REGULARIZATION STRATEGY AND FOCUSING ENERGY WITH A WAVE EQUATION FOR AN INVERSE PROBLEM

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Doctoral dissertation

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The thesis consists of this overview and the following articles:

Publications

[I] J. Korpela, M. Lassas, L. Oksanen: Regularization strategy for an inverse problem for a 1+1 dimensional wave equation. *Inverse Problems* 32 (2016), N. 6, 065001.

[II] J. Korpela, M. Lassas, L. Oksanen: Discrete regularization and convergence of the inverse problem for 1+1 dimensional wave equation. *Inverse Problems & Imaging*, 2019, 13(3): 575–596.

[III] A. Kirpichnikova, J. Korpela, M. Lassas, L. Oksanen: Construction of artificial point sources for a linear wave equation in unknown medium. ArXiv:2009.10371

The author's contribution to the publications

[I] The authors had an equal role in proving the theoretical results.

[II] The authors had an equal role in proving the theoretical results. J.K. had a leading role in implementing the numerical results of the paper.

[III] The authors and J.K had an equal role in the analytical parts of the proof. J.K. had a leading role in implementing the numerical results of the paper.

Summary

This thesis is about inverse problem. Inverse problems have rich mathematical theory that employs modern methods in partial differential equations, numerical analysis, probability theory, harmonic analysis, and differential geometry. Inverse problems research lies at the intersection of pure and applied mathematics. Traditionally, inverse problems are application oriented, although there are also pure mathematical problems that are considered to be inverse problems.

In this study, the wave equation is the physical model for analysis. The wave equation basically tells us how disturbances travel through a medium, transporting energy from one location to another location without transporting matter. It is a mathematical model that describes many physical phenomena in a reasonably manner. In many situations, the initial and boundary value problem for the wave equation is a quite convenient structure when solving inverse problems. Thus it is the central framework for analysis here, added to with the concept of measurements on the boundary, a so-called Neumann-to-Dirichlet map. This dissertation consists of three publications.

In Publication I, an inverse boundary value problem for a 1+1-dimensional wave equation with wave speed $c(x)$ is considered. We give a regularization strategy for inverting the map $\mathcal{A} : c \mapsto \Lambda$, where Λ is the hyperbolic Neumann-to-Dirichlet map corresponding to the wave speed c.

In Publication II, an inverse boundary value problem for the 1+1-dimensional wave equation $(\partial_t^2 - c(x)^2\partial_x^2)u(x,t) = 0, \quad x \in \mathbb{R}_+$ is considered. We give a discrete regularization strategy recovering the wave speed $c(x)$ when we are given the boundary value of the wave, $u(0,t)$, that is produced by a single pulse-like source. The regularization strategy gives an approximative wave speed $\widetilde{c}$, satisfying a Hölder type estimate $\|\widetilde{c} - c\| \leq C\epsilon^\gamma$, where ϵ is the noise level.

In Publication III, We studied the wave equation on a bounded domain of $\mathbb{R}^m$ and on a compact Riemannian manifold M with a boundary. We assumed, that the coefficients of the wave equation are unknown but that we are given the hyperbolic Neumann-to-Dirichlet map Λ that corresponds to the physical measurements on the boundary.

With the knowledge of Λ we construct a sequence of Neumann boundary values so that, at a time T, the corresponding waves converge to zero while the time derivative of the waves converges to a delta distribution. Such waves are called *artificial point sources.* The convergence of a wave takes place in the function spaces naturally related to the energy of the wave. We apply the results for inverse problems and demonstrate the focusing of the waves numerically in the one-dimensional case.

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1. An introduction to inverse problems

1.1. **General aspects.** Inverse problems have rich mathematical theory that employs modern methods in partial differential equations, numerical analysis, probability theory, harmonic analysis, and differential geometry. Inverse problems research lies at the intersection of pure and applied mathematics. Traditionally, inverse problems are application oriented, although there are also pure mathematical problems that are considered to be inverse problems.

Within the past three decades, the interest in the field of inverse problems has experienced explosive growth. The mathematical community studying inverse problems has gained better knowledge and understanding theoretically and practically. One perspective is that the field of inverse problems is heavily driven by applications. Inverse problems arise from the need to interpret indirect and incomplete measurements. The vast improvements when it comes to more accurate measurements, the recent development of powerful computers, and fast, reliable numerical methods are some of the reasons for the growing interest in inverse problems.

Inverse problems research concentrates on the mathematical theory and practical implementation of indirect measurements. An inverse problem can be seen as the opposite of a direct problem. In the context of inverse problems, a mathematical model of the measurements is often called *the direct problem.* The forward problem corresponding to an inverse problem is usually a mathematically well-defined problem and more stable with respect to noise. Applications are found in numerous research fields involving medical imaging, image processing, mathematical finance, elasticity, astronomy, geophysics, remote sensing, and non-destructive testing. A common feature for the applications is that it is either not possible or not practical to make direct observations of the target.

For example, if we think of computerized tomography, one could ask the following direct problem: If we precisely knew the structure of the inner organs of a patient, what kind of images we would get when doing X-ray measurements? On the other hand, the opposite question, the inverse problem, is: Given a set of X-ray images of a patient, can we determine the three-dimensional structure of the patient's inner organs derived by using this knowledge? The reconstructing of the inner structure of a patient from X-ray projection images is a classical example of an inverse problem.

Another example of an inverse problem is to reconstruct an unknown object from the scattering pattern that is produced by a certain input wave. For example, an ultrasonic scanner produces a sound wave by using a transducer. The wave propagates in the body and echoes back to the transducer that records the echo. Then one can use a wave equation to model the speed of echoes inside the domain and figure out the corresponding mixture of different tissues inside the human body.

In mathematical terms, there is a fundamental difference between direct and inverse problems. The distinction is that an inverse problem is ill posed or improperly posed in the sense of Hadamard, while a direct problem is well posed. According to Jacques Hadamard, a *well-posed* problem satisfies all of the following three conditions:

1. There exists a solution for the problem (existence).
2. There is at most one solution for the problem (uniqueness).
3. The solution continuously depends on the data (stability).

Let us consider a forward map, expressed as

$$\mathcal{A} : X \mapsto Y, \qquad \mathcal{A}f = m, \tag{1}$$

where $\mathcal{A}$ is an operator between the suitable model space X and data space Y. The direct problem must be well posed, in the sense of Hadamard. We call the problem of inverting (1), that is, "given m, find f" an *inverse problem* provided that it violates at least one of the following conditions: $\mathcal{A}^{-1}$ does not exist or $\mathcal{A}^{-1}$ is not continuous. Consequently, inverse problems, as they are considered in this work, are *ill-posed* problems.

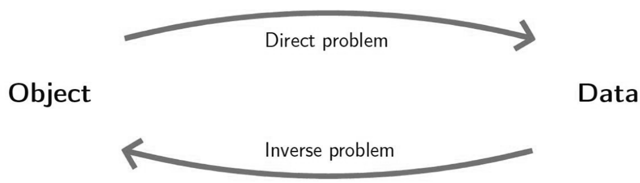

FIGURE 1. An inverse problem is always the counterpart to a direct problem, as shown in the schematic diagram above. One way to formulate these concepts is to define the direct problem as going from object to data, and the inverse problem as finding the object from the data.

Many times we face problems where it is not practically possible to make direct measurements. Thus indirect measurements are the only way to recover something that is unreachable, inside, or hidden in some way. Many times the measurements can only be done from the boundary of the object of interest or even from some distance away from the object.

In an ideal situation, $\mathcal{A}^{-1}$ exists and there are no measurement errors, that is, $m = \mathcal{A}f$. Then, having the measurements m and operator $\mathcal{A}^{-1}$, we can solve the inverse problem directly by

$$f = \mathcal{A}^{-1}(\mathcal{A}f) = \mathcal{A}^{-1}m.$$

In many circumstances this is not the case and this kind of approach cannot be used. In practice we can not avoid the measurement error and its effects on our mathematical models. The situation might be even more challenging when the problem is finding a good candidate to mathematically model a situation with noises from many different sources.

Mathematically, one starting point is to model an actual measurement (data) m_δ by

$$m_\delta = m + \epsilon, \quad \|\epsilon\| \le \delta,$$

where the exact measurement is denoted by m and the measurement error by δ. It is common to assume some kind of a priori bounds for the size of the error $\delta > 0$. Even in the situation, where $\mathcal{A}^{-1}$ is linear and we have noisy measurements $m_\delta = \mathcal{A}f + \epsilon$, the direct use of inverse operator $\mathcal{A}^{-1}$ might not give the best result. There we have

$$\widetilde{f} = \mathcal{A}^{-1}m_\delta = \mathcal{A}^{-1}(\mathcal{A}f + \epsilon) = f + A^{-1}\epsilon. \tag{2}$$

Depending on the operator A^{-1}, along with the measurement error ϵ, the difference between f and $\widetilde{f}$ might be too big or difficult to handle.

Mathematically, the lack of stability can be seen in the case where the inverse $\mathcal{A}^{-1}$ exists but is not continuous. These small errors in the data can cause arbitrarily large errors in the solution, and thus the use of $\mathcal{A}^{-1}$ to solve a problem is not a good idea.

Inverse problems are characterized by the property that the solutions are sensitive to measurement and modelling errors. A consequence of this is that small changes in the data may lead to arbitrarily large changes in the solution. Data errors are inevitable in the numerical treatment of inverse problems, and practically, the data is always influenced by measurements errors and other types of inconsistency. From the viewpoint of trying to find a numerical solution to inverse problems, it is usually the violation of Hadamards condition (3)(i.e, the lack of stability) that causes most of the difficulties. To solve an ill-posed inverse problem, we need to have a method that is robust against modelling errors and noise in the data. There must be stabilizing procedures when dealing with these kind of ill-posed problems, and the classical theory used for tackling these difficulties is called *regularization theory.*

In the continuous case, the Hilbert space functions, like $f \in L^2$, are often used for reasons in deep and versatile mathematical theorems. In a computational sense, we have to use discretization when solving the problem. One aspect is the challenge of defining the discretized measurement. In this way one can move away from more abstract mathematical definitions and define the discretized measurements that comes along with the original definitions and ideas at a reasonable level.

In our work, the change from measurement as an operator between Hilbert spaces to mapping between sets of point values is not an easy problem. During the working process, we had to stop and think how to define our discretized measurement many times.

When making concrete measurements on the boundary or simulations on a computer, we can only handle a finite number of points for the measurements. For example, having $2N$ different measurements, denoted by f_n, on the one-dimensional boundary $(0, 2T)$, we define the basis functions by

$$\phi_{n,N}(t) = \left(\frac{N}{T}\right)^{\frac{1}{2}} 1_{[\frac{(n-1)T}{N}, \frac{nT}{N})}(t), \quad t \in [0, 2T),$$

where $n \in \{1, 2, 3, ..., 2N\}$. The functions $\phi_{n,N}$ are orthonormal in $L^2(0, 2T$. We define the space of piecewise constant functions as

$$\mathcal{P}^N = span\{\phi_{1,N}, ..., \phi_{2N,N}\} \subset L^2(0, 2T).$$

Wr define an orthogonal projection P^N by using point values f_n as

$$P^N : L^2(0, 2T) \to \mathcal{P}^N, \quad P^N(f) = \sum_{j=1}^{2N} f_n \phi_{j,N}(t).$$

This causes disretization noise that needs to be estimated along with other kinds of noise.

Thus, one could say that an inverse problem is ill posed because the contaminated data and the model do not contain sufficient information for solving the problem in a reasonable manner in practice. Hence, the idea of a computational regularization method can be seen as bringing some additional a priori information about the solution to the inversion.

1.2. **An introduction to an inverse problem for a wave equation.** In this study, the wave equation is the physical model for analysis. The wave equation basically tells us how disturbances travel through a medium, transporting energy from one location to another location without transporting matter. It is a mathematical model that describes many physical phenomena in a reasonably manner. In many situations, the initial and boundary value problem for the wave equation is a quite convenient structure when solving inverse problems. Thus it is the central framework for analysis here, added to with the concept of measurements on the boundary, a so-called Neumann-to-Dirichlet map. For a Neumann-to-Dirichlet map, some technical assumptions must be involved. The structure of the initial and boundary value problem having some assumptions allows us to define the direct map $\mathcal{A}$ and the inverse map $\mathcal{A}^{-1}$. On top of that, the regularization strategy helps us to deal with measurement errors and discretization in order to gain the sound speed $c(x)$ inside the media and to focus energy on the small region inside the object.

The technical approach is based on the boundary control method [1, 36, 37, 6, 38, 32, 45, 49, 53, 57, 66, 85]. More precisely speaking, we use a variant of the boundary control method, called the *iterative time-reversal control method*, which was introduced in [10]. The method was later modified in [20] in order to focus the energy of a wave at a fixed time, and it was used in [67] to solve an inverse obstacle problem for a wave equation.

Classical regularization theory is explained in [27]. Iterative regularization of both linear and non-linear inverse problems and convergence rates are discussed in a Hilbert space setting in [11, 31, 35, 60, 63] and in a Banach space setting in [34, 42, 43, 48, 74, 75, 76].

