential. Applying (7) implies

$$[u_{sc}]_{\Sigma_i}(x+s\theta_i,\tilde{\boldsymbol{\theta}}) = \frac{1}{2}a_1(x+s\theta_i,\theta_i)$$

Therefore, the data at points $\mathbf{x} \in \prod_{j=1}^k \mathbb{R}^n$ in directions $\boldsymbol{\theta} \in \prod_{j=1}^k \mathbb{S}^{n-1}$ satisfies

$$\begin{aligned} D^k(\mathbf{x},\boldsymbol{\theta}) &= \mathbb{E}\Big(\lim_{s\to\infty}\prod_{j=1}^k [u_{sc}]_{\Sigma_1}(x_j+s\theta_j,\tilde{\boldsymbol{\theta}})\Big) \\ &= \frac{1}{2^k}\mathbb{E}\Big(\prod_{j=1}^k \int_{\mathbb{R}} V(x_j+s_j\theta_j)ds_j\Big) \\ &= \frac{1}{2^k}\int_{\mathbb{R}^k}\mathbb{E}\Big(\prod_{j=1}^k V(x_j+s_j\theta_j)\Big)ds_1\dots ds_k \\ &= \frac{1}{2^k}\int_{L(\mathbf{x},\boldsymbol{\theta})}\mathbb{E}\Big(\prod_{j=1}^k V(z_j)\Big)dl(z_1,\dots,z_k), \end{aligned} \tag{8}$$

where $L(\mathbf{x},\boldsymbol{\theta})$ is the affine subspace

$$L(\mathbf{x},\boldsymbol{\theta}) := \{\mathbf{x}+\boldsymbol{h} \in \prod_{j=1}^k \mathbb{R}^n : \boldsymbol{h} \in H_{\boldsymbol{\theta}}\}, \tag{9}$$

where

$$H_{\boldsymbol{\theta}} := \{(s_1\theta_1,\dots,s_k\theta_k) \in \prod_{j=1}^k \mathbb{R}^n : s_1,\dots,s_k \in \mathbb{R}\},$$

and $dl(z_1,\dots,z_k)$ denotes the pull-back volume form which is induced from the canonical volume form via the inclusion map $L(\mathbf{x},\boldsymbol{\theta}) \hookrightarrow \prod_{j=1}^k \mathbb{R}^n$.

The main idea of the next step is as follows: Based on the identity (8) and the fact that every $(nk-1)$-dimensional shifted hyperplane in $\prod_{j=1}^k \mathbb{R}^n$ is obtained by stacking up spaces $L(\mathbf{x},\boldsymbol{\theta})$ with different parameters $\mathbf{x}$, $\boldsymbol{\theta}$ we can reconstruct the Radon transform of the k^{th} moment map, $M^k(\mathbf{x}) := \mathbb{E}\Big(\prod_{j=1}^k V(x_j)\Big)$ from the data. That is to say, for arbitrary

$(r, \boldsymbol{\eta}) \in \mathbb{R} \times \mathbb{S}^{nk-1}$ one divides the hyperplane

$$\Gamma(r, \boldsymbol{\eta}) := \left\{ \boldsymbol{z} \in \prod_{j=1}^{k} \mathbb{R}^n : \boldsymbol{z} \cdot \boldsymbol{\eta} = r \right\} \tag{10}$$

into distinct subspaces of the form $L(\mathbf{x}, \boldsymbol{\theta})$ given in equation (9), then applies the data together with (8) to obtain integrals $\int_L M^k dl$ of the moment map over each of the subspaces and finally computes the superposition of them.

Let us formulate this idea rigorously. First, we provide a representation of the hyperplane $\Gamma(r, \boldsymbol{\eta})$ as an orthogonal decomposition involving the subspace L in (9). Construct arbitrary smooth functions

$$\theta_j : \mathbb{S}^{nk-1} \to \mathbb{S}^{n-1}$$

to satisfy

$$\eta_j \cdot \theta_j(\boldsymbol{\eta}) = 0$$

for any $j = 1, ..., k$. Next, let $\boldsymbol{\theta} := (\theta_1, \dots, \theta_k)$ and define the set

$$P(\boldsymbol{\eta}, \boldsymbol{\theta}) \subset \prod_{j=1}^{k} \mathbb{R}^n$$

