FINE STRUCTURE OF MEASURES

TUOMAS SAHLSTEN

Academic dissertation

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In Helsinki, September 2012 Tuomas Sahlsten

LIST OF INCLUDED ARTICLES

This thesis consists of an introductory part and the following three articles:

[A] T. SAHLSTEN, P. SHMERKIN, V. SUOMALA, *Dimension, entropy, and the local distribution of measures*, to appear on J. London Math. Soc. (2012).

[B] T. ORPONEN, T. SAHLSTEN, *Tangent measures of non-doubling measures*, Math. Proc. Cambridge Philos. Soc. **152** (2012), 555–569.

[C] T. SAHLSTEN, *Tangent measures of typical measures*, to appear on Real Analysis Exchange (2012).

The article [C] consists of author's independent research, with the exception of [C, Proposition 5.1]. The author had a central role in the research and writing of the articles [A] and [B].

1. OVERVIEW

During the birth of fractal geometry, several irregular sets, now commonly known as *fractals*, emerged from physical phenomena and became popular due to their rich fine structure. It turned out that *measures* are powerful in the study of fractal sets since measures can be used to distinguish local irregular structures. The basic intuition behind a measure is that one is given a fixed amount of *mass*, and then one *distributes* the mass to a space according to some rule. If we distribute mass as uniformly as possible around a fractal, the measure becomes complicated itself and actually characterizes the geometry of the fractal. These examples of measures arising from fractal geometry have led to the need to understand the fine structure of measures in general.

Several tools have been introduced to understand the local properties of measures. The theory of rectifiability has been a catalyst in the rise of such concepts. Here notions such as *densities*, *conical densities*, and *tangent measures* were introduced and became central, and have also recently become useful in the context of irregular sets. Moreover, *dimension* and *porosity* of measures, and a more recent related concept of *local homogeneity* have also risen when studying fractals with a lot of fine structure. In this thesis, we provide new results related to a number of these notions.

In Section 2 we study the geometric nature of fractal dimensions and summarize the work of article [A]. Dimensions of sets and measures are global quantities that capture the distribution of the set and measure under study. During recent decades there has been much research aiming to calculate the dimension of sets and measures. Often the objects in study have arisen from physical and dynamical contexts as attractors of some dynamical systems, where the dimension of the measure could be calculated using tools from dynamics. Here the exact value of the dimension becomes important as it gives also information about the behavior of the underlying dynamical system.

The intuition behind dimension can be understood from the following example. Consider the *Cantor dust* $C_\lambda \times C_\lambda$, where $C_\lambda \subset [0,1]$ is the classical Cantor set with contraction ratio $\lambda \in (0, 1/2]$. When we increase λ, the approximations of the set will occupy more space, see Figure 1.1. In fact, the self-similarity of the dust guarantees that such phenomenon happens everywhere no matter how close we look at the dust. The contraction ratio λ completely characterizes the fractal dimension of the Cantor-set: the theory of self-similar sets implies that any reasonable fractal dimension of the Cantor-dust is equal to

$$\dim(C_\lambda \times C_\lambda) = \frac{\log 4}{\log(1/\lambda)}\,.$$

Thus the dimension grows when λ increases. When λ approaches its maximal value $1/2$, the dimension grows to 2. In the case $\lambda = 1/2$, the dust fills everything and becomes the whole unit square $[0,1]^2$.

We are witnessing the geometric nature of dimension growth. Objects with large "dimension" should be more uniformly distributed in arbitrarily small scales. Concepts such as local homogeneity, conical densities and porosity are precisely made to serve us in the quantification of this phenomenon. Connecting dimension to these geometric notions has been a central part of research for the last century, all the way up to recent years. In the paper [A], we generalize and simplify a number of previous results related to them by providing a new unifying theory based on local entropy averages.

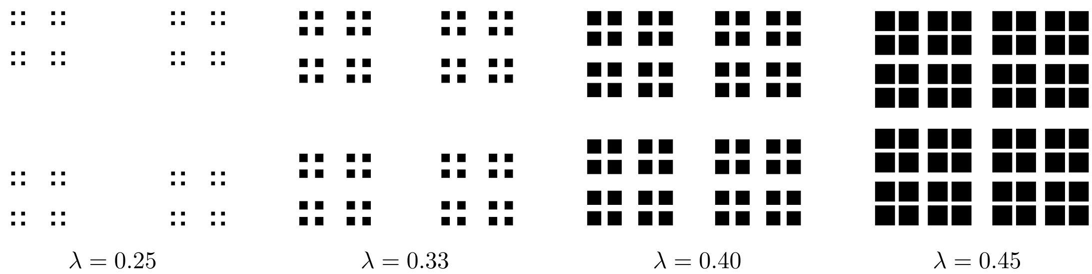

FIGURE 1.1. Third generation approximations of the Cantor dust $C_\lambda \times C_\lambda$ when the contraction ratio λ attains values 0.25, 0.33, 0.40 and 0.45. Higher the contraction ratio, or the "dimension" of the Cantor dust is, more space the dust ought to occupy.

