Teaching an Introductory Course in Logic to Undergraduate Students Using Extreme Apprenticeship Method

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Helsinki April 3, 2012

Master's thesis

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This master’s thesis explores the use of the Extreme Apprenticeship method in teaching the course Logic I to undergraduates. The Extreme Apprenticeship method derives from modern interpretations of apprenticeship-based teaching where teaching is conducted in a student centered fashion.

Passive activities, such as listening to lectures, are reduced while the active work of students is increased in the form of weekly exercises. At the same time the students receive support from an instructor who adjusts the weekly tasks according to students progress.

The study was done by monitoring a group of students (N = 18) and by comparing their performance to students took the course in a traditional lecture course setting (N = 171). The instructor gather information in particular about what kind of exercises were needed and how the instruction should be performed.

The results show that students in the experimental group benefited from the teaching method. The experiences gathered during the study can be used to scale up the teaching method to deliver courses with several hundred students.

AMS 2010 Mathematics Subject Classification (MSC2010):
- 97 [Mathematics education],
- 97.E [Foundations of mathematics]
- 97.E.30 [Logic]
- 97.D [Education and instruction in mathematics]
- 97.D.40 [Teaching methods and classroom techniques]
Acknowledgements

I am deeply grateful to my advisor, Professor Juha Oikkonen for his support during the whole process of initiating the research, carrying out the teaching experiment and writing this thesis. I am also greatly thankful for the comments and reviews of the text by Ph.D. Taina Kaivola, Ph.D. Jaakko Kurhila and Professor Jouko Väänänen.

I am also grateful for the help by B.Sc. Kaarlo Reipas who acted as an substitute instructor for a week during the teaching experiment. I am also thankful for B.Sc. Pekka Mikkola for his help with making charts for this thesis.

This thesis would not have seen daylight without the assistance and countless discussions with many people. My gratitude goes in particular to M.Sc. Terhi Hautala, Ph.D. Matti Luukkainen, M.Sc. Piia Nissinen, B.Sc. Matti Paksula, Ms. Tiina Romu, Ph.D. Johanna Rämö and M.Sc. Arto Vihavainen for inspirational discussions and support.
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1 Post-course online survey
1 Introduction

Freshmen dropout rates in higher education are a major concern for many universities, see for example (Bennett, 2003; Di Pietro and Cutillo, 2008; Ramsden, 2003). In mathematics early studies play an important role for student’s further success. This is because of the cumulative nature of the science, where new information is added on previously learned topics.

Lots of freshmen education of mathematicians at universities is conducted in large scale lecture courses because of tradition, low cost and scalability. This form of teaching is not supported by modern educational research where lectures are not seen as an ideal mean of educating students (Ramsden, 2003; Laurillard, 2002).

Mass-lectures lack the ability to give personally adjusted teaching to students. Educational research show that personalized teaching yields better results than traditional mass-teaching (Chamberlin and Powers, 2010) but the problems lie in scalability and cost. Lots of the research in computer aided education tries to tackle this problem where the software used to teach students tries to perform in a personalized way when interacting with the learner, for example (Kurhila, 2003).

Another approach has been to try to make lectures more activating and incorporating elements in them where students have to interact more actively in them (Oikkonen, 2009). This has yielded good results in particular in physics (Mazur, 1997).

The approach studied in this thesis is based on the old method of apprenticeship, where a student learns a craft under the supervision of a skilled master. This apprenticeship based learning has been successful in teaching computer programming to freshmen computer scientist (Vihavainen et al., 2011a,b) and the question is if the method is applicable to teaching mathematics.

Traditional apprenticeship learning focuses on teaching a craft that could be done by hand but learning abstract thinking such as that needed in reading, writing or mathematics needs a new way of looking at apprenticeship-based teaching. This kind of approach has been suggested in the form of a teaching framework: Cognitive Apprenticeship (Collins et al., 1989, 1991; Schoenfeld, 1992).

The common factor with traditional and new apprenticeship-based teaching is that in the end the apprentice has to actively do all the learning by himself and the task for the master becomes to show the road ahead and help the apprentice make the necessary steps to learn the topics at hand. The goal is that the apprentice becomes
In this thesis an apprenticeship learning method called the Extreme Apprenticeship method (Vihavainen et al., 2011a) was studied in order to find out if it is applicable to teaching under-graduate mathematics. This was done by teaching a group of students \((N = 18)\) a one semester introductory course in logic, Logic I, in an experimental setting. Comparisons were made between the same course taught through lectures \((N = 317)\).

The thesis gives insight to the use of this method in higher education. The results could be used to scale up the methods used to a full one semester under-graduate course in mathematics. The experiment conducted during the period were put into action by the author.

The thesis is organized as follows: First the pedagogical and practical motivation behind the teaching methodology in the teaching experiment is introduced in section 2. Then an overview of the course and the institution where it takes place is given in section 3. In section 4, the purpose of the study will be discussed. How the teaching experiment was conducted in spring 2011 will be shown in section 5. The performance of the students taking the course and their thoughts about the course will be explored in section 5. Section 6 discusses the findings of the thesis alongside with thoughts on the future of implementing the Extreme Apprenticeship method into teaching undergraduate mathematics.
2 Theoretical background

In this chapter we study the pedagogical background of the teaching experiment studied in this thesis. We begin by describing the traditional university teaching method, that of teaching through lecturing. Then we look into apprenticeship based learning methods employed in the teaching experiment.

2.1 Teaching through lectures

One of the earliest illustrations of teaching by lecturing is by Laurentius de Voltolina from around 1350, seen in Figure 1. The layout of the lecture room is the same as can be found in universities today. The seats are organized in elevated rows in order to make it possible for the lecturer teach and the students to listen. The students are apparently listening, sleeping or joined in conversations with each others.

Before the existence of printed and mass produced books lecturing was thought to be a good method of transmitting knowledge to a large group of students (Morrison, 1986). For 800 years this method of teaching has prevailed as the main method of organizing higher education teaching (Sheely, 2006).

Nowadays teaching in universities through lecturing faces a lot of criticism. Ramsden (2003, p. 6) gives a grim vision of a typical setting:

“Perhaps its nadir is reached in the vision of an outstanding scholar standing before a class of brilliant, handpicked first year students. He or she mumbles lifelessly from a set of wellworn notes while half the class snoozes and the other makes desultory jottings, or maybe – if this an engineering or medicine lecture especially – tests new aerodynamics theories by constructing and launching paper projectiles. Everyone longs to get the hour over and get back to something serious.

The greatest fault of this sort of ’teaching’ is not that it is inefficient or ineffective as a way of helping students to learn (though it is that as well) but that it is a tragic waste of knowledge, experience, youth, time and ability.”

This kind of teacher centered approach to knowledge and learning styles is still used a lot in higher education. In the light of the up-to date paradigms of learning, that puts the learner instead of the teacher in the forefront, sees this kind of attempts
to transform information uneffective (Biggs and Tang, 2007). Typically in a lecture the student sits and listens passively to the information given to him by the lecturer. Why then is higher education still so focused on teaching through lectures?

- **Students want lectures.** As they enter university students expect to sit and listen to lectures (Sheely, 2006). Lectures allows students a degree of anonymity and a degree of order in their studies (Sheely, 2006). Students also expect the lecturer to give specific tips on what should be known in exams, making the students think that listening to lectures will make them get better grades (Mazur, 1997).

- **Teachers want lectures.** Lecturing does not typically give room for surprise.
Everybody has a clear role to play (Laurillard, 2002) and lectures are therefore predictable, making it easy for the lecturer to prepare them well in beforehand. The teachers were subjected to lecturing while studying and therefore replicate their own teaching and learning experience (Errington, 2001; Toohey, 1999). To give lectures is to follow a tradition (Mazur, 1997) and lecturer is often taken as a synonym for teacher at universities.

- **The infrastructure is built around lectures.** As seen in the picture by de Voltolina, lecture halls have been around for a long time. Nowadays, lecture halls are still the primary concern of campus building programs (Sheely, 2006). University campuses are built around lecture halls, classroom booking is done with lectures in mind and researcher’s teaching responsibility is often set in lecture hours.

There have been attempts on making lectures more activating. In teaching university level physics, lecture-based teaching has been shown to give unsatisfactory results. When listening to lectures students become passive receivers of information, resulting in little learning. The issue has been discussed (Mazur, 2009; Stokstad, 2001) and studied (Deslauriers et al., 2011) in *Science magazine*, where the authors stress the importance of the introductory courses and active engagement of students in these courses.

Halloun and Hestenes (1985) studied the impact of traditional lectures on common sense beliefs in physics students. They showed the traditional lecture-based education has a small impact on the gain on the basic knowledge state of the students. Hake (1998) studied students taking their first physics course at different universities ($N = 6542$). His results indicated that those universities which incorporated active engagement of student in their courses showed a significantly higher average gain in student’s pre-post test-scores than those who participated in traditionally lectured courses.

These findings have lead to advances in the methods in how physics is lectured, most notably the success of the Peer Instruction method developed by Mazur (Crouch and Mazur, 2001; Mazur, 1997). Peer Instruction recognizes that even if some lecturers think that their lectures are activating for students, as they incorporate a lot of informal questioning during lectures, that they still engage only a few highly motivated individuals (Crouch and Mazur, 2001).

Peer Instruction lectures involves asking students structured questions, called Con-
CEPTTest, during lectures. These questions are preceded by a peer-discussion by fellow students sitting next to each other. The innovations of Peer Instruction have led to more activating lectures and better student performance (Crouch and Mazur, 2001; Mazur, 1997).

In what situations should lectures be used? In the sixties at the University of Toronto, a committee was selected to change the curriculum and structure of undergraduate education. The committee took a strong stance against lectures, calling it a medieval teaching method which lacks the capacity to give students individualized teaching (Greenleaf, 2010). In the report (Macpherson and Presidential Advisory Committee on Undergraduate Instruction in the Faculty of Arts and Science, 1967), the Macpherson committee\(^1\) listed reasons why lectures should be given, as seen in Figure 2.

Figure 2: Reasons for lecturing, according to the Macpherson committee in 1967 (Atkinson, 1970, p. 562).

| (1) providing an overview of a subject, or branch of a subject, not readily obtainable in any one or a few printed works |
| (2) conveying to students the professor’s enthusiasm and zeal for his subject in a way that cannot be done in print |
| (3) showing the students how to tackle problems of interpretation, and theoretical or experimental problems generally, so that they can tackle some on their own |
| (4) showing a scholar’s mind at work grappling with ideas, theoretical relations, intractable problems |
| (5) conveying to the students theoretical insights and advances in knowledge that are unique to a particular professor |
| (6) transmitting information (whether an orderly structure of facts, of theorems, or of theoretical interpretation) which the student must know in order to comprehend the subject |

We argue therefore that some lectures are justified if they perform almost any one of the first five functions, or any combination of them. But, ... we do not find that the present system of lecture courses encourages the performance of these functions. What it does is encourage the performance of the sixth function ... the one function which we regard as least desirable. ...

In mathematics, the lectures are often about the sixth function in the Macpherson report. The other five functions poses big requirements for the lecturer. In order to be able to give overviews of the subjects and to show how experts deal with

\(^1\)Actually called the Presidential Advisory Committee on Undergraduate Instruction in the Faculty of Arts & Science.
the problems under study, demand that the lecturer must be aware of the recent advances in his respectible field. This should however not be a problem in research universities, where the teaching staff are also researchers. The problem is how the get the five first functions in the Macpherson report into lectures.

2.2 Cognitive Apprenticeship

As an alternative to traditional mass education apprenticeship-based teaching has been proposed to reading, writing and mathematics by Collins et al. (1989, 1991). Their approach, called *Cognitive Apprenticeship*, forms its view on teaching from the ancient process where an apprentice is taught a craft under the supervision of a skilled master. They use the findings of Lave (later published in (Lave and Wenger, 1991)) who has studied West African tailors who learn their craft by observing and practicing in tailoring shops under the instruction of master tailors. Schoenfeld (1992, p. 85-86) summaries Lave’s observations as follows:

“Being a tailor is more than having a set of tailoring skills. It includes a way of thinking, a way of seeing, and having a set of values and perspectives. In Tailors’ Alley, learning the curriculum of tailoring and learning to be a tailor are inseparable: the learning takes place in the context of doing real tailors’ work, in the community of tailors. Apprentices are surrounded by journeymen and master tailors, from whom they learn their skills – and among whom they live, picking up their values and perspectives as well. These values and perspectives are not part of the formal curriculum of tailoring, but they are a central defining feature of the environment, and of what the apprentices learn. The apprentice tailors are apprenticing themselves into a community, and when they have succeeded in doing so, they have adopted a point of view as well as a set of skills – both of which define them as tailors.”

The aim of instruction in Cognitive Apprenticeship is to make the thinking visible (Collins et al., 1991). If mathematics is presented as an orderly collection of definitions and theorems, the thinking behind them is not revealed. This leads to students studying procedures which only help them solve textbook problems without actually understanding the problems and the solutions (Schoenfeld, 1985).

The challenge of Cognitive Apprenticeship is to create meaningful exercises for the students that are both systematic and diverse and to make the students to reflect
on what they are doing (Collins et al., 1991). In learning writing this means, for example to reduce the students’ information-processing burden by breaking down the writing process into small pieces before tackling the whole complex process of writing (Collins et al., 1991).

Even if working with small goals, Cognitive Apprenticeship emphasises that students need to have a global vision on what they are doing (Collins et al., 1991). The students should all the time know why they are working with the smaller goals in the same way as an apprentice tailor knows that the small part he is working on will eventually become a jacket. Mathematical rules and symbols should not be taught as if they were arbitrary conventions having little to do with larger phenomenas of regularities and relationships that also shape the physical world (Resnick and National Research Council (U.S.). Committee on Research in Mathematics, Science, and Technology Education, 1987). This same view of education was shared by Whitehead in 1929 when he wrote (Whitehead, 1996, p. 127):

“Whatever interest attaches to your subject-matter must be evoked here and now; whatever powers you are strengthening in the pupil, must be exercised here and now; whatever possibilities of mental life your teaching should impart, must be exhibited here and now. That is the golden rule of education, and a very difficult rule to follow.”

“You cannot put life into any schedule of general education unless you succeed in exhibiting its relation to some essential characteristic of all intelligent or emotional perception. It is a hard saying, but it is true; and I do not see how to make it any easier.”

An important issue in Cognitive Apprenticeship is that learning takes place in a social environment in which the apprentice becomes a member. In this study, this environment is the academic community and the community of mathematicians. The students should be taught in such an environment that they are surrounded by other learners, not in segregated learning environments that do not show a link to the actual place where the skills acquired are to be used (Brown et al., 1989; Ball, 1991; Schoenfeld, 1992). This growing into a group of people who share a common craft or profession, has been studied by Lave and Wenger (1991, p. 23):

“Cognitive apprenticeship must find a way to create a culture of expert practice for students to participate in, and aspire to, as well as
device meaningful benchmarks and incentives for progress. [...] Drawing students into a culture of expert practice in cognitive domains involves teaching them how to ‘think like experts’.”

To become a member of the community of mathematicians of course involves learning a lot of subject knowledge. Sfard (1998) stresses the importance of not neglecting any of these two views and see them as supporting each other. Schoenfeld (1992, p. 33) summarizes these views as a “pedagogical imperative”:

“If one hopes for students to achieve the goals specified here – in particular, to develop the appropriate mathematical habits and dispositions of interpretation and sense-making as well as the appropriately mathematical modes of thought – then the communities of practice in which they learn mathematics must reflect and support those ways of thinking. That is, classrooms must be communities in which mathematical sense-making, of the kind we hope to have students develop, is practiced.” [Italics in the original.]

Teaching through Cognitive Apprenticeship proceeds as follows: first the master, teaching the apprentice, teaches by modeling the task at hand. Modeling is the process where the apprentice observes the master while the master completes the task while explaining how he does it. After the modeling phase the apprentice practices to accomplish the task while the master coaches him by selecting necessary sub goals and through scaffolding the student’s learning.

Scaffolding is the support given by the master. It refers to the temporary constructions that are needed to be set up when building buildings. In this case it means the temporary support given by a master so that the student is able to learn. The aim is to make it possible for the student to progress on his own.

Bruner (1986, p. 24-25) who first used the term scaffolding in learning context describes it as follows:

“If the child is enabled to advance by being under the tutelage of an adult or a more competent peer, then the tutor or the aiding peer serves the learner as a vicarious form of consciousness until such a time as the learner is able to master his own action through his own consciousness and control. When the child achieves that conscious control over a new function or conceptual system, it is then that he is able to use it as a
tool. Up to that point, the tutor in effect performs the critical function of ‘scaffolding’ the learning task to make it possible for the child, in Vygotsky’s words, to internalize external knowledge and convert it into a tool for conscious control.”