to be the maximal linear subspace orthogonal to $\boldsymbol{\eta}$ and vectors

$$(0, \dots, 0, \underbrace{\theta_j(\boldsymbol{\eta})}_{j^{\text{th}} \text{ slot}}, 0, \dots, 0) \in \prod_{j=1}^{k} \mathbb{R}^n, \ j = 1, \dots, k,$$

simultaneously. Clearly, each $\mathbf{x} \in \Gamma(r, \boldsymbol{\eta})$ is uniquely written in form $\mathbf{x} = \mathbf{x}^L + \mathbf{x}^P$, where

$$\mathbf{x}^L = (x_1^L, \dots, x_k^L) \in L(r\boldsymbol{\eta}, \boldsymbol{\theta}) := L(r\eta_1, \dots, r\eta_k, \boldsymbol{\theta}),$$

$x_j^L \in \mathbb{R}^n$ for every $j = 1, \dots, k$, and

$$\mathbf{x}^P = (x_1^P, \dots, x_k^P) \in P(\boldsymbol{\eta}, \boldsymbol{\theta}),$$

where $x_j^P \in \mathbb{R}^n$ for $j = 1, \dots, k$. In consequence, we have that

$$\Gamma(r, \boldsymbol{\eta}) = L(r\boldsymbol{\eta}, \boldsymbol{\theta}) \oplus P(\boldsymbol{\eta}, \boldsymbol{\theta}).$$

Now, by identity (8), the Radon transform of the moment function M^k at $(r, \boldsymbol{\eta}) \in \mathbb{R} \times \mathbb{S}^{nk-1}$ takes the form

$$\begin{aligned}
R[M^k](r, \boldsymbol{\eta}) &= \int_{\Gamma(r,\boldsymbol{\eta})} M^k(\mathbf{x}) d\nu(\mathbf{x}) \\
&= \int_{P(\boldsymbol{\eta},\boldsymbol{\theta})} \int_{L(r\boldsymbol{\eta},\boldsymbol{\theta})} \mathbb{E}\Big(\prod_{j=1}^{k} V(x_j^L + x_j^P) \Big) dl(\mathbf{x}^L) dP(\mathbf{x}^P) \\
&= \int_{P(\boldsymbol{\eta},\boldsymbol{\theta})} \int_{L(r\boldsymbol{\eta}+\mathbf{x}^P,\boldsymbol{\theta})} \mathbb{E}\Big(\prod_{j=1}^{k} V(z_j) \Big) dl(\boldsymbol{z}) dP(\mathbf{x}^P) \\
&= 2^k \int_{P(\boldsymbol{\eta},\boldsymbol{\theta})} D^k(r\boldsymbol{\eta} + \mathbf{x}^P, \boldsymbol{\theta}(\boldsymbol{\eta})) dP(\mathbf{x}^P), \qquad (11)
\end{aligned}$$

where dP, dl, and $d\nu$ are the canonical volume forms, induced by inclusions into $\prod_{j=1}^k \mathbb{R}^n$. Let $\iota_{\boldsymbol{\theta}(\boldsymbol{\eta})} : T_{\boldsymbol{\theta}(\boldsymbol{\eta})} \prod_{j=1}^k \mathbb{S}^{n-1} \hookrightarrow T \prod_{j=1}^k \mathbb{S}^{n-1}$ be the trivial inclusion. Considering the exterior data as a function $D^k \in C_c^\infty\big(T \prod_{j=1}^k \mathbb{S}^{n-1}\big)$ reduces (11) to

$$R[M^k](r,\boldsymbol{\eta}) = \big\langle \delta\big(r - \boldsymbol{\eta}\cdot \iota_{\boldsymbol{\theta}(\boldsymbol{\eta})}(\boldsymbol{v})\big), D^k \circ \iota_{\boldsymbol{\theta}(\boldsymbol{\eta})}(\boldsymbol{v})\big\rangle = \big\langle (\iota_{\boldsymbol{\theta}(\boldsymbol{\eta})})_* \delta\big(r - \boldsymbol{\eta}\cdot \iota_{\boldsymbol{\theta}(\boldsymbol{\eta})}\big), D^k\big\rangle, \tag{12}$$

where we denote $\boldsymbol{v} \in T_{\boldsymbol{\theta}(\boldsymbol{\eta})} \prod_{j=1}^k \mathbb{S}^{n-1}$ and identify $T_\theta \mathbb{S}^{n-1} = \{\theta\}^\perp \times \{\theta\} \subset \mathbb{R}^n \times \mathbb{R}^n$. Finally, to obtain M^k from the transformed quantity one applies the Radon inversion formula,

$$\Delta^{(nk-1)/2} R^* R = id,$$

where R^* is the adjoint of the Radon transform and

$$\Delta^{(nk-1)/2} f(x) := \int_{\mathbb{R}^{nk}} e^{ix\cdot\xi} |\xi|^{nk-1} \widehat{f}(\xi) d\xi.$$

