Weak Solutions for Maxwell's Equations

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25.8.2021

HELSINGIN YLIOPISTO – HELSINGFORS UNIVERSITET – UNIVERSITY OF HELSINKI

Tiedekunta/Osasto — Fakultet/Sektion — Faculty

Faculty of Science

Laitos — Institution — Department

The Department of Mathematics and Statistics

Tekijä — Författare — Author

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Työn nimi — Arbetets titel — Title

Weak Solutions for Maxwell's Equations

Oppiaine — Läroämne — Subject

Mathematics

Työn laji — Arbetets art — Level

Master's thesis

Aika — Datum — Month and year

August 2021

Sivumäärä — Sidoantal — Number of pages

50 p.

Tiivistelmä — Referat — Abstract

Maxwell's equations are a set of equations which describe how electromagnetic fields behave in a medium or in a vacuum. This means that they can be studied from the perspective of partial differential equations as different kinds of initial value problems and boundary value problems. Because often in physically relevant situations the media are not regular or there can be irregular sources such as point sources, it's not always meaningful to study Maxwell's equations with the intention of finding a direct solution to the problem. Instead in these cases it's useful to study them from the perspective of weak solutions, making the problem easier to study.

This thesis studies Maxwell's equations from the perspective of weak solutions. To help understand later chapters, the thesis first introduces theory related to Hilbert spaces, weak derivates and Sobolev spaces. Understanding curl, divergence, gradient and their properties is important for understanding the topic because the thesis utilises several different Sobolev spaces which satisfy different kinds of geometrical conditions. After going through the background theory, the thesis introduces Maxwell's equations in section 2.3. Maxwell's equations are described in both differential form and time-harmonic differential forms as both are used in the thesis.

Static problems related to Maxwell's equations are studied in Chapter 3. In static problems the charge and current densities are stationary in time. If the electric field and magnetic field are assumed to have finite energy, it follows that the studied problem has a unique solution. The thesis demonstrates conditions on what kind of form the electric and magnetic fields must have to satisfy the conditions of the problem. In particular it's noted that the electromagnetic field decomposes into two parts, out of which only one arises from the electric and magnetic potential.

Maxwell's equations are also studied with the methods from spectral theory in Chapter 4. First the thesis introduces and defines a few concepts from spectral theory such as spectrums, resolvent sets and eigenvalues. After this, the thesis studies non-static problems related to Maxwell's equations by utilising their time-harmonic forms. In time-harmonic forms the Maxwell's equations do not depend on time but instead on frequencies, effectively simplifying the problem by eliminating the time dependency. It turns out that the natural frequencies which solve the spectral problem we study belong to the spectrum of Maxwell's operator $i\mathscr{A}$. Because the spectrum is proved to be discrete, the set of eigensolutions is also discrete. This gives the solution to the problem as the natural frequency solving the problem has a corresponding eigenvector with finite energy. However, this method does not give an efficient way of finding the explicit form of the solution.

Avainsanat — Nyckelord — Keywords

Maxwell's equations, weak solutions, Sobolev spaces

Säilytyspaikka — Förvaringsställe — Where deposited

Kumpula Campus Library

Muita tietoja — Övriga uppgifter — Additional information

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Chapter 1

Introduction

When studying differential equations, it is sometimes impossible to find solutions to the problem. A particular function with otherwise promising features might for example be discontinuous on the boundary or non-differentiable at interior points, in which case it can't be a solution to the problem. However, weak solutions to differential equations offer a way to find information about the problem even in these cases. Even though weak solutions aren't always solutions to the original problem, they can tell a lot about the nature of the differential equation. Additionally, they are often physically relevant as they can for example represent electromagnetic fields in irregular media or fields generated by irregular sources such as point sources.

Maxwell's equations are a set of important equations in physics which describe how electromagnetic radiation behaves in a medium or in a vacuum. In this thesis we will go through the theory of Sobolev spaces and other relevant topics of functional analysis. Utilising this knowledge, we will present boundary value problems related to Maxwell's equations and examine their weak solutions.

We will start by going through the theory of Hilbert spaces, weak derivatives and Sobolev spaces. For the study of this particular problem, we will need to study curl and divergence and how they behave in different Sobolev spaces. We also need to study corresponding trace theorems. After going through the background theory, we will first study static boundary value problems related to Maxwell's equations. Afterwards we will also examine non-static cases using some elements of spectral theory. As a primary source for information, we follow Mathematical Analysis and Numerical Methods for Science and Technology series by Robert Dautray and Jacques-Louis Lions, but often filling details and presenting additional information to suit our needs.

Chapter 2

Background and useful concepts

2.1 Functional analysis

In this section we will recall some of the most important definitions and notations. In particular many concepts from functional analysis are central to understanding weak solutions to Maxwell's equations. These topics include Hilbert and Sobolev's spaces, weak derivatives and weak solutions to differential equations.

2.1.1 Hilbert spaces

The euclidean spaces $\mathbb{R}^n$ have many useful properties and thankfully many results can be generalized to other spaces too. As an example we introduce Hilbert spaces which share many useful geometric and analytic properties of $\mathbb{R}^n$ and $\mathbb{C}^m$ with $n, m \in \mathbb{N}$.

We will start by defining inner product spaces for the scalar field $\mathbb{C}$.

Definition 2.1. (Inner product space)

Let E be a vector space with the scalar field $\mathbb{C}$. We say that a function $f\colon E \times E \to \mathbb{C}$ is an inner product if

(i) $f(x_1 + x_2, y) = f(x_1, y) + f(x_2, y)$ for all $x_1, x_2, y \in E$

(ii) $f(\lambda x, y) = \lambda f(x, y)$ for all $x, y \in E$ and $\lambda \in \mathbb{C}$

(iii) $f(y, x) = \overline{f(x, y)}$ for all $x, y \in E$ where $\overline{f(x, y)}$ is the conjugate of the $f(x, y) \in \mathbb{C}$.

(iv) $f(x, x) \geq 0$ for all $x \in E$ and $f(x, x) = 0$ if and only if $x = 0$

We then denote the inner product with $(x, y) = f(x, y)$ and call the space E endowed with the inner product $(\cdot)$ an inner product space $(E, (\cdot))$.

Note that the definition can be generalised for other scalar fields $\mathbb{K}$ as long as they are endowed with an involution or a conjugate. If $\mathbb{K} = \mathbb{R}$, the condition (iii) is instead of the form $f(y, x) = f(x, y)$ for all $x, y \in E$. [9]

As a special case of inner products, we denote the dot or scalar product of elements $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ with the usual notation $x \cdot y$. We can easily define the inner product in $\mathbb{C}^n$ in a similar manner as in $\mathbb{R}^n$. For $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$ and $y = (y_1, \ldots, y_n)$ we define (x, y) as

$$(x, y) = \sum_{j=1}^{n} x_j \bar{y_j}.$$

It's also worth pointing out that the notation (x, y) should not be confused with the notation $\langle x, y \rangle$ which we use to denote the duality between x and y.

Let $(E, (\cdot))$ be an inner product space. Inner products induce a norm defined by

$$\|x\| = \sqrt{(x, x)}, \;\; x \in E.$$

Definition 2.2. (Hilbert space)

Let $(E, (\cdot))$ be an inner product space. Then we say that $(E, (\cdot))$ is *a Hilbert space* if E is also a Banach space, that is, it is complete with respect to the norm induced by the inner product.

For example, we know that in a domain $\Omega \subset \mathbb{R}^n$ the vector space consisting of equivalency classes of square integrable functions, denoted by $L^2(\Omega)$, is a Hilbert space in terms of its corresponding inner product

$$(f, g) = \int_\Omega fg \; dx$$

for all real functions $f, g \in L^2(\Omega)$. For complex valued functions $f, g \in L^2(\Omega)$ on the other hand, the inner product is

$$(f, g) = \int_\Omega f\bar{g} \; dx.$$

Other important examples of Hilbert spaces include Sobolev spaces introduced later.

2.1.2 Weak derivatives

We start the study of weak derivatives by defining a set of test functions. There are multiple sets of test functions that we could use, but for convenience we fix them as follows.

Definition 2.3. Let $\Omega \subset \mathbb{R}^n$ be an open subset. Then we define

$$\mathscr{D}(\Omega) = C_c^\infty(\Omega) = \Big\{\psi \in C^\infty(\mathbb{R}^n) \mid \text{supp}(\psi) \subset \Omega \text{ is compact}\Big\}.$$

A function belonging to $\mathscr{D}(\Omega)$ is called *a test function.*

Recall that the support of a function ψ is defined as $\text{supp}(\psi) = \overline{\Big\{x \in \mathbb{R} \mid \psi(x) \neq 0\Big\}}$, that is, a closure of the set of all those elements $x \in \mathbb{R}$ such that the $\psi(x) \neq 0$. We often use shorter notation $\mathscr{D}(\Omega)^n$ for $(\mathscr{D}(\Omega))^n$ consisting of elements $(\psi_1, \dots, \psi_n)$ with $\psi_1, \dots, \psi_n \in \mathscr{D}(\Omega)$.

The next part is based on corresponding section in Partial Differential Equations by Lawrence Evans [1]. Let there be a function $u \in C^1(\Omega)$ and a test function $\varphi \in \mathscr{D}(\Omega)$. By using integration by parts we get

$$\int_\Omega u\varphi_{x_i}\, dx = -\int_\Omega u_{x_i}\varphi\, dx$$

for every $i = 1, \dots, n$. Because φ has a compact support in Ω, there are no boundary terms and thus it vanishes near $\partial\Omega$.

In the previous formula u is assumed to be differentiable. If $u \notin C^1$, then the expression u_{x_i} is not defined at all. However, even if u is not differentiable, we can define a concept that has similar properties as a derivative. We define weak derivatives to solve this issue.

Definition 2.4. Let Ω be an open subset of $\mathbb{R}^n$ and $f \in L^1_{loc}(\Omega)$. A function $g \in L^1_{loc}(\Omega)$ is the *weak derivative* of a function f with respect to x_i if

$$\int_\Omega g\varphi\, dx = -\int_\Omega f\varphi_{x_i}\, dx$$

with all test functions $\varphi \in \mathscr{D}(\Omega)$.

We can also extend the definition to higher order derivatives. We use the multi-index notation for derivatives. If $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ then we denote

$$D^\alpha\varphi = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}}\frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}}\cdots\frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}\varphi.$$

The order of α is denoted by $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. Then the definition of weak derivatives can be represented as follows.

Definition 2.5. Let $u, v \in L^1_{loc}$, that is, let u and v be locally integrable functions. We say that v is the αth *weak partial derivative* of u if

$$\int_\Omega u D^\alpha \varphi \, dx = (-1)^{|\alpha|} \int_\Omega v\varphi \, dx$$

for all test functions $\varphi \in \mathscr{D}(\Omega)$. We then denote $D^\alpha u = v$.

Weak derivatives are especially useful because they agree uniquely with the ordinary derivative if the ordinary derivatives exist. In addition, weak derivative is determined.

Theorem 2.6. *Let $f \in L^1_{loc}(\Omega)$ where Ω is an open subset of $\mathbb{R}^n$. If $g_1 \in L^1_{loc}(\Omega)$ and $g_2 \in L^1_{loc}(\Omega)$ satisfy*

$$(-1)^{|\alpha|} \int_\Omega g_1\varphi \, dx = \int_\Omega f D^\alpha \varphi \, dx = (-1)^{|\alpha|} \int_\Omega g_2\varphi \, dx$$

with all $\varphi \in \mathscr{D}(\Omega)$, then $g_1 = g_2$ almost everywhere.

Recall that almost everywhere means the same as up to a set of measure zero. Before proving this theorem, let's prove the following lemma.

Lemma 2.7. *If $f \in L^1_{loc}(\Omega)$ satisfies the condition $\int_\Omega f\varphi \, dx = 0$ for all $\varphi \in \mathscr{D}(\Omega)$, then $f = 0$ almost everywhere on Ω.*

Proof. Let G be any compact subset of Ω. Let's take $\psi \in \mathscr{D}(\Omega)$ such that $\psi = 1$ on the set G. Now we define the function f_ψ as

$$f_\psi(x) = \begin{cases} 0 \text{ when } x \notin \Omega \\ f(x)\psi(x) \text{ when } x \in \Omega. \end{cases}$$

The function f_ψ extends f to the whole $\mathbb{R}^n$ and also $f_\psi \in L^1(\mathbb{R}^n)$. Now, let $\varphi \in \mathscr{D}(\Omega)$ be a mollifier. We can for example use the standard mollifier $\varphi \in \mathbb{C}^\infty(\mathbb{R}^n)$ defined in Partial Differential Equations by Lawrence Evans [1].

$$\varphi(x) = \begin{cases} Ce^{\left(\frac{1}{|x|^2-1}\right)} \text{ if } |x| < 1 \\ 0 \text{ if } |x| \geq 1. \end{cases}$$

with a constant $C > 0$ such that $\displaystyle\int_{\mathbb{R}^n} \varphi \, dx = 1$. This is a function which has a compact support.

In that case we know that the convolution $\varphi_\varepsilon * f$ satisfies $\varphi_\varepsilon * f \to f$ in $L^1_{loc}(\mathbb{R}^n)$ as $\varepsilon \to 0$, where φ_ε is defined by $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$.

Now

$$\varphi_\varepsilon * f_\psi(x) = \int_{\mathbb{R}^n} f(y)\psi(y)\varphi_\varepsilon(x-y)\ dy.$$

We notice that the mapping $g(y) = \psi(y)\varphi_\varepsilon(x-y) \in \mathscr{D}(\Omega)$ for any fixed x, and thus $\varphi_\varepsilon * f_\psi = 0$ by assumption. As $\varepsilon \to 0$, the convolution satisfies $\varphi_\varepsilon * f_\psi \to f_\psi$ in $L^1_{loc}(\mathbb{R}^n)$ as mentioned before, pointwise almost everywhere. Thus it follows that $f_\psi = 0$ almost everywhere on Ω, and specifically $f = 0$ almost everywhere on G. Because G is any arbitrary compact subset of Ω, we have that $f = 0$ almost everywhere on Ω. This proves the claim. □

Now we are ready to prove Theorem 2.6.

Proof. Assume that $g_1, g_2 \in L^1_{loc}(\Omega)$ satisfy

$$(-1)^{|\alpha|}\int_\Omega g_1\varphi\ dx = \int_\Omega fD^\alpha\varphi\ dx = (-1)^{|\alpha|}\int_\Omega g_2\varphi\ dx$$

for all $\varphi \in \mathscr{D}(\Omega)$. Then it holds that

$$\int_\Omega (g_1 - g_2)\varphi\ dx = 0$$

for all $\varphi \in \mathscr{D}(\Omega) = C_c^\infty(\Omega)$.

From the Lemma 2.7 it follows that $g_1 - g_2 = 0$ almost everywhere on Ω. This proves that $g_1 = g_2$ almost everywhere on Ω. □

Finally, let's prove that the weak derivative agrees with the ordinary derivative if the ordinary derivative exists.