In the beginning more scaffolding is needed. As the apprentice becomes more skilled, help given by the master fades away. Scaffolding should be temporary, as the idea is not to make the apprentice dependent of the master. It is also important that the apprentices would be scaffolded by many masters and not to expect to be taught by one single authority (Collins et al., 1989).

The theory of scaffolding is interlinked with Vygotsky’s theory of the zone of proximal development (Vygotsky, 1978). The zone of proximal development is the distance between what a learner can learn by himself and how much he can learn with help from others. According to Vygotsky, a child learns a language from his parents while being in the zone of proximal development (Vygotsky, 1978). It is difficult for a child to learn a language by himself. The parents model the use of language by modifying it to be on a difficulty level that the child can learn from listening to it. At the same time they give guidance and challenge the child to perform on a higher level (Bruner, 1986).

Monitoring of the apprentice and adjusting the scaffolding accordingly is crucial in apprenticeship-based learning (Collins et al., 1989). It is also important because the teacher has to encourage the students to articulate his knowledge and thinking (Collins et al., 1991). This enables the student to compare his performance with others and through this reflection highlight differences in thinking. Apprenticeship-based learning is seen as an interplay between the apprentice and the master, where the apprentice learns from the master and the master learns from the apprentice.

The final stage is when the master fades away and the learner is able to commit exploration on his own (Collins et al., 1991). This stage demands that the student is capable of conducting problem-solving and tackling open questions on his own. In this stage the student already performs as an expert and uses exploration and learning strategies he has learned during the whole process.

### 2.3 Extreme Apprenticeship Method

Building on Cognitive Apprenticeship, Vihavainen et al. (2011a,b); Kurhila and Vihavainen (2011) have implemented apprenticeship-based learning in teaching com-
puter programming for undergraduates. Their method, called Extreme Apprenticeship\(^2\), provides a set of values and practices used as a basis for the approach taken in the teaching experiment of this study. For the values, see Figure 3.

Figure 3: Values of Extreme Apprenticeship method (Vihavainen et al., 2011a, p. 94-95).

- **Learning by doing.** The craft will only be mastered by actually practicing it.
- **Continuous feedback.** Continuous feedback must be implemented in both directions. The student receives multi-level feedback from his progress and instructors, and the instructor receives feedback by monitoring the students progress and challenges.
- **No compromise.** The skills to be learned are practiced as long as it takes for each individual.
- **An apprentice becomes a master.** The ultimate goal of instruction should be that the student will eventually become the master.

While supporting collaborative aspects, the Extreme Apprenticeship method focuses on the individual effort done by students (Vihavainen et al., 2011a). The method centers on exercises being completed under constant supervision of instructors. The aim is to raise the amount of actively conducted individual effort of students, while minimizing their passive activities such as listening to lectures.

In computer science, as in mathematics, a typical lecture course is based on lectures and take-home exercises. The weekly take-home exercises, that the students complete, are based on and complement the lectures. In Extreme Apprenticeship

\(^2\)The label “Extreme” comes from the software engineering industry, where Extreme Programming (Beck, 1999) is a method where good programming practices are taken to the extreme. For example, code review is considered a good programming practice and Extreme Programming takes this to the extreme by enforcing continuous code review in the form of pair programming. Extreme Apprenticeship tries to do the same for educational best practices, for example as monitoring student progress is seen as a good teaching practice, the Extreme Apprenticeship method enforces continuous monitoring of students progress (Vihavainen et al., 2011a).
this relationship is switched upside down. As the exercises are the main teaching material and conducting the exercises the main method of learning, the lectures, if any, are based on and complement the exercises.

Extreme Apprenticeship puts a lot of emphasis on how the exercises are constructed, conducted and evaluated (Vihavainen et al., 2011a). Students begin conducting exercises from day one of the course. Instead of take-home exercises, the exercises are done in a suitable exercise room where it is easy for the students and instructors to interact. This enables instructors to help and monitor the student’s progress.

There are lots of exercises, but at least in the beginning, they are quickly solvable. This makes the students feel that they are making progress from the start. The exercises are also repetitive in nature and in many cases train the routine of the students.

The teaching method should train the students to obtain the main tools mathematicians use: abstraction, symbolic representation and symbolic manipulation (Schoenfeld, 1992). Knowing how to use these tools does not conceptually differ from the knowledge of using craftsman’s tools. Learning should concentrate on solving problems, exploring patterns and formulating conjectures instead of memorizing and doing exercises simply based on examples given by the teacher.

Extreme Apprenticeship exercises have to be correctly answered. In computer programming this means that the exercises have to be correctly solved and also follow good programming practices (Vihavainen et al., 2011a), for example Clean Code (Martin, 2009). In mathematics this translates to the principle that solutions should be finished with clear notation and structure.

The instructors, in the role of masters, accept or reject the solutions of the students. In case of a rejection, the instructors help the students to correct the exercise at hand. Rejection should not be seen as a failure but as an opportunity for the student to learn and correct possible misunderstandings. Therefore, students are allowed to hand in exercises many times, until they come up with an acceptable solution.

The Extreme Apprenticeship method gives clear guidelines on how the instructors should behave in the role of masters (Vihavainen et al., 2011a,b; Kurhila and Vihavainen, 2011). They are not allowed to give direct answers to the students. Instead, instructors are supposed to help the students as little as possible as the students should get the feeling that they manage to solve the exercises by themselves. Moments of success and raising confidence level of students should be enforced from
day one as it will enhance the student’s performance and self-esteem in their studies (Boud et al., 1985).

The benefits of scaffolding are greatest when the students feel comfortable but also challenged while doing the exercises (Vihavainen et al., 2011a). This is when the zone of proximal development pushes the performance of the students further while the task still feels meaningful. Kurhila and Vihavainen (2011) give specific instructions for the teaching assistants conducting scaffolding, see Figure 4.

Figure 4: Instructions for teaching assistants in the Extreme Apprenticeship method (Kurhila and Vihavainen, 2011, p. 6).

- You will advise everyone in trouble
- You will not give out solutions but guide the student as much as needed in order to nudge the student to find the solutions herself.
- Advisors do constant round-robin in the lab. Observe and comment on students’ progress even if no-one asks anything.
- You will pay attention to the code style: students will learn to program according to Clean Code principles.
- Correct solutions is not enough. You need to push the style towards more understandable and maintainable code.
- Even if there is a slow moment in the lab, you as an advisor cannot sit still minding your own business!

It is important that the instructors monitor how the students perform. If the instructors notice that plenty of students have problems with some type of exercises this is addressed during upcoming lectures and exercises. Therefore, the exercises that the students will carry out in the coming weeks is not set long beforehand, but it changes during the progress of the course. This of course demands that administration, the way teaching assistants who serve as instructors are employed and how the lecturer or the person in charge of the course, is done in a very flexible manner (Kurhila and Vihavainen, 2011).
3 Conducting the study

The purpose of the study was to find out if the Extreme Apprenticeship method is applicable to teaching mathematics at university-level. The method has proven to raise student’s performance in learning computer programming (Vihavainen et al., 2011a) and the aim was to find out if similar results could be gained in mathematics. The method was to monitor the teaching experiment and gather necessary information for scaling up the teaching method to large basic and intermediate courses. The research questions were:

1. What kind of exercises are needed for applying the method to mathematics?
2. What kind of instruction is most beneficial for students?
3. What kind of learning environment is required by the teaching methodology?
4. How to transform lectures to benefit learning through the methodology?

The answers to these questions can be used when designing the structure for larger course implementations. It is important to know what kinds of exercises at which time of the course suit the teaching methodology. The physical learning environment must also be thought over as the normal classroom setting is not suitable for the student centered approach taken in this study.

The answers will also help the recruitment process of finding suitable teaching assistants performing as instructors. The Peer Instruction method (Crouch and Mazur, 2001; Mazur, 1997) also includes discussion sessions where a portion of the weeks’ homework is done in groups. In the group, a teaching assistant is asking questions about the exercises and helping the students to find the right solutions without giving them straight answers. At the end of the week, each student hands in individual solutions to each exercise and the exercises are graded individually. The developers of Peer Instruction method stress the importance of the choosing the right teaching assistants as (Crouch and Mazur, 2001, p. 975):

“... many TAs [teaching assistants] are excited by the opportunity to engage their students more actively, some resist innovations and may communicate a negative attitude to students.”

How teaching assistants should be instructed is an important issue for this study. If scaled up, th Extreme Apprenticeship method requires teaching assistants per-
forming as learning instructors for the apprentices. The experiences gathered in this study should give insight into what are required of teaching assistants and how they should be instructed and used.

The teaching experiment in this study was done without lectures. As argued by the Macpherson committee (Macpherson and Presidential Advisory Committee on Undergraduate Instruction in the Faculty of Arts and Science, 1967), there are reasons for using lectures. These reasons are however not those typically addressed in mathematics lectures, where often the lecture consists of orderly structure of facts and theorems. The study also explores what kind of lectures would be suitable for the teaching methodology.

A concrete example of the use of the Extreme Apprenticeship method will be studied in detail. The example will show how Tarski’s truth definition for predicate logic was introduced in the experimental group. Tarski’s truth definition is interesting for the course Logic I because it is a central piece in the theory of predicate logic. If students are unable to comprehend the definition, there is not much the course can teach them about predicate logic but routines and empty procedures.

The study was conducted applying participatory action research (Mills, 2000; Creswell, 2005) at the Kumpula campus at the University of Helsinki. The author of the research conducted the teaching experiment. The principal aim of the research was not the professional development of the author, but on the development of new practices for further development and study.

As this was the first attempt on using the Extreme Apprenticeship method to teach mathematics, it was unclear in the beginning of the experiment what the difficulties would be. Therefore the findings of the research are mostly due to collaboration with the students. The variable studied quantitatively was the students’ performance in the course exams.

The ethical principles (Hopkins and Ahtaridou, 2008) of the research were that the author performing as the instructor believed that it would enhance learning, that it was favorable to push ahead with the new teaching methods and that the methods were suitable for the students. The students taking part in the experiment were not selected randomly but chose themselves to take part in the group.
3.1 Overview of the educational setting

In Finland tertiary education is divided into universities and polytechnics\(^3\). A special feature of the Finnish tertiary education is that there are no tuition fees. University teaching resulting in a master’s degree is intended to take five years with three years of bachelor studies and two years of master studies.

Subject teachers, who teach their subjects at K-12 level, receive their qualification to teach the subject through getting a master’s degree. They study the subject they teach as a major or minor and study pedagogy as a minor subject. Therefore departments teaching subjects taught in school, such as mathematics, have many preservice teacher among their students.

The University of Helsinki is the biggest university in Finland with over 35 000 students and almost 4000 faculty members. The university defines in its strategy for 2010-2012 its philosophy on teaching as follows (University of Helsinki, 2009):

“One of the long-term strategic objectives of the University of Helsinki is to promote research-based teaching. The quality of teaching at a research-intensive university is founded on top-level, multidisciplinary research and teachers who are also researchers in their own field of specialization and who use teaching methods that enable inquiry-based learning.

Teaching is based on academic research and applies the results of multidisciplinary research on university-level teaching, studying and learning. Each member of the academic community engages in both teaching and research. In addition to possessing knowledge of their own research field, each university teacher must disseminate extensive information about the latest research in their discipline through their teaching. Research-based teaching also means that students are familiarized with and participate in the work of the research community. The objective of teaching and academic advising is student-oriented, profound learning in line with the principle of life-long learning. The university community fosters the development of students into versatile and responsible experts in their field.”

This emphasis on participation in the research community, on student-oriented

\(^3\)also called universities of applied sciences.
teaching and profound learning are well aligned with the principles of the apprenticeship-based teaching method explored in this study. However, Extreme Apprenticeship is not an inquiry-based learning method as it is very structured and involves much guidance. It can be seen as a teaching method helping the students learn essential skills needed for inquiry-based learning later needed in their studies and future carriers. Criticism on inquiry-based learning has been discussed by Kirschner et al. (2006); Sweller et al. (2007) where the authors stress the need for guided environments in order to help students learn better.

The experimental setting of this study was situated at the Department of Mathematics and Statistics at the University of Helsinki. The department is the biggest department in its field in Finland with over 1300 students. The students to the department are selected through their performance in the secondary high-school matriculation examination, an entry exam or both.

The yearly intake to the department can be seen in Table 3.1 (Faculty of Science, 2012). As seen in the table there are plenty of students who do not begin their studies even if accepted. A big problem for the department is also the students who dropout, as the amount of beginners compared to the amount of students who graduate is large. In 2011 there were 249 beginners and 62 students graduated.

Table 1: The yearly intake and graduation at the Department of Mathematics and Statistics, University of Helsinki.

<table>
<thead>
<tr>
<th>Year</th>
<th>2011</th>
<th>2010</th>
<th>2009</th>
</tr>
</thead>
<tbody>
<tr>
<td>Accepted</td>
<td>565+50*</td>
<td>527+49*</td>
<td>611+50*</td>
</tr>
<tr>
<td>Began their studies</td>
<td>212+37*</td>
<td>206+28*</td>
<td>242+34*</td>
</tr>
<tr>
<td>Graduated as Bachelors of Science</td>
<td>73</td>
<td>92</td>
<td>65</td>
</tr>
<tr>
<td>Graduated as Masters in Science</td>
<td>39+23**</td>
<td>29+20**</td>
<td>26+19**</td>
</tr>
</tbody>
</table>

* The first number represents students accepted for the program of mathematics and the second number those accepted for the program of mathematics subject teacher. Students accepted for the program of statistics are left out.

** The first number represents students graduated from the program of mathematics or applied mathematics and the second number those graduating from the program of mathematics subject teacher. Students who graduated from the program of statistics are left out.
There are ongoing attempts on tackling the issue of first year dropouts, for example the first year analysis course has undergone big changes (Oikkonen, 2009) and a guidance tutoring system for beginning students has been established (Hautala, 2010).

The educational development of the Department of Mathematics and Statistics is lead by the Working group of education development. The duties of the working group are, for example, to prepare degree requirements and other study matters to the council of the department, to start and coordinate projects for education development, to prepare presentations regarding teaching effectiveness and the development of teaching methods. It promotes continuous interaction between teachers and students in order to maintain a good and open teaching and learning environment at the department.

The proposal to implement the Extreme Apprenticeship method was introduced to the Working group of educational development in autumn 2010 by the author. After the working group decided that the teaching experiment should go ahead the teaching experiment was set to start in spring 2011.

3.2 Placement in the mathematics curriculum

Mathematics education is organized into three (possibly overlapping) phases. Students begin by studying basic studies and intermediate studies for a degree as Bachelor of Science. Then they study advanced studies leading to a degree as Masters of Science. Subject teachers can include intermediate studies in their advanced studies.

The introductory course of logic, Logic I, lies in the category of intermediate studies where it is taught as a full semester course worth 10 ECTS. The course gives students a basic understanding of concepts in mathematical logic such as models, truth and deduction. The emphasis is on introducing the concepts, without necessarily giving proofs to all results discovered.

The content of the course and a rough description of the pace of the course Logic I is described in Table 2.

For students wishing to continue studies in logic the department offers an advanced studies course named Mathematical logic. It is possible to graduate as mathematics major from the subprogram of mathematical logic where the course is a mandatory part of the degree. The course topics of the advanced course are similar at the beginning of the course, with a focus on giving proofs to the results discovered in
Table 2: Course content and course pace of the course Logic I in spring 2011.

<table>
<thead>
<tr>
<th>Timeline</th>
<th>Main topic</th>
<th>Content</th>
<th>Exam</th>
</tr>
</thead>
<tbody>
<tr>
<td>Week 1</td>
<td>Propositional Logic</td>
<td>Propositional formulas</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Truth tables</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Classification: tautology, contradiction, etc.</td>
<td></td>
</tr>
<tr>
<td>Week 2</td>
<td>Logic</td>
<td>Truth functions (generalized connectives)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Logical equivalence</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Logical consequence</td>
<td></td>
</tr>
<tr>
<td>Week 3</td>
<td>Formal proofs: Natural Deduction</td>
<td>Exams 1</td>
<td></td>
</tr>
<tr>
<td>Week 4</td>
<td>Soundness of Natural Deduction</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Formal proofs: Tableaux</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Week 5</td>
<td>Predicate Logic</td>
<td>FO structure</td>
<td>Exam 1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Vocabulary</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Atomic formulas</td>
<td></td>
</tr>
<tr>
<td>Week 6</td>
<td></td>
<td>Assignments</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Satisfaction for atomic formulas</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Quantifiers</td>
<td></td>
</tr>
<tr>
<td>Week 7</td>
<td>General formulas</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Satisfaction for general formulas</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Validity</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Logical consequence</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Free and bound variables</td>
<td></td>
</tr>
<tr>
<td>Week 8</td>
<td>Sentences</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Definable relations</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Substitution</td>
<td></td>
</tr>
<tr>
<td>Week 9</td>
<td>Natural Deduction for predicate logic</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Week 10</td>
<td>Soundness of ND for predicate logic</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Axioms and theories</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Identity axioms</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Week 11</td>
<td>Tableaux for PL</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Soundness of the tableau method</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Classification of sentences in PL</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Function symbols</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Isomorphism</td>
<td>Exam 2</td>
<td></td>
</tr>
</tbody>
</table>
Logic I. The rest of the advanced course goes further to Peano arithmetics and the proof of Gödel’s Incompleteness Theorem (Väänänen, 2010b).