More precisely, by (12), the Radon inversion formula implies

$$\Psi^k\{D^k\}(\mathbf{x}) = M^k(\mathbf{x})$$

where

$$\Psi^k := \Delta^{(n-1)/2} R^t A^k, \tag{13}$$

and

$$A^k : C_c^\infty\left(\prod_{j=1}^k T\mathbb{S}^{n-1}\right) \to C^\infty(\mathbb{R}\times \mathbb{S}^{kn-1}),\ A^k(r,\boldsymbol{\eta}) := (\iota_{\boldsymbol{\theta}(\boldsymbol{\eta})})_* \delta\big(r - \boldsymbol{\eta}\cdot\iota_{\boldsymbol{\theta}(\boldsymbol{\eta})}\big).$$

This concludes the proof. □

We will refer to the map Ψ^k as a reconstruction operator for smooth potentials.

2.3. **Proof of Theorem 1.1, The General Setting.** We shall first study the system for a single random parameter $\omega_0 \in \Omega$. Fix the potential $V(x) = V(x,\omega_0) \in L^\infty(\mathbb{R}^n) \cap H^2(\mathbb{R}^n)$ that is supported in $B(0,R)$, $R > 0$ and write the solution of (1) with a possibly non-smooth potential in the form

$$u_{sc}(x,t,\tilde{\boldsymbol{\theta}}) = \sum_{j=1}^N \frac{1}{2} a_1(x,\tilde{\theta}_j) H(t - x\cdot\tilde{\theta}_j) + u_R(x,t,\tilde{\boldsymbol{\theta}}), \tag{14}$$

where

$$a_1(x,\tilde{\theta}_j) := \int_{-\infty}^0 V(x + s\tilde{\theta}_j) ds$$

and $u_R(x,t,\tilde{\boldsymbol{\theta}})$ is the residual term. We will need the following lemmas:

Lemma 2.1. *Let $V(x) \in L^\infty(\mathbb{R}^n) \cap H^2(\mathbb{R}^n)$ be supported in $B(0,R)$. It holds that $x \mapsto H(t - x\cdot\theta)\Delta a_1(x,\theta) \in L^2(\mathbb{R}^n)$.*

Proof. Consider $\Delta a_1(x,\theta)$ as a linear functional

$$\Delta a_1(\cdot,\theta) : L^2(\mathbb{R}^n) \to \mathbb{R} : \phi(x) \mapsto \int_{-\infty}^0 \int_{\mathbb{R}^n} \Delta V(x + s\theta)\phi(x) dx ds.$$

It follows by the Cauchy–Schwarz inequality that

$$\begin{aligned}
|\langle H(t-x\cdot\theta)\Delta a_1(x,\theta),\phi(x)\rangle| &= \left|\int_{-\infty}^{0}\int_{\mathbb{R}^n}\Delta V(x+s\theta)\phi(x)H(t-x\cdot\theta)dxds\right| \\
&\leq \int_{-t-2R}^{0}\int_{\mathbb{R}^n}|\Delta V(x+s\theta)||\phi(x)|dxds \\
&\leq \int_{-t-2R}^{0}\|\Delta V(\cdot+s\theta)\|_{L^2(\mathbb{R}^n)}\|\phi\|_{L^2(\mathbb{R}^n)}ds \\
&= (2R+t)\|\Delta V\|_{L^2(\mathbb{R}^n)}\|\phi\|_{L^2(\mathbb{R}^n)},
\end{aligned}$$

that is, the functional $H(t-x\cdot\theta)\Delta a_1(x,\theta): L^2(\mathbb{R}^n)\to\mathbb{R}$ is bounded. The claim follows by duality.