Theorem 2.8. *Let $\Omega \subset \mathbb{R}^n$ and $g \in L^1_{loc}(\Omega)$ be the x_ith weak derivative of a function $f \in L^1_{loc}(\Omega)$. If f is differentiable, $f \in C^1(\Omega)$, then for the derivative f_{x_i} it holds that $f_{x_i} = g$.*

Proof. Since $f \in C^1(\Omega)$, the derivatives f_{x_i} exists for all $i = 1, \ldots, n$ in the usual sense. Let $\varphi \in \mathscr{D}(\Omega)$. Now by integration by parts,

$$\int_\Omega f\varphi_{x_i}\ dx = -\int_\Omega f_{x_i}\varphi\ dx.$$

However, this is exactly the definition of a weak derivative, and thus $g = f_{x_i}$.

□

2.1.3 Sobolev spaces

Sobolev spaces are subspaces of L^p spaces consisting of functions having weak derivatives up to a given order in the corresponding L^p space.

Definition 2.9. Let Ω be an open subset of $\mathbb{R}^n$. The set consisting of locally integrable functions $u \in L^p(\Omega)$ such that for each multi-index α with $|\alpha| \leq k$, weak derivatives $D^\alpha u$ exist and belong to $L^p(\Omega)$, is called a *Sobolev space*. We then denote the space $W^{k,p}(\Omega)$.

The $W^{k,p}(\Omega)$ spaces are normed spaces when endowed with a norm

$$\|u\|_W^{k,p}(\Omega) = \Big(\sum_{|\alpha| \leq k} \int_\Omega |D^\alpha u|^p \, dx \Big)^{1/p}$$

if $1 \leq p < \infty$ and

$$\|u\|_W^{k,\infty}(\Omega) = \sum_{|\alpha| \leq k} \||D^\alpha u|\|_\infty$$

if $p = \infty$. Here $\| \cdot \|_\infty$ denotes the essential supremum. Equipped with these norms, $W^{k,p}$ spaces are also Banach spaces [1].

In particular we are interested in a Sobolev space with $p = 2$ and $k = 1$, that is, $W^{1,2}(\Omega) \subset L^2(\Omega)$. Such a space is denoted $H^1(\Omega)$ as this space is also a Hilbert space with the inner product

$$(f, g) = \int_\Omega f(x)\overline{g(x)} + f'(x)\overline{g'(x)} \, dx \quad f, h \in H^1(\Omega).$$

This space consist of functions in $L^2(\Omega)$ that have weak first order partial derivatives in $L^2(\Omega)$. In a similar manner we'll denote $H^k(\Omega) = W^{k,2}$ for positive integers k. We'll also denote the closure of C_c^∞ in the space $W^{k,p}$ as $W_0^{k,p}$, and $H_0^k(\Omega) = W_0^{k,2}(\Omega)$.

Let's now consider the behaviour of functions $u \in W^{1,p}(\Omega)$ on the boundary $\partial\Omega$. If $u \in C(\overline{\Omega})$, then u has values on $\partial\Omega$ in the usual sense. However, if u is not continuous, we need the concept of traces. For this purpose, let's introduce the trace theorems in $W^{1,p}$.

Theorem 2.10. *Let's assume $\Omega \subset \mathbb{R}^n$ is bounded and the boundary $\partial\Omega$ is C^1. Let $1 \leq p < \infty$. Then there exists a bounded linear operator T*

$$T : W^{1,p}(\Omega) \to L^p(\partial\Omega)$$

such that

a) $Tu = u|_{\partial\Omega}$ *if* $u \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$

b) $\|Tu\|_{L^p(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}$.

The operator T is called the trace operator *and Tu* the trace *of u on $\partial\Omega$.*

Proof. Proof can be found for example in Partial Differential Equations by Lawrence Evans [1]. □

Trace theorem also gives a convenient way to define the spaces $H^{\frac{1}{2}}(\partial\Omega)$. For this purpose, let's consider the trace operator T from $H^1(\Omega)$ to $L^2(\partial\Omega)$. Then $H^{\frac{1}{2}}(\partial\Omega)$ can be defined as the range of T, that is

$$H^{\frac{1}{2}}(\partial\Omega) = \{u \in L^2(\partial\Omega) \mid u = T(\varphi) \text{ with some } \varphi \in H^1(\Omega)\}.$$

This is not the only way to define $H^{\frac{1}{2}}(\partial\Omega)$, but for our purposes it is enough. The space $H^{\frac{1}{2}}(\partial\Omega)$ is a normed space with a norm defined as

$$\|u\|_{H^{\frac{1}{2}}(\partial\Omega)} = \inf\{\|f\|_{H^1(\Omega)} \mid Tf = u\}$$

where T is the trace operator from $H^1(\Omega)$ to $L^2(\partial\Omega)$ as defined above.

Next, let's discuss the duality of Sobolev spaces. We start by recalling the definition of dual spaces. Let E be a normed space with scalar field $\mathbb{K}$. Then *the dual space* of E, denoted E^*, contains the bounded linear operators from E to $\mathbb{K}$ [9]. Thus

$$E^* = \mathscr{L}(E, \mathbb{K}) = \{T : E \to \mathbb{K} \mid \text{T is a continuous linear operator}\}. \tag{2.11}$$

Let $u \in E$ and $f \in E^*$. We'll denote the pairing between E and E^* as $f(u) = \langle f, u\rangle$. We can now study the duals of Sobolev spaces.

Definition 2.12. Denote by $H^{-1}(\Omega)$ the dual space to $H_0^1(\Omega)$.

The space H^{-1} is a normed space with norm

$$\|f\|_{H^{-1}(\Omega)} = \sup\{\langle f, u\rangle \mid u \in H_0^1(\Omega), \|u\|_{H_0^1(\Omega)} \leq 1\}.$$

However, with this definition we do not identify $H_0^1(\Omega)$ with its dual, $H^{-1}(\Omega)$. Instead the inclusion $H_0^1(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)$ holds [1]. In a similar manner, we define $H^{-\frac{1}{2}}(\partial\Omega)$ as the dual space of $H^{\frac{1}{2}}(\partial\Omega)$.

In the later chapters we'll also need to consider values of Sobolev functions on the boundary. But what exactly does it mean for Sobolev function to have boundary values? We already mentioned the existence of traces, but what can we say about the weak derivatives on the boundary? The definition of weak derivatives only applies to the interior of the Ω. Let's study the behaviour on $\partial\Omega$ with the use of Green's formulas, and start by studying the expression

$$\begin{aligned}\int_\Omega (\nabla \times B) \cdot \varphi \, dx = \int_\Omega \Big(\frac{\partial B_3}{\partial x_2}\varphi_1 - \frac{\partial B_1}{\partial x_2}\varphi_3 \Big) \, dx + \int_\Omega \Big(\frac{\partial B_1}{\partial x_3}\varphi_2 - \frac{\partial B_2}{\partial x_3}\varphi_1 \Big) \, dx \\ + \int_\Omega \Big(\frac{\partial B_2}{\partial x_1}\varphi_3 - \frac{\partial B_3}{\partial x_1}\varphi_2 \Big) \, dx.\end{aligned}$$

We can apply Green's formulas to this expression to find out that

$$\begin{aligned}\int_\Omega (\nabla \times B) \cdot \varphi \, dx = &- \int_\Omega \Big(B_3 \frac{\partial \varphi_1}{\partial x_2} - B_1 \frac{\partial \varphi_3}{\partial x_2} \Big) \, dx + \int_{\partial\Omega} \Big(B_3\varphi_1\nu_2 - B_1\varphi_3\nu_2 \Big) \, dS \\ &- \int_\Omega \Big(B_1 \frac{\partial \varphi_2}{\partial x_3} - B_2 \frac{\partial \varphi_1}{\partial x_3} \Big) \, dx + \int_{\partial\Omega} \Big(B_1\varphi_2\nu_3 - B_2\varphi_1\nu_3 \Big) \, dS \\ &- \int_\Omega \Big(B_2 \frac{\partial \varphi_3}{\partial x_1} - B_3 \frac{\partial \varphi_2}{\partial x_1} \Big) \, dx + \int_{\partial\Omega} \Big(B_2\varphi_3\nu_1 - B_3\varphi_2\nu_1 \Big) \, dS.\end{aligned}$$

We notice that we can again combine these terms back to the form

$$\begin{aligned}\int_\Omega (\nabla \times B) \cdot \varphi \, dx &= \int_\Omega B_1 \Big(\frac{\partial \varphi_3}{\partial x_2} - \frac{\partial \varphi_2}{\partial x_3} \Big) + B_2 \Big(\frac{\partial \varphi_1}{\partial x_3} - \frac{\partial \varphi_3}{\partial x_1} \Big) + B_3 \Big(\frac{\partial \varphi_2}{\partial x_1} - \frac{\partial \varphi_1}{\partial x_2} \Big) \, dx \\ &+ \int_{\partial\Omega} (B_3\varphi_1\nu_2 - B_1\varphi_3\nu_2) + (B_1\varphi_2\nu_3 - B_2\varphi_1\nu_3) + (B_2\varphi_3\nu_1 - B_3\varphi_2\nu_1) \, dS \\ &= \int_\Omega B \cdot (\nabla \times \varphi) \, dx + \int_{\partial\Omega} (B \times \varphi) \cdot n \, dS\end{aligned}$$

which is by the rules of the triple product equivalent to

$$\int_\Omega B \cdot (\nabla \times \varphi) \, dx + \int_{\partial\Omega} (n \times B) \cdot \varphi \, dS.$$

Thus we get

$$\int_{\partial\Omega} (n \times B) \cdot \varphi \, dS = \int_\Omega (\nabla \times B) \cdot \varphi \, dx - \int_\Omega B \cdot (\nabla \times \varphi) \, dx.$$

We can use this equation to define the boundary values of Sobolev functions' weak derivatives. To do this, let's define the linear functional from $\mathscr{D}(\Omega) \to L^p(\Omega)$ by

$$\varphi \to \int_\Omega (\nabla \times B) \cdot \varphi \, dx - \int_\Omega B \cdot (\nabla \times \varphi) \, dx.$$

This is a continuous linear operator on the boundary as long as $\nabla \times B$ and $\nabla \times \varphi$ are regular enough. We identify the weak derivative of B on $\partial\Omega$ with the aforementioned operator.

As a conclusion, with the concept of weak derivatives, one can examine weak solutions in Sobolev spaces. While the equation might not have solutions in the usual sense because of a discontinuity or in general non-differentiability, it might still have solutions in the weak sense. The weak solutions often carry information about the equation.

2.2 Curl and divergence spaces

In later chapters we'll study Maxwell equations by studying the so-called curl and divergence spaces, so we present some definitions and results in this section. We'll be studying concepts and results which can be found in [1, 3, 4]. Let us denote

$$\mathscr{D}'(\Omega) = \mathscr{L}(\mathscr{D}(\Omega)),$$

that is, a set of continuous linear operators from $\mathscr{D}(\Omega)$ to $\mathbb{R}$. The definition of $\mathscr{D}(\Omega)$ is the same as in the previous section. We use the shorter notation $\mathscr{D}'(\Omega)^n$ for $(\mathscr{D}'(\Omega))^n$.

The set $\mathscr{D}'(\Omega)^n$ consists of continuous linear forms on Ω, that is, distributions on Ω. It's important to note that the set $\mathscr{D}(\Omega)$ is not a normed space, and thus the continuity of the linear operators isn't defined as the boundedness of the operator. Instead we define the continuity with the use of convergence.

If $T \in \mathscr{D}'(\Omega)$, we denote its value on $\varphi \in \mathscr{D}(\Omega)$ as $\langle T, \varphi \rangle$ or sometimes simply by $T(\varphi)$. The linear operator T is continuous on $\mathscr{D}(\Omega)$ if for every sequence $\{\varphi_p\}_{p\in\mathbb{N}}$ in $\mathscr{D}(\Omega)$ such that $\varphi_p \to \varphi$ as $p \to \infty$, we have that $\langle T, \varphi_p \rangle \to \langle T, \varphi \rangle$.

Alternatively, one can show the continuity with semi-norms $\|\varphi\|_N := \max_{|\alpha|\leq N, x\in\Omega} |D^\alpha(\varphi)|$ in $\mathscr{D}(K)$ for all compact $K \subset \Omega$. In this case $L \in \mathscr{D}'(\Omega)$ is continuous if and only if for each compact $K \subset \Omega$ there is $N \in \mathbb{N}$ and $C \in \mathbb{R}$ such that $|L\varphi| \leq C\|\varphi\|_N$ for all $\varphi \in \mathscr{D}(K)$.

Let's now recall the definitions of divergence and curl. Let $\Omega \subset \mathbb{R}^n$ and $v \in \mathscr{D}'(\Omega)^n$. Then *the divergence* of v is

$$\nabla \cdot v = \text{div } v = \sum_{i=1}^{n} \frac{\partial v_i}{\partial x_i}$$

where $v = (v_1, v_2, \dots, v_n)$. Let us now assume that $n = 3$, that is, $\Omega \subset \mathbb{R}^3$ and $v \in \mathscr{D}'(\Omega)^3$. Then *the curl* of v is

$$\nabla \times v = \text{curl } v = \Big(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}, \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}, \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\Big).$$

We now define the following Sobolev spaces that have useful properties for the study of our problem.

Definition 2.13. Let Ω be an open subset of $\mathbb{R}^n$. We define

$$H(\text{div}, \Omega) = \{v \in L^2(\Omega)^n \mid \nabla \cdot v \in L^2(\Omega)\}.$$

Similarly we define the following.

Definition 2.14. Let Ω be an open subset of $\mathbb{R}^3$. We define

$$H(\text{curl}, \Omega) = \{v \in L^2(\Omega)^3 \mid \nabla \times v \in L^2(\Omega)^3\}.$$

We denote as $H_0(\text{div}, \Omega)$ and $H_0(\text{curl}, \Omega)$ the closures of $\mathscr{D}(\Omega)^3$ in $H(\text{div}, \Omega)$ and $H(\text{curl}, \Omega)$ respectively. We notice that both $H(\text{div}, \Omega)$ and $H(\text{curl}, \Omega)$ are Hilbert spaces with inner products

$$(w, v)_{H(\text{div},\Omega)} := (w, v) + (\nabla \cdot w, \nabla \cdot v)$$

and

$$(w, v)_{H(\text{curl},\Omega)} := (w, v) + (\nabla \times w, \nabla \times v)$$

respectively, with $(\ ,\ )$ denoting the inner product in L^2 spaces. Similarly, they can then be equipped with norms corresponding to these inner products,

$$\|v\|_{H(\text{div},\Omega)} := (\|v\|^2 + \|\nabla \cdot v\|^2)^{1/2}$$

and

$$\|v\|_{H(\text{curl},\Omega)} := (\|v\|^2 + \|\nabla \times v\|^2)^{1/2}$$

respectively.

2.2.1 Trace theorems

As discussed in the section 2.1.3, the trace operator is a tool that can be used to define the restriction of the function to the boundary.

Next we will introduce the trace theorems for $H(\text{div}, \Omega)$ and $H(\text{curl}, \Omega)$ presented in [4]. We will start by recalling the definition of *Lipschitz boundary* which loosely refers to a boundary that can be thought of as a graph of a Lipschitz continuous function. This kind of boundary is often regular enough for our needs.

