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Neutron Oscillations and the Parity Doubling Theorem

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Abstract: We review several aspects of parity and CP violation in the framework of neutron-antineutron oscillations. We focus on the parity doubling theorem, which provides a criterion for neutron oscillation in the general theory with $\Delta B = 2$ baryon number-violating interactions. We prove by explicit calculations that the violation of the conventional parity symmetry with $P^2 = 1$ is the necessary condition for neutron oscillations to happen. While the CP violation is not manifest in the oscillation, it is nevertheless intrinsic to the system, and it is transferred, by the mixing matrix, to the neutron interactions and potentially observable as a contribution to the electric dipole moment.

Keywords: neutron-antineutron oscillations; parity of Majorana fermions; CP violation

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1. Introduction

The matter-antimatter asymmetry in the Universe is one of the major challenges of modern particle physics, driving a huge amount of theoretical and experimental work. The framework of baryogenesis was distilled in the seminal work of Sakharov [1] in 1957 and captured in three general conditions: (i) the existence of processes with baryon number violation; (ii) charge and charge-parity violation (namely C and CP violation); and (iii) charge-parity-time reversal (CPT) violation, due to the expansion of the Universe, customarily modeled by thermal nonequilibrium. By now, it is quite clear that although all these conditions can be met within the Standard Model [2], a successful baryogenesis scenario implies physics beyond the Standard Model. A recurrent idea is the involvement of oscillations—usually of neutrinos in leptogenesis schemes, but also of baryons.

The very first ideas that neutrons might oscillate into antineutrons appeared specifically in the context of baryogenesis [3–10] (for a recent review, see [11]). The neutron-antineutron ($n - \bar{n}$) oscillations violate baryon number by two units, $\Delta B = 2$, as well as $B - L$. This is a very appealing feature of $n - \bar{n}$ oscillations, because the primordial baryon asymmetry created by this process would survive non-perturbative weak interaction effects (in which $\Delta(B - L) = 0$), unlike the ones responsible for the leading modes of proton decay—the other baryon number-violating candidate process, for which the bounds are, however, very stringent. More recently, the proposal of neutron-mirror neutron ($n - n'$) oscillations has been put forward [12].

Recently, the interest in experimental searches for $n - \bar{n}$ and $n - n'$ oscillations has been strongly revived. Experimental searches for neutron-antineutron conversion have been performed both with free neutron beams and within nuclei [13–16]. At the European Spallation Source (ESS), new high-precision, world-leading neutron conversion experiments are being planned (HIBEAM leading to NNBAR), aiming at improving by three orders of magnitude or more [17–19] the best bound on the oscillation time (0.86×10^8 s at 90% C.L.) obtained at ILL-Grenoble. The phenomenon will also be investigated at the DUNE experiment [20]. Searches for neutron-mirror neutron oscillations are also under consideration at the Oak Ridge National Laboratory (ORNL) [21]. Quite interestingly, the lower bounds on baryon number-violating dinucleon decays $nn \to e^+e^-$ and $nn \to \mu^+\mu^-$

obtained from $\bar{p}p$ and e^+e^- annihilation data are, at the moment, considerably stronger than those obtained from direct experimental searches [22].

To be an efficient ingredient for baryogenesis, neutron oscillations have to somehow involve C and CP violations. The subject has been thoroughly investigated recently in a series of papers [23–31]. In this paper, we review some general results on parity and CP symmetry in neutron-antineutron conversion.

2. Parity and CP Symmetry in Neutron-Antineutron Conversion

To fix the ideas and conventions, we start from the quadratic effective hermitian Lagrangian for the neutron field $n(x)$ with general small $\Delta B = 2$ terms added:

$$\begin{aligned}\mathcal{L} &= \overline{n}(x)i\gamma^\mu\partial_\mu n(x) - m\overline{n}(x)n(x) + \overline{n}(x)(i\gamma_5\delta m)n(x)\\ &- \frac{1}{2}[m_1 n^T(x)Cn(x) + m_1^\dagger\overline{n}(x)C\overline{n}^T(x)]\\ &- \frac{1}{2}[m_5 n^T(x)C\gamma_5 n(x) - m_5^\dagger\overline{n}(x)C\gamma_5\overline{n}^T(x)],\end{aligned} \tag{1}$$

where m and δm are real positive parameters, while m_1 and m_5 are complex parameters. The Majorana-type of mass terms in (1) break the baryon number symmetry. Our notational conventions follow [32]; in particular, the charge conjugation matrix is defined by $C = i\gamma^2\gamma^0$. The only assumption is that the magnitudes of $|m_1|$ and $|m_5|$ are very small compared to the neutron mass m. It is known [11] that the main aspects of the possible neutron-antineutron oscillations are described by the above Lagrangian.