The inverse problem for the wave equation can also be solved by using complex geometrical optics solutions. These solutions were developed in the context of elliptic inverse boundary value problems [84], and in [65] they were employed to solve an inverse boundary spectral problem. Local stability results can be proven using (real) geometrical optics solutions [8, 79, 83], and in [59], a local stability result was proved by using ideas from the boundary control method together with complex geometrical optics solutions. Alternative methods for hyperbolic inverse problems are studied in [18, 23, 24, 81]. Finally we mention the important method based on Carleman estimates [16] that can be used to show stability results when the initial data for the wave equation is non-vanishing.

1.3. **An introduction to the thesis.** In Section 2 we introduce the main steps that are needed for solving the inverse problems in the thesis. We start by defining the concept of a wave equation in Subsection 2.1. Subsection 2.2 consists of defining the initial and boundary value problem for a wave equation. In Subsection 2.3 we define a Neumann-to-Dirichlet map for our measurements on the boundary. In Subsection 2.4 we define the direct problem for the wave equation. In Subsection 2.5 we introduce the idea of a boundary control method and see how to calculate inner products inside the object by using boundary measurements. In Subsection 2.6 we define the inverse problem for the wave equation and prove that it is possible to calculate velocity functions by using the boundary data. Subsection 2.7 reviews the basics of the regularization strategy used to solve the inverse problem with perturbed measurements. In Subsection 2.8 we discusses the numerical aspects of solving the inverse problem. In Subsection 2.9 we introduce the ideas behind focusing energy by using boundary sources. Section 3 reviews the main mathematical steps and the results of Publications I–III.

2. The inverse problem for a wave equation

Here we introduce the main aspects that characterize the inverse problems in this thesis. The presentation is kept short, and we go through the essential steps in order to give better insights into to the thesis.

2.1. **The wave equation.** The initial and boundary value problem for the wave equation is the main framework here. Thus, the first thing is to mathematically define what we mean by that. Often, it is convenient to use a wave equation as one representative from a very broad class of differential equations, that is, hyperbolic partial differential equations. Many times the physical situations differ from each other, and thus, having one concrete model for a wave equation in analysis can be too restrictive. For example, the basic model that describes how the wave travels through the media in the isotropic case may not be the best model when handling anisotropic media.

The wave equation in an isotropic medium is defined by the formula

$$\partial_t^2 u(t,x) - c(x)^2 \Delta u(t,x) = 0 \quad \text{in } \mathbb{R} \times \Omega, \tag{3}$$

where $\Omega \subset \mathbb{R}^n$ and $c(x)$ is the wave speed. In anisotropic materials the wave speed depends on the direction of propagation. This means that the scalar wave speed $c(x)$ is replaced by a positive definite symmetric matrix $(g^{jk}(x))_{j,k=1}^n$. The wave equation in an anisotropic medium is defined by the formula

$$\partial_t^2 u(t,x) - \Delta_g u(t,x) = 0 \quad \text{in } \mathbb{R} \times M, \tag{4}$$

where M is a compact manifold or a bounded domain of $\mathbb{R}^n$. The Laplace–Beltrami operator Δ_g that corresponds to a smooth time-independent Riemannian metric g on M is defined in the formula

$$\Delta_g u = \sum_{j,k=1}^n \det(g)^{-1/2} \frac{\partial}{\partial x^j} \left(\det(g)^{1/2} g^{jk} \frac{\partial}{\partial x^k} u \right),$$

where $g(x) = [g_{jk}(x)]_{j,k=1}^n$, $\det(g) = \det(g_{jk}(x))$, and $[g^{jk}]_{j,k=1}^n = g(x)^{-1}$.

It is also possible to define a more general wave equation. Let $\mathcal{A}$ represent the most general, formally self-adjoint elliptic partial differential operator of the second order. In local coordinates, $\mathcal{A}$ has the form

$$\mathcal{A}u = -\sum_{j,k=1}^m \frac{1}{\mu(x)|g(x)|^{1/2}} \frac{\partial}{\partial x^j} \left(\mu(x)|g(x)|^{1/2} g^{jk}(x) \frac{\partial u}{\partial x^k} \right) + q(x)u,$$

where q is a smooth function $q\colon M \to \mathbb{R}$. Then $L^2(M)$ is defined by the inner product

$$\langle u, v \rangle = \int_M u(x)v(x)\,\mathrm{d}V,$$

where $\mathrm{d}V = \mu \mathrm{d}V_g$ and $\mu \in C^\infty(M)$ is a fixed, strictly positive function on M. For example, if $\mu = 1$ and $q = 0$ then $\mathcal{A}$ reduces to the Riemannian Laplace operator. Using this, the wave equation is defined by the formula

$$\partial_t^2 u(t,x) + \mathcal{A}u(t,x) = 0, \quad \text{in } \mathbb{R}\times M, \tag{5}$$

where M is a compact manifold or a bounded domain of $\mathbb{R}^n$.

2.2. **An initial and boundary value problem.** Let us consider the wave equation in M that is a bounded domain of $\mathbb{R}^m$ or a compact manifold. We define the initial and boundary value problem for the wave equation by

$$\begin{cases} u_{tt}(x,t) - \Delta_g u(x,t) = 0, & \text{in } M\times\mathbb{R}_+, \\ u|_{t=0} = 0, \quad u_t|_{t=0} = 0, \\ \partial_\nu u|_{\partial M\times\mathbb{R}_+} = f. \end{cases} \tag{6}$$

We use (4) to define the wave equation, but we could have replaced that with the definition from (3) or (5) as well. The smoothness of the boundary ∂M, the smoothness of the Riemannian metric g, and the smoothness of the Neumann boundary source f all have an influence on the solvability of this problem. We assume that the boundary ∂M has enough smoothness without trying to specify the limit for that smoothness. We also assume that the Riemannian metric g on M is smooth and time independent. In addition to that, having the Neumann boundary value $f \in L^2(\partial M \times (0,2T))$, there is a unique solution for the problem presented in (6)(see [54, Thm. A(1)]). For that solution, u, we use the notation $u = u^f$ to emphasize the relationship between the source and the solution of the problem. Having done this, we define a map from the space of boundary sources to the space of solutions for the problem presented in (6) by

$$\begin{aligned} &U : L^2(\partial M \times (0,2T)) \to C([0,2T]; H^{5/6-\epsilon}(M)), \quad \epsilon > 0, \\ &U : f \mapsto u^f. \end{aligned} \tag{7}$$

We take this result as a given. For the proof of the definition of (7), see [54, Thm. A].

2.3. **The Neumann-to-Dirichlet map.** The boundary control method is based on using sources on the boundary and measuring the effect they caused on the boundary. Thus, we need to define and understand the properties of these measurements. Let u^f be the solution for the initial and boundary value problem depicted in (6). We define the Neumann-to-Dirichlet operator Λ as mapping

$$\Lambda : f \mapsto \Lambda f \;=\; u^f|_{\partial M\times\mathbb{R}_+}, \tag{8}$$

where $f \in \partial M \times \mathbb{R}_+$ is a boundary source function and $u^f|_{\partial M\times\mathbb{R}_+}$ is the restriction of the solution to (6) to the boundary of the object. We need to restrict the size of the boundary for practical reasons. We want the option to only take measurements in part of the physical boundary. We define $\Gamma \subset \partial M$ as an non-empty and open subset. We also use the fixed time interval $(0,2T)$ when taking measurements. Let $T>0$ and $supp(f) \in \Gamma \times (0,2T)$. We define the Neumann-to-Dirichlet map Λ_{2T}^{Γ} by

$$\begin{aligned} &\Lambda_{2T}^{\Gamma}\colon L^2(\Gamma \times (0,2T)) \to L^2(\Gamma \times (0,2T)), \\ &\Lambda_{2T}^{\Gamma} f = u^f|_{\Gamma\times(0,2T)}. \end{aligned} \tag{9}$$

The map Λ_{2T}^{Γ} is well defined and bounded (see [54, Thm. A], [86, Thm. 9], and [54, Thm. 4.1]). However, the existence of Λ_{2T}^{Γ} does not ensure that we can recover the Riemannian metric g or the velocity function $c(x)$ when having the measurement operator shown in (9). We need to ask: Does the velocity function $c(x)$ uniquely define the Neumann-to-Dirichlet map Λ_{2T}^{Γ}? If so and we have the map $c \mapsto \Lambda_{2T}^{\Gamma}$, what kind of properties does this map have? Could it be possible, that there is another velocity function $\widetilde{c}(x)$ that also defines the same Neumann-to-Dirichlet map Λ_{2T}^{Γ}? For these questions, we need to study the properties of the direct problem.

2.4. **The direct problem for the wave equation.** Let us have 1+1-dimensional wave equation, as was the case in Publication I. The initial and boundary value problem can be defined as

$$\begin{cases} \left(\frac{\partial^2}{\partial t^2} - c(x)^2\frac{\partial^2}{\partial x^2}\right)u(x,t) = 0 \quad \text{in } M\times(0,2T), \\ u|_{t=0} = 0, \quad \partial_t u|_{t=0} = 0, \\ \partial_x u(0,t) = f(t), \end{cases} \tag{10}$$

where M is the half axis $M=[0,\infty) \subset \mathbb{R}$. The smoothness of the velocity function c is defined by

$$\|c\|_{C^k(M)} = \sum_{p=0}^{k} \sup_{x\in(0,\infty)} |\frac{\partial^p c}{\partial x^p}(x)|. \tag{11}$$

The set of bounded $C^k(M)$ functions is defined by

$$C_b^k(M) = \{c \in C^k(M); \|c\|_{C^k(M)} < \infty\}.$$

Let $C_0, C_1, L_0, L_1, m > 0$ and the space of k times differentiable velocity functions is defined by

$$\begin{aligned} \mathcal{V}^k = &\{c \in C^k(M); C_0 \le c(x) \le C_1, \\ &\|c\|_{C^k(M)} \le m,\ c-1 \in C_0^k([L_0,L_1])\}. \end{aligned} \tag{12}$$

Here, $C_0^k([L_0,L_1])$ is the subspace of functions in $C_b^k(M)$ that are supported on $[L_0,L_1]$. When $c \in \mathcal{V}^2$ and $f \in L^2(0,2T)$, the boundary value problem has the unique solution $u = u^f \in H^1(M\times(0,2T))$. Using the solution u^f of (10), we define a map as

$$U_T : L^2(0,2T) \mapsto H^1(M\times(0,2T)), \quad U_T f = u^f. \tag{13}$$

The map defined in (13) is well defined and continuous (see Publication I, Appendix A). Having that, we define the Neumann-to-Dirichlet operator by

$$\Lambda : L^2(0, 2T) \to L^2(0, 2T), \quad \Lambda f = u^f|_{x=0}. \tag{14}$$

We simplify the notation and denote $\Lambda = \Lambda^{\Gamma}_{2T}$. Next we analyse the relationships between velocity functions c and measurements Λ. Let $X = L^\infty(M)$, $Z = C^2_b(M)$, and $Y = \mathcal{L}(L^2(0, 2T))$, and let us define a direct map as

$$\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset Z \to \mathcal{R}(\mathcal{A}) \subset Y, \quad \mathcal{A}(c) = \Lambda, \tag{15}$$

where the domain $\mathcal{D}(\mathcal{A}) = \mathcal{V}^2$. The notation in (15) means that the range $\mathcal{R}(\mathcal{A}) = \mathcal{A}(\mathcal{V}^2)$ and the domain $\mathcal{D}(\mathcal{A})$ are equipped with the topologies of Y and Z, respectively. For proofs that the maps depicted in (14) and (15) are well defined and continuous, see Publication I, Appendix A. Note that the map $\mathcal{A}$ is non-linear.

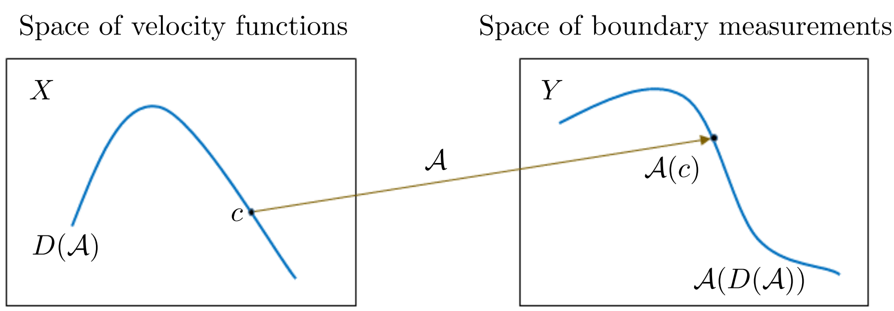

FIGURE 2. The direct map $\mathcal{A} : c \to \Lambda$ is defined by using the solutions for the initial and boundary value problem for the wave equation (10). For this we need to assume some smoothness from the velocity functions $c \in D(\mathcal{A}) \subset L^\infty(M) = X$ in order to get the existence and continuity of the map $\mathcal{A}$.

The existence of the direct map depicted in (15) says that, for a given velocity function $c \in \mathcal{D}(\mathcal{A})$, there is a unique boundary operator $\Lambda = \Lambda(c)$. The continuous property of the map $\mathcal{A}$ says that when there is small changes in the velocity function c, this causes controllable changes in the measurements $\Lambda(c)$. It is possible that two different velocity functions produces the same measurement. We thus need to analyse whether the map $\mathcal{A}$ can be inverted.