Definition 2.15. Let $\Omega \subset \mathbb{R}^n$. Then Ω is called a *Lipschitz domain* if for every point $x \in \partial\Omega$ there exists a hyperplane H of dimension $n-1$ which goes through x, a Lipschitz continuous function $g\colon H \to \mathbb{R}$ over the hyperplane H, and positive real numbers $r, h \in \mathbb{R}$ such that

$$\begin{cases} \Omega \cap C = \{y + zn \mid y \in B_r(x) \cap H,\ -h < z < g(y)\} \\ \partial\Omega \cap C = \{y + zn \mid y \in B_r(x) \cap H,\ g(y) = z\} \end{cases} \tag{2.16}$$

where n is the unit normal vector to H, $B_r(x)$ is the open ball of radius r centered around x and $C = \{y + zn \mid y \in B_r(x) \cap H, -h < z < h\}$. The boundary of Lipschitz domain is called a *Lipschitz boundary*.

Theorem 2.17. *Let $\Omega \subset \mathbb{R}^n$ be an open subset with a bounded Lipschitz boundary $\partial\Omega = \Gamma$. Then*

(i) The trace map $\gamma_n\colon \mathscr{D}(\Omega)^n \to \mathbb{R}$ defined by $\gamma_n(v) = v \cdot n|_{\partial\Omega}$ where n denotes the unit normal to $\partial\Omega$ towards the exterior of Ω, extends to a continuous linear mapping from $H(\mathrm{div}, \Omega)$ onto $H^{-\frac{1}{2}}(\partial\Omega)^n$.

(ii) Let γ_n be as in (i). Then the kernel $\ker(\gamma_n)$ is the space $H_0(\mathrm{div}, \Omega)$

(iii) The space $D(\bar{\Omega})^n$ is dense in $H(div, \Omega)$.

Recall the usual notation $H^{-\frac{1}{2}}(\partial\Omega)^n = (H^{-\frac{1}{2}}(\partial\Omega))^n$. Alternatively we might in this case assume that Ω has a smooth boundary, for example $C^{1,1}$-boundary.

Proof. We start with the point (iii).
(iii) Let $w \in H(\mathrm{div}, \Omega)$ such that w is orthogonal to the $\mathscr{D}(\bar{\Omega})^n$. Define

$$((w, v)) := (w, v) + (\nabla \cdot w, \nabla \cdot v).$$

Then $((w, v)) = 0$ for all $v \in \mathscr{D}(\bar{\Omega})^n$.

Let $w_0 = \nabla \cdot w$. We denote by $\tilde{w}$ and $\tilde{w_0}$ the extensions of w and w_0 to the whole $\mathbb{R}^n$ such that w and w_0 are zero outside of Ω. Similar to before, we have that

$$(\tilde{w}, \varphi) + (\tilde{w_0}, \nabla \cdot \varphi) = 0$$

for all $\varphi \in \mathscr{D}(\mathbb{R}^n)$. It follows that $\tilde{w} = \nabla \tilde{w_0}$ in $\mathscr{D}'(\mathbb{R}^n)$ and especially $\tilde{w_0} \in H^1(\mathbb{R}^n)$. This on the other hand implies that $w_0 \in H_0^1(\Omega)$. Because $\mathscr{D}(\Omega)$ is dense in $H_0^1(\Omega)$, there exists a sequence $(\varphi_p)_{p\in\mathbb{N}}$ of elements $\varphi_p \in \mathscr{D}(\Omega)$ such that $\varphi_p \to w_0$ in $H^1(\Omega)$ as $p \to \infty$.

Now it holds that

$$((w, v)) = \lim_{p\to\infty} \Big((\nabla\varphi_p, v) + (\varphi_p, \nabla \cdot v)\Big) = 0$$

for all $v \in H(\mathrm{div}, \Omega)$, and thus $w = 0$. This proves the point (iii).
(i) Let's recall the Green's formula which states that

$$(v, \nabla\varphi) + (\nabla \cdot v, \varphi) = \int_\Gamma v \cdot n\varphi \, d\Gamma \tag{2.18}$$

for all $v \in \mathscr{D}(\bar{\Omega})^n$ and all $\varphi \in \mathscr{D}(\bar{\Omega})$. Similarly this holds for all $\varphi \in H^1(\Omega)$ because $\mathscr{D}(\bar{\Omega})$ is dense in $H^1(\Omega)$. We get the inequality

$$\left| \int_\Gamma v \cdot n\varphi \, d\Gamma \right| \le \|v\|_{H(\mathrm{div},\Omega)} \|\varphi\|_{H^1(\Omega)}$$

for all $v \in \mathscr{D}(\overline{\Omega})^n$ and $\varphi \in H^1(\Omega)$.

From the last inequality we obtain that

$$\left| \int_\Gamma v \cdot n\varphi \, d\Gamma \right| \leq \|v\|_{H(\mathrm{div},\Omega)} \left\| \varphi|_\Gamma \right\|_{H^{1/2}(\Gamma)}$$

where $\varphi|_\Gamma$ is the trace of φ on Γ. This can be deduced by noting that

$$\left\| \varphi|_\Gamma \right\|_{H^{1/2}(\Gamma)} = \inf_{\psi \in H^1(\Omega), \psi|_\Gamma = \varphi|_\Gamma} \|\psi\|_{H^1(\Omega)}.$$

Let's now consider the mapping $\gamma_n \colon v \mapsto v{\cdot}n|_\Gamma$ which is defined on $\mathscr{D}(\overline{\Omega})^n$ and equipped with the norm of $H(\mathrm{div}, \Omega)$. It follows from previous remarks that γ_n has values in $H^{\frac{1}{2}}(\Gamma)$ and that γ_n is also continuous. Thus it extends by continuity to the space $H(\mathrm{div}, \Omega)$ because, as mentioned in point (iii), $\mathscr{D}(\overline{\Omega})^n$ is dense in $H(\mathrm{div}, \Omega)$.

Let's now denote by γ_n the trace map defined from $H(\mathrm{div}, \Omega)$ to $H^{-\frac{1}{2}}(\Gamma)$, again defined by the rule $\gamma_n v = v \cdot n|_\Gamma$. Our goal is to show that γ_n is surjective. Let $\mu \in H^{-\frac{1}{2}}(\Gamma)$ in which case the Neumann problem

$$\begin{cases} -\Delta u + u = 0 \text{ on } \Omega \\ \frac{\partial u}{\partial n}\Big|_\Gamma = \mu \text{ on } \Gamma. \end{cases} \tag{2.19}$$

has a solution $u \in H^1(\Omega)$ [3].

With the function u gotten from the Neumann problem, one can define $v = \nabla u$. In that case $v \in H(\mathrm{div}, \Omega)$ and $\gamma_n v = \mu$. This proves the point (i).
(ii) Finally, let w belong to the kernel of γ_n, $w \in \ker(\gamma_n)$. Additionally, let w be orthogonal to $\mathscr{D}(\Omega)^n$ in $H(\mathrm{div}, \Omega)$, that is,

$$((v, w)) = (v, w) + (\nabla \cdot v, \nabla \cdot w) = 0 \tag{2.20}$$

for all $v \in \mathscr{D}(\Omega)^n$.

This implies, with the notation $w_0 = \nabla \cdot w$, that $w = \nabla w_0$. Thus we find that $w_0 \in H^1(\Omega)$. We can then apply Green's formula of the form

$$(v, \nabla\varphi) + (\nabla \cdot v, \varphi) = \langle \gamma_n, \varphi|_\Gamma \rangle$$

where $\langle \cdot, \cdot \rangle$ denotes the duality between $H^{-\frac{1}{2}}(\Gamma)$ and $H^{\frac{1}{2}}(\Gamma)$. By replacing φ in Green's formula with w_0 and v by $\ker(\gamma_n)$, the result 2.20 gives

$$((v, w)) = \langle \gamma_n v, w_0|_\Gamma \rangle = 0,$$

and thus $w = 0$. In other words, $\ker(\gamma_n) = H_0(\mathrm{div}, \Omega)$ which proves the claim.

□

Theorem 2.21. *Let* $\Omega \subset \mathbb{R}^3$ *be an open subset with a bounded Lipschitz boundary* $\partial\Omega = \Gamma$. *Then*

(i) The trace map $\gamma_n \colon \mathscr{D}(\Omega)^3 \to \mathbb{R}$ *defined by* $\gamma_n(v) = v \wedge n|_{\partial\Omega}$ *where* n *denotes the unit normal to* $\partial\Omega$ *towards the exterior of* Ω*, extends to a continuous linear mapping from* $H(\text{curl}, \Omega)$ *into* $H^{-\frac{1}{2}}(\partial\Omega)^3$.

(ii) Let γ_n *be as in (i). Then the kernel* $\ker(\gamma_n)$ *is the space* $H_0(\text{curl}, \Omega)$.

(iii) The space $\mathscr{D}(\bar{\Omega})^n$ *is dense in* $H(\text{curl}, \Omega)$.

As mentioned before, $\wedge$ operator used on the 2.21 (i) denotes the exterior product (see 2.25). We omit the proof for the second trace theorem but it can be found in Robert Dautray and Jacques-Louis Lions's Mathematical Analysis and Numerical Methods for Science and Technology: Volume 3, Spectral Theory and Applications [4].

2.2.2 Properties of curl and divergence

In this section we will go through some of the properties and connections between curl, divergence, gradient and their respective Hilbert spaces. First we note some connections between spaces defined in the last section. In the following result *regularity* of the subset Ω means that it is of class $\mathscr{C}^2$. This means that for every point $x_0 \in \Gamma$ there exists $r > 0$ and a C^k function $\gamma \colon \mathbb{R}^{n-1} \to \mathbb{R}$ such that

$$U \cap B(x_0, r) = \{x \in B(x_0, r) \mid x_n > \gamma(x_1, \ldots, x_n)\}.$$

What this means in practice is that the boundary is quite smooth. Similarly we can define C^k boundaries and C^∞ boundaries if γ is C^k or C^∞ instead of C^2 [1].

Theorem 2.22. *Let* $\Omega \subset \mathbb{R}^3$ *be a bounded, open and regular subset of* $\mathbb{R}^3$. *Let the boundary of* Ω *be* $\Gamma = \partial\Omega$ *and define the Sobolev spaces*

$$\begin{cases} H^1_{t0}(\Omega)^3 = \{v \in H^1(\Omega)^3 \mid v \wedge n|_\Gamma = 0\} \\ H^1_{n0}(\Omega)^3 = \{v \in H^1(\Omega)^3 \mid v \cdot n|_\Gamma = 0\}. \end{cases} \tag{2.23}$$

Then for these spaces it holds that

$$H^1_{t0}(\Omega)^3 = H_0(\text{curl}, \Omega) \cap H(\text{div}, \Omega)$$

and

$$H^1_{n0}(\Omega)^3 = H(\text{curl}, \Omega) \cap H_0(\text{div}, \Omega).$$

Note that to define the boundary values in 2.23 you need trace theorems which we present in section 2.2.1.

In the last result $\wedge$ operator is called *an exterior product* which is loosely related to the cross product in three dimensional spaces. Let $u, v \in \mathbb{R}^3$ have the following representations with the standard basis $\{e_1, e_2, e_3\}$.

$$\begin{cases} u = u_1 e_1 + u_2 e_2 + u_3 e_3 \\ v = v_1 e_1 + v_2 e_2 + v_3 e_3. \end{cases} \tag{2.24}$$

Then the exterior product $u \wedge v$ is a bivector defined by

$$u \wedge v = (u_1 v_2 - u_2 v_1)(e_1 \wedge e_2) + (u_2 v_3 - u_3 v_2)(e_2 \wedge e_3) + (u_3 v_1 - u_1 v_3)(e_3 \wedge e_1). \tag{2.25}$$

Note that the coefficients are the same as in the cross product. While cross product can be interpreted as a vector perpendicular to both u and v, the exterior product can be interpreted as the parallelogram spanned by the vectors u and v. It's also important to note that the exterior product does not depend on the choice of orientation. Exterior product is also associative, whereas cross product is not.

The spaces $H^1_{t0}(\Omega)^3$ and $H^1_{n0}(\Omega)^3$ defined above represent those elements in $H^1(\Omega)^3$ that are either not tangential or normal vector fields of the surface Γ.

Another interesting connection is the decomposition of L^2 spaces using divergence spaces and the gradient of Sobolev spaces. This is often called the Helmholtz decomposition. Recall how we defined Lipschitz boundary on 2.15.

Lemma 2.26. *Let Ω be a connected open set in $\mathbb{R}^3$ with a Lipschitz boundary Γ. $L^2(\Omega)^3$ has the following orthogonal decompositions*

$$\begin{cases} L^2(\Omega)^3 = \nabla H^1(\Omega) \oplus H_0(\text{div } 0, \Omega) \\ L^2(\Omega)^3 = \nabla H^1_0(\Omega) \oplus H(\text{div } 0, \Omega) \end{cases}$$

where we use spaces $H(\text{div } 0, \Omega)$ and $H_0(\text{div } 0, \Omega)$ defined by

$$H(\text{div } 0, \Omega) = \Big\{ u \in L^2(\Omega)^3 \mid \nabla \cdot u = 0 \Big\},$$
$$H_0(\text{div } 0, \Omega) = \Big\{ u \in L^2(\Omega)^3 \mid \nabla \cdot u = 0, \ n \cdot u = 0 \ \textit{on } \Gamma \Big\}.$$

Proof. (of Lemma 2.26)

It's enough for us to prove that $\nabla H^1(\Omega)$ is orthogonal to $H_0(\text{div } 0, \Omega)$ and similarly that $\nabla H^1_0(\Omega)$ is orthogonal to $H(\text{div } 0, \Omega)$.

Now we get the Helmholtz decomposition by noting that $H^1(\Omega)$ and respectively $H_0^1(\Omega)$ are closed in $L^2(\Omega)^3$. This is because for Hilbert spaces E and their closed subspaces M it holds that $E = M \oplus M^\perp$ [9].

Let $v \in L^2(\Omega)^3$ be orthogonal to the space $\nabla H^1(\Omega)$. Then

$$(v, \nabla\varphi) = -\langle \nabla \cdot v, \varphi \rangle = 0$$

with all $\varphi \in \mathscr{D}(\Omega)$. Here $(v, \nabla\varphi)$ denotes the scalar product of v and $\nabla\varphi$ in $L^2(\Omega)^3$. Similarly here $\langle \nabla \cdot v, \varphi \rangle$ denotes the duality between corresponding spaces of the $\nabla \cdot v$ and φ.

Thus $\nabla \cdot v = 0$ and $v \in H(\mathrm{div}, \Omega)$. Let's use the Green's formula of the form

$$(v, \nabla\varphi) + (\nabla \cdot v, \varphi) = \langle v \cdot n, \varphi|_\Gamma \rangle$$

or in other terms,

$$\int_\Omega v \nabla\varphi \, dx + \int_\Omega (\nabla \cdot v)\varphi \, dx = \int_\Gamma (v \cdot n)\varphi \, d\Gamma$$

where n is the unit normal on $\partial\Omega = \Gamma$. Then we get that $v \cdot n|_\Gamma = 0$. Thus $v \in H_0(\mathrm{div}\ 0, \Omega)$ and we have proven that

$$(\nabla H^1(\Omega))^\perp \subset H_0(\mathrm{div}\ 0, \Omega).$$

Now, let $v \in L^2(\Omega)^3$ be orthogonal to $H_0(\mathrm{div}\ 0, \Omega)$. Then $(v, \varphi) = 0$ with all $\varphi \in H_0(\mathrm{div}\ 0, \Omega)$. Because for φ it holds that $\nabla \cdot \varphi = 0$ and $n \cdot \varphi = 0$ on Γ, the Green's formula

$$\int_\Omega \varphi \nabla v \, dx + \int_\Omega (\nabla \cdot \varphi) v \, dx = \int_\Gamma (\varphi \cdot n) v \, d\Gamma$$

implies that $\nabla v = 0$ and thus $v \in \nabla H^1$. This proves that

$$H_0(\mathrm{div}\ 0, \Omega) \subset (\nabla H^1(\Omega))^\perp.$$

We have now proven that $\nabla H^1(\Omega)$ and $H_0(\mathrm{div}\ 0, \Omega)$ are orthogonal. In a similar manner one can deduce that $\nabla H_0^1(\Omega)$ is orthogonal to $H(\mathrm{div}\ 0, \Omega)$.

