Amenable Banach Algebras and Johnson’s Theorem for the Group Algebra $L^1(G')$
Amenability is a notion that occurs in the theory of both locally compact groups and Banach algebras. The research on translation-invariant measures during the first half of the 20th century led to the definition of amenable locally compact groups. A locally compact group $G$ is called amenable if there is a positive linear functional of norm 1 in $L^\infty(G)$ that is left-invariant with respect to the given group operation.

During the same time the theory of Hochschild cohomology for Banach algebras was developed. A Banach algebra $A$ is called amenable if the first Hochschild cohomology group $H^1(A, X^*) = \{0\}$ for all dual Banach $A$-bimodule $X^*$, that is, if every continuous derivation $D : A \to X^*$ is inner. In 1972 B. E. Johnson proved that the group algebra $L^1(G)$ for a locally compact group $G$ is amenable if and only if $G$ is amenable. This result justifies the terminology amenable Banach algebra.

In this Master's thesis we present the basic theory of amenable Banach algebras and give a proof of Johnson's theorem.
## Contents

1 Introduction 1

2 Preliminaries 2
   2.1 Functional Analysis 2
   2.2 Measure Theory 6
      2.2.1 $L^p$-spaces 7
      2.2.2 Complex Measures 8
      2.2.3 Fubini’s Theorem 12

3 Introduction to Abstract Harmonic Analysis 15
   3.1 Locally Compact Groups 15
   3.2 Haar Measures 18
   3.3 The Group Algebra $L^1(G)$ 24
   3.4 The Measure Algebra $M(G)$ 28

4 Amenable and Contractible Banach Algebras 33
   4.1 Banach Bimodules and Hochschild Cohomology 33
   4.2 Characterization of Contractible Banach Algebras 38
   4.3 Characterization of Contractible Algebras 41
   4.4 Bounded Approximate Identities for Banach Algebras 44
      4.4.1 Bounded Approximate Identities for $L^1(G)$ 47
      4.4.2 The Cohen-Hewitt Factorization Theorem 51
   4.5 Pseudo-Unital Banach Algebras 54
   4.6 Characterization of Amenable Banach Algebras 58

5 Amenable Locally Compact Groups 61
   5.1 Left-Invariant Means 61
   5.2 Johnson’s Theorem for the Group Algebra $L^1(G)$ 68
Chapter 1

Introduction

The question of the existence of finitely additive set functions that are invariant under some group action was studied extensively in the days of Banach and Tarski. Groups that possess a left-invariant mean turned out to have very nice, "amenable", properties and they were called thereafter, as a pun, as amenable by M. M. Day in 1950. In a completely different area of mathematics, G. Hochschild introduced in [Hoch] a certain type of cohomology, later to be called Hochschild cohomology, for studying properties in abstract algebra. Later H. Kamowitz introduced in [Kam] Hochschild cohomology in the theory of Banach algebras.

In 1972 B. E. Johnson proved that the amenability of a locally compact group $G$ can be characterized by a certain homological property in the Hochschild cochain complex of the Banach algebra $L^1(G)$ and Banach algebras satisfying this property were to be called amenable. Hence, the theorem by Johnson acts as a connection between these seemingly different objects of study. Clearly the theory of both amenable Banach algebras and amenable locally compact groups benefits from this connection. Indeed, it gives a large supply of important examples of amenable Banach algebras and on the other hand, the homological aspects of amenable Banach algebras give a powerful machinery for investigating amenability on locally compact groups.

Two goals were set before beginning the writing process of this Master’s thesis. Firstly, to give an introduction to the vast theory of amenable Banach algebras, and secondly, to give a proof of the theorem by Johnson. The proof presented here is not the original one by Johnson but rather the elegant proof presented in the lecture notes Lectures on Amenability, [Run], by V. Runde.

A main focus was to make the mathematical journey leading to the proof of Johnson’s theorem relatively self-contained. Hence, this Master’s thesis is intended to be understandable by any reader with basic knowledge of functional analysis and measure theory. In particular, the abstract harmonic analysis which is needed for the theory of amenable locally compact groups is developed almost from scratch.
Chapter 2

Preliminaries

2.1 Functional Analysis

In this section we recall necessary concepts and results of functional analysis which will be frequently used in this Master’s thesis. Proofs will be omitted but can be found in (almost) any book on functional analysis. As a convention, all spaces and algebras will be over \( \mathbb{C} \).

Since the central objects of study in this thesis are Banach algebras we begin by recalling the definition of a Banach algebra.

**Definition 2.1.** Let \( A \) be a Banach space. If \( A \) is an associative algebra such that the algebra multiplication satisfies

\[
||ab|| \leq ||a|| ||b|| \quad (a, b \in A),
\]

then \( A \) is called a Banach algebra.

**Remark 2.2.** We observe that the algebra multiplication \( A \times A \to A, (a, b) \mapsto ab \), is jointly continuous.

In general, a Banach algebra does not have a unit. However, there is a canonical way to embed a non-unital Banach algebra into a unital one. Indeed, if \( A \) is a non-unital Banach algebra, we then consider the set \( A^\#: = \mathbb{C} \times A \) with pointwise addition and scalar multiplication and with algebra multiplication defined by

\[
(\alpha, a) \cdot (\beta, b) = (\alpha \beta, \alpha b + \beta a).
\]

Then \( A^\# \) is a unital algebra with unit \( 1_A := (1, 0) \) and by defining a norm on \( A^\# \) through

\[
||| (\alpha, a)||| = |\alpha| + ||a|| \quad (\alpha \in \mathbb{C}, a \in A)
\]

it is a Banach algebra. Indeed for \( (\alpha, a), (\beta, b) \in A^\# \) we have

\[
||| (\alpha, a) \cdot (\beta, b)||| = |\alpha \beta| + ||a\beta + \alpha b + \beta a|| \leq |\alpha| ||\beta|| + ||a|| ||b|| + |\alpha| ||b|| + ||\beta|| ||a||
\]

\[
= (|\alpha| + ||a||) \cdot (||\beta|| + ||b||) = ||| (\alpha, a)||| \cdot ||| (\beta, b)|||.
\]

Clearly \( i : a \mapsto (0, a) \) is an isometric embedding of \( A \) into \( A^\# \) and since

\[
(\alpha, a) = \alpha 1_A + i(a) \quad (\alpha \in \mathbb{C}, a \in A),
\]
we may and will write \((a, a)\) as the direct sum \(a + a\). Note that \(|1_A| = 1\) for the unit \(1_A\) in \(A^\#\). However, the unit of an arbitrary unital Banach algebra does not necessarily have norm 1.

If \(A\) is a unital Banach algebra with unit \(1_A\) and \(a \in A\) such that \(|a| < 1\) then \(1_A - a\) is invertible and the inverse \((1_A - a)^{-1}\) can be written in the following power series

\[(1_A - a)^{-1} = \sum_{n=0}^{\infty} a^n\]

called the Neumann series for \((1_A - a)^{-1}\).

We go on by defining and recalling concepts of locally convex topologies on vector spaces. A topological vector space is a vector space \(X\) such that addition and scalar multiplication are continuous operations, that is, the mappings

\[X \times X \to X, \quad (x, y) \mapsto x + y \quad \text{and} \quad \mathbb{C} \times X \to X, \quad (\lambda, x) \mapsto \lambda x\]

are jointly continuous. Let \(X\) be a vector space and let \(\mathcal{P}\) be a family of seminorms on \(X\). Then the locally convex topology on \(X\) defined by \(\mathcal{P}\) is the coarsest topology on \(X\) such that \(X\) is a topological vector space and every \(p \in \mathcal{P}\) is continuous. A vector space \(X\) with a locally convex topology is called a locally convex topological vector space, or in an abbreviated form LCS. Here, \(X\) is a first-countable space if and only if the family \(\mathcal{P}\) is countable and thus the use of nets rather than sequences is needed for investigating topological properties of locally convex vector spaces. If \((x_\alpha)_{\alpha \in I}\) is a net in a topological space converging to some point \(x\), we write \(\lim_{\alpha} x_\alpha = x\) or if we are dealing with a Hausdorff space, we usually denote \(x_\alpha \to x\) to emphasize the uniqueness of the limit point. We also sometimes write \((x_\alpha)_{\alpha \in I}\) to emphasize that the index set of the net \((x_\alpha)_{\alpha}\) is \(I\). We recall that a net \((y_\beta)_{\beta \in J}\) in a topological space \(X\) is a subnet of a net \((x_\alpha)_{\alpha \in I}\) in \(X\) if there exists a map \(N : J \to I\) such that \(y_\beta = x_{N(\beta)}\) for all \(\beta \in J\) and for any \(\alpha \in I\) there exists \(\beta \in J\) such that \(N(\beta) \geq \alpha\) whenever \(\beta' \geq \beta\). Note that if a net converges to some point, then also every subnet converges to the same point. Finally, a net \((x_\alpha)_{\alpha}\) in a LCS \(X\) that is defined by a seminorm family \(\mathcal{P}\) converges to a point \(x \in X\) if and only if \(p(x_\alpha - x) \to 0\) for all \(p \in \mathcal{P}\).

The dual of a normed space \(X\), that is, the space of all continuous functionals on \(X\), is denoted by \(X^*\) and for \(x \in X\), \(f \in X^*\) we usually denote \(\langle x, f \rangle := f(x)\). The \(w^*\)-topology (weak star topology) on \(X^*\) is the locally convex topology induced by the seminorms

\[X^* \to \mathbb{R}, \quad f \mapsto |\langle x, f \rangle| \quad (x \in X).\]

It follows that a net \((f_\alpha)_{\alpha}\) in \(X^*\) converges to a point \(f \in X^*\) in the \(w^*\)-topology if and only if \(\langle x, f_\alpha \rangle \to \langle x, f \rangle\) for all \(x \in X\). Similarly the \(u\)-topology (weak topology) on \(X\) is defined through the seminorms

\[X \to \mathbb{R}, \quad x \mapsto |\langle x, f \rangle| \quad (f \in X^*)\]

and a net \((x_\alpha)_{\alpha}\) in \(X\) converges to \(x \in X\) in the weak topology on \(X\) if and only if \(\langle x_\alpha, f \rangle \to \langle x, f \rangle\) for all \(f \in X^*\).

We recall the Hahn-Banach theorem for normed spaces.

**Theorem 2.3.** (Hahn-Banach) Let \(M\) be a linear subspace of a normed space \(X\). Then for each \(f \in M^*\) there exists an extension \(f' \in X^*\) of \(f\) such that \(||f'|| = ||f||\).
CHAPTER 2. PRELIMINARIES

It follows from the Hahn-Banach theorem that the canonical embedding \( x \mapsto \hat{x} \) from \( X \) into its second dual \( X^{**} \) defined by

\[
\langle f, \hat{x} \rangle = \langle x, f \rangle \quad (x \in X, f \in X^*)
\]

is isometric. The canonical embedding of a vector \( x \) into its second dual is always denoted by \( \hat{x} \). Another consequence of the Hahn-Banach theorem is that \( X^* \) separates the points in \( X \), that is, for each non-zero \( x \in X \) there exists a \( f \in X^* \) such that \( \langle x, f \rangle \neq 0 \). More generally, it follows from the Hahn-Banach theorem that the dual \( X^* \) of a Hausdorff LCS \( X \) separates the points in \( X \) (see [Rud, (3.4) Corollary]).

Let \( M \) be a closed subspace of a normed space \( X \). The annihilator, \( M^\perp \), of \( M \) is defined as

\[
M^\perp = \{ f \in X^* : \langle x, f \rangle = 0 \text{ for all } x \in M \}.
\]

We also define \( \hat{M}^\perp := \{ x \in X : \langle x, f \rangle = 0 \text{ for all } f \in M^\perp \} \). The following fact is also a consequence of the Hahn-Banach theorem.

**Theorem 2.4.** Let \( M \) be a closed subspace of a normed space \( X \). Then \( \hat{M}^\perp = M \).

If \( X \) is a normed space with a closed subspace \( M \), then the quotient space \( X/M \) is a normed space with the quotient norm defined by \( \| x + M \| = \inf \{ \| x - z \| : z \in M \} \). Furthermore, the dual of the quotient space has the following characterization.

**Theorem 2.5.** Let \( M \) be a closed subspace of a normed space \( X \) and let \( \pi : X \to X/M \) be the quotient mapping. Then \( T : (X/M)^* \to M^\perp, T(f) = f \circ \pi \) is an isometric isomorphism.

Let \( X \) and \( Y \) be normed spaces. The space of all bounded linear operators \( T : X \to Y \) is denoted by \( \mathcal{L}(X,Y) \). For each \( T \in \mathcal{L}(X,Y) \) we define the unique adjoint operator \( T^* \in \mathcal{L}(Y^*,X^*) \) of \( T \) through

\[
\langle x, T^* y^* \rangle = \langle Tx, y^* \rangle \quad (x \in X, y^* \in Y^*).
\]

In a similar fashion we define \( T^{**} := (T^*)^* \in \mathcal{L}(X^{**},Y^{**}) \). Note that for \( x \in X, y^* \in Y^* \) we have

\[
\langle y^*, T^{**} \hat{x} \rangle = \langle T^* y^*, \hat{x} \rangle = \langle x, T^* y^* \rangle = \langle \hat{x}, y^* \rangle.
\]

The following theorem is needed in section 4.6.

**Theorem 2.6.** Let \( X \) and \( Y \) be Banach spaces and \( T \in \mathcal{L}(X,Y) \). Then \( T^* \) is \( w^* - w^* \)-continuous and if \( T \) is surjective then \( (\ker T)^{**} = \ker T^{**} \).

From the theory of weak topologies on normed spaces we need the following theorem.

**Theorem 2.7.** (Mazur) The closure and the weak closure of a convex subset of a normed space are the same.

For a normed space \( X \) and \( m > 0 \) let \( b_m(X) := \{ x \in X : \| x \| \leq m \} \). Here, if \( X \) is an infinite-dimensional normed space then \( b_1(X) \) is not compact. However the \( w^* \)-topology has the following very useful property.

**Theorem 2.8.** (Alooglu) Let \( X \) be a normed space. Then \( b_1(X^*) \) is \( w^* \)-compact.

Another useful property of the \( w^* \)-topology which is frequently used is due to H. Goldstine.
CHAPTER 2. PRELIMINARIES

Theorem 2.9. (Goldstine) Let $X$ be a Banach space. Then for each $\phi \in X^{**}$ there exists a net $(x_\alpha)_\alpha$ in $X$, bounded by $||\phi||$, such that $\hat{x}_\alpha \to \phi$ in the $w^*$-topology on $X^{**}$.

For normed spaces $X$ and $Y$, an $n$-linear map $T : X^n \to Y$ is called bounded if there exists $M > 0$ such that

$$ ||T(x_1, \ldots, x_n)|| \leq M||x_1|| \cdots ||x_n|| \quad (x_i \in X, i \in \mathbb{N}_n), $$

where $\mathbb{N}_n = \{1, \ldots, n\}$. All bounded $n$-linear maps $T : X^n \to Y$ form a normed space, $\mathcal{L}^n(X, Y)$, where the norm is given by

$$ ||T|| = \sup\{||T(x_1, \ldots, x_n)|| : x_i \in b_1(X) \text{ for all } i \in \mathbb{N}_n \}. $$

If $Y$ is a Banach space, then $\mathcal{L}^n(X, Y)$ is a Banach space.

A final argument in Theorem 5.16 will use Gelfand’s theorem for commutative $C^*$-algebras and thus we conclude this section by recalling some basic concepts of the theory of $C^*$-algebras and Gelfand’s theorem which states that there are essentially only one kind of commutative $C^*$-algebras.

Definition 2.10. Let $A$ be a Banach algebra. Then $A$ is called a $C^*$-algebra if there exists a map $^* : A \to A, \quad x \mapsto x^*$, called the involution of $A$, such that for each $x, y \in A, \alpha, \beta \in \mathbb{C}$ we have

$$ (x^*)^* = x, \quad (xy)^* = y^*x^*, \quad (\alpha x + \beta y)^* = \bar{\alpha}x^* + \bar{\beta}y^* \quad \text{and} \quad ||x^*x|| = ||x||^2. $$

If $A$ is a commutative $C^*$-algebra we call a non-zero linear multiplicative functional $\tau : A \to \mathbb{C}$ a character on $A$. The space of all characters on $A$ endowed with the $w^*$-topology is called the character space of $A$ and is denoted by $\Omega(A)$. Characters preserves adjoints, that is, $\tau(a^*) = \overline{\tau(a)}$ for all $a \in A$ and $\tau \in \Omega(A)$. The Gelfand transform of an element $a \in A$ is the function $\hat{a}$ defined by

$$ \hat{a} : \Omega(A) \to \mathbb{C}, \quad \hat{a}(\tau) = \tau(a). $$

For a topological space $X$ let $C_0(X)$ denote the space of all continuous functions on $X$ vanishing at infinity, that is, $f \in C_0(X)$ if for all $\varepsilon > 0$ there exists a compact subset $K \subset X$ such that $|f(x)| < \varepsilon$ for all $x \in X \setminus K$. By using Alaoglu’s theorem one can show that $\hat{a} \in C_0(\Omega(A))$ for all $a$ in a commutative $C^*$-algebra $A$.

Theorem 2.11. (Gelfand) Let $A$ be a commutative $C^*$-algebra. Then the Gelfand representation

$$ A \to C_0(\Omega(A)), \quad a \mapsto \hat{a} $$

is an isometric $^*$-isomorphism. If $A$ is unital, then $\Omega(A)$ is compact.
2.2 Measure Theory

In this section we briefly define the necessary concepts of measure theory and present theorems which will be needed in the exposition of abstract harmonic analysis in chapter 3. In particular, we will need Riesz’ representation theorem for identifying the space of complex measures on a locally compact Hausdorff space \( X \) with the dual of \( C_0(X) \) (recall that a topological space \( X \) is locally compact if every point \( x \in X \) has a compact neighbourhood). Moreover, as a consequence of the Radon-Nikodym theorem, we get an important connection between integrable functions and absolutely continuous measures. We also need Fubini’s theorem to define the convolution operator that will serve as the multiplication operator in the group algebra \( L^1(G) \) for a locally compact group \( G \). Note that locally compact groups are not in general \( \sigma \)-finite which is usually assumed in Fubini’s theorem. However, Fubini’s theorem is indeed valid in locally compact Hausdorff spaces for a sufficiently large class of functions for our purposes. All the information in this section (and much more) can be found in [Coh].

**Definition 2.12.** Let \( X \) be a set and \( \Gamma \) a \( \sigma \)-algebra on \( X \). The pair \((X, \Gamma)\) is called a measurable space. A measure on \((X, \Gamma)\) (or if the \( \sigma \)-algebra is clear from context, a measure on \( X \)) is a countably additive function \( \mu : \Gamma \to [0, +\infty] \) such that \( \mu(\emptyset) = 0 \) and the triplet \((X, \Gamma, \mu)\) is called a measure space. The \( \sigma \)-algebra of all Borel sets on \( X \) is denoted by \( \mathcal{B}(X) \). The phrase \(^n\)almost everywhere with respect to the measure \( \mu \)\(^n\) will be abbreviated as \( \mu\)\(^n\)-a.e. Let \( X \) be a set and \( \mathcal{A} \) a family of subsets of \( X \). Then \( \sigma(\mathcal{A}) \) denotes the smallest \( \sigma \)-algebra on \( X \) that includes \( \mathcal{A} \). Let \((X, \Gamma)\) and \((Y, \Gamma')\) be measurable spaces and let \( \mathcal{A} \) be such that \( \sigma(\mathcal{A}) = \Gamma' \). Then a mapping \( f : X \to Y \) is called measurable (with respect to \( \Gamma \) and \( \Gamma' \)) if \( f^{-1}(A) \in \Gamma \) for all \( A \in \Gamma' \), or equivalently, for all \( A \in \mathcal{A} \).

For a topological space \( X \) we denote the space of compactly supported continuous functions on \( X \) by \( C_c(X) \) and the subset consisting of all non-negative non-zero functions in \( C_c(X) \) is denoted by \( C^+_c(X) \). If \( X \) is a normed space, \( C_b(X) \) will stand for the space of all bounded continuous functions on \( X \). All functions are complex-valued if not stated otherwise.

If \((X, \Gamma, \mu)\) is a measure space we recall that the integral of a function \( f : X \to \mathbb{C} \) with respect to \( \mu \) is defined as \( \int f \, d\mu = \int f_1 \, d\mu + i \int f_2 \, d\mu \) where \( f_1 \) and \( f_2 \) are the real-valued functions such that \( f = f_1 + i f_2 \). The following property of complex-valued functions will often be used without explicit mentioning.

**Proposition 2.13.** Let \((X, \Gamma, \mu)\) be a measure space and \( f : X \to \mathbb{C} \) a measurable function with respect to \( \Gamma \) and \( \mathcal{B}(\mathbb{C}) \). Then \( f \) is integrable if and only if \( |f| \) is integrable and in that case \( \int |f| \, d\mu \leq \int |f| \, d\mu \).

**Definition 2.14.** Let \((X, \Gamma, \mu)\) be a measure space and let \((Y, \Gamma')\) be a measurable space. If \( f : X \to Y \) is a measurable mapping then the image of \( \mu \) under \( f \), denoted by \( \mu f^{-1} \), is defined by

\[
\mu f^{-1}(A) = \mu(f^{-1}(A)) \quad (A \in \Gamma')
\]

It is easy to verify that \( \mu f^{-1} \) is a measure on \((Y, \Gamma')\). Furthermore:

**Proposition 2.15.** Let \((X, \Gamma, \mu)\), \((Y, \Gamma')\) and \( f \) be as above. Let \( g : Y \to [0, +\infty] \) be a \( \Gamma' \)-measurable function on \( Y \). Then \( g \) is \( \mu f^{-1} \)-integrable if and only if \( g \circ f \) is \( \mu \)-integrable and in that case

\[
\int_Y gd(\mu f^{-1}) = \int_X (g \circ f) \, d\mu.
\]
CHAPTER 2. PRELIMINARIES

2.2.1 $L^p$-spaces

Let $(X, \Gamma, \mu)$ be a measure space, and $p \in [1, +\infty)$. Let $L^p(X, \Gamma, \mu)$ denote the set of $\Gamma$-measurable functions $f : X \to \mathbb{C}$ such that $|f|^p$ is integrable with respect to $\mu$. The relation

$$f \sim g \iff f = g \mu\text{-a.e.},$$

defines an equivalence relation on $L^p(X, \Gamma, \mu)$ and thus we may define

$$L^p(X, \mathcal{A}, \mu) := \{[f] \in L^p(X, \Gamma, \mu) / \sim\},$$

that is, we identify two functions in $L^p(X, \Gamma, \mu)$ that agree almost everywhere. By defining addition and scalar multiplication in $L^p(X, \Gamma, \mu)$ through

$$(af + bg) = a[f] + b[g] \quad (a, b \in \mathbb{C}, [f], [g] \in L^p(X, \Gamma, \mu)),$$

we see that $L^p(X, \Gamma, \mu)$ is a vector space. Furthermore, the mapping

$$|| \cdot ||_p : L^p(X, \Gamma, \mu) \to \mathbb{R}, \quad ||[f]||_p = \left(\int |f|^p d\mu\right)^{1/p}$$

is well-defined, that is, does not depend on the function representing the equivalence class, and defines a norm on $L^p(X, \Gamma, \mu)$. With standard abuse of notation we will from now on denote $f \in L^p(X, \Gamma, \mu)$ for an arbitrary element in the equivalence class $[f] \in L^p(X, \Gamma, \mu)$.

In order to define the space $L^\infty(X, \Gamma, \mu)$ we will use a slightly stronger identification of functions: A set $A \in \Gamma$ is called locally $\mu$-null if $A \cap B$ is $\mu$-null for every $B \in \Gamma$. A property that holds on $X$ except for a locally $\mu$-null set is said to hold locally $\mu$-a.e.

A function $f : X \to \mathbb{C}$ is called essentially bounded if for some $M > 0$ the set $\{x \in X : f(x) > M\}$ is locally $\mu$-null. The set of essentially bounded functions is denoted by $L^\infty(X, \Gamma, \mu)$. Similarly as above, we let $L^\infty(X, \Gamma, \mu)$ denote the quotient space $L^\infty(X, \Gamma, \mu) / \sim$, where

$$f \sim g \iff f = g \text{ locally } \mu\text{-a.e.}$$

Again, $L^\infty(X, \Gamma, \mu)$ is a vector space and the mapping

$$|| \cdot ||_\infty : L^\infty(X, \Gamma, \mu) \to \mathbb{R}, \quad ||[f]||_\infty = \inf\{M > 0 : |f(x)| \leq M \text{ locally } \mu\text{-a.e.}\}$$

is well-defined and defines a norm on $L^\infty(X, \Gamma, \mu)$. Also, by $f \in L^\infty(X, \Gamma, \mu)$ we will mean an arbitrary element in the equivalence class $[f] \in L^\infty(X, \Gamma, \mu)$.

We recall that the norm on $L^p$-spaces is complete.

**Proposition 2.16.** Let $p \in [1, +\infty]$ and let $(X, \Gamma, \mu)$ be a measure space. Then $L^p(X, \Gamma, \mu)$ is a Banach space.

**Remark 2.17.** From the properties of complex conjugation it follows that $L^\infty(X, \Gamma, \mu)$ is in fact a $C^*$-algebra with pointwise multiplication and involution defined by $f^*(x) = \overline{f(x)}$. Note also that $L^\infty(X, \Gamma, \mu)$ is commutative and unital and thus Gelfand's theorem applies to $L^\infty(X, \Gamma, \mu)$.

The reason for identifying functions in $L^\infty(X, \Gamma, \mu)$ that coincide locally $\mu$-a.e. is that for a locally compact group $G$ we can identify $L^1(G)^*$ as a dual space with $L^\infty(G)$ by associating $\phi \in L^\infty(G)$ with the functional $f \mapsto \int_G \phi f dm_G$ where $m_G$ is a left Haar measure on $G$. Note moreover that if $(X, \Gamma, \mu)$ is a $\sigma$-finite measure space then a set $A$ is $\mu$-null if and only if $A$ is locally $\mu$-null and hence "local" identification of functions coincides with the usual one when dealing with a $\sigma$-finite measure space.
CHAPTER 2. PRELIMINARIES

Definition 2.18. Let $X$ be a Hausdorff space. A Borel measure $\mu$ on $X$ is regular if the following conditions are satisfied.

(i) $\mu(U) = \sup\{\mu(K) : K \text{ is compact and } K \subset U\}$ for all open $U \subset X$.
(ii) $\mu(A) = \inf\{\mu(V) : V \text{ is open and } A \subset V\}$ for all $A \in \mathcal{B}(X)$.
(iii) $\mu(K) < +\infty$ for all compact sets $K \subset X$.

Conditions (i) and (ii) are referred to as the inner-regularity and outer-regularity of $\mu$ respectively.

A measure $\mu$ with range in $[0, +\infty)$ is called finite. Condition (i) in the previous definition can be strengthened for finite measures.

Proposition 2.19. Let $X$ be a Hausdorff space and let $\mu$ be a finite regular Borel measure on $X$. Then $\mu(A) = \sup\{\mu(K) : K \text{ is compact and } K \subset A\}$ for every Borel set $A \subset X$.

Proof. Let $\varepsilon > 0$ and let $A$ be a Borel set. By the regularity of $\mu$ we can choose an open set $U$ and after that a compact set $K$ such that $A \subset U$, $K \subset U$ and

$$
\mu(U) - \varepsilon < \mu(K) \leq \mu(U) < \mu(A) + \varepsilon
$$

We have $\mu(U \setminus A) < \varepsilon$ and we can again use the regularity of $\mu$ to choose an open $V$ such that $U \setminus A \subset V$ and $\mu(V) < \varepsilon$. Now $K \setminus V$ is a closed subset contained in $K$ and hence compact. Note also that $K \setminus V \subset A$. Now, since

$$
\mu(K \setminus V) = \mu(K) - \mu(K \cap V) > \mu(U) - \varepsilon - \varepsilon \geq \mu(A) - 2\varepsilon
$$

we obtain that $\mu(A) = \sup\{\mu(K) : K \text{ is compact and } K \subset A\}$. \qed

Let $X$ be a set. Recall that a function $f : X \to \mathbb{C}$ of the form

$$
f = \sum_{i=1}^{n} a_{i} \chi_{A_{i}} \quad (A_{i} \subset X, a_{i} \in \mathbb{C}, n \in \mathbb{N})
$$

where the sets $A_{i}$ are pairwise disjoint is called a simple function on $X$. If $(X, \Gamma)$ is a measurable space, then $f$ is measurable if and only if each $A_{1}, \ldots, A_{n}$ is measurable. The set of all $\Gamma$-measurable simple functions form a dense subspace of $L^{p}(X, \mu)$ for $p \in [0, +\infty]$.

Furthermore, if $X$ is a locally compact Hausdorff space, then $C_{c}(X)$ is dense in $C_{0}(X)$ and if $\mu$ is a regular Borel measure, then $C_{c}(X)$ is also dense in $L^{p}(X, \mathcal{B}(X), \mu)$ for $p \in [1, +\infty)$.

2.2.2 Complex Measures

Let $(X, \Gamma)$ be a measurable space. A complex measure on $(X, \Gamma)$ is a countable additive function $\mu : \Gamma \to \mathbb{C}$ such that $\mu(\emptyset) = 0$. A complex measure with range in $(-\infty, +\infty)$ is called a finite signed measure. Note that the set of all complex measures on $(X, \Gamma)$ denoted by $M(X, \Gamma)$ form a vector space.

The variation $|\mu|$ of a complex measure $\mu$ on $(X, \Gamma)$ is defined by

$$
|\mu|(E) = \sup\{\sum_{i} |\mu(E_{i})| : \{E_{i}\} \text{ is a finite } \Gamma\text{-partition of } E\}.
$$
As one might perhaps guess, $|\mu|$ is a finite measure on $X$ and hence, the total variation $||\mu||$ of $\mu$ defined by

$$||\mu|| = |\mu|(X)$$

lies in $[0, +\infty)$ for all $\mu \in M(X, \Gamma)$. It follows that $|| \cdot || : \mu \mapsto ||\mu||$ defines a norm on $M(X, \Gamma)$. Moreover, the norm is complete.

**Proposition 2.20.** Let $(X, \Gamma)$ be a measurable space. Then $M(X, \Gamma)$ is a Banach space under the total variation norm.

Let $(X, \Gamma)$ be a measurable space and $\nu$ a finite signed measure on $(X, \Gamma)$. We note that $|\nu|(A) \geq |\nu(A)|$ for each $A \in \Gamma$ and thus

$$\nu_1 := \frac{1}{2}(|\nu| + \nu) \quad \text{and} \quad \nu_2 := \frac{1}{2}(|\nu| - \nu)$$

are finite measures. Clearly $\nu = \nu_1 - \nu_2$ and thus every finite signed measure can be written as the difference of two finite measures.

Moreover, every complex measure $\mu$ on $(X, \Gamma)$ can be written as $\mu = \mu_1 + i\mu_2$ where $\mu_1$ and $\mu_2$ are finite signed measures. Hence $\mu$ can be written as

$$\mu = \mu_1 - \mu_2 + i\mu_3 - i\mu_4$$

where each $\mu_1, \ldots, \mu_4$ is a finite measure. This representation of a complex measure $\mu$ will be referred to as the Jordan decomposition of $\mu$.

Let $\mu$ be a complex measure and let $\mu = \mu_1 - \mu_2 + i\mu_3 - i\mu_4$ be the Jordan decomposition of $\mu$. Let $B(X, \Gamma)$ denote the set of all bounded complex-valued $\Gamma$-measurable functions. For $f \in B(X, \Gamma)$, the integral of $f$ with respect to $\mu$ is defined by

$$\int f d\mu = \int f d\mu_1 - \int f d\mu_2 + i \int f d\mu_3 - i \int f d\mu_4.$$ 

It is easy to see that $|\int f d\mu| < \infty$ for any $f \in B(X, \Gamma)$ by first considering $\Gamma$-measurable characteristic functions and then by using the linearity of the integral and the dominated convergence theorem for an arbitrary function in $B(X, \Gamma)$.

A complex Borel measure $\mu$ is called regular if $|\mu|$ is regular, or equivalently, if each of the finite measures in the Jordan decomposition of $\mu$ is regular. We denote the set of all regular complex Borel measures on $X$ by $M_r(X)$.

The following result will be used in section 3.4.

**Proposition 2.21.** Let $X$ be a locally compact Hausdorff space, let $\mu$ be a regular Borel measure on $X$ and let $f : X \to (0, +\infty)$ be a continuous function. Then $\nu : \mathcal{B}(G) \to [0, +\infty]$ defined by

$$\nu(A) = \int_A f d\mu$$

is a regular Borel measure on $X$.

