# On analytic and geometric regularity of mappings of finite distortion

Olli Hirviniemi

Doctoral dissertation

To be presented for public discussion, with the permission of the Faculty of Science of the University of Helsinki, in Auditorium PII, Porthania, on the 13th of November, 2021 at 10 o'clock.

Department of Mathematics and Statistics
Faculty of Science
University of Helsinki
2021

ISBN 978-951-51-7682-0 (paperback)
ISBN 978-951-51-7683-7 (PDF)
http://ethesis.helsinki.fi
Unigrafia Oy
Helsinki 2021

# Acknowledgements

Above all, I would like to thank my advisors, Eero Saksman and István Prause, for introducing me to the quasiworld and patiently offering me their guidance though these years. I have enjoyed listening to the many stories they have shared, and learned so much about how the problems I study relate to the wider mathematical community.

I am especially grateful for having had the opportunity to collaborate with Lauri Hitruhin. Our joint work has made me more familiar with different methods, and he gave me encouragement when I felt stuck.

I would also like to thank Aapo Kauranen and Albert Clop for taking their time to give me feedback as the pre-examiners. I also wish to thank Kai Rajala for acting as the opponent at the defence.

I am thankful for the financial support by the Academy of Finland and the Doctoral Programme in Mathematics and Statistics.

I wish to thank all my friends and colleagues I have met during my time in Kumpula for creating the pleasant atmosphere. I want to especially thank Otte Heinävaara for all the shared good times and enthusiasm. Likewise, I thank the board game group of the department for making the period of remote working during pandemic more manageable by hosting remote game nights.

Finally, I want to thank my parents for always being supportive of my dreams, and my sister for being someone I can truly count on.

Helsinki, October 2021
Olli Hirviniemi

# List of included articles

This thesis consists of an introduction and the following three articles:

[**A**] O. Hirviniemi, I. Prause and E. Saksman. *Localized regularity of planar maps of finite distortion*, to appear in Revista Matemática Iberoamericana, doi: 10.4171/RMI/1297

[**B**] O. Hirviniemi, I.Prause and E. Saksman. *Stretching and rotation of planar quasiconformal mappings on a line.* arXiv 2007.07735, submitted

[**C**] O. Hirviniemi, L. Hitruhin, I. Prause and E. Saksman. *On mappings of finite distortion that are quasiconformal in the unit disk.* arXiv 2104.10961, to appear in Journal d'Analyse Mathématique

All respective authors played equal roles in research and writing of the joint articles.

# Contents

# Chapter 1

# Quasiconformality and distortion

Quasiconformal mappings have been studied by many mathematicians for almost a century. Originally introduced by Grötzch in 1928 and named by Ahlfors in 1935, they generalize the notion of conformal mappings. Especially in the plane, they have many important connections to not only complex analysis but also to other areas of mathematics such as elliptic partial differential equations, differential geometry, Teichmüller theory and complex dynamics.

The notion of quasiconformality generalizes to higher dimensions where the class of conformal mappings is rather limited compared to planar case. Other generalizations include quasiregular mappings, which drop the assumption of injectivity, comparable to holomorphic mappings in conformal case. One can also weaken the assumptions on the mappings further, arriving to the notion of mappings of finite distortion. In this thesis, the main focus is on planar mappings, both quasiconformal and of finite distortion.

Let $f : \Omega \to \Omega'$ be a homeomorphism between two planar domains. Define the *linear distortion* of $f$ at $z_0 \in \Omega$ as

$$H_f(z_0) := \limsup_{r\to 0} \frac{\max_{|z-z_0|=r} |f(z) - f(z_0)|}{\min_{|z-z_0|=r} |f(z) - f(z_0)|}.$$

If $f$ is differentiable at $z_0$, it easily follows that the directional derivatives satisfy

$$\max_{\alpha} |\partial_\alpha f(z_0)| = H_f(z_0) \min_{\alpha} |\partial_\alpha f(z_0)|.$$

If $f$ is a conformal mapping, then $|\partial_\alpha f(z_0)| = |f'(z_0)|$ for all directions $\alpha$ and therefore $H_f(z_0) = 1$ everywhere in $\Omega$. For an affine $f$, i.e. $f(z) = \alpha z + \beta \overline{z} + w_0$ with $|\alpha| \neq |\beta|$, we have

$$H_f(z_0) = \frac{|\alpha| + |\beta|}{\big||\alpha| - |\beta|\big|}.$$

Such an affine $f$ maps a circle of radius $r$ to an ellipse whose major axis is $(|\alpha| + |\beta|)r$ and minor axis $||\alpha| - |\beta||r$. In general, the distortion can be interpreted as describing the shape of the infinitesimally small ellipses that are the images of infinitesimal circles.