With these results and the knowledge that $H^1(\Omega)$ and $H_0^1(\Omega)$ are closed in $L^2(\Omega)^3$, it follows that

$$\begin{cases} L^2(\Omega)^3 = \nabla H^1(\Omega) \oplus H_0(\mathrm{div}\ 0, \Omega) \\ L^2(\Omega)^3 = \nabla H_0^1(\Omega) \oplus H(\mathrm{div}\ 0, \Omega). \end{cases}$$

□

Next, let's take a look at some other results related to curl, divergence and gradient. The following lemma is an important result that gives sufficient conditions for existence of gradient or curl representation for a vector field.

Lemma 2.27. *(Poincare's lemma) Let Ω be an open set in $\mathbb{R}^3$ and let $u = (u_i)$, $i \in \{1, \dots 3\}$. Let $u_i \in C^1(\Omega)$ with every $i \in \{1, \dots 3\}$.*

(i) If $\nabla \times u = 0$ then there exists $p \in \mathscr{C}^2$ such that $u = \nabla p$ locally, that is, every point $x_0 \in \Omega$ has a neighborhood $U(x_0)$ and a differentiable p defined in $U(x_0)$ such that $u = \nabla p$ in $U(x_0)$. Note that p may change when moving from one neighborhood to another.

(ii) If $\nabla \cdot u = 0$ then there exists w such that for the component functions it holds that $w_i \in C^1$ and $u = \nabla \times w$ locally.

Proof. Let $u \in C^1(\Omega)^3$ such that $\nabla \times u = 0$. Let $x' = (x'_1, x'_2, x'_3) \in \Omega$. Now, let's define the function p by using integration in a smart way.

$$p(x) = \int_{x'_1}^{x_1} u_1(x_1, x_2, x_3)\ dx_1 + \int_{x'_2}^{x_2} u_2(x_1, x_2, x_3)\ dx_2 + \int_{x'_3}^{x_3} u_3(x_1, x_2, x_3)\ dx_3$$

where $x = (x_1, x_2, x_3)$. Now if we calculate the gradient of p, we notice that

$$\nabla p = u$$

which proves the first part.

In the second part, let $u \in C^1(\Omega)^3$ such that $\nabla \cdot u = 0$. Let $x' = (x'_1, x'_2, x'_3) \in \Omega$. Let's define the functions $v = (v_1, v_2, v_3)$ and $v' = (v'_1, v'_2, v'_3)$ in a similar manner as before by

$$\begin{aligned}
v_1(x) &= \int_{x'_2}^{x_2} u_3(x_1, x_2, x_3)\ dx_2 - \int_{x'_3}^{x_3} u_2(x_1, x_2, x_3)\ dx_3 \\
v_2(x) &= \int_{x'_3}^{x_3} u_1(x_1, x_2, x_3)\ dx_3 - \int_{x'_1}^{x_1} u_3(x_1, x_2, x_3)\ dx_1 \\
v_3(x) &= \int_{x'_1}^{x_1} u_2(x_1, x_2, x_3)\ dx_1 - \int_{x'_2}^{x_2} u_1(x_1, x_2, x_3)\ dx_2
\end{aligned}$$

and

$$\begin{aligned}
v'_1(x) &= \int_{x'_2}^{x_2} u_3(x_1, x_2, x'_3)\ dx_2 - \int_{x'_3}^{x_3} u_2(x_1, x'_2, x_3)\ dx_3 \\
v'_2(x) &= \int_{x'_3}^{x_3} u_1(x'_1, x_2, x_3)\ dx_3 - \int_{x'_1}^{x_1} u_3(x_1, x_2, x'_3)\ dx_1 \\
v'_3(x) &= \int_{x'_1}^{x_1} u_2(x_1, x'_2, x_3)\ dx_1 - \int_{x'_2}^{x_2} u_1(x'_1, x_2, x_3)\ dx_2.
\end{aligned}$$

Now $\nabla \times v = -3u + u'$ and $\nabla \times v' = -2u'$ where we define u' by

$$u'(x_1, x_2, x_3) = (u_1(x_1', x_2, x_3), u_2(x_1, x_2', x_3), u_3(x_1, x_2, x_3')).$$

Then we notice that with $w = -\frac{1}{3}\left(v + \frac{1}{2}v'\right)$ it holds that $\nabla \times w = u$. This proves the second claim and ends the proof. □

Poincare's lemma also has the following global form presented in [4].

Corollary 2.28. *(Poincare's lemma) Let Ω be an open and simply connected set in $\mathbb{R}^3$ and let $u \in L^2(\Omega)^3$.*

(i) If $\nabla \times u = 0$ then there exists $p \in H^1(\Omega)$ such that $u = \nabla p$.

(ii) If $\nabla \cdot u = 0$ then there exists $v \in H^1(\Omega)$ such that $u = \nabla \times v$.

For our purposes it will be useful to consider elements of $L^2(\Omega)^3$ by decomposing them into a gradient and curl part. The following lemma guarantees the uniqueness of such decomposition [4].

Lemma 2.29. *Let $\Omega \subset \mathbb{R}^3$ such that Ω can be made simply connected by a finite number of regular cuts. Then each element of $u \in L^2(\Omega)^3$ has the unique decomposition*

$$u = \nabla p + (\nabla \times w)$$

where $p \in H^1(\Omega)$ is unique up to a constant and $w \in H^1(\Omega)^3$ satisfies on the boundary Γ

$$n \cdot (\nabla \times w)|_\Gamma = 0$$

where n is the normal vector of Γ.

Here the set Ω can be made simply connected by a finite number of regular cuts. The reqularity of the cuts means that for $\Omega \subset \mathbb{R}^n$ there exists cuts $\Sigma_1, \ldots, \Sigma_N$ of dimension $n-1$ that are of class $\mathscr{C}^r$ with $r \geq 2$ such that $\Sigma_i \cap \Sigma_j = \emptyset$ for $i \neq j$ and $i, j \in \{1, \ldots, N\}$. We also assume that Σ_i are non-tangential to the boundary $\Gamma = \partial\Omega$. Thus $\Omega \setminus (\bigcup_{i=1 \text{ to } N} \Sigma_i)$ is simply connected.

The following result is also used later when studying magnetic and electric fields and their decompositions.

Lemma 2.30. *Let $\Omega \subset \mathbb{R}^n$ be an open subset with $n = 2$ or $n = 3$ such that it's bounded and connected, with $n-1$ dimensional boundary of sufficient regularity, say, class $\mathscr{C}^2$. We also assume that Ω can be made simply connected with a finite number of regular cuts and that the boundary $\Gamma = \partial\Omega$ has a finite number of connected components.*

Then, the kernel of curl in $L^2(\Omega)^n$, denoted by $H(\text{curl}\ 0, \Omega)$, is the sum of two orthogonal spaces $\nabla H^1(\Omega)$ and $\mathbb{H}_1(\Omega)$, defined by

$$\nabla H^1(\Omega) = \{\nabla u \mid u \in H^1(\Omega)\}$$

and

$$\mathbb{H}_1(\Omega) = \{u \in L^2(\Omega)^2 \mid \nabla \times u = 0, \nabla \cdot u = 0, u \cdot n|_\Gamma = 0\}.$$

Proof. We omit the proofs for these two results related to decompositions here but they can be found in [4]. □

2.3 Introduction to Maxwell's equations

Maxwell's equations are an important set of equations that describe the behaviour of electromagnetic radiation in a vacuum or in a medium. The equations in a vacuum are

(i) $\nabla \cdot E = \frac{\rho}{\varepsilon_0}$ (Gauss's law)

(ii) $\nabla \cdot B = 0$ (Gauss's law for magnetic field)

(iii) $\nabla \times E = -\dfrac{\partial B}{\partial t}$ (Faraday's law)

(iv) $\nabla \times B = \mu_0 \left(J + \varepsilon_0 \dfrac{\partial E}{\partial t} \right)$ (Ampère's law),

where E denotes the electric field, B denotes magnetic flux density, J denotes current density, ε_0 denotes vacuum permittivity and μ_0 denotes vacuum permeability [7, 2, 6].

In a medium the situation is a bit different. Let's assume that the studied media are isotropic so that they react to electric or magnetic fields in the same way no matter the direction of the fields. In a medium the Maxwell's equations differ from the vacuum equations so that Gauss's law obtains a form

$$\nabla \cdot D = \rho$$

and Ampere's law a form

$$\nabla \times H = J + \varepsilon_0 \frac{\partial D}{\partial t},$$

where $D = \varepsilon E$ is the electric displacement field and $H = B/\mu$ is the magnetic field. We say that the media is perfect if $D = \varepsilon E$ and $H = B/\mu$ hold for some *constants* ε and μ.

Above ε is called the permittivity of the medium and μ is called the permeability of the medium. In a vacuum the situation reverts to the forms mentioned earlier, so in the future we will often present the equations with D and H.

Time-harmonic forms

Often it's useful to present Maxwell's equations in a form that doesn't depend on time but instead of the angular frequencies of the fields. Thus we need to define so-called *time-harmonic* Maxwell's equations.

For simplicity, we consider sinusoidal signals, that is, signals that have time dependancy shaped like sine wave. However, in general case any arbitrary signal could be formed with a sum of sinusoidal waves. Let's use complex representation and thus our signal is of the form

$$f(x,t) = u(x)e^{i\omega t+\theta}$$

where $u(x) \in \mathbb{C}$ does not depend on time. We can denote $u = u_R + iu_I$ where $u_R \in \mathbb{R}$ is the real part and $u_I \in \mathbb{R}$ is the imaginary part.

Time derivative acts only on the term $e^{i\omega t+\theta}$ in the Maxwell's equations. Thus the time derivative in the equations can be replaced with $i\omega$, and the time-harmonic Maxwell's equations are

(i) $\nabla \cdot D(x) = \rho$

(ii) $\nabla \cdot B(x) = 0$

(iii) $\nabla \times E(x) = -i\omega B(x)$

(iv) $\nabla \times H(x) = J + i\omega\varepsilon_0 E(x)$.

We notice that the equation (ii) follows from the equation (iii) by taking a divergence. Similarly equation (i) follows from equation (iv) by taking into account the conservation of charges, $\nabla \cdot J = -i\omega\rho$. Thus it's enough to view the equations

$$\begin{cases} \nabla \times E = -i\omega\mu H \\ \nabla \times H = i\omega\varepsilon E + J \end{cases} \tag{2.31}$$

assuming that the medium is perfect.

Chapter 3

Magnetostatics and Electrostatics

3.1 Magnetostatics of a surface current

Let's consider the following problem presented and studied in Mathematical Analysis and Numerical Methods for Science and Technology: Volume 3 by Robert Dautray and Jacques-Louis Lions which we utilise as a source during this chapter [4]. A media is called perfect magnetic media if it holds that $B = \mu H$ with some constant μ. Now, let there be two perfect magnetic media with permeability μ and μ'.

Let these media be such that the first one is taking up the domain Ω which is open, bounded and regular subset of $\mathbb{R}^3$, while the latter is taking up the domain Ω' where $\Omega' = \mathbb{R}^3 \setminus \bar{\Omega}$. For the regularity, we can for example assume that the boundary of Ω is of class $\mathscr{C}^2$. Let's also assume that there is no current through Ω and Ω', and thus there can only be currents in the surface between these two media. In particular, what happens to the magnetic field in the surface that separates the media? Let's denote this surface by $\Gamma = \partial\Omega$.

Problems such as this where we study the magnetic field in a system where the currents are steady are called magnetostatic problems. We will go through the details later, but we notice that there might be a jump in the magnetic field on the boundary Γ where we know currents can exist. This discontinuity is exactly the reason why we need to study the problem from the perspective of Sobolev spaces. Let's then study the problem further.

Let Ω, Ω' and Γ be as defined above. Let there be no current in Ω or Ω'. Now Maxwell's equations imply that

$$\begin{cases} \nabla \cdot B = 0 \text{ in } \mathbb{R}^3 \\ \nabla \times B = 0 \text{ in } \Omega \text{ and in } \Omega' \end{cases} \tag{3.1}$$

We might also assume that our solution needs to have finite energy. This is so that the solution will be consistent with the real-world phenomenon where energy needs to be finite but it also guarantees unique solvability. Nonetheless, it's important to note that our system is extremely idealized as we use perfect media. The finite energy means that

$$W = \frac{1}{2\mu}\int_{\Omega} |B|^2 \, dx + \frac{1}{2\mu'}\int_{\Omega'} |B|^2 \, dx < \infty. \tag{3.2}$$

Let's study the problem in two parts. First, let the surface current J_Γ be known. Then the problem is equivalent to the problem of finding a function $B \in L^2(\mathbb{R}^3)^3$ which satisfies the following conditions.

$$\begin{cases} \nabla \cdot B = 0 \text{ in } \mathbb{R}^3 \\ \nabla \times B = 0 \text{ in } \Omega \text{ and } \Omega' \\ \Big[-\dfrac{B}{\mu} \wedge n\Big]_\Gamma = J_\Gamma \text{ in } \Gamma \end{cases} \tag{3.3}$$

where n is a normal vector on Γ orientated to the exterior of Ω, and J_Γ satisfies $(\nabla \cdot J_\Gamma)|_\Gamma = 0$. Recall how we defined the boundary values of Sobolev functions in Chapter 2.1. Here $\Big[-\frac{B}{\mu} \wedge n\Big]_\Gamma$ denotes the jump in $-\frac{B}{\mu} \wedge n$ on Γ. In other words,

$$\Big[-\frac{B}{\mu} \wedge n\Big]_\Gamma = -\Big(\frac{B_{\Omega'}|_\Gamma}{\mu} - \frac{B_\Omega|_\Gamma}{\mu}\Big) \wedge n.$$

Now, let's denote the following spaces containing fields equipped with zero divergence as

$$V_{W^1} = \{A \in W^1(\mathbb{R}^3)^3 \mid \nabla \cdot A = 0\}$$

and

$$V_{L^2} = H(\text{div } 0, \mathbb{R}^3) = \{B \in L^2(\mathbb{R}^3)^3 \mid \nabla \cdot B = 0\}.$$

Here the notation $W^1(\mathbb{R}^3)$ refers to Beppo-Levi space of $\mathbb{R}^3$. For any arbitrary open set $\Omega \subset \mathbb{R}^3$ we define the semi-norm $\|\nabla\varphi\|_{L^2(\Omega)^n}$ for $\varphi \in \mathscr{D}(\overline{\Omega})$. Let us also assume Ω is connected. Then we define the space $W^1(\Omega)$ as the closure of $\mathscr{D}(\overline{\Omega})$ under $\|\nabla\varphi\|_{L^2(\Omega)^n}$. Because Ω is connected, if we assume that $\nabla\varphi = 0$ for $\varphi \in \mathscr{D}(\overline{\Omega})$, then it actually holds that $\varphi = 0$ [4]. Thus $\|\nabla\varphi\|_{L^2(\Omega)^n}$ is a norm on $\mathscr{D}(\overline{\Omega})$. We can then define the closure of $\mathscr{D}(\overline{\Omega})$ under the norm in the usual way as the intersection of all closed sets containing $\mathscr{D}(\overline{\Omega})$. Analogously to the definition of $W^1(\mathbb{R}^3)$ we define $W^1_0(\Omega)$ as the closure of $\mathscr{D}(\Omega)$ under the norm $\|\nabla\varphi\|_{L^2(\Omega)^n}$.