**Proof.** Firstly, for pairwise disjoint Borel sets $(A_1, A_2, \ldots)$ we have by Beppo Levi’s theorem that

$$\nu\left(\bigcup_{n=1}^{\infty} A_n\right) = \int_{\bigcup_{n=1}^{\infty} A_n} f d\mu = \int \sum_{n=1}^{\infty} \chi_{A_n} f d\mu = \sum_{n=1}^{\infty} \int \chi_{A_n} f d\mu = \sum_{n=1}^{\infty} \nu(A_n).$$
Also, since $\nu(\emptyset) = \int \chi_\emptyset f d\mu = 0$ we may conclude that $\nu$ is a Borel measure. To see that $\nu$ is regular, we first define

$$U_n = \{x \in X : \frac{1}{n} < f(x) < n\} \quad (n \in \mathbb{N}).$$

Here, every $U_n$ is open by the continuity of $f$. To prove inner-regularity, let $U$ be open. Since $U \cap U_1 \subset U \cap U_2 \subset \cdots \subset U$ and $\bigcup_{n=1}^{\infty} (U \cap U_n) = U$ we have that $\nu(U) = \lim_{n \to \infty} \nu(U \cap U_n)$ and hence, it is enough to show that

$$\nu(U \cap U_n) = \sup\{\nu(K) : K \text{ is compact and } K \subset U \cap U_n\}$$

for each $n \in \mathbb{N}$. Now let $n \in \mathbb{N}$ and $\varepsilon > 0$. Suppose first that $\mu(U \cap U_n) < +\infty$. Then we can, by regularity of $\mu$, choose a compact $K$ such that $K \subset U \cap U_n$ and $\mu((U \cap U_n) \setminus K) < \varepsilon/n$. It follows that

$$\nu((U \cap U_n) \setminus K) = \int_{(U \cap U_n) \setminus K} f d\mu \leq n\mu((U \cap U_n) \setminus K) < \varepsilon.$$

Next, suppose that $\mu(U \cap U_n) = +\infty$ and let $M > 0$. Again, by the regularity of $\mu$, we can choose a compact set $K$ such that $K \subset (U \cap U_n)$ and $\mu(K) > nM$. It follows that

$$\nu(K) = \int_{K} f d\mu \geq \frac{1}{n}\mu(K) > M.$$ 

Hence, $\nu$ is inner-regular.

In order to show outer-regularity of $\nu$ let $A$ be an arbitrary Borel set. If $\nu(A) = +\infty$ then the open set $X$ satisfies $\nu(X) = +\infty$ and thus we can restrict our attention to the case when $\nu(A) < +\infty$. Then for any $n \in \mathbb{N}$ we have

$$\mu(U_n \cap A) \leq n \int_{U_n \cap A} f d\mu = n\nu(U_n \cap A) < +\infty$$

and hence, for $\varepsilon > 0$ we may choose, using the outer-regularity of $\mu$, an open subset $V_n$ that includes $U_n \cap A$ and that satisfies $\mu(V_n) < \mu(U_n \cap A) + \varepsilon/n2^n$. By considering the open set $V_n \cap U_n$ if necessary, we may suppose that $V_n \subset U_n$. We obtain that

$$\nu(V_n \cap A) = \nu(V_n \setminus (U_n \cap A)) = \int_{V_n \setminus (U_n \cap A)} f d\mu \leq n\mu(V_n \setminus (U_n \cap A))$$

$$= n(\mu(V_n) - \mu(U_n \cap A)) < \varepsilon/2^n.$$ 

Now $V := \bigcup_{n=1}^{\infty} V_n$ is open and $A$ is contained in $V$. Moreover,

$$\nu(V \setminus A) = \nu((\bigcup_{n=1}^{\infty} V_n) \setminus A) = \nu(\bigcup_{n=1}^{\infty} (V_n \setminus A)) \leq \sum_{n=1}^{\infty} \nu(V_n \setminus A) = \sum_{n=1}^{\infty} \varepsilon/n2^n = \varepsilon,$$

and hence, $\nu$ is outer-regular.

Finally, for any compact set $K \subset X$ let $M = \max_{x \in K} f(x)$. Then we have

$$\nu(K) = \int_{K} f d\mu \leq M\mu(K) < +\infty$$

by the regularity of $\mu$. We conclude that $\nu$ is regular. \qed
**Definition 2.22.** Let $X$ and $Y$ be sets and $E \subseteq X \times Y$. Then the *sections* of $E$ are defined as

$$E_x := \{y \in Y : (x, y) \in E\} \quad \text{and} \quad E^y := \{x \in X : (x, y) \in E\}.$$ 

Similarly we define the sections of a function $f$ on $X \times Y$ by

$$f_x(y) = f(x, y) \quad \text{and} \quad f^y(x) = f(x, y).$$

We proceed by presenting the Radon-Nikodym theorem and looking at some of its immediate consequences.

Let $(X, \Gamma, \mu)$ be a measure space and let $f \in L^1(X, \mu)$. For each $A \in \Gamma$ define

$$\nu_f(A) := \int_A f \, d\mu.$$ 

Then $\nu_f$ is a complex measure on $X$ and we have

$$|\nu_f|(A) = \int_A |f| \, d\mu,$$

so in particular $||\nu_f|| = ||f||_1$ (see [Coh] chapter 4). Recall that a measure $\mu$ on a measurable space $X$ is *σ-finite* if there are measurable set $A_1, A_2, \ldots$ with finite measure such that $X = \bigcup_{n=1}^\infty A_n$.

**Theorem 2.23.** (Radon-Nikodym) Let $\mu$ be a $\sigma$-finite measure on $X$, and let $\nu$ be a complex measure on $X$ that is absolutely continuous with respect to $\mu$ (denoted $\nu << \mu$). Then there is a unique $f \in L^1(X, \mu)$ (that is, unique up to $\mu$-a.e. equality) such that

$$\nu(A) = \int_A f \, d\mu.$$ 

The function $f$ in the Radon-Nikodym theorem is called the *Radon-Nikodym derivative* of $\nu$ with respect to $\mu$ and is denoted by $d\nu/d\mu$.

Clearly $\mu << |\mu|$ for a complex measure $\mu$ (the converse need not to be true) and hence, the Radon-Nikodym theorem says that

$$\mu(A) = \int_A \left( \frac{d\mu}{d|\mu|} \right) |\mu|$$

for all measurable sets $A$. By going from characteristic functions to simple functions and using the dominated convergence theorem one can verify that

$$\int f d\mu = \int f \left( \frac{d\mu}{d|\mu|} \right) |\mu|$$

hold for each $f \in B(X, \mu)$. Furthermore, $|d\mu/d|\mu|| = 1$ $|\mu|$-a.e. (see [Coh, (4.2.5) Corollary]) and thus we get the following useful inequality:

$$|\int f d\mu| = \left| \int f \frac{d\mu}{d|\mu|} |\mu| \right| \leq \int \left| f \frac{d\mu}{d|\mu|} \right| |\mu| = \int |f| |\mu| \quad (f \in B(X, \mu)).$$

Let $X$ be a locally compact Hausdorff space. For a regular Borel measure $\mu$ on $X$ we denote by $M_\sigma(X, \mu)$ the Banach space of all complex measures $\nu$ such that $\nu << \mu$. Recall that $\nu_f$
defined by $\nu_f(A) = \int_A f\,d\mu$ for $f \in L^1(X, \mu)$ is a complex measure. In fact, since $X$ is a locally compact Hausdorff space and $\mu$ is regular, it follows that also $\nu_f$ is regular (see [Coh, (7.3.7) Proposition]). Now, clearly $\nu << \mu$ and hence the map $f \mapsto \nu_f$ is a linear isometric map from $f \in L^1(X, \mu)$ to $M_\sigma(X, \mu)$. It can be shown by using the Radon-Nikodym theorem that the map $f \mapsto \nu_f$ is in fact surjective and hence an isometric isomorphism.

**Proposition 2.25.** Let $X$ be a locally compact Hausdorff space and let $\mu$ be a regular Borel measure on $X$. For a complex regular measure $\nu$ on $X$, the following are equivalent:

(i) $\nu << \mu$.

(ii) There is a function $f \in L^1(X, \mu)$ such that

$$\nu(A) = \int_A f\,d\mu.$$ 

Thus, by the previous proposition we can identify $L^1(X, \mu)$ with $M_\sigma(X, \mu)$ as Banach spaces.

We proceed by presenting Riesz’ representation theorem which gives an important connection between measures and functionals. In particular, the complex-valued version of Riesz’ representation theorem is widely used in chapter 3.

**Theorem 2.26.** (Riesz’ representation theorem) Let $X$ be a locally compact Hausdorff space, and let $I$ be a positive linear functional on $C_c(X)$, that is, $I(f) \geq 0$ for all $f \in C_c(X)$ such that $f \geq 0$. Then there is a unique regular Borel measure $\mu$ on $X$ such that

$$I(f) = \int_X f\,d\mu \quad (f \in C_c(X)).$$

**Theorem 2.27.** (Riesz’ representation theorem for complex measures) Let $X$ be a locally compact Hausdorff space. For each complex regular Borel measure $\mu$ on $X$ define a linear functional $\Phi_\mu : C_0(X) \to \mathbb{C}$ by

$$\Phi_\mu(f) = \int_X f\,d\mu \quad (f \in C_0(X)).$$

Then the map $\mu \mapsto \Phi_\mu$ is an isometric isomorphism from $M_r(X)$ onto $C_0(X)^*$.

### 2.2.3 Fubini’s Theorem

Let $(X, \Gamma_1, \mu)$ and $(Y, \Gamma_2, \nu)$ be measure spaces with a locally compact Hausdorff topology such that $\mu$ and $\nu$ are regular. We recall that if $X$ and $Y$ are $\sigma$-finite, then there exists a unique regular measure $\mu \times \nu$ on $\Gamma_1 \times \Gamma_2$ such that

$$(\mu \times \nu)(A \times B) = \mu(A)\nu(B) \quad (A \in \Gamma_1, B \in \Gamma_2).$$

In general, if $X$ and $Y$ are not $\sigma$-finite we cannot expect such a good behaviour of product measures. For instance $\mathcal{B}(X) \times \mathcal{B}(Y)$ does not necessarily contain all Borel subsets of $X \times Y$, and so a measure defined on $\mathcal{B}(X) \times \mathcal{B}(Y)$ is not a Borel measure. There is, however, a natural way to define the product of two regular Borel measures through Riesz’ representation theorem. For that, we will need Fubini’s theorem for functions in $C_c(X)$.

**Theorem 2.28.** Let $X$ and $Y$ be locally compact Hausdorff spaces, and let $\mu$ and $\nu$ be regular Borel measures on $X$ and $Y$ respectively. Then for $f \in C_c(X \times Y)$ the functions

$$x \mapsto \int_Y f_x(y)\,d\nu(y) \quad \text{and} \quad y \mapsto \int_X f_y(x)\,d\mu(dx)$$
belong to $C_c(X)$ and $C_c(Y)$ respectively and
\[ \int_X \int_Y f(x,y)\nu(dy)\mu(dx) = \int_Y \int_X f(x,y)\mu(dx)\nu(dy). \]

Now, we may define a positive functional $I$ on $C_c(X \times Y)$ by
\[ I(f) = \int_X \int_Y f(x,y)\nu(dy)\mu(dx) \quad (f \in C_c(X \times Y)) \]
and thus by Riesz representation theorem there exists a unique regular Borel measure on $X \times Y$, which we will denote by $\mu \times \nu$, such that $\int_{X \times Y} f d(\mu \times \nu) = \int_X \int_Y f(x,y)\nu(dy)\mu(dx)$.

The following proposition gives a way to calculate the regular product measure of, in many cases, sufficiently many Borel sets.

**Proposition 2.29.** Let $X$ and $Y$ be locally compact Hausdorff spaces, and let $\mu$ and $\nu$ be regular Borel measures on $X$ and $Y$ respectively. Let $E$ be an open subset of $X \times Y$ or a Borel subset of $X \times Y$ such that there exists $\sigma$-finite Borel subsets $A \subset X$ and $B \subset Y$ for which $E \subset A \times B$. Then the functions
\[ x \mapsto \nu(E_x) \quad \text{and} \quad y \mapsto \mu(E^y) \]
are measurable and
\[ (\mu \times \nu)(E) = \int_X \nu(E_x)\mu(dx) = \int_Y \mu(E^y)\nu(dy). \]

From the above proposition we get Fubini’s theorem for locally compact Hausdorff spaces.

**Theorem 2.30.** (Fubini 1) Let $X$ and $Y$ be locally compact Hausdorff spaces with regular Borel measures $\mu$ and $\nu$ on $X$ and $Y$ respectively. Let $f$ be a non-negative real Borel measurable function on $X \times Y$ and suppose that there exists $\sigma$-finite Borel sets $A \in \mathcal{B}(X)$ and $B \in \mathcal{B}(Y)$ such that $f(x,y) = 0$ if $(x,y) \notin A \times B$. Then the functions
\[ x \mapsto \int_Y f(x,y)\nu(dy) \quad \text{and} \quad y \mapsto \int_X f(x,y)\mu(dx) \]
are measurable and
\[ \int_{X \times Y} f d(\mu \times \nu) = \int_Y \int_X f(x,y)\mu(dx)\nu(dy) = \int_X \int_Y f(x,y)\nu(dy)\mu(dx). \]

**Proof.** Suppose first that $f = \chi_E$ for some $E \in \mathcal{B}(X \times Y)$. By assumption, there exists $\sigma$-finite Borel sets $A \in \mathcal{B}(X)$ and $B \in \mathcal{B}(Y)$ such that $E \subset A \times B$. We note that $\nu(E_x) = \int_Y \chi_E(x,y)\nu(dy)$ and $\mu(E^y) = \int_X \chi_E(x,y)\mu(dx)$. Hence, by Proposition 2.29, the mappings
\[ x \mapsto \int_Y \chi_E(x,y)\nu(dy) \quad \text{and} \quad y \mapsto \int_X \chi_E(x,y)\mu(dx) \]
are measurable and
\[ \int_{X \times Y} \chi_E d(\mu \times \nu) = (\mu \times \nu)(E) = \int_Y \int_X \chi_E(x,y)\nu(dy)\mu(dx) = \int_X \int_Y \chi_E(x,y)\mu(dx)\nu(dy). \]
Now, by using the linearity of the integrals and the monotone convergence theorem, the theorem holds for general functions. \[\square\]
In a similar way one can prove:

**Theorem 2.31.** (Fubini 2) Let $X$ and $Y$ be locally compact Hausdorff spaces, let $\mu$ and $\nu$ be regular Borel measures on $X$ and $Y$ respectively. Let $f \in L^1(X \times Y)$ and suppose that there exists $\sigma$-finite Borel sets $A \in \mathcal{B}(X)$ and $B \in \mathcal{B}(Y)$ such that $f(x, y) = 0$ if $(x, y) \notin A \times B$. Then

a) $f_x$ is $\nu$-integrable for $\mu$-almost every $x$, and $f^y$ is $\mu$-integrable for $\nu$-almost every $y$.

b) The functions defined $\mu$-a.e. and $\nu$-a.e. respectively by

$$x \mapsto \int_Y f(x, y)\nu(dy) \quad \text{and} \quad y \mapsto \int_X f(x, y)\mu(dx)$$

are $\mu$- and $\nu$-integrable respectively.

c) $\int_{X \times Y} f d(\mu \times \nu) = \int_Y \int_X f(x, y)\mu(dx)\nu(dy) = \int_X \int_Y f(x, y)\nu(dy)\mu(dx)$. 
This chapter serves as a rather self-contained introduction to the theory of locally compact groups. Local compactness of the topology together with the group structure gives a rich theory without being too restrictive for a large variety of examples. For instance, a locally compact group $G$ is always normal as a topological space, which yields a large supply of continuous function on $G$ by Urysohn’s lemma. A locally compact group also possess a natural regular left-invariant Borel measure, a left Haar measure, which allows for a rich theory of integration.

3.1 Locally Compact Groups

In this short introductory section we recall important topological properties of locally compact groups.

Definition 3.1. A topological group is a group $G$ with a Hausdorff topology such that the group operations

\[ m : G \times G \to G, \quad (x, y) \mapsto xy \quad \text{and} \quad \iota : G \to G, \quad x \mapsto x^{-1} \]

are continuous. If the topology of a topological group $G$ is locally compact then $G$ is called a locally compact group.

From now on, $e$ will always denote the neutral element of a topological group $G$. For two subset $U$ and $V$ of $G$ we denote $UV = \{ uv : u \in U, v \in V \}$ and $V^n = \underbrace{VV \cdots V}_n$. The following properties of topological groups follow immediately from the continuity of the group operations.

Proposition 3.2. Let $G$ be a topological group.

(i) For each $y \in G$, the left and right translations

\[ L_y : x \mapsto yx \quad \text{and} \quad R_y : x \mapsto xy \]

respectively, and the inverse mapping $\iota : x \mapsto x^{-1}$ are homeomorphisms from $G$ onto $G$.

(ii) The product of two compact subsets of $G$ is compact.
(iii) For an open subset $U \subset G$, the sets $U^{-1}$, $AU$ and $UA$ are open for each $A \subset G$.

(iv) If $U$ is a neighbourhood of $e$ then there exists a neighbourhood $V$ of $e$ such that $V^2 \subset U$. Moreover, we can choose $V$ to be symmetric, that is, $V = V^{-1}$.

(v) Every open subgroup $H$ of $G$ is also closed.

(vi) For any neighbourhood $U$ of $e$ we have $U \subset U^2$.

**Proof.** (i) For each $y \in G$, the mapping $L_y$ can be written as the composition of two continuous maps, namely

$$G \rightarrow \{y\} \times G \rightarrow G, \quad x \mapsto (y, x) \mapsto yx$$

and hence, $L_y$ is continuous. It is also immediate that the inverse of $L_y$ is $L_{y^{-1}}$, which is continuous by the same argument. Hence $L_y$ is a homeomorphism. A similar argument holds for the right translations $R_y$, $y \in G$. Finally, since $e = e^{-1}$ and $i$ is continuous by definition, it is a homeomorphism.

(ii) If $K \subset G$ and $L \subset G$ are compact, then $KL$ is the image of the compact set $K \times L \subset G \times G$ under the multiplication map and thus compact.

(iii) Observing that $U^{-1} = i(U)$, $AU = \bigcup_{a \in A} L_a(U)$ and $UA = \bigcup_{a \in A} R_a(U)$ proves the claims together with (i).

(iv) Let $U$ be a neighbourhood of $e$. Since the multiplication map is continuous at $e$, there exists a neighbourhood $V$ of $e$ such that $V^2 \subset U$. Moreover, by (iii), $V^{-1}$ is a neighbourhood and clearly $e \in V^{-1}$. Hence $V \cap V^{-1}$ is a symmetric neighbourhood of $e$ and $(V \cap V^{-1})^2 \subset V^2 \subset U$.

(v) Let $H$ be an open subgroup of $G$. Then $xH$ is open for each $x \in G$ by (i) and since

$$H = G \setminus \bigcup_{x \in G \setminus H} xH$$

we see that $H$ is closed in $G$.

(vi) If $x \in \overline{U}$ then $xU^{-1}$ is a neighbourhood of $x$ by (i) and (iii). It follows that $xU^{-1} \cap U \neq \emptyset$ and hence, $x \in U^2$ which shows that $U \subset U^2$. \qed

It follows from Proposition 3.2 (i) and the Hausdorff property that $G$ is locally compact if and only if there exists a relatively compact neighbourhood of $e$. Recall that a topological space $X$ is called regular if given a point $x \in X$ and a closed subset $A \subset X$ such that $x \notin A$, there exist disjoint open subsets $U, V$ of $X$ such that $x \in U$ and $A \subset V$. Equivalently, $X$ is regular if given any point $x \in X$ and an open neighbourhood $U$ of $x$ there exist an open neighbourhood $V$ of $x$ such that $V \subset U$. A topological space $X$ is called normal if given two disjoint closed subsets $A, B \subset X$ there exist disjoint open subsets $U, V \subset X$ such that $A \subset U$ and $B \subset V$, or equivalently, there exists a continuous function $f : X \rightarrow [0, 1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$. The latter characterization of a normal space is referred to as Urysohn’s lemma. Finally, a topological space $X$ is called Lindelöf if every open cover of $X$ has a countable subcover.

**Corollary 3.3.** Every topological group $G$ is regular.
Proof. Let $U$ be a neighbourhood of $e$. By Proposition 3.2 (iv) there exists a neighbourhood $V$ such that $V^2 \subset U$. By Proposition 3.2 (vi) $\overline{V} \subset V^2 \subset U$ and by Proposition 3.2 (i) this is true in every point $x \in G$. Hence $G$ is regular.

From now on we shall only consider locally compact groups and we proceed by showing that every locally compact group is topologically normal. For this, we will use the following standard fact from topology (for a proof see [Kel, (4.1) Lemma]).

**Proposition 3.4.** Every regular Lindelöf space is normal.

Since every topological group is regular by Corollary 3.3, it remains to show that every locally compact group is Lindelöf. We begin with the following lemma.

**Lemma 3.5.** Let $X$ be a Hausdorff space and suppose that there exists a chain of relatively compact open sets $U_1 \subset U_2 \subset \cdots$ such that $X = \bigcup_{i=1}^{\infty} U_i$. Then $X$ is a Lindelöf space.

**Proof.** Let $U$ be an open cover of $X$. Define $U_0 := \emptyset$ and let $n \in \mathbb{N}$. Clearly $U_n \setminus U_{n-1}$ is a closed subset of $U_n$ and thus also compact since $X$ is Hausdorff. Therefore there exists a finite subcover $U_n$ of $U$ such that $U_n$ covers $U_n \setminus U_{n-1}$. It follows that $\bigcup_{n=1}^{\infty} U_n$ is a countable subcover of $U$ that covers $X$. 

Note that if $H$ is a subgroup of a topological group $G$ then $H$ is a topological group with respect to the relative topology on $H$. Furthermore:

**Proposition 3.6.** If $H$ is a closed subgroup of a locally compact group $G$, then $H$ is locally compact.

**Proof.** Let $U$ be a relatively compact neighbourhood of $e$. Now, $U \cap H$ is a neighbourhood of $e$ in $H$. Since $H$ is closed, the closure of $U \cap H$ in $H$ equals the closure of $U \cap H$ in $G$. Since $U \cap H$ is a compact subset of $U$, it is compact. Thus, $H$ is locally compact.

**Proposition 3.7.** Let $G$ be a locally compact group. Then $G$ has a subgroup $H$ which is both open and closed such that $H = \bigcup_{i=1}^{\infty} U_i$ where $U_1 \subset U_2 \subset \cdots$ is a chain of relatively compact open sets.

**Proof.** Let $U$ be an open symmetric neighbourhood of $e$ such that $U$ is compact (such a neighbourhood exists by the local compactness and the Hausdorff property combined with Propositions 3.2 (iv) and (vi)). Proposition 3.2 (iii) implies that $U^n$ is open for each $n \in \mathbb{N}$. It follows that $H := \bigcup_{n=1}^{\infty} U^n$ is an open subgroup of $G$ ($h \in H$ implies that $h^{-1} \in H$ by the symmetry of the sets $U^n$) and hence, by Proposition 3.2 (v), $H$ is also closed.

**Corollary 3.8.** Every locally compact group has an open and a closed subgroup which is topologically normal.

**Proof.** Follows directly from 3.2 (v), 3.3, 3.4 3.5 and 3.7.

We collect another corollary of Proposition 3.7 that will be needed in the future.

**Corollary 3.9.** Let $G$ be a locally compact group. Then $G$ has a subgroup that is open, closed and $\sigma$-compact (recall that a set is called $\sigma$-compact if it is a countable union of compact sets).
Proof. Let $U$ and $H$ be as in the proof of Proposition 3.7. Since $U^n \subset (\overline{U})^n \subset U^{2n}$ for each $n \in \mathbb{N}$, where the latter inclusion follows from Proposition 3.2 (vi), we have that

$$H = \bigcup_{n=1}^{\infty} (\overline{U})^n.$$  

Since $(\overline{U})^n$ is compact for each $n \in \mathbb{N}$ by Proposition 3.2 (ii) we conclude that $H$ is open, closed and $\sigma$-compact. \hfill $\Box$

**Proposition 3.10.** Let $G$ be a topological group and $H$ an open subgroup of $G$ which is topologically normal. Then $G$ is normal.

Proof. Let $A$ be a subset of $G$ such that the sets $aH$, $a \in A$ form a pairwise disjoint partition of $G$. Since the left translation $L_a$ is a homeomorphism for each $a \in A$, it follows that all cosets $aH$ are normal open topological subspaces of $G$. Now, let $C_1$ and $C_2$ be two disjoint closed neighbourhoods in $G$ and fix $a \in A$. Then $C_1 \cap aH$ and $C_2 \cap aH$ are disjoint closed neighbourhoods in $aH$. Thus, there exist two open disjoint neighbourhoods $U_a$ and $V_a$ in $aH$ such that $C_1 \cap aH \subset U_a$ and $C_2 \cap aH \subset V_a$. Since $aH$ is an open subset of $G$, it follows that $U_a$ and $V_a$ are open in $G$. Let

$$U := \bigcup_{a \in A} U_a \quad \text{and} \quad V := \bigcup_{a \in A} V_a.$$  

Then $U$ and $V$ are open subsets of $G$ and since the sets $aH$ form a pairwise disjoint partition of $G$ we have that $C_1 \subset U$, $C_2 \subset V$ and $U \cap V = \emptyset$. Hence $G$ is normal. \hfill $\Box$

Now, by combining Corollary 3.8 and Proposition 3.10 we get:

**Corollary 3.11.** Every locally compact group is normal.

### 3.2 Haar Measures

In this section we will introduce natural left-invariant measures on a locally compact group called left Haar measures. But first, we begin by presenting nice properties for compactly supported continuous functions on locally compact groups.

**Definition 3.12.** Let $G$ be a locally compact group. For a function $f$ on $G$ we define $L_x f = f \circ L_x$ and $R_x f = f \circ R_x$ for each $x \in G$, that is, $(L_x f)(y) = f(xy)$ and $(R_x f)(y) = f(yx)$ for $x, y \in G$. Note that if $f$ is continuous then $L_x f$ and $R_x f$ are continuous for every $x \in G$.

We say that a function $f$ on $G$ is left (respectively right) uniformly continuous if

$$\sup_{y \in G} |L_x f(y) - L_x f(y)| \to 0 \quad \text{(respectively} \sup_{y \in G} |R_x f(y) - R_x f(y)| \to 0)$$

whenever $(x_\alpha)_\alpha$ is a net in $G$ converging to some point $x \in G$.

**Remark 3.13.** In Definition 3.12 it is enough to consider nets converging to $e$. Indeed, if $x_\alpha \to x$ then $x_\alpha x^{-1} \to e$ and

$$\sup_{y \in G} |L_x f(y) - L_x f(y)| = \sup_{y \in G} |L_{x_\alpha x^{-1}} f(xy) - f(xy)| = \sup_{y \in G} |L_{x_\alpha} f(y) - f(y)|.$$

A similar argument holds for right uniform continuity.
Proposition 3.14. Let \( f \in \mathcal{C}_c(G) \). Then \( f \) is left and right uniformly continuous.

Proof. Let \( \varepsilon > 0 \) and let \( K \) denote the support of \( f \). Now, for every \( x \in K \) there exists, by continuity of \( f \), a neighbourhood \( U_x \) of \( x \) such that \( |f(z) - f(x)| < \varepsilon / 2 \) for all \( z \in xU_x \). For every \( x \in K \), let \( V_x \) be a symmetric neighbourhood of \( x \) such that \( V_x^2 \subset U_x \). Then \( \{V_x\}_{x \in K} \) is an open cover of \( K \), so for some \( x_1, \ldots, x_n \in K \), the subcovering \( \{x_iV_{x_i}\}_{i \in \mathbb{N}_n} \) is an open cover of \( K \). Let \( V := \bigcap_{i=1}^n V_{x_i} \). Then \( V \) is clearly a symmetric neighbourhood of \( x \).

Now, let \( x \in G \) and \( y \in V \) be arbitrary. We consider separately the three cases: \( x \in K \), \( xy \in K \) and \( x, xy \in K \).

If \( x \in K \) we have \( x \in x_iV_{x_i} \subset x_iU_{x_i} \) for some \( i \in \mathbb{N}_n \). Then also

\[
x y \in x_iV_{x_i} \subset x_iU_{x_i}.
\]

and so

\[
|R_y f(x) - f(x)| \leq |f(xy) - f(x_i)| + |f(x_i) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

If \( xy \in K \) then \( xy \in x_iV_{x_i} \subset x_iU_{x_i} \) for some \( i \in \mathbb{N}_n \). It follows by the symmetry of \( V \) that

\[
x \in x_iV_{x_i} \subset x_iV_{x_i} \subset x_iU_{x_i}.
\]

Hence,

\[
|R_y f(x) - f(x)| \leq |f(xy) - f(x_i)| + |f(x_i) - f(x)| < \varepsilon.
\]

Finally, if both \( x \) and \( xy \) are not in \( K \), then \( f(x) = f(xy) = 0 \).

Thus \( \sup_{x \in G} |R_y f(x) - f(x)| \to 0 \) when \( (y_\alpha)_\alpha \) is a net in \( G \) such that \( y_\alpha \to e \) and hence, \( f \) is right uniformly continuous by Remark 3.13. A similar argument shows that \( f \) is left uniformly continuous.

Proposition 3.15. Let \( G \) be a locally compact group with a regular Borel measure \( \mu \), and let \( f \in \mathcal{C}_c(G) \). Then the functions

\[
x \mapsto \int_G L_x f \, d\mu \quad \text{and} \quad x \mapsto \int_G R_x f \, d\mu,
\]

are continuous.

Proof. Let \( (x_\alpha)_\alpha \) be a net in \( G \) converging to some point \( x \in G \). Denote the compact support of \( f \) by \( K \) and let \( W \) be a compact neighbourhood of \( x \). Now, the function \( y \mapsto L_x f(y) \) is continuous and is supported by the compact set \( W^{-1}K \). Let \( \alpha_0 \) be an index such that \( x_\alpha \in W \) for all \( \alpha \geq \alpha_0 \). It follows that for all \( \alpha \geq \alpha_0 \) we have

\[
|\int_G L_{x_\alpha} f \, d\mu - \int_G L_x f \, d\mu| \leq \int_G |L_{x_\alpha} f - L_x f| \, d\mu \leq \mu(W^{-1}K) \sup_{y \in G} |L_{x_\alpha} f(y) - L_x f(y)|
\]

and since \( f \) is left uniformly continuous by Proposition 3.14, the right side tends to zero. A similar argument holds for the mapping \( x \mapsto \int_G R_x f \, d\mu \).

A left Haar measure on \( G \) is a non-zero regular Borel measure \( m_G \) on \( G \) which is left-invariant, that is, \( m_G(xE) = m_G(E) \) for every Borel set \( E \subset G \) and every \( x \in G \) (since \( L_x \) is a homeomorphism note that \( xE \) is also a Borel set). The following theorem is of fundamental importance in abstract harmonic analysis.
**Theorem 3.16.** Every locally compact group possesses a left Haar measure.

The proof of Theorem 3.16 is rather technical and lengthy and is omitted here (see [Coh, (9.2.1) Theorem] for a proof). Note that if $m_G$ is a left Haar measure on $G$, we have

$$\int_G L_x f dm_G = \int_G f dm_G \quad (x \in G)$$

for simple Borel functions on $G$ and thus, by using the monotone convergence theorem, also for all $f \in L^1(G)$. The following is a basic property of left Haar measures.

**Proposition 3.17.** Let $G$ be a locally compact group with a left Haar measure $m_G$. Let $U$ be a non-empty open set in $G$. Then $m_G(U) > 0$. Moreover, $\int_G f dm_G > 0$ for each $f \in C^+_{\text{UC}}(G)$.

**Proof.** Choose a compact set $K$ such that $m_G(K) > 0$ (such a set exists by the regularity of $m_G$). Now, clearly \( \{xU\}_{x \in G} \) is an open cover of $K$ and hence, for some elements $x_1, \ldots, x_n \in G$, \( \{x_iU\}_{i \in \mathbb{N}_n} \) is an open cover of $K$. Hence

$$m_G(K) \leq \sum_{i=1}^n m_G(x_iU) = nm_G(U)$$

where the latter equality follows from the left-invariance of $m_G$.

Moreover, for any $f \in C^+_{\text{UC}}(G)$ there exists a non-empty open set $U$ such that $f \geq \lambda 1_U$ for some $\lambda > 0$. Then $\int_G f dm_G \geq \lambda m_G(U) > 0$. \( \square \)

It follows that for continuous functions $f, g : G \to \mathbb{C}$ we have $f = g$ $m_G$-a.e. precisely when $f = g$. Also for $f \in C_b(G)$, the $m_G$-essential supremum norm $||f||_\infty$ defined in Chapter 2 coincides with the usual supremum norm given by $\sup_{x \in G} |f(x)|$.

Hence, a function $f$ in $C_b(G)$ is left (resp. right) uniformly continuous if the map

$$G \to C_b(G), \quad x \mapsto L_x f \quad (\text{resp. } x \mapsto R_x f)$$

is continuous. We denote the set of all left and right uniformly continuous functions $f \in C_b(G)$ by $\text{LUC}(G)$ and $\text{RUC}(G)$ respectively. A function $f : G \to \mathbb{C}$ is uniformly continuous if $f \in \text{UC}(G) := \text{LUC}(G) \cap \text{RUC}(G)$.

Clearly, if $m_G$ is a left Haar measure then also for any $c > 0$, the measure $cm_G$ defined by $(cm_G)(A) = cm_G(A)$ for all $A \in \mathfrak{B}(G)$ is a left Haar measure. The following property shows that there are actually no other kind of left Haar measures. From now on let $m_G$ always denote a fixed left Haar measure on a locally compact group $G$.