Any diffeomorphism $f$ between planar domains has $H_f < \infty$ everywhere. In order to develop meaningful theory, one has to impose further conditions on the distortion. One desirable property for the mappings would be Lusin property $\mathcal{N}$, which is to say that for any Lebesgue measurable set $E$ with $m(E) = 0$ we have $m(f(E)) = 0$. Assuming $H_f \leq M$ almost everywhere does not imply that $f$ has the property $\mathcal{N}$. To see this, consider $f : (0,1)^2 \to (0,2) \times (0,1)$,

$$f(x, y) = x + c(x) + y.$$

Here $c$ is the Cantor function that is constant in each component of the complement of the middle-third Cantor set $C_{1/3}$ that maps the Cantor set $C_{1/3}$ onto unit interval. This function maps the set $C_{1/3} \times (0,1)$ with measure zero onto a set with positive measure.

To impose some weak but sufficient regularity on the mapping, one can require that the mapping lies in local Sobolev space $W^{1,p}_{loc}(\Omega)$. This is the space of locally $p$-integrable functions on $\Omega$ whose distributional partial derivatives are locally $p$-integrable. Usually it is desirable to have $p = 2$, but $p = 1$ suffices for our purposes since it implies the square-integrability in homeomorphic setting. In Sobolev spaces, the functions are a priori only assumed to have partial derivatives at almost every point, but

$$\max_{\alpha} |\partial_\alpha f(z_0)| = H_f(z_0) \min_{\alpha} |\partial_\alpha f(z_0)|$$

remains valid at the points of differentiability. Assuming only that partial derivatives exist at $z_0$, it is consistent to use

$$\partial_\alpha f(z_0) = \cos(\alpha)\partial_x f(z_0) + \sin(\alpha)\partial_y f(z_0)$$

as the definition of the directional derivatives even when $f$ is not differentiable at $z_0$.

A planar homeomorphism can be either orientation-preserving or orientation-reversing. As most results for orientation-reversing mappings can be derived from the corresponding results for orientation-preserving mappings by composing with a reflection map, we can assume without loss of generality that our mappings are orientation-preserving.

A mapping $f : \Omega \to \Omega'$ between planar domains is $K$*-quasiconformal* for $K \geq 1$ if $f$ is an orientation-preserving homeomorphism in the Sobolev class $W^{1,1}_{loc}(\Omega)$ and the directional derivatives satisfy

$$\max_{\alpha} |\partial_\alpha f(z)| \leq K \min_{\alpha} |\partial_\alpha f(z)|$$

for almost every point $z \in \Omega$.

For any $K$-quasiconformal mapping, the pointwise distortion function

$$K_f(z) := \inf\{L \geq 1 : \max_{\alpha} |\partial_\alpha f(z)| \leq L \min_{\alpha} |\partial_\alpha f(z)|\}$$

is bounded from above by the quasiconformality constant $K$ at almost every point.

Homeomorphic mappings of finite distortion generalize the notion of quasiconformality by only requiring that $K_f(z) < \infty$ almost everywhere. Two important subclasses of such mappings are mappings of exponentially $p$-integrable distortion and mapping of $p$-integrable distortion.

A mapping $f : \Omega \to \Omega'$ between planar domains is a *homeomorphic mapping of finite distortion* if $f$ is an orientation-preserving homeomorphism in the Sobolev class $W^{1,1}_{loc}(\Omega)$ and the pointwise distortion function $K_f$ is finite almost everywhere.

A homeomorphic mapping of finite distortion $f$ has *exponentially $p$-integrable distortion* if $\exp(pK_f) \in L^1_{loc}(\Omega)$. It has *$p$-integrable distortion* if $K_f \in L^p_{loc}(\Omega)$.