In theories where the fermion number is conserved, the discrete symmetry operations can be generalized by introducing phase freedom [32,33]. We emphasize that any parity operator defined by including a phase factor is *generally* broken in the generic baryon number-violating Lagrangian (1). However, a properly defined parity is important for the analysis of neutron oscillation, and we call the most conventional definition "γ^0 parity". It turns out that the γ^0 parity violation is an indicator of the mass splitting of Majorana-type fermions and thus of neutron oscillation. The mass eigenvalues, which are exactly solved physical quantities, as well as the functional determinants in path integrals are obviously independent of the definitions of parity. The crucial property is that γ^0 parity provides decisive information about the presence or absence of neutron oscillation. The parity of operators and states is not uniquely specified, particularly in the theory with fermion number violation [33].

The "γ^0 parity" satisfies $P^2 = 1$. It is well-known that the parity of a single Majorana particle is defined by "$i\gamma^0$ parity" [33] and thus $P^2 = 1$ is not maintained, as is seen for the case of a single real Majorana fermion in the original Majorana representation, where γ^0 is hermitian but purely imaginary, and thus, γ^0 parity cannot be defined by keeping the reality of the fermion.

Let us see what the implications of γ^0 parity are for neutron oscillation in connection with the definition of intrinsic parity for the Majorana fermion. One may represent a Dirac fermion $\Psi(x)$ in terms of two Majorana fermions $\psi_1(x)$ and $\psi_2(x)$, namely,

$$\Psi(x) = \frac{1}{\sqrt{2}}[\psi_1(x) + \psi_2(x)],\ \ \Psi^c(x) \equiv \frac{1}{\sqrt{2}}[\psi_1(x) - \psi_2(x)], \tag{2}$$

where

$$\Psi^c(x) = C\overline{\Psi}^T(x),\quad C = i\gamma^2\gamma^0. \tag{3}$$

The Dirac fermion forms a doublet $\{\Psi(x), \Psi^c(x)\}$ under the charge conjugation. Under the γ^0 parity, one has

$$\Psi^p(x) = \gamma^0\Psi(t, -\vec{x}),\ \ \Psi^{cp}(x) = -\gamma^0\Psi^c(t, -\vec{x}), \tag{4}$$

namely, a Dirac fermion can be an eigenstate of γ^0 parity.

On the other hand, one can write Majorana fermions as

$$\psi_1(x) = \frac{1}{\sqrt{2}}[\Psi(x) + \Psi^c(x)], \;\; \psi_2(x) = \frac{1}{\sqrt{2}}[\Psi(x) - \Psi^c(x)]. \tag{5}$$

Under the charge conjugation, $\psi_1^c = \psi_1$ and $\psi_2^c = -\psi_2$; namely, they are the exact eigenstates of C. Under the γ^0 parity we have, using (4),

$$\begin{aligned}\psi_1^p(x) &= \frac{1}{\sqrt{2}}[\gamma_0\Psi(t,-\vec{x}) - \gamma_0\Psi^c(t,-\vec{x})] = \gamma^0\psi_2(t,-\vec{x}),\\ \psi_2^p(x) &= \frac{1}{\sqrt{2}}[\gamma_0\Psi(t,-\vec{x}) + \gamma_0\Psi^c(t,-\vec{x})] = \gamma^0\psi_1(t,-\vec{x}),\end{aligned} \tag{6}$$

where they form *a doublet* $\{\psi_1(x), \psi_2(x)\}$. When the masses of the two Majorana particles are *degenerate*, these symmetry operations are consistent, and $C^2 = 1$ and $P^2 = 1$; namely, the eigenvalues of those symmetry operators are $C = \pm 1$ and $P = \pm 1$. However, a single Majorana particle, which is an eigenstate of C, cannot be an eigenstate of γ^0 parity, and a modified definition of parity is required [33], which we call "$i\gamma^0$ parity" and specify by the replacement rule

$$\Psi(x) \to i\gamma^0\Psi(t,-\vec{x}), \quad \Psi^c(x) \to i\gamma^0\Psi^c(t,-\vec{x}) \tag{7}$$

in the present context. In this case, using again (4),

$$\begin{aligned}\psi_1^p(x) &= \frac{1}{\sqrt{2}}[i\gamma_0\Psi(t,-\vec{x}) + i\gamma_0\Psi^c(t,-\vec{x})] = i\gamma^0\psi_1(t,-\vec{x}),\\ \psi_2^p(x) &= \frac{1}{\sqrt{2}}[i\gamma_0\Psi(t,-\vec{x}) + i\gamma_0\Psi^c(t,-\vec{x})] = i\gamma^0\psi_2(t,-\vec{x}).\end{aligned} \tag{8}$$

This modified parity operator satisfies $P^2 = -1$.

In neutron-antineutron oscillation, one encounters the neutron expressed by a linear combination of two Majorana-type particles with different masses. One thus recognizes that the above doublet structure under γ^0-parity operation (6) is not consistent; this doublet structure persists even if Majorana-type particles are not the eigenstates of charge conjugation. This shows that γ^0-parity violation (i.e., the violation of the conventionally defined intrinsic parity) is a *necessary condition* of neutron oscillation. We demonstrate this fact in greater detail in the following. It is important that this γ^0-parity violation is readily recognized in the effective Lagrangian for the neutron.