**Theorem 3.18.** Let $G$ be a locally compact group and let $\nu$ be a left Haar measure on $G$. Then $\nu = cm_G$ for some $c > 0$.

**Proof.** Let $g \in C^+_{\text{UC}}(G)$. Then also $R_x g \in C^+_{\text{UC}}(G)$ for each $x \in G$ and so $\int_G R_x g dm > 0$ by Proposition 3.17. Thus $x \mapsto \int_G R_x g dm$ is a continuous function with non-zero values (Proposition 3.15). Hence, for an arbitrary $f \in C_b(G)$ we can define a continuous function $h : G \times G \to \mathbb{C}$ by

$$h(x, y) = \frac{f(x)g(yx)}{\int_G R_x g dm}$$

Now, let $K$ and $L$ denote the supports of $f$ and $g$ respectively. Then $h$ is supported by the compact set $K \times LK^{-1}$. Indeed, if $x \notin K$ then clearly $h(x, y) = 0$. On the other hand, if $x \in K$ and $y \notin LK^{-1}$ then $yx \notin L$ and so $h(x, y) = 0$. 
Hence, by denoting $c$ (Theorem 2.28), the order of integration in the double integral $\int_G \int_G h(x, y) m_G(dx) \nu(dy)$ does not matter. We see that

$$\int_G \int_G h(x, y) m_G(dx) \nu(dy) = \int_G \int_G \frac{f(x) g(yx)}{\int_G g(tx) \nu(dt)} m_G(dx) \nu(dy)$$

$$= \int_G \frac{f(x) g(yx)}{\int_G g(tx) \nu(dt)} m_G(dx) \nu(dy) = \int_G f(x) m_G(dx).$$

On the other hand, by using the left-invariance of $m_G$ and $\nu$ and changing the order of integration we obtain

$$\int_G \int_G h(x, y) m_G(dx) \nu(dy) = \int_G \int_G \frac{f(y^{-1} x) g(x)}{\int_G g(ty^{-1} x) \nu(dt)} m_G(dx) \nu(dy)$$

$$= \int_G \int_G \frac{f((xy)^{-1} x) g(x)}{\int_G g(t(xy)^{-1} x) \nu(dt)} \nu(dy) m_G(dx)$$

$$= \int_G \int_G \frac{f(y^{-1} x) g(x)}{\int_G g(ty^{-1}) \nu(dt)} m_G(dx) \nu(dy)$$

$$= \int_G g(x) m_G(dx) \int_G \frac{f(y^{-1})}{\int_G g(ty^{-1}) \nu(dt)} \nu(dy).$$

It follows that

$$\frac{\int_G f(x) m_G(dx)}{\int_G g(x) m_G(dx)} = \int_G \frac{f(y^{-1})}{\int_G g(ty^{-1}) \nu(dt)} \nu(dy).$$

In particular, the ratio $\frac{\int_G fdv}{\int_G gdv}$ does not depend on $m_G$ and therefore $\frac{\int_G fdm_G}{\int_G gdm_G} = \frac{\int_G fdv}{\int_G gdv}$.

Hence, by denoting $c := \frac{\int_G gdv}{\int_G gdm_G}$ we have that $\int_G fdv = c \int_G fdm_G$.

Finally, it follows from Riesz’ representation theorem that $\nu = cm_G$ is the unique measure such that $\int_G fdv = c \int_G fdm_G$ for all $f \in C_c(G)$. \qed

A left Haar measure is not in general a right Haar measure, that is, a right-invariant non-zero regular Borel measure on $G$. However, as the following proposition shows, right translations of functions behave quite well when integrating with respect to a left Haar measure.

**Proposition 3.19.** Let $G$ be a locally compact group. Then there is a continuous homomorphism $\Delta : G \to \mathbb{R}_+$ where $\mathbb{R}_+$ denotes the multiplicative group of positive real numbers, such that

$$\int_G R_y f dm_G = \Delta(y^{-1}) \int_G f dm_G \quad (f \in L^1(G), y \in G).$$

**Proof.** For $x \in G$ we may define a new measure $m^x_G$ by $m^x_G(E) = m_G(Ex)$ for $E \in \mathcal{B}(G)$. Since $R_x$ is a homomorphism $m^x_G$ is regular and by the left-invariance of $m_G$ we have $m^x_G(yE) = m_G(yEx) = m_G(Ex) = m^x_G(E)$ which shows that $m^x_G$ is also a left Haar measure. Hence, by
Theorem 3.18. \( m_G^x = \Delta(x)m_G \) for some positive number \( \Delta(x) \). Suppose that \( n_G \) is another left Haar measure on \( G \) such that \( n_G = cm_G \) for some \( c > 0 \). By defining \( n_G^x \) similarly as \( m_G^x \), we also have \( n_G^x = cm_G^x \) and thus
\[
n_G^x = cm_G^x = c\Delta(x)m_G = \Delta(x)n_G,
\]
which shows that \( \Delta(x) \) does not depend on the left Haar measure in question. Furthermore, let \( U \in \mathcal{B}(G) \) have non-zero measure. Then
\[
\Delta(xy)m_G(U) = m_G(Uxy) = \Delta(y)m_G(Ux) = \Delta(y)\Delta(x)m_G(U) \quad (x, y \in G),
\]
which shows that \( \Delta \) is a homomorphism. Furthermore, by observing that \( \chi_U(xy) = \chi_Uy^{-1}(x) \) we have
\[
\int_G \chi_U(xy)m_G(dx) = m_G(Uy^{-1}) = \Delta(y^{-1})m_G(U) = \Delta(y^{-1}) \int_G \chi_U(x)m_G(dx),
\]
and thus by linearity and the monotone convergence theorem, the equation (3.20) holds for any \( f \in L^1(G) \).

To show continuity of \( \Delta \) we recall from Proposition 3.15 that \( y \mapsto \int_G R_y f dm_G \) is continuous for \( f \in C_c(G) \) and hence, by (3.20), \( y \mapsto \Delta(y^{-1}) \int_G f dm_G \) is continuous. Since the inverse mapping and scalar multiplication are continuous, we conclude that \( \Delta \) is continuous. \( \square \)

Remark 3.21. (i) A locally compact group \( G \) is called unimodular if \( \Delta \equiv 1 \), that is, if left Haar measures are right-invariant. Clearly, Abelian groups are unimodular. Also, any compact group \( G \) is unimodular. Indeed, since \( \Delta \) is a continuous homomorphism, \( \Delta(G) \) is a compact subgroup of \( \mathbb{R} \) and hence \( \Delta(G) = \{1\} \).

(ii) A Haar measure is finite if and only if \( G \) is compact. In such a case it is customary to speak of the Haar measure on \( G \) which is the normalized Haar measure \( m_G \) such that \( m_G(G) = 1 \).

(iii) It is easy to construct a right Haar measure from a left Haar measure and vice versa. Indeed, since \( \iota: G \to G, g \mapsto g^{-1} \) is a homeomorphism, it is easy to see that \( m_G \) is a left Haar measure and only if \( m_G \) defined by \( \tilde{m}_G(A) = m_G(A^{-1}) \) is a right Haar measure. Hence, by this observation and Theorem 3.18 it follows that also all right-invariant Haar measures are equal up to a multiplicative positive constant. Furthermore, we will need the fact that all left and right Haar measures are equivalent, that is, they have the same sets of measure zero. To see this, we will need the following characterization of \( \tilde{m}_G \).

Proposition 3.22. Let \( G \) be a locally compact group. Then
\[
\tilde{m}_G(A) = \int_A \Delta(x^{-1})m_G(dx) \quad (A \in \mathcal{B}(G)).
\]

Proof. Define \( \nu: \mathcal{B}(G) \to [0, +\infty] \) by
\[
\nu(A) = \int_A \Delta(x^{-1})m_G(dx) \quad (A \in \mathcal{B}(G)).
\]
In Proposition 2.21 we showed that \( \nu \) is a regular Borel measure. It follows by using Proposition
3.19 that

\[ \nu(A) = \int_G \chi_{A_y}(x) \Delta(x^{-1}) m_G(dx) = \int_G \chi_{A_y}(x) \Delta(y^{-1}) \Delta(y) \Delta(x^{-1}) m_G(dx) \]

\[ = \int_G \chi_{A_y}(x) \Delta(y^{-1}) \Delta((xy)^{-1}) m_G(dx) \]

\[ = \Delta(y^{-1}) \int_G \chi_{A_x}(x) \Delta((xy)^{-1}) m_G(dx) \]

\[ = \Delta(y^{-1}) \Delta(y) \int_G \chi_{A}(x) \Delta(x^{-1}) m_G(dx) \]

\[ = \nu(A) \quad (A \in \mathfrak{B}(G)). \]

Hence, \( \nu \) is right-invariant and thus \( \nu = cm_G \) for some \( c > 0 \) by Remark 3.21. Now, since \( \Delta \) is continuous and \( \Delta(e) = 1 \) we can choose for any \( \varepsilon > 0 \) a symmetric neighbourhood \( A \) of \( e \) such that \( |\Delta(x) - 1| < \varepsilon \) for all \( x \in A \). Hence

\[ c = \frac{\nu(A)}{m_G(A)} = \frac{\nu(A)}{m_G(A^{-1})} = \frac{1}{m_G(A)} \int_A \Delta(x^{-1}) m_G(dx) < \frac{m_G(A)}{m_G(A)}(1 + \varepsilon) = 1 + \varepsilon. \]

Similarly \( c > 1 - \varepsilon \) and so \( c = 1 \). Hence, \( \nu = \hat{m}_G. \)

Corollary 3.23. All left and right Haar measures are pairwise equivalent.

Proof. Clearly all left Haar measures are pairwise equivalent, as are the right Haar measures and hence it suffices to show that if \( m_G \) is a left Haar measure then \( m_G \) and \( \hat{m}_G \) are equivalent. To see this, suppose that \( m_G \) is a left Haar measure such that \( m_G(A) = 0 \) for some Borel set \( A \). Then by Proposition 3.22, \( \hat{m}_G(A) = \int_A \Delta(x^{-1}) m_G(dx) = 0. \) Conversely, if \( \hat{m}_G(A) = \int_A \Delta(x^{-1}) m_G(dx) = 0 \) for some Borel set \( A \), then \( m_G(A) = 0 \) since \( \Delta > 0 \) everywhere.

We end this section by showing that the left and right translates of a fixed function in \( L^1(G) \) behave well with respect to the \( L^1 \)-norm.

Proposition 3.24. Let \( G \) be a locally compact group and \( f \in L^1(G) \). Then the mappings

\[ G \to L^1(G), \quad x \mapsto L_x f \] and \( x \mapsto R_x f \]

are continuous.

Proof. We first show continuity at \( e \). For that, let \((y_\alpha)\) be a net in \( G \) such that \( y_\alpha \to e \) and let \( V \) be a compact symmetric neighbourhood of \( e \) and \( g \in C_c(G) \). Denote \( K := (\text{supp } g)V \cup V(\text{supp } g) \). Then \( K \) is compact by Proposition 3.2 and the fact that a union of two compact sets is compact. Now, clearly \( L_y g \) and \( R_y g \) are supported by \( K \) for all \( y \in V \). Let \( \alpha_0 \) be such that \( y_\alpha \in V \) for all \( \alpha \geq \alpha_0 \). Then we obtain for all \( \alpha \geq \alpha_0 \) that

\[ ||L_{y_\alpha} g - g||_1 = \int_G |L_{y_\alpha} g(x) - g(x)| m_G(dx) \leq ||L_{y_\alpha} g - g||_\infty m_G(K) \to 0 \]

since \( g \) is left uniformly continuous by Proposition 3.14. Analogously, \( ||R_{y_\alpha} g - g||_1 \to 0. \)

Next, let \( f \in L^1(G) \) and \( \varepsilon > 0 \). By left-invariance, we have \( ||L_y f||_1 = ||f||_1 \) and by Proposition 3.19 we have \( ||R_y f||_1 = \Delta(y^{-1}) ||f||_1 \leq C ||f||_1 \) for every \( y \in V \) where \( C = \)
max \{ \Delta(y^{-1}) : y \in V \}$. Since \( C_c(G) \) is dense in \( L^1(G) \) there exists \( g \in C_c(G) \) such that \( ||f-g||_1 < \varepsilon \). Hence, for \( \alpha \geq \alpha_0 \) we have

\[
||R_{y_n}f-f||_1 \leq ||R_{y_n}f-R_{y_n}g||_1 + ||R_{y_n}g-g||_1 + ||g-f||_1 \\
\leq C||f-g||_1 + ||R_{y_n}g-g||_1 + ||g-f||_1 < (C+1)\varepsilon + ||R_{y_n}g-g||_1,
\]

where the last term tends to zero by the calculation above. Thus \( ||R_{y_n}f-f||_1 \to 0 \). With a similar argument in which we do not have to consider the modular function we get \( ||L_{y_n}f-f||_1 \to 0 \) which shows continuity of \( x \mapsto L_xf \) at \( e \).

Now, let \( x \in G \) be arbitrary and let \( (x_\alpha)_\alpha \) be a net in \( G \) such that \( x_\alpha \to x \). By continuity we have \( x^{-1}x_\alpha \to e \) and \( x_\alpha x^{-1} \to e \) and since continuity at \( e \) was already verified, we have

\[
||L_{x^{-1}x_\alpha}f-f||_1 + ||R_{x^{-1}x_\alpha}f-f||_1 \to 0.
\]

Since \( m_G \) is left-invariant, we have

\[
||L_{x_\alpha}f-L_xf||_1 = ||L_x(L_{x_\alpha}x^{-1}f-f)||_1 = ||L_{x_\alpha}x^{-1}f-f||_1 \to 0.
\]

Also, by Proposition 3.19 we have

\[
||R_{x_\alpha}f-R_x||_1 = ||R_x(R_{x_\alpha}x^{-1}f-f)||_1 = \Delta(x^{-1})||R_{x^{-1}x_\alpha}f-f||_1 \to 0,
\]

which completes the proof. \( \square \)

### 3.3 The Group Algebra \( L^1(G) \)

In this section we shall introduce the convolution product in \( L^1(G) \) and prove that \( L^1(G) \) is a Banach algebra with convolution acting as the algebra multiplication. We begin with some technicalities.

**Lemma 3.25.** Let \( f \in L^1(G) \). Then \( A := \{ x \in G : f(x) \neq 0 \} \) is contained in a \( \sigma \)-compact set.

**Proof.** We first note that \( A \) is \( \sigma \)-finite. To see this, for each \( n \in \mathbb{N} \) define

\[
A_n := \{ x \in G : |f(x)| \geq \frac{1}{n} \}.
\]

Now, since \( f \in L^1(G) \) we have \( m_G(A_n) < +\infty \) for all \( n \in \mathbb{N} \). Clearly \( A = \bigcup_{n \in \mathbb{N}} A_n \) and hence, \( A \) is \( \sigma \)-finite. Now, by the outer-regularity of \( m_G \) there exist open subsets \( U_n \subset G \) such that \( m_G(U_n) < +\infty \) for each \( n \in \mathbb{N} \) and \( A \subset \bigcup_{n \in \mathbb{N}} U_n \). Let \( H \) be an open \( \sigma \)-compact subgroup of \( G \) (see Corollary 3.9). Let \( J \subset G \) be such that the sets \( xH, x \in J \), form a pairwise disjoint partition of \( G \). Now, the sets \( U_n \cap xH \) are open for all \( n \in \mathbb{N}, x \in J \) and each \( U_n \) is the pairwise disjoint union of the sets \( U_n \cap xH, x \in J \). Hence for each finite subset \( J_0 \subset J \) and \( n \in \mathbb{N} \) we have

\[
\sum_{x \in J_0} m_G(U_n \cap xH) = m_G(\bigcup_{x \in J_0} (U_n \cap xH)) \leq m_G(U_n).
\]

Thus \( \sum_{x \in J} m_G(U_n \cap xH) < +\infty \). Since each non-empty open set has non-zero measure, \( U_n \cap xH \neq \emptyset \) for at most countably many \( x \in J \). Hence, for each \( n \in \mathbb{N} \) there exists a countable index set \( I_n \subset J \) such that \( U_n \subset \bigcup_{x \in I_n} xH \). Thus

\[
A \subset \bigcup_{n \in \mathbb{N}} \bigcup_{x \in I_n} xH
\]

which proves the claim since each \( xH \) is \( \sigma \)-compact. \( \square \)
Lemma 3.26. Let $G$ be a locally compact group. Then the mapping $F : G \times G \to G \times G$, $F(x,y) = (x,xy)$ is a homeomorphism such that

$$(m_G \times m_G)(F(A)) = (m_G \times m_G)(A)$$

for each $A \in \mathcal{B}(G \times G)$.

Proof. Clearly, the mapping $F^{-1} : (x,y) \mapsto (x,x^{-1}y)$ is the inverse of $F$ and by continuity of the group operations both $F$ and $F^{-1}$ are continuous, hence $F$ is a homeomorphism. Now for each open subset $U \subset G$ and $x \in G$ we have

$$s \in F(U)_x \iff (x,s) \in F(U) \iff s = xy, \text{ for some } y \in U_x \iff s \in xU_x$$

(the section of a set was defined in Definition 2.22). Hence $F(U)_x = xU_x$ and so by Proposition 2.29 and the left-invariance of $m_G$ we have

$$(m_G \times m_G)(U) = \int_G m_G(U_x)m_G(dx) = \int_G m_G(xU_x)m_G(dx) = \int_G m_G(F(U)_x)m_G(dx) = (m_G \times m_G)(F(U)).$$

Since $m_G \times m_G$ is regular and $F$ is a homeomorphism, the measure $(m_G \times F)$ is also regular. Thus we have

$$(m_G \times m_G)(A) = \inf \{(m_G \times m_G)(U) : U \text{ is open and } A \subset U\}$$

$$= \inf \{(m_G \times m_G)(F(U)) : U \text{ is open and } A \subset U\} = (m_G \times m_G)(F(A))$$

for all $A \in \mathcal{B}(G \times G)$. \hfill \Box

The following proposition implies that the convolution operator can be defined on $L^1(G)$.

Proposition 3.27. Let $G$ be a locally compact. Let $f,g \in L^1(G)$ and define $\phi : G \times G \to \mathbb{C}$ by $\phi(x,y) = f(x)g(x^{-1}y)$. Then the function $x \mapsto \phi(x,y)$ is integrable for $m_G$-almost every $y \in G$.

Proof. Define $\phi' : G \times G \to G$ by $\phi'(x,y) \mapsto f(x)(g(y))$. Then $\phi' : (x,y) \mapsto (f(x),g(y)) \mapsto f(x)g(y)$ is a composition of two measurable functions and hence, measurable. By Lemma 3.25 there are compact sets $K_n$ and $K_n'$, $n \in \mathbb{N}$ such that $f$ and $g$ vanishes outside $A := \bigcup_{n=1}^{\infty} K_n$ and $B := \bigcup_{n=1}^{\infty} K_n'$ respectively. Then $\phi'(x,y) = f(x)g(y) = 0$ when $(x,y) \notin \bigcup_{n,m=1}^{\infty} (K_n \times K_m') = A \times B$. We also have that

$$\int_G \int_G |\phi'(x,y)| m_G(dx)m_G(dy) = \int_G \int_G |f(x)g(y)| m_G(dx)m_G(dy) = ||f||_1||g||_1 < \infty$$

and hence, by Fubini’s theorem 2.30, $|\phi'|$ belongs to $L^1(G \times G)$ and thus also $\phi' \in L^1(G \times G)$. Define $F$ as in Lemma 3.26. Then $\phi = \phi' \circ F^{-1}$ and since $F$ is a measure preserving homeomorphism it follows that

$$\int_{G \times G} |\phi| d(m_G \times m_G) = \int_{G \times G} |\phi' \circ F^{-1}| d(m_G \times m_G) = \int_{G \times G} |\phi'| d((m_G \times m_G)F)$$

$$= \int_{G \times G} |\phi'| d(m_G \times m_G) < +\infty$$
and thus $\phi \in L^1(G \times G)$. Denote $A_{n,m} := K_n \times K'_m$. The sets $A_{n,m}$ are compact and thus they have finite measure. Since $F$ is a homeomorphism we have

$$(x,y) \in \bigcup_{n,m=1}^{\infty} F(A_{n,m}) \iff F^{-1}(x,y) \in \bigcup_{n,m=1}^{\infty} A_{n,m}$$

and so $\phi(x,y) = 0$ when $(x,y) \notin \bigcup_{n,m=1}^{\infty} F(A_{n,m})$. Since $m_G(F(A_{n,m})) = m_G(A_{n,m})$ for each $n, m \in \mathbb{N}$ it follows that $\phi$ vanishes outside a $\sigma$-compact set. Hence, by Fubini’s theorem 2.31 $x \mapsto \phi(x,y)$ belongs to $L^1(G)$ for almost every $y \in G$. \hfill \Box

It follows from Proposition 3.27 that we may define the convolution $f * g$ of $f,g \in L^1(G)$ through

$$f * g(y) = \int_G f(x)g(x^{-1}y)m_G(dx)$$

for $m_G$-almost every $y \in G$. By the linearity of integration we note that the mapping

$$L^1(G) \times L^1(G) \to L^1(G), \quad (f,g) \mapsto f * g$$

is bilinear. Furthermore, by the left-invariance of the Haar measure and Fubini’s theorem we obtain

$$||f * g||_1 = \int_G |f * g| dm_G = \int_G | \int_G f(x)g(x^{-1}y)m_G(dx)m_G(dy) |
\leq \int_G \int_G |f(x)g(x^{-1}y)|m_G(dx)m_G(dy) = \int_G \int_G |f(x)||g(x^{-1}y)|m_G(dy)m_G(dx)
\leq \int_G |f(x)| \left( \int_G |g(y)|m_G(dy) \right)m_G(dx) = ||f||_1 ||g||_1 \quad (f,g \in L^1(G)).$$

The estimate above implies that $f * g \in L^1(G)$. To see that the convolution is indeed well-defined on $L^1(G)$, let $f, f', g, g' \in L^1(G)$ such that $f \sim f'$ and $g \sim g'$. Then

$$||f * g - f' * g'||_1 \leq ||f * (g - g')||_1 + ||(f - f') * g'||_1 \leq ||f||_1 ||g - g'||_1 + ||f - f'||_1 ||g'||_1 = 0.$$ 

Hence $f * g \sim f' * g'$ (in fact, one can show that $f * g(x) = f' * g'(x)$ for all $x \in G$). It is also immediate from the estimation above that for a fixed $g \in L^1(G)$, the mappings

$$L^1(G) \to L^1(G), \quad f \mapsto f * g \text{ and } f \mapsto g * f$$

define bounded linear operators on $L^1(G)$.

**Remark 3.28.** By using the left-invariance of the Haar measure and Proposition 3.22 we can express the convolution $f * g$ in the following ways:

$$f * g(x) = \int_G f(y)g(y^{-1}x)m_G(dy)
= \int_G f(xy)g(y^{-1})m_G(dy)
= \int_G f(y^{-1})g(yx)\Delta(y^{-1})m_G(dy)
= \int_G f(xy^{-1})g(y)\Delta(y^{-1})m_G(dy).$$
We also have the following identities
\[ L_x(f * g) = (L_x f) * g \]
\[ R_x(f * g) = f * (R_x g) \]
\[ f * L_x g = \Delta(x)(R_x f * g). \]

For instance, by using Proposition 3.19 the third identity is seen from:

\[ (f * L_x g)(y) = \int_G f(h)g(xh^{-1}y)m_G(dh) = \Delta(x)^{-1} \int_G R_x(f(h)g(xh^{-1}y))m_G(dh) \]
\[ = \Delta(x) \int_G f(hx)g(h^{-1}y)m_G(dh) = \Delta(x)(R_x f * g)(y) \quad (x, y \in G). \]

**Remark 3.29.** We can also convolve \( f * \phi \) for \( f \in L^1(G) \) and \( \phi \in L^\infty(G) \), and the convolution \( f * \phi \) defines a function in \( L^\infty(G) \). Indeed we have

\[ |f * \phi(x)| = \left| \int_G f(h)\phi(h^{-1}x)m_G(dh) \right| \leq \int_G |f(h)|\|\phi(h^{-1}x)m_G(dh) \]
\[ \leq \|f\|_1 \|\phi\|_\infty, \]

and hence \( \|f * \phi\|_\infty \leq \|f\|_1 \|\phi\|_\infty < \infty \). Moreover, if \((x_\alpha)_\alpha\) is a net in \( G \) that converges to some \( x \in G \), we see by using Proposition 3.24 that

\[ |f * \phi(x_\alpha) - f * \phi(x)| \leq \int_G |f(x_\alpha y) - f(xy)\phi(y^{-1})|m_G(dy) \leq \|\phi\|_\infty \|L_{x_\alpha} f - L_x f\|_1 \rightarrow 0. \]

Hence \( f * \phi \) is continuous on \( G \).

Changing the order of the functions in the convolution is more problematic, however for \( f \in L^1(G) \) and \( \phi \in L^\infty(G) \) the mapping \( x \mapsto \phi * f(x) \) defines a continuous function on \( G \) where \( f \) is defined by \( f(x) = f(x^{-1}) \) (actually \( \phi * f \in L^1(U(G)) \), [H-R, (20.19)])..

If \( \phi \in L^\infty(G) \) and \( f \in C_c(G) \) with \( K := \text{supp } f \), then

\[ |\phi * f(x)| \leq \int_G |\phi(xy^{-1})f(y)\Delta(y^{-1})|m_G(dy) \leq C\|f\|_1 \|\phi\|_\infty, \]

where \( C = \sup_{y \in K} |\Delta(y^{-1})| \), which shows that \( \phi * f \in L^\infty(G) \). Furthermore, for a net \((x_\alpha)_\alpha\) in \( G \) with \( x_\alpha \rightarrow x \) we have, again using Proposition 3.24, that

\[ |\phi * f(x_\alpha) - \phi * f(x)| \leq \int_G |\phi(y^{-1})\Delta(y^{-1})||f(xy_\alpha) - f(xy)|m_G(dy) \leq C\|\phi\|_\infty \|R_{x_\alpha} f - R_x f\|_1 \rightarrow 0 \]

which shows that \( \phi * f \) is continuous.

As a final remark, we observe that if \( K := \text{supp } f \) and \( L := \text{supp } \phi \) are both compact then \( \text{supp } f * \phi \) is contained in the compact set \( KL \). Indeed, if \( x \notin KL \) then \( y^{-1}x \notin L \) for any \( y \in K \) and hence \( f * \phi(x) = \int_G f(y)\phi(y^{-1}x)m_G(dy) = 0 \).

**Proposition 3.30.** \( L^1(G) \) is a Banach algebra with convolution as the algebra multiplication.
Proof. By Theorem 2.16 \( L^1(G) \) is a Banach space. The mapping
\[
L^1(G) \times L^1(G) \to L^1(G), \quad (f, g) \mapsto f \ast g
\]
is bilinear and the inequality \( \|f \ast g\|_1 \leq \|f\|_1 \|g\|_1 \) was shown on page 26. Associativity of convolution is still to be shown. For that, let \( f, g, h \in C_c(G) \) with \( K = \operatorname{supp} f \) and \( L = \operatorname{supp} g \). Then for each \( x \in G \) the function \( \psi : G \times G \to \mathbb{C}, \psi(s, t) = f(s)g(s^{-1}t)h(t^{-1}x) \) is supported by \( K \times KL \) and hence belongs to \( C_c(G \times G) \). Then by using the left-invariance of the Haar measure and Fubini’s theorem 2.28 for compactly supported continuous functions we obtain
\[
f \ast (g \ast h)(x) = \int_G f(s)(g \ast h)(s^{-1}x)m_G(ds) = \int_G \int_G f(s)g(t)h(t^{-1}s^{-1}x)m_G(dt)m_G(ds)
= \int_G \int_G f(s)g(t)(st)^{-1}x)m_G(dt)m_G(ds)
= \int_G f(s)g(s^{-1}t)h(t^{-1}x)m_G(dt)m_G(ds)
= \int_G \int_G f(s)g(s^{-1}t)h(t^{-1}x)m_G(ds)m_G(dt)
= \int_G (f \ast g)(t)h(t^{-1}x)m_G(dt)
= (f \ast g) \ast h(x) \quad (x \in G).
\]
Next, let \( f, g, h \in L^1(G) \). Since \( C_c(G) \) is dense in \( L^1(G) \) there exist sequences \( (f_n)_{n=1}^{\infty}, (g_n)_{n=1}^{\infty} \) and \( (h_n)_{n=1}^{\infty} \) in \( C_c(G) \) such that \( f_n \to f, g_n \to g \) and \( h_n \to h \). It follows that
\[
\|f \ast (g \ast h) - f_n \ast (g_n \ast h_n)\|_1 \leq \|f \ast (g \ast h) - f_n \ast (g \ast h)\|_1 + \|f_n \ast (g \ast h) - f_n \ast (g_n \ast h)\|_1 + \|f_n \ast (g_n \ast h) - f_n \ast (g_n \ast h)\|_1
\leq \|f - f_n\|_1 \|g \ast h\|_1 + \|f_n\|_1 \|g - g_n\|_1 \|h\|_1 + \|f_n\|_1 \|g_n\|_1 \|h - h_n\|_1 \to 0
\]
when \( n \to \infty \). Similarly one verifies that \( \|(f \ast g) \ast h - (f_n \ast g_n) \ast h_n\|_1 \to 0 \) when \( n \to \infty \). Hence
\[
\|f \ast (g \ast h) - (f \ast g) \ast h\|_1 \leq \|f \ast (g \ast h) - f_n \ast (g \ast h)\|_1 + \|f_n \ast (g \ast h) - (f_n \ast g_n) \ast h_n\|_1 \to 0
\]
and therefore \( f \ast (g \ast h) = (f \ast g) \ast h \). \( \square \)

3.4 The Measure Algebra \( M(G) \)

Another important Banach algebra in abstract harmonic analysis, besides the group algebra \( L^1(G) \) for a locally compact group \( G \), is the measure algebra \( M(G) \) which consists of all complex regular Borel measures on \( G \). In Proposition 3.35 we shall show that \( M(G) \) is indeed a Banach algebra with the convolution of two measures as the algebra multiplication. Another important fact about \( M(G) \) is that \( L^1(G) \) can be regarded as a closed ideal of \( M(G) \). This will be shown in Proposition 3.36. But first, we begin with a technical lemma.
Lemma 3.31. Let \( \mu \) and \( \nu \) be finite regular Borel measures on \( G \) and let \( \mu \times \nu \) be the regular Borel product of \( \mu \) and \( \nu \). Furthermore, let \( m \) be the multiplication operation on \( G \), that is, \( m(a,b) = ab \) for all \( a,b \in G \). Then \( \psi(A) := (\mu \times \nu)(m^{-1}(A)) \), where \( A \in \mathcal{B}(G) \), defines a regular Borel measure on \( G \).

Proof. The continuity of \( m \) implies that \( \psi \) is a measure on \( G \) (see Definition 2.14). Now let \( \varepsilon > 0 \) and \( A \in \mathcal{B}(G) \). By Proposition 2.19 there exists a compact \( K \) such that \( K \subset m^{-1}(A) \) and \( (\mu \times \nu)(K) > (\mu \times \nu)(m^{-1}(A)) - \varepsilon \). Then for the compact set \( m(K) \) we have \( m(K) \subset A \) and since \( K \subset m^{-1}(m(K)) \) we obtain that

\[
\psi(m(K)) > \psi(A) - \varepsilon.
\]

This proves inner-regularity of \( \psi \).