A planar quasiconformal mapping or mapping of finite distortion $f$ satisfies the *Beltrami equation*

$$\overline{\partial} f(z) = \mu(z) \partial f(z)$$

at almost every point. Here $\mu$ is a measurable complex function with $|\mu(z)| < 1$ almost everywhere and $\overline{\partial}$ and $\partial$ are the complex partial differential operators

$$\overline{\partial} = \frac{1}{2}\big(\partial_x + i\partial_y\big), \qquad \partial = \frac{1}{2}\big(\partial_x - i\partial_y\big).$$

There is no analogous expression in higher dimensions, making this a key difference between dimension 2 and $n \geq 3$. The absolute value of $\mu$ and the pointwise distortion are related as

$$K_f(z) = \frac{1 + |\mu(z)|}{1 - |\mu(z)|}.$$

The Beltrami equation has unique normalized solution if the corresponding distortion function is exponentially $p$-integrable [10], and it in fact suffices to have slightly weaker integrability condition [7].

Quasiconformality, quasiregularity and distortion naturally generalize in higher dimensions, where the notions of BMO-quasiconformality and quasiregularity have been studied extensively [35, 23]. For some results in higher dimensions, see for example [21, 24, 26, 27, 28].

For more background in planar quasiconformal mappings and mappings of finite distortion, we refer reader to [3] and [29].

# Chapter 2

# Stretching and rotation

Consider the following two quasiconformal mappings:

$$f_1(z) = \frac{z}{|z|}|z|^{\alpha}, \qquad f_2(z) = \frac{z}{|z|}|z|^{1+i\gamma}.$$

Here $\alpha > 0$ and $\gamma \in \mathbb{R}$ are real numbers. Let us investigate the images of the rays starting from the origin.

For $f_1$, the mapping preserves the argument, so all rays map onto themselves. However, $|f_1(z)| = |z|^{\alpha}$. This means that the exponent $\alpha$ determines the stretching near origin.

In contrast, $f_2$ preserves the distance from the origin. Each ray is mapped onto a logarithmic spiral of form $\{t \exp(i(a + \gamma \log t)) : t > 0\}$. In this sense, $\gamma$ determines the rate of rotation when approaching the origin.

Observe that composing the two maps $g = f_2 \circ f_1$ yields a new mapping that stretches like $f_1$ and rotates the rays logarithmically like $f_2$. Explicitly,

$$g(z) = \frac{z}{|z|}|z|^{\alpha(1+i\gamma)}.$$

Rotation could be defined in a few different ways, for example by starting from the treatment of spirals in [15]. We describe both the stretching and rotation phenomena by the notion of complex stretching exponents as

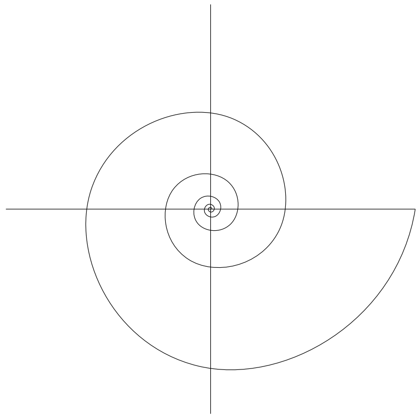

Figure 2.1: The image of unit interval $0 < t < 1$ under the mapping $f_2$ for $\gamma = 2\pi$.

introduced in [4] using the notation from [**B**]. As both can be interpreted as the limit behaviour of complex logarithms, it is not particularly surprising that the modulus of continuity and rotation are connected to each other.

For a mapping $f : \Omega \to \Omega'$ between complex domains, the *set of complex stretching exponents* at the point $\zeta \in \Omega$ is

$$\mathcal{X}_f(\zeta) := \bigcap_{r>0, B(\zeta,r)\subset\Omega} \overline{\left\{ \frac{\log(f(\zeta+t) - f(\zeta))}{\log(t)} : 0 < t < r \right\}},$$

i.e. the set of limit points of the quotient $\log(f(\zeta+t)-f(\zeta))/\log t$ as $t \to 0$. The choice of the branch for the complex logarithm $\log(f(w+t) - f(w))$ does not affect the set of limit points. For more discussion on the notions of stretching and rotation, see [4]. Analogously, the notions make sense in more general setting, see [18, 19, 20] for similar treatment of the mappings of finite distortion.

For any locally $\alpha$-Hölder continuous mapping $f$, it immediately follows that for $z \in \mathcal{X}_f(\zeta)$ we have $\operatorname{Re} z \geq \alpha$ everywhere. Classical quasiconformal estimates imply further that for a $K$-quasiconformal $f$ it holds that $\mathcal{X}_f(\zeta) \subset \{\frac{1}{K} \le \operatorname{Re} z \le K\}$ at every $\zeta$.