Having a choice among different definitions of parity, the question appears: which one is the "good" one? In the case when the Lagrangian is invariant under a specific choice of the parity operator, say, $e^{i\xi}\gamma_0$, one can conclude that parity is conserved. If the Lagrangian is not invariant under the action of any parity operator, then parity is physically violated.

The Lagrangian (1) breaks mirror symmetry no matter how we choose the parity operator; consequently, from this point of view, one cannot say that γ_0 or $i\gamma_0$ parity is more relevant for analysing it. One may have a preference for the $i\gamma_0$ parity in view of the fact that this is the natural choice for Majorana fields [30]. However, in [27], we proved in detail that the γ_0 parity provides a *criterion* for discriminating between the parameter choices in (1) which lead to oscillations or do not lead to oscillations. We called this a "parity doubling theorem" [27], stating that the γ_0-parity violation of the Lagrangian written in terms of the neutron fields $n(x)$ is a necessary condition for neutron-antineutron oscillations. If the γ_0-parity is not violated, baryon number-violating conversion of neutron to antineutron is still possible, but the effect could, in principle, be observed only in a medium. We shall clarify this statement below by considering some explicit solutions.

3. Explicit Solutions

Neutron oscillation is a subtle phenomenon, and thus, it is useful to explicitly solve our quadratic Lagrangian, which is regarded as describing asymptotic fields. γ^0 parity has a major role in distinguishing between oscillation and the lack of it; therefore, we shall split the analysis with respect to the γ^0 parity transformation properties of the baryon number-violating term.

3.1. γ^0-Parity Violating ($i\gamma_0$-Parity and CP Conserving) Case

We analyze the hermitian Lorentz invariant local Lagrangian

$$\mathcal{L} = \overline{n}(x)i\gamma^\mu\partial_\mu n(x) - m\overline{n}(x)n(x) - \frac{1}{2}\epsilon_1[e^{-i\alpha}n^T(x)Cn(x) - e^{i\alpha}\overline{n}(x)C\overline{n}^T(x)], \tag{9}$$

namely the case when we take in (1) $m_1 = e^{i\alpha}\epsilon_1$ and fix $\delta m = 0$, $m_5 = 0$. We are allowed to rephase the field $n(x)$, such that the Lagrangian becomes

$$\mathcal{L} = \overline{n}(x)i\gamma^\mu\partial_\mu n(x) - m\overline{n}(x)n(x) - \frac{1}{2}\epsilon_1[n^T(x)Cn(x) - \overline{n}(x)C\overline{n}^T(x)]. \tag{10}$$

This is the customarily encountered version of the neutron-antineutron oscillation Lagrangian. It preserves C and $i\gamma_0$ parity, while γ_0 parity is broken. We can therefore conclude that, physically, all the discrete symmetries (C, P, CP) are conserved.

We obtain the equations of motion from (9):

$$\begin{aligned}&[i\gamma^\mu\partial_\mu - m]n(x) - \epsilon_1 n^c(x) = 0,\\ &[i\gamma^\mu\partial_\mu - m]n^c(x) - \epsilon_1 n(x) = 0,\end{aligned} \tag{11}$$

with $n^c = C\overline{n}^T$, which are rewritten as

$$[i\gamma^\mu\partial_\mu - m](n(x) \pm n^c(x)) \mp \epsilon_1(n(x) \pm n^c(x)) = 0. \tag{12}$$

We define the combinations

$$\psi_\pm(x) = \frac{1}{2}[n(x) \pm n^c(x)], \tag{13}$$

which satisfy Dirac equations with *different masses*,

$$[i\gamma^\mu\partial_\mu - (m \pm \epsilon_1)]\psi_\pm(x) = 0. \tag{14}$$

We thus have

$$\begin{aligned}n(x) &= \psi_+(x) + \psi_-(x),\\ n^c(x) &= \psi_+(x) - \psi_-(x).\end{aligned} \tag{15}$$

By comparing the definition of $n^c(x) = \psi_+^c(x) + \psi_-^c(x)$ with the second expression in (15), we obtain

$$\psi_\pm^c(x) = \pm\psi_\pm(x), \tag{16}$$

showing that $\psi_+(x)$ and $\psi_-(x)$ are Majorana fields with different masses, and γ^0 parity is broken, as in (6).