□

Lemma 2.2. *The residual term u_R in* (14) *satisfies $u_R(\cdot,t,\tilde{\boldsymbol{\theta}})\in H^1_{loc}(\mathbb{R}^{n+1})$.*

Proof. Let $T>R$. Substituting the ansatz (14) into (1) yields

$$(\Box - V(x))u_R(x,t,\tilde{\boldsymbol{\theta}}) = \frac{1}{2}\sum_{j=1}^{N}H(t-x\cdot\tilde{\theta}_j)(V(x)+\Delta)a_1(x,\tilde{\theta}_j). \tag{15}$$

The idea is to construct a sufficiently regular solution of the equation (15) by extending a local solution that satisfy initial and boundary conditions of the residual. The right side of (15) is a sum of L^2-functions supported in $B(0,R+t)$ for each t. In particular,

$$(\Box - V(x))w_R(x,t,\tilde{\boldsymbol{\theta}})\in L^1\big([-R,T];L^2(B(0,r))\big) \tag{16}$$

where w_R refers to a solution of (15) in a smaller set $(x,t)\in B(0,r)\times[-R,T]$, with sufficiently large $r>2R+T$. As no scattering occurs before the incident wave hits the potential, we consider the initial conditions

$$\begin{cases} w_R(x,t,\tilde{\boldsymbol{\theta}}) = u_R(x,t,\tilde{\boldsymbol{\theta}}) = 0, & t=-R, \\ \partial_t w_R(x,t,\tilde{\boldsymbol{\theta}}) = \partial_t u_R(x,t,\tilde{\boldsymbol{\theta}}) = 0, & t=-R. \end{cases} \tag{17}$$

In $\big(\mathbb{R}^n\setminus\overline{B(0,R)}\big)\times\mathbb{R}$ the scattered wave satisfies the free wave equation $\Box u_{sc}=0$ so due to the initial values $u_{sc}(x,t,\tilde{\boldsymbol{\theta}})=0=\partial_t u_{sc}(x,t,\tilde{\boldsymbol{\theta}})$, $t<-R$, and unit propagation speed of disturbances the scattered wave is supported within the set

$$C_R := \{(x,t)\in\mathbb{R}^{n+1} : x\in B(0,2R+t)\}$$

which contains also the support of $a(x,\theta)H(t-x\cdot\theta)$ for each $\theta\in\mathbb{S}^{n-1}$. Consequently, $\operatorname{supp}(u_R)\subset C_R$. It is therefore suitable to associate w_R with the zero boundary conditions,

$$w_R(x,t,\tilde{\boldsymbol{\theta}})=0,\ \text{for } (x,t)\in\partial B(0,r)\times[-R,T]$$

By [37, Ch. IV §3] the initial and boundary conditions above ensure existence and uniqueness of $w_R(x,t,\tilde{\boldsymbol{\theta}})\in H^1(B(0,r)\times[-R,T])$. Within the space $\big(B(0,r)\setminus B(0,T+R)\big)\times[-R,T]$ the function w_R satisfies the free wave equation, $\Box w_R=0$, so applying the energy estimate together with zero initial and boundary conditions one extends the domain of w_R into $\mathbb{R}^{n+1}$.

This can be seen by first applying the estimate to find a zero extension $\overline{w}_R \in H^1(\mathbb{R}^n \times [-R,T])$ of w_R with respect to variable x and, second, to extend the domain to the whole timeline $t \in \mathbb{R}$ in the weak sense. Finally, uniqueness of u_{sc} implies that the extension equals u_R. □

Due to regularity of u_R the peak $[u_R]_{\Sigma_i}$ vanishes. As in the smooth case, the amplitude of the peak singularity carried by the scattered wave reduces to $\frac{1}{2}a_1(\cdot,\theta_i)$ on distant regions within the boundary Σ_i:

$$\begin{aligned}[u_{sc}]_{\Sigma_i}(x+s\theta_i,\tilde{\boldsymbol{\theta}}) &= \left[\sum_{j=1}^N \frac{1}{2}a_1(x,\tilde{\theta}_j)H(t-x\cdot\tilde{\theta}_j)\right]_{\Sigma_i}(x+s\theta_i,\tilde{\boldsymbol{\theta}}) \\ &= \frac{1}{2}a_1(x+s\theta_i,\theta_i),\end{aligned}$$

Consequently, we have

$$\lim_{s\to\infty}[u_{sc}]_{\Sigma_i}(x+s\theta_i) = \frac{1}{2}\int_{\mathbb{R}} V(x+s\theta_i)ds.$$