One question to ask is how large can the set be where there is non-trivial complex stretching. By differentiability, $\mathcal{X}_f(\zeta) = \{1\}$ for almost every $\zeta$ with respect to planar Lebesgue measure for any quasiconformal $f$, so studying the multifractal spectrum of such mappings is natural. This pertains investigating the upper bound for the Hausdorff dimension of the set $S_{\alpha,\gamma} := \{\zeta : \alpha(1+i\gamma) \in \mathcal{X}_f(\zeta)\}$.

A complete characterization of the complex multifractal spectra, given in [4] states that for a $K$-quasiconformal $f$ we have

$$\dim S_{\alpha,\gamma} \le 1 + \alpha - \frac{1}{k}\sqrt{(1-\alpha)^2 + (1-k)^2\alpha^2\gamma^2},$$

where $k = (K-1)/(K+1)$. An equivalent formulation in terms of the stretching exponents can be stated as follows. For a $K$-quasiconformal $f$ and $0 \le s \le 2$, let $B_K$ be the closed disk with geometric diameter

$$\left[\frac{1-k}{1+k} + \frac{k}{1+k}s, \frac{1+k}{1-k} - \frac{k}{1-k}s\right]$$

where $k = (K-1)/(K+1)$. Then we have

$$\mathcal{X}_f(\zeta) \subset B_K$$

for almost every $\zeta$ with respect to $s$-dimensional Hausdorff measure.

With sufficient symmetry, this general bound can be improved considerably. For a line, the following result is stronger than the one implied by general 1-dimensional upper bound.

**Theorem 1** (**B**, Theorem 1)**.** *Let $f : \mathbb{C} \to \mathbb{C}$ be a $K$-quasiconformal mapping. For almost every $x \in \mathbb{R}$ with respect to 1-dimensional Lebesgue measure we have $\mathcal{X}_f(x) \subset \overline{B}(1/(1-k^4), k^2/(1-k^4))$. Here $k = (K-1)/(K+1)$.*

This implies the following rotational properties on the line.

**Corollary 2** (**B**, Corollary 2)**.** *Let $f : \mathbb{C} \to \mathbb{C}$ be a $K$-quasiconformal mapping. Then for almost every $x \in \mathbb{R}$, we have*

$$\limsup_{t \to 0+} \left| \frac{\arg(f(x+t) - f(x))}{\log t} \right| \leq \frac{k^2}{\sqrt{1-k^4}}.$$

*Here again $k = (K-1)/(K+1)$.*

The proof of Theorem 1 relies on finding an estimate for suitable weighted averages for the fractions in the definition of complex stretching exponents. For any subset of the line $A \subset \mathbb{R}$ with positive 1-dimensional Lebesgue measure, then necessarily $A$ has infinite $\delta$-dimensional Hausdorff measure. In particular, for any $M > 0$ there is $\varepsilon > 0$ such that for any countable covering $\{B(x_i, r_i)\}$ of $A$ with the radii satisfying $r_i < \varepsilon$ we have

$$\sum (r_i)^\delta > M.$$

The key is to prove that for sufficiently large $M$ and with $\delta$ close to 1, there is a weighted average of the fractions in the definition of complex stretching exponents that lies close to the desired disk.

Supposing we have sufficiently large collection of disks, we embed the mapping $f$ in holomorphic motion and consider the Cantor sets generated by the images of these disks, similarly to [1]. The Cantor set in holomorphic motion lies on some quasicircle $\Gamma$, and we can therefore estimate the Hausdorff dimension of the Cantor set from above with the bound $\dim(\Gamma) \leq 1 + k^2$ [36].

To find the suitable weights, we use pressure estimates. The pressure function is defined as

$$P_\lambda(d) := \log \sum_j |r_j(\lambda)|^d,$$

where $r_j(\lambda)$ are complex radii of the disks in holomorphic motion. Jensen's inequality implies that

$$P_\lambda(d) = \sup_p \sum_j p_j \log \frac{|r_j(\lambda)|^d}{p_j},$$

where the supremum is taken over all positive probability distributions $p$.

The same techniques also give a lower bound for the Hausdorff dimension of the image of 1-dimensional subset of the real line, which is a counterpart to the upper bound given in [36]. This generalizes the result from [32] and [34] by dropping the assumption of the line mapping onto itself. Improving the upper bound would give more information, as it has been shown that $1 + k^2$ is not sharp [16, 22]. Moreover, the proof yields estimates that are effective when the dimension is near 1, and this should be compared to the estimates for dimensions near 0 [33].