3.2. γ^0-Parity and CP Conserving Case

We next analyze the hermitian Lorentz invariant local Lagrangian

$$\begin{aligned}\mathcal{L} &= \overline{n}(x)i\gamma^{\mu}\partial_{\mu}n(x) - m\overline{n}(x)n(x) \\ &- \frac{i}{2}\epsilon_5[e^{-i\alpha}n^T(x)C\gamma_5 n(x) + e^{i\alpha}\overline{n}(x)C\gamma_5\overline{n}^T(x)];\end{aligned} \tag{17}$$

namely, we take in the Lagrangian (1) $m_1 = 0$, $m_5 = i\epsilon_5 e^{-i\alpha}$. We can again rephase the neutron field and remove the phase α, which leads to

$$\begin{aligned}\mathcal{L} &= \overline{n}(x)i\gamma^{\mu}\partial_{\mu}n(x) - m\overline{n}(x)n(x) \\ &- \frac{i}{2}\epsilon_5[n^T(x)C\gamma_5 n(x) + \overline{n}(x)C\gamma_5\overline{n}^T(x)].\end{aligned} \tag{18}$$

The Lagrangian (18) preserves γ_0 parity, as well as C and CP symmetries. Still, the baryonic number symmetry is broken.

We obtain the equations of motion from (18):

$$\begin{aligned}&[i\gamma^{\mu}\partial_{\mu} - m]n(x) - i\epsilon_5\gamma_5 n^c(x) = 0, \\ &[i\gamma^{\mu}\partial_{\mu} - m]n^c(x) - i\epsilon_5\gamma_5 n(x) = 0,\end{aligned} \tag{19}$$

with $n^c(x) \equiv C\bar{n}^T(x)$. The equation (19) is solved by rewriting it as

$$[i\gamma^{\mu}\partial_{\mu} - m](n(x) \pm n^c(x)) \mp i\epsilon_5\gamma_5(n(x) \pm n^c(x)) = 0 \tag{20}$$

and defining

$$m \pm i\epsilon_5\gamma_5 = Me^{\pm 2i\phi\gamma_5} \tag{21}$$

with

$$M = \sqrt{m^2 + \epsilon_5^2} \tag{22}$$

and

$$\sin 2\phi \equiv \epsilon_5/\sqrt{m^2 + \epsilon_5^2}. \tag{23}$$

Namely, we have

$$[i\gamma^{\mu}\partial_{\mu} - M]e^{\pm i\phi\gamma_5}(n(x) \pm n^c(x)) = 0. \tag{24}$$

We thus identify the combinations

$$\begin{aligned}\psi_+ &= \frac{1}{2}e^{i\phi\gamma_5}(n(x) + n^c(x)), \\ \psi_- &= \frac{1}{2}e^{-i\phi\gamma_5}(n(x) - n^c(x)),\end{aligned} \tag{25}$$

which satisfy the standard Dirac equation

$$[i\gamma^{\mu}\partial_{\mu} - M]\psi_{\pm} = 0. \tag{26}$$

One can confirm that

$$\psi^p_{\pm}(x^0, \vec{x}) = \gamma^0\psi_{\mp}(x^0, -\vec{x}), \tag{27}$$

and the consistent doublet representation of γ^0 parity is with $P^2 = 1$. Thus we have the exact solutions of the field equations (19),

$$n(x) = [e^{-i\phi\gamma_5}\psi_+(x) + e^{i\phi\gamma_5}\psi_-(x)],$$
$$n^c(x) = [e^{-i\phi\gamma_5}\psi_+(x) - e^{i\phi\gamma_5}\psi_-(x)]. \tag{28}$$

One defines $N_\pm(x)$ with a shifted mass $M = \sqrt{m^2 + \epsilon_5^2}$ by

$$N_\pm(x) \equiv \psi_+(x) \pm \psi_-(x), \tag{29}$$

which have γ^0 parity properties of the Dirac fermion $N_\pm^p(x) = \pm\gamma^0 N_\pm(x^0, -\vec{x})$ and $N_+^c = N_-$. Then, one can rewrite (28) as

$$n(x) = \cos\phi N_+(x) - \sin\phi(i\gamma_5)N_-(x),$$
$$n^c(x) = \cos\phi N_-(x) - \sin\phi(i\gamma_5)N_+(x). \tag{30}$$

It is natural to assume that weak interactions are described in terms of flavor fields $n(x)$ and $n^c(x)$. In the present case, we have *no oscillation* because of the degeneracy of the masses of the fields $\psi_+(x)$ and $\psi_-(x)$ in (26), but the expression in (30) shows that one observes both the decay $N_+ \to p + e^- + \bar{\nu}_e$ and the decay $N_+ \to \bar{p} + e^+ + \nu_e$ through a small mixture of $n^c(x)$. Additionally, the pair annihilation of the neutron takes place when $N_+(x)$ collides with bulk matter. This picture is consistent with the off-shell propagator from the neutron to the anti-neutron

$$\langle T^\star n(x)\overline{n^c}(y)\rangle = \frac{1}{(i\gamma^\mu\partial_\mu)^2 - M^2 + i\epsilon_5}\epsilon i\gamma_5\delta(x-y). \tag{31}$$

The difference in physical implications of the presence of oscillation and its absence is that, if the oscillation should take place, the decay $n \to \bar{p} + e^+ + \nu_e$, for example, would happen exclusively if one observes the neutron at the proper moment of complete oscillation, while we do not have any such "bunching effect" without the oscillation.