3.3 The traditional course setting in spring 2011

In spring 2011 Logic I was organized in the same manner as during previous years. Enrollment for the course started in the beginning of January and 337 students enlisted for the course. As they enlisted for the course, students registered themselves to exercise groups which were held by teaching assistants. In spring 2011 the course had 9 exercise groups. The lectures lasted 13 weeks with eleven weeks of exercises. Parallel to the large course in Finnish a smaller course was also given in English. The course followed the large course with the same exercises and course exams.

Teaching consisted of five times 45 minutes lectures and six take-home exercises a week. The take-home exercises were discussed in the exercise groups lasting two hours. In the exercise groups one student at a time presented their solution to one of the six exercises, so that each of the exercise was discussed once.

The course material was available online. The course literature consisted of material provided by the previous lecturer of the course (Väänänen, 2010a). Many students had also access to the book (Salminen and Väänänen, 1992) used to teach the course prior to 2010. The lecturers of the Finnish and English speaking course provided jointly the exercises for the course. The teaching assistants also provided the students with model solutions for the exercises which were posted on the course web page.

Student performance was evaluated by two exams, one in the middle of the term and one at the end of the course. Some extra points were awarded to those students who had completed a certain percentage of their take-home assignments. The students who passed the course were given a grade on the scale 1 to 5.

3.4 The experimental setup

In addition to the traditional course format an experimental course setup was introduced. This exercise group was instructed by the author and served as the experimental theater for the research of this thesis. The group’s working methods were based on the Extreme Apprenticeship method with focus on hands-on doing and solving exercises under constant supervision.
The group met six hours a week instead of the normal two in traditional groups. The students had the opportunity to visit normal lectures. This meant that students following the traditional teaching setting attended five hours of lectures and two hours of exercise group per week. Students in the experimental setup had the opportunity for five hours of lectures and six hours of exercise group instruction a week. However, surprisingly none of the students in the experimental setting attended any but a few lectures during the whole course, so the amount of contact teaching they received is difficult to evaluate.

Apprenticeship-based education needs an environment where it is easy for the participants to engage in discussions and exchange ideas. For mathematics, often some surface to write down the ideas discussed is needed.

The classroom where the teaching experiment took place was reorganized to fit the needs of the teaching methodology. The sitting order of the classroom was changed from a row formation to table groups so that up to eight students could sit at one table group at a time. The tables where covered with acrylic glass\(^4\) so that all tables could be used as whiteboards. Figure 5 show photographs of the rearranged classroom used in the experiment.

The room was already equipped with blackboards covering one of the walls and another wall. Another wall, covered with glass, was also used as a whiteboard. The classroom was equipped with an interactive video projector which allowed the course material, which was published online, to be available and used flexibly in the classroom.

### 3.5 Participants

The demographic of the attendants of the course Logic I differs from other mathematics courses. Mathematics majors are a minority as the course attracts a lot of minor students, in particular computer science majors. Many student studying to become subject teachers in mathematics take the course as logic is taught as a voluntary course called *Number theory and Logic* (Kangasaho et al., 2008) at secondary high-school-level. The participants of the experimental setup and the traditional course can be seen in Table 3.5.

A typical phenomena for undergraduate mathematics courses is that many students register for the course but do not show up or only visit the first lectures. Therefore

\(^4\)also called Plexiglas.
Figure 5: The classroom used by the Extreme Apprenticeship group. The tables are arranged into groups and surfaces covered with acrylic glass.
Table 3: Enrollment figures for the course Logic I in spring 2011

<table>
<thead>
<tr>
<th>Study program</th>
<th>Math. majors</th>
<th>Subject teacher majors</th>
<th>Math. minors</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Experimental</strong></td>
<td>Registered %</td>
<td>9.5</td>
<td>61.9</td>
<td>28.6</td>
</tr>
<tr>
<td>Participated %</td>
<td>1</td>
<td>13</td>
<td>4</td>
<td>18</td>
</tr>
<tr>
<td><strong>Traditional</strong></td>
<td>Registered %</td>
<td>32.0</td>
<td>19.5</td>
<td>48.5</td>
</tr>
<tr>
<td>Participated %</td>
<td>58</td>
<td>33</td>
<td>80</td>
<td>171</td>
</tr>
</tbody>
</table>

only those students who marked at least on exercise as done are considered participants in the traditional setting. As the first exercise group meetings took place on week two of the traditional course, only those students who handed in at least one exercise from week two are considered participants in the experimental setting of this study.

The experimental setup was promoted on the course page and in the students e-mail list as follows [e-mail by author originally in Finnish]:

“In [the Extreme Apprenticeship group], the course Logic I is performed with the same exams as other attendants of the course. Everybody can enlist to [the experimental group] according to the participation limit [25 attendants].

[The Extreme Apprenticeship group] will meet three times a week to address course topics. [The experimental group] is always attended by an instructor and the main focus is on guided exercises in where exercises are solved in under personal instruction. The amount of exercises is much bigger than in traditional ‘take-home assignment’ exercises and the aim is that the participants will learn by doing by themselves. The exercises are also handed in every week to the instructor.

Due to the structure of [the experimental group], it is desirable that the workshop subscribers will have the opportunity to participate in
all [the experimental group] sessions. Due to the separate exercises, [the experimental group] cannot be visited by students in other exercise groups.”

The Extreme Apprenticeship group was advertised as a exercise group where more work would be carried. It was also advertised as giving more support for the students. This could have an impact on the demographic of the group. Students who would anticipate that the course would be difficult for them, were probably more willing to join the course.

As mathematics majors and minors were so few in the experimental setup their performance cannot be published without danger of jeopardizing their anonymity. Therefore the students in the experimental setup will hereafter be considered a single group and the variations between the performances of different student clusters has to be done in further research.

As seen in Table 3.5, the demographic of the participants in the experimental setup varied a lot from the large course. Subject teacher students counted for 72.2% of the participants in the experimental setup and mathematics minors 29.4% in the traditional setup.

The lecturers of the traditional course were two members of the faculty of the department. One gave the lectures in English and the other gave the course in Finnish. The exercises for the traditional course were done in collaboration with the two lecturers.

The teaching assistants of the traditional course were hired amongst the students and doctoral students of the department. The teaching assistant serving as the instructor of the Extreme Apprenticeship group had previously served as teaching assistant for six exercise groups during the years 2006-2010. A student of department served as a substitute instructor for one week of the experiment.

The participants of the English speaking lectures were mainly foreign students. Their performance will not be considered in this study as other factors not visible in the Finnish speaking Extreme Apprenticeship group might influence their course outcome.
3.6 Scaffolding students learning

Scaffolding refers to the temporary support given to students in order to help them learn. This section gives a description of the teaching methodologies used for scaffolding in the experimental setup. The focus was on what the students do: that they should do plenty of exercises, that the exercises should be evaluated and that the students would have an opportunity to correct those exercises that were wrong. The working process involved constant monitoring and assistance by an instructor.

3.6.1 Exercises

In the Extreme Apprenticeship method the exercises play a crucial role. They are the main teaching material and should therefore cover the topics of the course, structure learning and establish the pace of the course. Together with instruction they should provide the students with the scaffolding needed to learn the learning objectives of the course.

The exercises given in the experimental group were different than the exercises that were given in the traditional course. However, as the exams and the learning objectives of the two course setups were the same, the exercises in the Extreme Apprenticeship group were heavily influenced by the exercises created by the lecturers of the traditional setting.

Creating exercises for the Extreme Apprenticeship method requires a lot of adjustments from the teacher. In creating the exercises an iterative process was needed. The basis of creating the exercises was the monitoring of the students performance while they solved the exercises. Any major difficulty that had risen during the week was tackled in the next set of exercises. This was possible as the creator of the exercises was also doing the monitoring, thereby avoiding the difficulties described by Vihavainen et al. (2011b) where the lecturer did not participate in the exercise sessions.

After trial and error a pattern for the structure of the exercises emerged. First a new concept of the week was introduced with many small easy exercises. This was intended to familiarize them with the concepts. The intention was to make it easy for them to ask specific questions of matters they did not understand. These questions were also intended to give the students practice in managing the technical finesses of the notation.
While solving the simpler exercises the students should get the feeling of “wanting to solve one more problem”. Soon they would realize that they had managed to do one third of the weekly exercises without frustration.

Secondly there were exercises where the students needed to use earlier learned concepts together with the concepts they had became familiar with in the easier exercises. Usually this meant deepening their understanding of the earlier learned concepts which often revealed misunderstandings.

In a lecture setting it is nearly impossible for the lecturer to judge what are the misunderstandings of the individual students. These misunderstandings might reveal themselves only after correcting the end exam of the course. This leaves little or no possibility for the lecturer to react.

Revealing misunderstanding is one of the strengths of the Extreme Apprenticeship method. The misunderstandings are revealed each week. This makes it possible to correct them and make the next set of exercises such that they can be used to evaluate if the misunderstandings still persist.

Thirdly the students had to answer exercises where they had to explain the concepts and to reflect on what they were doing. When a student did not formulate his thinking in way that showed his understanding of the topic, he had to reformulate his answer. With the help of the instructor this process served as a part of the articulation and reflection stages that were enforced by Cognitive Apprenticeship (Collins et al., 1991).

Articulation and reflection helped to prevent the students of just learning empty procedures. As witnessed by the author on previous occasions when being an teaching assistant for the course Logic I, many students take an exam-centric approach to learning the content. This approach is taken in order to mimic examples and solutions in order to be able to answer questions in an exam and involves little understanding of the topics under study. In Logic I this kind of approach has been witnessed previously by the author, for example in learning the rules of inference in natural deduction and the rules for the logical connectives when dealing with the tableau method.

After the students had solved the exercises they handed in their solutions for inspection. Their solutions were evaluated fail or pass. In the case of a fail, students had to correct their solution until a correct solution was delivered. The students had one week time to finish their first attempt on doing the exercises and an additional
week to finish the final version of the solution. As new exercises were published each week, the students were working on two sets of exercises each week.

The solutions were corrected by the instructor of the experimental setup. The students were told not to expect exam-like precision with the correction as the correction procedure was done in a rapid pace. The aim of the correction was to give the students opportunities to learn by forcing them to think again about their solutions if incomplete.

Even if done in a quick manner, correcting the exercises was a time consuming process. The level of detail required in the solutions influenced a lot the work load of the students doing the exercises and also influenced the amount needed to correct the returned exercises.

There is a big difference in doing the exercises correctly or almost correctly. The experience was that the students valued the opportunity to correct their solution more than they would have valued being awarded too easy points.

This is illustrated by a real situation that appeared during the course: As the students had to hand in their exercises, one student became frustrated after his/her fourth attempt to get the one of the assignments right. The student asked if he/she could just get the right answer as the task seemed too difficult to solve. It would have been very attempting for the instructor at a situation like this to feel sorry for the student and give him/her an easy pass.

After ones more going through the exercise the student finally managed to solve the exercise correctly. This gave the students a huge feeling of accomplishment as he/she had managed to solve the problem by himself/herself. It was also an extremely rewarding moment for the instructor.

Of course, if the exercises were too difficult to solve, then the fault was that of the instructor who had made them.

In computer programming correcting the exercises has been transferred to automatic testing environments but in mathematics this is more difficult. In the future alternative correction strategies where not all exercises are corrected or students correct each others solutions have to be evaluated.
3.6.2 Instruction

The exercises and the instruction form the basis of scaffolding and are key elements in the Extreme Apprenticeship method. As laid out in the theory of the zone of proximal development (Vygotsky, 1978) the instructor is a key element in this process.

To instruct as little as possible was difficult to learn for the instructor. Previously as working as an teaching assistant the job of the instructor was to evaluate and explain model solutions in front of class. In the Extreme Apprenticeship method the role of the instructor is completely different. The instructor should refrain from helping the students too much. It should always be the student who comes up with the solution and not the instructor.

A typical situation was where the student was frustrated and claimed not to understand anything of what the exercise was about. In a situation like this the instructor might feel tempted to explain the whole exercise and what it was about. The lesson learned was that the students do not benefit from these kinds of explanations.

The students should always be made to ask specific question. If the student was not capable of forming the question he was probably not going to benefit from hearing the answer either. The task of the instructor should be to get the student to first form the question and only then answer it. When the student had been helped to pinpoint the difficulty in the exercise, he could often answer his own without the help of the instructor.

According to Collins et al. (1991), it is important to get the students to articulate and reflect on what they are doing. This enables them to become self-directed and teaches them to find flaws in their own reasoning. Schoenfeld (1992) asks his students questions, which forces them to reflect on their own activities, see Figure 6.

The aim of the Extreme Apprenticeship method sessions was not only to practice the skill of doing mathematics but also to introduce them to a culture where mathematics is done. This meant that the instructor also discussed mathematical topics that were outside the scope of the exercises. Especially the beginning students are not all familiar with an atmosphere where mathematics is discussed casually.

In a typical lecture setting there is only one way of communication from lecturer to student. Neither is the typical demo session style of take-home exercise sessions a place where much discussion happens. It is more about showing students model
solutions and show how a mathematical presentation is given.

The important task for the Extreme Apprentice method instructor is therefore to discuss mathematics with the students. This also gives the opportunity to create an atmosphere of doing and discussing mathematics that motivates the students to discuss more mathematics amongst themselves. The ratio of around 15-20 students per instructor seemed to fit this goal. The instructor was typically occupied with helping a student all the time. This often forced the students to discuss the problem by themselves before asking the instructor for help. This gave the instructor the opportunity to listen to such explanations and only intervene if something was explained incorrectly.

Scaffolding should be temporary and the fading process is important in apprenticeship-based learning. The students should learn how to read the course literature, doodle, sketch and discuss mathematics and become less and less dependent of the instructor. The fading can also be done iteratively. A good practice turned up to be to help the student a little bit. Then tell him to think about the exercise by himself and return after a while to see if progress had been done. If progress had been done this made it possible to give well earned credit to the student. If not, the instructor could give additional advise.

Fading is a process that takes longer than one course. To be able to use mathematics in an upcoming profession is the aim of the university degree in mathematics. To be able to conduct research independently in mathematics is even beyond to scope
of a master’s degree. The fading done in the course Logic I should therefore be seen in this perspective. The students should be able to perform alone in the end exam and be well prepared for the studies ahead.
4 An example: Introducing Tarski’s truth definition for first-order predicate logic

In this section we\textsuperscript{5} will give a description of the introduction of Tarski’s truth definition for (first-order) predicate logic that was done in the Extreme Apprentice group.

While instructing previously six exercise groups of Logic I as a teaching assistant, Tarski’s truth definition has stood up as a major challenge for students. Students are attempted to tackle these difficulties by learning the definition by heart.

We begin by giving the truth definition. This is done in a more thorough manner than in the actual course Logic I. However, as the Extreme Apprenticeship method involves lots of discussion with the students many of the questions discussed below aroused in class.

4.1 Tarski’s truth definition

Alfred Tarski published in 1933 a paper where he discusses the semantic conception of truth, for a translation to English see for example (Tarski, 1983). This review of Tarski’s definition follows his later publication The Semantic Conception of Truth: and the Foundations in Semantics (Tarski, 1944).

By semantics Tarski means the discipline that investigates (Tarski, 1944, p. 345)

\begin{quote}
“certain relations between expressions of language and the objects (or ‘states of affairs’) ‘referred to’ by those expressions.” [Text formation from the original]
\end{quote}

Tarski defines sentences as declarative statements and views truth as a property of sentences and explores how this property could be defined. He defines truth with the help of another semantic notion, satisfaction, which he describes as an “expression of relations” between sentences and objects that the sentences refer to.

Tarski’s truth definition can only be used for such languages whose structure is exactly specified, in other words where the sentences of the language are recursively\textsuperscript{5}

\textsuperscript{5}This section is primarily formal mathematics. Those parts that are formal mathematics are written using the academic we as is a tradition. The academic we refers to the author and the mathematical community at large. The parts that are formal mathematics are also written in present tense for the same reason.
build from *primitive sentences* and from combinations of sentences. The forming of combinations follow defined rules.

The truth definition is not applicable to natural languages as they do not strictly follow rules for creating sentences in the languages.