2.5. **The boundary control method.** We show that there is an inverse map $\mathcal{A}^{-1}$ for the map $\mathcal{A}$ and find a mathematical formula that connects the boundary measurements and the velocity function c together.

For this we use a strategy that is one variant of the boundary control method. The method was pioneered by M. Belishev [36] and developed by M. Belishev and Y. Kurylev [7, 38] in the late 1980s and early 1990s. Of crucial importance for this method were the results of D. Tataru [85] that concerned a Holmgren-type uniqueness theorem for non-analytic coefficients. In the center of this approach is the ability to calculate inner products inside the object by using measurements at the boundary, namely two Blagovestchenskii identities.

Let M be a bounded domain of $\mathbb{R}^m$ or a compact manifold. The first identity can be formulated as

$$\langle u^f(T), 1\rangle_{L^2(M)} = -\langle f, \widehat{P}\Phi\rangle_{L^2(\partial M\times(0,2T))}, \tag{16}$$

which can also be written as

$$\int_M u^f(x,T)\cdot 1\, dV_g(x) = -\int_{\partial M\times(0,2T)} f(x,t)\widehat{P}\Phi(x,t)\, dS_g(x)dt.$$

There, $f \in L^2(\partial M\times(0,2T))$ is a boundary source function and $u^f(x,T)$ is the solution for the initial and boundary value problem depicted in (6) at time $t = T$. The operator $\Phi(x,t) = T - t$ and

$$\begin{aligned} \widehat{P}&\colon L^2(\partial M\times(0,2T)) \to L^2(\partial M\times(0,2T)),\\ \widehat{P}&\colon f(x,t) \mapsto 1_{\partial M\times(0,T]}(x,t)\, f(x,t). \end{aligned} \tag{17}$$

The second identity can be formulated as

$$\langle u^f(T), u^f(T)\rangle_{L^2(M)} = \langle f, Kf\rangle_{L^2(\partial M\times[0,2T])}, \tag{18}$$

which can also be written as

$$\int_M u^f(x,T)u^f(x,T)\, dV_g(x) = \int_{\partial M\times[0,2T]} (Kf)(x,t)f(x,t)\, dS_g(x)dt.$$

There, the operator K is defined by

$$\begin{aligned} K&\colon L^2(\partial M\times(0,2T)) \to L^2(\partial M\times(0,2T)),\\ K &= J\Lambda_{2T} - R\Lambda_{2T}RJ. \end{aligned} \tag{19}$$

The Neumann-to-Dirichlet operator Λ_{2T} is as defined in (9), $Rf(x,t) = f(x,2T-t)$ is the time reversal operator, and

$$\begin{aligned} J&\colon L^2(\partial M\times(0,2T)) \to L^2(\partial M\times(0,2T)),\\ J &: f(x,t) \mapsto \frac{1}{2}\int_t^{2T-t} f(x,s)ds \end{aligned}$$

is a time filter operator. The identities depicted in (16) and (18) originate from [12], and their proofs can be found (e.g, in [10]).

Let us define the domain of influence of Γ at time s by the set

$$M(\Gamma,s) = \{x\in M :\ d(x,\Gamma)\le s\}, \tag{20}$$

where $s > 0$, $\Gamma\subset\partial M$ is a non-empty open subset, and $d(x,y)$ is the geodesic distance that corresponds to g (the travel time distance).

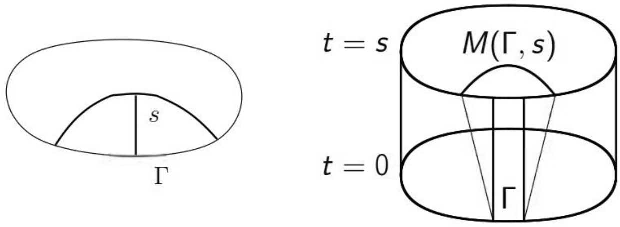

FIGURE 3. The domain of influence of Γ at time s. The property of the finite propagation speed for the wave equation indicates that when a source function f is supported on Γ at time $t = 0$, then at time $t = s$ the solution is supported on the domain of influence, that is $supp(u^f(s)) \subset M(\Gamma, s)$.

It turns out that for the reconstruction method we consider, the optimal solution would be to find a boundary source function $h \in L^2(\Gamma \times (0, T))$ for which

$$u^{Ph}(T) = 1_{M(\Gamma,s)}.$$

The reason for this is the connection between the weighted norm of indicator function $1_{M(\Gamma,s)}$ and the velocity parameter c. Unfortunately this strategy is not usable in our case, because $u^{Ph}(T) \in H^{5/6-\epsilon}(M))$ and $1_{M(\Gamma,s)} \notin H^{5/6-\epsilon}(M))$. This leads us to seek a boundary source h for which

$$u^{Ph}(T) \approx 1_{M(\Gamma,s)}.$$

For that, the seminal result implying controllability is Tataru's unique continuation result (see [85, 87]).

Proposition 1 (Tataru)**.** *Let $u \in H^1_{\mathrm{loc}}(M \times \mathbb{R}_+)$ be a solution of the wave equation*

$$u_{tt}(x, t) + \mathcal{A}u(x, t) = 0.$$

Assume that

$$u|_{\Sigma\times(0,2s)} = 0, \quad \partial_\nu u|_{\Sigma\times(0,2s)} = 0,$$

where $\Sigma \subset \partial M$ is an non-empty open set and $s > 0$. Then

$$u(x, s) = 0, \ \partial_t u(x, s) = 0, \qquad \text{for } x \in M(\Sigma, s).$$

By using Tataru's unique continuation result, one can prove the following two controllability results (see, e.g, [20]). The first, the approximative local controllability result, is that the linear subspace

$$\left\{u^{Ph}(T) : h \in L^2(\Gamma \times (0, T))\right\} \tag{21}$$

is dense in $L^2(M(\Gamma, s))$. There, $B = \Gamma \times (T - s, T)$ and

$$P\colon f(x, t) \ \mapsto \ 1_B(x, t)\, f(x, t). \tag{22}$$

The second, the approximative global controllability result, is that the linear subspace

$$\Big\{(u^f(T), u^f_t(T)) : \, f \in C_0^\infty(\Gamma \times (0,T))\Big\} \tag{23}$$

is dense in $H^1(M) \times L^2(M)$. We search for a boundary function h that minimizes the following problem

$$\min_{h \in L^2(\partial M \times [0,2T])} \|1_{M(\Gamma,s)} - u^{Ph}(T)\|^2_{L^2(M)}. \tag{24}$$

By using the approximative local controllability result from (23), we know that there is a sequence of boundary source functions $\{h_j\}_{j=1}^\infty$, for which $u^{Ph_j}(T) \to 1_{M(\Gamma,s)}$ as $j \to \infty$. The minimization problem depicted in (24) usually has no solution and is ill posed. Thus, instead of that, we consider the following regularized minimization problem

$$\min_{h \in L^2(\partial M \times [0,2T])} \Big(\|1_{M(\Gamma,s)} - u^{Ph}(T)\|^2_{L^2(M)} + \alpha \|h\|^2_{L^2(\partial M \times [0,2T])}\Big), \tag{25}$$

where $\alpha \in (0,1)$. By using the identities depicted in (16) and (18), the problem depicted in (25) can equivalently be written as

$$\min_{h \in V} \Big(2\langle Ph, \widehat{P}\Phi\rangle_V + \langle Ph, KPh\rangle_V + \alpha\langle h, h\rangle_V\Big), \tag{26}$$

where $V = L^2(\partial M \times [0,2T])$. The formulation of (26) only uses boundary measurements, boundary sources, and boundary operators. The minimization problem depicted in (25) for waves inside V can be solved when the boundary measurements are known. By using the Fréchet derivative, for every $\alpha \in (0,1)$ we have the unique solution

$$h_\alpha = (PKP + \alpha)^{-1} P\Phi, \tag{27}$$

a minimizer for the problem depicted in (26). These solutions, a sequence of boundary source functions $\{h_\alpha\}_{\alpha \in (0,1)}$, are exactly the correct ones. We have

$$\|1_{M(\Gamma,s)} - u^{Ph_\alpha}(T)\|^2_{L^2(M)} \to 0, \quad \text{when} \quad \alpha \to 0.$$

2.6. **The inverse problem for the wave equation.** We define an inverse problem as recovering the velocity function c by using the unperturbed boundary measurements Λ.In Publication I we follow a 1+1-dimensional case. It is well known, that these kinds of direct maps $\mathcal{A}$ are invertible (see e.g, [77, 78]). The same holds in our case and we define

$$\mathcal{A}^{-1} : \Lambda \mapsto c.$$

In addition to the existence result, we give a new formula to compute c from Λ. Let us start by defining the domain of influence as

$$M(r) = \{x \in M; d(x,0) \leq r\}, \tag{28}$$

where

$$d(x,0) = \int_0^x \frac{1}{c(t)} dt$$

is the travel time of the waves from 0 to the point x. Let $\alpha > 0$. The regularized minimization problem depicted in (26) has a unique minimizer

$$h_{\alpha,r} = (P_r K P_r + \alpha)^{-1} P_r \Phi, \tag{29}$$

where $\Phi(x,t) = T - t \in L^2(\partial M \times (0,2T))$, K is as defined in (19), and

$$P_r \colon L^2(\partial M \times (0,2T)) \to L^2(\partial M \times (0,2T)),$$
$$P_r f(t) = 1_{(T-r,T)}(t) f(t),$$

where

$$1_{(T-r,T)}(t) = \begin{cases} 1, & t \in (T-r,T), \\ 0, & \text{otherwise.} \end{cases}$$

Moreover $u^{h_{\alpha,r}}(T)$ converges into to the indicator function of the domain of influence,

$$\lim_{\alpha \to 0} \left\| u^{h_{\alpha,r}}(T) - 1_{M(r)} \right\|_{L^2(M;dV)} = 0. \tag{30}$$

The proofs for (29) and (30) are given in Publication I. We define the travel time coordinates for $x \in M$ by

$$\tau : [0,\infty) \to [0,\infty), \quad \tau(x) = d(x,0).$$

The function τ is strictly increasing and we denote its inverse by

$$\chi = \tau^{-1} : [0,\infty) \to [0,\infty).$$

We have

$$\chi(0) = 0, \quad \chi'(t) = \frac{1}{\tau'(\chi(t))} = c(\chi(t)). \tag{31}$$

Thus, denoting $v(t) = c(\chi(t))$ and using $V(r)$ to denote the volume of $M(r)$ with respect to the measure dV, we have

$$V(r) = \left\| 1_{M(r)} \right\|^2_{L^2(M;dV)} = \int_0^{\chi(r)} \frac{dx}{c(x)^2} = \int_0^r \frac{\chi'(t)dt}{v(t)^2} = \int_0^r \frac{dt}{v(t)}. \tag{32}$$

Note that $M(r) = [0, \chi(r)]$. In particular, $V(r)$ determines the wave speed in the travel time coordinates:

$$v(r) = \frac{1}{\partial_r V(r)}, \tag{33}$$

and also in the original coordinates since

$$c(x) = v(\chi^{-1}(x)), \quad \chi(t) = \int_0^t v(t')dt'. \tag{34}$$

Using (16), (29), and (30), we have a method with which to compute the volumes of the domains of influence

$$V(r) = \left\|1_{M(r)}\right\|^2_{L^2(M;dV)} = \lim_{\alpha\to 0}\langle h_{\alpha,r}, P_r\Phi\rangle_{L^2(0,2T)}, \tag{35}$$

where $r \in [0,T]$. For a given measurement Λ, equations (33), (34), and (35) gives us a way in which to calculate the value of the velocity function

$$c(x) = v(\chi^{-1}(x)) = \mathcal{A}^{-1}(\Lambda)(x)$$

for all $x \in (0,L)$. As we assumed that outside of the interval $(0,L)$ the value of the function c is identically one, we have a formula for the inverse map $\mathcal{A}^{-1}$. When we restrict $\mathcal{A}$ to the set $\mathcal{V}^3 \subset \mathcal{V}^2$, the map $\mathcal{A}|_{\mathcal{V}^3} : \mathcal{V}^3 \subset Z \to \mathcal{A}(\mathcal{V}^3)$ has a continuous inverse operator in the following sense: The inverse map

$$\mathcal{A}^{-1} : \mathcal{A}(\mathcal{V}^3) \subset Y \to \mathcal{V}^3 \subset Z, \quad \mathcal{A}^{-1}(\Lambda) = c$$

is continuous. The continuity of $\mathcal{A}^{-1}$ is abstract in the sense that it does not contain quantitative estimates.

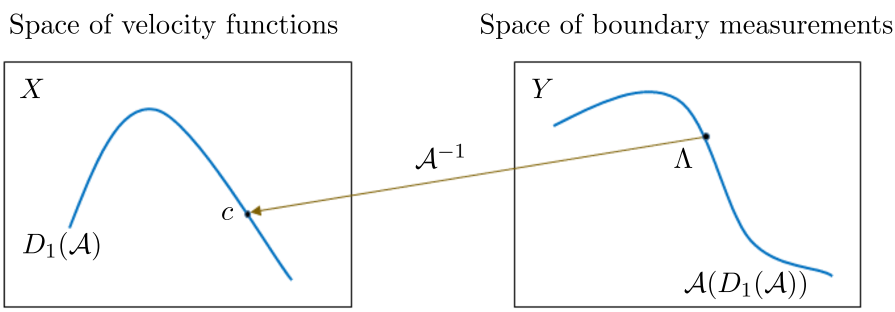

FIGURE 4. The formulation of the inverse map $\mathcal{A}^{-1} : \Lambda \to c$. When we carefully choose the domain $D_1(\mathcal{A}) \subset L^\infty(M)$, the map $\mathcal{A}^{-1}$ is continuous.