# Chapter 3

# Regularity of the derivative

Let us turn our attention to the local integrability of the derivative in the plane. Quasiconformal mappings were defined to have square-integrable distributional partial derivatives. However, it holds that for a $K$-quasiconformal $f$ one has actually even stronger integrability for the derivatives [1], namely

$$|Df| \in L^p_{loc}$$

for $p < 2K/(K-1)$. This is not true for $p = 2K/(K-1)$, as can be seen by radial stretching map.

Likewise, a homeomorphic planar mapping of exponentially $p$-integrable distortion has a priori only locally integrable derivatives, but even stronger integrability holds. In [2], improving on [9], it was shown that

$$|Df|^2 \log^\beta(e + |Df|) \in L^1_{loc}$$

for $\beta < p - 1$.

We can further sharpen the above bound by the following theorem.

**Theorem 3** (**A**, Theorem 2)**.** *Let $f$ be a homeomorphic planar mapping of exponentially $p$-integrable distortion. Then for $\varepsilon > 0$ we have*

$$|Df|^2(\log(e + |Df|))^{p-1}(\log\log(10 + |Df|))^{-(1+3p+\varepsilon)} \in L^1_{loc}.$$

This can be obtained by finding the dependence on $\beta$ in the classical estimate and using the fact that the estimate holds for any $\beta < p - 1$.

The failure of $p$-integrability at $p = 2K/(K-1)$ in quasiconformal case cannot happen in the set where the mapping is conformal [6]. In fact, a weighted variant holds in general [5]: letting $K_f$ be pointwise distortion, a $K$-quasiconformal mapping has

$$(K - K_f(z))|Df(z)|^{2K/(K-1)} \in L^1_{loc}.$$

We have an analogous result for the mappings of exponentially integrable distortion. Of particular interest is the borderline case $p = 1$ where the mapping fails to have square-integrable derivatives, where we have the following theorem.

**Theorem 4** (**A**, Theorem 1)**.** *Let $f$ be a homeomorphic planar mapping of exponentially integrable distortion with the distortion function $K_f$. Then for $\varepsilon > 0$ we have*

$$\frac{1}{\log^{4+\varepsilon}(e + K_f)}|Df|^2 \in L^1_{loc}.$$

This is proved by estimating the area distortion of $f$ carefully and interpolating suitably constructed holomorphic families of mappings of finite distortion. For a radial mapping, we show the optimal area distortion $|f(E)| \leq C \log^{-p}(e + 1/|E|)$ and can refine Theorem 4:

**Theorem 5** (**A**, Theorem 3, part i)**.** *Let $f$ be a homeomorphic planar mapping of exponentially integrable distortion with the distortion function $K_f$. Assume also that $f$ is radial, so that $f(z) = \frac{z}{|z|}\rho(|z|)$. Then for $\varepsilon > 0$ we have*

$$\frac{1}{\log^{1+\varepsilon}(e + K_f)}|Df|^2 \in L^1_{loc}.$$

We conjecture this exponent to be optimal for the general case as well.

For the case where the mapping of exponentially integrable distortion is conformal inside some domain, we have even more regularity. By modifying Theorem 4 suitably, one obtains

$$|Df|^2 \log^p(e + |Df|) \in L^1(\Omega'),$$

where $\Omega'$ is a bounded domain where the mapping of exponentially $p$-integrable distortion $f$ is conformal.

Assuming more geometric properties from the image domain, this can be further improved. For simplicity, let $f$ be a mapping of exponentially $p$-integrable distortion that is conformal inside the unit disk $\mathbb{D}$. Suppose that the boundary of $f(\mathbb{D})$ is $C^{1+\varepsilon}$-regular apart from finitely many outward $C^{1+\varepsilon}$-cusps, intuitively defined to be intersections of two $C^{1+\varepsilon}$-curves. In this situation, we have the following theorem.

**Theorem 6** (**C**, Theorem 5)**.** *Let $f : \mathbb{C} \to \mathbb{C}$ be conformal inside unit disk and have exponentially $p$-integrable distortion. Assume that $f(\mathbb{D})$ has $C^{1+\varepsilon}$-regular boundary apart from finitely many points that are outward $C^{1+\varepsilon}$-cusps. Then*

$$\int_{\mathbb{D}} |f'(z)|^2 \log^{2p+1-\varepsilon}(e + |f'(z)|) dA(z) < \infty$$

*for any $\varepsilon > 0$.*

The proof of this result is based on the idea of comparing the integral of the derivative to the worst-case scenario for such an outward cusp, and concluding the integrability from the corresponding integrability for the worst-case cusp. This cusp is analogous to the extremal angle in quasiconformal case [33].