*Formulas*\(^6\) are also recursively defined. A formula is either an *atomic formula*\(^7\) or is a compound formula of other formulas, where the combination is done according to defined combination rules. Formulas are otherwise like sentences but they can include *free occurrences of variables*\(^8\), for example the variables \(x\) and \(y\) occur free in the formula \("x\) is greater than \(y"\). This means that the meaning of a formula depends on which objects, or actually names of objects, replace the free occurrences of the variables. The meaning of sentences in other hand does not depend on the interpretations of the free variables.

Tarski defines the *satisfaction of a formula* also recursively. First he defines which objects that satisfy the atomic formulas and then the satisfaction of compound formulas are defined according to their structure. This leads to Tarski’s truth definition (Tarski, 1944, p. 353):

> “a sentence is true if it is satisfied by all objects, and false otherwise.”

[Italics in the original.]

### 4.2 The truth definition for predicate logic

The definition given for predicate logic\(^9\) here follows that of Tarski and Vaught (1956). Here it is presented with the notation used by Väänänen (2007, 2010b), with small modifications. Differences in terminology from that of Tarski and Vaught are given in footnotes.

We use the notation \(\{a_1, \ldots, a_n\}\) for finite sets, where \(a_i\) are elements of sets and \(i\) is a natural number. If \(a_i\) are elements of the set \(A\), we denote by a \(n\)-tuple

\[
\langle a_1, \ldots, a_n \rangle
\]

an (ordered) sequence, with length \(n\), of elements in \(A\). A special case is the empty sequence \(\emptyset\), which corresponds to the case \(n = 0\). We denote \(a\) is an element if the

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\(^6\)Tarski uses the word *sentential functions* for formulas.

\(^7\)Tarski talks about the *simplest sential functions* instead of atomic formulas.

\(^8\)Tarski and Vaugh call these *free and bound variables* instead of free and bound occurrences of variables.

\(^9\)Tarski and Vaught use the label *truth definition of model-theoretic languages*. 

set $A$ by $a \in A$ and the Cartesian product
\[
\underbrace{A \times \cdots \times A}_{\text{n times}}
\]
as $A^n$ and thus $A^n$ is the set of all $n$-tuples of elements in $A$.

**4.1 Definition.** A relation $R$ of arity $n$ is a subset of $A^n$.

We think of a function $f : A \rightarrow A$ as a relation $\{(x, f(x)) : x \in A\}$ on $A$.

**4.2 Definition.** A vocabulary is a set $L$, that consists of symbols, in other words names of relations, functions and constants. The relation symbols are denoted by $R$, function symbols by $f$ and constant symbols by $c$.

All symbols in $L$ have an arity in the vocabulary $L$ given by the function $\sharp_L : L \rightarrow \mathbb{N}$. The arity of constant symbols is zero. The arity of a relation symbol may be zero. We use $x_0, x_1, \ldots$ to denote variables.

**4.3 Definition.** A $L$-model or $L$-structure $\mathcal{M}^{12}$ is a non-empty set $M$, which is the universe or domain of $\mathcal{M}$, endowed with

1. an element $c^M$ for each $c \in L$,
2. an $\sharp_L(R)$-ary relation $R^M$ on $M$ for each $R \in L$, and
3. an $\sharp_L(f)$-ary function $f^M$ on $M$ for each $f \in L$.

We write
\[
\mathcal{M} = \langle M, I_M \rangle,
\]
where $I_M = \{ R^M_1, \ldots, f^M_1, \ldots, c^M_1, \ldots \}^{13}$ is the interpretation of each symbol in the vocabulary $L$ to the model $\mathcal{M}$.

**4.4 Definition.** We define $L$-terms inductively as follows:

If $L$ is a vocabulary then the $L$-terms are

---

10 Tarski and Vaugh allow $n$ to be replaced by an possibly infinite ordinal $\omega$ and denote the set $A^\omega$ as the set of all sequences of $A^n$ with are eventually constant, in other words such that for some $m \in \mathbb{N}$, $x_n = x_m$ for all $n \geq m$.

11 Tarski and Vaugh uses the word rank instead of arity.

12 Tarski and Vaugh uses the word (relational) system instead of model or structure.

13 Väänänen (2010b) defines $I_M$ as a function.
1. the variable symbol $x$ is a $L$-term;

2. the constant symbol $c$ is a $L$-term;

3. if $f \in L, \sharp_L(f) = n$ and $t_1, \ldots, t_n$ are $L$-terms, then the function symbol $f(t_1, \ldots, t_n)$ is a $L$-term

**4.5 Definition.** We call any function $s$ that maps elements from a finite set of variables $X$ to elements of the universe of a model an assignment\textsuperscript{14} $s : X \rightarrow M$.

**4.6 Definition.** The *value* of a $L$-term assigned in a structure $M$ with the assignment $s$, noted by

$$ t^M (s) $$

is defined according to the inductive definition of $L$-terms as follows

1. if $t$ is a variable symbol $x_i$ then $t^M (s) = s(x_i)$;

2. if $t$ is a constant symbol $c$ then $t^M (s) = c^M$;

3. if $t$ is a function symbol $f(t_1, \ldots, t_n)$ then $t^M (s) = f^M(t_1^M (s), \ldots, t_n^M (s))$.

Terms without variables are called *constant terms* and their value do not depend on the assignment $s$ but have a definite value in the model so

$$ t^M (s) = t^M, \text{ for constant terms.} $$

**4.7 Definition.** We call the assignment $s(k/x_j)$ the *modified assignment* of $s$ where $k \in M$ and

$$ s(k/x_j)(x_i) = \begin{cases} 
  k, & \text{as } j = i \\
  s(x_i), & \text{as } j \neq i.
\end{cases} $$

**4.8 Definition.** Statements built using equations and relations of $L$-terms, logical connectives (and brackets for unambiguous reading) are called $L$-*formulas*. We define $L$-formulas as follows

1. if $t_1$ and $t_2$ are $L$-terms then $t_1 = t_2$ is an $L$-formula;

\textsuperscript{14}Tarski and Vaughan do not use the notion of assignment but speak of sequences $x \in A^{(n)}$. 
2. If $R \in L$, $\sharp_L(R) = n$ and $t_1, \ldots, t_n$ are $L$-terms, then $R(t_1, \ldots, t_n)$ is an $L$-formula;

3. If $\varphi$ is a $L$-formula, then $\neg \varphi$ is a formula;

4. If $\varphi$ and $\psi$ are $L$-formulas, then $(\varphi \land \psi)$ is a $L$-formula;

5. If $\varphi$ and $\psi$ are $L$-formulas, then $(\varphi \lor \psi)$ is a $L$-formula;

6. If $\varphi$ and $\psi$ are $L$-formulas, then $(\varphi \rightarrow \psi)$ is a $L$-formula;

7. If $\varphi$ and $\psi$ are $L$-formulas, then $(\varphi \leftrightarrow \psi)$ is a $L$-formula;

8. If $\varphi$ is a $L$-formula and $i$ a natural number, then $\exists x_i \varphi$ is a $L$-formula;

9. If $\varphi$ is a $L$-formula and $i$ a natural number, then $\forall x_i \varphi$ is a $L$-formula.

Formulas of type 1 and 2 are called atomic formulas. Formulas defined this way are called first-order.

4.9 Definition. Every occurrence of a variable $x_i$ in a formula $\varphi$ of the form $\exists x_i \varphi$ or $\forall x_i \varphi$ is called a bound occurrence. Occurrences which are not bound are called free occurrences.

Formulas with no free occurrences of variables are called sentences.

4.10 Definition (Tarski’s truth definition). If $\varphi$ is a $L$-formula and $s$ is an assignment, we say that $s$ satisfies $\varphi$ in a model $M = \langle M, I_M \rangle$ if one of the following conditions hold:

1. $\varphi$ is of the form $t_1 = t_2$, where $t_1$ and $t_2$ are $L$-terms, and $t_1^M(s) = t_2^M(s)$;

2. $\varphi$ is of the form $R(t_1, \ldots, t_n)$, where $t_i$ are $L$-terms, and $\langle t_1^M(s), \ldots, t_n^M(s) \rangle \in R^M$;

3. $\varphi$ is of the form $\neg \psi$, where $\psi$ is not satisfied by $s$;

4. $\varphi$ is of the form $(\psi' \land \psi'')$, where $\psi'$ and $\psi''$ are $L$-formulas which are both satisfied by $s$;

5. $\varphi$ is of the form $(\psi' \lor \psi'')$, where $\psi'$ and $\psi''$ are $L$-formulas and $\psi'$ or $\psi''$ are satisfied by $s$;

6. $\varphi$ is of the form $(\psi' \rightarrow \psi'')$, where $\psi'$ and $\psi''$ are $L$-formulas and $\psi'$ is not satisfied by $s$ or $\psi''$ is satisfied by $s$;
7. \( \varphi \) is of the form \((\psi' \leftrightarrow \psi'')\), where \( \psi' \) and \( \psi'' \) are \( L \)-formulas where \( \psi' \) and \( \psi'' \) are satisfied by \( s \), or \( \psi' \) and \( \psi'' \) are not satisfied by \( s \);

8. \( \varphi \) is of the form \( \exists x_k \psi \), \( \psi \) is a \( L \)-formula, and there is an element \( a \in M \) such that \( s(a/x_k) \) satisfies \( \psi \);

9. \( \varphi \) is of the form \( \forall x_k \psi \), \( \psi \) is a \( L \)-formula, and for all elements \( a \in M \) holds that \( s(a/x_k) \) satisfies \( \psi \).

We denote satisfaction of a formula \( \varphi \) in a model \( M \) with the assignment \( s \) by

\[ M \models_s \varphi. \]

We notice that if an assignment \( s \) satisfies a formula in a structure or not, depends only on the values of \( s \) on variables that occur free in the formula.

4.11 Definition. A sentence \( \sigma \) is said to be true in the structure \( M \) if every assignment \( s \) satisfies \( \sigma \) in \( M \). We denote it by

\[ M \models \sigma. \]

Under the same conditions we say that \( M \) is a model of \( s \). Sentences are true or false in a model, according to whether some (equivalently, all) assignments satisfy them, as they do not have free occurrences of variables.

4.12 Definition. The \( L \)-sentence \( \sigma \) is the logical consequence of the \( L \)-sentence \( \varsigma \). If for all \( L \)-models \( M \) and assignments \( s \) holds that

\[ \text{if } M \models_s \varsigma \text{ then } M \models_s \sigma \]

Two \( L \)-sentences are said to be logically equivalent if they are the logical consequences of each other.

4.13 Definition. The \( L \)-sentence \( \sigma \) is said to be valid if it is true in all models, in other words if

\[ M \models_s \sigma \text{ for all } L \text{-models } M \text{ and assignments } s. \]

4.3 Uniqueness of Tarski’s Truth Definition

Some of the philosophical difficulties with Tarski’s truth definition relate to the observation that the definition appears circular. It looks as if the definition only changes words from the natural language to logical connectives and quantifiers.
Here we give a proof of the uniqueness of Tarski’s truth definition. The definition
is a least fixed-point of a truth operator and the only consistent fixed-point of that
truth operator. In the proof we follow Väänänen (2011). For the proof of the
Knaster-Tarski theorem we follow Libkin (2004).

We use the following set theoretic notation: Let \( A \) and \( B \) be sets. \( A \) is a subset of
\( B \) is denoted by \( A \subset B \), a proper subset by \( A \subsetneq B \), \( \mathcal{P}(A) = \{B: B \subset A\} \) is the
power set of \( A \) and \( A \setminus B = \{x \in A: x \notin B\} \) is the relative complement of \( A \) in \( B \).
The union of \( A \) and \( B \) is \( A \cup B = \{x: x \in A \text{ or } x \in B\} \). The intersection of \( A \) and
\( B \) is \( A \cap B = \{x: x \in A \text{ and } x \in B\} \). The general intersection of the sets \( A_i \), where
\( i \in \mathbb{N} \), is \( \bigcap_{i=0}^{\infty} A_i = \bigcap A_i \).

We notice that if \( \mathcal{M} \) is a model, \( s \) an assignment and \( \varphi, \psi \) \( L \)-formulas, then
\[
\mathcal{M} \models_s (\varphi \land \psi) \text{ is logically equivalent to } \mathcal{M} \models_s \neg(\neg\varphi \lor \neg\psi);
\]
\[
\mathcal{M} \models_s (\varphi \rightarrow \psi) \text{ is logically equivalent to } \mathcal{M} \models_s (\neg\varphi \lor \psi);
\]
\[
\mathcal{M} \models_s (\varphi \leftrightarrow \psi) \text{ is logically equivalent to } \mathcal{M} \models_s \neg(\neg(\neg\varphi \lor \psi) \lor \neg(\varphi \lor \neg\psi));
\]
\[
\mathcal{M} \models_s \forall x \varphi \text{ is logically equivalent to } \mathcal{M} \models_s \neg\exists x \neg\varphi.
\]

Therefore we can restrict ourselves to formulas containing only the connectives \( \neg \) and \( \lor \) and the quantifier \( \exists \).

4.14 Definition. We fix a model \( \mathcal{M} \). Let \( S_0 \) be a set of triples such that
\[
S_0 = \{(s, \varphi, d) : s \text{ is an assignment, } \varphi \text{ is a formula and } d \in \{0, 1\}\}.
\]

We define the \textit{truth operator} \( \Gamma \) as a function inductively as follows:
\[
\Gamma: \mathcal{P}(S_0) \rightarrow \mathcal{P}(S_0).
\]

Let \( S \subset S_0 \), \( \varphi \) and \( \psi \) \( L \)-formulas and \( t, u, t_i \) \( L \)-terms. Let

1. \( (s, t = u, 1) \in \Gamma(S) \) if \( t^\mathcal{M}(s) = u^\mathcal{M}(s) \);
2. \( (s, t = u, 0) \in \Gamma(S) \) if \( t^\mathcal{M}(s) \neq u^\mathcal{M}(s) \);
3. \( (s, R(t_1, \ldots, t_m), 1) \in \Gamma(S) \) if \( (t_1^\mathcal{M}(s), \ldots, t_m^\mathcal{M}(s)) \in R^\mathcal{M} \);
4. \( (s, R(t_1, \ldots, t_m), 0) \in \Gamma(S) \) if \( (t_1^\mathcal{M}(s), \ldots, t_m^\mathcal{M}(s)) \notin R^\mathcal{M} \);
5. \( (s, \varphi \lor \psi, 1) \in \Gamma(S) \) if \( (s, \varphi, 1) \in S \) or \( (s, \psi, 1) \in S \);
6. \( (s, \varphi \lor \psi, 0) \in \Gamma(S) \) if \( (s, \varphi, 0) \in S \) and \( (s, \psi, 0) \in S \);
7. \( \langle s, \neg \varphi, 1 \rangle \in \Gamma(S) \) if \( \langle s, \varphi, 0 \rangle \in S \);

8. \( \langle s, \neg \varphi, 0 \rangle \in \Gamma(S) \) if \( \langle s, \varphi, 1 \rangle \in S \);

9. \( \langle s, \exists x \varphi, 1 \rangle \in \Gamma(S) \) if \( \langle s(a/x), \varphi, 1 \rangle \in S \) for some \( a \in M \);

10. \( \langle s, \exists x \varphi, 0 \rangle \in \Gamma(S) \) if \( \langle s(a/x), \varphi, 0 \rangle \in S \) for all \( a \in M \).

This means that the triples with atomic formulas are in \( \Gamma(S) \) depending on whether they hold in \( M \) or not. If the judgment \( M \models_s \varphi \) can be made on the basis of triples in \( S \), then \( \langle s, \varphi, 1 \rangle \in \Gamma(S) \). If the judgment \( M \models_s \varphi \) can be made on the basis of triples in \( S \), then \( \langle s, \varphi, 0 \rangle \in \Gamma(S) \).

4.15 Definition. Let \( U \) be a set. A monotone operator \( F \) on \( U \) is a mapping \( F : \mathcal{P}(U) \to \mathcal{P}(U) \) such that for all \( A, B \subseteq U \), it holds, that \( A \subseteq B \) implies \( F(A) \subseteq F(B) \).

4.16 Lemma. The truth operator \( \Gamma \) is a monotone operator on \( S_0 \).

Proof. Let \( A, B \subseteq S_0 \) be sets of triples as defined in definition 4.14. Let \( A \subseteq B \) and a triplet, \( a \in \Gamma(A) \). Now, there exists such a set \( C \subseteq A \) such that the judgment \( a \in \Gamma(A) \) can be made from triples in \( C \). As \( C \subseteq A \subseteq B \) then the judgment \( a \in \Gamma(B) \) can be made. Therefore \( \Gamma(A) \subseteq \Gamma(B) \) and \( \Gamma \) is a monotone operator on \( S_0 \).\[ \square \]

4.17 Definition. Given an operator \( F : \mathcal{P}(U) \to \mathcal{P}(U) \), a set \( A \subseteq U \) is a fixed-point of \( F \) if \( A = F(A) \). A set \( B \subseteq A \) is a least fixed-point of \( F \) if it is a fixed point, and for every other fixed point \( C \) of \( F \) we have \( B \subseteq C \). We denote the least fixed-point of \( F \) by \( \text{lfp}(F) \).