Incidentally, the propagator (31) signals anomalous canonical quantization relations for the neutron field $n(x)$ and its charge conjugate. Using the Bjorken–Johnson–Low procedure, we established them in [28]:

$$\begin{aligned} \delta(x^0 - y^0)\{n(t,\vec{x}), \overline{n}(t,\vec{y})\} &= \gamma^0\delta^4(x-y), \\ \delta(x^0 - y^0)\{\partial_{x^0} n(t,\vec{x}), \overline{n}(t,\vec{y})\} &= (-\gamma^k\partial_k - iM\cos 2\phi)\delta^4(x-y). \end{aligned} \tag{32}$$

This modification has an important physical implication; namely, the neutron decays through the baryon number-violating channels into two modes as specified above, even in the case where the oscillation between the neutron and antineutron is absent due to the degenerate Majorana fermion masses.

3.3. P and CP Violating Case

If we consider again the general Lagrangian (1), we can make a chiral transformation to remove the parity-violating mass term containing δm and a rephasing of the field $n(x)$, which brings it to the form:

$$\begin{aligned} \mathcal{L} &= \overline{n}(x)i\gamma^\mu\partial_\mu n(x) - m\overline{n}(x)n(x) \\ &- \frac{i\epsilon_1}{2}[e^{i\alpha}n^T(x)Cn(x) + e^{-i\alpha}\overline{n}(x)C\overline{n}^T(x)] \\ &- \frac{i\epsilon_5}{2}[n^T(x)C\gamma_5 n(x) + \overline{n}(x)C\gamma_5\overline{n}^T(x)], \end{aligned} \tag{33}$$

where $m, \epsilon_1, \epsilon_5, \alpha$ are real parameters. There is no possibility to simplify the Lagrangian more if we have started with a general set of parameters.

We assume the charge conjugation transformation as in (3). For the parity analysis of (33), one can use either γ_0 parity, namely

$$n(x) \to n^p(x) = \gamma_0\, n(t, -\vec{x}), \quad n^c(x) \to (n^c)^p(x) = -\gamma_0\, n^c(t, -\vec{x}),$$

or the $i\gamma_0$ parity transformation, namely

$$n(x) \to n^p(x) = i\gamma_0\, n(t, -\vec{x}), \quad n^c(x) \to (n^c)^p(x) = i\gamma_0\, n^c(t, -\vec{x}).$$

The first line in (33) corresponding to a massive Dirac field is invariant under either γ_0 or $i\gamma_0$ parity. The terms containing the parameter ϵ_1 are even under $i\gamma_0$ parity, but odd under γ_0 parity. The terms containing ϵ_5, on the other hand, are even under γ_0 parity, but odd under $i\gamma_0$ parity. Thus, it is confirmed that the Lagrangian (33) breaks γ_0 parity as well as $i\gamma_0$ parity.

Under charge conjugation (3), the terms containing ϵ_5 are even, while those containing ϵ_1 are not. If $\alpha = 0$, then the terms with ϵ_1 are odd under C-transformation, and even under CP (with P defined as $i\gamma_0$ parity). As a result, in the Lagrangian (33), CP is necessarily broken, irrespective of the definition of parity, unless α is *fixed* to 0.

There is still a more intuitive way to see that the CP violation of the Lagrangian (33) cannot be eliminated. We can bring the Lagrangian to the form (for details, see [27]):

$$\begin{aligned} \mathcal{L} &= \overline{N}(i\,\partial\!\!\!/ - M)N - i\epsilon_1 \sin\alpha \sin 2\phi \overline{N}\gamma_5 N \\ &- (i/2)\epsilon_1 e^{i\tilde{\alpha}} \sqrt{1 - (\sin\alpha \sin 2\phi)^2}\, \overline{N^c} N + h.c., \end{aligned} \tag{34}$$

where $\sin\tilde{\alpha} = \sin\alpha \cos 2\phi / \sqrt{1 - (\sin\alpha \sin 2\phi)^2}$, $M = \sqrt{m^2 + \epsilon_5^2}$ and $\sin\phi$ is as in (23). This is achieved by performing the transformation (30) in (33), followed by setting $N_+ = N$. The Lagrangian (34) is clearly P and CP violating through the γ_5 mass term, meaning that it actually contributes to the neutron electric dipole moment. Consequently, this form of the Lagrangian has still another merit; namely, it shows that the *CP violation is not connected to the* $\Delta B = 2$ *processes, though it may appear in a* $\Delta B = 2$ *Lagrangian term*. As a result, we can anticipate that the neutron-antineutron oscillation is CP conserving, even in the presence of CP-violating Lagrangian terms. We shall return to this aspect soon.