The better regularity in Theorem 6 happens inside the domain where the mapping of finite distortion is conformal. In the next section, we will consider the image of the unit disk under similar situation, but relax the condition to only require quasiconformality inside the disk, and obtain results analogous to the case of quasidisks.

# Chapter 4

# Quasidisks and generalizations

A quasidisk is the image of the unit disk under some quasiconformal mapping of the plane. To generalize this, we consider domains that are images of the unit disk under a quasiconformal mapping on the unit disk that can be extended to the whole plane as a mapping of finite distortion. Unless stated otherwise, we assume that the distortion is exponentially $p$-integrable.

The classical three point condition of Ahlfors gives the geometric characterization of quasidisks [11]. Namely, a Jordan domain $\Omega$ is a quasidisk if and only if there is a constant $C$ such that for any distinct points $x, y \in \partial\Omega$ we have

$$\min_{i\in\{1,2\}} \operatorname{diam}(\gamma_i) \leq C|x-y|,$$

where $\gamma_1, \gamma_2$ are the connected components of $\partial\Omega \setminus \{x, y\}$. To understand why this is called three point condition, note that it is equivalent to the following property. There exists a constant $C' > 0$ such that for any point in the smaller arc $z \in \gamma_i$ one has

$$|x-z| + |z-y| \leq C'|x-y|,$$

a type of reverse triangle inequality.

This condition has been extended to the more general setting. This leads to the notion of three point condition with control function [14]. A domain $\Omega$ satisfies three point condition with the increasing control function $h$ if for any distinct points $x, y \in \partial\Omega$ we have

$$\min_{i\in\{1,2\}} \operatorname{diam}(\gamma_i) \leq h(|x-y|),$$

where $\gamma_1, \gamma_2$ are again the connected components of $\partial\Omega \setminus \{x, y\}$.

A sufficient condition for the generalized setting is known [12, 14]. A domain $\Omega$ is a quasiconformal image of unit disk under a mapping that extends to a mapping of exponentially integrable distortion if it satisfies the three point condition with a control function $h(t) = Ct[\log\log(e + \frac{1}{t})]^{1/2-\varepsilon}$. An earlier example of a cusp [30] shows that the control function $h(t) = Ct\log^{1+\varepsilon}(\frac{1}{t})$ is not sufficient [13]. The following theorem consirably sharpens this example.

**Theorem 7** (**C**, Theorem 1)**.** *For any $K \geq 1$ and $\varepsilon > 0$, there exists a Jordan domain $\Omega$ such that $\Omega$ satisfies three point condition with a control function*

$$h(t) = Ct\log^{1/2+\varepsilon}\left(\frac{1}{t}\right),$$

*and there is no $K$-quasiconformal mapping $f : \mathbb{D} \to \Omega$ that extends to a mapping of exponentially integrable distortion to $\mathbb{C}$.*

We also give an analogous example for the case of just $p$-integrable distortion.

**Theorem 8** (**C**, Theorem 2)**.** *For any $0 \leq s \leq 1$, there exists a Jordan domain $\Omega$ such that $\Omega$ satisfies three point condition with a control function $h(t) = Ct^s$, and there is no conformal mapping $f : \mathbb{D} \to \Omega$ that extends to a mapping of $p$-integrable distortion to $\mathbb{C}$ for $p > \frac{s}{2(1-s)}$.*

The example for both of these theorems is a certain snake-like domain, as pictured in Figure 4.1. To prove that there are no such mappings taking $\mathbb{D}$ to $\Omega$, we derive a contradiction from modulus estimates.

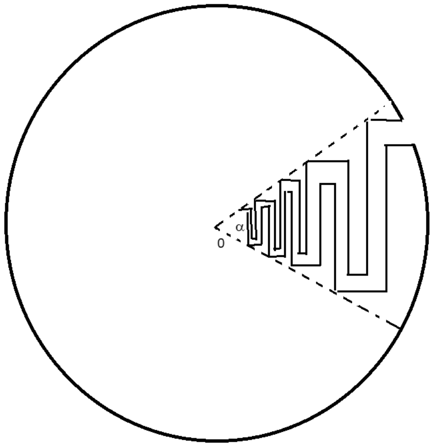


Figure 4.1: The snake-like domain for Theorems.