4.18 Theorem (Knaster-Tarski). Every monotone operator \( F : \mathcal{P}(U) \to \mathcal{P}(U) \) has a least fixed-point \( \text{lfp}(F) \) which can be defined as

\[
\text{lfp}(F) = \bigcap \{ Y : Y = F(Y) \}.
\]

Proof. Let \( W_F = \{ Y : F(Y) \subseteq Y \} \) and \( X_F = \bigcap W_F \). We notice that \( W_F \neq \emptyset \) as \( U \in W_F \). We show first that \( X_F \) is a fixed-point of \( F \).

Let \( A \in F(X_F) \). As \( X_F \subseteq Y \) for all \( Y \in W_F \) and because \( F \) is a monotone operator, we have that \( F(X_F) \subseteq F(Y) \). Now \( A \in F(Y) \) for all \( Y \in W_F \). Therefore \( A \in X_F \) and \( F(X_F) \subseteq X_F \).
Because \( X_F \in \mathcal{W}_F \) we have that \( F(X_F) \subset X_F \). Because \( F \) is a monotone operator we have that \( F(F(X_F)) \subset F(X_F) \). Therefore \( F(X_F) \in \mathcal{W}_F \). Because \( X_F \subset Y \) for all \( Y \in \mathcal{W}_F \) we have that \( X_F \subset F(X_F) \).

We have shown that \( X_F = F(X_F) \).

We then show that \( X_F \) is a least fixed-point. Let \( \mathcal{W}_F' = \{ Y : F(Y) = Y \} \). Now \( X_F \in \mathcal{W}_F' \), hence \( X_F' \subset X_F \). On the other hand, as \( \mathcal{W}_F' \subset \mathcal{W}_F \) we have that \( X_F' \in \mathcal{W}_F \) and hence \( X_F \subset X_F' \). We have shown that \( X_F = X_F' \), hence \( X_F \) is a fixed-point.

As \( X_F \) is the intersection of all fixed-points it is the least fixed-point, and

\[
\text{lfp}(F) = \bigcap \{ Y : F(Y) = Y \} = \bigcap \{ Y : F(Y) \subset Y \}.
\]

\[
\square
\]

4.19 Definition. We can reformulate Tarski’s truth definition as follows. \( T \) is the set of triples such that

\[
T = \{ \langle s, \varphi, 1 \rangle : \mathcal{M} \models_s \varphi \} \cup \{ \langle s, \varphi, 0 \rangle : \mathcal{M} \not\models_s \varphi \}.
\]

4.20 Theorem. The set \( T \) is the least fixed-point of the truth operator \( \Gamma \).

Proof. We show that

\[
T = \text{lfp}(\Gamma) = \bigcap \{ S \subset S_0 : \Gamma(S) \subset S \}.
\]

Let \( \mathcal{W}_T = \{ S \subset S_0 : \Gamma(S) \subset S \} \) and \( \langle s, \varphi, d \rangle \in T \).

We show by induction on \( \varphi \), that if \( \langle s, \varphi, d \rangle \in T \) then \( \langle s, \varphi, d \rangle \in X_\Gamma \).

Let \( t, u, t_i \) be \( L \)-terms and \( \psi, \gamma \) be \( L \)-formulas.

Case \( \varphi \) is \( t = u \): Let \( d = 1 \). As \( \langle s, \varphi, 1 \rangle \in T \) then \( \mathcal{M} \models_s t = u \). Now \( t^\mathcal{M} \langle s \rangle = u^\mathcal{M} \langle s \rangle \), hence \( \langle s, t = u, 1 \rangle \in \Gamma(S) \). As \( \Gamma(S) \subset S \) for all \( S \in \mathcal{W}_T \) we have that \( \langle s, t = u, 1 \rangle \in X_\Gamma \). The case for \( d = 0 \) is similar.

Case \( \varphi \) is \( R(t_1, \ldots, t_m) \), \( 1 \leq i \leq m \in \mathbb{N} \): Let \( d = 1 \). As \( \langle s, \varphi, 1 \rangle \in T \) then \( \mathcal{M} \models_s R(t_1, \ldots, t_m) \), \( 1 \leq i \leq m \in \mathbb{N} \). Then \( t_i^\mathcal{M} \langle s \rangle, \ldots, t_m^\mathcal{M} \langle s \rangle \in R^\mathcal{M} \), hence \( \langle s, R(t_1, \ldots, t_m), 1 \rangle \in \Gamma(S) \). As \( \Gamma(S) \subset S \) for all \( S \in \mathcal{W}_T \) we have that \( \langle s, R(t_1, \ldots, t_m), 1 \rangle \in \mathcal{W}_T \). The case for \( d = 0 \) is similar.

Case \( \varphi \) is \( (\psi \lor \gamma) \): We make the induction hypothesis that if \( \langle s, \psi, d_0 \rangle, \langle s, \gamma, d_0 \rangle \in T \), then \( \langle s, \psi, d_0 \rangle, \langle s, \gamma, d_0 \rangle \in X_\Gamma \). Let \( d = 1 \) and \( \langle s, (\psi \lor \gamma), 1 \rangle \in T \). Now \( \mathcal{M} \models_s (\psi \lor \gamma) \), so \( \mathcal{M} \models_s \psi \) or \( \mathcal{M} \models_s \gamma \). By the induction hypothesis \( \langle s, \psi, 1 \rangle \in \mathcal{W}_T \) and \( \langle s, \gamma, 1 \rangle \in \mathcal{W}_T \). Therefore \( \langle s, (\psi \lor \gamma), 1 \rangle \in \mathcal{W}_T \) and \( \langle s, (\psi \lor \gamma), 1 \rangle \in \Gamma(S) \). As \( \Gamma(S) \subset S \) for all \( S \in \mathcal{W}_T \) we have that \( \langle s, (\psi \lor \gamma), 1 \rangle \in \mathcal{W}_T \). The case for \( d = 0 \) is similar.
$X_\Gamma$ or $\langle s, \gamma, 1 \rangle \in X_\Gamma$. Then for all $S \supset \Gamma(S)$ holds that $\langle s, (\psi \vee \gamma), 1 \rangle \in \Gamma(S)$ and therefore $\langle s, (\psi \vee \gamma), 1 \rangle \in S$. Now we have that $\langle s, (\psi \vee \gamma), 1 \rangle \in X_\Gamma$. The case for $d = 0$ is similar.

**Case $\varphi$ is $\neg \psi$:** We make the induction hypothesis that if $\langle s, \psi, d_0 \rangle \in T$, then $\langle s, \psi, d_0 \rangle \in X_\Gamma$. Let $d = 1$ and $\langle s, \neg \psi, 1 \rangle \in T$. Now $M \models_s \neg \psi$, so $M \not\models_s \psi$ and therefore $\langle s, \psi, 0 \rangle \in T$. By the induction hypothesis $\langle s, \psi, 0 \rangle \in X_\Gamma$. Then for all $S \supset \Gamma(S)$ holds that $\langle s, \neg \psi, 1 \rangle \in \Gamma(S)$ and therefore $\langle s, \neg \psi, 1 \rangle \in S$. Now we have that $\langle s, \neg \psi, 1 \rangle \in X_\Gamma$. The case for $d = 0$ is similar.

**Case $\varphi$ is $\exists x \psi$:** We make the induction hypothesis that if $\langle s, \psi, d_0 \rangle \in T$, then $\langle s, \psi, d_0 \rangle \in X_\Gamma$. Let $d = 1$ and $\langle s, \exists x \psi, 1 \rangle \in T$. Now $M \models_s \exists x \psi$, so for some $a \in M$ $M \models (a/x) \psi$ and therefore $\langle s(a/x), \psi, 1 \rangle \in T$. By the induction hypothesis $\langle s(a/x), \psi, 1 \rangle \in X_\Gamma$. Then for all $S \supset \Gamma(S)$ holds that $\langle s(a/x), \psi, 1 \rangle \in \Gamma(S)$ and therefore $\langle s, \exists x \psi, 1 \rangle \in S$. Now we have that $\langle s, \neg \psi, 1 \rangle \in X_\Gamma$. The case for $d = 0$ is similar.

We have that $T \subset X_\Gamma$.

We then show that $X_\Gamma \subset T$. Let $\langle s, \varphi, d \rangle \in X_\Gamma$. We show by induction on $\varphi$, that if $\langle s, \varphi, d \rangle \in X_\Gamma$, then $\langle s, \varphi, d \rangle \in T$. Let $t, u, t_i$ be $L$-terms and $\psi, \gamma$ be $L$-formulas.

**Case $\varphi$ is $t = u$:** Let $d = 1$. We use a proof by contradiction. Suppose $\langle s, t = u, 1 \rangle \in X_\Gamma$ but $\langle s, t = u, 1 \rangle \not\in T$. Now, $M \not\models_t t = u$, hence $t^M \models \langle s \rangle \not\equiv u^M \langle s \rangle$. Therefore for all $S \in W_\Gamma$, $\langle s, t = u, 1 \rangle \not\in \Gamma(S)$. Let $S_\varphi = S_0 \setminus \{ \langle s, t = u, 1 \rangle \}$. As $\langle s, t = u, 1 \rangle \not\in \Gamma(S_\varphi)$, we have $\Gamma(S_\varphi) \subset S_\varphi$. So, $\langle s, t = u, 1 \rangle \not\in X_\Gamma$ which is a contradiction. We have shown that $\langle s, t = u, 1 \rangle \in T$. The case for $d = 0$ is similar.

**Case $\varphi$ is $R(t_1, \ldots, t_m), 1 \leq i \leq m \in \mathbb{N}$:** Let $d = 1$. We use a proof by contradiction. Suppose $\langle s, R(t_1, \ldots, t_m), 1 \rangle \in X_\Gamma$ but $\langle s, R(t_1, \ldots, t_m), 1 \rangle \not\in T$. Now, $M \not\models_s R(t_1, \ldots, t_m)$, hence $\langle t_1^M, \ldots, t_m^M \rangle \not\in R^M$. Therefore for all $S \in W_\Gamma$, $\langle s, R(t_1, \ldots, t_m), 1 \rangle \not\in \Gamma(S)$. Let $S_\varphi = S_0 \setminus \{ \langle s, R(t_1, \ldots, t_m), 1 \rangle \}$. As $\langle s, t = u, 1 \rangle \not\in \Gamma(S_\varphi)$, we have $\Gamma(S_\varphi) \subset S_\varphi$. So, $\langle s, R(t_1, \ldots, t_m), 1 \rangle \not\in X_\Gamma$ which is a contradiction. We have shown that $\langle s, R(t_1, \ldots, t_m), 1 \rangle \in T$. The case for $d = 0$ is similar.

**Case $\varphi$ is $(\psi \vee \gamma)$:** We make the induction hypothesis that if $\langle s, \psi, d_0 \rangle, \langle s, \gamma, d_0 \rangle \in X_\Gamma$ then $\langle s, \psi, d_0 \rangle, \langle s, \gamma, d_0 \rangle \in T$. Let $d = 1$. We use a proof by contradiction.
Suppose \( \langle s, (\psi \lor \gamma), 1 \rangle \in X_\Gamma \) but \( \langle s, (\psi \lor \gamma), 1 \rangle \notin T \). Therefore \( M \not\models_s (\psi \lor \gamma) \) leads to \( M \not\models_s \gamma \) and \( M \not\models_s \psi \), hence \( \langle s, \psi, 1 \rangle, \langle s, \gamma, 1 \rangle \notin T \). By the induction hypothesis
\[
\langle s, \psi, 1 \rangle, \langle s, \gamma, 1 \rangle \notin X_\Gamma.
\] (4.21)

Let \( S_\varphi = X_\Gamma \setminus \{ \langle s, (\psi \lor \gamma), 1 \rangle \} \). We show by induction on triples \( x \in S_\varphi \) that \( \Gamma(S_\varphi) \subset S_\varphi \). Let \( \theta_i \) be \( L \)-formulas.

**Case** \( x \) is \( \langle s, t = u, 1 \rangle \): Let \( \langle s, t = u, 1 \rangle \in \Gamma(S_\varphi) \). Then this is because \( t^M \langle s \rangle = u^M \langle s \rangle \). Clearly \( t = u \neq (\psi \lor \gamma) \) hence \( x \in \Gamma(X_\Gamma) \). As \( \Gamma(X_\Gamma) \subset X_\Gamma \) then \( \langle s, t = u, 1 \rangle \in X_\Gamma \). Since \( t = u \neq (\psi \lor \gamma) \), we get \( \langle s, t = u, 1 \rangle \in S_\varphi \). Case \( x \) is \( \langle s, t = u, 0 \rangle \) is similar.

**Case** \( x \) is \( \langle s, R(t_1, ..., t_m), 1 \rangle \): Let \( \langle s, R(t_1, ..., t_m), 1 \rangle \in \Gamma(S_\varphi) \). Then this is because
\[
\langle t_1^M \langle s \rangle, ..., t_m^M \langle s \rangle \rangle \in R^M. \quad \text{Clearly } R(t_1, ..., t_m) \neq (\psi \lor \gamma) \text{ hence } x \in \Gamma(X_\Gamma). \quad \text{As } \Gamma(X_\Gamma) \subset X_\Gamma \text{ then } \langle s, R(t_1, ..., t_m), 1 \rangle \in X_\Gamma. \quad \text{Since } R(t_1, ..., t_m) \neq (\psi \lor \gamma), \text{ we get } \langle s, R(t_1, ..., t_m), 1 \rangle \in S_\varphi. \quad \text{Case } x \text{ is } \langle s, R(t_1, ..., t_m), 0 \rangle \text{ is similar.}
\]

**Case** \( x \) is \( \langle s, \neg \theta_1, 1 \rangle \): Let \( \langle s, \neg \theta_1, 1 \rangle \in \Gamma(S_\varphi) \). Then this is because \( \langle s, \theta_1, 0 \rangle \in S_\varphi \). Clearly \( \neg \theta_1 \neq (\psi \lor \gamma) \) hence \( x \in \Gamma(X_\Gamma) \). As \( \Gamma(X_\Gamma) \subset X_\Gamma \) then \( \langle s, \neg \theta_1, 1 \rangle \in X_\Gamma \). Since \( \neg \theta_1 \neq (\psi \lor \gamma) \), we get \( \langle s, \neg \theta_1, 1 \rangle \in S_\varphi \). Case \( x \) is \( \langle s, \neg \theta_1, 0 \rangle \) is similar.

**Case** \( x \) is \( \langle s, (\theta_1 \lor \theta_2), 1 \rangle \): Let \( \langle s, (\theta_1 \lor \theta_2), 1 \rangle \in \Gamma(S_\varphi) \). Then this is because \( \langle s, \theta_1, 1 \rangle \in S_\varphi \) or \( \langle s, \theta_2, 1 \rangle \in S_\varphi \). By (4.21), in either case \( (\theta_1 \lor \theta_2) \neq (\psi \lor \gamma) \) hence \( x \in \Gamma(X_\Gamma) \). As \( \Gamma(X_\Gamma) \subset X_\Gamma \) then \( (\theta_1 \lor \theta_2) \in X_\Gamma \). Since \( (\theta_1 \lor \theta_2) \neq (\psi \lor \gamma) \), we get \( \langle s, (\theta_1 \lor \theta_2), 1 \rangle \in S_\varphi \). Case \( x \) is \( \langle s, (\theta_1 \lor \theta_2), 0 \rangle \) is similar.

**Case** \( x \) is \( \langle s, \exists x \theta_1, 1 \rangle \): Let \( \langle s, \exists x \theta_1, 1 \rangle \in \Gamma(S_\varphi) \). Then this is because for some \( a \in M \), \( \langle s(a/x), \theta_1, 1 \rangle \in S_\varphi \). Clearly \( \exists x \theta_1 \neq (\psi \lor \gamma) \) hence \( x \in \Gamma(X_\Gamma) \). As \( \Gamma(X_\Gamma) \subset X_\Gamma \) then \( \langle s, \exists x \theta_1, 1 \rangle \in X_\Gamma \). Since \( \exists x \theta_1 \neq (\psi \lor \gamma) \), we get \( \langle s, \exists x \theta_1, 1 \rangle \in S_\varphi \). Case \( x \) is \( \langle s, \exists x \theta_1, 0 \rangle \) is similar.

Now we have shown that \( \Gamma(S_\varphi) \subset S_\varphi \). Then \( S_\varphi \in W_\Gamma \) but \( S_\varphi \not\subset X_\Gamma \) which is a contradiction. Therefore \( \langle s, \psi, 1 \rangle \in T \). The case for \( d = 0 \) is similar.