We know by Poincare's lemma 2.27 that for each $\widetilde{B} \in V_{L^2}$ there exists a unique $\widetilde{A} \in V_{W^1}$ such that $\widetilde{B} = \nabla \times \widetilde{A}$. Let us define the following operators.

$$a(B, \widetilde{B}) = \frac{1}{2\mu} \int_{\Omega} B \cdot \widetilde{B} \, dx + \frac{1}{2\mu'} \int_{\Omega'} B \cdot \widetilde{B} \, dx$$

and

$$a_0(A, \widetilde{A}) = a(\nabla \times A, \nabla \times \widetilde{A})$$

with all $B, \widetilde{B} \in V_{L^2}$ and $A, \widetilde{A} \in V_{W^1}$. Note that a corresponds to the magnetic energy defined in 3.2 if $B = \widetilde{B}$. Using these operators we can find equivalent form to the problem 3.3 and also show the uniqueness of the solution.

Lemma 3.4. *Let J_Γ be a given surface current such that $J_\Gamma \in H^{-\frac{1}{2}}(\Gamma)^3$, $J_\Gamma \cdot n = 0$ almost everywhere on Γ and $\nabla \cdot J_\Gamma = 0$ on Γ. Then the problem 3.3 is equivalent to the problem of finding $B \in V_{L^2}$ (and respectively $A \in V_{W^1}$ for $B = \nabla \times A$) satisfying the condition*

$$a(B, \widetilde{B}) = a_0(A, \widetilde{A}) = \frac{1}{2} \int_{\Gamma} J_\Gamma \cdot \widetilde{A}|_\Gamma \, d\Gamma \tag{3.5}$$

for all $\widetilde{B} = \nabla \times \widetilde{A} \in V_{L^2}$, $\widetilde{A} \in V_{W^1}$. This problem has exactly one solution.

Proof. Let's show that if 3.5 holds for $B \in V_{L^2}$ then the second and third equations in 3.3 are satisfied. Because $B \in V_{L^2}$, it follows that $\nabla \cdot B = 0$ and the first equation is already clear.

Let's choose $\tilde{A}$ in 3.5 such that $\tilde{A} \in \mathscr{D}(\Omega)^3$. In particular $\nabla \cdot \tilde{A} = 0$ because $\tilde{A} \in V_{L^2}$. Equation 3.5 then gives $\nabla \times B = 0$ in Ω. Then, let's choose $\tilde{A}$ in 3.5 such that $\tilde{A} \in \mathscr{D}(\Omega')^3$. Again, 3.5 then gives $\nabla \times B = 0$ in Ω'. Thus the second equation in 3.3 holds.

Now, let's apply Green's formula in Ω and Ω' for $\tilde{A} \in \mathscr{D}(\mathbb{R}^3)^3$ with $\nabla \cdot \tilde{A} = 0$, or $\tilde{A} \in V_{W^1}$. Now 3.5 gives

$$a(B, \tilde{B}) = a_0(A, \tilde{A}) = \frac{1}{2\mu} \int_{\Gamma} (B_\Omega \wedge n) \cdot \tilde{A} \, d\Gamma - \frac{1}{2\mu'} \int_{\Gamma} (B_{\Omega'} \wedge n) \cdot \tilde{A} \, d\Gamma. \tag{3.6}$$

Thus

$$a(B, \tilde{B}) = a_0(A, \tilde{A}) = \frac{1}{2\mu} \int_{\Gamma} \Big[-\frac{B}{\mu} \wedge n \Big]_\Gamma \cdot \tilde{A}|_\Gamma \, d\Gamma. \tag{3.7}$$

We notice that the set of functions in $f \in H^{-\frac{1}{2}}(\Gamma)^3$ that satisfy $\int_\Gamma f \cdot n \, d\Gamma = 0$ is exactly the same as the set that contains traces $f = \tilde{A}|_\Gamma$ for $\tilde{A} \in V_{W^1}$. Now the equation 3.7 gives that

$$J_\Gamma = \Big[-\frac{B}{\mu} \wedge n \Big]$$

and thus the third condition in 3.3 holds.

Finally, let's show the uniqueness and existence of the problem, guaranteeing the unique solvability. We notice that the mapping $L\colon \tilde{A} \to \frac{1}{2}\int_\Gamma J_\Gamma \cdot \tilde{A}|_\Gamma \, d\Gamma$ is a continuous linear form because $J_\Gamma \in H^{-\frac{1}{2}}(\Gamma)^3$. We also notice that the bilinear forms a and a_0 are coercive. Let $\mu_1 = \inf\{\mu, \mu'\}$ and $\mu_2 = \sup\{\mu, \mu'\}$. Then

$$\frac{1}{2\mu_2}\int_{\mathbb{R}^3} |B|^2 \, dx \le a(B, B) \le \frac{1}{2\mu_1}\int_{\mathbb{R}^3} |B|^2 \, dx$$

for all $B \in V_{L^2}$. This proves that a is V_{L^2}-coercive. Analogously a_0 is V_{W^1}-coercive.

Because it holds that the bilinear form a_0 is V_{W^1}-coercive (and similarly a is V_{L^2}-coercive), we can use Lax-Milgram theorem which states that there exists exactly one solution $B = \nabla \times A \in V_{L^2}$ to the problem 3.5 [9]. □

Now, let's consider a general case of the problem and try to find all of the solutions to the problem 3.1 that also have finite energy. Our goal is to prove the following result.

Theorem 3.8. *The set of solutions to the problem 3.1 that also satisfy the finite energy requirement consists of such functions $B \in L^2(\mathbb{R}^3)^3$ that for $B_\Omega = B|_\Omega$ and $B_{\Omega'} = B|_{\Omega'}$ it holds:*

$$\begin{cases} \nabla \cdot B_\Omega = 0 \text{ in } \Omega \\ \nabla \times B_\Omega = 0 \text{ in } \Omega \\ \nabla \cdot B_{\Omega'} = 0 \text{ in } \Omega' \\ \nabla \times B_{\Omega'} = 0 \text{ in } \Omega' \end{cases} \tag{3.9}$$

and

$$n \cdot B_\Omega|_\Gamma = n \cdot B_{\Omega'}|_\Gamma \text{ on } \Gamma \tag{3.10}$$

where n is any normal vector of Γ, orientated to the exterior of Ω.

Proof. Let B satisfy the conditions 3.1 and 3.2. Now B_Ω and $B_{\Omega'}$ satisfy the equations 3.9. We can now use Green's formula. Let $\varphi \in \mathscr{D}(\mathbb{R}^3)$. Then by Green's formula of the form

$$(v, \nabla\varphi) + (\nabla \cdot v, \varphi) = \int_\Gamma (v \cdot n)\varphi \, d\Gamma \tag{3.11}$$

we get that

$$\begin{aligned}(B, \nabla\varphi) &= \int_\Omega B\nabla\varphi\; dx + \int_{\Omega'} B\nabla\varphi\; dx \\ &= -\int_\Omega (\nabla\cdot B)\varphi\; dx - \int_{\Omega'} (\nabla\cdot B)\varphi\; dx + \int_\Gamma [B\cdot n]_\Gamma \varphi\; d\Gamma \\ &= \int_\Gamma [B\cdot n]_\Gamma \varphi\; d\Gamma\end{aligned}$$

since $\nabla \cdot B = 0$ in $\mathbb{R}^3$. This implies that $n \cdot B_\Omega|_\Gamma = n \cdot B_{\Omega'}|_\Gamma$ on Γ since test functions are dense in Sobolev spaces and the duality is non-degenerate.

For the other direction, let $B_\Omega \in L^2(\Omega)^3$ and $B_{\Omega'} \in L^2(\Omega')^3$ satisfy the equations 3.9 and 3.10. Let's show that then the function $B \in L^2(\mathbb{R}^3)^3$ such that $B|_\Omega = B_\Omega$ and $B|_{\Omega'} = B_{\Omega'}$ satisfy 3.1.

By Green's formula, for all $\varphi \in \mathscr{D}(\mathbb{R}^3)$ it holds that

$$\begin{aligned}\langle \nabla\cdot B, \varphi\rangle = -(B, \nabla\varphi) &= -\int_\Omega B_\Omega \nabla\varphi\; dx - \int_{\Omega'} B_{\Omega'}\nabla\varphi\; dx \\ &= \int_\Omega (\nabla\cdot B_\Omega)\varphi\; dx + \int_{\Omega'} (\nabla\cdot B_{\Omega'})\varphi\; dx \\ &+ \int_\Gamma (n\cdot B_{\Omega'} - n\cdot B_\Omega)\varphi\; d\Gamma = 0\end{aligned}$$

Thus $\nabla \cdot B = 0$ in $\mathbb{R}^3$. Additionally $\nabla \times B = 0$ in Ω and Ω', so the set of equations in problem 3.1 hold. □

3.1.1 Decompositions of the solutions

Now by the results mentioned in chapter 2.2.2, especially Lemma 2.30 which quaranteed a decomposition of $H(\mathrm{curl}\ 0, \Omega)$ into H^1 part and $\mathbb{H}_1$ part under certain conditions, B_Ω and $B_{\Omega'}$ have the forms

$$\begin{cases} B_\Omega = \nabla\phi_\Omega + \widetilde{B}_\Omega \text{ with } \phi_\Omega \in \mathscr{H}^1(\Omega),\ \widetilde{B}_\Omega \in \mathbb{H}_1(\Omega) \\ B_{\Omega'} = \nabla\phi_{\Omega'} + \widetilde{B}_{\Omega'} \text{ with } \phi_{\Omega'} \in \mathscr{H}^1(\Omega'),\ \widetilde{B}_{\Omega'} \in \mathbb{H}_1(\Omega') \end{cases} \tag{3.12}$$

where the sets $\mathscr{H}^1(\Omega)$ and $\mathbb{H}_1(\Omega)$ are defined by

$$\mathscr{H}^1(\Omega) \coloneqq \{u \in H^1(\Omega) \mid \Delta u = 0 \text{ in } \Omega\} \tag{3.13}$$

and

$$\mathbb{H}_1(\Omega) \coloneqq H(\text{curl } 0, \Omega) \cap H(\text{div } 0, \Omega) = \{u \in L^2(\Omega)^n \mid \nabla \times u = 0, \nabla \cdot u = 0, u \cdot n|_\Gamma = 0\}. \tag{3.14}$$

Now, because $n \cdot B_\Omega|_\Gamma = n \cdot B_{\Omega'}|_\Gamma$ on Γ, we notice that

$$\frac{\partial \phi_\Omega}{\partial n}\Big|_\Gamma = \frac{\partial \phi_{\Omega'}}{\partial n}\Big|_\Gamma.$$

Let's denote $g = \frac{\partial \phi_\Omega}{\partial n}\Big|_\Gamma = \frac{\partial \phi_{\Omega'}}{\partial n}\Big|_\Gamma$. From the last condition we get that $\int_{\partial\Omega_i} g \; d\Gamma = 0$ and $\int_{\partial\Omega'_i} g \; d\Gamma = 0$ for each bounded and connected component of Ω and Ω'. Specifically, $\int_{\Gamma_i} g \; d\Gamma = 0$ for each connected component of Γ.

Here the functions ϕ_Ω and $\phi_{\Omega'}$ are called magnetic scalar potentials. Scalar potentials represent the difference in potential energy that depend only on location.

Additionally, by using the decomposition 3.12 we can express the surface current J_Γ in a new way. We notice that

$$J_\Gamma = \Big[-\frac{B}{\mu} \wedge n\Big]_\Gamma = \Big[-\frac{\nabla\phi}{\mu} \wedge n\Big]_\Gamma + \Big[-\frac{\widetilde{B}}{\mu} \wedge n\Big]_\Gamma. \tag{3.15}$$

Let's denote

$$m_\Gamma \coloneqq \Big[\frac{\phi}{\mu}\Big]_\Gamma = \frac{\phi_{\Omega'}|_\Gamma}{\mu'} - \frac{\phi_\Omega|_\Gamma}{\mu}.$$

and note that $m_\Gamma \in H^{\frac{1}{2}}(\Gamma)$. Let's now define the operator curl_Γ as

$$\text{curl}_\Gamma \; h \coloneqq (\nabla \widetilde{h}) \wedge n|_\Gamma \tag{3.16}$$

for all $h \in H^{\frac{1}{2}}(\Gamma)$ where $\widetilde{h}$ denotes the extension of h in a neighbourhood of Γ. In particular, $\text{curl}_\Gamma \; h \in H^{-\frac{1}{2}}(\Gamma)^3$.

Then the surface current in 3.15 can be expressed as

$$J_\Gamma = -\text{curl}_\Gamma \; m_\Gamma - \Big[\frac{\widetilde{B}}{\mu}\Big]_\Gamma \wedge n$$

with the condition $\nabla \cdot J_\Gamma = 0$ on Γ, as was assumed in Lemma 3.4.

Let's now denote $J_\Gamma^1 \coloneqq -\mathrm{curl}_\Gamma\ m_\Gamma$ and $J_\Gamma^2 \coloneqq -\left[\frac{\widetilde{B}}{\mu}\right]_\Gamma \wedge n$ to get the decomposition $J_\Gamma = J_\Gamma^1 + J_\Gamma^2$. Here J_Γ^1 is the surface current that gives the magnetic potential while J_Γ^2 corresponds to a surface current that does not contribute to the magnetic potential.

Similarly we can decompose B in both Ω and Ω' by using equations 3.12. Let's denote

$$\begin{cases} B_\Omega^1 \coloneqq \nabla\phi_\Omega \\ B_\Omega^2 \coloneqq \widetilde{B}_\Omega \\ B_{\Omega'}^1 \coloneqq \nabla\phi_{\Omega'} \\ B_{\Omega'}^2 \coloneqq \widetilde{B}_{\Omega'} \end{cases} \tag{3.17}$$

so that $B_\Omega = B_\Omega^1 + B_\Omega^2$ and $B_{\Omega'} = B_{\Omega'}^1 + B_{\Omega'}^2$. By expressing the magnetic field as a pair $B = \{B_\Omega, B_{\Omega'}\}$, we can simply denote

$$B = B^1 + B^2 \tag{3.18}$$

when $B^1 = \{B_\Omega^1, B_{\Omega'}^1\}$ and $B^2 = \{B_\Omega^2, B_{\Omega'}^2\}$.