Similarly, there exists a compact set \( K' \) such that \( K' \subset A^c \) and \( \psi(K') > \psi(A^c) - \varepsilon \). Hence,

\[
\psi((K')^c) = \psi(G) - \psi(K') < \psi(G) - \psi(A^c) + \varepsilon = \psi(A) + \varepsilon
\]

which proves outer-regularity, since \((K')^c \) is open and \( A \subset (K')^c \). \( \square \)

By Proposition 2.29 we have \( \psi(A) = \int_G \nu((m^{-1}A)x)\mu(dx) = \int_G \mu((m^{-1}A)^y)\nu(dy) \) and since \((m^{-1}A)x = x^{-1}A \) and \((m^{-1}A)^y = Ay^{-1} \) we obtain that

\[
\psi(A) = \int_G \nu(x^{-1}A)\mu(dx) = \int_G \mu(Ay^{-1})\nu(dy) \quad (A \in \mathcal{B}(G)).
\]

Hence, by considering the Jordan decomposition for arbitrary complex measures \( \mu, \nu \in M(G) \) we can by Lemma 3.31 define the convolution measure \( \mu * \nu \in M(G) \) through

\[
\mu * \nu(A) = \int_G \nu(x^{-1}A)\mu(dx) = \int_G \mu(Ay^{-1})\nu(dy) \quad (A \in \mathcal{B}(G)).
\]

Remark 3.32. We also have the following convenient formula for integrating a bounded Borel function with respect to the convolution measure \( \mu * \nu \).

\[
(3.33) \quad \int_G f d(\mu * \nu) = \int_G \int_G f(xy)\mu(dx)\nu(dy) = \int_G \int_G f(xy)\nu(dy)\mu(dx).
\]

Indeed,

\[
\int_G \chi_A(\mu * \nu) = (\mu * \nu)(A) = \int_G \nu(x^{-1}A)\mu(dx) = \int_G \int_G \chi_A(xy)\nu(dy)\mu(dx)
\]

and similarly

\[
\int_G \chi_A(\mu * \nu) = (\mu * \nu)(A) = \int_G \mu(Ay^{-1})\nu(dy) = \int_G \int_G \chi_A(xy)\mu(dx)\nu(dy).
\]

Thus, by linearity of the integral the formula holds for all simple functions and by the dominated convergence theorem, it holds for any bounded Borel function.
CHAPTER 3. INTRODUCTION TO ABSTRACT HARMONIC ANALYSIS

Remark 3.34. The Dirac measures $\delta_x$ for $x \in G$ defined by

$$\delta_x(A) = \begin{cases} 
1, & x \in A \\
0, & x \notin A 
\end{cases}$$

for all $A \in \mathfrak{B}(G)$ behave nicely with respect to the convolution: For $x, y \in G$, $A \in \mathfrak{B}(G)$ we have

$$\delta_x * \delta_y(A) = \int_G \delta_y(t^{-1}A)\delta_x(dt) = \chi_{x^{-1}A}(y) = \chi_A(xy) = \delta_{xy}(A)$$

and hence, $\delta_x * \delta_y = \delta_{xy}$ for all $x, y \in G$. Also, for any $\mu \in M(G)$ we have

$$\delta_x * \mu(A) = \int_G \mu(x^{-1}A)\delta_x(dx) = \mu(e^{-1}A) = \mu(A) = \mu(Ae^{-1})$$

$$= \int_G \mu(Ay^{-1})\delta_x(dy) = \mu * \delta_x(A).$$

Moreover, by regarding a function $f \in L^1(G)$ as a measure on $M(G)$ we have

$$\delta_x * f(A) = \int_G \delta_x(Ay^{-1})f(y)m_G(dy) = \int_G \chi_A(xy)f(y)m_G(dy)$$

$$= \int_G \chi_A(y)f(x^{-1}y)m_G(dy) = \int_A L_{x^{-1}}f(y)m_G(dy)$$

$$= L_{x^{-1}}f(A) \quad (x \in G, A \in \mathfrak{B}(G)).$$

and using formula (3.20) we obtain

$$f * \delta_x(A) = \int_G \delta_x(y^{-1}A)f(y)m_G(dy) = \int_G \chi_A(xy)f(y)m_G(dy)$$

$$= \Delta(x^{-1})\int_G \chi_A(y)f(yx^{-1})m_G(dy) = \Delta(x^{-1})\int_A R_{x^{-1}}f(y)m_G(dy)$$

$$= \Delta(x^{-1})R_{x^{-1}}f(A) \quad (x \in G, A \in \mathfrak{B}(G)).$$

Hence $\delta_x * f = L_{x^{-1}}f$ and $f * \delta_x = \Delta(x^{-1})R_{x^{-1}}f$ for all $x \in G$.

Proposition 3.35. $M(G)$ is a unital Banach algebra with convolution of measures as the algebra multiplication.

Proof. According to Proposition 2.20, $M(G)$ is a Banach space and the bilinearity of the mapping

$$M(G) \times M(G) \rightarrow M(G), \quad (\mu, \nu) \mapsto \mu * \nu$$

follows from the linearity of the integral. The calculation in Remark 3.34 shows that $\delta_x$ is the unit on $M(G)$. In order to show associativity of the convolution let $\mu_1, \mu_2, \mu_3 \in M(G)$. Then by using the formula (3.33) we have

$$(\mu_1 * \mu_2) * \mu_3(A) = \int_G \mu_3(z^{-1}A)(\mu_1 * \mu_2)(dz) = \int_G \int_G \mu_3((xy)^{-1}A)\mu_2(dy)\mu_1(dx)$$

$$= \int_G \int_G \mu_3(y^{-1}(x^{-1}A))\mu_2(dy)\mu_1(dx) = \int_G (\mu_2 * \mu_3)(x^{-1}A)\mu_1(dx)$$

$$= \mu_1 * (\mu_2 * \mu_3)(A) \quad (A \in \mathfrak{B}(G)).$$
Finally, let \( \{A_i\} \) be a finite Borel partition of \( G \). Then \( \{A_iy^{-1}\} \) is also a finite Borel partition of \( G \) for each \( y \in G \) and hence, using the inequality (2.24), we obtain

\[
\sum_i |(\mu \ast \nu)(A_i)| = \sum_i \left| \int_G \mu(A_iy^{-1})\nu(dy) \right| \\
\leq \sum_i \int_G |\mu(A_iy^{-1})| |\nu|(dy) = \int_G \sum_i |\mu(A_iy^{-1})| |\nu|(dy) \\
\leq \int_G ||\mu|| |\nu| = ||\mu|| |\nu|.
\]

It follows that \( ||\mu \ast \nu|| \leq ||\mu|| |\nu| \) and we conclude that \( M(G) \) is a unital Banach algebra.

Recall from pages 11 and 12 that the measure \( \nu_f \) for \( f \in L^1(G) \) defined by \( \nu_f(A) = \int_A f dm_G \) is a complex regular measure and that \( f \mapsto \nu_f \) is an isometric isomorphism from \( L^1(G) \) onto \( M_a(G) := M_a(G, m_G) \), that is, the Banach space of all complex measures \( \mu \) on \( G \) such that \( \mu \ll m_G \). Furthermore, using the left-invariance of \( m_G \) and Fubini’s theorem we have

\[
\nu_{f \ast g}(A) = \int_A (f \ast g) dm_G = \int_A \chi_A(y) \int_G f(x)g(x^{-1}y) m_G(dx)m_G(dy) \\
= \int_G f(x) \left( \int_G \chi_A(y)g(x^{-1}y) m_G(dy) \right) m_G(dx) \\
= \int_G f(x) \left( \int_G \chi_A(xy)g(y) m_G(dy) \right) m_G(dx) = \int_G \left( \int_G \chi_A(xy) \nu_g(dy) \right) \nu_f(dx) \\
= \int_G \int_G \chi_{x^{-1}A}(y) \nu_g(dy) \nu_f(dx) = \int_G \nu_{g(x^{-1}A)} \nu_f(dx) \\
= (\nu_{f \ast g})(A) \quad (f, g \in L^1(G)).
\]

Hence, \( f \mapsto \nu_f \) is an isometric algebra isomorphism from \( L^1(G) \) onto \( M_a(G) \) and we may thus identify \( L^1(G) \) with \( M_a(G) \). Moreover:

**Proposition 3.36.** \( M_a(G) \) is an ideal in \( M(G) \).

**Proof.** Let \( \nu_1 \in M(G) \), \( \nu_2 \in M_a(G) \) and let \( A \) be a Borel set such that \( m_G(A) = 0 \). Then also \( m_G(x^{-1}A) = 0 \) for each \( x \in G \) and hence \( \nu_2(x^{-1}A) = 0 \) for each \( x \in G \). It follows that

\[
(\nu_1 \ast \nu_2)(A) = \int_G \nu_2(x^{-1}A) \nu_1(dx) = 0.
\]

Hence \( \nu_1 \ast \nu_2 \in M_a(G) \).

For the case \( \nu_2 \ast \nu_1 \) we obtain from Corollary 3.23 that \( m_G(yA^{-1}) = 0 \) for each \( y \in G \) and hence also

\[
(\nu_2 \ast \nu_1)(A) = \int_G \nu_2(Ay^{-1}) \nu_1(dy) = 0.
\]

Thus \( M_a(G) \) is an ideal in \( M(G) \).

Recall that \( M(G) \) is isometrically isomorphic to \( C_0(G)^* \) by Riesz’ representation theorem. It is often more convenient to work in \( C_0(G)^* \) than in \( M(G) \) and we will do so from now on. Thus \( f \in L^1(G) \) will be identified with the linear functional

\[
\phi \mapsto \int_G \phi f dm_G \quad (\phi \in C_0(G))
\]
in cases when $L^1(G)$ is regarded as the ideal $M_0(G)$ of $M(G)$.

There are some difficulties in working with $M(G)$. For instance, a simple mapping such as

$$G \to M(G), \ x \mapsto \delta_x$$

is continuous only when $G$ is discrete. To see this, let $x$ and $y$ be two distinct points in $G$ and let $U$ be a relatively compact open neighbourhood of $x$ such that $y \notin U$. Now, by Urysohn’s lemma there exists a continuous function $f : G \to [0,1]$ such that $f(x) = 1$ and $f(G \setminus U) = \{0\}$. Clearly $f \in C_0(G)$ and therefore

$$||\delta_x - \delta_y|| = \sup_{||f|| \leq 1} |f(x) - f(y)| \geq 1.$$

Therefore, the inverse image of $\delta_x + b_{1/2}(M(G))$ under the map $z \mapsto \delta_z$ is just $\{x\}$, which is a neighbourhood in a topological group only when $G$ is discrete.

However, $x \mapsto \delta_x$ is continuous with respect to the given topology on $G$ and both the $w^*$-topology on $M(G)$ and the strict topology on $M(G)$ with respect to $L^1(G)$. For two Banach algebras $A$ and $B$ where $A$ is a closed ideal of $B$ we define the strict topology on $B$ (with respect to $A$) to be the locally convex topology induced by the seminorms

$$p_a(b) := ||ab|| + ||ba|| \quad (a \in A, b \in B).$$

**Proposition 3.37.** Let $G$ be a locally compact group. Then the map

$$G \to M(G), \ x \mapsto \delta_x$$

is continuous both with respect to the $w^*$-topology and the strict topology on $M(G)$.

**Proof.** Let $(x_\alpha)_{\alpha}$ be a net in $G$ converging to some $x \in G$. Then for all $f \in C_0(G)$,

$$(f, \delta_{x_\alpha}) = f(x_\alpha) \to f(x) = (f, \delta_x),$$

since $f$ is continuous on $G$. Therefore $x \mapsto \delta_x$ is continuous with respect to the $w^*$-topology on $M(G)$.

In order to show strict continuity, let $f \in L^1(G)$. Then by Proposition 3.24 and Remark 3.34 we obtain

$$||(\delta_{x_\alpha} - \delta_x) * f||_1 = ||L_{x_\alpha^{-1}} f - L_{x^{-1}} f||_1 \to 0$$

and

$$||f * (\delta_{x_\alpha} - \delta_x)||_1 = ||\Delta(x_\alpha^{-1}) R_{x_\alpha^{-1}} f - \Delta(x^{-1}) R_{x^{-1}} f||_1$$

$$\leq ||\Delta(x_\alpha^{-1}) R_{x_\alpha^{-1}} f - \Delta(x_\alpha^{-1}) R_{x_\alpha^{-1}} f||_1 + ||\Delta(x_\alpha^{-1}) R_{x_\alpha^{-1}} f - \Delta(x^{-1}) R_{x^{-1}} f||_1$$

$$\leq ||\Delta(x_\alpha^{-1})|| ||R_{x_\alpha^{-1}} f - R_{x_\alpha^{-1}} f||_1 + ||\Delta(x_\alpha^{-1}) - \Delta(x^{-1})|| ||R_{x^{-1}} f||_1 \to 0$$

since $x_\alpha^{-1} \to x^{-1}$. Hence, $x \mapsto \delta_x$ is continuous with respect to the strict topology on $M(G)$. \(\square\)
Chapter 4

Amenable and Contractible Banach Algebras

As mentioned in the introduction, amenability (and related properties) is a homological property and hence in order to understand the theory of amenable Banach algebras properly, it is advisable to be acquainted with homology theory. A comprehensive source for the theory of homology of Banach algebras is the monograph by Helemskii [Hel]. One may however investigate amenability outside the realm of homology theory and there are in fact nice intrinsic characterizations of amenable and contractible Banach algebras, respectively, which avoid the use of the homological machinery.

In section 4.3 we will briefly discuss the notion of contractibility for general algebras without topological considerations. Interestingly, there is a close connection between contractible and semisimple algebras.

4.1 Banach Bimodules and Hochschild Cohomology

The Hochschild cochain complex is defined through certain Banach bimodules and thus we begin by introducing different types of Banach bimodules that will be of importance for us.

First, we recall the definition of a bimodule as an algebraic object.

**Definition 4.1.** Let $A$ be an algebra over $C$ and $X$ a vector space. If there exist bilinear maps

$$A \times X \to X, \quad (a,x) \mapsto a \cdot x$$

and

$$X \times A \to X, \quad (x,a) \mapsto x \cdot a$$

such that for each $a, b \in A$ and $x \in X$ the following conditions hold:

$$\begin{align*}
(M1) \quad & a \cdot (b \cdot x) = ab \cdot x \\
(M2) \quad & (x \cdot a) \cdot b = x \cdot ab \\
(M3) \quad & a \cdot (x \cdot b) = (a \cdot x) \cdot b,
\end{align*}$$

then $X$ is called an $A$-bimodule.

A Banach bimodule is defined in such a way so that the module operations are well-behaved.
Definition 4.2. Let $A$ be a Banach algebra. A Banach space $X$ which is algebraically an $A$-module is called a Banach $A$-bimodule if there exists $\kappa \geq 0$ such that

$$
||a \cdot x|| \leq \kappa ||a|| ||x|| \quad \text{and} \quad ||x \cdot a|| \leq \kappa ||a|| ||x|| \quad (a \in A, x \in X).
$$

The constant $\kappa$ is called a module constant for $X$ and in calculations, the letter $\kappa$ will always refer to a module constant for the bimodule in question.

Remark 4.4. It follows from (4.3) that the module actions

$$
A \times X \to X, \quad (a, x) \mapsto a \cdot x 
$$

and

$$
X \times A \to X, \quad (x, a) \mapsto x \cdot a
$$

are jointly continuous.

We now present examples of canonical Banach bimodule structures on Banach spaces which will be used in this text.

Example 4.5. (i) If $A$ is a Banach algebra and $X = A$ as a Banach space then $X$ is clearly a Banach $A$-bimodule with the module actions defined by

$$
a \cdot x := ax \quad \text{and} \quad x \cdot a := xa \quad (a, x \in A).
$$

Such module actions on $X = A$ are referred to as the canonical Banach $A$-bimodule actions on $A$.

(ii) The zero left action of a Banach algebra $A$ on a Banach space $X$ is defined by

$$
a \cdot x := 0 \quad (a \in A, x \in X).
$$

Similarly, we define the zero right action on $X$. If $X$ is a Banach $A$-bimodule then by replacing one (or both) of the module actions by the zero action, $X$ remains a Banach $A$-bimodule.

(iii) If $X$ is a Banach $A$-bimodule we obtain a canonical Banach $A$-bimodule structure on the dual space $X^*$ where the left and right actions are defined by

$$
\langle x, a \cdot f \rangle := \langle x, a, f \rangle \quad \text{and} \quad \langle x, f \cdot a \rangle := \langle x, f, a \rangle \quad (x \in X, a \in A, f \in X^*).
$$

Indeed, (M1) and (M2) are clearly satisfied from the analogous properties for $X$. To see that (M3) holds, let $x \in X, f \in X^*$ and $a, b \in A$. Then

$$
\langle x, a \cdot (f \cdot b) \rangle = \langle x, a, f \cdot b \rangle = \langle b \cdot (x \cdot a), f \rangle = \langle (b \cdot x) \cdot a, f \rangle = \langle b \cdot x, a \cdot f \rangle = \langle x, (a \cdot f) \cdot b \rangle.
$$

Note that the calculation above also convinces us that the more "obvious" operations $\langle x, a \cdot f \rangle := \langle a \cdot x, f \rangle$ and $\langle x, f \cdot a \rangle := \langle x, a, f \rangle$ do not in general satisfy (M3). We also have

$$
||a \cdot f|| = \sup_{||x|| \leq 1} ||\langle x, a \cdot f \rangle|| = \sup_{||x|| \leq 1} ||\langle x \cdot a, f \rangle|| 
$$

$$
\leq \sup_{||x|| \leq 1} \kappa ||x|| ||a|| ||f|| = \kappa ||a|| ||f|| \quad (a \in A, f \in X^*).
$$

Analogously, we see that $||f \cdot a|| \leq \kappa ||a|| ||f||$ for all $a \in A, f \in X^*$. 
The dual $X^*$ equipped with the canonical $A$-bimodule structure is called a dual Banach $A$-bimodule and if not otherwise stated, a Banach $A$-bimodule structure on $X^*$ will always mean the canonical one. Note that the dual module actions on $X^*$ are $w^*$-continuous. Similarly we obtain a canonical Banach $A$-bimodule structure on the bidual $X^{**}$. Note also that if a Banach $A$-bimodule $X$ has the zero left (respectively right) action, then the dual Banach $A$-bimodule $X^*$ has the zero right (respectively left) action.

(iv) If $X$ is a Banach $A$-bimodule and $X_1$ is a closed submodule of $X$, then the quotient bimodule $X/X_1$ with module actions $a \cdot (x + X_1) = a \cdot x + X_1$ and $(x + X_1) \cdot a = x \cdot a + X_1$ is a Banach $A$-bimodule. Indeed, we know from algebra that $X/X_1$ is a Banach module with the above actions. Furthermore

$$||a \cdot (x + X_1)|| = \inf \{||a \cdot x - z|| : z \in X_1\} \leq \inf \{||a \cdot x - a \cdot z|| : z \in X_1\}$$

$$= \inf \{||a \cdot (x - z)|| : z \in X_1\} \leq \inf \{\kappa ||a|| ||x - z|| : z \in X_1\}$$

$$= \kappa ||a|| ||x + X_1|| \quad (a \in A, x \in X).$$

Similarly it is verified that $||((x + X_1) \cdot a)|| \leq \kappa ||a|| ||x + X_1||$ for all $a \in A, x \in X$. Hence, $X/X_1$ is a Banach $A$-bimodule.

In the next example of a Banach bimodule it is quite tedious to verify the axioms of a Banach $A$-bimodule. It is however of no significant importance for us here and hence a verification is omitted.

(v) Let $A$ be a Banach algebra and $X$ a Banach $A$-bimodule. Then $\mathcal{L}^a(A, X)$ is a Banach $A$-bimodule with the following module actions:

$$(a \cdot T)(a_1, \ldots, a_n) := a \cdot T(a_1, \ldots, a_n) \quad (a \in A, T \in \mathcal{L}^a(A, X))$$

and

$$(T \cdot a)(a_1, \ldots, a_n) := T(aa_1, \ldots, a_n) + \sum_{j=1}^{n-1} (-1)^j T(a, a_1, \ldots, a_j a_{j+1}, \ldots, a_n)$$

$$+ (-1)^n T(a, a_1, \ldots, a_{n-1}) \cdot a_n.$$ 

We conclude by defining a natural Banach $L^1(G)$-bimodule structure on $L^\infty(G)$ for a locally compact group $G$.

(vi) Let $G$ be a locally compact group. Then

$$f \cdot \phi := f \ast \phi \quad \text{and} \quad \phi \cdot f := \left( \int_G fdm_G \right) \phi \quad (f \in L^1(G), \phi \in L^\infty(G))$$

define a $L^1(G)$-bimodule structure on $L^\infty(G)$. Indeed, it follows from the bilinearity of the convolution operator and the linearity of integration that the module actions are bilinear. Thus, also for $f_1, f_2 \in L^1(G), \phi \in L^\infty(G)$ we have

$$(f_1 \ast \phi) \cdot f_2 = \left( \int_G f_2 dm_G \right) (f_1 \ast \phi) = f_1 \ast \left( \int_G f_2 dm_G \right) \phi = f_1 \cdot (\phi \cdot f_2).$$

Furthermore, we have

$$(f_1 \ast f_2) \ast \phi = f_1 \ast (f_2 \ast \phi)$$
by associativity of the convolution operator, and using Fubini’s theorem and the left-invariance of $m_G$ we obtain that

$$
\phi \cdot (f_1 * f_2) = \left( \int_G (f_1 * f_2)(x)m_G(dx) \right) \phi = \left( \int_G \int_G f_1(y)f_2(y^{-1}x)m_G(dy)m_G(dx) \right) \phi
$$

$$
= \left( \int_G f_1(y) f_2(y^{-1}x)m_G(dx)m_G(dy) \right) \phi = \left( \int_G f_1(y) \int_G f_2(x)m_G(dx)m_G(dy) \right) \phi
$$

$$
= \left( \int_G f_1(y)m_G(dy) \int_G f_2(x)m_G(dx) \right) \phi = (\phi \cdot f_1) \cdot f_2.
$$

Finally, $\|f \cdot \phi\|_{\infty} \leq \|f\|_1 \|\phi\|_{\infty}$ by Remark 3.29 and it is clear that $\|\phi \cdot f\|_1 \leq \|f\|_1 \|\phi\|_{\infty}$. Hence, $L^\infty(G)$ is a Banach $L^1(G)$-bimodule with the module actions defined as above.

We proceed by introducing the Hochschild cohomology and finally we arrive at the definition of an amenable Banach algebra.

**Definition 4.6.** Let $A$ be a Banach algebra and $X$ a Banach $A$-bimodule. For each $x \in X$, $n \in \{−1, 0, 1, 2, \ldots\}$, we define maps $\delta^n$ in the following way:

$\delta^{-1} : \{0\} \rightarrow X$, $0 \mapsto 0$, $\delta^0 : X \rightarrow \mathcal{L}(A, X)$, $(\delta^0x)(a) = a \cdot x - x \cdot a$

and for $n \geq 1$, $\delta^n : \mathcal{L}^n(A, X) \rightarrow \mathcal{L}^{n+1}(A, X)$ is defined by

$$(\delta^nT)(a_1, \ldots, a_{n+1}) = a_1 \cdot T(a_2, \ldots, a_{n+1}) + \sum_{j=1}^{n} (-1)^j T(a_1, \ldots, a_ja_{j+1}, \ldots, a_{n+1})$$

$$+ (-1)^{n+1} T(a_1, \ldots, a_n) \cdot a_{n+1}.$$

In particular, for $n = 1$ we have that

$$(\delta^1T)(a_1, a_2) = a_1 \cdot T(a_2) - T(a_1a_2) + T(a_1) \cdot a_2.$$

Let $Z^n(A, X) := \ker \delta^n$ and $B^n(A, X) := \text{Im} \delta^{n-1}$. A direct, but tedious, calculation shows that $\delta^{n+1} \delta^n \equiv 0$, for all $n \in \mathbb{N}_0$. Hence

$$B^n(A, X) \subset Z^n(A, X) \quad (n \in \mathbb{N}_0)$$

and so the pair $(\mathcal{L}^n(A, X), \delta^n)$ defines a cochain complex

$$\cdots \leftarrow \mathcal{L}^n(A, X) \xleftarrow{\delta^{n-1}} \mathcal{L}^{n-1}(A, X) \xleftarrow{\delta^n} \mathcal{L}^1(A, X) \xleftarrow{\delta^0} X \leftarrow 0,$$

called the **Hochschild cochain complex**. Hence we can define the $n$-th **Hochschild cohomology group**

$H^n(A, X) := Z^n(A, X)/B^n(A, X)$. We will however only be concerned with the first Hochschild cohomology group, where it is easy to verify that

$$(4.7) \quad B^1(A, X) \subset Z^1(A, X).$$

Note that the cohomology groups $H^n(A, X)$ are in fact vector spaces but they are for historical reasons called cohomology groups.

It is easily verified that $Z^1(A, X)$ consists of all $D \in \mathcal{L}(A, X)$ such that

$$D(ab) = a \cdot D(b) + D(a) \cdot b, \quad (a, b \in A).$$
Such a map is called a \textit{derivation} from \( A \) into \( X \). Moreover, we see that \( B^1(A, X) \) consists of all \( T \in \mathcal{L}(A, X) \) such that, for some \( x \in X \),

\[
T(a) = a \cdot x - x \cdot a.
\]

Such a map is called an \textit{inner derivation} from \( A \) into \( X \). The inner derivation defined by \( x \in X \) will be denoted by \( \text{ad}_x \), that is, \( \text{ad}_x : a \mapsto a \cdot x - x \cdot a \). All inner derivations are derivations by (4.7). Hence for a Banach \( A \)-bimodule \( X \), we have \( H^1(A, X) = \{0\} \) if and only if all derivations from \( A \) into \( X \) are inner derivations. The class of Banach algebras \( A \) such that \( H^1(A, X) = \{0\} \) for all Banach \( A \)-modules \( X \) will be studied in section 4.2.

\textbf{Definition 4.8.} Let \( A \) be a Banach algebra. Then \( A \) is called \textit{contractible} (or super-amenable) if \( H^1(A, X) = \{0\} \) for all Banach \( A \)-modules \( X \).

We will however show that this definition is somewhat restrictive to allow a rich variety of examples. For instance all contractible Banach algebras turn out to be unital.

\textbf{Proposition 4.9.} \textit{Let \( A \) be a contractible Banach algebra. Then \( A \) is unital.}

\textit{Proof.} Consider \( X = A \) as a Banach space and let \( X \) be a \( A \)-bimodule with the canonical left module action and the zero right module action, that is,

\[
a \cdot x = ax \quad \text{and} \quad x \cdot a = 0 \quad (a \in A, x \in X).
\]

Then the identity map is a derivation from \( A \) into \( X \) and since \( A \) is contractible there exists \( x \in X \) such that \( a = a \cdot x - x \cdot a = ax \) for all \( a \in A \). Thus, \( x \) is a right unit for \( A \). In an analogous fashion, by considering \( A \) as an \( A \)-bimodule with the canonical left and the zero right module actions we obtain a left unit \( y \) for \( A \). Finally, \( 1 := x + y - xy \) is the unit on \( A \). □

However, if we restrict ourselves to those Banach algebras \( A \) such that the first Hochschild cohomology group is trivial for all dual Banach \( A \)-modules, it turns out that we get a fruitful theory with interesting examples and applications.

\textbf{Definition 4.10.} Let \( A \) be a Banach algebra. Then \( A \) is called \textit{amenable} if \( H^1(A, X^*) = \{0\} \) for all Banach \( A \)-modules \( X \).

Interestingly, the following reduction-of-dimension formula holds: for \( n, k \in \mathbb{N}_0 \) we have \( H^{n+k}(A, X) \cong H^n(A, \mathcal{L}^k(A, X)) \) where \( \mathcal{L}^k(A, X) \) have the canonical \( A \)-module structure defined in Example 4.5 (v) (see [Run, section 2.4] for details). In particular, \( H^n(A, X) = \{0\} \) for all Banach \( A \)-modules \( X \) if and only if \( H^{n+k}(A, X) = \{0\} \) for all Banach \( A \)-modules \( X \) and all \( k \in \mathbb{N}_0 \). One can also show that \( \mathcal{L}^k(A, X^*) \) is isomorphic to a dual Banach \( A \)-bimodule and thus the following theorem holds.

\textbf{Theorem 4.11.} \textit{For a Banach algebra \( A \) the following are equivalent:}

(i) \( A \) is amenable (respectively contractive).

(ii) \( H^n(A, X^*) = \{0\} \) (respectively \( H^n(A, X) = \{0\} \)) for each Banach \( A \)-module \( X \) and for all \( n \in \mathbb{N} \).

Contractible and amenable Banach algebras are not the only interesting classes of Banach algebras that one can derive from the Hochschild cochain complex. Another class of Banach algebras of interest is the class of \textit{weakly amenable} Banach algebras, where a Banach algebra \( A \) is weakly amenable if \( H^1(A, A^*) = \{0\} \). For instance, all \( C^* \)-algebras and the group algebra \( L^1(G) \) for a locally compact group \( G \) turn out to be weakly amenable. However, as we shall see in chapter 5, \( L^1(G) \) is amenable precisely when \( G \) is amenable as a locally compact group.
4.2 Characterization of Contractible Banach Algebras

Before presenting important properties of amenable Banach algebras we will first briefly consider the notion of contractible Banach algebras and in section 4.3 contractible algebras without any topology. In this section we will show an intrinsic characterization of contractible Banach algebras and for this, we will need some basic facts about projective tensor products of Banach algebras. For a rather self-contained treatise on this topic we recommend [Rya].

Definition 4.12. Let $X \otimes Y$ denote the algebraic tensor product of two Banach spaces $X$ and $Y$. Then the projective norm $\| \cdot \|_\pi$ on $X \otimes Y$ is defined by

$$\| x \|_\pi = \inf \{ \sum_{i=1}^n \| x_i \| \| y_i \| : x = \sum_{i=1}^n x_i \otimes y_i, n \in \mathbb{N} \}.$$ 

It can be shown that $\| \cdot \|_\pi$ is a cross norm, that is, $\| x \otimes y \|_\pi = \| x \| \| y \|$ for all $x \in X, y \in Y$. The completion of $X \otimes Y$ with respect to the projective norm is called the projective tensor product and is denoted by $X \hat{\otimes} Y$.

The following Proposition shows that the elements of the projective tensor product $X \hat{\otimes} Y$ can be represented in a natural way that reflects the representation of the elements in $X \otimes Y$ (see [Rya, (2.8) Proposition] for a proof).

Proposition 4.13. Let $X$ and $Y$ be Banach spaces and $u$ an element in $X \hat{\otimes} Y$. Then there exist bounded sequences $(x_1, x_2, \ldots)$ and $(y_1, y_2, \ldots)$ in $X$ and $Y$ respectively such that

$$u = \sum_{i=1}^\infty x_i \otimes y_i \quad \text{where} \quad \sum_{i=1}^\infty \| x_i \| \| y_i \| < \infty.$$ 

Furthermore, we have

$$\| u \|_\pi = \inf \{ \sum_{i=1}^\infty \| x_i \| \| y_i \| : u = \sum_{i=1}^\infty x_i \otimes y_i \}.$$ 

From now on we simply denote $\| \cdot \|$ for the projective norm on $X \hat{\otimes} Y$.

Recall that for any normed space $X$ and a Banach space $Y$ the bounded operator $T \in \mathcal{L}(X, Y)$ determines a unique bounded operator $\hat{T} \in \mathcal{L}(\hat{X}, Y)$, where $\hat{X}$ denotes the completion of $X$. Bearing this in mind, we will only give values for the set of basic tensors $\{ a \otimes b : a, b \in A \}$ of $A \otimes A$ when defining bounded linear operators with $A \hat{\otimes} A$ as the domain.

Let $A$ be a Banach algebra. Then there is a canonical $A$-bimodule structure on $A \hat{\otimes} A$ defined by the linear extension of

$$a \cdot (x \otimes y) := ax \otimes y \quad \text{and} \quad (x \otimes y) \cdot a := x \otimes ya \quad (a, x, y \in A).$$

Definition 4.15. Let $A$ be a Banach algebra. The diagonal operator $\pi$ on $A$ is the bounded linear operator defined by the linear extension of

$$\pi : A \hat{\otimes} A \to A, \quad x \otimes y \mapsto xy.$$ 

An element $m \in A \hat{\otimes} A$ is called a projective diagonal for $A$ if

$$a \cdot m - m \cdot a = 0 \quad \text{and} \quad a\pi(m) = a \quad (a \in A).$$
By using continuity of \( \pi \) and the module actions, we see that \( \pi \) is a bounded module homomorphism with respect to the canonical \( A \)-bimodule structure on \( A \hat{\otimes} A \), that is, a bounded linear map such that \( \pi(a \cdot u) = a\pi(u) \) and \( \pi(u \cdot a) = \pi(u)a \) for all \( a \in A, \ u \in A \hat{\otimes} A \). It follows from this observation that if \( A \) has a projective diagonal \( m \), then \( A \) is unital with unit \( \pi(m) \).

Indeed, for \( a \in A \) we have \( \pi(m)a = \pi(m \cdot a) = \pi(a \cdot m) = a\pi(m) = a \).

Note also that \( \ker \pi \) is a closed submodule of \( A \hat{\otimes} A \).

Recall that every contractible Banach algebra is unital. The following proposition shows that a contractible Banach algebra also have a projective diagonal.

**Proposition 4.16.** Let \( A \) be a unital Banach algebra with unit \( 1 \in A \) and \( \pi \) the diagonal operator on \( A \). If \( H^1(A, \ker \pi) = \{0\} \), then \( A \) has a projective diagonal.

**Proof.** Let \( D : A \to \ker \pi, \ D(a) = a \otimes 1 - 1 \otimes a \). Clearly \( D \) is linear and also bounded since

\[
\|D(a)\| = \|a \otimes 1 - 1 \otimes a\| \leq \|a\| \|1\| + \|1\| \|a\| = 2\|1\| \|a\|.
\]

Also, for \( a, b \in A \) we have

\[
D(ab) = ab \otimes 1 - 1 \otimes ab = ab \otimes 1 - a \otimes b + a \otimes b - 1 \otimes ab
\]

\[
= a \cdot (b \otimes 1 - 1 \otimes b) + (a \otimes 1 - 1 \otimes a) \cdot b = a \cdot D(b) + D(a) \cdot b.
\]

Hence \( D \in Z^1(A, \ker \pi) \). Since \( H^1(A, \ker \pi) = \{0\} \) there exists \( x \in \ker \pi \) such that

\[
a \otimes 1 - 1 \otimes a = a \cdot x - x \cdot a \quad (a \in A),
\]

which is equivalent to

\[
a \cdot (1 \otimes 1 - x) - (1 \otimes 1 - x) \cdot a = 0 \quad (a \in A).
\]

Also, by linearity, we have

\[
\pi(1 \otimes 1 - x) = \pi(1 \otimes 1) - \pi(x) = 1^2 - 0 = 1.
\]

Thus \( \pi(1 \otimes 1 - x) \) is the unit on \( A \) and so we may conclude that \( 1 \otimes 1 - x \) is a projective diagonal for \( A \). \( \square \)

The following proposition shows that the existence of a unit and a projective diagonal actually characterize contractibility of Banach algebras. For the proof we use the following observation.

