Topology and its Applications 297 (2021) 107700

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Intermediate-value property for Assouad dimension by its inversion invariance ☆

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ARTICLE INFO

Article history:
Received 29 October 2016
Received in revised form 6 April 2021
Accepted 15 April 2021
Available online 20 April 2021

MSC:
54F45
28A80

Keywords:
Assouad dimension
Intermediate-value property
Inversion

ABSTRACT

We prove that if X is a metric space of Assouad dimension $s \in (0, \infty)$, then for every $\alpha \in [0, s]$ there is a countable subset of X of Assouad dimension α. We reduce this to the special case of bounded X due to Wang and Wen in 2016 by using the inversion invariance of Assouad dimension.

1. Introduction

Let $\dim_A X$ denote the Assouad dimension of a metric space (X, d) in the sense of [3]. That is, $\dim_A X$ is the infimum in $[0, \infty]$ of the numbers $s \geq 0$ (if any) for which there is $C \geq 0$ such that $\operatorname{card} F \leq C(R/r)^s$ whenever $0 < r \leq R$ are numbers and $F \subset X$ a set with $r \leq d(x, y) \leq R$ if $x, y \in F$ and $x \neq y$. Recently Wang and Wen [4] proved that if X is a bounded metric space with $s = \dim_A X \in (0, \infty)$, then for every $\alpha \in (0, s)$ there is a countable subset F of X with $\dim_A F = \alpha$. They used in an essential way the fact that such a space X is totally bounded. (For the definition of $\dim_A X$ in [4, p. 121] replace "$R < \operatorname{diam}(X)$" by "$R < \infty$". In fact, otherwise even $\dim_A X = 0$ for the bounded but not totally bounded space X in [4, Remark 1].) We observe that here the requirement of countability is redundant as every metric space of finite Assouad dimension is separable and as the closure operator preserves the Assouad dimension of a subset by [3, A.5(2)]. Hence, the result also holds for $\alpha = s$. It holds trivially for $\alpha = 0$ as $\dim_A \emptyset = 0$.

☆ This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.
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https://doi.org/10.1016/j.topol.2021.107700
0166-8641/

It is desirable to get the result of Wang and Wen also for an arbitrary, possibly unbounded metric space X and to have just such a proof for the general case which is a reduction to the special case of bounded X. But it is not immediate how to do this reduction as it is possible that the supremum of the Assouad dimensions of the bounded subsets of X is less than s. For example, such is the case with the space $\mathbb{Z}$, whose bounded subsets are finite and thus of Assouad dimension zero, while $\dim_{\mathrm{A}} \mathbb{Z} = 1$.

However, there is a basic property of Assouad dimension that is decisive here: the invariance of Assouad dimension under inversions of normed spaces in spheres. To be more specific, let E be a normed space and let $u\colon E \setminus \{0\} \to E \setminus \{0\}$ be the inversion $x \mapsto x/|x|^2$ of E in the unit sphere of E. Then $\dim_{\mathrm{A}} uX = \dim_{\mathrm{A}} X$ for every set $X \subset E \setminus \{0\}$ by [3, A.10(1)]. This fact can be applied for an arbitrary metric space X as by the Kuratowski embedding theorem [2, Theorem XIII.5.2] there exist a normed space E and an isometric embedding $f\colon X \to E$ with fX disjoint from the open unit ball of E (to get the latter property of f translate the Kuratowski embedding by the constant function 1).

2. Applying the inversion invariance of Assouad dimension

Theorem 1. *Let X be a metric space of finite positive Assouad dimension s. Then for every $\alpha \in [0, s]$ there is a countable subset of X of Assouad dimension α.*

Proof. We may assume that X is a subset of a normed space E disjoint from the open unit ball B of E. Let u be the inversion of E in the unit sphere of E. Then $uX \subset \overline{B}$ is bounded and $\dim_{\mathrm{A}} uX = s$. Now let $\alpha \in [0, s]$. Then by [4, Theorem 1] there is a countable set $F_0 \subset uX$ with $\dim_{\mathrm{A}} F_0 = \alpha$. Hence the set $F = uF_0 \subset X$ is countable and $\dim_{\mathrm{A}} F = \alpha$. □

Remark 1. Theorem 1 is also contained in [1, Theorem 1], whose proof is however different.

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