We shall now apply the previous observations to the probabilistic setting. As earlier, define the data for k base points by

$$D^k(\mathbf{x},\boldsymbol{\theta}) := \mathbb{E}\left(\lim_{s\to\infty}\prod_{j=1}^k [u_{sc}]_{\Sigma_j}(x_j+s\theta_j,\tilde{\boldsymbol{\theta}})\right). \tag{18}$$

For $k=1$, taking convolution with the mollifier $\phi_\epsilon(x) = \epsilon^{-n}\phi(\epsilon^{-1}x)$, $\phi \in C_c^\infty(\mathbb{R}^n)$, $\int_{\mathbb{R}^n}\phi(x)dx = 1$ yields

$$\begin{aligned}[\phi_\epsilon * D^1(\cdot,\theta)](x) &= \int_{\mathbb{R}^n}\phi_\epsilon(z)\mathbb{E}\left(\frac{1}{2}\int_{\mathbb{R}} V(x-z+s\theta)ds\right)dz \\ &= \mathbb{E}\left(\frac{1}{2}\int_{\mathbb{R}} V_\epsilon(x+s\theta)ds\right) \\ &= D^1_\epsilon(x,\theta), \qquad (19)\end{aligned}$$

where $D^1_\epsilon(x,\theta)$ is the exterior data associated to the mollified smooth potential $V_\epsilon(x) := (\phi_\epsilon * V)(x)$. Similarly, one derives

$$[\phi_\epsilon^k * D^k(\cdot,\boldsymbol{\theta})](\mathbf{x}) = D^k_\epsilon(\mathbf{x},\boldsymbol{\theta}),$$

where $\phi_\epsilon^k(z_1,\ldots,z_k) := \phi_\epsilon(z_1)\cdots\phi_\epsilon(z_k)$, and $D_\epsilon(\mathbf{x},\boldsymbol{\theta})$ is the data (8) at $\mathbf{x} \in \prod_{j=1}^k \mathbb{R}^n$ in directions $\boldsymbol{\theta} \in \prod_{j=1}^k \mathbb{S}^{n-1}$ for the smooth potential V_ϵ. The mollified moment M^k_ϵ approximates

M^k in the sense of generalized functions:

$$\begin{aligned}
\lim_{\epsilon\to 0} M_\epsilon^k(\mathbf{x}) &= \lim_{\epsilon\to 0}\mathbb{E}\int_{\prod_{j=1}^k \mathbb{R}^n} \phi_\epsilon^k(x_1-z_1,\dots,x_k-z_k)V(z_1)\cdots V(z_k)dz \\
&= \lim_{\epsilon\to 0}\int_{\prod_{j=1}^k \mathbb{R}^n} \phi_\epsilon^k(x_1-z_1,\dots,x_k-z_k)\mathbb{E}(V(z_1)\cdots V(z_k))dz \\
&= \lim_{\epsilon\to 0}\phi_\epsilon^k * M^k(\mathbf{x}) \qquad (20)\\
&= \delta_0 * M^k(\mathbf{x}) \\
&= M^k(\mathbf{x}) \in \mathcal{D}'\left(\prod_{j=1}^k \mathbb{R}^n\right),
\end{aligned}$$

see e.g. [18, Ch 5]. The limit in (20) is to be understood by means of sequential convergence in the topology of distributions.

In summary, we obtain a reconstruction strategy which consists of the following three steps:

(Step 1) **Regularisation of the data:** The operator

$$\Phi^k : C^\infty\Big(\prod_{j=1}^k \mathbb{S}^{n-1}; \mathcal{D}'\Big(\prod_{j=1}^k \mathbb{R}^n\Big)\Big) \to C_c^\infty\Big((0,\infty)\times T\prod_{j=1}^k \mathbb{S}^{n-1}\Big),$$

given by

$$\Phi^k(v)(\epsilon,\boldsymbol{y}) := [\phi_\epsilon * v(\cdot,\boldsymbol{\theta})](\boldsymbol{y}),\ \text{for } \epsilon\in(0,\infty),\ \boldsymbol{y}\in T_{\boldsymbol{\theta}}\prod_{j=1}^k \mathbb{S}^{n-1},$$

transforms the exterior data into a parametrised family of data which corresponds to the regularised potentials:

$$\Phi^k(D^k)(\epsilon,\boldsymbol{y}) = D_\epsilon^k(\boldsymbol{y},\boldsymbol{\theta}),\ \text{for } \boldsymbol{y}\in T_{\boldsymbol{\theta}}\prod_{j=1}^k \mathbb{S}^{n-1}.$$

Above, the regularised data is identified with a smooth function on $T\prod_{j=1}^k \mathbb{S}^{n-1}$ according to the invariance (3).