**Remark 4.17.** Let \( A \) be a Banach algebra and \( X \) a Banach \( A \)-bimodule. If \( S \in \mathcal{L}(A,X) \) we can define a bounded left module homomorphism \( T \in \mathcal{L}(A \hat{\otimes} A, X) \) by the linear extension of

\[
T(a \otimes b) = a \cdot S(b) \quad (a, b \in A).
\]

We note that for \( a, b \in A, \ \lambda \in \mathbb{C} \), we have

\[
T(a \otimes \lambda b) = a \cdot S(\lambda b) = a \cdot \lambda S(b) = \lambda a \cdot S(b) = T(\lambda a \otimes b)
\]
and therefore $T$ is well-defined. $T$ is also bounded since
\[ ||T(a \otimes b)|| = ||a \cdot S(b)|| \leq \kappa ||a|| ||S(b)|| \leq \kappa ||S|| ||a|| ||b|| \]
for all $a, b \in A$. We also have $c \cdot T(a \otimes b) = c \cdot a \cdot S(b) = ca \cdot S(b) = T(ca \otimes b)$ for all $a, b, c \in A$. Thus by linearity and continuity of the left module action, we have
\[ (4.19) \quad c \cdot T(u) = T(c \cdot u) \quad (c \in A, u \in A \hat{\otimes} A). \]

**Theorem 4.20.** Let $A$ be a Banach algebra. Then the following are equivalent:

(i) $A$ is contractible.

(ii) $A$ is unital and possesses a projective diagonal.

**Proof.** (i)⇒(ii): This is shown in Propositions 4.9 and 4.16.

(ii)⇒(i): Let $X$ be a Banach $A$-bimodule and let $u = \sum_{i=1}^{\infty} a_i \otimes b_i$ be a projective diagonal for $A$ and so $\pi(u)$ is the unit 1 on $A$. Let $D \in Z^1(A, X)$. Since
\[ D(a) = D(1a) = 1 \cdot D(a) + D(1) \cdot a \]
for all $a \in A$, it suffices by linearity, to show that the maps $a \mapsto 1 \cdot D(a)$ and $a \mapsto D(1) \cdot a$ are inner derivations.

Define $T \in \mathcal{L}(A \hat{\otimes} A, X)$ by the linear extension of
\[ T(a \otimes b) = a \cdot D(b). \]

Since $D$ is linear, $T$ is well-defined by Remark 4.17. Now, for $a \in A$ note that $b_i \cdot D(a) = D(b_i a) - D(b_i) \cdot a$ for all $i \in \mathbb{N}$. Hence, by continuity and linearity, the fact that $u$ is a projective diagonal and (4.19) we obtain
\[ 1 \cdot D(a) = \pi(u) \cdot D(a) = \lim_{n \to \infty} \sum_{i=1}^{n} a_i b_i \cdot D(a) = \lim_{n \to \infty} \sum_{i=1}^{n} a_i \cdot (D(b_i a) - D(b_i) \cdot a) \]
\[ = \lim_{n \to \infty} \left( T(\sum_{i=1}^{n} a_i b_i a) - T(\sum_{i=1}^{n} a_i \otimes b_i) \cdot a \right) \]
\[ = T\left( \lim_{n \to \infty} \sum_{i=1}^{n} a_i \otimes b_i \right) \cdot a - T\left( \lim_{n \to \infty} \sum_{i=1}^{n} a_i \otimes b_i \right) \cdot a \]
\[ = T(u \cdot a) - T(a \cdot u) = T(a \cdot u) - T(u \cdot a) = a \cdot T(a) - T(a) \cdot a. \]

Moreover, again since $D$ is a derivation we have
\[ 1 \cdot D(1) \cdot a = 1 \cdot (D(1a) - 1 \cdot D(a)) = 1 \cdot D(a) - 1 \cdot D(a) = 0 \quad (a \in A). \]

Thus, we obtain that
\[ D(1) \cdot a = a \cdot D(1) - a \cdot D(1) - 1 \cdot D(1) \cdot a + D(1) \cdot a \]
\[ = a \cdot (1 \cdot D(1) - D(1)) = (1 \cdot D(1) - D(1)) \cdot a \quad (a \in A). \]

Thus, the maps $a \mapsto 1 \cdot D(a)$ and $a \mapsto D(1) \cdot a$ are inner derivations and therefore $D \in B^1(A, X)$. Hence, $A$ is contractible. \qed
4.3 Characterization of Contractible Algebras

In this section we will briefly discuss the notion of contractible algebras without any topology. As one might expect, we get an analogous characterization of contractible algebras as for contractible Banach algebras. More interestingly, all contractible algebras turn out to be semisimple and finite-dimensional. We begin by defining the analogies for general algebras of the main concepts from sections 4.1 and 4.2.

**Definition 4.21.** A derivation from an algebra $A$ into an $A$-bimodule $X$ is a linear map $D : A \to X$ satisfying

$$D(ab) = a \cdot D(b) + D(a) \cdot b \quad (a, b \in A).$$

Moreover, $D$ is an inner derivation if $D(a) = a \cdot x - x \cdot a$ for some $x \in X$. An algebra $A$ is called contractible if for all $A$-bimodules $X$, every derivation from $A$ into $X$ is an inner derivation.

We define the diagonal operator $\pi$ on $A$ through

$$\pi : A \otimes A \to A, \quad x \otimes y \mapsto xy,$$

and an element $m \in A \otimes A$ is a diagonal for $A$ if

$$a \cdot m - m \cdot a = 0 \quad \text{and} \quad a\pi(m) = a \quad (a \in A).$$

**Example 4.22.** As the name perhaps indicates, the algebra $M_n$ of complex $n \times n$ matrices has a diagonal for all $n \in \mathbb{N}$. In fact, for $i, j \in \mathbb{N}_n$, let $\epsilon_{i,j}$ denote the basis vector

$$\epsilon_{i,j}(k,l) = \begin{cases} 1, & (k,l) = (i,j) \\ 0, & (k,l) \neq (i,j). \end{cases}$$

Now, $m := \sum_{i=1}^{n} \epsilon_{i,1} \otimes \epsilon_{1,i}$ is a diagonal for $M_n$ since

$$\pi(m) = \sum_{i=1}^{n} \epsilon_{i,1} \epsilon_{1,i} = \sum_{i=1}^{n} \epsilon_{i,i} = I_n$$

and for any $j, k \in \mathbb{N}_n$ we have

$$\epsilon_{j,k} \cdot m = \epsilon_{j,k} \cdot \sum_{i=1}^{n} \epsilon_{i,1} \otimes \epsilon_{1,i} = \sum_{i=1}^{n} \epsilon_{j,k} \epsilon_{i,1} \otimes \epsilon_{1,i} = \epsilon_{j,1} \otimes \epsilon_{1,k}$$

$$= \sum_{i=1}^{n} \epsilon_{i,1} \otimes \epsilon_{1,i} \epsilon_{j,k} = \left( \sum_{i=1}^{n} \epsilon_{i,1} \otimes \epsilon_{1,i} \right) \cdot \epsilon_{j,k} = m \cdot \epsilon_{j,k}.$$

Thus by linearity,

$$a \cdot m = m \cdot a \quad (a \in A).$$

Furthermore, if $A = M_{n_1} \oplus \cdots \oplus M_{n_k}$ where $n_1, \ldots, n_k \in \mathbb{N}$ and $m_i$ is a diagonal for each $M_{n_i}$, where $i \in \mathbb{N}_k$, then $\bigoplus_{i=1}^{k} m_i$ is a diagonal for $A$.

The analog of Proposition 4.20 holds for algebras.

**Proposition 4.23.** Let $A$ be an algebra. Then the following are equivalent:

(i) $A$ is contractible.

(ii) $A$ is unital and possesses a diagonal.
Proof. (i)⇒(ii): The proofs of Propositions 4.9 and 4.16 are purely algebraic and thus they also hold for contractible algebras.

(ii)⇒(i): The proof is the same as in Proposition 4.20 without the limiting process.

It follows that all algebras of the form $M_{n_1} \oplus \cdots \oplus M_{n_k}$ where $n_1, \ldots, n_k \in \mathbb{N}$, are contractible. The main theorem of this section shows that all contractible algebras are actually of that form.

For this, we recall some concepts of abstract algebra.

**Definition 4.24.** Let $A$ be a unital algebra with unit $1 \in A$. A proper left ideal $I$ of $A$ is called **left modular** if there exists $a \in A$ such that $A(1 - a) \subset I$. The intersection of all maximal left modular ideals in $A$, denoted by $\text{rad } A$, is also a left ideal and is called the **radical** of $A$. If $\text{rad } A = \{0\}$ then $A$ is called **semisimple**.

From the fact that $1 - a \in I$ it follows that $a \notin I$, otherwise $1 - a + a = 1 \in I$ and hence $I = A$ which contradicts the properness of $I$. Using a standard application of Zorn’s lemma one can show that the set of all left ideals $J$ such that $I \subset J$ and $a \notin J$ has a maximal element $M$. We have $A(1 - a) \subset M$ and $a \notin M$ and hence, $M$ is also a left modular ideal. Thus every left modular ideal is contained in a maximal left modular ideal.

The following simple lemma states that there are no non-trivial idempotent elements in $\text{rad } A$.

**Lemma 4.25.** Let $A$ be a unital algebra and $a \in \text{rad } A$. If $a^2 = a$ then $a = 0$.

Proof. Let $a \in \text{rad } A$ such that $a^2 = a$. Let $I := A(1 - a)$ and suppose that $a \notin I$. Then $I \neq A$ and thus $I$ is clearly a left modular ideal. Let $M$ be a maximal left modular ideal such that $I \subset M$ and $a \notin M$. Thus $a \notin \text{rad } A$ which is a contradiction.

Hence, $a \in I$ and thus there exists $b \in A$ such that $a = b(1 - a)$. Hence,

$$a = a^2 = b(1 - a)a = b(a - a^2) = b0 = 0.$$  

In the proof of Theorem 4.27 we will use the following deep classical theorem due to J. Wedderburn (see [Dal, (1.3.9) Theorem] for a proof).

**Theorem 4.26.** (The Wedderburn structure theorem) Let $A \neq \{0\}$ be a semisimple unital and finite-dimensional algebra. Then there exist $n_1, \ldots, n_k \in \mathbb{N}$ such that

$$A = M_{n_1} \oplus \cdots \oplus M_{n_k}.$$  

We are now ready for the main theorem of this section.

**Theorem 4.27.** Let $A$ be an algebra. Then the following are equivalent:

(i) $A$ is contractible.

(ii) $A$ is unital and has a diagonal.

(iii) $A$ is semisimple and finite-dimensional.

Proof. The equivalence of (i) and (ii) was proven in Proposition 4.20.

(ii)⇒(iii): Let $m = \sum_{i=1}^{n} a_i \otimes b_i$ be a diagonal for $A$ and define $B := \text{span} \{a_ib_j : i, j \in \mathbb{N}_n\}$. We will show that $A = B$ from which it follows that $A$ is finite-dimensional.
Let $\phi$ be a linear projection of $A$ onto span $\{b_1, \ldots, b_n\}$. To see that such a projection exists, recall that any linearly independent subset $\{v_i : i \in I\} \subset A$ can be extended, using Zorn’s lemma, to a Hamel basis $\{v_i : i \in I\}$ of $A$ such that $I \subset J$. Hence every $x \in A$ has a unique representation $x = \sum_{i \in J} \lambda_i v_i$ where only finitely many $\lambda_i$ are non-zero. Then the mapping

$$\phi(x) = \sum_{i \in I} \lambda_i v_i$$

is clearly a linear projection on to the subspace spanned by $\{v_i : i \in I\}$. By Remark 4.17, we may define a linear map $T : \bigotimes_{a \in A} b \mapsto a \cdot \phi(b)$ $(a, b \in A)$.

Now, since $m$ is a diagonal for $A$, we obtain for $a \in A$ that

$$a = a \pi(m) = a \sum_{i=1}^n a_i b_i = a \sum_{i=1}^n a_i a_i \phi(b_i)$$

$$= \sum_{i=1}^n T(aa_i \otimes b_i) = T(a \sum_{i=1}^n a_i \otimes b_i)$$

$$= T(a \cdot m) = T(m \cdot a)$$

$$= \sum_{i=1}^n T(a_i \otimes b_i a) = \sum_{i=1}^n a_i \phi(b_i a) \in B.$$  

Hence $A \subset B$. The inclusion $B \subset A$ is trivial and thus $A = B$ which shows that $A$ is finite-dimensional.

In order to show that $A$ is semisimple, let $\psi$ be a linear projection of $A$ onto rad $A$ and define $T \in L(A \otimes A, A)$ by the linear extension of

$$T : a \otimes b \mapsto a \cdot \psi(b) \quad (a, b \in A).$$

Then

$$S : a \mapsto T(m \cdot a) \quad (a \in A)$$

also defines a linear projection of $A$ onto rad $A$. Indeed, for $a \in \text{rad } A$ we have

$$S(a) = T(m \cdot a) = \sum_{i=1}^n T(a_i \otimes b_i a) = \sum_{i=1}^n a_i \psi(b_i a) = \sum_{i=1}^n a_i b_i a = \pi(m) a = a,$$

where $\psi(b_i a) = b_i a$ holds for each $i \in \mathbb{N}_n$ because rad $A$ is a left ideal and thus $b_i a \in \text{rad } A$ for each $i \in \mathbb{N}_n$. Again, using the facts that $m$ is a diagonal for $A$ and that rad $A$ is a left ideal in $A$ we get

$$a_j \psi(b_j) \left( \sum_{i=1}^n a_i \psi(b_i) \right) = \sum_{i=1}^n a_j \psi(b_j) a_i \psi(b_i) = T(\sum_{i=1}^n a_j \psi(b_j) a_i \otimes b_i)$$

$$= T(a_j \psi(b_j) \cdot m) = T(m \cdot a_j \psi(b_j))$$

$$= \sum_{i=1}^n a_i \psi(b_i a_j \psi(b_j)) = \sum_{i=1}^n a_i b_i a_j \psi(b_j)$$

$$= \pi(m) a_j \psi(b_j) = a_j \psi(b_j) \quad (j \in \mathbb{N}_n).$$
Consequently
\[
T(m)^2 = \left( \sum_{i=1}^{n} a_i \psi(b_i) \right)^2 = a_1 \psi(b_1) \left( \sum_{i=1}^{n} a_i \psi(b_i) \right) + \ldots + a_n \psi(b_n) \left( \sum_{i=1}^{n} a_i \psi(b_i) \right) = a_1 \psi(b_1) + \ldots + a_n \psi(b_n) = T(m).
\]

Again, since \( \text{rad } A \) is a left ideal we have that \( T(m) = \sum_{i=1}^{n} a_i \psi(b_i) \in \text{rad } A \) and thus Lemma 4.25 implies that \( T(m) = 0 \). It follows that
\[
S(a) = T(m \cdot a) = T(a \cdot m) = a T(m) = 0 \quad (a \in A),
\]
and so \( \text{rad } A = 0 \). Hence \( A \) is semisimple.

(iii)\( \Rightarrow \) (ii): By the Wedderburn structure theorem 4.26 there exist \( n_1, \ldots, n_r \in \mathbb{N} \) such that \( A = M_{n_1} \oplus \cdots \oplus M_{n_r} \). Hence \( A \) is unital and possesses a diagonal by Example 4.22. \( \square \)

It is clear that the proof that all contractible algebras are finite-dimensional cannot be applied to contractible Banach algebras. However, all known contractible Banach algebras are finite-dimensional and an infinite-dimensional one would have to have a highly pathological Banach space structure. For more details see [Run, chapter 4].

### 4.4 Bounded Approximate Identities for Banach Algebras

The lack of a unit in arbitrary Banach algebras create some difficulties in working with them. For instance, there is no obvious way to factorize arbitrary elements in Banach algebras. However, many important Banach algebras have a bounded approximate identity which allows an "almost" factorization of an element into a product of two elements. More precisely, any element is arbitrarily close to a product of two elements. Cohen’s factorization theorem, due to P. Cohen, states that the existence of a bounded approximate identity for a Banach algebra remarkably implies that every element can be represented as a product of two elements. In this section we will prove a slightly generalized version of Cohen’s factorization theorem, often referred to as the Cohen-Hewitt factorization theorem, which will be needed later on when we will give a similar characterization of amenable Banach algebras as for the contractible ones. We begin by presenting some basic facts about bounded approximate identities for Banach algebras.

**Definition 4.28.** Let \( A \) be a Banach algebra. A bounded left approximate identity for \( A \) is a bounded net \( (e_{\alpha})_{\alpha} \) in \( A \) such that
\[
\lim_{\alpha} e_{\alpha} a = a \quad (a \in A).
\]

Similarly one defines a bounded right approximate identity. A bounded net which is both a left and a right approximate identity for \( A \) is called a bounded approximate identity for \( A \) (or simply, a bounded approximate identity). Note that a bound \( m \) for a bounded left (respectively right) approximate identity cannot be less than 1. If we want to emphasize the bound, we speak of a left (respectively right) approximate identity of bound \( m \).

Let \( X \) be a Banach \( A \)-bimodule and suppose that \( A \) has a bounded left approximation identity \( (e_{\alpha})_{\alpha} \). If also
\[
\lim_{\alpha} e_{\alpha} \cdot x = x \quad (x \in X)
\]
CHAPTER 4. AMENABLE AND CONTRACTIBLE BANACH ALGEBRAS

holds, then \((e_{a_i})_a\) is called a bounded left approximate identity for \(X\). Similarly one defines a bounded right approximate identity for \(X\) and again, a bounded net that is both a left and a right approximate identity for \(X\) is called a bounded approximate identity for \(X\).

**Remark 4.29.** Let \(A\) be a Banach algebra. Suppose that \((e_{a_i})_{a_i} \in J\) is a bounded left approximate identity for \(A\) and that \((f_{\beta_i})_{\beta \in J}\) is a bounded right approximate identity for \(A\) with bounds \(M > 0\) and \(N > 0\) respectively. Then the net \((g(a_{(b)}))_{(a,b) \in I \times J}\), where \(g(a_{(b)}) = e_a + f_{b_{\beta}} - f_{b_{\beta} e_a}\) is a bounded approximate identity for \(A\) where \(I \times J\) is directed by the relation

\[
(a, \beta) \leq (a', \beta') \iff a \leq a' \text{ and } \beta \leq \beta'.
\]

Indeed, for each \(a \in A\),

\[
\|g(a_{(b)} a - a)\| \leq \|e_a a - a\| + \|f_{b_{\beta}}(a - e_{a})\| \leq \|e_a a - a\| + N\|a - e_{a}\| \to 0
\]

and similarly one shows that \(g(a_{(b)}) a - a\) for all \(a \in A\). We also have for each \((a, \beta) \in I \times J\) that

\[
\|g(a_{(b)})\| \leq \|e_a\| + \|f_{\beta}\| + \|f_{b_{\beta} e_a}\| \leq M + N + MN,
\]

which proves the boundedness.

Let \(A\) be a Banach algebra and \(m \geq 1\). Suppose that for each \(a \in A\) and \(\varepsilon > 0\) there exists \(u \in b_m(A)\) such that \(\|a - u\| < \varepsilon\). Then \(A\) is said to have left approximate units of bound \(m\).

In a similar fashion we define right approximate units of bound \(m\).

Clearly if \(A\) has a left (respectively right) approximate identity of bound \(m\) then it has left (respectively right) approximate units of bound \(m\). In fact, the converse is also true.

**Proposition 4.30.** Let \(A\) be a Banach algebra with left (resp. right) approximate identity of bound \(m \geq 1\). Then \(A\) has a bounded left (resp. right) approximate identity of bound \(m\).

**Proof.** The claim is trivially true if \(A\) is unital and hence we can suppose that \(A\) is non-unital.

We first show that for any finite subset \(I \subset A\) and \(\varepsilon > 0\) there exists \(u \in b_m(A)\) such that

\[
\|v - uv\| < \varepsilon\text{ for all } v \in I.
\]

For that purpose, let \(I := \{a_1, \ldots, a_n\} \subset A\), \(\varepsilon > 0\) and denote \(M := \max\{|a_i| : i \in \mathbb{N}\}\). Now, for any \(u \in b_m(A)\) and \(v \in A\) we have that

\[
\|a_i - va_i\| \leq \|a_i - va_i\| + \|va_i - va_i\| + \|va_i - va_i\| \leq (1 + \|a_i\|)(|a_i - va_i| + \|v - uv\| |a_i|)
\]

\[
\leq (1 + m)|a_i - va_i| + M|v - uv|\quad \text{for all } i \in \mathbb{N}.
\]

Hence, it is enough to show that there exists \(v \in A\) such that \(|a_i - va_i| < \varepsilon/2(1 + m)|a_i|\) for all \(i \in \mathbb{N}\) because we can then by assumption choose an element \(u \in b_m(A)\) such that

\[
\|v - uv\| < \varepsilon/2M\text{ from which it follows that }\|a_i - uv\| < \varepsilon\text{ for all } i \in \mathbb{N}.
\]

Consider the isometric embedding of \(A\) into \(A^\#\) with unit \(1_A\) and rewrite \(a - ua\) as \((1_A - u)\alpha\) for \(a, u \in A\). We see that we can, by assumption, successively choose elements \(u_1, \ldots, u_n \in b_m(A)\) such that

\[
\|((1_A - u_i) \cdots (1_A - u_1)\alpha_i| \leq \frac{\varepsilon}{2(1 + m)^{a_i+1}}
\]

for each \(i \in \mathbb{N}\). Note that, \(\prod_{i=1}^n (1_A - u_i) = 1_A - v\) for some \(v \in A\) and we have

\[
\|a_i - va_i\| = \|((1_A - v)\alpha_i| \leq \|((1_A - u_n) \cdots (1_A - u_{i+1})\| \|a_i - (a_i - u_{i+1})\| \leq \|1_A - u_n| \cdots |1_A - u_{i+1}| \|a_i - (a_i - u_{i+1})\| \leq (1 + m)m^{n-i} \frac{\varepsilon}{2(1 + m)^{a_i+1}} = \frac{\varepsilon}{2(1 + m)^{n-i+1}} \quad (i \in \mathbb{N}).
\]
As mentioned above, by choosing \( u \in b_m(A) \) such that \( ||v - uv|| < \varepsilon/2M \) thus proves the claim.

Finally, let \( F := \{ I \subset A : |I| < \infty \} \). Then \( F \) is a directed set by the relation
\[
I \leq J \iff I \subset J.
\]
For each \( I \in F \) there exists, as we have shown, \( u_I \in b_m(A) \) such that \( ||u - u_Ia|| < |I|^{-1} \) for each \( a \in I \). Then \( (u_I)_{I \in F} \) is clearly a left approximate identity for \( A \) of bound \( m \).

The right-sided version is proved in a similar fashion. \( \square \)

The following is a consequence of Mazur’s theorem and Proposition 4.30.

**Theorem 4.31.** Let \( A \) be a Banach algebra and suppose that \( (e_\alpha)_\alpha \) is a bounded weak left (respectively right) approximate identity for \( A \), meaning that \( w\text{-lim}_\alpha e_\alpha a = a \) for all \( a \in A \) (\( w\text{-lim}_\alpha ae_\alpha = a \) for all \( a \in A \)), then there exists a bounded left (respectively right) identity for \( A \).

**Proof.** Again, the left-sided and the right-sided version are proved similarly, so we only prove the left-sided one.

Suppose that \( (e_\alpha)_\alpha \) is a bounded weak left approximate identity for \( A \). Let \( m := \sup_\alpha ||e_\alpha|| \) and \( a \in A \). By denoting the weak closure of a set \( U \subset A \) by \( U^w \) we have that \( a \in b_m(A) \cdot a^w \).

Now, it is easy to see that \( b_m(A) \cdot a \) is a convex set and so, by Mazur’s theorem, we have \( a \in b_m(A) \cdot a \). Hence, there exists left approximate units of bound \( m \) for \( A \) and so, by Proposition 4.30, \( A \) has a bounded left approximate identity. \( \square \)

The following proposition shows the significance of approximate identities in the theory of amenable Banach algebras.

**Theorem 4.32.** Let \( A \) be an amenable Banach algebra. Then \( A \) has a bounded approximate identity.

**Proof.** We first show that \( A \) has a bounded right approximate identity. Choose \( X = A \) as a Banach space and define the left and right bimodule actions through
\[
a \cdot x = ax \quad \text{and} \quad x \cdot a = 0 \quad (a \in A, x \in X).
\]
Now, the canonical injection \( i : A \hookrightarrow X^* \) is a derivation. Indeed, we have
\[
(f, i(ab)) = \langle ab, f \rangle = \langle a \cdot b, f \rangle = \langle b, f \cdot a \rangle = \langle f \cdot a, i(b) \rangle = \langle f, a \cdot i(b) \rangle = \langle f, a \cdot i(b) \rangle + \langle f, i(a) \cdot b \rangle \quad (a, b \in A, f \in X^*)
\]
since \( X^* \) has the zero right action. Since \( A \) is amenable there exists \( \phi \in X^* \) such that
\[
i(a) = a \cdot \phi - \phi \cdot a = a \cdot \phi \quad (a \in A).
\]
By Goldstine’s theorem there exists a bounded net \( (e_\alpha)_\alpha \) in \( X \) such that \( w^*\text{-lim} i(e_\alpha) = \phi \). Thus \( w^*\text{-lim} i(a \cdot e_\alpha) = a \cdot \phi = i(a) \) for all \( a \in A \) which is equivalent to \( w\text{-lim} ae_\alpha = a \) for all \( a \in A \). Thus \( (e_\alpha)_\alpha \) is a bounded weak right approximate identity for \( A \) and so by Proposition 4.31 there exists a bounded right approximate identity for \( A \). Similarly, by considering \( A \) with the module operations
\[
a \cdot x = 0 \quad \text{and} \quad x \cdot a = xa \quad (a \in A, x \in X),
\]
we obtain a bounded left approximate identity for \( X \). Thus by Remark 4.29 there exists a bounded approximate identity for \( X \). \( \square \)
4.4.1 Bounded Approximate Identities for $L^1(G)$

We now return for a brief moment to abstract harmonic analysis and verify that the group algebra $L^1(G)$ has a bounded approximate identity for every locally compact group $G$ and present some immediate consequences of this fact.

**Proposition 4.33.** Let $G$ be a locally compact group with left Haar measure $m_G$ and let $U$ be a neighbourhood base of $e$ directed by the relation $U \leq V \Leftrightarrow V \subset U$.

Denote $P(G) := \{ f \in L^1(G) : f \geq 0, \|f\|_1 = 1 \}$. Let $(e_U)_{U \in \mathcal{U}}$ be a net in $P(G)$ such that for each $U \in \mathcal{U}$ the support of $e_U$ is compact and contained in $U$ and $e_U(x) = e_U(x^{-1})$ for all $x \in G$. Then $(e_U)_{U \in \mathcal{U}}$ is a bounded approximate identity for $L^1(G)$.

**Proof.** Let $U \in \mathcal{U}$. By the properties of $e_U$ we have

\[
f \ast e_U(x) - f(x) = \int_G f(xy)e_U(y^{-1})m_G(dy) - f(x) \int_G e_U(y)m_G(dy) = \int_G f(xy)e_U(y)m_G(dy) - f(x) \int_G e_U(y)m_G(dy) = \int_G (R_yf(x) - f(x))e_U(y)m_G(dy).
\]

Hence, using (4.34) and Fubini’s theorem we obtain

\[
\|f \ast e_U - f\|_1 = \int_G |\int_G (R_yf(x) - f(x))e_U(y)m_G(dy)|m_G(dx) \\
\leq \int_G \int_G |R_yf(x) - f(x)|e_U(y)m_G(dy)m_G(dx) = \int_G \|R_yf - f\|_1 e_U(y)m_G(dy) \\
\leq \sup_{y \in U} \|R_yf - f\|_1 \int_G e_U(y)m_G(dy) = \sup_{y \in U} \|R_yf - f\|_1, 
\]

where the right side is smaller than any given $\varepsilon > 0$ for sufficiently small neighbourhoods $U \in \mathcal{U}$ by Proposition 3.24. Similarly we see that

\[
e_U \ast f(x) - f(x) = \int_G (L_yf(x) - f(x))e_U(y)m_G(dy)
\]

and by a similar argument as above we get

\[
\|e_U \ast f(x) - f(x)\|_1 \leq \sup_{y \in U} \|L_yf - f\|_1 < \varepsilon,
\]

for sufficiently small $U \in \mathcal{U}$. Hence, $(e_U)_{U \in \mathcal{U}}$ is a bounded approximate identity for $L^1(G)$.

**Remark 4.36.** There are various ways to construct bounded approximate identities for $L^1(G)$. Perhaps the easiest way is to define a neighbourhood base $\mathcal{U}$ of $e$ consisting of compact and symmetric neighbourhoods $U$ and define $e_U := m_G(U)^{-1}\chi_U$ for all $U \in \mathcal{U}$. Note that such a neighbourhood base exists by the local compactness and the Hausdorff property of $G$ combined with Propositions 3.2 (iv) and 3.2 (vi). Furthermore, $m_G(U) > 0$ for all $U \in \mathcal{U}$ by Proposition 3.17.
One may also construct bounded approximate identities consisting of continuous functions on $G$, for instance in the following way: For each $U \in \mathcal{U}$ let $V$ be a symmetric open neighbourhood such that $V \subset U$. Since $G$ is normal, there exists by Urysohn’s lemma a continuous function $f_1 : G \to [0,1]$ such that $f_1(e) = 1$ and $f_1(G \setminus V) = \{0\}$. Define $f_2 : G \to [0,2]$ by $f_2(x) = f_1(x) + f_1(x^{-1})$. Since $V$ is symmetric, $f_2$ is supported by $V$. Thus, by normalising, $e_U := \frac{f_2}{\int f_2}$ clearly satisfies the conditions in Proposition 4.33.

**Remark 4.37.** One could in an analogous fashion prove that all spaces $L^p(G)$ where $p \in [1, +\infty)$ have an approximate identity. The only "non-trivial" step would be the calculations in (4.35) where a reference to Minkowski’s inequality for integrals is needed (see [Fol, (6.19) Theorem]).

We now use the first bounded approximate identity for $L^1(G)$ constructed in Remark 4.36 to show that span $\{\delta_x : x \in G\}$ is dense in $M(G)$ with respect to the strict topology on $M(G)$. For this, we need the following facts which are also interesting in themselves.

**Proposition 4.38.** Let $I$ be a closed linear subspace of $L^1(G)$ such that $L_\alpha f \in I$ for all $x \in G$ and $f \in I$. Then $I$ is a left ideal in $L^1(G)$.

**Proof.** Recall that $L^\infty(G) \cong L^1(G)^*$ with

$$
\langle f, \phi \rangle = \int_G f(x)\phi(x)m_G(dx) \quad (f \in L^1(G), \phi \in L^\infty(G)).
$$

Let $\phi \in I^\perp$ and define $\check{\phi} \in L^\infty(G)$ by $\check{\phi}(x) = \phi(x^{-1})$ for all $x \in G$. Then we have

$$
f \ast \check{\phi}(x) = \int_G f(xy)\check{\phi}(y^{-1})m_G(dy) = \int_G L_\alpha f(y)\phi(y)m_G(dy) = 0 \quad (x \in G, \phi \in I^\perp)
$$

since $L_\alpha f \in I$ for all $x \in G$ by assumption. Hence

$$
\int_G (g \ast f)(y)\phi(y)m_G(dy) = \int_G \left( \int_G g(x)f(x^{-1}y)m_G(dx) \right) \check{\phi}(y^{-1})m_G(dy)
$$

$$
= \int_G g(x) \left( \int_G f(x^{-1}y)\check{\phi}(y^{-1})m_G(dy) \right) m_G(dx)
$$

$$
= \int_G g(x)(f \ast \check{\phi})(x^{-1})m_G(dx) = 0 \quad (f \in I, g \in L^1(G), \phi \in I^\perp),
$$

and so $g \ast f \in I^\perp(I^\perp)$ for all $f \in I$, $g \in L^1(G)$. Thus, by Theorem 2.4, we have that $g \ast f \in I$ for all $f \in I, g \in L^1(G)$. \[\square\]

**Corollary 4.39.** Let $I$ be a closed linear subspace of $L^1(G)$ such that $L_\alpha f \in I$ for all $x \in G$ and $f \in I$. Then $I$ is a closed left ideal in $M(G)$.

**Proof.** Let $(e_\alpha)_{\alpha \in J}$ be a bounded approximate identity for $L^1(G)$. By Proposition 4.38 $I$ is a left ideal in $L^1(G)$ and since $L^1(G)$ is an ideal in $M(G)$ we obtain that $(\mu \ast e_\alpha) \ast f \in I$ for all $\alpha \in J$, $f \in I, \mu \in M(G)$. Since

$$
(\mu \ast e_\alpha) \ast f = \mu \ast (e_\alpha \ast f) \iff \mu \ast f \quad (f \in I, \mu \in M(G))
$$

and since $I$ is closed in $L^1(G)$ and thus also in $M(G)$, it follows that $\mu \ast f \in I$ for all $f \in I$, $\mu \in M(G)$. Thus $I$ is a closed left ideal in $M(G)$. \[\square\]
**Proposition 4.40.** The linear span of $\{\delta_x : x \in G\}$ is strictly dense in $M(G)$.