Finally, the Lagrangian (33) is brought to the diagonal form (see [27])

$$\mathcal{L} = (1/2)\bar{\Phi}_+(x)[i\,\partial\!\!\!/ - M_+]\Phi_+(x) + (1/2)\bar{\Phi}_-(x)[i\,\partial\!\!\!/ - M_-]\Phi_-(x), \tag{35}$$

where $\Phi_\pm(x)$ are Majorana fields and

$$M_\pm = \left([M \pm \epsilon_1 \sqrt{1 - (\tilde{\epsilon}_1/\epsilon_1)^2}]^2 + (\tilde{\epsilon}_1)^2\right)^{1/2}. \tag{36}$$

Here, $\tilde{\epsilon}_1 \equiv \epsilon_1 \sin\alpha \sin 2\phi$, with $\sin\phi$ given by (23).

The flavour neutron field $n(x)$ and its charge conjugate are expressed in terms of the mass eigenfields by

$$\begin{aligned} n(x) &= (1/\sqrt{2})[\cos\phi e^{-i\tilde{\alpha}/2} + \gamma_5 \sin\phi e^{i\tilde{\alpha}/2}]e^{-i\theta_+\gamma_5}\Phi_+(x) \\ &+ (1/\sqrt{2})[\cos\phi e^{-i\tilde{\alpha}/2} - \gamma_5 \sin\phi e^{i\tilde{\alpha}/2}]e^{-i\theta_-\gamma_5}\Phi_-(x), \\ n^c(x) &= (-i/\sqrt{2})[\cos\phi e^{i\tilde{\alpha}/2} - \gamma_5 \sin\phi e^{-i\tilde{\alpha}/2}]e^{-i\theta_-\gamma_5}\Phi_+(x) \\ &+ (i/\sqrt{2})[\cos\phi e^{i\tilde{\alpha}/2} + \gamma_5 \sin\phi e^{-i\tilde{\alpha}/2}]e^{-i\theta_+\gamma_5}\Phi_-(x), \end{aligned} \tag{37}$$

where $\theta_\pm$ are given by the relation

$$(M \pm \epsilon_1 \sqrt{1 - (\tilde{\epsilon}_1/\epsilon_1)^2}) + i\tilde{\epsilon}_1\gamma_5 \equiv M_\pm e^{2i\theta_\pm\gamma_5} \tag{38}$$

and

$$\sin\tilde{\alpha} = \sin\alpha\cos 2\phi/\sqrt{1-(\sin\alpha\sin 2\phi)^2}. \tag{39}$$

In this general parity violating case, the values of the two mass eigenvalues $M_\pm$ are always different (as long as $\epsilon_1 \neq 0$) and oscillations take place. The formulas become much simpler for:

(i) $\alpha = 0$ (CP-even case): $M_\pm = \sqrt{m^2+\epsilon_5^2} \pm \epsilon_1$;

(ii) $\alpha = -\pi/2$ (CP-violating case): $M_\pm = \sqrt{(m\pm\epsilon_1)^2+\epsilon_5^2}$.

Naturally, when $\epsilon_1 = 0$, we obtain the γ_0-parity preserving case treated above (22), and no oscillations.

The oscillation probability was found in [27] to be

$$\mathcal{P}(n(\vec{p}) \to \bar{n}(\vec{p});t) = (1-\sin^2 2\phi\cos^2\tilde{\alpha})\cos^2\theta\sin^2\left(\frac{1}{2}\Delta E t\right). \tag{40}$$

From the definitions of $\sin\tilde{\alpha}$, $\theta_\pm$ and $M_\pm$, we can see that the CP transformation, which is equivalent to $\alpha \to -\alpha$, corresponds to

$$\tilde{\alpha} \to -\tilde{\alpha}, \quad \theta = \theta_+ - \theta_- \to -\theta, \tag{41}$$

and the above oscillation probability (40) and the energy difference $\Delta E = \sqrt{\vec{p}^2+M_+^2} - \sqrt{\vec{p}^2+M_-^2}$ are all invariant. Although α modifies the magnitudes of ΔE (and thus the oscillation time) and probability $\mathcal{P}$ themselves, we do not regard these modifications as a manifestation of CP violation in oscillation, which is typically expressed by $\mathcal{P}(n(\vec{p}) \to n^c(\vec{p});t) \neq \mathcal{P}(n^c(\vec{p}) \to n(\vec{p});t)$. We observe no direct CP violation in the neutron oscillation in vacuum.

Nevertheless, the CP violation survives. It is transferred, by the mixing transformations (37), to the interaction terms of the neutron $n(x)$. This is perhaps better seen in the chiral notation, which will be detailed in the next section.