2.7. **The regularization strategy.** The situation where we only have access to perturbed data $\widetilde{\Lambda}$ means that the data is contaminated with noise by inaccurate measurement or model errors. In this situation, we also use the same symbols as in (15) for the direct map $\mathcal{A}$ and define $\mathcal{A} : c \to \widetilde{\Lambda}$. In general, the inverse problems for the wave equation are unstable with respect to measurement errors and non-linear. Hence they are hard to solve, especially computationally. A good theoretical understanding plays a crucial role in designing practical solution methods.

One model for measurements that concerns perturbations of the Neumann-to-Dirichlet operator is defined by

$$\widetilde{\Lambda} : L^2(0,2T) \to L^2(0,2T), \quad \widetilde{\Lambda} = \Lambda + \mathcal{E}, \tag{36}$$

where Λ is as defined in (14) and linear operator $\mathcal{E}$ models the measurement error. We assume that $\|\mathcal{E}\|_{L^2(0,2T)\to L^2(0,2T)} \leq \epsilon$, where $\epsilon > 0$ is known. Now, we cannot use the map $\mathcal{A}^{-1}$ to calculate function c since $\widetilde{\Lambda}$ may not be in the range $\mathcal{R}(\mathcal{A})$. To overcome this, we use a method called *regularization.*

Tikhonov regularization, one classical example of a regularization methods, finds the solution for problem $\mathcal{A}(c) = \widetilde{\Lambda}$ as minimizer of

$$\arg\min_c \{\|\mathcal{A}(c) - \widetilde{\Lambda}\|^2 + \alpha\|c\|^2\}, \quad \alpha > 0.$$

Here the purpose of the first term of the objective functional is to ensure that the model $\mathcal{A}(c) = m$ is satisfied approximately, while the second term works from the prior information that the norm of the solution is not too large. The regularization parameter α is used to tune the balance between these two requirements.

Theoretically the regularization strategy that we use here is defined (e.g, in [48]) as a family of bounded mappings $\mathcal{R}_{\alpha(\epsilon)}, \alpha > 0$, that approximate the inverse of $\mathcal{A}$ in the sense that

$$\lim_{\alpha\to 0} \mathcal{R}_{\alpha(\epsilon)}(A(c)) = c \quad , \forall c \in \mathcal{D}(\mathcal{A}).$$

Moreover, the choice of $\alpha = \alpha(\epsilon)$ should depend on the noise level $\epsilon > 0$ such that $\alpha(\epsilon) \to 0$ and

$$R_{\alpha(\epsilon)}(\widetilde{\Lambda}) \to \mathcal{A}^{-1}(\Lambda),$$

as $\epsilon \to 0$. That is, the regularized solution should tend to towards the true solution as the noise level tends to towards zero. In Publication I we defined such a family of operators $\{\mathcal{R}_{\alpha(\epsilon)}\}$ by using a construction that is motivated by the methods that we use for the calculation of $\mathcal{A}^{-1}$. We recall the definition of a regularization strategy, (see, e.g, [27] and [48]). Figure 5 is a schematic illustration of regularization from Publication I.

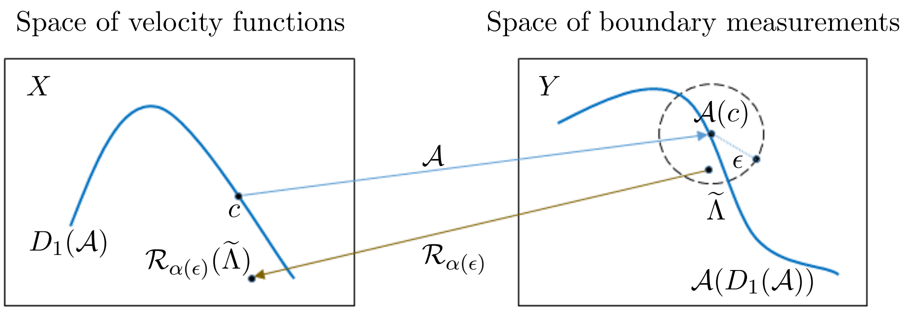

FIGURE 5. The idea of regularization is to construct a family $\mathcal{R}_{\alpha(\epsilon)}$ of continuous maps from the data space Y to the model space X in such a way that c can be approximately recovered from noisy data $\widetilde{\Lambda}$. For a smaller noise level ϵ, the approximation $\mathcal{R}_{\alpha(\epsilon)}(\widetilde{\Lambda})$ is closer to c. More details can be found in [64, Fig. 11.5].

2.8. **Numerical approximation.** Furthermore, when dealing with real-life measurements or doing simulations on a computer, discretization to a finite dimensional setting is an important issue. In particular, how does the problem at first hand behave if we only have finite, possibly noisy data available? What is our unperturbed measurement on the boundary and how do we define the measurement noise? In Publication II we gave a solution for the inverse problem when our measurement is discrete and noisy. Here we show how to define a noisy measurements operator using point values at the boundary.

Let $T > 0$ and $N \in \mathbb{Z}_+$. For $n \in \{1, 2, 3, ..., 2N\}$ we define the basis functions at the boundary of the object as

$$\phi_{n,N}(t) = \left(\frac{N}{T}\right)^{\frac{1}{2}} 1_{[\frac{(n-1)T}{N}, \frac{nT}{N})}(t), \quad t \in [0, 2T). \tag{37}$$

Note that the functions $\phi_{n,N}$ are orthonormal in $L^2(0, 2T)$. Having (37), we define the space of piecewise constant functions as

$$\mathcal{P}^N = span\{\phi_{1,N}, ..., \phi_{2N,N}\} \subset L^2(0, 2T) \tag{38}$$

and an orthogonal projection in $L^2(0, 2T)$ by

$$P^N : L^2(0, 2T) \to \mathcal{P}^N, \quad P^N(f) = \sum_{j=1}^{2N} \langle f, \phi_{j,N} \rangle_{L^2(0,2T)} \phi_{j,N}(t). \tag{39}$$

Let the measurements Λ be as in (14). Using (39), we define discretized measurements by

$$\Lambda_N = P^N \Lambda P^N. \tag{40}$$

We define a measurement error by

$$\mathcal{E}_N : \mathcal{P}^N \to \mathcal{P}^N.$$

We define a discrete and noisy measurement operator as

$$\begin{aligned} &\widetilde{\Lambda}_N : L^2(0,2T) \to L^2(0,2T), \\ &\widetilde{\Lambda}_N f = \begin{cases} (\Lambda_N + \mathcal{E}_N) f = (P^N \Lambda P^N + \mathcal{E}_N) f, & f \in \mathcal{P}^N, \\ 0, & f \in (\mathcal{P}^N)^\perp. \end{cases} \end{aligned} \tag{41}$$

Note that $\mathcal{P}^N \cup (\mathcal{P}^N)^\perp = L^2(0,2T)$. Let $N \in \mathbb{Z}_+$. Let Λ be as in (14) and Λ_N as in (40). Then we can show

$$\|\Lambda_N - \Lambda\|_Y \leq C N^{-\frac{1}{4}},$$

for more on this, see Publication II. Let $\epsilon > 0$, $N \in \mathbb{Z}_+$, and $N \geq \epsilon^{-4}$. Assume that $\left\|\widetilde{\Lambda}_N - \Lambda_N\right\|_Y = \|\mathcal{E}_N\|_Y \leq \epsilon$, then

$$\left\|\widetilde{\Lambda}_N - \Lambda\right\|_Y \leq C\epsilon = \widetilde{\epsilon}.$$

We show how to construct a family of operators $\{\mathcal{R}_{N,\alpha}\}$ for the case when we have perturbed and discretized measurements $\widetilde{\Lambda}_N$ on the boundary, (see Publication II). This is motivated by the methods that we used in Publication I to define operators $\{\mathcal{R}_{\alpha(\epsilon)}\}$ for the case when we have perturbed measurements $\widetilde{\Lambda}$ on the boundary. In Publication II we show that, if $\widetilde{\epsilon} \to 0$ — that is $\left\|\widetilde{\Lambda}_N - \Lambda\right\|_Y \to 0$ — then

$$\left\|\mathcal{R}_{N,\alpha}(\widetilde{\Lambda}_N) - c\right\|_Z \to 0.$$

Figure 6 shows a schematic illustration of regularization from Publication II.

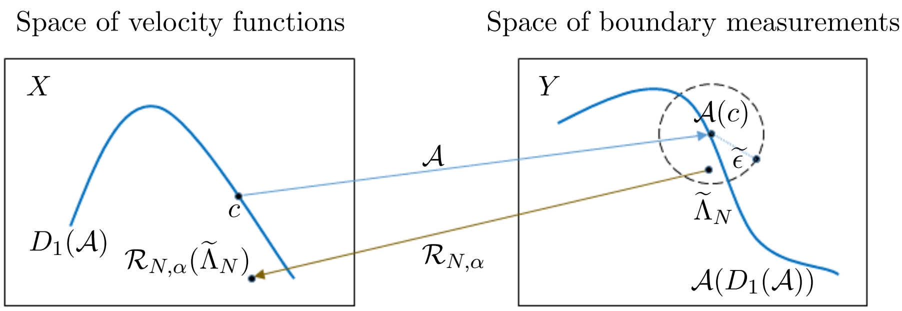

FIGURE 6. The idea of regularization is to construct a family $\{\mathcal{R}_{N,\alpha}\}$ of continuous maps from the data space Y to the model space X in such a way that c can be approximately recovered from noisy and discretized $\widetilde{\Lambda}$. For a smaller noise level $\widetilde{\epsilon}$, the approximation $\mathcal{R}_{\alpha(\epsilon)}(\widetilde{\Lambda})$ is closer to c.

2.9. **Focusing energy.** We explain the main ideas behind focusing energy by using boundary control methods. Let us consider the wave equation in M that is a bounded domain of $\mathbb{R}^m$ or a compact manifold. We define the initial boundary value problem for the wave equation as

$$\begin{cases} u_{tt}(x,t) - \Delta_g u(x,t) = 0, & \text{in } M \times \mathbb{R}_+, \\ u|_{t=0} = 0, \quad u_t|_{t=0} = 0, \\ \partial_\nu u|_{\partial M \times \mathbb{R}_+} = a. \end{cases} \tag{42}$$

Let $a \in L^2(\partial M \times (0, 2T))$ be the Neumann boundary value. Then there is a unique solution for problem, and we denote that solution $u = u^a$ and define the map

$$U : a \mapsto u^a. \tag{43}$$

The basic idea to get control for the energy of the solution at time $t = T$ and start by looking for a boundary source function a for which

$$\partial_t u^a(x,T) \approx 1_{M(\Gamma,s)} \quad \textit{and} \quad u^a(x,T) \approx 0. \tag{44}$$

Having $a \in H^1((0,2T); L^2(\partial M))$ as the Neumann boundary value, then for $\epsilon > 0$, the map

$$U : H^1((0,2T); L^2(\partial M)) \to C([0,2T]; H^{3/2-\epsilon}(M)) \cap C^1([0,2T]; H^{1/2-\epsilon}(M))$$

is bounded according to the thinking of Lasiecka, Triggiani (see [55, Thm. 3.1(iii)]). Having the boundary source $a \in H^1((0,2T); L^2(\partial M))$, then for the solution at time $t = T$ we have $\partial_t u^a(x,T) \in H^{1/2-\epsilon}(M)$ and $1_{M(\Gamma,s)} \notin H^{1/2-\epsilon}(M)$. Using the approximate global controllability result, we can control u^a and $\partial_t u^a$ at the same time. Thus for $\epsilon > 0$ we have $a = a(\epsilon)$, for which

$$\|1_{M(\Gamma,s)} - \partial_t u^a(T)\|^2_{L^2(M)} + \|0 - u^a(T)\|^2_{H^1(M)} < \epsilon. \tag{45}$$

When using a Blagovestchenskii identity (16), the solution is expected to be supported at time $t = T$ on the domain of influence. For this reason, we need to use an extra source function, $h(\epsilon)$. We demand that for $\epsilon > 0$ we have sources h and a, for which

(46)
$$\|1_{M(\Gamma,s)} - u^{Ph}(T)\|^2_{L^2} + \|u^{Ph}(T) - \partial_t u^a(T)\|^2_{L^2(M)} + \|u^a(T)\|^2_{H^1(M)} < \epsilon.$$

This minimization problem usually has no solution and is ill posed. Thus we consider the following regularized minimization problem: Let boundary sources $h(\alpha), a(\beta) \in V \times Y$ be found by minimizing the following functional with the regularization parameters $\alpha \in (0,1)$ and $\beta \in (0,1)$:

$$\begin{aligned} \mathcal{F}(h,a,\alpha,\beta) &= \|1_{\mathcal{N}} - u^{Ph}(T)\|^2_{L^2(\mathcal{M})} + \alpha\|h\|^2_V \\ &\quad + \|u^{Ph}(T) - u^a_t(T)\|^2_{L^2(\mathcal{M})} + \|u^a(T)\|^2_{H^1(\mathcal{M})} \\ &\quad + \beta\|a\|^2_Y. \end{aligned} \tag{47}$$