**Proof.** Let $X$ denote the linear span of $\{\delta_x : x \in G\}$ and $X'$ the closure of $X$ with respect to the strict topology on $M(G)$. Let $\mu \in M(G)$ be arbitrary and $(e_\alpha)_{\alpha}$ a bounded approximate identity for $L^1(G)$. Then $\mu * e_\alpha \to \mu$ in the strict topology on $M(G)$. Indeed, for any $f \in L^1(G)$ we have

$$(\mu * e_\alpha) * f = \mu * (e_\alpha * f) \to \mu * f$$

and

$$f * (\mu * e_\alpha) = (f * \mu) * e_\alpha \to f * \mu.$$ 

Hence, it is enough to show that a) there exists a bounded approximate identity for $L^1(G)$ contained in the linear subspace $X' \cap L^1(G)$ and that b) $X' \cap L^1(G)$ is a left ideal in $M(G)$.

a) We will first show that $\chi_K \in X'$ for a compact, symmetric neighbourhood $K$ of $e$. For that, let $f \in L^1(G)$ and $\epsilon > 0$. Since $C_c(G)$ is dense in $L^1(G)$ there exists $g \in C_c(G)$ such that $||f - g||_1 < \epsilon / 4m_G(K)$. Moreover, by the compactness of $K$, there exists for any open neighbourhood $U$ of $e$ elements $x_1, \ldots, x_n \in G$ such that $\{x_1 U, x_2 U, \ldots, x_n U\}$, where $x_1 = e$, is an open cover of $K$. Now, by defining $V_1 = x_1 U \cap K$ and for $i = 2, \ldots, n$

$$V_i := \left( \bigcup_{j=1}^{i-1} V_j \right) \cap x_i U \cap K,$$

the sets $\{V_1, \ldots, V_n\}$ form a Borel partition of $K$ with $V_i \subseteq x_i U$ for each $i \in \mathbb{N}_n$. Since $g$ is uniformly continuous by Proposition 3.14 and $\Delta$ is continuous, we obtain for a sufficiently small $U$ that

\[
\begin{align*}
|g(t^{-1}s) - g(t_1^{-1}s)| &< \frac{\epsilon}{2m_G(K)Lm_G(K)} \\
|g(st^{-1}) - g(st_1^{-1})| &< \frac{\epsilon}{4m_G(K)Lm_G(K)M} \\
|\Delta(t^{-1}) - \Delta(t_1^{-1})| &< \frac{\epsilon}{4m_G(K)Lm_G(K)||g||_\infty} (s \in G, t \in V_i, i \in \mathbb{N}_n)
\end{align*}
\]

where $M = \sup_{t \in V_1} |\Delta(t^{-1})|$, $L = \sup g$ and $t_i$ is a fixed element in $V_i$ for all $i \in \mathbb{N}_n$. It follows from (4.42) and (4.43) that

$$|g(st^{-1})\Delta(t^{-1}) - g(st_1^{-1})\Delta(t_1^{-1})| \leq |g(st^{-1})\Delta(t^{-1}) - g(st_1^{-1})\Delta(t^{-1})|$$

$$+ |g(st_1^{-1})\Delta(t^{-1}) - g(st_1^{-1})\Delta(t_1^{-1})|$$

$$\leq |g(st^{-1}) - g(st_1^{-1})| |\Delta(t^{-1})| + |g(st_1^{-1})| |\Delta(t^{-1}) - \Delta(t_1^{-1})|$$

$$< \frac{\epsilon}{2m_G(K)Lm_G(K)} (s \in G, t \in V_i, i \in \mathbb{N}_n).$$

Now, define $\nu := \sum_{i=1}^n m_G(V_i)\delta_i$. Then by definition $\nu \in X$. Moreover, for any finite Borel partition $\{E_1, \ldots, E_r\}$ of $G$ we have

$$\sum_{j=1}^r |\nu(E_j)| = \sum_{j=1}^r \left| \sum_{i=1}^n m_G(V_i)\delta_i(E_j) \right| = \sum_{j=1}^r \sum_{i=1}^n m_G(V_i)\delta_i(E_j)$$

$$= \sum_{i=1}^n \sum_{j=1}^r m_G(V_i)\delta_i(E_j) = \sum_{i=1}^n m_G(V_i) = m_G(K) = ||\chi_K||_1.$$
and thus $||\nu|| = ||\chi_K|| = m_G(K)$. Since
\[
(\nu * g)(s) = \sum_{i=1}^{n} m_G(V_i)g(t_i^{-1}s) = \sum_{i=1}^{n} \int_{V_i} g(t_i^{-1}s)m_G(dt) \quad (s \in G)
\]
and
\[
(\chi_K * g)(s) = \int_{K} g(t^{-1}s)m_G(dt) = \sum_{i=1}^{n} \int_{V_i} g(t_i^{-1}s)m_G(dt) \quad (s \in G)
\]
we obtain that
\[
||\chi_K * g - \nu * g||_1 = \int_{G} \left| \sum_{i=1}^{n} \int_{V_i} (g(t^{-1}s) - g(t_i^{-1}s))m_G(dt) \right| m_G(ds)
= \int_{KL} \left| \sum_{i=1}^{n} \int_{V_i} (g(t^{-1}s) - g(t_i^{-1}s))m_G(dt) \right| m_G(ds)
\]
since the support of the function $s \mapsto |\sum_{i=1}^{n} \int_{V_i} (g(t^{-1}s) - g(t_i^{-1}s))m_G(dt)|$ is contained in $KL$.

Hence by (4.41) we have that
\[
||\chi_K * g - \nu * g|| \leq \int_{KL} \sum_{i=1}^{n} \int_{V_i} \frac{\varepsilon}{2m_G(K)m_G(K)} m_G(dt)m_G(ds)
= m_G(K)m_G(K) \frac{\varepsilon}{2m_G(K)} = \varepsilon/2.
\]

It follows that
\[
||\chi_K * f - \nu * f||_1 \leq ||\chi_K * f - \chi_K * g||_1 + ||\chi_K * g - \nu * g||_1 + ||\nu * g - \nu * f||_1
\]
\[
\leq ||\chi_K||_1|f - g||_1 + \varepsilon/2 + ||\nu||_1|g - f||_1
\]
\[
\leq m_G(K) \frac{\varepsilon}{4m_G(K)} + \varepsilon/2 + m_G(K) \frac{\varepsilon}{4m_G(K)} = \varepsilon.
\]

In an analogous fashion we obtain that $||f * \chi_K - f * \nu||_1 \leq \varepsilon$.

Now, let $F := \{F \subset L^1(G) : |F| < \infty\}$. For $F = \{f_1, \ldots, f_n\} \in F$ and $\varepsilon > 0$ let $g_1, \ldots, g_n \in C_c(G)$ be such that $||f_i - g_i|| < \varepsilon/4m_G(K)$ for each $i \in \mathbb{N}_n$. Moreover, for each $i \in \mathbb{N}_n$ let $\{V_{i_1}, \ldots, V_{i_m}\}$ be a Borel partition of $K$ such that the estimates (4.41-4.43) hold for $g_i$ where $L$ now denotes the compact union of the supports of $g_i$, $i \in \mathbb{N}_n$. Then the estimates (4.41-4.43) also hold for all $g_i$ with respect to the Borel partition
\[
\{V_{i_1} \cap \cdots \cap V_{i_m} : i_1 \in \mathbb{N}_m_1, \ldots, i_m \in \mathbb{N}_m, \text{ and } V_{i_1} \cap \cdots \cap V_{i_m} \neq \emptyset\}
\]
of $K$. Let $\nu_{(F,\varepsilon)}$ denote the the measure in $X$ obtained above such that $||\chi_K * f_i - \nu_{(F,\varepsilon)} * f_i||_1 \leq \varepsilon$ and $||f_i * \chi_K - f_i * \nu_{(F,\varepsilon)}||_1 \leq \varepsilon$ for all $i \in \mathbb{N}_n$.

It is easily verified that $F \times \mathbb{R}^+$ is a directed set with the partial order defined by
\[
(F_1, \varepsilon_1) \leq (F_2, \varepsilon_2) \iff F_1 \subset F_2 \text{ and } \varepsilon_1 \geq \varepsilon_2.
\]

It follows that $(\nu_{(F,\varepsilon)})_{(F,\varepsilon)}$ is a net in $X$ such that $\nu_{(F,\varepsilon)} \rightarrow \chi_K$ with respect to the strict topology on $M(G)$ and hence, $\chi_K \in X^\prime$. Thus also $\chi_K/m_G(K) \in X^\prime$ for each compact and
CHAPTER 4. AMENABLE AND CONTRACTIBLE BANACH ALGEBRAS

symmetric neighbourhood $K$ of $e$ and so, by Remark 4.36, $L^1(G)$ has a bounded approximate identity $(e_n)_n$ contained in $\overline{X}^s \cap L^1(G)$.

b) Clearly $\overline{X}^s$ is closed in $M(G)$. Indeed, for a net $(\mu_n)_n$ in $\overline{X}^s$ such that $\mu_n \to \mu$ for some $\mu \in M(G)$ we have $f * \mu_n \to f * \mu$ and $\mu_n * f \to \mu * f$ for all $f \in L^1(G)$ and thus $\mu_n \to \mu$ in the strict topology. Thus $\overline{X}^s$ is closed in $M(G)$ and hence, $\overline{X}^s \cap L^1(G)$ is closed in $L^1(G)$.

It follows from Corollary 4.39 that it is sufficient to show that $L_x f \in \overline{X}^s \cap L^1(G)$ for each $x \in G$ and $f \in X^r \cap L^1(G)$. By the left-invariance of $m_G$ we have $L_x f \in L^1(G)$ for each $x \in G$ and $f \in L^1(G)$ and so it remains to show that $L_x f \in \overline{X}^s$, or equivalently, that $\delta_x * f \in \overline{X}^s$ (see Remark 3.34) for each $x \in G$ and $f \in X^r \cap L^1(G)$. We first note that for $f = \sum_{i=1}^n \lambda_i \delta_x \in X$ where $x_1, \ldots, x_n \in G$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ we have

$$
(4.44) \quad \delta_x * f = \delta_x * \sum_{i=1}^n \lambda_i \delta_{x_i} = \sum_{i=1}^n \lambda_i (\delta_x * \delta_{x_i}) = \sum_{i=1}^n \lambda_i \delta_{x_i} \in X \quad (x \in G).
$$

Now, for $f \in \overline{X}^s \cap L^1(G)$ let $(f_n)_n$ be a net in $X$ such that $f_n \to f$ in the strict topology. Then also $\delta_x * f_n \to \delta_x * f$ in the strict topology for any $x \in G$. Indeed, we have

$$
\phi * (\delta_x * f_n) = (\phi * \delta_x) * f_n \to (\phi * \delta_x) * f = \phi * (\delta_x * f) \quad (x \in G, \phi \in L^1(G))
$$

and

$$
(\delta_x * f_n) * \phi = \delta_x * (f_n * \phi) \to \delta_x * (f * \phi) = (\delta_x * f) * \phi \quad (x \in G, \phi \in L^1(G)).
$$

Since the net $(\delta_x * f_n)_n$ is contained in $X$ by (4.44) for any $x \in G$, it follows that $\delta_x * f \in \overline{X}^s$. Thus we may conclude that $L_x f \in \overline{X}^s \cap L^1(G)$ for all $x \in G$ and $f \in X^r \cap L^1(G)$. Hence, $\overline{X}^s \cap L^1(G)$ is a left ideal in $M(G)$ and thus we are finished with the proof.

4.4.2 The Cohen-Hewitt Factorization Theorem

To conclude this section we will, as mentioned, prove the Cohen-Hewitt factorization theorem.

**Theorem 4.45.** (Cohen-Hewitt) Let $A$ be a Banach algebra with a bounded left (respectively right) approximate identity $(e_n)_n$ with bound $m \geq 1$, and let $X$ be a Banach $A$-bimodule. Then $A \cdot X$ (respectively $X \cdot A$) is a closed submodule of $X$.

**Proof.** We will prove the case where $(e_n)_n$ is a left approximate identity of bound $m \geq 1$. The right-sided version can be proved in a similar fashion.

First, we observe that

$$
A \cdot X = \{ x \in X : \lim_{\alpha} e_\alpha \cdot x = x \}.
$$

To see this, let $(a_1, x_1, a_2, x_2, \ldots)$ be a sequence in $A \cdot X$ such that $a_n \cdot x_n \to x$ for some $x \in X$. Then for each $\alpha$ we also have that $e_\alpha \cdot (a_n \cdot x_n) \to e_\alpha \cdot x$. Hence, for any $\varepsilon > 0$ and a fixed sufficiently large $n \in \mathbb{N}$ we obtain

$$
||e_\alpha \cdot x - x|| \leq ||e_\alpha \cdot x - e_\alpha \cdot (a_n \cdot x_n)|| + ||e_\alpha \cdot (a_n \cdot x_n) - a_n \cdot x_n|| + ||a_n \cdot x_n - x||
\leq (k \cdot m + 1) \cdot ||a_n \cdot x_n - x|| + ||e_\alpha \cdot (a_n \cdot x_n) - a_n \cdot x_n||
\leq (k \cdot m + 1) \cdot ||a_n \cdot x_n - x|| + k||e_\alpha a_n - a_n|| ||x_n|| < \varepsilon
$$
for all sufficiently large $\alpha$ and so $\overline{A \cdot X} \subset \{ x \in X : \lim_\alpha e_\alpha \cdot x = x \}$. The other inclusion is clearly true and hence $\overline{A \cdot X} = \{ x \in X : \lim_\alpha e_\alpha \cdot x = x \}$. By continuity of the module actions, one easily verifies that $\overline{A \cdot X}$ is a closed submodule of $X$. For instance for $x \in \overline{A \cdot X}$ and $a \in A$ we have $\lim_\alpha e_\alpha \cdot (x \cdot a) = (\lim_\alpha e_\alpha \cdot x) \cdot a = x \cdot a$ and thus $x \cdot a \in \overline{A \cdot X}$. It remains to show that $A \cdot X$ is closed.

Firstly, if $A$ is unital with unit 1, then by noting that $A \cdot X = \{ x \in X : 1 \cdot x = x \}$ it is easily verified that $A \cdot X$ is closed for instance by using (4.46).

For the non-trivial non-unital case, consider the unitization $A^\#$ of $A$ with unit $1_A \in A^\#$ and fix $x \in \overline{A \cdot X}$. The idea is to construct a sequence $(E_1, E_2, \ldots)$ of invertible elements in $A^\#$ such that the products $E_n^{-1} \cdots E_1^{-1}$ and $E_1 \cdots E_n \cdot x$ converge to some elements $a \in A$ and $y \in X$ respectively and thus by continuity of the module operations we obtain

\[
x = \lim_{n \to \infty} \left( E_n^{-1} \cdots E_1^{-1} \right) \cdot \left( E_1 \cdots E_n \cdot x \right) = a \cdot y.
\]

Set $\lambda = m^{-1}$. Then for any $a \in b_m(A)$ we have $||\lambda(1 + \lambda)^{-1}a|| \leq (1 + \lambda)^{-1} < 1$. Hence $1_A - \lambda(1 + \lambda)^{-1}a$ is invertible and by the series expansion for inverses, the Neumann series, we observe that

\[
(1_A - \lambda(1 + \lambda)^{-1}a)^{-1} = \sum_{k=0}^{\infty} (\lambda(1 + \lambda)^{-1}a)^k = 1_A + b
\]

for some $b \in A$. Now $(1 + \lambda)1_A - \lambda a = (1 + \lambda)(1_A - \lambda(1 + \lambda)^{-1}a)$ is also invertible and from the observation in (4.47) we see that

\[
((1 + \lambda)1_A - \lambda a)^{-1} = (1 + \lambda)^{-1} \sum_{k=0}^{\infty} (\lambda(1 + \lambda)^{-1}a)^k = (1 + \lambda)^{-1}1_A + c
\]

for some $c \in A$. Note also that

\[
||((1 + \lambda)1_A - \lambda a)^{-1}|| = (1 + \lambda)^{-1}|| \sum_{k=0}^{\infty} (\lambda(1 + \lambda)^{-1}a)^k || \leq (1 + \lambda)^{-1} \sum_{k=0}^{\infty} (\lambda(1 + \lambda)^{-1} \cdot m)^k
\]

\[
= (1 + \lambda)^{-1} \sum_{k=0}^{\infty} ((1 + \lambda)^{-1})^k = (1 + \lambda)^{-1} \frac{1}{1 - (1 + \lambda)^{-1}} = \lambda^{-1} = m.
\]

It follows from the calculations above that for any sequence $(e_n)_n$ in $b_m(A)$ we may define

\[
b_n = \left( (1 + \lambda)1_A - \lambda e_n \right)^{-1} \cdots \left( (1 + \lambda)1_A - \lambda e_1 \right)^{-1}
\]

and by (4.48) we have for some $c_1, \ldots, c_n \in A$ that

\[
b_n = \left( (1 + \lambda)^{-1}1_A + c_n \right) \cdots \left( (1 + \lambda)^{-1}1_A + c_1 \right) = (1 + \lambda)^{-n}1_A + a_n
\]

for some $a_n \in A$. Now, set $a_0 = 0$ and $b_0 = 1_A$ and choose $e_1, e_2, \ldots$ inductively from the set \{ $e_\alpha : \alpha \in I$ \} such that

\[
||e_{n+1}a_n - a_n|| < m(1 + \lambda)^{-n-1}
\]

\[
||x - e_{n+1} \cdot x|| < ||b_n^{-1}||^{-1}m(1 + \lambda)^{-n-1}
\]
for all \( n \in \mathbb{N} \) where \( a_n \) and \( b_n \) are as in (4.50) and (4.51). Note that the second estimate can be satisfied by (4.46).

We claim that \(((1 + \lambda)1_A - \lambda e_1, (1 + \lambda)1_A - \lambda e_2, \ldots)\) is the desired sequence. For that, we first show that \((a_1, a_2, \ldots)\) is a Cauchy sequence. We observe that

\[
a_{n+1} = b_{n+1} - (1 + \lambda)^{-n}1_A
= \left((1 + \lambda)1_A - \lambda e_{n+1}\right)^{-1}b_n - (1 + \lambda)^{-n}1_A
= \left((1 + \lambda)1_A - \lambda e_{n+1}\right)^{-1}\left(b_n - ((1 + \lambda)1_A - \lambda e_{n+1})(1 + \lambda)^{-n}1_A\right)
= \left((1 + \lambda)1_A - \lambda e_{n+1}\right)^{-1}\left(b_n - ((1 + \lambda)^{-n}1_A - (1 + \lambda)^{-n}1_A\right)
\]

and that

\[
a_n = b_n - (1 + \lambda)^{-n}1_A
= \left((1 + \lambda)1_A - \lambda e_{n+1}\right)^{-1}\left((1 + \lambda)1_A - \lambda e_{n+1}\right)\left(b_n - (1 + \lambda)^{-n}1_A\right)
= \left((1 + \lambda)1_A - \lambda e_{n+1}\right)^{-1}\left((1 + \lambda)1_A - \lambda e_{n+1}\right)b_n - (1 + \lambda)^{-n+1}1_A + (1 + \lambda)^{-n}\lambda e_{n+1}.
\]

Hence

\[
\left((1 + \lambda)1_A - \lambda e_{n+1}\right)(a_{n+1} - a_n) = (1 - (1 + \lambda)1_A + \lambda e_{n+1})b_n - (1 + \lambda)^{-n}1_A
+ (1 + \lambda)^{-n-1}1_A - (1 + \lambda)^{-n}\lambda e_{n+1}
\]

and by writing out the product

\[
(1_A - (1 + \lambda)1_A + \lambda e_{n+1})b_n = (1_A - (1 + \lambda)1_A + \lambda e_{n+1})(1 + \lambda)^{-n}1_A + a_n
= (1 + \lambda)^{-n}1_A - (1 + \lambda)^{-n+1}1_A + (1 + \lambda)^{-n}\lambda e_{n+1} + \lambda(e_{n+1}a_n - a_n)
\]

we obtain after cancellation of terms that

\[
\left((1 + \lambda)1_A - \lambda e_{n+1}\right)(a_{n+1} - a_n) = (1 + \lambda)^{-n-1}\lambda e_{n+1} + \lambda(e_{n+1}a_n - a_n).
\]

It follows, by using the estimates in (4.49) and (4.52), that

\[
||a_{n+1} - a_n|| \leq m \cdot ||(1 + \lambda)^{-n}1_A|| + m \cdot ||\lambda(e_{n+1}a_n - a_n)||
\leq m \cdot (1 + \lambda)^{-n-1} \lambda m + m \cdot (1 + \lambda)^{-n-1}
= 2m(1 + \lambda)^{-n-1}.
\]

Finally, for \( m > n \) we have

\[
||a_n - a_m|| \leq ||a_n - a_{n+1}|| + ||a_{n+1} - a_{n+2}|| + \cdots + ||a_{m-1} - a_m||
\leq 2m(1 + \lambda)^{-n-1} + 2m(1 + \lambda)^{-n-2} + \cdots + 2m(1 + \lambda)^{-m}
= 2m \sum_{k=n+1}^{m} (1 + \lambda)^{-k} \to 0
\]

when \( n \to \infty \). Hence, \((a_1, a_2, \ldots)\) is a Cauchy sequence and thus \( a_n \to a \) for some \( a \in A \). Since \( ||b_n - a_n|| = ||(1 + \lambda)^{-n}1_A|| \to 0 \) we also have that \( b_n \to a \).
To see that the sequence \((b_1^{-1} \cdot x, b_2^{-1} \cdot x, \ldots)\) converges we note that
\[
b_{n+1}^{-1} = b_n^{-1}((1 + \lambda)1_A - \lambda e_{n+1})
\]
and thus
\[
|b_{n+1}^{-1} \cdot x - b_n^{-1} \cdot x| = |b_n^{-1}((1 + \lambda)1_A - \lambda e_{n+1} - 1_A) \cdot x|
\]
\[
= |b_n^{-1} \cdot (\lambda 1_A (x - e_{n+1} \cdot x))| \leq |b_n^{-1}| \|\lambda \rho| x - e_{n+1} \cdot x|\\
\leq \kappa (1 + \lambda)^{-n-1}
\]
where the last inequality follows from (4.53). By a similar argument as in (4.54) we obtain that \((b_1^{-1} \cdot x, b_2^{-1} \cdot x, \ldots)\) is a Cauchy sequence and thus it converges to some element \(y \in X\).

We conclude that \(x = \lim_{n \to \infty} b_n \cdot (b_n^{-1} \cdot x) = a \cdot y \in A \cdot X\), and hence \(A \cdot X = A \cdot X\).  

As a corollary we obtain Cohen's factorization theorem.

**Corollary 4.55.** (Cohen's factorization theorem). Let \(A\) be a Banach algebra and \(X\) a Banach \(A\)-bimodule and suppose that \((e_\alpha)\alpha\) is a bounded left approximate identity for \(X\). Then \(A \cdot X = X\).

**Proof.** By the assumption on \((e_\alpha)\alpha\) and Theorem 4.45 we have
\[
X = \{x \in X : \lim_\alpha e_\alpha \cdot x = x\} = A \cdot X.
\]

**Remark 4.56.** In particular, if \(A\) is a Banach algebra with a bounded approximate identity, then by considering \(A\) with the canonical bimodule structure, Cohen's factorization theorem states that \(A \cdot A = A\).

### 4.5 Pseudo-Unital Banach Algebras

If \(A\) is a Banach algebra with a bounded approximate identity \((e_\alpha)\alpha\) and \(X\) is a Banach \(A\)-bimodule, it would be convenient, bearing Cohen's factorization theorem in mind, if \((e_\alpha)\alpha\) would also be a bounded approximate identity for \(X\). In order to achieve that, we introduce the following class of Banach bimodules.

**Definition 4.57.** Let \(A\) be a Banach algebra and \(X\) a Banach \(A\)-bimodule. If \(X = A \cdot X \cdot A\) we say that \(X\) is pseudo-unital.

**Remark 4.58.** It follows directly from the definition that if \((e_\alpha)\alpha\) is a bounded approximate identity for \(A\) then \((e_\alpha)\alpha\) is a bounded approximate identity for \(X\) if \(X\) is a pseudo-unital Banach \(A\)-bimodule. Note also the special case: If \(A\) is unital with unit \(1_A \in A\) then \(1_A \cdot x = x = x \cdot 1_A\) for all \(x \in X\).

Pseudo-unital Banach bimodules have the following extension property.

**Proposition 4.59.** Let \(A\) and \(B\) be Banach algebras such that \(A\) is a closed ideal of \(B\) and let \(X\) be a pseudo-unital Banach \(A\)-bimodule. If \(A\) has a bounded approximate identity then \(X\) is a Banach \(B\)-bimodule in a canonical way.
Chapter 4. Amenable and Contractible Banach Algebras

Proof. Suppose that \((e_\alpha)_\alpha\) is a bounded approximate identity for \(A\). Now, for \(x \in X\), let \(a \in A\) and \(y \in X\) be such that \(x = a \cdot y\). For \(b \in B\), define \(b \cdot x := ba \cdot y\). The module action is well-defined since for \(a' \in A\), \(y' \in X\) such that \(x = a' \cdot y'\) we have

\[
ba' \cdot y' = \lim_\alpha (be_\alpha) \cdot (a' \cdot y') = \lim_\alpha (be_\alpha) \cdot (a \cdot y) = ba \cdot y.
\]

Since \(A\) is an ideal in \(B\) we have \(b \cdot x \in A \cdot X\) for all \(b \in B\) and so, since \(X\) is an \(A\)-bimodule it follows that \(X\) is a left \(B\)-bimodule with the defined operation. In an analogous fashion, by defining for \(x \in X\) where \(a' \in A\) and \(y \in X\) such that \(x = y \cdot a'\) the right module action through \(x \cdot b := y \cdot a'b\), \(X\) turns into a right Banach \(B\)-bimodule. For showing associativity of the module actions, let \(x \in X\) and \(a, a' \in A\), \(y \in X\) such that \(x = a \cdot y \cdot a'\). Then for \(b, b' \in B\) we have

\[
(b \cdot x) \cdot b' = (ba \cdot y \cdot a') \cdot b' = ba \cdot y \cdot a'b' = b \cdot (a \cdot y \cdot a'b') = b \cdot (x \cdot b').
\]

We thus conclude that \(X\) is a Banach \(B\)-bimodule.

The following continuation of Proposition 4.59 is needed in Theorem 5.16.

Proposition 4.60. Let \(A\) and \(B\) and \(X\) be as in Proposition 4.59 and let \(D \in Z^1(A, X^*)\). Then there is a unique extension \(D' \in Z^1(B, X^*)\) of \(D\) such that

(i) \(D'|_A = D\).

(ii) \(D'\) is continuous with respect to the strict topology on \(B\) and the \(w^*\)-topology on \(X^*\).

Proof. Let \((e_\alpha)_\alpha\) be an approximate identity on \(A\) and let \(x \in X\) be arbitrary. Since \(X\) is pseudo-unital we see using Remark 4.58 that

\[
(x, D(a)) = \lim_\alpha (e_\alpha \cdot x, D(a)) = \lim_\alpha (x, D(a) \cdot e_\alpha) = \lim_\alpha (x, D( ae_\alpha) - a \cdot D(e_\alpha))
\]

and thus

\[
D(a) = \lim_\alpha \left( D( ae_\alpha) - a \cdot D(e_\alpha) \right) \quad (a \in A).
\]

Let \(b \in B\). Since \(X\) is pseudo-unital, we can write \(x \in X\) as \(x = y \cdot a\) where \(a \in A\) and \(y \in X\). It follows that

\[
\langle x, D(be_\alpha) - b \cdot D(e_\alpha) \rangle = \langle y \cdot a, D(be_\alpha) - b \cdot D(e_\alpha) \rangle = \langle y, a \cdot D(be_\alpha) - ab \cdot D(e_\alpha) \rangle
\]

\[
= \langle y, D(ab \cdot be_\alpha - D(ab)e_\alpha) + D(ab) \cdot e_\alpha \rangle
\]

\[
= \langle e_\alpha \cdot y, D(ab) \rangle - \langle be_\alpha \cdot y, D(a) \rangle
\]

\[
\rightarrow \langle y, D(ab) \rangle - \langle b \cdot y, D(a) \rangle
\]

by continuity of the bimodule actions. Since the limit of a net is unique in \(\mathbb{C}\), the calculations above do not depend on the factorization of \(x\). Hence \(D' : B \to X^*\) defined by

\[
\langle x, D'(b) \rangle = \lim_\alpha (x, D(be_\alpha) - b \cdot D(e_\alpha)) \quad (x \in X)
\]

is well-defined and by (4.62) if \(x = y \cdot a\) where \(a \in A\), \(y \in X\) we have

\[
\langle x, D'(b) \rangle = \langle y, D(ab) \rangle - \langle b \cdot y, D(a) \rangle \quad (b \in B).
\]
We now show that \( \tilde{D} \in Z^1(B, X^*) \). For this, let \( b, c \in B \) and let \( x = y \cdot a \) be as above. Then we have

\[
\langle x, b \cdot \tilde{D}(c) + \tilde{D}(b) \cdot c \rangle = \langle y \cdot a, b \cdot \tilde{D}(c) + \tilde{D}(b) \cdot c \rangle
\]

\[
= \langle y \cdot ab, \tilde{D}(c) \rangle + \langle c \cdot y, \tilde{D}(b) \rangle
\]

\[
= \langle y, D(abc) \rangle - \langle c \cdot y, D(ab) \rangle + \langle c \cdot y, D(ab) \rangle - \langle bc \cdot y, D(a) \rangle
\]

\[
= \langle y, D(abc) \rangle - \langle bc \cdot y, D(a) \rangle
\]

\[
= \langle x, \tilde{D}(bc) \rangle.
\]

The linearity of \( \tilde{D} \) follows easily from the linearity of \( D \) and the module operations. Finally, let \( m \geq 1 \) be a bound for \( (e_\alpha)_\alpha \). Then for any \( b \in B, x \in X \) we have

\[
\|\langle x, \tilde{D}(b) \rangle\| = \lim_{\alpha} \|\langle x, D(be_\alpha) - b \cdot D(e_\alpha) \rangle\|
\]

\[
\leq \lim_{\alpha} \left( \|\langle x, D(be_\alpha) \rangle\| + \|\langle x, b \cdot D(e_\alpha) \rangle\| \right)
\]

\[
\leq (\kappa + 1) m \|x\| \|D\| \|b\| < \infty
\]

and thus \( \tilde{D} \) is bounded and we conclude that \( \tilde{D} \in Z^1(B, X^*) \). Also, by (4.61), \( \tilde{D}|_A = D \).

For showing the desired continuity in (ii), let \( (b_\alpha)_\alpha \) be a net in \( B \) such that \( b_\alpha \to b \) for some \( b \in B \) in the strict topology on \( B \) with respect to \( A \), that is,

\[
\|a(b_\alpha - b)\| + \|b_\alpha - ba\| \to 0
\]

for all \( a \in A \). Since \( X \) is pseudo-unital we get by writing any \( y \in X \) as \( y = a_1 \cdot y_1 \), where \( a_1 \in A \) and \( y_1 \in X \), that

\[
\|(b_\alpha - b) \cdot y\| = \|(b_\alpha - b)a_1 \cdot y_1\| \leq \kappa \|(b_\alpha - b)a_1\| \|y_1\| \to 0.
\]

Hence, by writing \( x = y \cdot a \) we obtain by (4.63) that

\[
\|\langle x, \tilde{D}(b_\alpha) - \tilde{D}(b) \rangle\| = \|\langle y, D(ab_\alpha) - D(ab) \rangle - \langle b_\alpha - b \cdot y, D(a) \rangle\|
\]

\[
\leq \|\langle y, D(ab_\alpha - b) \rangle\| + \|\langle b_\alpha - b \cdot y, D(a) \rangle\|
\]

\[
\leq \|y\| \|D(ab_\alpha - b)\| + \|\langle b_\alpha - b \cdot y\| \|D(a)\| \to 0
\]

by continuity of \( D \). Thus \( \tilde{D} \) is continuous with respect to the strict topology on \( B \) and the \( w^* \)-topology on \( X^* \).

\[\square\]

**Lemma 4.64.** Let \( A \) be a Banach algebra with a bounded approximate identity and let \( X \) be a Banach \( A \)-bimodule with a trivial left or right module action. Then \( H^1(A, X^*) = \{0\} \).

**Proof.** Suppose that the left module action is trivial, that is, \( A \cdot X = \{0\} \) and hence, for \( D \in Z^1(A, X^*) \) we have

\[
D(ab) = a \cdot D(b) + D(a) \cdot b = a \cdot D(b) \quad (a, b \in A).
\]

Let \( (e_\alpha)_\alpha \) be a bounded approximate identity for \( A \). Then \( (D(e_\alpha))_\alpha \) is a bounded net in \( X^* \) and therefore, by Alaoglu’s theorem, there exists a \( w^* \)-accumulation point \( f \) of \( (D(e_\alpha))_\alpha \).
Since any subnet of \((e_\alpha)_\alpha\) is also a bounded approximate identity for \(A\), we may suppose that 
\[ w^*\lim_\alpha D(e_\alpha) = f. \]
Consequently, for each \(a \in A\)
\[
\langle x, D(a) \rangle = \lim_\alpha \langle x, D(ae_\alpha) \rangle = \lim_\alpha \langle x, a \cdot D(e_\alpha) \rangle = \lim_\alpha \langle x, a \cdot D(e_\alpha) \rangle = \langle x \cdot a, D(e_\alpha) \rangle
\]
\[
= \langle x \cdot a, f \rangle = \langle x, a \cdot f \rangle = \langle x, a \cdot f - f \cdot a \rangle \quad (x \in X)
\]
since \(f \cdot a = 0\). Hence \(D \in B^1(A, X^*)\) and thus \(H^1(A, X^*) = \{0\}\).