Our theorems consider the problem of extending a mapping defined in the unit disk. For a different problem about extending homeomorphisms on the real line into a mapping of finite distortion in half-plane, see the recent work in [25].

In this kind of generalized setting, the modulus of continuity estimate was discovered in [37], namely that

$$|f(z) - f(w)| \leq \frac{C}{\log^p \left( \frac{1}{|z-w|} \right)}$$

for $z, w \in \overline{\mathbb{D}}$ where $f$ is a $K$-quasiconformal mapping in the unit disk that extends to a mapping of exponentially $p$-integrable distortion. We give the corresponding lower estimate, yielding the modulus of continuity for the inverse mapping. Our proof is based on estimating moduli of path families.

**Theorem 9** (**C**, Theorem 3)**.** *Let $f$ be a mapping of exponentially $p$-integrable distortion that is $K$-quasiconformal inside the unit disk $\mathbb{D}$. Then for* *any* $\varepsilon > 0$ *there is a constant* $c > 0$ *such that*

$$|f(z) - f(w)| \geq c|z - w|^{2K(1+\varepsilon)}$$

*for all* $z, w \in \overline{\mathbb{D}}$.

The upper bound can be compared to the general modulus of continuity estimate for mappings of exponentially integrable distortion in [31], where the exponent of the logarithm would be $p/2$. Contrasting this, our lower bound is of similar type as the bound $|f(z) - f(w)| \geq c|z - w|^K$ for $K$-quasiconformal mapping $f$. The methods also yield improvement for when the extension has subexponentially integrable distortion, see [8, 17] for general results.

As we saw in Chapter 2, the modulus of continuity is closely related to the rotation. We can derive the following corollary.

**Theorem 10** (**C**, Corollary 4)**.** *Let* $f$ *be a mapping of exponentially* $p$*-integrable distortion that is* $K$*-quasiconformal in the upper half-plane, and normalized by* $f(0) = 0$, $f(1) = 1$*. Then for sufficiently small* $z \in \mathbb{R}$*, we have*

$$|\arg(f(z))| \leq cK \log\left(\frac{1}{|z|}\right),$$

*where* $c$ *is a constant.*

# Bibliography

[1] K. Astala, *Area distortion of quasiconformal mappings,* Acta Math. 173 (1994), no. 1, 37–60.

[2] K. Astala, J. T. Gill, S. Rohde and E. Saksman, *Optimal regularity for planar mappings of finite distortion*, Ann. Inst. H. Poincaré Anal. Non Lineaire 27 (2010), 1–19.

[3] K. Astala, T. Iwaniec and G. J. Martin, *Elliptic partial differential equations and quasiconformal mappings in the plane*, Princeton University Press, 2009.

[4] K. Astala, T. Iwaniec, I. Prause and E. Saksman, *Bilipschitz and quasiconformal rotation, stretching and multifractal spectra*, Publ. Math. Inst. Hautes Études Sci. 121 (2015), 113–154.

[5] K. Astala, T. Iwaniec, I. Prause and E. Saksman, *Burkholder integrals, Morrey's problem and quasiconformal mappings,* J. Amer. Math. Soc. 25 (2012), 507–531.

[6] K. Astala and V. Nesi, *Composites and quasiconformal mappings: new optimal bounds in two dimensions*, Calc. Var. Part. Diff. Eq. 18 (2003), 335–355.

[7] M. A. Brakalova and J. A. Jenkins, *On solutions of the Beltrami equation*, J. Anal. Math. 76 (1998), 67–92.

[8] A. Clop and D. Herron, *Mappings with subexponentially integrable distortion: Modulus of continuity, and distortion of Hausdorff measure and Minkowski content*, Illinois J. Math. 57 (2013), no. 4, 965–1008.

[9] D. Faraco, P. Koskela and X. Zhong, *Mappings of finite distortion: The degree of regularity*, Adv. Math. 190 (2005), 300–318.

[10] G. David, *Solutions de l'equation de Beltrami avec* $\|\mu\|_\infty = 1$, Ann. Acad. Sci. Fenn. Ser. AI Math. 13 (1988), 67–92.