3.2 Electrostatics of a surface charge

Similar to the previous section, we consider electric fields caused by surface charges between two media. Let's consider two perfect dielectric media that are occupying the domains Ω and Ω' such that $\Omega' = \mathbb{R}^3 \setminus \bar{\Omega}$ and Ω is a regular bounded set in $\mathbb{R}^3$. Here regularity means that the boundary $\partial\Omega = \Gamma$ is of class $\mathscr{C}^2$. Let these media have permittivities ε and ε'. Perfect dielectricity means that the material has zero conductivity.

We assume that there isn't any charge in either of the media. It follows that only surface charges can exist on Γ that separates Ω and Ω'. Our goal is to solve the electric field E in $\mathbb{R}^3$. We know that E must satisfy Maxwell's equations

$$\begin{cases} \nabla \times E = 0 \text{ in } \mathbb{R}^3 \\ \nabla \cdot E = 0 \text{ in } \Omega \text{ and } \Omega'. \end{cases} \tag{3.19}$$

Similar to the case of magnetostatics, we only consider solutions with finite energy. Thus it must also hold that

$$W = \frac{\varepsilon}{2} \int_{\Omega} |E|^2 \, dx + \frac{\varepsilon'}{2} \int_{\Omega'} |E|^2 \, dx < \infty. \tag{3.20}$$

However, this condition is equivalent to $E \in L^2(\mathbb{R}^3)^3$ as we know that ε and ε' are finite, positive constants.

As we did in the last section, we consider the problem in two parts. First, let's study the case where the surface charge is known. Let's denote the surface charge with ρ_Γ. Now the problem 3.19 is equivalent to finding a solution to the problem

$$\begin{cases} \nabla \times E = 0 \text{ in } \mathbb{R}^3 \\ \nabla \cdot E = 0 \text{ in } \Omega \text{ and } \Omega' \\ \left[\varepsilon E \cdot n\right]_\Gamma = \rho_\Gamma \end{cases} \tag{3.21}$$

with finite energy condition 3.20 applying, and $\left[\varepsilon E \cdot n\right]_\Gamma$ denotes the jump in $\varepsilon E \cdot n$ on Γ. In this case,

$$\left[\varepsilon E \cdot n\right]_\Gamma = (\varepsilon E_\Omega|_\Gamma - \varepsilon' E_{\Omega'}|_\Gamma) \cdot n.$$

Again, the boundary values of Sobolev functions are defined in the same way as explained in Chapter 2.1.

Similar to the case of magnetostatics and the space V_{L^2}, we define the space

$$V_{curl} = H(\text{curl } 0, \mathbb{R}^3) = \{E \in L^2(\mathbb{R}^3)^3 \mid \nabla \times E = 0\}.$$

We note that by Poincare's lemma 2.27, for all $\widetilde{E} \in V_{curl}$ there exists exactly one function $\widetilde{\varphi} \in W^1(\mathbb{R}^3)$, often called electric potential of the field $\widetilde{E}$, such that

$$\widetilde{E} = -\nabla\widetilde{\varphi}. \tag{3.22}$$

Recall how the Beppo-Levi spaces W^1 were defined in section 3.1. Equation 3.22 states that the potential φ satisfies a scalar Neumann problem. Now, let's define a bilinear form a corresponding to the electric energy, in the analogous way as in the chapter 3.1.

$$a(E, \widetilde{E}) = \frac{\varepsilon}{2}\int_{\Omega} E \cdot \widetilde{E}\, dx + \frac{\varepsilon'}{2}\int_{\Omega'} E \cdot \widetilde{E}\, dx \tag{3.23}$$

for any $E, \widetilde{E} \in V_{curl}$. Similarly we define for φ, $\widetilde{\varphi} \in W^1(\mathbb{R}^3)$ the operator

$$a_0(\varphi, \widetilde{\varphi}) = a(-\nabla\varphi, -\nabla\widetilde{\varphi}). \tag{3.24}$$

Now we can express the problem 3.21 in an equivalent way by using these operators.

Lemma 3.25. *Let ρ_Γ be a surface charge such that $\rho_\Gamma \in H^{-\frac{1}{2}}(\Gamma)$. Then the problem 3.21 is equivalent to the problem of finding $E \in V_{curl}$ and respectively $\phi \in W^1(\mathbb{R}^3)$ where $E = -\nabla\phi$, that satisfy the condition*

$$a(E, \widetilde{E}) = a_0(\phi, \widetilde{\phi}) = \frac{1}{2}\int_{\Gamma} \rho_\Gamma \widetilde{\phi}|_\Gamma\, d\Gamma \tag{3.26}$$

for all $\widetilde{E} = \nabla\widetilde{\phi} \in V_{curl}$, $\widetilde{\phi} \in W^1(\mathbb{R}^3)$. This problem has exactly one solution.

Proof. The result is analogous to the Lemma 3.4 and the proof follows a similar structure as well.

Because $E \in V_{curl}$, then $\nabla \times E = 0$ and the first equation in 3.21 is automatically satisfied. Then, if E satisfies 3.26, then the second and third claim in 3.21 are satisfied. We first choose $\widetilde{\phi}$ in 3.26 such that $\widetilde{\phi} \in \mathscr{D}(\Omega)$. Then by 3.26 it holds that $\nabla \cdot E = 0$ in Ω. Similarly, we choose $\widetilde{\phi}$ in 3.26 such that $\widetilde{\phi} \in \mathscr{D}(\Omega')$. Then by 3.26 it holds that $\nabla \cdot E = 0$ in Ω'. Thus the second claim holds.

To prove the third part of 3.21, we use Green's formula. Let $\widetilde{\phi} \in \mathscr{D}(\mathbb{R}^3)$ or $\widetilde{\phi} \in W^1(\mathbb{R}^3)$. Now 3.26 gives

$$a(E,\widetilde{E}) = a_0(\phi,\widetilde{\phi}) = -\frac{1}{2}\int_\Gamma \varepsilon E_\Omega \cdot n\widetilde{\phi}\, d\Gamma + \frac{1}{2}\int_\Gamma \varepsilon' E_{\Omega'} \cdot n\widetilde{\phi}\, d\Gamma. \tag{3.27}$$

Thus it holds that

$$a(E,\widetilde{E}) = a_0(\phi,\widetilde{\phi}) = \frac{1}{2}\int_\Gamma \Big[\varepsilon E\cdot n\Big]_\Gamma \widetilde{\phi}|_\Gamma\, d\Gamma. \tag{3.28}$$

This implies that $\rho_\Gamma = \Big[\varepsilon E\cdot n\Big]_\Gamma$ which proves the third condition in 3.21.

Finally, let's show the uniqueness. The mapping $L_p : \widetilde{\phi} \in W^1(\mathbb{R}^3) \to \frac{1}{2}\int_\Gamma \rho_\Gamma \widetilde{\phi}|_\Gamma\, d\Gamma$ is a continuous linear form since $\rho_\Gamma \in H^{-\frac{1}{2}}(\Gamma)$. We also notice that the bilinear forms a and a_0 are coercive. Let $\varepsilon_1 = \inf\{\varepsilon,\varepsilon'\}$ and $\varepsilon_2 = \sup\{\varepsilon,\varepsilon'\}$. Then clearly

$$\frac{\varepsilon_1}{2}\int_{\mathbb{R}^3} |E|^2\, dx \le a(E,E) \le \frac{\varepsilon_2}{2}\int_{\mathbb{R}^3} |E|^2\, dx$$

for all $E \in V_{curl}$. This proves that a is V_{curl}-coercive. In a similar manner the bilinear form a_0 is W^1-coercive.

Because the bilinear form a_0 is coercive on $W^1(\mathbb{R}^3)$, the Lax-Milgram theorem shows that there exists exactly one solution $E = -\nabla\phi \in L^2(\mathbb{R}^3)^3$ to the problem 3.26 [9].

□

Let's now return to the general case of the problem 3.19 where the surface charge may not be known.

Theorem 3.29. *The set of solutions E of 3.19 with finite energy consists of such functions $E \in L^2(\mathbb{R}^3)^3$ that for $E_\Omega = E|_\Omega$ and $E_{\Omega'} = E|_{\Omega'}$ it holds*

$$\begin{cases} \nabla\times E_\Omega = 0 \ \textit{in}\ \Omega \\ \nabla\cdot E_\Omega = 0 \ \textit{in}\ \Omega \\ \nabla\times E_{\Omega'} = 0 \ \textit{in}\ \Omega' \\ \nabla\cdot E_{\Omega'} = 0 \ \textit{in}\ \Omega' \end{cases} \tag{3.30}$$

and

$$n\wedge E_\Omega|_\Gamma = n\wedge E_{\Omega'}|_\Gamma \ \textit{on}\ \Gamma. \tag{3.31}$$

Proof. As the result is similar to Theorem 3.8, the following proof is also quite similar. Let E satisfy 3.19 and also the finite energy condition. Then $E_\Omega = E|_\Omega$ and $E_{\Omega'} = E|_{\Omega'}$ satisfy the equations 3.30. In addition, for all $v, \varphi \in \mathscr{D}(\mathbb{R}^3)^3$, we have the Green's formula of the form

$$(v, \nabla \times \varphi) - (\nabla \times v, \varphi) = \int_\Gamma (v \wedge n) \cdot \varphi \, d\Gamma \tag{3.32}$$

or equivalently for both Ω and Ω'

$$\int_\Omega v \cdot (\nabla \times \varphi) \, dx - \int_\Omega (\nabla \times v) \cdot \varphi \, dx = \int_\Gamma (v \wedge n) \cdot \varphi \, d\Gamma. \tag{3.33}$$

This gives that

$$\begin{aligned}
(E, \nabla \times \varphi) &= \int_\Omega E \cdot (\nabla \times \varphi) \, dx + \int_{\Omega'} E \cdot (\nabla \times \varphi) \, dx \\
&= \int_\Omega (\nabla \times E) \cdot \varphi \, dx + \int_{\Omega'} (\nabla \times E) \cdot \varphi \, dx + \int_\Gamma \Big[E \wedge n \Big]_\Gamma \varphi \, d\Gamma \\
&= \int_\Gamma \Big[E \wedge n \Big]_\Gamma \varphi \, d\Gamma
\end{aligned}$$

as $\nabla \times E = 0$ in $\mathbb{R}^3$. This proves the equation

$$n \wedge E_\Omega|_\Gamma = n \wedge E_{\Omega'}|_\Gamma \text{ on } \Gamma. \tag{3.34}$$

For the other direction, let $E_\Omega \in L^2(\Omega)^3$ and $E_{\Omega'} \in L^2(\Omega)^3$ such that they satisfy the equations 3.30 and 3.31. Let's construct a function $E \in L^2(\mathbb{R}^3)^3$ such that $E|_\Omega = E_\Omega$ and $E|_{\Omega'} = E_{\Omega'}$. Then the function E satisfies 3.19, as for any $\varphi \in \mathbb{R}^3$ the Green's formula gives

$$\begin{aligned}
\langle \nabla \times E, \varphi \rangle = (E, \nabla \times \varphi) &= \int_\Omega E_\Omega \cdot (\nabla \times \varphi) \, dx + \int_{\Omega'} E_{\Omega'} \cdot (\nabla \times \varphi) \, dx \\
&= \int_\Omega (\nabla \times E_\Omega) \varphi \, dx + \int_{\Omega'} (\nabla \times E_{\Omega'}) \varphi \, dx \\
&+ \int_\Gamma (E_\Omega \wedge n - E_{\Omega'} \wedge n) \cdot \varphi \, d\Gamma \\
&= 0
\end{aligned}$$

which implies that $\nabla \times E = 0$ in $\mathbb{R}^3$ in addition to the previously assumed $\nabla \cdot E = 0$ in Ω and Ω'. Thus the conditions in 3.19 are satisfied.

□

3.2.1 Decompositions of the solutions

Analogously to the case of magnetostatics, we can use the results such as Lemma 2.30 mentioned in chapter 2.2.2 to show that E_Ω and $E_{\Omega'}$ have the forms

$$\begin{cases} E_\Omega = -\nabla\phi_\Omega + \widetilde{E}_\Omega \text{ with } \phi_\Omega \in \mathscr{H}^{11}(\Omega),\ \widetilde{E}_\Omega \in \mathbb{H}_2(\Omega) \\ E_{\Omega'} = -\nabla\phi_{\Omega'} + \widetilde{E}_{\Omega'} \text{ with } \phi_{\Omega'} \in \mathscr{H}^{11}(\Omega'),\ \widetilde{E}_{\Omega'} \in \mathbb{H}_2(\Omega'). \end{cases} \tag{3.35}$$

since in fact E is of the form $E = -\nabla\phi$ with $\phi \in W^1(\mathbb{R}^3)$ as mentioned in Lemma 3.25, and thus $E_\Omega \in \nabla\mathscr{H}^1(\Omega)$ and $E_{\Omega'} \in \nabla\mathscr{H}^1(\Omega')$. Note that since $E = -\nabla\phi$, the potential ϕ does not only consist of ϕ_Ω and $\phi_{\Omega'}$ if $\widetilde{E}_\Omega$ or $\widetilde{E}_{\Omega'}$ are non-zero. Here the sets $\mathscr{H}^{11}(\Omega)$ and $\mathbb{H}_2(\Omega)$ are defined by

$$\mathscr{H}^{11}(\Omega) = \{u \in \mathscr{H}^1(\Omega) \mid \int_{\Gamma_i} \frac{\partial u}{\partial n}\, d\Gamma = 0 \text{ with } i = 1 \text{ to } n(\Gamma)\}$$

and

$$\mathbb{H}_2(\Omega) = \{u = \nabla\varphi \mid \varphi \in H^1(\Omega), \Delta\varphi = 0, \varphi|_{\Gamma_i} \text{ is constant for } i = 1 \text{ to } n(\Gamma)\}$$

where Γ_i are the connected components of Γ and $n(\Gamma)$ denotes the amount of them. Recall how spaces $\mathscr{H}^1(\Omega)$ and $\mathbb{H}_1(\Omega)$ were defined in 3.13 and 3.14.

Now, because $n \wedge E_\Omega|_\Gamma = n \wedge E_{\Omega'}|_\Gamma$ on Γ by 3.31, we notice that

$$n \wedge \nabla\phi_\Omega|_\Gamma = n \wedge \nabla\phi_{\Omega'}|_\Gamma.$$

We can then write with the notation from 3.16

$$\operatorname{curl}_\Gamma \phi_\Omega = \operatorname{curl}_\Gamma \phi_{\Omega'}. \tag{3.36}$$

This means that the tangential derivatives of the functions ϕ_Ω and $\phi_{\Omega'}$ are equal on Γ. Thus on each connected component Γ_i the traces $\phi_\Omega|_\Gamma$ and $\phi_\Omega|_\Gamma$ can only differ by a constant.

We can choose the functions ϕ_Ω and $\phi_{\Omega'}$ in such a way that the traces $\phi_\Omega|_\Gamma$ and $\phi_\Omega|_\Gamma$ are equal. In this case the choice is actually unique. This is not trivial at all but we omit the details here. Details can be found in Mathematical Analysis and Numerical Methods for Science and Technology: Volume 3 by Robert Dautray and Jacques-Louis Lions [4]. This gives us a well-defined potential ϕ on the whole $\mathbb{R}^3$.