The proof is analogous for the case when the right module action is trivial. 

We conclude this section by showing an important property of pseudo-unital Banach bi-modules in the theory of amenability for Banach algebras.

**Proposition 4.65.** Let \(A\) be a Banach algebra with a bounded approximate identity. Suppose that \(H^1(A, X^*) = \{0\}\) for each pseudo-unital Banach \(A\)-bimodule \(X\). Then \(A\) is amenable.

**Proof.** Let \(X\) be a Banach \(A\)-bimodule and put \(X_1 = A \cdot X\) and \(X_0 = A \cdot X \cdot A\). By Theorem 4.45, \(X_1\) is a closed submodule of \(X\) and \(X_0\) is a closed submodule of \(X_1\). Furthermore, it follows from Remark 4.56 that \(X_0\) is pseudo-unital. Indeed,
\[
A \cdot X_0 \cdot A = A \cdot A \cdot X \cdot A = A \cdot X = X_0.
\]

We will first show that \(H^1(A, X^+_1) = \{0\}\). For this, let \(D \in Z^1(A, X^+_1)\) and let \(\pi_0 : X^+_1 \to X^*_0\) be the restriction map. Hence, \(H^1(A, X^+_1)\) is a bounded module homomorphism and hence,
\[
(\pi \circ D)(ab) = \pi(a \cdot D(b) + D(a) \cdot b) = a \cdot (\pi \circ D)(b) + (\pi \circ D)(a) \cdot b \quad (a, b \in A).
\]

Also \(\|\pi_0 \circ D\| \leq \|D\|\) and thus we conclude that \(\pi_0 \circ D \in Z^1(A, X^*_0)\). Since \(H^1(A, X^*_0) = \{0\}\) by assumption, there exists \(f_0 \in X^*_0\) such that \(\pi_0 \circ D = \operatorname{ad} f_0\). By the Hahn-Banach theorem there exists \(f_1 \in X^*_1\) such that \(f_1|X_0 = f_0\). Then also \((\operatorname{ad} f_1)(a)|X_0 = \operatorname{ad} f_0(a)\) for all \(a \in A\) since
\[
\langle x_0, a \cdot f_1 - f_1 \cdot a \rangle = \langle x_0, a - a \cdot x_0, f_1 \rangle = \langle x_0, a - a \cdot x_0, f_0 \rangle = \langle x_0, a \cdot f_0 - f_0 \cdot a \rangle \quad (a \in A, x_0 \in X_0).
\]

It follows that
\[
(\pi_0 \circ D)(ab) = \langle x_0, (\pi_0 \circ D - \operatorname{ad} f_0)(a) \rangle = 0 \quad (a \in A, x_0 \in X_0).
\]

Thus \((\pi_0 \circ D)(a) \in X^*_0\) for all \(a \in A\). Recall that \(X^*_0 \cong (X_1/X_0)^*\) as Banach spaces by Theorem 2.6 and that \(X_1/X_0\) is a Banach \(A\)-bimodule as observed in Example 4.5 (iv).

We may thus conclude that \(D-\operatorname{ad} f_1 \in Z^1(A, X^*_0)\). Now, since \(X_1 \cdot A = X_0\) we obtain that \((X_1/X_0) \cdot A = \{0\}\). Hence by Lemma 4.64 we have \(H^1(A, X^*_1) = \{0\}\) and thus there exists \(\phi_1 \in X^*_0\) such that \(D - \operatorname{ad} f_1 = \operatorname{ad} \phi_1\), which is equivalent to \(D = \operatorname{ad} f_1 + \phi_1\). Consequently \(D \in B^1(A, X^*_1)\) and thus \(H^1(A, X^*_1) = \{0\}\).

Finally, let \(D \in Z^1(A, X^*)\) and let \(\pi_1 : X^* \to X^*_1\) be the restriction map. Again, \(\pi_1\) is a contractive module homomorphism and hence, \(\pi_1 \circ D \in Z^1(A, X^*_1)\). Since we have showed that \(H^1(A, X^*_1) = \{0\}\) there exists \(f_1 \in X^*_1\) such that \(\pi_1 \circ D = \operatorname{ad} f_1\). Again, let \(f\) denote the extension of \(f_1\) to \(X^*\) and so \(D - \operatorname{ad} f \in Z^1(A, X^*_1)\) similarly as in (4.66). Since \(X^*_1 \cong (X/X_1)^*\) and \(A \cdot (X/X_1) = \{0\}\) there exists by Lemma 4.64 an element \(\phi \in X^*_1\) such that \(D - \operatorname{ad} f = \operatorname{ad} \phi\) and thus \(D = \operatorname{ad} f + \phi\). Hence \(D \in B^1(A, X^*)\) and thus we have \(H^1(A, X^*) = \{0\}\), which shows that \(A\) is amenable.

\[
\Box
\]
CHAPTER 4. AMENABLE AND CONTRACTIBLE BANACH ALGEBRAS

4.6 Characterization of Amenable Banach Algebras

In this section we will give an intrinsic characterization of amenable Banach algebras in terms of asymptotic versions of a projective diagonal in a similar fashion as we did in section 4.2 for contractible Banach algebras.

Definition 4.67. Let $A$ be a Banach algebra and let $\pi : A \hat{\otimes} A \to A$ be the diagonal operator on $A$ (see Definition 4.15). A bounded net $(m_\alpha)_\alpha$ in $A \hat{\otimes} A$ is called an approximate diagonal for $A$ if for every $a \in A$ we have

\[
\begin{align*}
\text{(AD1)} & \quad \lim_\alpha a \cdot m_\alpha - m_\alpha \cdot a = 0 \\
\text{(AD2)} & \quad \lim_\alpha a \pi (m_\alpha) = a.
\end{align*}
\]

An element $M \in (A \hat{\otimes} A)^{**}$ is a virtual diagonal for $A$ if for every $a \in A$ we have

\[
\begin{align*}
\text{(VD1)} & \quad a \cdot M - M \cdot a = 0 \\
\text{(VD2)} & \quad a \cdot \pi^{**}(M) = \hat{a}.
\end{align*}
\]

Remark 4.68. We notice that if $(m_\alpha)_\alpha$ is an approximate diagonal for $A$, then (AD2) says that $(\pi(m_\alpha))_\alpha$ is a right approximate identity for $A$. In fact, it is also a left approximate identity since

\[
\|\pi(m_\alpha)a - a\| \leq \|a \pi(m_\alpha) - \pi(m_\alpha)a\| + \|a \pi(m_\alpha) - a\| = \|\pi(a \cdot m_\alpha - m_\alpha \cdot a)\| + \|a \pi(m_\alpha) - a\| \to 0.
\]

The definitions of approximate and virtual diagonals are very much alike, and so the following theorem may not come as a surprise.

Theorem 4.69. Let $A$ be a Banach algebra. Then $A$ has an approximate diagonal if and only if $A$ has a virtual diagonal.

Proof. Suppose that $(m_\alpha)_\alpha$ is an approximate diagonal for $A$. Then $(\hat{m}_\alpha)_\alpha$ is a bounded net in $(A \hat{\otimes} A)^{**}$ and thus by Alaoglu’s theorem there exists a $w^*$-accumulation point, $M \in (A \hat{\otimes} A)^{**}$, of $(\hat{m}_\alpha)_\alpha$. Again, since any subnet of $(m_\alpha)_\alpha$ is also an approximate diagonal for $A$, we may assume that $w^*\lim_\alpha \hat{m}_\alpha = M$. Then

\[
a \cdot M - M \cdot a = w^*\lim_\alpha a \cdot \hat{m}_\alpha - \hat{m}_\alpha \cdot a = w^*\lim_\alpha a \cdot m_\alpha - m_\alpha \cdot a = 0 \quad (a \in A)
\]

where the last equality follows from (AD1). Hence, (VD1) is satisfied.

Furthermore

\[
a \cdot \pi^{**}(M) = w^*\lim_\alpha a \cdot \pi^{**}(\hat{m}_\alpha) = w^*\lim_\alpha a \pi (m_\alpha) = \hat{a} \quad (a \in A)
\]

and so (VD2) is satisfied (recall, if necessary, the properties of adjoint operators mentioned in section 2.1).

Conversely, suppose that $M$ is a virtual diagonal for $A$. By Goldstine’s theorem there exists a bounded net $(m_\alpha)_\alpha$ in $A \hat{\otimes} A$ such that $M = w^*\lim_\alpha \hat{m}_\alpha$. Then

\[
w^*\lim_\alpha a \cdot m_\alpha - m_\alpha \cdot a = w^*\lim_\alpha a \cdot \hat{m}_\alpha - \hat{m}_\alpha \cdot a = a \cdot M - M \cdot a = 0 \quad (a \in A)
\]
and
\[ \lim_{\alpha} a \pi(m_{\alpha}) = \lim_{\alpha} a \cdot \pi^*(\tilde{m}_{\alpha}) = a \cdot \pi^{**}M = a \quad (a \in A). \]

Let \( F := \{ F \subset A : |F| < \infty \} \). Then for every \( F = \{ a_1, \ldots, a_n \} \in F \) and \( \varepsilon > 0 \) the bounded net
\[
(a_1 \cdot m_{\alpha} - m_{\alpha} \cdot a_1, a_1, a_1 \pi(m_{\alpha}) - a_1, \ldots, a_n \cdot m_{\alpha} - m_{\alpha} \cdot a_n, a_n \pi(m_{\alpha}) - a_n)_{\alpha}
\]
in the product space \( ((A \hat{\otimes} A) \times A)^{\infty} \) converges to 0 in the weak topology. Now, by denoting the convex hull of \( \{ m_{\alpha} : \alpha \in I \} \) by \( H \) we see that
\[ 0 \in (a_1 \cdot \overline{\pi}^w - \overline{\pi}^w \cdot a_1) \cap (a_i \pi(\overline{\pi}^w) - a_i) \quad (i \in \mathbb{N}). \]
By Mazur’s theorem \( \overline{\pi}^w = \overline{\pi} \) and hence there exists \( u_{F,\varepsilon} \in H \) such that
\[ ||a_1 \cdot u_{F,\varepsilon} - u_{F,\varepsilon} \cdot a_1|| < \varepsilon \quad \text{and} \quad ||a_i \pi(u_{F,\varepsilon})|| < \varepsilon \quad (i \in \mathbb{N}). \]
As in Proposition 4.40, \( F \times \mathbb{R}^+ \) is a directed set with the partial order defined by
\[ (F_1, \varepsilon_1) \leq (F_2, \varepsilon_2) \iff F_1 \subset F_2 \text{ and } \varepsilon_1 \geq \varepsilon_2 \]
and by construction, the net \( (u_{F,\varepsilon}) \) is an approximate diagonal for \( A \).

After the following fundamental characterization of amenable Banach algebras we will put an end to chapter 4 and turn again our attention on analysis on locally compact groups.

**Theorem 4.70.** Let \( A \) be a Banach algebra. Then the following are equivalent:

(i) \( A \) is amenable.

(ii) \( A \) has a bounded approximate identity and \( H^1(A, X^*) = \{ 0 \} \) for each pseudo-unital Banach \( A \)-bimodule \( X \).

(iii) \( A \) has a bounded approximate diagonal.

(iv) \( A \) has a virtual diagonal.

**Proof.** (ii)⇔(i) and (iii)⇔(iv): By Propositions 4.65 and 4.69, respectively.

(i)⇒(iv): Let \( (e_{\alpha})_{\alpha} \) be a bounded approximate identity for \( A \) (see Theorem 4.32) and consider the bounded net \( (e_{\alpha} \otimes e_{\alpha})_{\alpha} \) in \( (A \hat{\otimes} A)^{**} \). Let \( I \) be a \( w^* \)-accumulation point of \( (e_{\alpha} \otimes e_{\alpha})_{\alpha} \). We may again suppose, by passing to a subnet if necessary, that \( w^* \)-lim\( a = e_{\alpha} \otimes e_{\alpha} \). Let \( \text{ad}_I : A \to (A \hat{\otimes} A)^{**}, \quad a \mapsto a \cdot I - I \cdot a \). Then
\[ \pi^{**}(\text{ad}_I(a)) = \lim_{\alpha} \pi^{**}(a \cdot (e_{\alpha} \otimes e_{\alpha}) - (e_{\alpha} \otimes e_{\alpha}) \cdot a) = \lim_{\alpha} \pi(a \cdot (e_{\alpha} \otimes e_{\alpha}) - (e_{\alpha} \otimes e_{\alpha}) \cdot a) \]
\[ = \lim_{\alpha} (ae_{\alpha}^2 - e_{\alpha}^2 a) = 0 \quad (a \in A) \]
since \( (e_{\alpha})_{\alpha} \) is also a bounded approximate identity for \( A \). Indeed,
\[ ||e_{\alpha}^2 a - a|| \leq ||e_{\alpha}^2 a - e_{\alpha} a|| + ||e_{\alpha} a - a|| \leq (||e_{\alpha}|| + 1)||e_{\alpha} a - a|| \to 0, \]
and similarly \( ||ae_{\alpha}^2 - a|| \to 0 \).

Furthermore, \( \pi(A \hat{\otimes} A) = A \cdot A = A \) by Corollary 4.55 so \( \pi \) is surjective and thus \( \ker \pi^{**} = (\ker \pi)^{**} \) by Theorem 2.6. Recall that \( \pi \) is a bounded module homomorphism and thus \( \ker \pi \) is a closed submodule of \( A \hat{\otimes} A \). Hence, \( \ker \pi^{**} \) is a dual Banach \( A \)-bimodule and hence, \( \text{ad}_I \in \)
CHAPTER 4. AMENABLE AND CONTRACTIBLE BANACH ALGEBRAS

$Z^1(A, \ker \pi^*)$. Thus there exists by assumption $J \in \ker \pi^*$ such that $\text{ad}_I = \text{ad}_J$. Let $M = I - J$. It follows that

$$M \cdot a - a \cdot M = I \cdot a - J \cdot a - a \cdot I + a \cdot J = \text{ad}_J(a) - \text{ad}_I(a) = 0 \quad (a \in A)$$

and

$$a \cdot \pi^*(M) = a \cdot (\pi^*(I) - \pi^*(J)) = a \cdot \pi^*(I) = w^* \lim_{\alpha} a \cdot \pi^*(e_{\alpha} \otimes e_{\alpha})$$

$$= w^* \lim_{\alpha} \pi(e_{\alpha} \otimes e_{\alpha}) = w^* \lim_{\alpha} e_{\alpha}^2 = \hat{a} \quad (a \in A).$$

Hence, $M$ satisfies the conditions (VD1) and (VD2) so it is a virtual diagonal.

$(iii) \Rightarrow (ii)$: Let $(m_{\alpha})_\alpha \subset A \hat{\otimes} A$ be a bounded approximate diagonal for $A$. By Remark 4.68, $(\pi(m_{\alpha}))_\alpha$ is a bounded approximate identity for $A$.

Let $X$ be a pseudo-unital Banach $A$-bimodule and $D \in Z^1(A, X^*)$. Once again, by Alaoglu’s theorem there exists a $w^*$-accumulation point $M$ of the net $(\tilde{m}_{\alpha})_\alpha \subset (A \hat{\otimes} A)^{**}$ and by passing to a subnet if necessary we may suppose that $M = w^* \lim_{\alpha} \tilde{m}_{\alpha}$. Given $x \in X$, we can by Remark 4.17 define $A_x \in (A \hat{\otimes} A)^*$ through the linear extension of

$$A_x(a \otimes b) = \langle x, a \cdot D(b) \rangle.$$ 

We define $f \in X^*$ through $\langle x, f \rangle = \langle A_x, M \rangle$ and claim that $D = \text{ad} f$.

To see this, let $c \in A, x \in X$. First of all, we have

$$\langle c \cdot (a \otimes b) - (a \otimes b) \cdot c, A_x \rangle + \langle x, ab \cdot D(c) \rangle$$

$$= \langle (ca \otimes b) - (a \otimes bc), A_x \rangle + \langle x, ab \cdot D(c) \rangle$$

$$= \langle x, ca \cdot D(b) - a \cdot D(bc) + ab \cdot D(c) \rangle$$

$$= \langle x, ca \cdot D(b) - a \cdot D(b) + ab \cdot D(c) \rangle$$

$$= \langle x, c \cdot x, ab \cdot D(b) \rangle$$

$$= \langle a \otimes b, A_{x \cdot c - c \cdot x} \rangle \quad (a, b \in A).$$

Hence, by linearity and continuity we have

$$\langle u, A_{x \cdot c - c \cdot x} \rangle = \langle u, A_x \cdot c - c \cdot A_x \rangle + \langle x, \pi(u) \cdot D(c) \rangle \quad (u \in A \hat{\otimes} A).$$

Therefore, for $x \in X, a \in A$ we have

$$\langle x, a \cdot f - f \cdot a \rangle = \langle x \cdot a - a \cdot x, f \rangle = \langle A_{x \cdot a - a \cdot x}, M \rangle$$

$$= \lim_{\alpha} \langle A_{x \cdot a - a \cdot x}, \tilde{m}_{\alpha} \rangle = \lim_{\alpha} \langle m_{\alpha}, A_{x \cdot a - a \cdot x} \rangle$$

$$= \lim_{\alpha} \left( \langle m_{\alpha}, A_x \cdot a - a \cdot A_x \rangle + \langle x, \pi(m_{\alpha}) \cdot D(a) \rangle \right)$$

$$= \lim_{\alpha} \left( \langle a \cdot m_{\alpha} - m_{\alpha} \cdot a, A_x \rangle + \langle x, \pi(m_{\alpha}) \cdot D(a) \rangle \right)$$

$$= \lim_{\alpha} \langle x, \pi(m_{\alpha}) \cdot D(a) \rangle = \lim_{\alpha} \langle x, \pi(m_{\alpha}), D(a) \rangle$$

$$= \langle x, D(a) \rangle$$

where the last equality holds since $X$ is pseudo-unital. Hence $D \in B^1(A, X^*)$ and thus $H^1(A, X^*) = \{0\}$. 

$\square$
Chapter 5

Amenable Locally Compact Groups

In this final chapter we will introduce some of the basic theory of amenable locally compact groups which will be needed, in addition to the theory developed in chapter 4, for the proof of Theorem 5.16 by Johnson. For a comprehensive treatise on amenability for locally compact groups, we refer to [Pat] and [Pie].

5.1 Left-Invariant Means

In what follows, we assume that a locally compact group $G$ comes equipped with a fixed left Haar measure $m_G$. Also, we will denote by $E$ one of the following spaces: $\text{L}^\infty(G)$, $\text{C}_{b}(G)$, $\text{LUC}(G)$, $\text{RUC}(G)$ and $\text{UC}(G)$.

**Definition 5.1.** Let $G$ be a locally compact group and suppose that $m \in E^*$ is a linear functional such that $||m|| = \langle 1, m \rangle = 1$. Then $m$ is called a mean on $E$. Furthermore $m$ is called left-invariant if

\[ (L_g f, m) = (f, m) \quad (g \in G, f \in E). \]

The set of means on $E$ is denoted by $\mathcal{M}(E)$.

The following characterization of a mean is often useful:

**Proposition 5.2.** Let $G$ be a locally compact group. For a linear functional $m : E \to \mathbb{C}$ such that $m(1) = 1$ the following are equivalent:

(i) $m$ is a mean.

(ii) $m$ is a contraction, that is, $|\langle f, m \rangle| \leq ||f||_\infty$ for all $f \in E$.

(iii) $m$ is positive, that is, if $f \in E$ and $f \geq 0$, then $(f, m) \geq 0$.

**Proof.** (i)$\Rightarrow$(ii): For any $f \in E$ we have $|\langle f, m \rangle| \leq ||f||_\infty ||m|| = ||f||_\infty$.

(ii)$\Rightarrow$(i): For any $f \in b_1(E)$ we have $|\langle f, m \rangle| \leq 1$. Thus $||m|| \leq 1$. Since $1 \in b_1(E)$ we obtain that $||m|| \geq |\langle 1, m \rangle| = 1$ and so $||m|| = 1$.

(ii)$\Rightarrow$(iii): First, suppose that $f \in E$ is real-valued. Clearly, if $||f||_\infty = 0$ then $(f, m) = 0$ and hence, we may suppose that $||f||_\infty > 0$. Now, $(f, m) = a + ib$ for some $a, b \in \mathbb{R}$. Suppose
that \( b \neq 0 \). Then, for any \( t \in \mathbb{R} \) such that \( b^2 + 2bt > \|f\|_\infty^2 \), we get that
\[
(b + t)^2 \leq |a + i(b + t)|^2 = |(f + it1, m)|^2 \\
\leq \|f + it1\|_\infty^2 = \|f\|_\infty^2 + t^2
\]
where the last equality holds because \( f \) is real-valued. Thus \( b^2 + 2bt \leq \|f\|_\infty^2 \) which is a contradiction. Therefore \( b = 0 \) and thus, \( (f, m) \in \mathbb{R} \).

Now, suppose that \( f \geq 0 \). Let \( g := \frac{f^2}{\|f\|_\infty^2} f - 1 \). Then clearly \( \|g\|_\infty \leq 1 \), so by assumption
\[
\langle g, m \rangle \leq 1.
\]
Since \( f \) is real-valued, \( g \) is also real-valued and hence \( \langle g, m \rangle \in [-1, 1] \) by the argument above. Hence
\[
\langle f, m \rangle = \frac{\|f\|_\infty}{2} (1 + \langle g, m \rangle) = \frac{\|f\|_\infty}{2} (1 + \langle g, m \rangle) \geq 0.
\]

(iii) \( \Rightarrow \) (ii): Again, suppose first that \( f \in E \) is real-valued. For \( g := \|f\|_\infty 1 - f \) we have \( g \geq 0 \), and thus \( \langle g, m \rangle \geq 0 \) by assumption. It follows that
\[
\langle f, m \rangle = \langle \|f\|_\infty 1, m \rangle - \langle g, m \rangle \leq \|f\|_\infty \langle 1, m \rangle = \|f\|_\infty.
\]
Also, by a similar argument for \( g := \|f\|_\infty 1 + f \), we get that \( \langle f, m \rangle \geq -\|f\|_\infty \). Hence,
\[
\|f, m\| \leq \|f\|_\infty.
\]

Now, for any \( f \in E \) let \( r \geq 0, \phi \in [0, 2\pi) \) be such that \( \langle f, m \rangle = re^{i\phi} \). Also, let \( f_1, f_2 \in E \) be real-valued functions such that \( e^{-i\phi}f = f_1 + if_2 \). Then
\[
|\langle f, m \rangle| = r = e^{-i\phi} \langle f, m \rangle = (e^{-i\phi}f, m) = (f_1, m) + i(f_2, m).
\]
Since \( \langle f_1, m \rangle, \langle f_2, m \rangle \in \mathbb{R} \), we must have that \( \langle f_2, m \rangle = 0 \). Therefore
\[
|\langle f, m \rangle| = \langle f_1, m \rangle \leq \|f_1\|_\infty \leq \|e^{-i\phi}f\|_\infty = \|f\|_\infty.
\]

\( \square \)

Next, we prove an important topological and geometrical property of \( M(E) \).

**Proposition 5.3.** The set \( M(E) \) is \( w^* \)-compact and convex.

**Proof.** Let \( (m_\alpha) \) be a net in \( M(E) \) such that \( w^*\text{-}\lim_\alpha m_\alpha = m \) for some \( m \in E^* \). Then \( 1 = \langle 1, m_\alpha \rangle \to \langle 1, m \rangle \), hence \( \langle 1, m \rangle = 1 \). Also for any \( f \in E \) we have
\[
|\langle f, m_\alpha \rangle| \to |\langle f, m \rangle|
\]
and thus \( |\langle f, m \rangle| \leq \|f\|_\infty \) since \( |\langle f, m_\alpha \rangle| \leq \|f\|_\infty \) for all \( m_\alpha \). Hence \( m \in M(E) \) and therefore \( M(E) \) is \( w^* \)-closed in \( E^* \). Since \( M(E) \) is contained in \( b_1(E^*) \), which is compact by Alaoglu’s theorem, we conclude that \( M(E) \) is compact.

For showing convexity, let \( m, n \in M(E) \) and \( t \in [0, 1] \). Then
\[
(1, tm + (1 - t)n) = t(1, m) + (1 - t)(1, n) = t + 1 - t = 1,
\]
and for any \( f \in E \) we have
\[
|\langle f, tm + (1 - t)n \rangle| \leq t|\langle f, m \rangle| + (1 - t)|\langle f, m \rangle| \leq t\|f\|_\infty + (1 - t)\|f\|_\infty = \|f\|_\infty.
\]
Thus, \( tm + (1 - t)n \in M(E) \). \( \square \)
For a set $A$ we denote the convex hull of $A$ by $\text{conv} A$. Now, for $g \in G$, the Dirac measure $\delta_g$ regarded as a functional on $C_b(G)$ by $(f, \delta_g) = f(g)$ is clearly a mean on $C_b(G)$. Indeed $\langle 1, \delta_g \rangle = 1(g) = 1$ and $(f, \delta_g) = f(g) \geq 0$ for any $f \geq 0$. Hence, also $\text{conv} \{ \delta_g : g \in G \} \subset \mathcal{M}(C_b(G))$ since $\mathcal{M}(C_b(G))$ is convex by Proposition 5.3. Furthermore:

**Proposition 5.4.** Let $G$ be a locally compact group. Then $\text{conv} \{ \delta_g : g \in G \}$ is $w^*$-dense in the set of means on $C_b(G)$.

**Proof.** It is easy to see that for any $f \in C_b(G)$ and $\varepsilon > 0$ there exists a finite Borel partition $\{A_1, \ldots, A_n\}$ of $G$ such that for

$$f_{\{A_i\}} := \sum_{i=1}^n f(g_i) \chi_{A_i},$$

where $g_i$ is any point in $A_i$ for each $i \in \mathbb{N}$, we have $||f_{\{A_i\}} - f||_{\infty} \leq \varepsilon$. Indeed, since $f$ is bounded there exists a finite disjoint Borel cover, $\{D_1, \ldots, D_n\}$, of $f(G)$ consisting of rectangles with a diameter equal to $\varepsilon$ and with sides that are either open, closed or half-open intervals. Then the Borel sets $A_i = f^{-1}(D_i)$ satisfies the conditions in our claim.

We denote the set of all finite Borel partitions of $G$ by $\mathcal{U}$. The relation

$$U \leq V \iff V \text{ is a Borel subpartition of } U$$

turns $\mathcal{U}$ into an ordered set where by a Borel subpartition of $U \in \mathcal{U}$ we mean a partition $V \in \mathcal{U}$ such that each $A \in U$ is a finite union of elements in $V$.

For each neighbourhood $U \subset G$, choose a point $g_U \in U$ and let $m$ be a mean on $C_b(G)$. Then for each $f \in C_b(G)$ and $\{A_i\} := \{A_1, \ldots, A_n\} \in \mathcal{U}$ define

$$m_{\{A_i\}} := \sum_{i=1}^n (\chi_{A_i}, m) \delta_{g_A_i} \quad \text{and} \quad f_{\{A_i\}} := \sum_{i=1}^n f(g_A_i) \chi_{A_i}.$$

We see that $m_{\{A_i\}} \in \text{conv} \{ \delta_g : g \in G \}$ since $\sum_{i=1}^n (\chi_{A_i}, m) = \langle 1, m \rangle = 1$ and $(\chi_{A_i}, m) \geq 0$ for each $i \in \mathbb{N}$.

Now, for any $\varepsilon > 0$ there exists, by the discussion above, a Borel partition $\{B_i\} \in \mathcal{U}$ of $G$ such that $||f_{\{B_i\}} - f||_{\infty} \leq \varepsilon$. Clearly $||f_{\{B_i\}} - f||_{\infty} \leq \varepsilon$ for each $\{B_i\} \geq \{B_i\}$. Hence $||f_{\{A_i\}} - f||_{\infty} \rightarrow 0$. Hence, by continuity of $m$ we obtain

$$\langle f, m_{\{A_i\}} \rangle = \langle f, \sum_{i=1}^n (\chi_{A_i}, m) \delta_{g_A_i} \rangle = \sum_{i=1}^n \langle \chi_{A_i}, m \rangle \langle f, \delta_{g_A_i} \rangle$$

$$= \sum_{i=1}^n (\chi_{A_i}, m) f(g_A_i) = \sum_{i=1}^n f(g_A_i) \chi_{A_i}, m$$

$$= \langle f_{\{A_i\}}, m \rangle \rightarrow \langle f, m \rangle$$

which completes the proof.

**Proposition 5.5.** For a locally compact group $G$ the following are equivalent:

(i) There is a left invariant mean on $L^\infty(G)$.

(ii) There is a left invariant mean on $C_b(G)$.

(iii) There is a left invariant mean on $LUC(G)$. 
Proof. (i)⇒(ii): The restriction to $C_b(G)$ of a left-invariant mean on $L^\infty(G)$ is clearly a left-invariant mean on $C_b(G)$.

(ii)⇒(iii): A similar argument as above proves the claim since $LUC(G) \subset C_b(G)$ by definition.

(iii)⇒(i): Let $K$ be a compact, symmetric neighbourhood of $e$ and $m$ a left-invariant mean on $LUC(G)$. Define

$$m' : L^\infty(G) \to \mathbb{C}$$

by $m'(\phi) = \langle \chi_K \ast \phi, m \rangle$. Here, $\chi_K \ast \phi$ is continuous on $G$ by Remark 3.29, and also

$$|\chi_K \ast \phi(x)| \leq \int_G |\chi_K(t) \phi(t^{-1}x)| m_G(dt) \leq ||\phi||_{L^\infty} m_G(K)$$

for all $x \in G$, $\phi \in L^\infty(G)$, and hence $\chi_K \ast \phi \in C_b(G)$. Moreover, for a net $(g_\alpha)_\alpha$ in $G$ such that $g_\alpha \to g \in G$ we have

$$||L_{g_\alpha}(\chi_K \ast \phi) - L_g(\chi_K \ast \phi)||_{L^\infty} = ||(L_{g_\alpha} \chi_K) \ast \phi - (L_g \chi_K) \ast \phi||_{L^\infty}$$

$$\leq ||L_{g_\alpha} \chi_K - L_g \chi_K||_{L^1} ||\phi||_{L^\infty} \to 0$$

by Proposition 3.24 and Remark 3.29. Hence $m'$ is well-defined.

By the linearity of the convolution operation, $m'$ is a linear functional on $L^\infty(G)$. We also observe that

$$\chi_K \ast 1(y) = \int_G \chi_K(t) 1(t^{-1}y) m_G(dt) = m_G(K) \quad (y \in G)$$

and hence $m'(1) = m_G(K)$. Moreover, if $\phi \in L^\infty(G)$ is non-negative, then

$$t \mapsto \chi_K(t) \phi(t^{-1}x) \geq 0 \quad (x \in G)$$

and hence $\chi_K \ast \phi(x) \geq 0$ for all $x \in G$. Since $m$ is a mean on $LUC(G)$, it follows that

$$m'(\phi) = \langle \chi_K \ast \phi, m \rangle \geq 0 \quad (\phi \in L^\infty(G), \phi \geq 0).$$

By the symmetry of $K$, we obtain that $R_g \chi_K = L_{g^{-1}} \chi_K$ for any $g \in G$ and hence

$$(\chi_K \ast L_g \phi)(x) = \int_G \chi_K(y) L_g \phi(y^{-1}x) m_G(dy) = \int_G R_g \chi_K(y) \phi(y^{-1}x) m_G(dy)$$

$$= \int_G L_{g^{-1}} \chi_K(y) \phi(y^{-1}x) m_G(dy) = (L_{g^{-1}} \chi_K) \ast \phi(x)$$

$$= (\chi_K \ast \phi)(x) \quad (g, x \in G, \phi \in L^\infty(G)).$$

Hence by the left-invariance of $m$ on $LUC(G)$ we have for each $g \in G, \phi \in L^\infty(G)$ that

$$m'(L_g \phi) = \langle \chi_K \ast L_g \phi, m \rangle = \langle L_{g^{-1}} (\chi_K \ast \phi), m \rangle = \langle \chi_K \ast \phi, m \rangle = m'(\phi).$$

Finally, after normalisation, $\tilde{m} := \frac{m'}{m_G(K)}$ is a left-invariant mean on $L^\infty(G)$. \qed
In fact, one could even prove that if $UC(G)$ has a left-invariant mean, then also $L^\infty(G)$ has a left-invariant mean.

**Definition 5.6.** A locally compact group is called amenable if there exists a left-invariant mean on $L^\infty(G)$.

We now look at some standard examples of amenable and non-amenable groups.

**Example 5.7.** Let $G$ be a compact topological group. Then for the normalized Haar measure $m_G$, the linear functional $m : f \mapsto \int_G fm_G$ is a left invariant mean on $L^1(G)$ and thus, the restriction of $m$ to $L^\infty(G) \subset L^1(G)$ is a left-invariant mean on $L^\infty(G)$. Hence $G$ is amenable.