(Step 2) **Reconstruction of moments from the regularised data:** Let Ψ^k be the reconstruction operator for smooth potentials, given by (13). The associated operator

$$\tilde\Psi^k : C_c^\infty\Big((0,\infty)\times T\prod_{j=1}^k \mathbb{S}^{n-1}\Big) \to C_c^\infty\Big((0,\infty)\times \prod_{j=1}^k \mathbb{R}^n\Big),$$

defined by

$$\tilde\Psi^k f(\epsilon,\mathbf{x}) := [\Psi^k f(\epsilon,\cdot)](\mathbf{x}),\ \mathbf{x}\in\prod_{j=1}^k \mathbb{R}^n,\ \epsilon\in(0,\infty)$$

transforms the regularised data into the corresponding family of moment maps, i.e.,

$$\tilde\Psi^k\Phi^k D^k(\epsilon,\mathbf{x}) = M_\epsilon^k(\mathbf{x})$$

(Step 3) **High-resolution limit:** After the first two steps, the moment map $M^k \in \mathcal{E}'\big(\prod_{j=1}^k \mathbb{R}^n\big)$ is obtained by taking the limit $\epsilon \longrightarrow 0$ within the space of distributions, that is,

$$\lim_{\epsilon\to 0} \Psi^k \Phi^k D^k(\epsilon, \mathbf{x}) = M^k(\mathbf{x}),$$

2.4. **Proof of Theorem 1.2.** Let $\mathbb{R}^{\mathbb{N}}$ stand for the set of infinite sequences $(x_j)_{j=1}^{\infty}$, $x_j \in \mathbb{R}$, $j \in \mathbb{N}$ endowed with the smallest topology for which the coordinate projections are continuous. We recall that $\mathbb{R}^{\mathbb{N}}$ is a Polish space. Also, let ℓ^2 stand for the space of all sequences bounded in the 2-norm. Let $\iota : \ell^2 \hookrightarrow \mathbb{R}^{\mathbb{N}}$ stand for the trivial inclusion and define

$$\iota^{-1}\mathcal{B}(\mathbb{R}^{\mathbb{N}}) := \sigma\{\ell^2 \cap B : \ B \in \mathcal{B}(\mathbb{R}^{\mathbb{N}})\}.$$

Before continuing to the proof we record the following lemma:

Lemma 2.3. *The Borel algebra of ℓ^2 is generated by ι, that is, $\mathcal{B}(\ell^2) = \iota^{-1}\mathcal{B}(\mathbb{R}^{\mathbb{N}})$.*

Proof. Let $\mathcal{F}^j = (\pi^{\{j\}})^{-1}\mathcal{B}(\mathbb{R})$ be the σ-algebra generated by the coordinate projection $\pi^{\{j\}} : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$, for each $j \in \mathbb{N}$. Since the Borel algebra of $\mathbb{R}$ is generated by open sets, preimages of the form $(\pi^{\{j\}})^{-1}U$, where $U \subset \mathbb{R}$ is open, generate $\mathcal{F}^j$. As the union $\bigcup_{j\in\mathbb{N}} \mathcal{F}_j$ generates $\mathcal{B}(\mathbb{R}^{\mathbb{N}})$, the algebra $\iota^{-1}\mathcal{B}(\mathbb{R}^{\mathbb{N}})$ is generated by the pre-images, $\iota^{-1}(\pi^{\{j\}})^{-1}U$, $j \in \mathbb{N}$, which are open in ℓ^2 by continuity of the coordinate projections $(\pi^{\{j\}} \circ \iota)$, $j \in \mathbb{N}$. In particular, $\iota^{-1}\mathcal{B}(\mathbb{R}^{\mathbb{N}}) \subset \mathcal{B}(\ell^2)$. On the other hand, each closed ball $\overline{B}_r$ of ℓ^2 with radius $r > 0$, centered at $f \in \ell^2$, is the limit set