The minimization problem depicted in (47) for h and a is not convenient for numerical analysis, thus, we split the functional depicted in (47) into two functionals, namely,

$$\mathcal{F}(h,a,\alpha,\beta) = \mathcal{F}_1(h,\alpha) + \mathcal{F}_2(a,\beta|h).$$

For $\alpha \in (0,1)$ and $h \in V$ we define the first functional in a quadratic form on $V \times (0,1)$ as

$$\mathcal{F}_1(h,\alpha) = \|1_{\mathcal{N}} - u^{Ph}(T)\|^2_{L^2(\mathcal{M})} + \alpha\|h\|^2_V. \tag{48}$$

For $\beta \in (0,1)$ and $a \in Y$ we define the second functional in a quadratic form on $Y \times (0,1)$ as

$$\mathcal{F}_2(a,\beta|h) = \|u^{Ph}(T) - u^a_t(T)\|^2_{L^2(\mathcal{M})} + \|u^a(T)\|^2_{H^1(\mathcal{M})} + \beta\|a\|^2_Y, \tag{49}$$

where $h = h(\alpha) \in V$ is fixed and is the solution of (48). The minimization procedure is two-staged: We first minimize the functional depicted in (48) and then, with this solution we minimize the second functional depicted in (49). For $\alpha \in (0,1)$, the solution $h = h(\alpha) \in V$ to the equation

$$(PKP + \alpha)\, h = -P\Phi_T \tag{50}$$

is a unique minimizer of $\mathcal{F}_1(h,\alpha)$ (48) in the space $(h,\alpha) \in V \times (0,1)$, namely, $h = \arg\min_{h\in V} \mathcal{F}_1(h,\alpha)$. Let $h_\alpha = h(\alpha) \in V$ be the solution to equation (50). For $\beta \in (0,1)$, the solution $a = a(\beta, h_\alpha) \in Y$ to the following equation

$$(L + \beta)a = -NQ\partial_t KPh_\alpha \tag{51}$$

is a unique minimizer of the functional $\mathcal{F}_2(a,\beta|h_\alpha)$ (49), namely, $a = \arg\min_{a\in Y} \mathcal{F}_2(a,\beta|h_\alpha)$, where

$$L\colon Y \to Y, \quad L = NQ\left(R\Lambda R\partial_t\widehat{P} - \widehat{P}\partial_t\Lambda + K\right). \tag{52}$$

For the definition of the operators used above, see Publication III. Solving these two minimizing problems give us the sequence $\{a_\alpha\}_{\alpha\in(0,1)}$, for which

$$\|1_{M(\Gamma,s)} - \partial_t u^{a_\alpha}(T)\|^2_{L^2(M)} + \|u^{a_\alpha}(T)\|^2_{H^1(M)} \to 0, \quad \text{when} \quad \alpha \to 0.$$

Next we heuristically describe our strategy to continue from here in the case of $\mathbb{R}^2$.
1) We can find a source a_1, for which $\partial_t u^{a_1}(T) \approx 1_{N_1}$ and $u^{a_1}(T) \approx 0$ in $L^2(M)$.

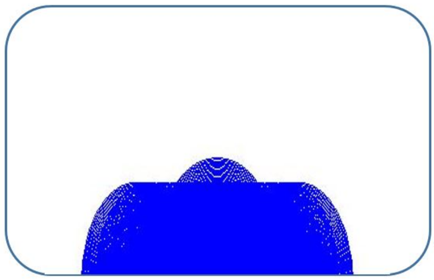

FIGURE 7. We can find a boundary source function a_1, for which $\partial_t u^{a_1}(T) \approx 1_{N_1}$ and $u^{a_1}(T) \approx 0$ in $L^2(M)$. Here we denote the blue area by N_1.

2) We can find a source a_2, for which $\partial_t u^{a_2}(T) \approx 1_{N_2}$ and $u^{a_2}(T) \approx 0$ in $L^2(M)$.

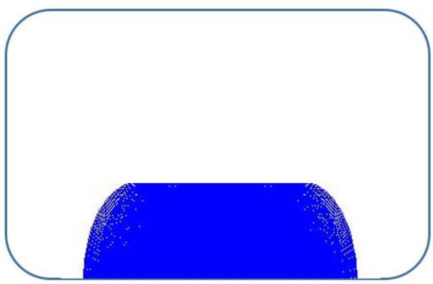

FIGURE 8. We can find a boundary source function a_2, for which $\partial_t u^{a_2}(T) \approx 1_{N_2}$ and $u^{a_2}(T) \approx 0$ in $L^2(M)$. Here we denote the blue area by N_2.

3) Then for source $a_1 - a_2$, we have $u^{a_1-a_2}(T) \approx 1_{N_1 \setminus N_2}$ and $u^{a_1-a_2}(T) \approx 0$ in $L^2(M)$.

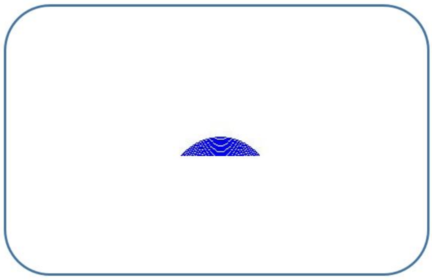

FIGURE 9. We have found a boundary source function $a_1 - a_2$, for which, $u^{a_1-a_2}(T) \approx 1_{N_1\setminus N_2}$ and $u^{a_1-a_2}(T) \approx 0$ in $L^2(M)$. Here we denote the blue area by $1_{N_1\setminus N_2}$.

By using these steps we can construct an artificial point source by solving a blind control problem. This is done in Publication III.

3. A review of the results of Publications I–III

In this section we briefly review the main ideas and results of Publications I–III.

3.1. Publication I. An inverse boundary value problem for a 1+1- dimensional wave equation with wave speed $c(x)$ is considered. We give a regularization strategy for inverting the map $\mathcal{A} : c \mapsto \Lambda$, where Λ is the hyperbolic Neumann-to-Dirichlet map corresponding to the wave speed c. That is, we consider the case when we are given a perturbation of the Neumann-to-Dirichlet map $\widetilde{\Lambda} = \Lambda + \mathcal{E}$, where $\mathcal{E}$ corresponds to the measurement errors, and reconstruct an approximative wave speed $\widetilde{c}$. We emphasize that $\widetilde{\Lambda}$ may not be in the range of the map $\mathcal{A}$. We show that the reconstructed wave speed $\widetilde{c}$ satisfies $\|\widetilde{c} - c\| \leq C\|\mathcal{E}\|^{1/54}$. Our regularisation strategy is based on a new formula for computing c from Λ.

Here we introduce another modification of the iterative time-reversal control method that is tailored for the 1+1-dimensional wave equation. This gives a direct regularization method for the non-linear inverse problem for the wave equation. The result contains an explicit (but not necessarily optimal) convergence rate. This direct method gives an explicit construction of a non-linear map that solves the problem without resorting to a local optimization method. The advantage of this direct approach is that it does not suffer from the possibility that the algorithm converges to a local minimum. In particular, this method does not require a priori knowledge that the solution is in a small neighbourhood of a given function. There are currently only a few regularized direct methods for non-linear inverse problems. Our main result (Publication I, Theorem 3) concerns perturbations of the Neumann-to-Dirichlet operator of the form

$$\widetilde{\Lambda} = \Lambda + \mathcal{E}, \tag{53}$$

where $\mathcal{E} \in Y$ models the measurement error. We assume that $\|\mathcal{E}\|_Y \leq \epsilon$, where $\epsilon > 0$ is known. In this situation we can not use the map $\mathcal{A}^{-1}$ to calculate function c since $\widetilde{\Lambda}$ may not be in the range $\mathcal{R}(\mathcal{A})$. We use the definition of a regularization strategy given in [27] and [48].

Definition 1. *Let Z, Y be Banach spaces and $\Omega \subset Z$. Let $\mathcal{A} : \Omega \subset Z \to Y$ be a continuous mapping. Let $\alpha_0 \in (0, \infty]$. A family of continuous maps $\mathcal{R}_\alpha : Y \to Z$, parametrized by $0 < \alpha < \alpha_0$, is called a regularization strategy for $\mathcal{A} : \Omega \to Y$ if*

$$\lim_{\alpha \to 0} \mathcal{R}_\alpha(\mathcal{A}(c)) = c$$

for every $c \in \Omega$. A regularization strategy is called admissible if the parameter α is chosen as a function of $\epsilon > 0$ so that $\lim_{\epsilon \to 0} \alpha(\epsilon) = 0$ and for every $c \in \Omega$

$$\lim_{\epsilon \to 0} \sup \left\{ \left\| \mathcal{R}_{\alpha(\epsilon)} \widetilde{\Lambda} - c \right\|_Z : \widetilde{\Lambda} \in Y, \left\| \widetilde{\Lambda} - \mathcal{A}(c) \right\|_Y \leq \epsilon \right\} = 0.$$

The main result says that we have an admissible regularization strategy inverting $\mathcal{A}$.

Theorem 1. *There exists an admissible regularization strategy $\mathcal{R}_\alpha$ with the choice of parameter*

$$\alpha(\epsilon) = 2^{\frac{13}{9}} T^{\frac{4}{9}} \epsilon^{\frac{4}{9}}$$

that satisfies the following: For every $c \in \mathcal{D}(\mathcal{A})$ there is ϵ_0, $C > 0$ such that

$$\sup \left\{ \left\| \mathcal{R}_{\alpha(\epsilon)} \widetilde{\Lambda} - c \right\|_X : \widetilde{\Lambda} \in Y, \left\| \mathcal{A}(c) - \widetilde{\Lambda} \right\|_Y \leq \epsilon \right\} \leq C \epsilon^{\frac{1}{18}},$$

for all $\epsilon \in (0, \epsilon_0)$.

In Publication I, we gave explicit choices of $\mathcal{R}_\alpha$ and ϵ_0. Here is a short summary on the regularization strategy. Assume that we are given $\widetilde{\Lambda} \in Y$, that is, the Neumann-to-Dirichlet map for the unknown wave speed $c(x)$ with measurements errors. Then the regularization strategy is obtainded by doing the following steps:

(1) We calculate the operator $\widetilde{H}_r = P_r(R\widetilde{\Lambda}RJ - J\widetilde{\Lambda})P_r$ for $r \in [0, T]$. This operator approximately determines the inner products of the waves by $\langle u^{f_1}(T), u^{f_2}(T) \rangle_{L^2(M)} \approx \langle \widetilde{H}_r f_1, f_2 \rangle_{L^2(M)}$ for all boundary sources $f_1, f_2 \in L^2(T - r, T)$.
(2) Using operator $\widetilde{H}_r$, we construct a source $\widetilde{f}_{\alpha,r}$ that approximates the solution $f_{\alpha,r}$ of the minimization problem. Here, the source $f_{\alpha,r}$ produces a wave such that $u^{f_{\alpha,r}}(t, x)|_{t=T}$ is close to the indicator function $1_{M(r)}(x)$ of the domain of influence $M(r)$.
(3) Using sources $\widetilde{f}_{\alpha,r}$ we approximately compute the volumes $V(r) = \mathrm{Vol}_c(M(r))$ of the domains of influences.
(4) Using finite differeces we compute approximate values of the derivative of the volume of the domain influences $\partial_r V(r)$.
(5) We interpolate the obtained values of $\partial_r V(r)$. This determines the approximate values of the wave speed $v(r)$ in the travel time coordinates.
(6) Finally, we change coordinates from the travel time coordinates to the Euclidean coordinates in order to obtain the approximate values of the wave speed $c(x)$ for $x \in M$.

3.2. **Publication II.** An inverse boundary value problem for the 1+1 - dimensional wave equation $(\partial_t^2 - c(x)^2\partial_x^2)u(x,t) = 0, \quad x \in \mathbb{R}_+$ is considered. We give a discrete regularization strategy recovering the wave speed $c(x)$ when we are given the boundary value of the wave, $u(0,t)$, that is produced by a single pulse-like source. The regularization strategy gives an approximative wave speed $\widetilde{c}$, satisfying a Hölder type estimate $\|\widetilde{c}-c\| \leq C\epsilon^\gamma$, where ϵ is the noise level.

The novelty in this paper is that we analyzed the effect of the discretization in the regularized solution of the inverse problem. We give a direct discrete regularization method for the non-linear inverse problem for the wave equation. The result contains an explicit (but not necessarily optimal) convergence rate. By referring to direct methods for non-linear problems we refer to the explicit construction of a non-linear map to solve the problem without resorting to a local optimization method. The advantage of direct approaches is that they do not suffer from the possibility that the algorithm converges to a local minimum. In particular, they do not require a priori knowledge that the solution is in a small neighbourhood of a given function.

The iterative regularization of both linear and non-linear inverse problems and convergence rates are discussed in a Hilbert space setting in [11, 31, 35, 60, 63] and in a Banach space setting in [34, 42, 43, 48, 74, 75, 76]. Numerical methods based on boundary control method are discussed in [25, 26, 69, 91]. In Publication II we compare our regularization strategy to Morozov's discrepancy principle (MDP).