The locally compact group $(\mathbb{R}, +)$ is amenable. To see this, we first observe that $$L^\infty(\mathbb{R}) \to \mathbb{R}, \quad m_n(\phi) = \frac{1}{2n} \int_{-n}^n \phi(x)dx$$ is a mean on $L^\infty(\mathbb{R})$ for every $n \in \mathbb{N}$. Clearly $m_n$ is not left-invariant for any $n \in \mathbb{N}$. However, for any $t \in \mathbb{R}, \phi \in L^\infty(\mathbb{R})$ we have that

$$|m_n(t+x) - m_n(x)| = \left| \frac{1}{2n} \int_{-n}^{n} (\phi(t+x) - \phi(x))dx \right| = \left| \frac{1}{2n} \left( \int_{-n-t}^{n-t} \phi(x)dx - \int_{-n}^{n} \phi(x)dx \right) \right|$$

$$= \frac{1}{2n} \int_{-n}^{n} \phi(x)dx + \int_{n}^{n+t} \phi(x)dx \leq \frac{2|t||\phi||_\infty}{2n} \to 0$$

when $n \to \infty$. Since $\mathcal{M}(L^\infty(\mathbb{R}))$ is $w^*$-compact by Proposition 5.3, it follows that every $w^*$-accumulation point of the sequence $(m_1, m_2, \ldots)$ is a left-invariant mean on $L^\infty(\mathbb{R})$.

Actually all abelian locally compact groups are amenable. For a proof of this non-trivial fact and for constructions of more left-invariant means for concrete cases we refer to [Pat].

**Example 5.8.** A standard example of a non-amenable group is the free group $\mathbb{F}_2$ in two generators equipped with the discrete topology.

To see this, let $a$ and $b$ denote the generators of $\mathbb{F}_2$ so that each element of $\mathbb{F}_2$ is a reduced word over the alphabet $A := \{a, b, a^{-1}, b^{-1}\}$. For any $x \in A$ let

$$W(x) := \{w \in \mathbb{F}_2 : w \text{ starts with } x\}.$$  

Thus we have the following disjoint union:

$$\mathbb{F}_2 = \bigcup_{x \in A} W(x) \cup \{e\}.$$  

If $w \in \mathbb{F}_2 \setminus W(x)$ where $x \in \{a, b\}$, then $x^{-1}w \in W(x^{-1})$ and thus $w \in xW(x^{-1})$. Therefore

$$\mathbb{F}_2 = W(x) \cup xW(x^{-1}) \quad (x \in \{a, b\}).$$

Suppose that $m$ is a left-invariant mean for $\mathbb{F}_2$. Then by using positivity and the left-invariance of $m$ and (5.9) we obtain that

$$1 = \langle 1, m \rangle = \sum_{x \in \mathcal{A}} \langle \chi_{W(x)}, m \rangle + \langle \chi_{\{e}\}}, m \rangle$$

$$\geq \sum_{x \in \mathcal{A}} \langle \chi_{W(x)}, m \rangle = \langle \chi_{W(a)} + \chi_{aW(a^{-1})}, m \rangle + \langle \chi_{W(b)} + \chi_{bW(b^{-1})}, m \rangle$$

$$\geq \langle \chi_{W(a)} + \chi_{aW(a^{-1})}, m \rangle + \langle \chi_{W(b)} + \chi_{bW(b^{-1})}, m \rangle = \langle 1, m \rangle = 1 + \langle 1, m \rangle = 2.$$  

This is a contradiction which shows that $\mathbb{F}_2$ is not amenable.
CHAPTER 5. AMENABLE LOCALLY COMPACT GROUPS

The following theorem gives a useful characterization of amenable locally compact groups.

**Theorem 5.10.** (Day’s fixed point theorem) Let $G$ be a locally compact group. Then the following are equivalent:

(i) $G$ is amenable.

(ii) If $G$ acts affinely on a compact, convex subset $K$ of a Hausdorff LCS $E$, that is,

\[ g \cdot (tx + (1 - t)y) = t(gx) + (1 - t)(gy) \quad (g \in G, x, y \in K, t \in [0, 1]), \]

such that the action

\[(5.11) \quad G \times K \to K, \quad (g, x) \mapsto gx \]

is separately continuous, then there exists $x \in K$ so that $gx = x$ for all $g \in G$.

**Proof.** (i)$\Rightarrow$(ii): For each $\phi \in E^*$, $x \in K$ and $g \in G$ define

\[ \phi_x : G \to \mathbb{C}, \quad g \mapsto \langle gx, \phi \rangle \quad \text{and} \quad \phi_g : K \to \mathbb{C}, \quad x \mapsto \langle gx, \phi \rangle. \]

Since we suppose that the action (5.11) is separately continuous, it follows that $\phi_x$ and $\phi_g$ are continuous. In particular, $\phi_x \in C_b(G)$ for all $x \in K$ since $K$ is compact. Let $m$ be a left-invariant mean on $C_b(G)$. By Proposition 5.4, $m$ is the $w^*$-limit in $C_b(G)^*$ of a net $(m_\alpha)_\alpha$ where

\[ m_\alpha = \sum_{i=1}^{n_\alpha} t_{i,\alpha} \delta_{g_{i,\alpha}} \]

for some $n_\alpha \in \mathbb{N}$, $g_{1,\alpha}, \ldots, g_{n_\alpha,\alpha} \in G$, and $t_{1,\alpha}, \ldots, t_{n_\alpha,\alpha} \geq 0$ such that $\sum_{i=1}^{n_\alpha} t_{i,\alpha} = 1$ for all $\alpha$.

Thus, for fixed $x_0 \in K, y_0 \in G$ we get that

\[(5.12) \quad \langle L_{g_0} \phi_{x_0}, m_\alpha \rangle = \sum_{i=1}^{n_\alpha} t_{i,\alpha} \langle L_{g_{i,\alpha}} \phi_{x_0}, \phi \rangle = \sum_{i=1}^{n_\alpha} t_{i,\alpha} \langle \phi_{g_{i,\alpha} x_0}, \phi \rangle = \langle \phi_{g_{0} \left( \sum_{i=1}^{n_\alpha} t_{i,\alpha} g_{i,\alpha} x_0 \right)}, \phi \rangle = \phi_{g_0}(\sum_{i=1}^{n_\alpha} t_{i,\alpha} g_{i,\alpha} x_0) \]

since the action is affine and $K$ is convex by definition. Since $K$ is compact, the net $(x_\alpha)_{\alpha} := (\sum_{i=1}^{n_\alpha} t_{i,\alpha} g_{i,\alpha} x_0)_\alpha$ in $K$ has an accumulation point $x \in K$. By passing to a subnet if necessary, we can suppose that $\lim_\alpha x_\alpha = x$. Since $w^*$-$\lim_\alpha m_\alpha = m$ in $C_b(G)^*$ and $m$ is left invariant, we have by using (5.12) that

\[ \phi_{g_0}(x) = \lim_\alpha \phi_{g_0}(x_\alpha) = \lim_\alpha \langle L_{g_0} \phi_{x_0}, m_\alpha \rangle = \langle L_{g_0} \phi_{x_0}, m \rangle = \langle \phi_{x_0}, m \rangle. \]

We see that the right hand side does not depend on $g_0$ and therefore $\phi_g(x) = \phi_e(x)$ for all $g \in G$ and hence,

\[ \langle gx, \phi \rangle = \phi_g(x) = \phi_e(x) = \langle ex, \phi \rangle = \langle x, \phi \rangle \quad (g \in G). \]

Since $E$ is a Hausdorff LCS by definition, $E^*$ separates the points in $E$ and hence, $gx = x$ for all $g \in G$. 

(ii)⇒(i): By taking a clue from Propositions 5.3 and 5.5 we define \( gm \in \mathcal{M}(LUC(G)) \) for \( g \in G, m \in \mathcal{M}(LUC(G)) \) through

\[
(f, gm) := \langle L_g f, m \rangle \quad (f \in LUC(G))
\]

and claim that the mapping

\[
\Psi : G \times \mathcal{M}(LUC(G)) \to \mathcal{M}(LUC(G)), \quad (g, m) \mapsto gm
\]

together with \( \mathcal{M}(LUC(G)) \) equipped with the \( w^* \)-topology satisfies the conditions in (ii).

a) The mapping \( \Psi \) is well-defined and defines an action on \( \mathcal{M}(LUC(G)) \): Fix \( g \in G \). First, we note that \( L_g f \in LUC(G) \) for all \( f \in LUC(G) \). Indeed, clearly \( L_g f \in C_\alpha(G) \) and if \( x_\alpha \to x \) in \( G \) we have that

\[
\|L_x \alpha(L_g f) - L_x(L_g f)\|_\infty = \|L_{gx_\alpha} f - L_{gx} f\|_\infty \to 0
\]

since \( gx_\alpha \to gx \). Thus \( L_g f \in LUC(G) \). Moreover,

\[
gm(\lambda_1 f_1 + \lambda_2 f_2) = \langle L_g(\lambda_1 f_1 + \lambda_2 f_2), m \rangle = (\lambda_1 L_gf_1 + \lambda_2 L_g f_2, m) = \lambda_1 gm(f_1) + \lambda_2 gm(f_2) \quad (\lambda_i \in \mathbb{C}, f_i \in LUC(G), i = 1, 2),
\]

and hence \( gm \) is linear. Finally, by observing that

\[
gm(1) = \langle L_g 1, m \rangle = (1, m) = 1
\]

and that

\[
|gm(f)| = |\langle L_g f, m \rangle| \leq \|L_g f\|_\infty = \|f\|_\infty
\]

we conclude that \( gm \in \mathcal{M}(LUC(G)) \) and so the mapping \( \Psi \) is well-defined. The see that \( \Psi \) defines an action on \( \mathcal{M}(LUC(G)) \), let \( g, h \in G \). Then

\[
\langle f, (gh)m \rangle = \langle L_{gh} f, m \rangle = \langle L_h(L_g f), m \rangle = \langle L_g f, hm \rangle = \langle f, g(hm) \rangle \quad (f \in LUC(G)),
\]

and since

\[
\langle f, cm \rangle = \langle L_c f, m \rangle = \langle f, m \rangle \quad (f \in LUC(G)),
\]

we conclude that \( \Psi \) defines an action on \( \mathcal{M}(LUC(G)) \).

b) \( \Psi \) is affine and separately continuous: Let \( g \in G, m, n \in \mathcal{M}(LUC(G)) \) and \( t \in [0, 1] \).

Then for any \( f \in LUC(G) \) we have

\[
\langle f, g(tm + (1 - t)n) \rangle = \langle L_g f, tm + (1 - t)n \rangle = t\langle L_g f, m \rangle + (1 - t)\langle L_g f, n \rangle = t\langle f, gm \rangle + (1 - t)\langle f, gn \rangle,
\]

which shows that \( \Psi \) is affine.

To show the required continuity, let \( m \in \mathcal{M}(LUC(G)) \) and let \( (g_\alpha)_\alpha \) be a net in \( G \) that converges to some point \( g \in G \). Then, \( \|L_{g_\alpha} f - L_g f\|_\infty \to 0 \) for any \( f \in LUC(G) \) by definition. Thus

\[
\langle f, (g_\alpha) m \rangle = \langle L_{g_\alpha} f, m \rangle \to \langle L_g f, m \rangle = \langle f, gm \rangle,
\]
and hence, $g \mapsto gm$ is continuous with respect to the given topology on $G$ and the $w^*$-topology on $LUC(G)^*$.

Next, let $g \in G$ and $(m_n)_n$ be a net in $M(LUC(G))$ such that $w^*$-$\lim_n m_n = m$ in $LUC(G)^*$ for some mean $m \in M(LUC(G))$. It follows directly from the definition of the $w^*$-topology that for any $f \in LUC(G)$ we have

$$(f, gm_n) = (L_g f, m_n)$$

Hence $m \mapsto gm$ is $w^*$-$w^*$-continuous.

Since $LUC(G)^*$ equipped with the $w^*$-topology is a Hausdorff LCS and $M(LUC(G))$ was shown in Proposition 5.3 to be $w^*$-compact and convex, there is by assumption a mean $m \in M(LUC(G))$ such that $gm = m$ for all $g \in G$, that is, $(L_g f, m) = (f, m)$ for all $f \in LUC(G)$ and $g \in G$. Hence $m$ is a left-invariant mean on $LUC(G)$ and thus $G$ is amenable by Theorem 5.5.

5.2 Johnson’s Theorem for the Group Algebra $L^1(G)$

In the final section of this Master’s thesis we prove the theorem by B. E. Johnson that connects the concepts of amenable Banach algebras and amenable locally compact groups. Before we are ready for the proof presented here we need one final notion from the theory of amenable locally compact groups, namely topological left-invariance of means.

**Definition 5.14.** Recall that $P(G) := \{f \in L^1(G) : f \geq 0, \|f\|_1 = 1\}$. A mean $m \in L^\infty(G)^*$ is called **topologically left-invariant** if

$$(f \ast \phi, m) = \langle \phi, m \rangle$$

for every $f \in P(G)$.

It is easy to see that the canonical embedding $f \in L^1(G)^{**} \cong L^\infty(G)^*$ is a mean on $L^\infty(G)$ for every $f \in P(G)$. More interestingly, the canonical embedding of $P(G)$ into $L^\infty(G)^*$ is $w^*$-dense in $M(L^\infty(G))$, see [Pat, (0.1) Proposition]. We will not however need this fact but rather the following lemma which connects the notions of topological left-invariance and left-invariance for means.

**Lemma 5.15.** Let $G$ be a locally compact group. If $m \in L^\infty(G)^*$ is a topologically left-invariant mean, then $m$ is left-invariant.

**Proof.** Let $\phi \in L^\infty(G)$, $g \in G$ and $f \in P(G)$. Then $\Delta(g) R_g f$ also belongs to $P(G)$ where $\Delta$ denotes the modular function of $m_G$. Indeed, using Proposition 3.20, we obtain

$$\Delta(g) R_g f \geq 0 \quad \text{and} \quad \int_G \Delta(g) R_g f dm_G = \Delta(g) \Delta(g^{-1}) \int_G f dm_G = 1.$$

Thus, by using Remark 3.28 and the topological left-invariance of $m$, we obtain that

$$(L_g \phi, m) = (f \ast L_g \phi, m) = (\Delta(g) R_g f \ast \phi, m) = (\phi, m)$$

and hence, $m$ is left-invariant.

Finally, we have gathered all the tools required for proving Johnson’s theorem.
Theorem 5.16. (Johnson) Let $G$ be a locally compact group. Then the following are equivalent:

(i) $G$ is amenable as a locally compact group.

(ii) $L^1(G)$ is an amenable Banach algebra.

Proof. (i)$\Rightarrow$ (ii): Let $X$ be a Banach $L^1(G)$-bimodule and let $D \in Z^1(L^1(G), X^*)$. By Proposition 4.33, $L^1(G)$ has a bounded approximate identity and so by Proposition 4.65 it is enough to suppose that $X$ is pseudo-unital. Since $L^1(G)$ is an ideal of $M(G)$ we can define the canonical $M(G)$-bimodule structure on $X$ obtained in Proposition 4.59. Furthermore, by Proposition 4.60 there is a unique extension of $D$, $\hat{D} \in Z^1(M(G), X^*)$, which is continuous with respect to the strict topology on $M(G)$ and the $w^*$-topology on $X^*$. We will use Day’s fixed point theorem to show that the mapping

$$\Phi : G \times X^* \to X^*, \quad (g, \phi) \mapsto \delta_g \cdot \phi \cdot \delta_g^{-1} + \hat{D}(\delta_g) \cdot \delta_g^{-1}$$

has a fixed point $\psi \in X^*$ such that $\Phi(g, \psi) = \psi$ for all $g \in G$. This is equivalent to the existence of $\psi \in X^*$ such that $\hat{D}(\delta_g) = \delta_g \cdot (-\psi) - (-\psi) \cdot \delta_g$ for all $g \in G$. Indeed,

$$\delta_g \cdot \psi \cdot \delta_g^{-1} + \hat{D}(\delta_g) \cdot \delta_g^{-1} = \psi \iff \delta_g \cdot \psi \cdot (\delta_g^{-1} \ast \delta_g) + \hat{D}(\delta_g) \cdot (\delta_g^{-1} \ast \delta_g) = \psi \cdot \delta_g$$

$$\iff \delta_g \cdot \psi \cdot \delta_g + \hat{D}(\delta_g) \cdot \delta_g = \psi \cdot \delta_g$$

$$\iff \delta_g \cdot \psi + \hat{D}(\delta_g) = \psi \cdot \delta_g$$

$$\iff \hat{D}(\delta_g) = \delta_g \cdot (-\psi) - (-\psi) \cdot \delta_g \quad (g \in G, \psi \in X^*)$$

where the third equivalence holds since $X$ is pseudo-unital.

Since span $\{\delta_x : x \in G\}$ is dense in $M(G)$ with respect to the strict topology by Proposition 4.40 and $\hat{D}$ is continuous with respect to the strict topology on $M(G)$, it follows by linearity and continuity that $\hat{D}(\mu) = \mu \cdot (-\psi) - (-\psi) \cdot \mu$ for all $\mu \in M(G)$. Thus $\hat{D} \in \mathcal{B}^1(M(G), X^*)$ and hence $D \in \mathcal{B}^1(L^1(G), X^*)$ by the uniqueness of $\hat{D}$.

Now, let $K$ be the $w^*$-closure of conv $\{[\hat{D}(\delta_g) \cdot \delta_g^{-1} : g \in G\}$ in $X^*$. We will show that $K$ and $\Phi$ satisfies the conditions in Day’s fixed point theorem.

a) $\Phi$ is an affine action of $G$ on $X^*$: For any $x \in X$ we have

$$\langle x, \hat{D}(\delta_e) \rangle = \langle x, \hat{D}(\delta_e \ast \delta_e) \rangle = \langle x, \delta_e \cdot \hat{D}(\delta_e) + \langle x, \hat{D}(\delta_e) \cdot \delta_e \rangle$$

$$\langle x, \delta_e, \hat{D}(\delta_e) \rangle + \langle \delta_e \cdot x, \hat{D}(\delta_e) \rangle = \langle x, \hat{D}(\delta_e) \rangle + \langle x, \hat{D}(\delta_e) \rangle,$$

which implies that $\hat{D}(\delta_e) = 0$. Hence,

$$\Phi(e, \phi) = \delta_e \cdot \phi \cdot \delta_e + \hat{D}(\delta_e) \cdot \delta_e = \phi \quad (\phi \in X^*).$$

Moreover, since $\hat{D} \in Z^1(M(G), X^*)$ we obtain that

$$\Phi(gh, \phi) = \delta_{gh} \cdot \phi \cdot \delta_{(gh)^{-1}} + \hat{D}(\delta_{gh}) \cdot \delta_{(gh)^{-1}}$$

$$= \delta_g \cdot (\delta_h \cdot \phi \cdot \delta_h^{-1}) \cdot \delta_g^{-1} + \langle \delta_g \cdot \hat{D}(\delta_h) \cdot \delta_h \rangle \cdot \delta_h^{-1} \ast \delta_g^{-1}$$

$$= \delta_g \cdot (\delta_h \cdot \phi \cdot \delta_h^{-1} + \hat{D}(\delta_h) \cdot \delta_h^{-1}) \cdot \delta_g^{-1} + \hat{D}(\delta_g) \cdot \delta_g^{-1}$$

$$= \Phi(g, \Phi(h, \phi)) \quad (g, h \in G, \phi \in X^*)$$
and therefore Φ defines a group action.

The action Φ is affine since, for a fixed \( g \in G \), the mapping

\[
X^* \to X^*, \quad \phi \mapsto \Phi(g, \phi) = \delta_g \cdot \phi \cdot \delta_{g^{-1}} + \tilde{D}(\delta_g) \cdot \delta_g
\]

is a sum of a linear and a constant mapping.

b) Φ is separately continuous: Let \( \tau \) denote the topology on \( G \). By Proposition 3.37 the map

\[
G \to M(G), \quad g \mapsto \delta_g
\]

is continuous with respect to \( \tau \) and the strict topology on \( M(G) \). It follows that the mapping \( G \to X^* \), \( g \mapsto \tilde{D}(\delta_g) \) is \( \tau \)-\( w^* \)-continuous. Now, let \( x \in X \) be arbitrary and let \( \phi \in X^* \) be fixed. Since \( X \) is pseudo-unital there exist \( f_1, f_2 \in L^1(G) \) and \( y \in X \) such that \( x = f_1 \cdot y \cdot f_2 \). Hence, if \( g_n \to g \) in \( G \) we obtain that

\[
|x, (\delta_{g_n} - \delta_g) \cdot \phi \cdot (\delta_{g_n^{-1}} - \delta_g^{-1})| = |\langle f_1 \cdot y \cdot f_2 \operatorname{\ast} (\delta_{g_n} - \delta_g), \phi \cdot (\delta_{g_n^{-1}} - \delta_g^{-1}) \rangle| + |\langle (\delta_{g_n^{-1}} - \delta_g^{-1}) \ast f_1 \cdot y \cdot f_2, \delta_g \cdot \phi \rangle| \\
\leq \kappa ||f_1 \cdot y|| ||f_2 \ast (\delta_{g_n} - \delta_g)|| ||\phi|| + \kappa ||(\delta_{g_n^{-1}} - \delta_g^{-1}) \ast f_1|| ||y \cdot f_2|| ||\phi|| \to 0,
\]

and

\[
|x, \tilde{D}(\delta_{g_n}) \cdot (\delta_{g_n^{-1}} - \delta_{g^{-1}})| + |x, (\tilde{D}(\delta_{g_n}) - \tilde{D}(\delta_g)) \cdot \delta_{g^{-1}}| \\
= |\langle (\delta_{g_n^{-1}} - \delta_{g^{-1}}) \ast f_1 \cdot y \cdot f_2, \tilde{D}(\delta_{g_n}) \rangle| + |\langle \delta_{g^{-1}} \cdot x, \tilde{D}(\delta_{g_n} - \delta_g) \rangle| \\
\leq \kappa ||(\delta_{g_n^{-1}} - \delta_{g^{-1}}) \ast f_1|| ||y \cdot f_2|| ||\tilde{D}|| + \kappa ||(\delta_{g_n} - \delta_g) \ast f_1|| ||x|| \to 0,
\]

since \( ||\delta_g|| = 1 \) for all \( g \in G \). Thus,

\[
|x, \Phi(g_n, \phi) - \Phi(g, \phi)| = |\langle x, \delta_{g_n} \cdot \phi \cdot \delta_{g_n^{-1}} + \tilde{D}(\delta_{g_n}) \cdot \delta_{g_n^{-1}} - (\delta_g \cdot \phi \cdot \delta_{g^{-1}} + \tilde{D}(\delta_g) \cdot \delta_{g^{-1}}) \rangle| \\
\leq |\langle x, (\delta_{g_n} - \delta_g) \cdot \phi \cdot \delta_{g_n^{-1}} \rangle| + |\langle x, \delta_{g^{-1}} \cdot \phi \cdot (\delta_{g_n^{-1}} - \delta_{g^{-1}}) \rangle| \\
+ |\langle x, \tilde{D}(\delta_{g_n}) \cdot (\delta_{g_n^{-1}} - \delta_{g^{-1}}) \rangle| + |\langle x, (\tilde{D}(\delta_{g_n}) - \tilde{D}(\delta_g)) \cdot \delta_{g^{-1}} \rangle| \to 0,
\]

and so \( g \mapsto \Phi(g, \phi) \) is \( \tau \)-\( w^* \)-continuous.

Next, let \( g \in G \) and \((\phi_n)\) be a net in \( X^* \) such that \( w^* \)-\( \lim \phi_n = \phi \) for some \( \phi \in X^* \).

Then, for any \( x \in X \) we have

\[
|x, \Phi(g, \phi_n) - \Phi(g, \phi)| = |\langle x, \delta_{g_n} \cdot \phi_n \cdot \delta_{g_n^{-1}} + \tilde{D}(\delta_{g_n}) \cdot \delta_{g_n^{-1}} - (\delta_g \cdot \phi \cdot \delta_{g^{-1}} + \tilde{D}(\delta_g) \cdot \delta_{g^{-1}}) \rangle| \\
= |\langle x, \delta_{g_n} \cdot (\phi_n - \phi) \cdot \delta_{g_n^{-1}} \rangle| = |\langle (\delta_{g_n^{-1}} \cdot x \cdot \delta_{g_n}, \phi_n - \phi) \rangle| \to 0
\]

which shows that \( \phi \mapsto \Phi(g, \phi) \) is \( w^* \)-\( w^* \)-continuous.

c) \( K \) is \( w^* \)-compact and convex: For each \( g \in G \) we have

\[
||\tilde{D}(\delta_g) \cdot \delta_{g^{-1}}|| \leq \kappa ||\tilde{D}(\delta_g)|| \leq \kappa ||\tilde{D}|| ||\delta_g|| = \kappa ||\tilde{D}||.
\]
CHAPTER 5. AMENABLE LOCALLY COMPACT GROUPS

Hence also

\[ ||\phi|| \leq \kappa ||\hat{D}|| \]

for all \( \phi \in \text{conv}\{\hat{D}(\delta) \cdot \delta^{-1} : g \in G\} \) and thus (5.18) holds also for all \( \phi \in K \). Hence, \( K \) is bounded and since \( K \) is \( w^*\)-closed by definition, it is \( w^*\)-compact by Alaoglu’s theorem.

In order to verify the convexity of \( K \), let \( \phi, \psi \in K \), \( t \in [0,1] \) and \( (\phi_\alpha)_{\alpha} \) and \( (\psi_\beta)_{\beta} \) be nets in \( \text{conv}\{\hat{D}(\delta) \cdot \delta^{-1} : g \in G\} \) that \( w^*\)-converges to \( \phi \) and \( \psi \) respectively. Then clearly the net \( (t\phi_\alpha + (1-t)\psi_\beta)_{(\alpha,\beta)} \) is contained in \( \text{conv}\{\hat{D}(\delta) \cdot \delta^{-1} : g \in G\} \) and \( w^*\)-converges to \( t\phi + (1-t)\psi \). Thus \( t\phi + (1-t)\psi \in K \) and hence, \( K \) is convex.

d) \( K \) is invariant under \( \Phi \): We first observe that \( \{\hat{D}(\delta) \cdot \delta^{-1} : g \in G\} \) is invariant under \( \Phi \). Indeed, for \( g, h \in G \), we have

\[
\Phi(g, \hat{D}(\delta_h) \cdot \delta_{h^{-1}}) = \delta_g \cdot (\hat{D}(\delta_h) \cdot \delta_{h^{-1}}) \cdot \delta_{g^{-1}} + \hat{D}(\delta_g) \cdot \delta_{g^{-1}}
\]

\[
= \delta_g \cdot \hat{D}(\delta_h) \cdot \delta_{g^{-1}} + \hat{D}(\delta_g) \cdot \delta_{g^{-1}}
\]

\[
= (\delta_g \cdot \hat{D}(\delta_h) + \hat{D}(\delta_g) \cdot \delta_{g^{-1}}) \cdot \delta_{g^{-1}}
\]

\[
= \hat{D}(\delta_h \cdot \delta_{g^{-1}}) = \hat{D}(\delta_{gh}) \cdot \delta_{gh^{-1}} \in \{\hat{D}(\delta) \cdot \delta^{-1} : g \in G\}
\]

Thus, for \( a_1, \ldots, a_n \in \{\hat{D}(\delta) \cdot \delta^{-1} : g \in G\} \) and \( \lambda_1, \ldots, \lambda_n \geq 0 \) such that \( \sum_{i=1}^n \lambda_i = 1 \) we obtain that

\[
g \cdot \left( \sum_{i=1}^n \lambda_i a_i \right) = \sum_{i=1}^n \lambda_i ga_i \in \text{conv}\{\hat{D}(\delta) \cdot \delta^{-1} : g \in G\}
\]

since \( \Phi \) is an affine action and \( K \) is convex. Hence \( \text{conv}\{\hat{D}(\delta) \cdot \delta^{-1} : g \in G\} \) is invariant under \( \Phi \). Finally, since \( K \) is \( w^*\)-closed and \( \phi \mapsto \Phi(g, \phi) \) is \( w^*\)-continuous for each \( g \in G \) it follows that \( K \) is invariant under \( \Phi \).

Hence, \( K \) and \( \Phi \) satisfies the conditions in Day’s fixed point theorem and thus there exists \( \psi \in K \) such that \( \Phi(g, \psi) = \psi \) for all \( g \in G \). It follows by the discussion in the beginning of the proof that \( D \in B^1(L^1(G), X^*) \) and hence, \( L^1(G) \) is amenable.

(ii)\( \Rightarrow \) (i): Define a \( L^1(G) \)-bimodule action on \( L^\infty(G) \) by

\[
f \cdot \phi := f \ast \phi \quad \text{and} \quad \phi \cdot f := \left( \int_G f dm_G \right) \phi \quad (f \in L^1(G), \phi \in L^\infty(G)).
\]

By Example 4.5 (vi) this action defines indeed a Banach bimodule action on \( L^\infty(G) \). Now, \( C_1 \) is clearly a closed submodule of \( L^\infty(G) \). Hence, by Example 4.5 (iv), \( E := L^\infty(G)/C_1 \) is a Banach \( L^1(G) \)-bimodule. Choose \( n \in L^\infty(G)^* \) such that \( (1, n) = 1 \) \( (L^\infty(G)^* \) separates the points in \( L^\infty(G) \). Then for \( ad_n : L^1(G) \to L^\infty(G)^* \), \( f \mapsto f \cdot n - n \cdot f \) we have

\[
(1, ad_n(f)) = (1, f \cdot n - n \cdot f) = (1, f, n) - (f, 1, n) = (\int_G f dm_G, 1, n) - (f \ast 1, n) = (\int_G f dm_G, n) - (\int_G f dm_G, n) = 0 \quad (f \in L^1(G)).
\]
Thus $ad_n(L^1(G)) \subset C^1 \cong E^*$ and so $ad_n \in Z^1(L^1(G), E^*)$. Since $L^1(G)$ is amenable by assumption there exists $\tilde{n} \in C^1$ such that

$$ad_n(f) = f \cdot \tilde{n} - \tilde{n} \cdot f \quad (f \in L^1(G)).$$

Let $\tilde{m} := n - \tilde{n} \in L^\infty(G)^*$. Then $\langle 1, \tilde{m} \rangle = \langle 1, n \rangle = 1$ and for $\phi \in L^\infty(G), f \in P(G)$ we observe that $\phi \cdot f = \left( \int_G f \, dm_G \right) \phi = \phi$. Furthermore, $f \cdot n - n \cdot f = f \cdot \tilde{n} - \tilde{n} \cdot f$ which implies that $f \cdot \tilde{m} = \tilde{m} \cdot f$. Thus

$$\langle f \cdot \phi, \tilde{m} \rangle = \langle f \cdot \phi, \tilde{n} \rangle = \langle \phi, \tilde{m} \cdot f \rangle = \langle \phi, f \cdot \tilde{m} \rangle = \langle \phi \cdot f, \tilde{m} \rangle = \langle \phi, \tilde{m} \rangle \quad (f \in P(G), \phi \in L^\infty(G)).$$

Hence, as in Lemma 5.15 we obtain that $\langle L_g \phi, \tilde{m} \rangle = \langle f, \tilde{m} \rangle$ for all $g \in G$ and $\phi \in L^\infty(G)$. Finally, since $A := L^\infty(G)$ is a commutative unital $C^*$-algebra, $A$ and $C_0(\Omega(A))$ are isometrically $\ast$-isomorphic by Gelfand’s theorem. Furthermore, by Riesz representation theorem, $C_0(\Omega(A))^*$ is isometrically isomorphic to $M(\Omega(A))$, the measure space of $\Omega(A)$. Viewing $\tilde{m}$ as a measure, the variation $\lvert \tilde{m} \rvert$ is a positive measure and thus also positive as a linear functional in $L^\infty(G)^*$. To see this, let $f \in L^\infty(G)$ such that $f(x) \geq 0$ for all $x \in G$. Then clearly $f = g^2$ where $g(x) = \sqrt{f(x)}$ for all $x \in G$. Now, $g = \mathcal{g}$ where $\mathcal{g}(x) = \mathcal{g}(x)$ for all $x \in G$ and thus

$$\hat{f}(\tau) = \tau(f) = \tau(g\mathcal{g}) = \tau(g)\tau(\mathcal{g}) = \tau(g)\overline{\tau(g)} \geq 0 \quad (\tau \in \Omega(A)).$$

Hence $\hat{f} \geq 0$ and so

$$\langle f, \lvert \tilde{m} \rvert \rangle = \int_{\Omega(A)} \hat{f} \, d\lvert \tilde{m} \rvert \geq 0.$$ 

Also since $\tilde{m}$ is left invariant, $\lvert \tilde{m} \rvert$ is left invariant and thus by normalising $\lvert \tilde{m} \rvert$, we get that $\lvert m \rvert := \frac{\lvert \tilde{m} \rvert}{\langle \tilde{m}, \tilde{m} \rangle} \in L^\infty(G)^*$ is left-invariant, positive and $\langle 1, m \rangle = 1$. Consequently, $m$ is a left-invariant mean on $L^\infty(G)$ which means that $G$ is amenable.
Bibliography