$$\overline{B}_r = \bigcap_{m=1}^{\infty} \Big\{ g \in \mathbb{R}^{\mathbb{N}} : \sum_{j=1}^{m} |f_j - \pi^{\{j\}} g|^2 \in [0, r^2] \Big\}, \ f_j := \pi^{\{j\}}\iota f$$

which is measurable in $\mathbb{R}^{\mathbb{N}}$ as a countable intersection of measurable sets. Consequently, as every open set of a separable metric space is a countable union of closed balls, the Borel algebra of $\mathbb{R}^{\mathbb{N}}$ contains the topology of ℓ^2 which implies $\mathcal{B}(\ell^2) \subset \iota^{-1}\mathcal{B}(\mathbb{R}^{\mathbb{N}})$. □

We shall now continue to the proof. Let the random field $V \in H^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ satisfy

$$\mathbb{E}e^{a\|V\|_{H^2(\mathbb{R}^n)}} < \infty. \tag{21}$$

Fix an orthonormal basis φ_j, $j \in \mathbb{N}$ of $H^2(\mathbb{R}^n)$ and let $\mu^J(V_{j_1}, \dots, V_{j_k})$ be the distribution

$$\mu^J(V_{j_1}, \dots, V_{j_k}) := (V_{j_1}, \dots, V_{j_k})_*\mathbb{P} : \mathcal{B}(\mathbb{R}^k) \to [0, 1],$$

with

$$(V_{j_1}, \dots, V_{j_k})_*\mathbb{P}(B) := \mathbb{P}((V_{j_1}, \dots, V_{j_k}) \in B),$$

generated by the finite collection of random variables $V_l := \langle V, \varphi_l \rangle$, $l \in J = \{j_1, \dots, j_k\} \subset \mathbb{N}$. By definition, the moments satisfy

$$\langle M^k, \varphi_{j_1} \otimes \cdots \otimes \varphi_{j_k} \rangle = \mathbb{E}(\langle V, \varphi_{j_1} \rangle \cdots \langle V, \varphi_{j_k} \rangle) = \mathbb{E}(V_{j_1} \cdots V_{j_k})$$

for every $j_1, \dots, j_k \in \mathbb{N}$. These quantities are determined from the data by Theorem 1.1. Since,

$$e^{a|(V_{j_1}, \dots, V_{j_k})|} \le e^{a\|(V_j)_{j=1}^{\infty}\|_{\ell^2}} = e^{a\|V\|_{H^2(\mathbb{R}^n)}},$$

the condition (21) yields $\mathbb{E}e^{a|(V_1, \dots, V_k)|} < \infty$, implying that the moments, and hence the data uniquely describes each measure $\mu^J(V_1, \dots, V_k)$ with finite $J \subset \mathbb{N}$, as shown in [14].

Suppose that $W \in H^2(\mathbb{R}^n)$ is a random field that satisfies (21) and yields the same data as V. In particular,

$$\mu^J(V_{j_1}, \ldots, V_{j_k}) = \mu^J(W_{j_1}, \ldots, W_{j_k}),$$

for every finite $J = \{j_1, \ldots, j_k\}$. It is a consequence of the Kolmogorov extension theorem [28, Thm. 6.16] that there is an unique probability distribution

$$\mu^{\mathbb{N}} : \mathcal{B}(\mathbb{R}^{\mathbb{N}}) \to [0, 1]$$

in $\mathbb{R}^{\mathbb{N}}$, satisfying $\mu^J(V_{j_1}, \ldots, V_{j_k}) = \pi^J_* \mu^{\mathbb{N}}$ for any finite $J \subset \mathbb{N}$ where $\pi^J : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{|J|}$ is the natural projection.

Finally, we can show that the probability distributions $V_*\mathbb{P}$ and $W_*\mathbb{P}$ equal as measures on Borel sets of ℓ^2. Let $I : H^2(\mathbb{R}^n) \to \ell^2$ be the isometry $I := (\langle \cdot, \varphi_j \rangle)_{j=1}^{\infty}$. It enough to prove the claim for elements $I(V)$ and $I(W)$. By uniqueness of $\mu^{\mathbb{N}}$,

$$\iota_* I(V)_*\mathbb{P} = \mu^{\mathbb{N}} = \iota_* I(W)_*\mathbb{P},$$

that is,

$$I(V)_*\mathbb{P}(\iota^{-1}A) = I(W)_*\mathbb{P}(\iota^{-1}A)$$

for every $A \in \mathcal{B}(\mathbb{R}^{\mathbb{N}})$. Thus we obtain $I(V)_*\mathbb{P} = I(W)_*\mathbb{P}$ by Lemma 2.3. This completes the proof of Theorem 1.2.