In Section 3 we study the theory of tangent measures and summarize the articles [B] and [C]. Tangent measures have been quite useful in problems related to rectifiability since often when tangent measures behave well at many points, also the original measure will be quite regular. Moreover, tangent measures often possess more regularity than the original measure, so they might be easier to analyze. However, such heuristics do not hold in general. An example by D. Preiss in [33, Example 5.9] demonstrated that a singular measure might have only constant multiples of the Lebesgue measure as tangent measures. This provided an argument for the fact that irregularity of a measure might not preserve to tangent measures at all. Moreover, an example by T. O'Neil in [32] demonstrated that the set of tangent measures can be very rich: O'Neil constucted a measure that has every non-zero measure as a tangent measure of it at almost every point with respect to the measure.

Articles [B] and [C] are devoted to study the possible extensions of the results by Preiss and O'Neil. In the paper [B], we consider very badly singular measures, *non-doubling* measures, and give an example of a non-doubling measure with very regular tangent measures. This example also has some implications to the theory of porous measures. In the paper [C] we prove that *typical* measures in the sense of Baire category satisfy the same property as the measure O'Neil constructed. This result is not new, it was already proved by O'Neil in his PhD thesis [31], but our contribution to the problem is to exhibit a different self-contained proof, provide some ramifications, and study the sharpness of the result.

2. Dimension and mass distribution

In this section we go through the results we obtained in the paper [A]. The main focus is the relationship between dimension of measures and different geometric notions that describe the local distribution of mass. There are several ways to define a dimension for a measure. Originally dimensions were only defined for sets and the classical notions here are the Hausdorff and packing dimension. They can also be used to define global dimensions of measures: the *Hausdorff dimension* of a measure μ on $\mathbb{R}^d$ is defined by

$$\dim_{\mathrm{H}} \mu = \inf\{\dim_{\mathrm{H}} A : A \subset \mathbb{R}^d \text{ is a Borel set with } \mu(A) > 0\};$$

and the *packing dimension* of μ is

$$\dim_{\mathrm{P}} \mu = \inf\{\dim_{\mathrm{P}} A : A \subset \mathbb{R}^d \text{ is a Borel set with } \mu(A) > 0\},$$

where $\dim_{\mathrm{H}} A$ and $\dim_{\mathrm{P}} A$ are the usual Haudorff and packing dimension of the set $A \subset \mathbb{R}^d$, respectively. Here a *measure* on $\mathbb{R}^d$ is always a Radon-measure. In other words, dimension of the measure is defined by the minimal dimensions of the sets the measure charges.

Another way to define a dimension of a measure is given by the behavior of $\mu(B(x,r))$ with respect to r^s, when $r \searrow 0$. Here $B(x,r)$ is a closed ball of radius $r > 0$ and center $x \in \mathbb{R}^d$. This gives rise to the notion of *upper-* and *lower local dimensions* of a measure μ, which at $x \in \mathbb{R}^d$ are defined by the quantities:

$$\overline{\dim}_{\mathrm{loc}}(\mu, x) = \limsup_{r \searrow 0} \frac{\log \mu(B(x,r))}{\log r} \quad \text{and} \quad \underline{\dim}_{\mathrm{loc}}(\mu, x) = \liminf_{r \searrow 0} \frac{\log \mu(B(x,r))}{\log r}.$$

It turns out that understanding $\mu(B(x,r))$ when $r \searrow 0$ at μ almost every point is essential in the evaluation of the Hausdorff and packing dimension:

$$\dim_{\mathrm{H}} \mu = \sup\{s > 0 : \underline{\dim}_{\mathrm{loc}}(\mu, x) \geq s \text{ for } \mu \text{ almost every } x \in \mathbb{R}^d\};$$

and

$$\dim_{\mathrm{P}} \mu = \sup\{s > 0 : \overline{\dim}_{\mathrm{loc}}(\mu, x) \geq s \text{ for } \mu \text{ almost every } x \in \mathbb{R}^d\}.$$

For a proof of this fact, see [9, Proposition 10.2]. Hence the local dimensions of measures are the key to understand global dimensions.