3.2.1. *A model for a single discrete and noisy measurement.* Let $\epsilon_0 > 0$ and let us define that

$$l_0(\epsilon_0) = \lfloor \frac{4}{7}\log_2 \epsilon_0^{-1} \rfloor \tag{54}$$

and

$$N_0(\epsilon_0) = 2^{l_0}. \tag{55}$$

Let $N = 2^l \geq N_0$, where $l \in \mathbb{Z}_+$. Let P^N be as in (39) and Λ be as in (14). Let us define that

$$H(t) = \begin{cases} 1, & t \geq 0, \\ 0, & t < 0. \end{cases} \tag{56}$$

Let us also define that

$$\widetilde{m}_{N,\epsilon_0} = P^N \Lambda H + n_{N,\epsilon_0}, \tag{57}$$

in which $n_{N,\epsilon_0} \in \mathcal{P}^N$ represents the error and $\|n_{N,\epsilon_0}\|_{L^2(0,2T)} \leq \epsilon_0$. We consider the quantity $(\epsilon_0, N_0, \widetilde{m}_{N,\epsilon_0})$ that we call a measurement. Let $Z = C_b^2(M)$ and $Y = \mathcal{L}(L^2(0,2T))$. The operator $\mathcal{A}$ is defined in the domain $\mathcal{D}(\mathcal{A}) = \mathcal{V}^3$ by setting

$$\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset Z \to \mathcal{R}(\mathcal{A}) \subset Y, \quad \mathcal{A}(c) = \Lambda_c. \tag{58}$$

The notation in (58) means that the range $\mathcal{R}(\mathcal{A}) = \mathcal{A}(\mathcal{V}^3)$ and the domain $\mathcal{D}(\mathcal{A})$ are equipped with the topologies of Y and Z respectively.

Let $\mathcal{A}$ be as in (58) and H be as in (56). We define that

$$\mathcal{A}_0 : \mathcal{D}(\mathcal{A}_0) \subset Z \to \mathcal{R}(\mathcal{A}_0) \subset L^2(0,2T), \quad \mathcal{A}_0(c) = \mathcal{A}(c)H = \Lambda H, \tag{59}$$

where $\mathcal{D}(\mathcal{A}_0) = \mathcal{V}^3$. Our main result for the reconstruction of $c(x)$ from the measurement $(\epsilon_0, N_0, \widetilde{m}_{N,\epsilon_0})$ is given by the following theorem.

Theorem 2. *For the operator $\mathcal{A}_0 : V^3 \subset Z \to L^2(0,2T)$, there exists an admissible regularization strategy $\mathcal{R}^{(0)}_{N_0,\alpha_0}$ with the choice of parameter*

$$\alpha_0(\epsilon_0) = a_0 \epsilon_0^{\frac{4}{45}},$$

satisfying the following: For every $c \in \mathcal{V}^3$ there are $\widetilde{\epsilon}_0 > 0$, $a_0 > 0$, and $C > 0$ such that

$$\sup\Big\{ \Big\|\mathcal{R}^{(0)}_{N_0,\alpha_0}\widetilde{m}_{N,\epsilon_0} - c\Big\|_Z : \widetilde{m}_{N,\epsilon_0} \in \mathcal{P}^N, N = 2^l \geq N_0(\epsilon_0),$$
$$\big\|\widetilde{m}_{N,\epsilon_0} - P^N \Lambda H\big\|_{L^2(0,2T)} \leq \epsilon_0 \Big\} \leq C\epsilon_0^{c\gamma_0},$$

for all $\epsilon_0 \in (0, \widetilde{\epsilon}_0)$. Here, $\gamma_0 = \frac{1}{270}$ and $N_0(\epsilon_0)$ is as in (55).

An explicit bound $\widetilde{\epsilon}_0$ and the value for constant a_0 are given in the proof. The proof of Theorem 2 is given in Publication II.

3.2.2. *A model for several discrete and noisy measurements.* Let

$$\mathcal{E}_{N_1} : \mathcal{P}^{N_1} \to \mathcal{P}^{N_1}, \tag{60}$$

where $N_1 \in \mathbb{Z}_+$. Having Λ_{N_1} as in (40), we define a discrete and noisy measurement operator

$$\widetilde{\Lambda}_{N_1} : L^2(0,2T) \to L^2(0,2T), \tag{61}$$
$$\widetilde{\Lambda}_{N_1} f = \begin{cases} \Lambda_{N_1} f + \mathcal{E}_{N_1} f, & f \in \mathcal{P}^{N_1}, \\ 0, & f \in (\mathcal{P}^{N_1})^{\perp}. \end{cases}$$

With data corresponding to several boundary measurements $(\epsilon_1, N_1, \widetilde{\Lambda}_{N_1})$, we get the following results with improved error estimates.

Theorem 3. *For the operator $\mathcal{A} : V^3 \subset Z \to Y$, there exists an admissible regularization strategy $\mathcal{R}^{(1)}_{N_1,\alpha_1}$ with the choice of parameter*

$$\alpha_1(\epsilon_1) = a_1 \epsilon_1^{\frac{4}{9}}$$

that satisfies the following: For every $c \in \mathcal{V}^3$ there are $\widetilde{\epsilon}_1 > 0$, $a_1 > 0$, and $C > 0$ such that

$$\sup\Big\{ \Big\|\mathcal{R}^{(1)}_{N_1,\alpha_1}\widetilde{\Lambda}_{N_1} - c\Big\|_Z : N_1 \geq \epsilon_1^{-4}, \quad \widetilde{\Lambda}_{N_1} \in \mathcal{L}(\mathcal{P}^{N_1}),$$
$$\Big\|\widetilde{\Lambda}_{N_1} - \Lambda_{N_1}\Big\|_Y \leq \epsilon_1 \Big\} \leq C\epsilon_1^{\gamma_1},$$

for all $\epsilon_1 \in (0, \widetilde{\epsilon}_1)$. Here $\gamma_1 = \frac{1}{54}$.

An explicit bound $\widetilde{\epsilon}_1$ and the value for constant a_1 are given in the proof. The proof of Theorem 3 is given in Publication II, where we give explicit choices for $\mathcal{R}^{(0)}_{N_0,\alpha_0}$ and for $\mathcal{R}^{(1)}_{N_1,\alpha_1}$. Here we give a short summary of the regularization strategy. Assume that we are given $\widetilde{\Lambda}_N \in \mathcal{L}(\mathcal{P}^{N_1}) \subset Y$, that is, the discrete Neumann-to-Dirichlet map for the unknown wave speed $c(x)$ with measurements errors. Then the regularization strategy is obtained by the following steps:

(1) Using operator $\widetilde{\Lambda}_{N_1}$ we construct a source that produces a wave such that $u^{f_{\alpha,r}}(t,x)|_{t=T}$ is close to the indicator function $1_{\mathcal{M}(r)}(x)$ of the domain of influence $\mathcal{M}(r)$.
(2) Using sources $\widetilde{f}^N_{\alpha,r}$, we approximately compute the volumes $V(r)$ of the domains of influences.
(3) Using finite differences we compute the approximate values of the derivatives of the volumes of the domain influences $\partial_r V(r)$.
(4) We interpolate the obtained values of $\partial_r V(r)$. This determines the approximate values of the wave speed $v(r)$ in the travel time coordinates.
(5) Finally, we change the coordinates from travel time coordinates to Euclidean coordinates in order to obtain the approximate values of the wave speed $c(x)$ for $x \in M$.

3.3. **Publication III.** We studied the wave equation on a bounded domain of $\mathbb{R}^m$ and on a compact Riemannian manifold M with a boundary. We assumed, that the coefficients of the wave equation are unknown but that we are given the hyperbolic Neumann-to-Dirichlet map Λ that corresponds to the physical measurements on the boundary.

In this paper we show that at a fixed time t_0, a wave can be cut off outside a suitable set. That is, if $N \subset M$ is a union of balls in the travel time metric having centers at the boundary, then we can modify a given Neumann boundary value of a wave such that, at time t_0, the modified wave is arbitrarily close to the original wave inside N, and arbitrarily small outside N. Also, at time t_0 the time derivative of the modified wave is arbitrarily small in all of M.

With the knowledge of Λ we construct a sequence of Neumann boundary values so that, at a time T, the corresponding waves converge to zero while the time derivative of the waves converges to a delta distribution. Such waves are called *artificial point sources*. The convergence of a wave takes place in the function spaces naturally related to the energy of the wave. We apply the results for inverse problems and demonstrate the focusing of the waves numerically in the one-dimensional case.

We consider the wave equation in M that is a bounded domain of $\mathbb{R}^m$, $m \geq 1$, or a compact manifold. Let $u = u^f(x,t)$ be the solution of the wave equation

$$\begin{cases} \partial_t^2 u(x,t) + \mathcal{A}u(x,t) = 0, & \text{in } M \times \mathbb{R}_+, \\ u|_{t=0} = 0, \quad \partial_t u|_{t=0} = 0, \\ \partial_\nu u|_{\partial M \times \mathbb{R}_+} = f, \end{cases} \tag{62}$$

where $\mathcal{A}$ is a self-adjoint second-order elliptic differential operator of the form

$$\mathcal{A} = -\Delta_g + \sum_{j=1}^m V_j(x)\partial_{x_j},$$

where Δ_g is the Laplace operator associated with a Riemannian metric g (see [5] for a precise definition). Moreover, $f \in L^2(\partial M \times \mathbb{R}_+)$ is a Neumann boundary value that physically corresponds to a boundary source, $u = u^f$ is the unique solution wave corresponding to the boundary source f, and ν is the interior pointing normal vector of the boundary ∂M. We assume that we are given the Neumann-to-Dirichlet map, $\Lambda f = u^f|_{\partial M \times \mathbb{R}_+}$. The map Λ corresponds to the knowledge of measurements made on the boundary of the domain, and it models the response $u^f|_{\partial M \times \mathbb{R}_+}$ of the medium to a source f, put on the boundary of M.

We show that using Λ we can find a sequence of Neumann boundary values f_i such that the wave and its time derivative at the large enough time T — that is, the pair $(u^{f_i}(\,\cdot\,,T), u_t^{f_i}(\,\cdot\,,T))$ — converge in the energy norm to $(0, \frac{1}{\mathrm{Vol}(V)}1_V)$, as $i \to \infty$. Here, $V(x)$ is the indicator function a small neighborhood of a point $\widehat{x} \in M$ and $\mathrm{Vol}(V)$ is the Riemannian volume of V in (M, g). More precisely, $\widehat{x} = \gamma_{\widehat{z},\nu}(\widehat{t})$ is a point on the normal geodesics emanating from a boundary point $\widehat{z}$.

Furthermore, when the neighborhood V converges to the point $\widehat{x}$, the limits $(0, \frac{1}{\mathrm{Vol}(V)}1_V)$ converges in suitable function space to $(0, \delta_{\widehat{x}})$, where $\delta_{\widehat{x}}$ is the Dirac delta distribution. We call the waves u^{f_i} that concentrate their energy in a small neighbourhood V of a point inside the domain the *focusing waves.* When $V \to \{\widehat{x}\}$, the waves $u^{f_i}(x,t)$ in the set $M \times (T, \infty)$ converge to $G(x,t;\widehat{x},T)$, where Green's function $G(x,t;x_0,t_0)$ is the solution of

$$\begin{cases} \left(\partial_t^2 - \mathcal{A}\right) G(x,t;x_0,t_0) = \delta_{x_0}(x)\delta_{t_0}(t) \quad \text{on } M \times \mathbb{R} \\ G(\cdot,\cdot;x_0,t_0)|_{t<t_0} = 0;\ \partial_\nu G(\cdot,\cdot;x_0,t_0)|_{\partial M \times \mathbb{R}} = 0. \end{cases} \tag{63}$$

Roughly speaking, the waves u^{f_i} in the set $M \times (T, \infty)$ converge to the wave that is produced at a point source located at $(\widehat{x}, T)$. Due to this, we say that when $V \to \{x\}$, the limit of the focusing waves produces an artificial point source at time $t = T$.