We can now continue studying the surface charge density ρ_Γ. By using third equation of 3.3 and 3.35 we get that

$$\rho = -\left[\varepsilon\frac{\partial\phi}{\partial n}\right]_\Gamma + \left[\varepsilon\widetilde{E}\cdot n\right]_\Gamma. \tag{3.37}$$

where ϕ is a function such that $\phi|_\Omega = \phi_\Omega$ and $\phi|_{\Omega'} = \phi_{\Omega'}$ and similarly ρ is a function such that $\rho|_\Omega = \rho_\Omega$ and $\rho|_{\Omega'} = \rho_{\Omega'}$.

Let's denote

$$\rho_\Gamma^1 := -\left[\varepsilon\frac{\partial\phi}{\partial n}\right]_\Gamma \text{ and } \rho_\Gamma^2 := \left[\varepsilon\widetilde{E}\cdot n\right] \tag{3.38}$$

in which case $\rho = \rho_\Gamma^1 + \rho_\Gamma^2$.

We can also decompose E in a similar manner by using 3.35. We denote

$$E^1 = -\nabla\phi \text{ and } E^2 = \widetilde{E}$$

where $\widetilde{E}$ is such that $\widetilde{E}|_\Omega = \widetilde{E}_\Omega$ and $\widetilde{E}|_{\Omega'} = \widetilde{E}_{\Omega'}$. Then $E = E^1 + E^2$.

3.3 A review of chapter 3

In this chapter we studied the following problem. Let Ω be an open domain in $\mathbb{R}^3$ with permittivity ε and permeability μ, and $\Omega' = \mathbb{R}^3 \setminus \overline{\Omega}$ with permittivity ε' and permeability μ'. Here we only consider static cases where there is zero current and charge density going through the media. Thus there can only exist surface currents or surface charges on the boundary Γ which separates Ω and Ω'.

Then the problem we study is to find electromagnetic field, that is, fields E, D, B and H that satisfy

$$\begin{cases} \nabla \times H = J \\ \nabla \cdot D = \rho \\ \nabla \times E = 0 \\ \nabla \cdot B = 0 \end{cases} \tag{3.39}$$

where J and ρ are given and J satisfies $\nabla \cdot J = 0$. Additionally we want the energy

$$\begin{aligned} W &= \frac{1}{2} \int_{\mathbb{R}^3} (DE + HB)\ dx \\ &= \frac{\varepsilon}{2} \int_{\mathbb{R}^3} |E|^2\ dx + \frac{1}{2\mu} \int_{\mathbb{R}^3} |B|^2\ dx \end{aligned}$$

to be finite.

Without going into too much detail here, in the end we proved the existence and uniqueness of the solution, as long as the current J and charges ρ belong to the dual space of the Beppo-Levi space $W^1(\mathbb{R}^3)$. Recall that $W^1(\mathbb{R}^3)$ is the closure of $\mathscr{D}(\mathbb{R}^3)$ under the semi-norm $\|\nabla\varphi\|_2$. Dual spaces were defined in 2.11.

These results allow the current and charge to be concentrated on surfaces. In particular we are interested in the case where the current or charges are concentrated on the surface $\Gamma = \partial\Omega$ which separates Ω and Ω'. Let n denote the normal of Γ orientated to the exterior of Ω. In this case the problem is equivalent to the problem of finding fields E, D, B and H which satisfy

$$\begin{cases} \nabla \times H = 0 \text{ in } \Omega \text{ and } \Omega' \\ \nabla \cdot D = 0 \text{ in } \Omega \text{ and } \Omega' \\ \nabla \times E = 0 \text{ in } \mathbb{R}^3 \\ \nabla \cdot B = 0 \text{ in } \mathbb{R}^3 \end{cases} \tag{3.40}$$

and it holds that

$$\begin{cases} (D_\Omega|_\Gamma - D_{\Omega'}|_\Gamma) \cdot n = \rho \\ (H_\Omega|_\Gamma - H_{\Omega'}|_\Gamma) \wedge n = -J \\ \nabla \cdot J = 0 \end{cases} \tag{3.41}$$

and

$$\begin{cases} (E_\Omega|_\Gamma - E_{\Omega'}|_\Gamma) \wedge n = 0 \\ (B_\Omega|_\Gamma - B_{\Omega'}|_\Gamma) \cdot n = 0. \end{cases} \tag{3.42}$$

This is to say that the jump in quantity $D \cdot n$ on the boundary corresponds to the charge density ρ and jump in quantity $H \wedge n$ corresponds to current $-J$. The last two conditions further explain how the fields E and B behave on the boundary.

Additionally we find that the electric and magnetic fields admit to decompositions

$$\begin{cases} E = E^1 + E^2 \\ B = B^1 + B^2 \end{cases} \tag{3.43}$$

where E^1 and B^1 arise from electric and magnetic potentials, while E^2 and B^2 satisfy the conditions

$$\begin{cases} E^2 \wedge n|_\Gamma = 0 \\ B^2 \cdot n|_\Gamma = 0. \end{cases} \tag{3.44}$$

Chapter 4

Non-static boundary value problem as a spectral problem

4.1 Spectral theory

Before moving to the problem related to Maxwell's equations, we'll discuss some concepts of spectral theory. The following definitions and results are explained in detail in [4].

Let X be a Banach space with a norm $\|\cdot\|_X$. We have the set of continuous linear mappings from X to itself, denoted by $\mathscr{L}(X)$. Equivalently, $\mathscr{L}(X)$ is the space of bounded operators on X. When equipped with a norm

$$\|B\| = \sup_{\|x\|_X < 1} \|Bx\|_X,$$

the space $\mathscr{L}(X)$ is a Banach space.

Recall that a subset $\Omega \subset X$ is *dense* in X if the closure of Ω in X is the whole X. In other words, every point in X either belongs to Ω or is a limit point of Ω. Furthermore, we call a linear operator B from topological vector space X to another topological vector space Y *densely defined* if it is defined on a dense linear subspace of X [10].

Let A be a densely defined unbounded operator on X. We may now define another operator A_λ in the following manner.

$$A_\lambda = \lambda I - A \tag{4.1}$$

where $\lambda \in \mathbb{C}$ and I is the identity on X.

We can now define the *resolvent sets* and start studying spectrums [9, 4].

Definition 4.2. The *resolvent set* for A, denoted by $\rho(A)$, is the set of those $\lambda \in \mathbb{C}$ such that

$$\begin{cases} A_\lambda(D(A)) \text{ is dense in } X \\ A_\lambda^{-1} \text{ exists and is continuous from } A_\lambda(D(A)) \text{ into } X \end{cases} \tag{4.3}$$

where $D(A)$ denotes the domain of A. For $\lambda \in \rho(A)$ we define the *resolvent operator* $R(\lambda, A)$ or *resolvent* of A as the inverse of A_λ, that is, $R(\lambda, A) := A_\lambda^{-1}$.

With the resolvent sets $\rho(A)$ we can define the spectrum of operator A.

Definition 4.4. The *spectrum* of A, denoted by $\sigma(A)$, is the complement in $\mathbb{C}$ of the set $\rho(A)$. For a bounded operator A the spectrum $\sigma(A)$ is the union of the point spectrum $\sigma_p(A)$, the continuous spectrum $\sigma_c(A)$ and the residual spectrum $\sigma_r(A)$ [4]. These are defined by

$$\begin{cases} \sigma_p(A) = \{\lambda \in \mathbb{C} \mid \lambda \text{ is an eigenvalue of } A\} \\ \sigma_c(A) = \{\lambda \in \mathbb{C} \mid A_\lambda^{-1} \text{ is unbounded on } X \text{ with a domain that is dense in } X\} \\ \sigma_r(A) = \{\lambda \in \mathbb{C} \mid A_\lambda^{-1} \text{ exists with a domain which is not dense in } X\}. \end{cases}$$

Recall that such $\lambda \in \mathbb{C}$ such that the equation

$$Ax = \lambda x \tag{4.5}$$

has at least one non-trivial solution x, is called an *eigenvalue* of A, while the solution x is called and *eigenvector* of A corresponding to the eigenvalue λ.

Remark 4.6. If $A_\lambda = A - \lambda I$ is non-injective, then λ is an eigenvalue and $\lambda \in \sigma_p(A) \subset \sigma(A)$. In fact, we could've just as well defined σ_p as those $\lambda \in \mathbb{C}$ such that A_λ is not injective.

The set of eigenvectors relative to the eigenvalue λ is called the *eigenspace* associated with λ and denoted by E_λ. We note that the set E_λ is in fact the kernel of A_λ. Additionally, the dimension of the eigenspace E_λ is called the *multiplicity* of λ.

4.2 Study of the Maxwell's equations as a spectral problem

We'll now move on to study a spectral problem related to Maxwell's equations. The structure and main results in this chapter follow the Mathematical Analysis and Numerical Methods for Science and Techonology: Volume 3, Spectral Theory and Applications by Robert Dautray and Jacques-Louis Lions [4]. In this chapter we'll study electromagnetic field in the interior of a bounded domain Ω which is occupied by either empty space or by a perfect medium. This is often referred to as a cavity. We assume Ω is bounded by a perfect conductor.

Recall the time-harmonic forms of the Maxwell's equations in Chapter 2.3. Now we can study the following spectral problem. Find the *angular frequencies* ω, and the corresponding electromagnetic fields E_0 and B_0 with finite, non-zero energy such that

$$\begin{cases} \nabla \times B_0 - i\omega E_0 = 0 \\ -\nabla \times E_0 - i\omega B_0 = 0 \\ \nabla \cdot E_0 = \nabla \cdot B_0 = 0 \\ E_0 \wedge n|_\Gamma = 0,\ B_0 \cdot n|_\Gamma = 0. \end{cases} \tag{4.7}$$

We use the same definitions for energy as in 3.2 and 3.20, essentially meaning that the fields E_0 and B_0 have finite, non-zero L^2 norms. The frequencies ω are called *eigenfrequencies* or *natural frequencies* of the cavity, while the corresponding electromagnetic fields are called the *eigenmodes* of the cavity. Note that if $\omega \neq 0$, then the third condition is implied by the first two conditions.

We'll need to discuss the Maxwell operator here, as the problem 4.7 is the spectral problem relative to the Maxwell operator $\mathscr{A}$.

Definition 4.8. Let $\Omega \in \mathbb{R}^3$ be an open subset. We define the Maxwell operator $\mathscr{A}$ by

$$\mathscr{A}((E,B)) = (-\nabla \times B, \nabla \times E)$$

where $\mathscr{A}$'s domain is

$$\begin{aligned} D(\mathscr{A}) &= \{(E,B) \in L^2(\Omega)^6 \mid (-\nabla \times B, \nabla \times E) \in L^2(\Omega)^6, E \wedge n|_\Gamma = 0\} \\ &= H_0(\mathrm{curl}, \Omega) \times H(\mathrm{curl}, \Omega). \end{aligned}$$

The Maxwell operator $\mathscr{A}$ has the following matrix representation

$$\mathscr{A} = \begin{pmatrix} 0 & -\mathrm{curl} \\ \mathrm{curl} & 0 \end{pmatrix} \tag{4.9}$$

where $(E_0, B_0) \in L^2(\Omega)^3 \times L^2(\Omega)^3 = L^2(\Omega)^6$ [7, 4].

The problem 4.7 is a spectral problem relative to $\mathscr{A}$ because 4.7 can be represented as a spectral problem

$$\mathscr{A}(E, B) = -i\omega(E, B) \quad \text{for } \omega \neq 0 \tag{4.10}$$

which gives

$$\begin{pmatrix} 0 & -\text{curl} \\ \text{curl} & 0 \end{pmatrix} \begin{pmatrix} E \\ B \end{pmatrix} = \begin{pmatrix} -i\omega E \\ -i\omega B \end{pmatrix}$$

,

that is,

$$\begin{cases} \nabla \times B - i\omega E = 0 \\ -\nabla \times E - i\omega B = 0. \end{cases} \tag{4.11}$$

We note that 4.10 is equivalent to the spectral problem of $i\mathscr{A}$ with the eigenvalues ω,

$$i\mathscr{A}(E, B) = \omega(E, B).$$

Before moving into studying the properties of Maxwell operator, let us recall how adjoint operator is defined and what the self-adjointness of an operator means.

Definition 4.12. Let $B\colon X \to Y$ be a linear operator between two Hilbert spaces X and Y with corresponding inner products $(x_1, x_2)_x$ and $(y_1, y_2)_Y$ with $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. Then the *adjoint operator* B^* is the linear operator $B^*\colon Y \to X$ that satisfies the condition

$$(Bx, y)_Y = (x, B^*y)_X$$

for $x \in X, y \in Y$. For the adjoint B^* to be uniquely determined, we also require it to be densely defined, so it must hold that $D(B)$ is dense in the Hilbert space X [5].

Definition 4.13. Let $x, y \in X$. A linear operator $A\colon X \to X$ is called *self-adjoint*, if it holds that $(Ax, y) = (x, Ay)$ for all $x, y \in D(A)$ (in this case A is called *symmetric*), and the domain of adjoint A^* (that is, those $y \in X$ such that $x \mapsto (Ax, y)$ is a continuous linear functional in $D(A)$) is dense, that is, A is densely defined. This means that the operator A is its own adjoint A^* [5, 9].

We'll shortly discuss the properties of the Maxwell operator $\mathscr{A}$.

Theorem 4.14. *Let $\Omega \in \mathbb{R}^3$ be an open set with a bounded and regular, say, Lipschitz boundary Γ. Then the operator $\mathscr{A}$ in $L^2(\Omega)^3 \times L^2(\Omega)^3 = L^2(\Omega)^6$ is a closed operator with dense domain such that*

$$\mathscr{A}^t = -\mathscr{A}$$

where $\mathscr{A}^t$ is the transpose of $\mathscr{A}$. The operator $i\mathscr{A}$ is self-adjoint in complex $L^2(\Omega)^6$.

Proof. The domain $D(\mathscr{A})$ can be shown to be dense in $L^2(\Omega)^6$ by noting that $\mathscr{D}(\Omega)^6 \subset D(\mathscr{A})$. Because $\mathscr{D}(\Omega)$ is dense in $L^2(\Omega)$, it follows that $\mathscr{D}(\Omega)^6$ is dense in $L^2(\Omega)^6$ and similarly $D(\mathscr{A})$ is dense in $L^2(\Omega)^6$.

Operator $\mathscr{A}$ is closed if $D(\mathscr{A})$ is closed under the graph norm

$$\|u\|_{D(\mathscr{A})} = (|u|^2 + |Au|^2)^{1/2}. \tag{4.15}$$

However, we notice that

$$D(\mathscr{A}) = H_0(\mathrm{curl}, \Omega) \times H(\mathrm{curl}, \Omega) \tag{4.16}$$

and that the closedness follows from the trace theorem 2.21 for $H(\mathrm{curl}, \Omega)$. Thus $\mathscr{A}$ is a closed operator.

Next, let's show that $\mathscr{A}^t = -\mathscr{A}$. We notice that $(E, B) \in \mathscr{A}^t$ is equivalent to the mapping $(\widetilde{E}, \widetilde{B}) \to (\mathscr{A}(\widetilde{E}, \widetilde{B}), (E, B)) = (-\nabla \times \widetilde{B}, E) + (\nabla \times \widetilde{E}, B)$ being continuous on $D(\mathscr{A})$ equipped with the topology of $L^2(\Omega)^6$.