**Case** \( \varphi \) is \( \neg \psi \): We make the induction hypothesis that if \( \langle s, \psi, d_0 \rangle \in X_\Gamma \) then \( \langle s, \psi, d_0 \rangle \in T \). Let \( d = 1 \). We use a proof by contradiction. Suppose
\( \langle s, -\psi, 1 \rangle \in X_{\Gamma} \) but \( \langle s, -\psi, 1 \rangle \notin T \). Therefore \( \mathcal{M} \not\models_s \psi \) leads to \( \mathcal{M} \models_s \neg \psi \), hence \( \langle s, -\psi, 1 \rangle \notin T \). By the induction hypothesis

\[
\langle s, \psi, 0 \rangle \notin X_{\Gamma}. \tag{4.22}
\]

Let \( S_{\varphi} = X_{\Gamma} \setminus \{ \langle s, -\psi, 1 \rangle \} \). We show by induction on triples \( x \in S_{\varphi} \) that \( \Gamma(S_{\varphi}) \subset S_{\varphi} \). Let \( \theta_i \) be \( L \)-formulas.

**Case** \( x \) **is** \( \langle s, t = u, 1 \rangle \): Let \( \langle s, t = u, 1 \rangle \in \Gamma(S_{\varphi}) \). Then this is because \( t^M \langle s \rangle = u^M \langle s \rangle \). Clearly \( t = u \neq -\psi \) hence \( x \in \Gamma(X_{\Gamma}) \). As \( \Gamma(X_{\Gamma}) \subset X_{\Gamma} \) then \( \langle s, t = u, 1 \rangle \in X_{\Gamma} \). Since \( t = u \neq -\psi \), we get \( \langle s, t = u, 1 \rangle \in S_{\varphi} \). Case \( x \) is \( \langle s, t = u, 0 \rangle \) is similar.

**Case** \( x \) **is** \( \langle s, R(t_1, \ldots, t_m), 1 \rangle \): Let \( \langle s, R(t_1, \ldots, t_m), 1 \rangle \in \Gamma(S_{\varphi}) \). Then this is because

\[
\langle t^m_1 \langle s \rangle, \ldots, t^m_m \langle s \rangle \rangle \in R^M. \]

Clearly \( R(t_1, \ldots, t_m) \neq -\psi \) hence \( x \in \Gamma(X_{\Gamma}) \). As \( \Gamma(X_{\Gamma}) \subset X_{\Gamma} \) then \( \langle s, R(t_1, \ldots, t_m), 1 \rangle \in X_{\Gamma} \). Since \( R(t_1, \ldots, t_m) \neq -\psi \), we get \( \langle s, R(t_1, \ldots, t_m), 1 \rangle \in S_{\varphi} \). Case \( x \) is \( \langle s, R(t_1, \ldots, t_m), 0 \rangle \) is similar.

**Case** \( x \) **is** \( \langle s, -\theta_1, 1 \rangle \): Let \( \langle s, -\theta_1, 1 \rangle \in \Gamma(S_{\varphi}) \). Then this is because \( \langle s, \theta_1, 0 \rangle \in S_{\varphi} \). By (4.22), \( -\theta_1 \neq -\psi \) hence \( x \in \Gamma(X_{\Gamma}) \). As \( \Gamma(X_{\Gamma}) \subset X_{\Gamma} \) then \( \langle s, -\theta_1, 1 \rangle \in X_{\Gamma} \). Since \( -\theta_1 \neq -\psi \), we get \( \langle s, -\theta_1, 1 \rangle \in S_{\varphi} \). Case \( x \) is \( \langle s, -\theta_1, 0 \rangle \) is similar.

**Case** \( x \) **is** \( \langle s, (\theta_1 \lor \theta_2), 1 \rangle \): Let \( \langle s, (\theta_1 \lor \theta_2), 1 \rangle \in \Gamma(S_{\varphi}) \). Then this is because \( \langle s, \theta_1, 1 \rangle \in S_{\varphi} \) or \( \langle s, \theta_1, 1 \rangle \in S_{\varphi} \). Clearly \( \theta_1 \lor \theta_2 \neq -\psi \) hence \( x \in \Gamma(X_{\Gamma}) \). As \( \Gamma(X_{\Gamma}) \subset X_{\Gamma} \) then \( \langle s, (\theta_1 \lor \theta_2), 1 \rangle \in X_{\Gamma} \). Since \( \theta_1 \lor \theta_2 \neq -\psi \), we get \( \langle s, (\theta_1 \lor \theta_2), 1 \rangle \in S_{\varphi} \). Case \( x \) is \( \langle s, (\theta_1 \lor \theta_2), 0 \rangle \) is similar.

**Case** \( x \) **is** \( \langle s, \exists a \theta_1, 1 \rangle \): Let \( \langle s, \exists a \theta_1, 1 \rangle \in \Gamma(S_{\varphi}) \). Then this is because for some \( a \in M \), \( \langle s(a/x), \theta_1, 1 \rangle \in S_{\varphi} \). Clearly \( \exists a \theta_1 \neq -\psi \) hence \( x \in \Gamma(X_{\Gamma}) \). As \( \Gamma(X_{\Gamma}) \subset X_{\Gamma} \) then \( \langle s, \exists a \theta_1, 1 \rangle \in X_{\Gamma} \). Since \( \exists a \theta_1 \neq -\psi \), we get \( \langle s, \exists a \theta_1, 1 \rangle \in S_{\varphi} \). Case \( x \) is \( \langle s, \exists a \theta_1, 0 \rangle \) is similar.

Now we have shown that \( \Gamma(S_{\varphi}) \subset S_{\varphi} \). Then \( S_{\varphi} \in W_{\Gamma} \) but \( S_{\varphi} \not\subset X_{\Gamma} \) which is a contradiction. Therefore \( \langle s, \psi, 1 \rangle \in T \). The case for \( d = 0 \) is similar.

**Case** \( \varphi \) **is** \( \exists x \psi \): We make the induction hypothesis that if \( \langle s, \psi, d_0 \rangle \in X_{\Gamma} \) then \( \langle s, \psi, d_0 \rangle \in T \). Let \( d = 1 \). We use a proof by contradiction. Suppose \( \langle s, \exists x \psi, 1 \rangle \in X_{\Gamma} \) but \( \langle s, \exists x \psi, 1 \rangle \notin T \). Therefore \( \mathcal{M} \not\models_s \exists x \psi \) leads to that
for all $a \in M$, $M \not\models_{s(a/x)} \psi$, hence $\langle s, \exists x \psi, 1 \rangle \not\in T$. By the induction hypothesis

$$\langle s(a/x), \psi, 1 \rangle \not\in X_\Gamma.$$  \hfill (4.23)

Let $S_\varphi = X_\Gamma \setminus \{ \langle s, \exists x \psi, 1 \rangle \}$. We show by induction on triples $x \in S_\varphi$ that $\Gamma(S_\varphi) \subset S_\varphi$. Let $\theta_i$ be $L$-formulas.

Case $x$ is $\langle s, t = u, 1 \rangle$: Let $\langle s, t = u, 1 \rangle \in \Gamma(S_\varphi)$. Then this is because $t^M \langle s \rangle = u^M \langle s \rangle$. Clearly $t = u \neq \exists x \psi$ hence $x \in \Gamma(X_\Gamma)$. As $\Gamma(X_\Gamma) \subset X_\Gamma$ then $\langle s, t = u, 1 \rangle \in X_\Gamma$. Since $t = u \neq \exists x \psi$, we get $\langle s, t = u, 1 \rangle \not\in S_\varphi$. Case $x$ is $\langle s, t = u, 0 \rangle$ is similar.

Case $x$ is $\langle s, R(t_1, ..., t_m), 1 \rangle$: Let $\langle s, R(t_1, ..., t_m), 1 \rangle \in \Gamma(S_\varphi)$. Then this is because $\langle t_1^M \langle s \rangle, ..., t_m^M \langle s \rangle \rangle \in R^M$. Clearly $R(t_1, ..., t_m) \neq \exists x \psi$ hence $x \in \Gamma(X_\Gamma)$. As $\Gamma(X_\Gamma) \subset X_\Gamma$ then $\langle s, R(t_1, ..., t_m), 1 \rangle \in X_\Gamma$. Since $R(t_1, ..., t_m) \neq \exists x \psi$, we get $\langle s, R(t_1, ..., t_m), 1 \rangle \not\in S_\varphi$. Case $x$ is $\langle s, R(t_1, ..., t_m), 0 \rangle$ is similar.

Case $x$ is $\langle s, -\theta_1, 1 \rangle$: Let $\langle s, -\theta_1, 1 \rangle \in \Gamma(S_\varphi)$. Then this is because $\langle s, \theta_1, 0 \rangle \in S_\varphi$. Clearly $-\theta_1 \neq \exists x \psi$ hence $x \in \Gamma(X_\Gamma)$. As $\Gamma(X_\Gamma) \subset X_\Gamma$ then $\langle s, -\theta_1, 1 \rangle \in X_\Gamma$. Since $-\theta_1 \neq \exists x \psi$, we get $\langle s, -\theta_1 \rangle \not\in S_\varphi$. Case $x$ is $\langle s, -\theta_1, 0 \rangle$ is similar.

Case $x$ is $\langle s, (\theta_1 \lor \theta_2), 1 \rangle$: Let $\langle s, (\theta_1 \lor \theta_2), 1 \rangle \in \Gamma(S_\varphi)$. Then this is because $\langle s, \theta_1, 1 \rangle \in S_\varphi$ or $\langle s, \theta_1, 1 \rangle \in S_\varphi$. Clearly $(\theta_1 \lor \theta_2) \neq \exists x \psi$ hence $x \in \Gamma(X_\Gamma)$. As $\Gamma(X_\Gamma) \subset X_\Gamma$ then $\langle s, (\theta_1 \lor \theta_2), 1 \rangle \in X_\Gamma$. Since $(\theta_1 \lor \theta_2) \neq \exists x \psi$, we get $\langle s, (\theta_1 \lor \theta_2), 1 \rangle \not\in S_\varphi$. Case $x$ is $\langle s, (\theta_1 \lor \theta_2), 0 \rangle$ is similar.

Case $x$ is $\langle s, \exists x \theta_1, 1 \rangle$: Let $\langle s, \exists x \theta_1, 1 \rangle \in \Gamma(S_\varphi)$. Then this is because for some $a \in M$, $\langle s(a/x), \theta_1, 1 \rangle \in S_\varphi$. By (4.23), $\exists x \theta_1 \neq \exists x \psi$ hence $x \in \Gamma(X_\Gamma)$. As $\Gamma(X_\Gamma) \subset X_\Gamma$ then $\langle s, \exists x \theta_1, 1 \rangle \in X_\Gamma$. Since $\exists x \theta_1 \neq \exists x \psi$, we get $\langle s, \exists x \theta_1, 1 \rangle \not\in S_\varphi$. Case $x$ is $\langle s, \exists x \theta_1, 0 \rangle$ is similar.

Now we have shown that $\Gamma(S_\varphi) \subset S_\varphi$. Then $S_\varphi \in W_\Gamma$ but $S_\varphi \not\subset X_\Gamma$, which is a contradiction. Therefore $\langle s, \psi, 1 \rangle \in T$. The case for $d = 0$ is similar.

We have shown that $X_\Gamma \subset T$, hence $T = X_\Gamma = \text{lfp}(\Gamma)$. \hfill \qed

**4.24 Definition.** A set of triples is *consistent* if it does not contain both $\langle s, \varphi, 1 \rangle$ and $\langle s, \varphi, 0 \rangle$ and *inconsistent* otherwise.
4.25 Theorem. $T$ is the only consistent fixed-point of $\Gamma$.

Proof. Suppose $T'$ is another fixed-point. As $T$ is the least fixed-point we have that $T \subset T'$. Suppose that $\langle s, \varphi, 1 \rangle \in T' \setminus T$. Then $\langle s, \varphi, 0 \rangle \in T$ and therefore $\langle s, \varphi, 0 \rangle \in T'$. So $T'$ is inconsistent. \(\qed\)

4.4 Examples of students’ difficulties

The difficulties students faced when encountering the Tarski’s truth definition can be characterized as technical and philosophical. That these difficulties emerged is the strength of the Extreme Apprenticeship method.

The technical problems lay with understanding the concept of interpretation of terms in a structure. More specifically the notion of interpreting a variable to a model with a value of an assignment, see definition 4.5. Especially this is the case when dealing with the satisfaction of a quantified $L$-formula that requires the use of the notion of a modified assignment, see definitions 4.7, 4.10.8 and 4.10.9.

This is called a technical problem as it was mostly related to the notation used. A typical problem was to distinguish the difference between the name of the assignment and a value of an assignment:

\[
s(k/x_2) \quad (4.26) \\
\]
\[
s(x_2) \quad (4.27) \\
\]
\[
s(k/x_2)(x_2) \quad (4.28) \\
\]

where (4.26) stands for a name of a modified assignment, (4.27) is the value of the assignment $s$ in respect to the variable $x_2$ and (4.28) is the value of the modified assignment $s(k/x_2)$ in respect to the variable $x_2$ (which is $k$).

These kinds of problems were typical to be revealed during instruction. Even if they seem trivial or small, they can be a major reason why students are unable to grasp a concept. With knowledge of what the problem was precisely these problems could be tackled with instruction.

Scaffolding, in order to solve the problem, was done as follows. During the week when Tarski’s truth definition was introduced, the beginning exercises were very easy. The students were expected to solve them by themselves and no hints about the possible solutions were given. Only the possible mistakes were discussed with the students. This meant, for example discussing why the student thought that the
truth definition worked in the way he used it and showing the student what his understanding of the definitions leads to.

As the exercises were easy and small the mistakes and misunderstandings were easy to point out. The small exercises also led to quick correction of the exercises that were handed in. The intent was to get to a point where the instructor would know that the students had mastered to use the notation by looking at the exercises they had handed in for inspection.

The philosophical difficulty students face is that even if sounding pompous, the definition of truth does not give any insight into the mysterious “essence” of truth. This has also to do with that the definition appears circular, as was unproven in the previous section. As Tarski (1944, p. 361) puts it:

“I have heard it remarked that the formal definition of truth has nothing to do with ‘the philosophical problem of truth’. How- ever, nobody has ever pointed out to me in an intelligible way just what this problem is. I have been informed in this connection that my definition, though it states necessary and sufficient conditions for a sentence to be true, does not really grasp the ‘essence’ of this concept. Since I have never been able to understand what the ‘essence’ of a concept is, I must be excused from discussing this point any longer.

In general, I do not believe that there is such a thing as ‘the philosophical problem of truth’. I do believe that there are various intelligible and interesting (but not necessarily philosophical) problems concerning the notion of truth, but I also believe that they can be exactly formulated and possibly solved only on the basis of a precise conception of this notion.”

It has to be pointed that calling this a “difficulty students face” should not be interpreted as something that should be avoided. The discussions with students about the philosophical issue were really a joy and something not experienced by the instructor in previous years while being an teaching assistant. A student who can discuss the philosophical problems with Tarski’s truth definition clearly has reached some of the learning objectives of the course. He has also probably overcome the technical problems with the issue.
4.5 The structure of the exercises

In this section is given the exercises of week seven\(^{15}\). During that week Tarski’s truth definition for predicate logic was introduced for the first time in the experimental group.

The attempt of helping the students learn the truth definition started by giving the students 18 basic exercises\(^{16}\):

Let \( M = \{a, b, c, d\} \) and \( R^M = \{\langle a, b \rangle, \langle b, a \rangle, \langle b, c \rangle, \langle c, d \rangle, \langle d, a \rangle, \langle d, d \rangle\} \).

Does the assignment \( s_i \) satisfy the formula \( (R(x, y) \rightarrow \neg R(x, z)) \) in the model \( \mathcal{M} = \langle M, \{R^M\} \rangle \)? Motivate your answer.