We emphasize that the boundary sources f_i that produce focusing waves can be determined without knowing the coefficients of the operator $\mathcal{A}$, that is, when the medium in M is unknown, and it is enough to only know the map Λ that corresponds to measurements done on the boundary of the domain. Our main result is the following:

Theorem 4. *Let $T > \frac{1}{2}\mathit{diam}\ (M)$ and $\widehat{x} = \gamma_{\widehat{z},\nu}(\widehat{t}) \in M$, $\widehat{z} \in \partial M$, $0 < \widehat{t} < T$. Let $\tau_{\partial M}(\widehat{z})$ be the critical distance along the normal geodesic $\gamma_{\widehat{z},\nu}$ (for a definition, see Publication III, eq. (7)).*

Then $(\partial M, g|_{\partial M})$ and the Neumann-to-Dirichlet map Λ determine Neumann boundary values $f_n(\alpha,\beta,k)$, $n,k \in \mathbb{Z}_+$, $\alpha,\beta > 0$, such that the following is true:

If $\widehat{t} < \tau_{\partial M}(\widehat{z})$ then

$$\lim_{\alpha\to 0^+} \lim_{\beta\to 0^+} \lim_{n\to\infty} \begin{pmatrix} u^{f_n(\alpha,\beta,k)}(\cdot,T) \\ \partial_t u^{f_n(\alpha,\beta,k)}(\cdot,T) \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{\mathit{Vol}(\Omega_k)} 1_{\Omega_k} \end{pmatrix} \tag{64}$$

in $H_0^1(M) \times L^2(M)$, *where* $\Omega_k \subset M$ *are neighborhoods of* $\widehat{x}$ *satisfying* $\lim_{k\to\infty} \Omega_k = \{\widehat{x}\}$. *Moreover,*

$$\lim_{k\to\infty}\left(\lim_{\alpha\to 0^+}\lim_{\beta\to 0^+}\lim_{n\to\infty}\begin{pmatrix} u^{f_n(\alpha,\beta,k)}(\cdot,T) \\ \partial_t u^{f_n(\alpha,\beta,k)}(\cdot,T)\end{pmatrix}\right) = \begin{pmatrix} 0 \\ \delta_{\widehat{x}}\end{pmatrix}, \tag{65}$$

where the inner limits with respect to n, β, α *are in* $H_0^1(M) \times L^2(M)$ *and the outer limit with respect to* k *is in the space* $H^{-s+1}(M) \times H^{-s}(M)$ *with* $s > \dim(M)/2$. *In addition, for* $t > T$

$$\lim_{k\to\infty}\left(\lim_{\alpha\to 0^+}\lim_{\beta\to 0^+}\lim_{n\to\infty} u^{f_n(\alpha,\beta,k)}(\,\cdot\,,t)\right) = G(\,\cdot\,,t;\widehat{x},T) \tag{66}$$

where the inner limits with respect to n, β, α *are in* $H_0^1(M)$ *and the outer limit with respect to* k *is in the space* $H^{-s+1}(M)$.

If $\widehat{t} > \tau_{\partial M}(\widehat{z})$, *then limits depicted in* (64), (65), *and* (66) *are equal to zero.*

The boundary sources $f_n(\alpha, \beta, k)$ in Theorem 4, that produce an artificial point source, are obtained using an iterative sequence of measurements. In this iteration, for $n = 1$ we first measure the boundary value, Λf_1, of the wave that is produced by a certain boundary source f_1. In each iteration step, we use the boundary source f_n and its response Λf_n to compute the boundary source f_{n+1} for the next iteration step. The iteration algorithm in this paper was inspired by time reversal methods (see [2, 3, 14, 13, 29, 30, 41, 62, 68, 70]). We note that when the traditional time-reversal algorithms are used in imaging, one typically needs to assume that the medium contains some point-like scatterers.

Generally, when the coefficients of the operator $\mathcal{A}$ are unknown, one can not specify the Euclidean coordinates of the point $\widehat{x}$ to which the waves focus, only the Riemannian boundary normal coordinates $(\widehat{z}, \widehat{t})$ (also called *the ray coordinates* in optics or *the migration coordinates* in Earth sciences) of $\widehat{x}$ can be specified. However, in the case when $M \subset \mathbb{R}^m$ and the operator $\mathcal{A}$ is of the form $\mathcal{A} = -c(x)^2\Delta$, we show in Publication III, Corollary 4.3, that the Euclidean coordinates of the point $\widehat{x}$ can be computed using the Neumann-to-Dirichlet map Λ.

The problem studied in the paper is motivated by recent advances in the applications of optimal control methods to *lithotripsy* and *hyperthermia*. In lithotripsy, one breaks down a kidney or bladder stone using a focusing ultrasonic wave. Likewise, in *hyperthermia* medical treatments, cancer tissue is destroyed by ultrasound induced heating that produces an excessive heat dose generated by a focusing wave [61]. Often, to apply these methods one needs to use other physical imaging modalities, for example X-rays tomography of MRI, to estimate the material parameters in M. However, for the wave equation there are various methods to estimate material parameters using the boundary measurements of waves. These methods are, however, quite unstable [1, 46]. Therefore they might not be suitable for hyperthermia, where safety is crucial. An important question is therefore how to focus waves in unknown media.

In the paper we further advance the techniques developed in [9] and [19]. In [19], a construction of focusing waves was considered in the analogous setting to this paper, but using the function space $L^2(M) \times L^2(M)$ instead of the natural energy space $H_0^1(M) \times L^2(M)$ depicted in (64). The use of the function space associated with energy makes it possible to concentrate the energy of the wave near a single point. For instance, in the above ultrasound induced heating problem, the use of the correct energy norm is crucial as otherwise the energy of the wave may not be concentrating at all.

The other novelties of the paper are that, in the case of an isotropic medium, that is, with the operator $\mathcal{A} = -c(x)^2\Delta$, we can focus the wave near a point $\widehat{x}$ for which Euclidean coordinates can be computed (a posteriori). We apply this to an inverse problem, that is, to determining the wave speed in the unknown medium.

The methodology in this paper arises from boundary control methods used to study inverse problems in hyperbolic equations [1, 4, 5, 47, 44] and on the focusing of waves for non-linear equations [22, 28, 33, 50, 52, 56, 90]. Similar problems have been studied using geometrical optics [71, 72, 73, 82] and the methods of scattering theory [18], see also the reviews of these methods in [88, 89]. For distance functions (see [15, 17, 21, 40, 58, 80]).

In particular, Theorem 4 provides an analogous construction of the artificial point sources for linear equations that is developed in [52] for non-linear hyperbolic problems with a time-dependent metric. We note that this technique is used as a surprising example of how the inverse problems for non-linear equations are sometimes easier than for the corresponding problems for linear equations. Thus Theorem 4 shows that some tools that are developed for inverse problems for non-linear equations can be generalized for linear equations.

Let us describe the key features of the algorithm in this paper. First, to focus a wave onto a point $\widehat{x}$ inside the media, we assume that *(i)* $\widehat{x}$ is on a normal geodesic from Γ that is distance minimizing and *(ii)* the travel time coordinates of $\widehat{x}$ are known from the boundary. Let us emphasize that we do not assume the medium in $\mathcal{M}$ to be known. However, if the medium is known, then condition *ii* always holds. Condition *ii* also means that focusing can be done with the same coordinates with which imaging is done. Thus, as the algorithm for focusing does not rely on media parameters obtained from imaging, errors in imaging do not accumulate into errors in focusing. Second, the algorithm can focus a wave onto an area having no scatterer. Third, the algorithm is computationally cheap. In a sense, all computations are done by the media; there is no need to solve the wave equation (cf. [39]). We will assume that the medium is linear, non-dispersive, non-dissipative, frequency-independent, and depends smoothly on location. However, we do not need any other approximations (like single scattering approximations) to prove that the algorithm works.

A limitation of the present algorithm is that we assume the self-adjointness of operator $\mathcal{A}$ and that time T, when the wave focuses, is large enough. We also impose the above geometric conditions on $\widehat{x}$.

The present work is a continuation of [9] where a similar iterative scheme was introduced, for which $u(T)$ focuses on a delta distribution, but the time derivative $u_t(T)$ was uncontrolled.

The present work can also be seen as also a generalization of so-called retrofocusing in control theory, where the aim is to produce boundary sources giving the same final state as the boundary sources sent before in the medium (see [41, 51]).

The outline of this work is as follows. In Publication III, Section 2, we introduced notation, boundary control operators, and review some relevant results from control theory. In Section 3 we stated and described the minimization problem for the boundary sources. In Section 4, we discussed the focusing of the waves and proved Theorem 1.1. In Section 5 we introduced the modified iteration time-reversal scheme to generate boundary sources using an iteration of simple operators and boundary measurements. In Section 6 we presented the results of the numerical experiment. In Section 7 we applied the results to inverse problems.

References

1. M. Anderson, A. Katsuda, Y. Kurylev, M. Lassas, and M. Taylor, *Boundary regularity for the ricci equation, geometric convergence, and gel'fand's inverse boundary problem*, Invent. Math. **158** (2004), 261–321.
2. G. Bal and L.Ryzhik, *Time reversal and refocusing in random media*, SIAM J. Appl. Math **63** (2003), 1475–1498.
3. G. Bal and O. Pinaud, *Time reversal based detection in random media*, Inverse Problems **21** (2005), 1593–1620.
4. M. Belishev, *An approach to multidimensional inverse problems for the wave equation. (russian)*, Dokl. Akad. Nauk SSSR **297** (1987), 524–527.
5. M. Belishev and Y. Kurylev, *To the reconstruction of a riemannian manifold via its spectral data (bc-method).*, Comm. Partial Differential Equations **17** (1992), 767–804.
6. M. I. Belishev, *Boundary control and tomography of riemannian manifolds (the bc-method)*, Russian Mathematical Surveys **72** (2017), 581–644.
7. M. I. Belishev and Ya. V. Kurylëv, *A nonstationary inverse problem for the multidimensional wave equation "in the large"*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **165** (1987), no. Mat. Vopr. Teor. Rasprostr. Voln. 17, 21–30, 189. MR 918955 (89e:35143)
8. Mourad Bellassoued and David Dos Santos Ferreira, *Stability estimates for the anisotropic wave equation from the Dirichlet-to-Neumann map*, Inverse Probl. Imaging **5** (2011), no. 4, 745–773. MR 2852371
9. K. Bingham, Y. Kurylev, M. Lassas, and S. Siltanen, *Iterative time-reversal control for inverse problems*, Inverse Problems and Imaging **2** (2008), no. 1.
10. Kenrick Bingham, Yaroslav Kurylev, Matti Lassas, and Samuli Siltanen, *Iterative time-reversal control for inverse problems*, Inverse Probl. Imaging **2** (2008), no. 1, 63–81. MR 2375323 (2009c:35470)
11. Nicolai Bissantz, Thorsten Hohage, and Axel Munk, *Consistency and rates of convergence of nonlinear Tikhonov regularization with random noise*, Inverse Problems **20** (2004), no. 6, 1773–1789. MR 2107236 (2005i:65089)
12. A. S. Blagoveščenskiĭ, *The inverse problem of the theory of seismic wave propagation*, Problems of mathematical physics, No. 1: Spectral theory and wave processes (Russian), Izdat. Leningrad. Univ., Leningrad, 1966, pp. 68–81. (errata insert). MR 0371360 (51 #7579a)
13. L. Borcea, G. Papanicolaou, and C. Tsogka, *Theory and applications of time reversal and interferometric imaging.*, Inverse Problems **19** (2003), 5139–5164.
14. L. Borcea, G. Papanicolaou, C. Tsogka, and J. Berryman, *Imaging and time reversal in random media.*, Inverse Problems **18** (2002), 1247–1279.
15. Roberta Bosi, Yaroslav Kurylev, and Matti Lassas, *Reconstruction and stability in gel'fand's inverse interior spectral problem*, (2017).
16. A. L. Bukhgeĭm and M. V. Klibanov, *Uniqueness in the large of a class of multidimensional inverse problems*, Dokl. Akad. Nauk SSSR **260** (1981), no. 2, 269–272. MR 630135 (83b:35157)
17. Dmitri Burago, Sergei Ivanov, Matti Lassas, and Jinpeng Lu, *Stability of the gel'fand inverse boundary problem via the unique continuation*, 12 2020.
18. Peter Caday, Maarten V. de Hoop, Vitaly Katsnelson, and Gunther Uhlmann, *Scattering control for the wave equation with unknown wave speed*, Arch. Ration. Mech. Anal. **231** (2019), no. 1, 409–464. MR 3894555
19. M. Dahl, A. Kirpichnikova, and M. Lassas, *Focusing waves in unknown media by modified time reversal iteration.*, SIAM Journal on Control and Optimization **48** (2009), 839–858.