4. Analysis of P and CP in Chiral Notation

The Lagrangian (1) is equivalent to

$$\begin{aligned}\mathcal{L} &= \overline{n_L}(x)i\gamma^\mu\partial_\mu n_L(x) + \overline{n_R}(x)i\gamma^\mu\partial_\mu n_R(x)\\ &- M_D\overline{n_L}(x)n_R(x) - \frac{1}{2}M_L n_L^T(x)Cn_L(x) - \frac{1}{2}M_R n_R^T(x)Cn_R(x) + h.c.,\end{aligned} \tag{42}$$

with $n_{R,L}(x) = [(1\pm\gamma_5)/2]n(x)$. In terms of the mass parameters in (1),

$$M_D = m + i\delta m, \quad M_L = m_1 - m_5, \quad M_R = m_1 + m_5. \tag{43}$$

The mass terms of the Lagrangian (42) can be written as

$$-2\mathcal{L}_{mass} = \begin{pmatrix} \overline{n_R} & n_L^T(x)C \end{pmatrix}\begin{pmatrix} M_R^\dagger & M_D \\ M_D & M_L \end{pmatrix}\begin{pmatrix} C\overline{n_R}^T \\ n_L \end{pmatrix} + h.c. \tag{44}$$

We diagonalize the *complex symmetric* mass matrix using a 2×2 unitary matrix

$$\tilde{U}^T\begin{pmatrix} M_R^\dagger & M_D \\ M_D & M_L \end{pmatrix}\tilde{U} = \begin{pmatrix} M_+ & 0 \\ 0 & M_- \end{pmatrix}, \tag{45}$$

where the mass eigenvalues, always positive, will be given by (36). The general form of the $U(2)$ matrix $\tilde{U}$ in two dimensions can be parametrized by one angle a and three phases $\gamma_1, \gamma_2, \beta$, in the Autonne–Takagi factorization [34,35]:

$$\tilde{U} = \begin{pmatrix} e^{i\gamma_1} & 0 \\ 0 & e^{i\gamma_2} \end{pmatrix} \begin{pmatrix} \cos a & \sin a \\ -\sin a & \cos a \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\beta} \end{pmatrix} = \begin{pmatrix} e^{i\gamma_1} & 0 \\ 0 & e^{i\gamma_2} \end{pmatrix} U(a, \beta). \tag{46}$$

Then, we can re-write (45) as

$$\begin{aligned} &U^T(a,\beta) \begin{pmatrix} e^{i\gamma_1} & 0 \\ 0 & e^{i\gamma_2} \end{pmatrix} \begin{pmatrix} M_R^\dagger & M_D \\ M_D & M_L \end{pmatrix} \begin{pmatrix} e^{i\gamma_1} & 0 \\ 0 & e^{i\gamma_2} \end{pmatrix} U(a,\beta) \\ &= U^T(a,\beta) \begin{pmatrix} m_R^\dagger & m_D \\ m_D & m_L \end{pmatrix} U(a,\beta) = \begin{pmatrix} M_+ & 0 \\ 0 & M_- \end{pmatrix}, \end{aligned} \tag{47}$$

where the phases γ_1, γ_2 are suitably fixed in terms of the parameters M_D, M_R, M_L. For example, m_D will be real if we put

$$\gamma_1 + \gamma_2 = -\text{Arg}\, M_D.$$

The arguments of m_L and m_R will have to satisfy, in general,

$$\begin{aligned} 2\gamma_1 - \text{Arg}\, M_R &= \arg m_R, \\ 2\gamma_2 + \text{Arg}\, M_L &= \arg m_L. \end{aligned}$$

One possible parametrization of m_D, m_L, m_R in (47), compatible with the formulas above and with the four-component notation in (33), is

$$m_D = m, \quad m_L = i(\epsilon_1 e^{i\alpha} - \epsilon_5), \quad m_R = i(\epsilon_1 e^{i\alpha} + \epsilon_5). \tag{48}$$

Originally, we had six real parameters in M_D, M_L, M_R, which were transformed by a change of variables to $m, \epsilon_1, \epsilon_5, \alpha, \gamma_1, \gamma_2$. It turns out, however, that the phases γ_1 and γ_2 are unessential, because they can be absorbed by rephasing $n_L(x)$ and $n_R(x)$.

Upon mass matrix diagonalization, (44) becomes

$$\begin{aligned} -2\mathcal{L}_{mass} &= \begin{pmatrix} \overline{n_R} & n_L^T(x)C \end{pmatrix} (\tilde{U}^T)^\dagger \begin{pmatrix} M_+ & 0 \\ 0 & M_- \end{pmatrix} \tilde{U}^\dagger \begin{pmatrix} C\overline{n_R}^T \\ n_L \end{pmatrix} + h.c. \\ &= \begin{pmatrix} \overline{n_R} & n_L^T(x)C \end{pmatrix} (U^T(a,\beta))^\dagger \begin{pmatrix} M_+ & 0 \\ 0 & M_- \end{pmatrix} U^\dagger(a,\beta) \begin{pmatrix} C\overline{n_R}^T \\ n_L \end{pmatrix} + h.c. \\ &= \begin{pmatrix} \bar{\Phi}_+ & \bar{\Phi}_- \end{pmatrix} \begin{pmatrix} M_+ & 0 \\ 0 & M_- \end{pmatrix} \begin{pmatrix} \Phi_+ \\ \Phi_- \end{pmatrix}, \end{aligned} \tag{49}$$

using in the last line the notation from (35). In going from the first to the second line above, we eliminated the unphysical γ_1, γ_2 by rephasing

$$e^{i\gamma_1} n_R \to n_R, \qquad e^{-i\gamma_2} n_L \to n_L.$$

We have also

$$\begin{pmatrix} \Phi_+ \\ \Phi_- \end{pmatrix}_L = U^\dagger(a,\beta) \begin{pmatrix} C\overline{n_R}^T \\ n_L \end{pmatrix}. \tag{50}$$