1. \( s_0(x) = a, s_0(y) = b, s_0(z) = c \)
2. \( s_1(x) = a, s_1(y) = d, s_1(z) = c \)
3. \( s_2(x) = a, s_2(y) = d, s_2(z) = a \)
4. \( s_3(x) = d, s_3(y) = d, s_3(z) = d \)
5. \( s_4(x) = b, s_4(y) = b, s_4(z) = c \)
6. \( s_5(x) = b, s_5(y) = a, s_5(z) = c \)
7. \( s_6(x) = c, s_6(y) = d, s_6(z) = c \)
8. \( s_7(x) = c, s_7(y) = d, s_7(z) = b \)
9. \( s_8(x) = a, s_8(y) = b, s_8(z) = b \)
10. \( s_9(x) = b, s_9(y) = d, s_9(z) = c \)
11. \( s_{10}(x) = c, s_{10}(y) = a, s_{10}(z) = b \)
12. Does it hold that \( \mathcal{M} \models s_0 \exists x (R(x, y) \rightarrow \neg R(x, z)) \)?
13. Does it hold that \( \mathcal{M} \models s_0 \exists y (R(x, y) \rightarrow \neg R(x, z)) \)?
14. Does it hold that \( \mathcal{M} \models s_0 \exists z (R(x, y) \rightarrow \neg R(x, z)) \)?
15. Does it hold that \( \mathcal{M} \models s_0 \exists y \exists x (R(x, y) \rightarrow \neg R(x, z)) \)?
16. Does it hold that \( \mathcal{M} \models s_0 \exists z \exists x (R(x, y) \rightarrow \neg R(x, z)) \)?
17. Does it hold that \( \mathcal{M} \models s_0 \exists z \exists y \exists x (R(x, y) \rightarrow \neg R(x, z)) \)?

\(^{15}\)The exercises used were heavily influenced by the exercises prepared by lecturers of the traditional course instances.

\(^{16}\)Some changes are made from the original to fit the notation used in the definitions earlier. The exercises are translated from Finnish by the author.
18. Does it hold that $\mathcal{M} \models_{s_0} \neg \forall x (R(x, y) \rightarrow \neg R(x, z))$?

These exercises should not be misunderstood as being only a form of teaching with drill and practice. The aim was to introduce a new concept by (1) beginning with easy assignments where the students should feel that “I can do one exercise more” and keep going even if the exercises become more difficult. By monitoring the work of the students the instructor (2) kept an eye on the notation the students used, so that they could learn the correct way of formulating their ideas from the beginning.

Even if the basic assignments were easy, students did mistakes while solving them. As the exercises were small it was quickly evident for the instructor and the student where a mistake took place. This gave the instructor an opportunity (3) to discuss and recognise the mistakes instantly when they occurred.

It should be remembered in discussions like these, that it is the student who should understand why his approach was wrong and what the correct approach should be. The instructor should not give the correct answer and steal the satisfactory feeling of revelation from the student.

An example of the use of the this method is exercise 18. The students had to think when a formula is not satisfied in a model. A typical mistake was to assume that in a situation like this

$$\mathcal{M} \models_{s} (A \lor B)$$

is according to definition 4.10

$$\mathcal{M} \not\models_{s} A \text{ or } \mathcal{M} \not\models_{s} B$$

when the correct answer would be

$$\mathcal{M} \not\models_{s} A \text{ and } \mathcal{M} \not\models_{s} B.$$  

If the instructor gives the right answer, then there is a big possibility that the student only learns a new rule. But if the student first understands why his initial thinking when wrong, he will get an unique opportunity to understand the whole idea of a model satisfying a formula with an assignment means.

It has to be remembered that the students handed in readymade exercise after every session, so for the instructor it was clear all the time if these simple exercises were understood and correctly completed.
After success in these assignments the students were faced with the assignments that require understanding of how Tarski’s truth definition works and how related concepts such as validity, logical consequence and logical equivalence (see definition 4.13) are related:

20. In an exam the students A and B formalized the claim that “there exists a man, who has long hair” as follows (we denote $M =$man and $P =$has long hair):

\begin{align*}
A: & \exists x (M(x) \land P(x)) \\
B: & \exists x (M(x) \rightarrow P(x)).
\end{align*}

Examine the truth of these sentences in the model $S = \langle \{a, b, c\}, \{M^S, P^S\} \rangle$ where $M^S = \{c\}$ and $P^S = \{b\}$. Which one of the formalization is correct?

21. Let $M = \langle N, \{R^M\} \rangle$, where $P^M = \emptyset$. Show, that $M \models \forall x (P(x) \rightarrow \neg x = x)$.

22. Let $R^M \subset M^2, M \neq \emptyset$. Show, that $R^M$ is symmetric if and only if

$\langle M, \{R^M\} \rangle \models \forall x \forall y (R(x, y) \rightarrow R(y, x))$.

23. Show, that $\forall x (A \rightarrow \exists x A)$ is valid.

24. Show, that $\forall x A \rightarrow \forall x B$ is the logical consequence of the sentence $\forall x (A \rightarrow B)$.

25. Show, that $\exists x P_0(x) \rightarrow \exists x P_1(x)$ is not the logical consequence of the sentence $\exists x (P_0(x) \rightarrow P_1(x))$.

26. Show, that $\exists x \neg A \rightarrow (B \rightarrow \neg \forall x A)$ is valid.

27. Show, that the formulas $\exists x \neg (A \land B)$ and $(\neg \forall x A \lor \neg \forall x B)$ are logically equivalent.

How students solved the exercises can be seen in Table 4.5. This shows that all students were able to solve correctly almost all the first basic assignments while being instructed. On the other hand while they were able to solve the basic exercises many failed to answer the more advanced questions.

Especially the exercise 21, which was the most difficult of the week, had a very low completion rate. According to the students this was because they found the concept of a symmetric relation difficult to prove with the arsenal of methods used at this point of the course. It also shows that many students strategically left out the most difficult exercises as the counted that they would still be awarded with full extra points for the exam. To be awarded full extra points a student had to correctly hand
Table 4: Exercises done in the week that introduced Tarski’s truth definition for predicate logic.

| Amount of exercises in week 7 | 26 |
| Students who handed in exercises in week 7 | 15 |
| Correctly made exercises, exercises 1-18 | 265/270 98% |
| Correctly made exercises, exercise 19 | 11/15 73% |
| Correctly made exercises, exercise 20 | 13/15 87% |
| Correctly made exercises, exercise 21 | 5/15 33% |
| Correctly made exercises, exercise 22 | 13/15 87% |
| Correctly made exercises, exercise 23 | 12/15 80% |
| Correctly made exercises, exercise 24 | 9/15 60% |
| Correctly made exercises, exercise 25 | 11/15 73% |
| Correctly made exercises, exercise 26 | 9/15 60% |
| Correctly made exercises, exercises 19-26 | 83/120 69% |
| Correctly made exercises, overall | 348/390 89% |

in 90% of the exercises each week. During week seven this meant that the student could skip two exercises without losing any points.

This should not be seen as a big problem. There should always be an exercise or two for those students who would like to tackle some of the more difficult problems. In this case the exercise was such that similar exercises would pop out in the coming weeks, so it was not critical for the students to be able to solve them immediately. In some weeks such critical exercises existed. For the completion of those exercises additional credit was awarded so that the students could not strategically leave them out.
5 Results

In this section data gathered during the experiment will be portrayed. First data from the students performance will be studied. The independent variable is whether a student belonged to the experimental group or the traditional course setting. The dependent variable is the students’ performance in the course exams. After this data from the post-course online survey will be given.

5.1 Student performance

The evaluation of the course consisted of two exams, one held at the middle of the course and the second at the end. The students participating in the Extreme Apprenticeship group took the same exams as the those in the traditional course setting and their performance was evaluated equally.

Students had two hours time to answer the questions in each mid-term exam. The exams were constructed by the lecturers of the Finnish and English speaking instances of the course. The exams consisted of four exam questions each. The questions were awarded with points 1-6. This made the total maximum of points scored through exams 48. The exams can be seen in figures 7 and 8.

Of the 272 students who registered for the traditional course format of Logic I, 177 students took part in the first exam. However, as this study only tracks those students who participated in the exercise groups, only 139 exams belonging to the students considered participants are studied. The students left out probably studied the course on their own or just came to see the exam. In the second exam, 124 students took part of which we consider the results of 102 students.

Of the 18 students who participated in the Extreme Apprenticeship course format of Logic I, 17 students took part in the first exam. In the second exam, 14 students took part.

Average points of exam questions the students performed can be seen in figures 9 and 10. The Extreme Apprenticeship group outscores the traditional group in all exam questions except the last question in the end-term exam.

Making exercises had an impact on student’s performance in both course formats. Students who made more exercises during the course scored better in the exams.
Figure 7: The mid-term exam for Logic I on 4th of March, 2011. The notation differs somewhat from that used in this study.

1. Present the formula \( \neg((\neg p_0 \rightarrow p_2) \land (\neg p_1 \rightarrow p_3)) \) in disjunctive normal form.

2. Derive the formula \( A \leftrightarrow (A \land C) \) by Natural Deduction from the formula \( (A \lor B) \rightarrow C \).

3. Give a tableau proof for the formula \( ((A \land B) \rightarrow C) \rightarrow ((A \rightarrow C) \lor (B \rightarrow C)) \).

4. Is it possible to derive the formula \( (\neg p_0 \rightarrow p_2) \land (\neg p_1 \rightarrow p_3) \) by Natural Deduction from the formula \( \neg(p_0 \lor p_1) \rightarrow p_3 \)?

Figure 8: The end-term exam for Logic I on 6th of May, 2011. The notation differs somewhat from that used in this study.

1. Give a tableau proof for \( \exists x (A \rightarrow B) \rightarrow (\forall x A \rightarrow \exists x B) \).

2. Let \( M \) be a structure, and let \( s \) be an \( M \)-assignment. Show directly by Tarski’s truth definition that

\[
M \models_s \exists x (A \rightarrow B) \rightarrow (\forall x A \rightarrow \exists x B).
\]

3. Derive by Natural Deduction the formula \( \exists x (A \rightarrow B) \rightarrow (\forall x A \rightarrow \exists x B) \).

4. Let \( M = (\{0, 1, 2, 3\}, E^M) \) and \( N = (\{0, 1, 2, 3\}, E^N) \) be graphs such that

\[
E^M = \{(0, 1), (1, 0), (1, 2), (2, 1), (2, 3), (3, 2)\}
\]

and

\[
E^N = \{(0, 1), (1, 0), (0, 2), (2, 0), (0, 3), (3, 0)\}.
\]

Show that \( M \) is not isomorphic to \( N \). (Hint: One possibility is to find a sentence that is true in one structure and false in the other, and to apply a theorem from the lectures.)

\[
\begin{array}{cccccc}
& & o & & & \\
& o & \quad & o & \quad & o \\
M & o & \quad & o & \quad & o \\
& o & \quad & o & \quad & o \\
& & & & & \\
& & & & & \\
N & & & & & \\
\end{array}
\]
Figure 9: Average points of exam questions for those students taking part in the mid-term exam of Logic I.

![Bar chart showing average points scored for exam questions 1 to 4, comparing traditional and experimental groups.]

Figure 10: Average points of exam questions for those students taking part in the end-term exam of Logic I.

![Bar chart showing average points scored for exam questions 1 to 4, comparing traditional and experimental groups.]

Students taking part in the traditional setting had eleven weeks of exercises with six assignment each. This amounts to a maximum of 66 exercises for the course.

Students taking part in the Extreme Apprenticeship method group started making exercises on week one of the course. This makes the amount of exercise weeks twelve for the group. The amount of exercises varied a lot between 6-40 assignments per week. However, the amount of exercises is a bad measure for weekly workload as the assignments were very different in character. But as the amount of exercises is fairly big, the metric for student’s effort is calculated through solved exercises. The amount of exercises for the experimental group was 208.

The exam points are plotted against the solved exercises in figures 11 and 12. The students with the same amount of solved exercises and scored exam points are plotted with a bigger circle. The area of the circle grows by the area of one normal circle for each overlapping data point.

Figure 11: Student exam performance and solved exercises in traditional Logic I ($N = 171$).

As seen by the regression line making exercises was beneficial for students. As those students who solved more exercises also got some extra points for their effort, their course grade was also higher.
5.2 Course feedback

After the course the students had an opportunity to answer an anonymous survey, seen in Appendix 1 (in Finnish). The survey was sent to all students who had registered for the course and answering it was voluntary.

From students in the traditional course setting 28 answered the survey. From students in the Extreme Apprenticeship group seven answered the survey. As the survey is anonymous, it is not possible to distinguish between those students who only registered for the course but did not participate. Therefore the answer rate for the traditional course was 9.8% and for the Extreme Apprenticeship group 33.3%. These answer rates are low, affecting what can be interpreted from the survey. We give here the answers to the questions relevant for this study. The numbering differs from that in the original survey.
1. During the course, did you form a view of why the topics of the course were studied?

<table>
<thead>
<tr>
<th>Value</th>
<th>Experimental</th>
<th></th>
<th>Traditional</th>
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</thead>
<tbody>
<tr>
<td>Yes</td>
<td>1</td>
<td>7</td>
<td>100</td>
<td>21</td>
</tr>
<tr>
<td>No</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>7</td>
</tr>
<tr>
<td>Total</td>
<td>7</td>
<td>100</td>
<td>28</td>
<td>100</td>
</tr>
<tr>
<td>Median:</td>
<td>1</td>
<td></td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Mean:</td>
<td>1</td>
<td></td>
<td>1.25</td>
<td></td>
</tr>
<tr>
<td>Standard deviation:</td>
<td>0</td>
<td></td>
<td>0.44</td>
<td></td>
</tr>
</tbody>
</table>

2. Did the exercises group assignments support understanding of course topics?

<table>
<thead>
<tr>
<th>Value</th>
<th>Experimental</th>
<th></th>
<th>Traditional</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Not at all</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>To some extent</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>7</td>
</tr>
<tr>
<td>Well</td>
<td>3</td>
<td>7</td>
<td>100</td>
<td>19</td>
</tr>
<tr>
<td>Total</td>
<td>7</td>
<td>100</td>
<td>28</td>
<td>100</td>
</tr>
<tr>
<td>Median:</td>
<td>3</td>
<td></td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>Mean:</td>
<td>3</td>
<td></td>
<td>2.61</td>
<td></td>
</tr>
<tr>
<td>Standard deviation:</td>
<td>0</td>
<td></td>
<td>0.63</td>
<td></td>
</tr>
</tbody>
</table>
3. Did the course exams measure the understanding of the central topics of the course?

<table>
<thead>
<tr>
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<th>Experimental</th>
<th>%</th>
<th>Traditional</th>
<th>%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Not at all</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>To some extent</td>
<td>2</td>
<td>28.6</td>
<td>4</td>
<td>14.3</td>
</tr>
<tr>
<td>Well</td>
<td>3</td>
<td>71.4</td>
<td>24</td>
<td>85.7</td>
</tr>
<tr>
<td>Total</td>
<td>7</td>
<td>100</td>
<td>28</td>
<td>100</td>
</tr>
</tbody>
</table>

Median: 3
Mean: 2.71
Standard deviation: 0.49

4. Evaluate the usefulness of the learning material\textsuperscript{17} of the course.

<table>
<thead>
<tr>
<th>Value</th>
<th>Experimental</th>
<th>%</th>
<th>Traditional</th>
<th>%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Completely useless</td>
<td>1</td>
<td>0</td>
<td>3</td>
<td>10.7</td>
</tr>
<tr>
<td>More useless than usefull</td>
<td>2</td>
<td>0</td>
<td>7</td>
<td>25</td>
</tr>
<tr>
<td>More usefull than useless</td>
<td>3</td>
<td>5</td>
<td>15</td>
<td>53.6</td>
</tr>
<tr>
<td>Very usefull</td>
<td>4</td>
<td>2</td>
<td>3</td>
<td>10.7</td>
</tr>
<tr>
<td>Total</td>
<td>7</td>
<td>100</td>
<td>28</td>
<td>100</td>
</tr>
</tbody>
</table>

Median: 3
Mean: 3.26
Standard deviation: 0.49

\textsuperscript{17}The students did probably not consider the exercises as learning material in this question.
5. How relevant for your own learning do you consider: Attending lectures?

<table>
<thead>
<tr>
<th>Value</th>
<th>Experimental</th>
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<tbody>
<tr>
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<td>Frequency</td>
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<td>1</td>
<td>4</td>
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<tr>
<td>Not relevant at all</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Relevant to some extent</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>Very relevant</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>Total</td>
<td>7</td>
<td>100</td>
</tr>
</tbody>
</table>

Median: 1  
Mean: 1.71  
Standard deviation: 0.95

6. How relevant for your own learning do you consider: Attending the exercise group?

<table>
<thead>
<tr>
<th>Value</th>
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</tr>
</thead>
<tbody>
<tr>
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<td>%</td>
</tr>
<tr>
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<td>0</td>
</tr>
<tr>
<td>Not relevant at all</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>Relevant to some extent</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>Very relevant</td>
<td>4</td>
<td>7</td>
</tr>
<tr>
<td>Total</td>
<td>7</td>
<td>100</td>
</tr>
</tbody>
</table>

Median: 4  
Mean: 4  
Standard deviation: 0.96
7. How relevant for your own learning do you consider: Studying the lecture material own your own?