3. Appendix: Exterior Data on $\prod_{j=1}^k T\mathbb{S}^{n-1}$

We shall show in detail how the representation of the exterior data $D^k(\mathbf{x}, \boldsymbol{\theta})$ on the tangent bundle $T \prod_{j=1}^k \mathbb{S}^{n-1}$ is constructed. Below, $\{\theta\}^\perp := \{x \in \mathbb{R}^n : x \cdot \theta = 0\}$ for $\theta \in \mathbb{S}^{n-1}$.

Lemma 3.1. *The vector bundles $T\mathbb{S}^{n-1}$ and $\bigcup_{\theta \in \mathbb{S}^{n-1}} \{\theta\}^\perp \times \{\theta\} \to \mathbb{S}^{n-1}$ with the trivial projection $pr_2(v, \theta) := \theta$ are isomorphic. The isomorphism is obtained from the tangent of the inclusion $\mathbb{S}^{n-1} \hookrightarrow \mathbb{R}^n$ by restricting the codomain.*

Proof. By expressing each tangent vector $v \in T_\theta \mathbb{S}^{n-1}$ as an equivalence class of curves $\gamma : (-\epsilon, \epsilon) \to \mathbb{S}^{n-1}$ with $\gamma(0) = \theta$, $\dot{\gamma}(0) = v$, the tangent map $T\iota : T\mathbb{S}^{n-1} \hookrightarrow T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$ associated to the inclusion $\iota : \mathbb{S}^{n-1} \hookrightarrow \mathbb{R}^n$ takes the form $T\iota[\gamma] = [\iota \circ \gamma]$. As $(\iota \circ \gamma)'(0)$ is perpendicular to the base point $\iota \circ \gamma(0)$, we obtain $T\iota(T_\theta \mathbb{S}^{n-1}) \subset \{\theta\}^\perp \times \{\theta\}$ which, together with injectivity and the rank-nullity theorem, implies that $T\iota$ defines a bundle isomorphism

$$\begin{array}{ccc} T\mathbb{S}^{n-1} & \xrightarrow{T\iota} & \bigcup_{\theta \in \mathbb{S}^{n-1}} \{\theta\}^\perp \times \{\theta\} \\ & \searrow & \downarrow \mathrm{pr}_2 \\ & & \mathbb{S}^{n-1} \end{array}$$

Note the untypical order of the base point θ and the fiber space $\{\theta\}^\perp$ in $\{\theta\}^\perp \times \{\theta\}$. □

The exterior data can be rewritten as a generalised function $\boldsymbol{\theta} \mapsto D^k(\boldsymbol{\theta}) \in \mathcal{D}'(T_{\boldsymbol{\theta}} \prod_{j=1}^k \mathbb{S}^{n-1})$ of the bundle $T \prod_{j=1}^k \mathbb{S}^{n-1}$ by setting

$$\langle D^k(\boldsymbol{\theta}), \varphi \rangle = \left\langle D^k, \varphi \circ (T\iota)^{-1}|_{\prod_{j=1}^k \{\theta_j\}^\perp \times \{\theta_j\}} \right\rangle, \ \varphi \in C_c^\infty \left(T_{\boldsymbol{\theta}} \prod_{j=1}^k \mathbb{S}^{n-1} \right)$$

where—with abuse of notation—we denote by $T\iota$ also the associated isomorphism

$$\prod_{j=1}^{k} T\mathbb{S}^{n-1} \to \prod_{j=1}^{k} \bigcup_{\theta\in\mathbb{S}^{n-1}} \{\theta\}^{\perp} \times \{\theta\},$$

generated by the tangent map above. It is a consequence of the invariance (3) that no information is lost in the identification. That is, the lift of $D^k(\cdot,\boldsymbol{\theta})$ to the space $T_{\boldsymbol{\theta}} \prod_{j=1}^{k} \mathbb{S}^{n-1} = \prod_{j=1}^{k} \{\theta_j\}^{\perp} \times \{\theta_j\}$ is well defined and unique for each $\boldsymbol{\theta} \in \prod_{j=1}^{k} \mathbb{S}^{n-1}$.

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