20. Matias F. Dahl, Anna Kirpichnikova, and Matti Lassas, *Focusing waves in unknown media by modified time reversal iteration*, SIAM J. Control Optim. **48** (2009), no. 2, 839–858. MR 2486096 (2010d:35394)
21. Maarten de Hoop and Teemu Saksala, *Inverse problem of travel time difference functions on a compact riemannian manifold with boundary*, The Journal of Geometric Analysis **29** (2018).
22. Maarten de Hoop, Gunther Uhlmann, and Yiran Wang, *Nonlinear interaction of waves in elastodynamics and an inverse problem*, Math. Ann. **376** (2020), no. 1-2, 765–795. MR 4055177
23. Maarten V. de Hoop, Sean F. Holman, Einar Iversen, Matti Lassas, and Bjorn Ursin, *Recovering the isometry type of a riemannian manifold from local boundary diffraction travel times*, Journal de Mathematiques Pures et Appliquées **103** (2015), no. 3, 830–848 (English).
24. Maarten V. de Hoop, Sean F. Holman, Einar Iversen, Matti Lassas, and Bjørn Ursin, *Reconstruction of a conformally euclidean metric from local boundary diffraction travel times*, SIAM Journal on Mathematical Analysis **46** (2014), no. 6, 3705–3726.
25. Maarten V. de Hoop, Paul Kepley, and Lauri Oksanen, *On the construction of virtual interior point source travel time distances from the hyperbolic neumann-to-dirichlet map*, SIAM Journal on Applied Mathematics **76** (2016), no. 2, 805–825.
26. ———, *Recovery of a smooth metric via wave field and coordinate transformation reconstruction*, SIAM Journal on Applied Mathematics **78** (2018), no. 4, 1931–1953.
27. Heinz W. Engl, Martin Hanke, and Andreas Neubauer, *Regularization of inverse problems*, Mathematics and its Applications, vol. 375, Kluwer Academic Publishers Group, Dordrecht, 1996. MR 1408680 (97k:65145)
28. Ali Feizmohammadi and Yavar Kian, *Recovery of nonsmooth coefficients appearing in anisotropic wave equations*, SIAM J. Math. Anal. **51** (2019), no. 6, 4953–4976. MR 4040792
29. M. Fink, *Time-reversal acoustics in complex environments.*, Geophysics **71** (2006), SI151–SI164.
30. M. Fink, D. Cassereau, A. Derode, C. Prada, P. Roux, M. Tanter, J.-L. Thomas, and F. Wu, *Time-reversed acoustics.*, Rep. Prog. Phys. **63** (2000), 1933–1995.
31. Martin Hanke, *Regularizing properties of a truncated Newton-CG algorithm for nonlinear inverse problems*, Numer. Funct. Anal. Optim. **18** (1997), no. 9-10, 971–993. MR 1485990 (99a:65077)
32. T. Helin, M. Lassas, L. Oksanen, and T. Saksala, *Correlation based passive imaging with a white noise source*, Journal de Mathématiques Pures et Appliquées **116** (2018), 132–160.
33. Peter Hintz and Gunther Uhlmann, *Reconstruction of Lorentzian manifolds from boundary light observation sets*, Int. Math. Res. Not. IMRN (2019), no. 22, 6949–6987. MR 4032181
34. B. Hofmann, B. Kaltenbacher, C. Pöschl, and O. Scherzer, *A convergence rates result for Tikhonov regularization in Banach spaces with non-smooth operators*, Inverse Problems **23** (2007), no. 3, 987–1010. MR 2329928 (2008e:65180)
35. Thorsten Hohage and Mihaela Pricop, *Nonlinear Tikhonov regularization in Hilbert scales for inverse boundary value problems with random noise*, Inverse Probl. Imaging **2** (2008), no. 2, 271–290. MR 2395144 (2009d:47075)
36. Belishev M. I., *An approach to multidimensional inverse problems for the wave equation*, Dokl. Akad. Nauk SSSR **297** (1987), no. 3, 524–527. MR 924687 (89c:35152)

37. ______, *Recent progress in the boundary control method*, Inverse Problems **23** (2007), no. 5, R1–R67. MR 2353313 (2008h:93001)
38. Belishev M. I. and Yaroslav V. Kurylev, *To the reconstruction of a Riemannian manifold via its spectral data (BC-method)*, Comm. Partial Differential Equations **17** (1992), no. 5-6, 767–804. MR 1177292 (94a:58199)
39. D. Isaacson, *Distinguishability of conductivities by electric current computed tomography.*, IEEE Trans. on Medical Imaging **MI-5** (1986), 92–95.
40. Sergei Ivanov, *Distance difference representations of riemannian manifolds*, Geometriae Dedicata **207** (2018), 167–192.
41. B.L.G. Jonsson, M. Gustafsson, V.H. Weston, and M.V. de Hoop, *Retrofocusing of acoustic wave fields by iterated time reversal.*, SIAM J. Appl. Math. **64** (2014), 1954–1986.
42. B. Kaltenbacher and A. Neubauer, *Convergence of projected iterative regularization methods for nonlinear problems with smooth solutions*, Inverse Problems **22** (2006), no. 3, 1105–1119. MR 2235657 (2007b:65055)
43. Barbara Kaltenbacher, Andreas Neubauer, and Otmar Scherzer, *Iterative regularization methods for nonlinear ill-posed problems*, Radon Series on Computational and Applied Mathematics, vol. 6, Walter de Gruyter GmbH & Co. KG, Berlin, 2008. MR 2459012 (2010c:65001)
44. A. Katchalov, Y. Kurylev, and M. Lassas, *Inverse boundary spectral problems.*, Chapman & Hall/CRC, 2001.
45. A. Katchalov, Y. Kurylev, M. Lassas, and N. Mandache, *Equivalence of time-domain inverse problems and boundary spectral probl*, Inverse Problems **20** (2004), 419.
46. A. Katsuda, Y. Kurylev, and M. Lassas, *Stability of boundary distance representation and reconstruction of riemannian manifolds.*, Inverse Problems and Imaging **1** (2007), 135–157.
47. Yavar Kian, Morgan Morancey, and Lauri Oksanen, *Application of the boundary control method to partial data Borg-Levinson inverse spectral problem*, Math. Control Relat. Fields **9** (2019), no. 2, 289–312. MR 3924852
48. Andreas Kirsch, *An introduction to the mathematical theory of inverse problems*, Springer-Verlag New York, Inc., New York, NY, USA, 1996.
49. Katsiaryna Krupchyk, Yaroslav Kurylev, and Matti Lassas, *Inverse spectral problems on a closed manifold*, Journal de Mathématiques Pures et Appliquées **90** (2008), no. 1, 42–59.
50. Katya Krupchyk and Gunther Uhlmann, *A remark on partial data inverse problems for semilinear elliptic equations*, Proc. Amer. Math. Soc. **148** (2020), no. 2, 681–685. MR 4052205
51. Y. Kurylev and M. Lassas, *Hyperbolic inverse boundary-value problem and time-continuation of the non-stationary dirichlet-to-neumann map.*, Proc. Roy. Soc. Edinburgh Sect. A **132** (2002), 931–949.
52. Yaroslav Kurylev, Matti Lassas, and Gunther Uhlmann, *Inverse problems for lorentzian manifolds and non-linear hyperbolic equations.*, Invent. Math. **212** (2018), no. 3, 781–857.
53. Yaroslav Kurylev, Lauri Oksanen, and Gabriel P. Paternain, *Inverse problems for the connection Laplacian*, Journal of Differential Geometry **110** (2018), no. 3, 457 – 494.
54. I. Lasiecka and R. Triggiani, *Regularity theory of hyperbolic equations with non-homogeneous Neumann boundary conditions. II. General boundary data*, J. Differential Equations **94** (1991), no. 1, 112–164. MR 1133544 (93c:35016)

55. I. Lasiecka and R. Triggiani, *Regularity theory of hyperbolic equations with non-homogeneous neumann boundary conditions. ii. general boundary data.*, J. Differential Equations **94** (1991), 112–164.
56. Matti Lassas, *Inverse problems for linear and non-linear hyperbolic equations*, Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. IV. Invited lectures, World Sci. Publ., Hackensack, NJ, 2018, pp. 3751–3771. MR 3966550
57. Matti Lassas and Lauri Oksanen, *Inverse problem for the riemannian wave equation with dirichlet data and neumann data on disjoint sets*, Duke Mathematical Journal **163** (2014), no. 6, 1071–1103 (English).
58. Matti Lassas and Teemu Saksala, *Determination of a riemannian manifold from the distance difference functions*, Asian Journal of Mathematics **23** (2019), no. 2, 173–199 (English).
59. Shitao Liu and Lauri Oksanen, *A lipschitz stable reconstruction formula for the inverse problem for the wave equation*, Submitted. Preprint arXiv:1210.1094 (2012).
60. Shuai Lu, Sergei V. Pereverzev, and Ronny Ramlau, *An analysis of Tikhonov regularization for nonlinear ill-posed problems under a general smoothness assumption*, Inverse Problems **23** (2007), no. 1, 217–230. MR 2302970 (2008d:65160)
61. Matti Malinen, Tomi Huttunen, and Jari P. Kaipio, *An optimal control approach for ultrasound induced heating.*, International Journal of Control **76** (2003).
62. T.D. Mast, A.I. Nachman, and R.C. Waag, *Focusing and imaging using eigenfunctions of the scattering operator.*, J. Acoust. Soc. Am. **102** (1997), 715–725.
63. Peter Mathé and Bernd Hofmann, *How general are general source conditions?*, Inverse Problems **24** (2008), no. 1, 015009, 5. MR 2384768 (2009b:65144)
64. Jennifer L Mueller and Samuli Siltanen, *Linear and nonlinear inverse problems with practical applications*, vol. 10, Siam, 2012.
65. Adrian Nachman, John Sylvester, and Gunther Uhlmann, *An n-dimensional Borg-Levinson theorem*, Comm. Math. Phys. **115** (1988), no. 4, 595–605. MR 933457 (89g:35082)
66. Lauri Oksanen, *Solving an inverse problem for the wave equation by using a minimization algorithm and time-reversed measurements*, Inverse problems and imaging **5** (2011), no. 3, 731–744 (English).
67. Lauri Oksanen, *Inverse obstacle problem for the non-stationary wave equation with an unknown background*, Comm. Partial Differential Equations **38** (2013), no. 9, 1492–1518. MR 3169753
68. George Papanicolaou, Leonid Ryzhik, and Knut Solna, *Statistical stability in time reversal.*, SIAM J. on Appl. Math. **64** (2004), 1133–1155.
69. Leonid Pestov, Victoria Bolgova, and Oksana Kazarina, *Numerical recovering of a density by the BC-method*, Inverse Probl. Imaging **4** (2010), no. 4, 703–712. MR 2726426
70. C. Prada, J.-L. Thomas, and M. Fink, *The iterative time reversal process: Analysis of the convergence.*, J. Acoust. Soc. Am. **97** (1995), 62–71.
71. Rakesh, *A linearised inverse problem for the wave equation*, Comm. Partial Differential Equations **13** (1988), no. 5, 573–601. MR 919443 (89f:35209)
72. ———, *Reconstruction for an inverse problem for the wave equation with constant velocity*, Inverse Problems **6** (1990), no. 1, 91–98. MR 1036380 (91d:35232)
73. Rakesh and Paul Sacks, *Uniqueness for a hyperbolic inverse problem with angular control on the coefficients*, J. Inverse Ill-Posed Probl. **19** (2011), no. 1, 107–126. MR 2794398 (2012c:35480)

74. Ronny Ramlau, *Regularization properties of Tikhonov regularization with sparsity constraints*, Electron. Trans. Numer. Anal. **30** (2008), 54–74. MR 2480069 (2010c:65085)
75. Ronny Ramlau and Gerd Teschke, *A Tikhonov-based projection iteration for nonlinear ill-posed problems with sparsity constraints*, Numer. Math. **104** (2006), no. 2, 177–203. MR 2242613 (2007e:65057)
76. Elena Resmerita, *Regularization of ill-posed problems in Banach spaces: convergence rates*, Inverse Problems **21** (2005), no. 4, 1303–1314. MR 2158110 (2006d:65060)
77. Blagoveščenskiĭ A. S., *A one-dimensional inverse boundary value problem for a second order hyperbolic equation*, Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **15** (1969), 85–90. MR 0282060 (43 #7774)
78. ______, *The inverse boundary value problem of the theory of wave propagation in an anisotropic medium*, Trudy Mat. Inst. Steklov. **115** (1971), 39–56. (errata insert). MR 0298226 (45 #7278)
79. Plamen Stefanov and Gunther Uhlmann, *Stability estimates for the hyperbolic Dirichlet to Neumann map in anisotropic media*, J. Funct. Anal. **154** (1998), no. 2, 330–358. MR 1612709 (99f:35120)
80. ______, *Stable determination of generic simple metrics from the hyperbolic dirichlet-to-neumann map*, International Mathematics Research Notices (2004).
81. ______, *Stable determination of generic simple metrics from the hyperbolic Dirichlet-to-Neumann map*, International Mathematics Research Notices **2005** (2005), no. 17, 1047–1061.
82. ______, *Stable determination of generic simple metrics from the hyperbolic Dirichlet-to-Neumann map*, Int. Math. Res. Not. (2005), no. 17, 1047–1061. MR 2145709 (2006a:58030)
83. Plamen Stefanov and Gunther Uhlmann, *Recovery of a source term or a speed with one measurement and applications*, (2011).
84. John Sylvester and Gunther Uhlmann, *A global uniqueness theorem for an inverse boundary value problem*, Ann. of Math. (2) **125** (1987), no. 1, 153–169. MR 873380 (88b:35205)
85. Daniel Tataru, *Unique continuation for solutions to PDE's; between Hörmander's theorem and Holmgren's theorem*, Comm. Partial Differential Equations **20** (1995), no. 5-6, 855–884. MR 1326909 (96e:35019)
86. ______, *On the regularity of boundary traces for the wave equation*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **26** (1998), no. 1, 185–206. MR 1633000 (99e:35129)
87. ______, *Unique continuation for operators with partially analytic coefficients*, J. Math. Pures Appl. (9) **78** (1999), no. 5, 505–521. MR 1697040 (2000e:35005)
88. Gunther Uhlmann, *Inverse boundary value problems for partial differential equations*, Proceedings of the International Congress of Mathematicians, Vol. III (Berlin, 1998), no. Extra Vol. III, 1998, pp. 77–86. MR 1648142
89. ______, *The Cauchy data and the scattering relation*, Geometric methods in inverse problems and PDE control, IMA Vol. Math. Appl., vol. 137, Springer, New York, 2004, pp. 263–287. MR 2169908 (2006f:58032)
90. Yiran Wang and Ting Zhou, *Inverse problems for quadratic derivative nonlinear wave equations*, Comm. Partial Differential Equations **44** (2019), no. 11, 1140–1158. MR 3995093
91. Tianyu Yang and Y. Yang, *A non-iterative reconstruction algorithm for the acoustic inverse boundary value problem*, arXiv: Analysis of PDEs (2020).