<table>
<thead>
<tr>
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<th>Frequency</th>
<th>%</th>
</tr>
</thead>
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<td>I did not participate</td>
<td>1</td>
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<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Not relevant at all</td>
<td>2</td>
<td>0</td>
<td>4</td>
<td>14.3</td>
</tr>
<tr>
<td>Relevant to some extent</td>
<td>3</td>
<td>6</td>
<td>85.7</td>
<td>53.6</td>
</tr>
<tr>
<td>Very relevant</td>
<td>4</td>
<td>1</td>
<td>14.3</td>
<td>28.6</td>
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<tr>
<td>Total</td>
<td>7</td>
<td>100</td>
<td>28</td>
<td>100</td>
</tr>
</tbody>
</table>

Median: 3
Mean: 3.14
Standard deviation: 0.38

8. How releveant for your own learning do you consider: Solving weekly exercises on your own?

<table>
<thead>
<tr>
<th>Value</th>
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<th>%</th>
<th>Frequency</th>
<th>%</th>
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</thead>
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<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Not relevant at all</td>
<td>2</td>
<td>0</td>
<td>3</td>
<td>10.7</td>
</tr>
<tr>
<td>Relevant to some extent</td>
<td>3</td>
<td>5</td>
<td>71.4</td>
<td>28.6</td>
</tr>
<tr>
<td>Very relevant</td>
<td>4</td>
<td>2</td>
<td>28.6</td>
<td>60.7</td>
</tr>
<tr>
<td>Total</td>
<td>7</td>
<td>100</td>
<td>28</td>
<td>100</td>
</tr>
</tbody>
</table>

Median: 4
Mean: 3.29
Standard deviation: 0.49
9. Studying with other students (elsewhere than at lectures or in exercise groups)?

<table>
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<td>%</td>
<td>Frequency</td>
<td>%</td>
</tr>
<tr>
<td>I did not participate</td>
<td>1</td>
<td>2</td>
<td>11</td>
<td>39.3</td>
</tr>
<tr>
<td>Not relevant at all</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>3.6</td>
</tr>
<tr>
<td>Relevant to some extent</td>
<td>3</td>
<td>5</td>
<td>6</td>
<td>21.4</td>
</tr>
<tr>
<td>Very relevant</td>
<td>4</td>
<td>0</td>
<td>10</td>
<td>35.7</td>
</tr>
<tr>
<td>Total</td>
<td>7</td>
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<td>28</td>
<td>100</td>
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</tbody>
</table>

Median: 3
Mean: 2.43
Standard deviation: 0.98

10. What was the demand-level of the course?

<table>
<thead>
<tr>
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<tbody>
<tr>
<td>Value</td>
<td>Frequency</td>
<td>%</td>
<td>Frequency</td>
<td>%</td>
</tr>
<tr>
<td>Very demanding</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>10.7</td>
</tr>
<tr>
<td>Demanding to some extent</td>
<td>2</td>
<td>4</td>
<td>12</td>
<td>42.9</td>
</tr>
<tr>
<td>Easy to some extent</td>
<td>3</td>
<td>2</td>
<td>11</td>
<td>39.3</td>
</tr>
<tr>
<td>Very easy</td>
<td>4</td>
<td>0</td>
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<td>7.1</td>
</tr>
<tr>
<td>Total</td>
<td>7</td>
<td>100</td>
<td>28</td>
<td>100</td>
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</tbody>
</table>

Median: 2
Mean: 2.14
Standard deviation: 0.69
11. How was the work load in comparison with the credit units of the course [10 ECTS] (1 ECTS = 27 hours of effort by the student)?

<table>
<thead>
<tr>
<th>Value</th>
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<th>%</th>
<th>Traditional</th>
<th>Frequency</th>
<th>%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Too heavy work load</td>
<td>1</td>
<td>1</td>
<td>14.3</td>
<td>1</td>
<td>3.6</td>
<td></td>
</tr>
<tr>
<td>Just right</td>
<td>2</td>
<td>6</td>
<td>85.7</td>
<td>26</td>
<td>92.9</td>
<td></td>
</tr>
<tr>
<td>Too little work load</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>3.6</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>7</td>
<td>100</td>
<td>100</td>
<td>28</td>
<td>100</td>
<td></td>
</tr>
</tbody>
</table>

Median: 2  
Mean: 1.86  
Standard deviation: 0.38
6 Discussion

The aim for this study was to find out if the Extreme Apprenticeship method can be used to teach undergraduate mathematics. The practical action research carried out shows that the method can be used.

Students in this limited experiment benefited from the method. The students in the Extreme Apprenticeship group had higher scores in their exam results than those students studying in a traditional course setting. They also had a clearer understanding of why the topics were studied.

Students in different stages in their studies seemed to enjoy the method, as seen in feedback given in the post-course survey:

“[...] my comprehension about mathematics turned more positive on this fifth year of studies. [...]”

“A good first course at the university. [Extreme Apprenticeship] made a soft landing into university studies. [...]”

[Transcribed from Finnish by author.]

These results encourage to carry out further research in the area of conducting mathematics education with the Extreme Apprenticeship method. The insights that were obtained through this study will be used in future attempts in scaling up the experiment to whole undergraduate courses. In these attempts the importance of the exercises, the instruction and the recruitment of suitable teaching assistants play a crucial role.

Making a good set of exercises for the Extreme Apprenticeship method is difficult. When employing the method for the first time it can be difficult to assess what areas of the topics studied are difficult for students. As seen in the example of introducing Tarski’s truth definition the major difficulty for students was the rather simple looking notation of a modified assignment.

That these kinds of difficulties are revealed is the strength of the Extreme Apprenticeship method as it makes it possible to tackle them. Reporting on the difficulties will also be a topic in further research.

A good set of exercises forces students to use concepts they have learned to use during previous weeks. Students were made to reflect and articulate on exercises
they had done in earlier weeks. For example, the students had to explain, without mathematical notation, a natural deduction solution they had come up with the previous week. This proved to be a good way of engaging students into mathematical discussions and to help them structure their thinking.

When planning the exercises the teacher should have a clear learning objective for each exercise. At least in the beginning when introducing a new concept, it is better to have one goal per exercise. As the amount of exercises is not set, it makes it possible for the teacher to make many small exercises with very small goals. The goal can be as simple as to check that everybody has understood how the notation is used with a particular concept. The smaller the exercise, the easier it is for the instructor and the student to see where the problem lies. This makes it possible for the student to ask the necessary questions in order to comprehend the studied topic by himself.

The instruction that the instructor gives the students is the key element in learning with the Extreme Apprenticeship method. The instructions makes it possible for the student to learn topics he would not otherwise learn. The findings of the approach to instruction can be summarized in four statements:

- **Do not to give students ready answers.** The joy of learning should not be taken away by the instructor. Instead the students should feel that they have made the effort and succeeded in it. This is a difficult task for the instructor as the instructor might feel a need to give “perfect” answers to students revealing all the fine detail which comes with answering the question.

- **Push student to the course material.** In the beginning many students asked about definitions or results which were clearly written in the course material. Answering these kind of questions that can be directly found in the course material should be avoided as it is a waste of the instructor’s time and it resembles the traditional lecture format, where theory is read straight from the written material, something the Extreme Apprenticeship method tries to avoid. The instructor should not put himself in a position where he is a substitute for the course material as this will in the end prohibit the students of becoming self-directed learners. Instead the instructor can help the students in learning how to use the material.

- **Push students to form their own question.** The instructors has his own view of how to understand the topics. This view might be very different from
that of the student and answering his own questions will therefore not help. Help the student to formulate his own question and first then answer them. This often leads to that the student answers his own question.

- **Be encouraging.** Pushing the students to learn and forcing them to deliver correct solutions is demanding for them. Therefore, when a student is frustrated be encouraging and when the student succeed after struggling give him the well earned credit he deserves.

The atmosphere in the Extreme Apprentice group was good. The students did a lot of work, but seemed to value the method. A student gave the following answer to the voluntary feedback form after the course:

“[The instructor] supported well the solving of exercises and did not give readymade answers but supported one’s own thinking and comprehension even when one had difficulties. Gave goof briefings about the most essential concepts in an understandable language and was always ready to explain and discuss the exercises and other topics regarding the concepts. Demanded a good answer and was not satisfied with an exercise being solved almost right, it helped to comprehend the concepts and to make clean and mathematically correct answers. Absolutely great, not just studying logic, but studying how to study as well. [...]” [Translated from Finnish by author.]

The emphasis on hands-on doing seems important for students learning. Although the answer rate of the post-course survey was small, it can be seen that students in both groups saw exercise groups as an important vehicle for their learning. They even seemed to see them more important than lectures. The answerers did not see the higher workload as problem. The student who answered that the workload was too big commented on his/her answer in the post-course online survey:

“I answered earlier that the workload was too big. That was expected as the working method was [the Extreme Apprenticeship method] but totally worth it. The workload was just bigger than 27 hours per [ECTS] credit unit and therefore I answered as I did. Overall the [Extreme Apprenticeship] group was totally great, big thanks!” [Translated from Finnish by author.]
This findings support the view that strong emphasis on lectures should be changed to strong emphasis on the work students carry out. This change of attitude has to come from the teachers responsible of courses. The work being carried out in exercise groups cannot be independent from what happens in the lectures. Therefore the responsible teachers has to take part in the exercise groups and see for themselves where the students struggle. If the teachers only monitor students from the front of a lecture hall, it is likely that their view of the student’s learning will be blinded by those few students taking actively part in the lectures.

There are occasions were lecturing is a motivated choice for teaching method as proposed by the Macpherson committee (1967). It just should not be used in the way it is often used today: as a method of transforming structured knowledge readily available in writing.

Lectures could benefit from the Extreme Apprenticeship method. Lambert (1963) urges lecturers to use the fact that printed books have been around for the last 600 hundred years. He advocates the Gutenberg method\(^\text{18}\) (Morrison, 1986) where students come to lectures after they have read a portion of the course literature before class. The lecture is then not about going through the literature, but about discussing it.

The Extreme Apprenticeship method could be used as an extension to the Gutenberg method. Not only would the students have studied the course literature on beforehand, but they would have spent hours on doing exercises that the course literature was about. This would enable the lecture to become a place where the big picture of the topics would be presented. It would also be a place where the difficulties the students faced could be brought up.

Lecturing should also be used in the modeling phase of the apprenticeship-based learning. The lecturer should give insight into how an expert thinks, for example by showing how a skilled computer programmer creates a hash table (Luukkainen et al., 2012).

The physical environment in where the learning takes place must also support teaching methodology. In Extreme Apprenticeship method this means that discussing mathematics and giving instruction should be as easy as possible. The experiences of this study point to the importance of having lots of surfaces nearly at hand where mathematical ideas can be written down and sketched.

\(^{18}\)Johannes Gensfleisch zur Laden zum Gutenberg was born in 1400 in Menz (Hanebutt-Benz, 2012). He was a pioneer of movable printing and the inventor of the printing press.
Scaling up the Extreme Apprenticeship method poses many challenges. One major concern is the recruitment of suitable teaching assistants working as instructors. In the recruitment it would be important not to only select assistant who have a good subject knowledge, but assistants who also are able to adapt to a new type of teaching. Abilities and skills needed from teaching assistants are at least:

- **Ability to do teamwork.** If the course is large then there will be a need to have multiple teaching assistants working at the same time in the same space. As the teaching assistants and the responsible teacher have to share the same learning goals and teaching methods, it is important that no one of the teaching staff acts solo.

- **Ability to teach in a changing environment.** The Extreme Apprenticeship method tries to adapt the teaching to fit the students’ individual needs. These needs can be very different and teaching assistant working as instructors can not expect to know on beforehand what kind of misunderstandings or questions the students have.

- **Understanding and adapting of the mentoring role of the instructor.** The teaching assistant should understand his role as mentor and instructor of the students. The students have to do the learning by themselves. The instructor can not take the role of a teacher who teacher in front of the class.

Scaling up requires cost effectiveness and an administration that supports the flexible nature of the Extreme Apprenticeship method. In the future working methods have to be more efficient. Especially correcting exercises has to be performed in a more effective way when scaling up. Teaching assistant allocation and recruitment are also issues for further research.

The findings in this thesis were used to scale up the method in a large course Linear algebra and matrices I&II with several hundred students (Hautala et al., 2012). The results of students’ learning were promising, which encourages to further research in using and developing the Extreme Apprenticeship method in higher education.
7 References


Faculty of Science (2012). Web page of the Faculty of Science, University of Helsinki. http://www.helsinki.fi/ml/.


Macpherson, C. and Presidential Advisory Committee on Undergraduate Instruction in the Faculty of Arts and Science (1967). Undergraduate instruction in arts and science: report of the Presidential Advisory Committee on Undergraduate Instruction in the Faculty of Arts and Science, University of Toronto, 1967.


Appendix 1. Post-course online survey

The standard post-course online survey that was used by the Department of Mathematics and Statistics in spring 2011. Each student who had registered for a course got an e-mail inviting to answer the survey. Answering the survey was voluntary and anonymous.
Tervetuloa antamaan kurssipalautta

Opetustapahtuma: 52774 Logiikka 1
Tyyppi: Luentokurssi
Organisaatio: Matematiikan ja tilastotieteen laitos
Opetustapahtuman opettajat: Tapani Hyttinen
Ajalla: 17.01.2011-08.05.2011

Yllä olevasta opintojakson tunnisteesta voit katsoa opintojakson tarkemmat tiedot. Tiedot aukiavat omaan selainikkunaan.

* = pakollinen kysymys

Matematiikan ja tilastotieteen laitos: kurssipalaute

<table>
<thead>
<tr>
<th>Koodi</th>
<th>Kysymys</th>
<th>Vastaus</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Mainitse kolme kurssin asiaa, jotka olivat mielestäsi keskeisiä. 1.</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2.</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>3.</td>
<td></td>
</tr>
</tbody>
</table>

Asteikko
1 = Kyllä
2 = Ei

<table>
<thead>
<tr>
<th>Koodi</th>
<th>Kysymys</th>
<th>Vastaus</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>Muodostuiko sinulle kurssin aikana käsitys siitä, miksi kurssin asioita opiskellaan?</td>
<td>1 2</td>
</tr>
</tbody>
</table>

Asteikko
1 = Täysin hyödytön
2 = Enemmän hyödytön kuin hyödyllinen
3 = Enemmän hyödyllinen kuin hyödytön
4 = Hyvin hyödyllinen
<table>
<thead>
<tr>
<th>Koodi</th>
<th>Kysymys</th>
<th>Vastaus</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>Arvioi kurssilla käytetyn oppimateriaalin hyödyllisyyttä.</td>
<td>1 2 3 4</td>
</tr>
</tbody>
</table>

Asteikko

1 = Ei lainkaan
2 = Jossain määrin
3 = Hyvin

<table>
<thead>
<tr>
<th>Koodi</th>
<th>Kysymys</th>
<th>Vastaus</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>Mittasiko kurssikoe tai -kokeet kurssin keskeisten asioiden ymmärtämistä?</td>
<td>1 2 3</td>
</tr>
<tr>
<td>7</td>
<td>Tukivatko laskuharjoitustehtävät kurssin asioiden ymmärtämistä?</td>
<td>1 2 3</td>
</tr>
<tr>
<td>8</td>
<td>Kommentteja laskuharjoitustilanteista.</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>Laskuharjoitusten pitäjän nimi ja kommentteja laskuharjoitustilanteista.</td>
<td></td>
</tr>
</tbody>
</table>

Asteikko

1 = En osallistunut/tehnyt
2 = Ei lainkaan oleellista
3 = Jossain määrin oleellista
4 = Hyvin oleellista

<table>
<thead>
<tr>
<th>Koodi</th>
<th>Kysymys</th>
<th>Vastaus</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>Luonneille osallistuminen</td>
<td>1 2 3 4</td>
</tr>
</tbody>
</table>

Kuinka oleellisena oman oppimisen kannalta pidit eri oppimistilaisuuksia/opiskelumuotoja:
11 Laskuharjoituksiin osallistuminen

12 Luontomateriaaliin tutustuminen itsenäisesti

13 Laskuharjoitusten tekeminen itsenäisesti

14 Opiskelu yhdessä muiden opiskelijoiden kanssa (muualla kuin luennoilla/laskuharjoituksissa/ohjauksissa)
17 Kuinka kurssin tiedotus (verkkosivut, tiedotus luentoilla/harjoituksissa, jne.) 1 2 3 4
toimi?

18 Mitä kurssilla olisi voitu tehdä toisin, jotta opiskelu ja oppiminen olisivat sujuen paremmin?

19 Vapaa sana! Anna ruusuja ja risuja!

☐ Palautettani voidaan käsitellä ja siitä voidaan muodostaa yhteenveto, vaikka alle viisi opiskelijaa on antanut palautetta. Tässä tapauksessa yhteenveto muodostetaan vain niiden opiskelijoiden vastauksista, jotka ovat antaneet luvan viiden henkilön minimivastausmäärästä poikkeamiselle.