By restricting ourselves to $\mathscr{D}(\Omega)^6$, this equivalency implies that $\nabla \times E \in L^2(\Omega)^3$ and $\nabla \times B \in L^2(\Omega)^3$. If on the other hand we restrict to $\mathscr{D}(\Omega)^3 \times \mathscr{D}(\bar{\Omega})^3$ and apply Green's formula, we get that $n \wedge E|_\Gamma = 0$.

Remembering the decomposition 4.16, we have shown that if $(E, B) \in D(\mathscr{A}^t)$, then $(E, B) \in D(\mathscr{A})$. If on the other hand $(E, B) \in D(\mathscr{A})$, then for all $(\widetilde{E}, \widetilde{B}) \in D(\mathscr{A})$ it holds that

$$(\mathscr{A}(\widetilde{E}, \widetilde{B}), (E, B)) = ((\widetilde{E}, \widetilde{B}), \mathscr{A}(E, B)) \tag{4.17}$$

from which the self-adjointness of $i\mathscr{A}$ also follows.

Thus $D(\mathscr{A}) \subset D(\mathscr{A}^t)$. Because it was shown before that also $D(\mathscr{A}^t) \subset D(\mathscr{A})$, it holds that $D(\mathscr{A}) = D(\mathscr{A}^t)$. Finally, it's easy to show that on the domain of $\mathscr{A}$ and $\mathscr{A}^t$ it holds that $\mathscr{A} = -\mathscr{A}^t$. Since

$$\mathscr{A} = \begin{pmatrix} 0 & -\mathrm{curl} \\ \mathrm{curl} & 0 \end{pmatrix},$$

it holds that

$$\mathscr{A}^t = \begin{pmatrix} 0 & \text{curl} \\ -\text{curl} & 0 \end{pmatrix}.$$

Thus for all $(E, B) \in D(\mathscr{A}) = D(\mathscr{A}^t)$ we can easily see that

$$\mathscr{A}(E, B) = (-\nabla \times B, \nabla \times E) = -(\nabla \times B, -\nabla \times E) = -\mathscr{A}^t(E, B)$$

which proves the claim. □

We also note that the kernel of $\mathscr{A}$ is

$$\begin{aligned} \ker(\mathscr{A}) &= \{(E, B) \in L^2(\Omega)^6 \mid \nabla \times E = 0, E \wedge n|_\Gamma = 0, \nabla \times B = 0\} \\ &= H_0(\text{curl } 0, \Omega) \times H(\text{curl } 0, \Omega). \end{aligned}$$

Now, let's denote

$$\mathscr{H} = H(\text{div } 0, \Omega) \times H_0(\text{div } 0, \Omega)$$

which is a closed subspace of $L^2(\Omega)^6$. In fact, $\mathscr{H} \subset \ker(\mathscr{A})$ and we say that $\mathscr{H}$ is a *stable* subspace for $\mathscr{A}$. Now we can define the restriction of $\mathscr{A}$ to the space $\mathscr{H}$ by

$$\begin{cases} \mathscr{A}_{\mathscr{H}}((E, B)) = \mathscr{A}((E, B)) \text{ where } (E, B) \in D(\mathscr{A}_{\mathscr{H}}) \\ D(\mathscr{A}_{\mathscr{H}}) = D(\mathscr{A}) \cap \mathscr{H}. \end{cases} \tag{4.18}$$

Similarly, we can define the $\mathscr{H}_c$ as the corresponding complex space to $\mathscr{H}$, or the *complexification* of $\mathscr{H}$.

Corollary 4.19. *Let $\Omega \in \mathbb{R}^3$ be a bounded open set which is also regular, in this case of class $\mathscr{C}^2$. Then the domain $D(\mathscr{A}_{\mathscr{H}}) = D(\mathscr{A}) \cap \mathscr{H}$ is a closed vector subspace of the Sobolev space $H^1(\Omega)^3 \times H^1(\Omega)^3$ when $D(\mathscr{A}_{\mathscr{H}})$ is endowed with the graph norm 4.15.*

Now we note that because the natural injection $H^1(\Omega) \to L^2(\Omega)$ is compact with the conditions of the last corollary, it follows that the injection $D(\mathscr{A}_{\mathscr{H}}) \to \mathscr{H}$ is also compact. Thus the operator $\mathscr{A}_{\mathscr{H}}$ has a compact resolvent $R(\lambda, \mathscr{A}_{\mathscr{H}})$. What this means is that there exists $\lambda \in \rho(\mathscr{A}_{\mathscr{H}})$ such that the resolvent $R(\lambda, \mathscr{A}_{\mathscr{H}})$ is a compact operator. Recall that compact operators are defined to map any bounded subset into a relatively compact subset, that is, the image has a compact closure. Such an operator is also bounded and thus continuous.

The previous deduction about $\mathscr{A}_{\mathscr{H}}$ having a compact resolvent gives us the main theorem of the chapter [4].

Theorem 4.20. *Let $\Omega \in \mathbb{R}^3$ be a regular, bounded and open subset of $\mathbb{R}^3$. Then the spectrum of the operator $i\mathscr{A}$ in the space $\mathscr{H}_c$ is a real point spectrum with no finite accumulation point. The vector space E_ω of eigenvectors of $\mathscr{A}$ for the eigenvalue $i\omega$ has finite dimension.*

The theorem 4.20 gives a solution to the problem 4.7. This is because the set of frequencies ω for which there exists a solution (E_0, B_0) to the problem 4.7 with finite, non-zero energy is exactly the same as the set $\sigma(i\mathscr{A})$. Thus the set of solutions is a discrete set of real values with finite multiplicity. Note that the assumption that $\nabla \cdot E_0 = \nabla \cdot B_0 = 0$ guarantees that zero is not included in the spectrum.

Given a frequency ω, there are two alternatives. If the frequency $\omega \notin \sigma(i\mathscr{A})$, there is no possibility of a stationary electromagnetic field with finite and non-null energy as a solution of 4.7. If on the other hand $\omega \in \sigma(i\mathscr{A})$, there is a corresponding eigenmode of finite and non-null energy.

4.2.1 Non-homogeneous case

In a similar manner we can treat non-homogeneous problems, in which case there exists non-zero charge densities and currents in Ω. In this case, the problem is to find $(E_0, B_0) \in L^2(\Omega)^6$ which satisfies

$$\begin{cases} \nabla \times B_0 - i\omega E_0 = j_0 \text{ in } \Omega \\ -\nabla \times E_0 - i\omega B_0 = 0 \text{ in } \Omega \\ \nabla \cdot E_0 = \rho_0, \nabla \cdot B_0 = 0 \text{ in } \Omega \\ E_0 \wedge n|_\Gamma = 0,\ B_0 \cdot n|_\Gamma = 0 \text{ on } \Gamma \end{cases} \tag{4.21}$$

with $\omega \in \mathbb{R}$ and given current and charge densities $j_0 \in L^2(\Omega)^3$ and $\rho_0 \in H^{-1}(\Omega)$ which satisfy the condition

$$i\omega\rho_0 + \nabla \cdot j_0 = 0.$$

This problem can also be written using the Maxwell operator $\mathscr{A}$ as

$$\mathscr{A}(E_0, B_0) + i\omega(E_0, B_0) = (-j_0, 0). \tag{4.22}$$

As $\mathscr{H}$ is a closed subspace of $L^2(\Omega)^6$, we can decompose the space $L^2(\Omega)^6$ into $\mathscr{H} \oplus \mathscr{H}^\perp$ and similarly elements of $L^2(\Omega)^6$ into two parts,

$$(E_0, B_0) = (E_1, B_1) + (E_2, B_2)$$

where $(E_1, B_1) \in \mathscr{H}$ and $(E_2, B_2) \in \mathscr{H}^\perp$. Similarly we may write

$$j_0 = j_1 + j_2$$

where $j_1 \in H(\text{div } 0, \Omega)$ and $j_2 \in H(\text{div } 0, \Omega)^\perp$.

The problem 4.22 can then be represented as

$$\begin{cases} \mathscr{A}(E_1, B_1) + i\omega(E_1, B_1) = (-j_1, 0) \;\; \text{in } \mathscr{H} \\ i\omega(E_2, B_2) = (-j_2, 0) \;\; \text{in } \mathscr{H}^\perp \end{cases} \tag{4.23}$$

The second equation of 4.23 then gives

$$E_2 = -\frac{j_2}{i\omega}$$

and

$$B_2 = 0.$$

We again denote $\mathscr{A}_{\mathscr{H}}$ as the restriction of $\mathscr{A}$ in $\mathscr{H}$. The behaviour of the first equation of 4.23 instead depends on whether $\omega \in \sigma(i\mathscr{A})$ or not. Let's study the two alternatives.

If $\omega \notin \sigma(i\mathscr{A})$, then for all $j_1 \in L^2(\Omega)^3$, the first equation of 4.23 has exactly one solution $(E_1, B_1) \in D(\mathscr{A}_{\mathscr{H}})$. Thus the problem 4.22 has exactly one solution

$$(E_0, B_0) = (E_1, B_1) + (-\frac{j_2}{i\omega}, 0).$$

If on the other hand $\omega \in \sigma(i\mathscr{A})$, then the first equation of 4.23 has only one solution if $(-j_1, 0)$ is orthogonal to the eigenspace corresponding to the frequency ω, denoted by E_ω. However, in this case E_ω must be known which is not always reasonable to expect.

Chapter 5

Appendix: Notation

5.1 Function spaces

Definition 5.1. Let $\Omega \subset \mathbb{R}^n$. Then we define

$$\mathscr{D}(\Omega) = C_c^\infty(\Omega) = \Big\{\psi \in C^\infty(\mathbb{R}^n) \mid \text{supp}(\psi) \subset \Omega \text{ is compact}\Big\}.$$

A function belonging to $\mathscr{D}(\Omega)$ is called *a test function*. We often use shorter notation $\mathscr{D}(\Omega)^n$ for $(\mathscr{D}(\Omega))^n$.

Definition 5.2. Let Ω be an open subset of $\mathbb{R}^n$. The set consisting of locally integrable functions $u \in L^p(\Omega)$ such that for each multi-index α with $|\alpha| \leq k$ weak derivatives $D^\alpha u$ exist and belong to $L^p(\Omega)$, is called a *Sobolev space*. We then denote the space $W^{k,p}(\Omega)$.

Definition 5.3. The Sobolev space $W^{1,2}(\Omega)$ is denoted $H^1(\Omega)$. In a similar manner we'll denote $H^k(\Omega) = W^{k,2}$. We'll also denote the closure of C_c^∞ in the space $W^{k,p}$ as $W_0^{k,p}$, and $H_0^k(\Omega) = W_0^{k,2}(\Omega)$.

Definition 5.4. For any arbitrary open set $\Omega \subset \mathbb{R}^3$ we define the Beppo-Levi space $W^1(\Omega)$ as the closure of $\mathscr{D}(\bar{\Omega})$ under the semi-norm $\|\nabla\varphi\|_{L^2(\Omega)^n}$ for $\varphi \in \mathscr{D}(\bar{\Omega})$. Analogously we define $W_0^1(\Omega)$ as the closure of $\mathscr{D}(\Omega)$ under the same semi-norm.

Definition 5.5. Let T be the trace operator from $H^1(\Omega)$ to $L^2(\partial\Omega)$. Then $H^{\frac{1}{2}}(\partial\Omega)$ can be defined as the range of T, that is

$$H^{\frac{1}{2}}(\partial\Omega) = \{u \in L^2(\partial\Omega) \mid u = T(\varphi) \text{ with some } \varphi \in H^1(\Omega)\}.$$

Definition 5.6. Let E be a normed space with scalar field $\mathbb{K}$. Then the dual space of E, denoted E^*, contains the bounded linear operators from E to $\mathbb{K}$. Thus

$$E^* = \mathscr{L}(E, \mathbb{K}) = \{T : E \to \mathbb{K} \mid \text{T is a continuous linear operator}\}.$$

Let $u \in E$ and $f \in E^*$. We'll denote the pairing between E and E^* as $f(u) = \langle f, u \rangle$.

Definition 5.7. Denote by $H^{-1}(\Omega)$ the dual space of $H_0^1(\Omega)$. The space H^{-1} is a normed space with norm

$$\|f\|_{H^{-1}(\Omega)} = \sup\{\langle f, u \rangle \mid u \in H_0^1(\Omega), \|u\|_{H_0^1(\Omega)} \leq 1\}.$$

In a similar manner, we define $H^{-\frac{1}{2}}(\partial\Omega)$ as the dual space of $H^{\frac{1}{2}}(\partial\Omega)$.

5.1.1 Divergence and curl spaces

The divergence and curl spaces are defined as follows.

Definition 5.8. Let Ω be an open subset of $\mathbb{R}^n$. We define

$$H(\text{div}, \Omega) = \{v \in L^2(\Omega)^n \mid \nabla \cdot v \in L^2(\Omega)\}.$$

Similarly we define

$$H(\text{curl}, \Omega) = \{v \in L^2(\Omega)^3 \mid \nabla \times v \in L^2(\Omega)^3\}.$$

We denote as $H_0(\text{div}, \Omega)$ and $H_0(\text{curl}, \Omega)$ the closures of $\mathscr{D}(\Omega)^3$ in $H(\text{div}, \Omega)$ and $H(\text{curl}, \Omega)$ respectively.

Definition 5.9. We define the spaces $H(\text{div } 0, \Omega)$ and $H_0(\text{div } 0, \Omega)$ as

$$\begin{aligned} H(\text{div } 0, \Omega) &= \Big\{u \in L^2(\Omega)^3 \mid \nabla \cdot u = 0\Big\}, \\ H_0(\text{div } 0, \Omega) &= \Big\{u \in L^2(\Omega)^3 \mid \nabla \cdot u = 0,\ n \cdot u = 0 \text{ on } \Gamma\Big\}. \end{aligned}$$

Similarly,

$$\begin{aligned} H(\text{curl } 0, \Omega) &= \Big\{u \in L^2(\Omega)^3 \mid \nabla \times u = 0\Big\}, \\ H_0(\text{curl } 0, \Omega) &= \Big\{u \in L^2(\Omega)^3 \mid \nabla \times u = 0,\ n \wedge u = 0 \text{ on } \Gamma\Big\}. \end{aligned}$$

Definition 5.10. The sets $\mathscr{H}^1(\Omega)$ and $\mathbb{H}_1(\Omega)$ are defined by

$$\mathscr{H}^1(\Omega) = \{u \in H^1(\Omega) \mid \Delta u = 0 \text{ in } \Omega\}$$

and

$$\begin{aligned} \mathbb{H}_1(\Omega) &= H(\text{curl } 0, \Omega) \cap H(\text{div } 0, \Omega) \\ &= \{u \in L^2(\Omega)^n \mid \nabla \times u = 0, \nabla \cdot u = 0, u \cdot n|_\Gamma = 0\}. \end{aligned}$$

In chapter 3 we also define the following spaces for convenience, but these are very specific to the task at hand.

$$\begin{cases} V_{W^1} = \{A \in W^1(\mathbb{R}^3)^3 \mid \nabla \cdot A = 0\} \\ V_{L^2} = H(\text{div } 0, \mathbb{R}^3) = \{B \in L^2(\mathbb{R}^3)^3 \mid \nabla \cdot B = 0\} \\ V_{curl} = H(\text{curl } 0, \mathbb{R}^3) = \{E \in L^2(\mathbb{R}^3)^3 \mid \nabla \times E = 0\} \end{cases} \tag{5.11}$$

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