Formula (50) is the inverted equivalent of the transformation (37) in four-component spinor notation.

To summarize, we started with six real parameters equivalent to the complex M_D, M_L, M_R and eliminated two of them by absorbing the phases γ_1, γ_2 of the $U(2)$ mixing matrix into n_L and n_R. We were left with four real parameters, $m, \epsilon_1, \epsilon_5, \alpha$ (see (48)), which were finally re-cast as the two mass eigenvalues $M_\pm$ and the two parameters of the mixing matrix U, namely the angle a and the phase β. The latter can be found by determining the eigenvalues and eigenvectors of the 4×4 real symmetric matrix

$$\mathcal{M} = \begin{pmatrix} \mathrm{Re}\, M & -\mathrm{Im}\, M \\ -\mathrm{Im}\, M & -\mathrm{Re}\, M \end{pmatrix}, \qquad M = \begin{pmatrix} m_R^\dagger & m_D \\ m_D & m_L \end{pmatrix}.$$

The eigenvalues of $\mathcal{M}$ are the masses $M_\pm$, and the eigenvectors represent the real and imaginary parts of the columns of the unitary diagonalizing matrix $U(a, \beta)$. For details of the procedure, see [36].

The phase β cannot be absorbed by a redefinition of the fields, because the eigenfields $\Phi_\pm$ are of Majorana type. As a result, the presence of the phase β signals a potential CP violation. The mixing matrix U transfers the formal CP violation to the interaction terms, where it can, in principle, develop into a genuine CP violation, provided that there would be several interfering channels. The oscillation probability is, however, oblivious to the CP Majorana phase, and is therefore CP invariant. The parity and CP violation effects of the baryon number-violating terms may appear as an addition to the electric dipole moment of the neutron [27]. Thus, all four parameters $M_\pm, a, \beta$ are essential for the neutron physics (oscillations and interactions).

5. Conclusions

While the general effective Lagrangian describing the neutron-antineutron conversion contains parity- and CP-violating Majorana mass terms, these features are not observable in the phenomenon of oscillation. Conversely, this analysis also shows that an electric dipole moment of the neutron produced from other sources (for example, the theta-term in the Standard Model) will not lead to an observable CP violation in neutron oscillation, though it slightly modifies the oscillation time. For this reason, in baryogenesis scenarios, it is customary to include CP violation effects in the baryon number-violating neutron oscillation by adopting the view of a neutron-dark matter coupling [25,37] by which the neutrons and antineutrons decay at different rates.

We analyzed the physical implications of the given quadratic effective action of the neutrons by solving it exactly, in some cases with a simplified model but also with the full model. We thus discussed CP and other discrete symmetries, assuming that the effective Lagrangian is an exact prediction of some underlying interactions. Alternatively, we also attempted to use the Pauli–Gürsey $U(2)$ transformation together with two Majorana fermion masses as a convenient parametrization of the most general quadratic model and discussed its possible implications elsewhere. We confirmed that these two approaches give the same answer as to the effects of CP-breaking phases on the neutron oscillation formula.

On the other hand, one might take the given neutron action as an analogue of the seesaw model of neutrinos, for example, and separate the $U(2)$ phase in the Pauli–Gürsey transformation. In this case, however, we have no definite prediction, since no model of weak interactions is given. This approach is attractive in the more fundamental formulation, but not practically useful for the present quadratic effective model of neutron oscillations.

Both the γ_0 and $i\gamma_0$ parities are broken in the Lagrangian (33); therefore, neither one is preferable to the other as a measure of actual symmetry of the full theory, including oscillations and interactions. However, we have established (see [27]) that the γ_0 parity violation can be used as a criterion for the occurrence of neutron-antineutron oscillation, given the fact that the Lagrangian (1) also includes the possibility of conversion without oscillation (through a mixing of mass-degenerate Majorana fields). Since the known physics related to the neutron, namely, hadron scattering and the entire nuclear physics, is based on the use of γ^0 parity (i.e., intrinsic parities of neutron and antineutron ± 1), the consistent description of neutron-antineutron oscillations by γ^0 parity is in fact gratifying. In that sense, the γ_0 parity is arguably more useful in the analysis of the original Lagrangian. Nevertheless, both definitions of parity have their respective merits and can be interchangeably utilized.

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