[11] F. W. Gehring, *Characteristic properties of quasidisks*, Presses de l'Université de Montréal, 1982.

[12] C. Guo, *Generalized quasidisks and conformality II*, Proc. Amer. Math. Soc. 143 (2013), no. 8, 3505–3517.

[13] C. Guo, P. Koskela and J. Takkinen, *Generalized quasidisks and conformality*, Publ. Mat. 58 (2014), 193–212.

[14] C. Guo and H. Xu, *Generalized quasidisks and conformality: progress and challenges*, Complex Anal. Syn. 7 (2021), 2.

[15] V. Gutlyanskiĭ and O. Martio, *Rotation estimates and spirals*, Conform. Geom. Dynam. 5 (2001), 6–20.

[16] H. Hedenmalm, *Bloch functions, asymptotic variance, and geometric zero packing*, Amer. J. Math. 142, no.1 (2020), 267–321

[17] D. Herron and P. Koskela, *Mappings of finite distortion: gauge dimension of generalized quasicircles*, Illinois J. Math. 47 (2003), 1243–1259.

[18] L. Hitruhin, *Joint rotational and stretching multifractal spectra of mappings with integrable distortion*, Rev. Mat. Iber. 35 (2019), no. 6, 1649–1675.

[19] L. Hitruhin, *Pointwise rotation for mappings with exponentially integrable distortion*, Proc. Amer. Math. Soc. 144 (2016), 5183–5195.

[20] L. Hitruhin, *Rotational properties of homeomorphisms with integrable distortion*, Conform. Geom. Dyn. 22 (2018), 78–98.

[21] I. Holopainen and P. Pankka, *Mappings of finite distortion: global homeomorphism theorem*, Ann. Acad. Sci. Fenn. Ser. AI Math. 29 (2004), 59–80.

[22] O. Ivrii, *Quasicircles of dimension* $1 + k^2$ *do not exist*, preprint, arXiv:1511.07240, 2015.

[23] T. Iwaniec, P. Koskela and G. Martin, *Mappings of BMO-distortion and Beltrami-type operators*, J. Anal. Math. 88 (2002), 337–381.

[24] T. Iwaniec, P. Koskela, G. Martin and C. Sbordone, *Mappings of finite distortion:* $L^n \log^\alpha L$ *integrability*, J. London Math. Soc. 67 (2003), no. 2, 123–136.

[25] C. Karafyllia and D. Ntalampekos, *Extension of boundary homeomorphisms to mappings of finite distortion*, preprint, arXiv:2102.00049, 2021.

[26] J. Kauhanen, P. Koskela and J. Malý, *Mappings of finite distortion: condition N*, Michigan Math. J. 49(1) (2001), 169–181.

[27] A. Kauranen, R. Luisto and V. Tengvall, *Mappings of finite distortion: compactness of the branch set*, J. Anal. Math. 143 (2021), 207–229.

[28] A. Kauranen, R. Luisto and V. Tengvall, *On BLD-mappings with small distortion*, Complex Anal. Syn. 7 (2021), 5.

[29] P. Koskela, *Planar mappings of finite distortion*, Comput. Methods Funct. Theory 10 (2010), 663–678.

[30] P. Koskela and J. Takkinen, *Mappings of finite distortion: formation of cusps*, Publ. Mat. 51 (2007), no. 1, 223–242.

[31] J. Onninen and X. Zhong, *A note on mappings of finite distortion: the sharp modulus of continuity*, Michigan Math. J. 53 (2005), no. 2, 329–335.

[32] I. Prause, *A remark on quasiconformal dimension distortion on the line*, Ann. Acad. Sci. Fenn. Math. 32 (2007), no. 2, 341–352.

[33] I. Prause, *Quasidisks and twisting of the Riemann map*, Int. Math. Res. Not. (2019), no. 7, 1936–1954.

[34] I. Prause and S. Smirnov, *Quasisymmetric distortion spectrum*, Bull. London Math. Soc. 43 (2011), 267–277.

[35] V. Ryazanov, U. Srebro and E. Yabukov, *BMO-quasiconformal mappings*, J. Anal. Math. 83 (2001), 1–20.

[36] S. Smirnov, *Dimension of quasicircles*, Acta Math. 205 (2010), no. 1, 189–197.

[37] A. Zapadinskaya, *Modulus of continuity for quasiregular mappings with finite distortion extension*, Ann. Acad. Sci. Fenn. Math. 33 (2008), 373–385.