In many examples, the measure μ under consideration often arises as an invariant measure for some dynamical system, such as in the case of self-similar sets. In these cases, the local dimension can be naturally derived from the theory of dynamics. However, calculating the local dimension directly can in general be very hard. If we are given some information on the distribution of μ, we can possibly try to rely on the technique of *local entropy averages*. Local entropy average is an alternative formula for the local dimension of a measure μ at μ almost every point. The formula is based on averaging entropies of the measure with respect to some cube grid. Considering problems related to mass distribution, this formulation turns out to be much more useful than the definition of local dimension as entropy takes into account the uniformity of the distribution of mass.

Let $\mathcal{Q}_k$ be the set of all *dyadic cubes* of generation $k \in \mathbb{N}$, that is, all the cubes of the form $[0, 2^{-k})^d + 2^{-k}b$ for some $b \in \mathbb{Z}^d$. Given $x \in \mathbb{R}^d$, we let $Q^{k,x}$ be the unique cube from $\mathcal{Q}_k$ containing x. When $a \in \mathbb{N}$, we denote $Q' \prec_a Q$ if Q' is an 2^a-adic *child* of Q, that is,

$Q \in \mathcal{Q}_k$, $Q' \in \mathcal{Q}_{k+a}$ and $Q' \subset Q$. If μ is a measure on $\mathbb{R}^d$, $a \in \mathbb{N}$, and Q is a cube with $\mu(Q) > 0$, the *a-entropy* of μ in the cube Q is

$$H^a(\mu, Q) = \sum_{Q' \prec_a Q} \tfrac{\mu(Q')}{\mu(Q)} \log \tfrac{\mu(Q)}{\mu(Q')}.$$

Proposition 2.1 (Local entropy averages). *Let μ be a measure on $\mathbb{R}^d$ and $a \in \mathbb{N}$. Then for μ almost every $x \in \mathbb{R}^d$:*

$$\overline{\dim}_{\mathrm{loc}}(\mu, x) = \limsup_{N \to \infty} \frac{1}{N \log 2^a} \sum_{k=1}^{N} H^a(\mu, Q^{k,x});$$

and

$$\underline{\dim}_{\mathrm{loc}}(\mu, x) = \liminf_{N \to \infty} \frac{1}{N \log 2^a} \sum_{k=1}^{N} H^a(\mu, Q^{k,x}).$$

The number $\log 2^a$ in the average could be considered as some kind of *Lyapunov exponent* of the measure μ with respect to the 2^a-adic dyadic partition of the cube $Q^{k,x}$. In fact, Shmerkin also considered in [37] a more general nested partition where the Lyapunov exponents depend on the generation k and the point x.

There are many variations of the exact statement of local entropy averages, see for example [10, 37, 36]. The particular formulation above is used in [A] and it is due to M. Hochman in a personal communication from 2011. The proof of the principle is an application of the law of large numbers for martingale differences when invoking the fact that the local dimension of μ can be calculated via dyadic filtrations μ almost everywhere, see [36, Theorem 5]. Local entropy averages were first considered by J. G. Llorente and A. Nicolau, but they relied on the law of the iterated logarithm rather than the law of large numbers. As a result, they get sharper results but under stronger assumptions on the measure, such as dyadic doubling, see [17, Corollary 6.2].

The main contribution of the paper [A] is to introduce local entropy averages as a unifying technique for the relationship between dimension and the geometric notions of local homogeneity, conical densities, and porosity. The idea is that local entropy averages can be used to obtain discrete information about the distribution of μ on a certain portion of scales. More precisely, if the entropy $H^a(\mu, Q^{k,x})$ is large for some generation k, then the measure μ is fairly uniformly distributed among 2^a-adic children of $Q^{k,x}$. This information is then used to derive bounds for local homogeneity, conical densities and porosity near x. This general strategy was introduced and used by Shmerkin in the study of mean porous measures in [37, 36]. However, the geometric arguments in our applications are rather more involved at all steps. In the next three sections we will introduce the concepts under study and our results.

2.1. **Local homogeneity.** Homogeneity of a measure was introduced by E. Järvenpää and M. Järvenpää in [12] as a tool which describes how far a measure is from uniform distribution. However, *a priori*, homogeneity by Järvenpääs is not a local notion in the sense that it would describe the behavior of a measure near a given point. For this purpose *local* homogeneity was introduced by A. Käenmäki, T. Rajala and V. Suomala in [13]. A local version of the homogeneity was a very welcomed concept since it served as a natural tool to study porosities and conical densities.

Definition 2.1. The *local homogeneity* of a measure μ at $x \in \mathbb{R}^d$ with parameters $\delta, \varepsilon, r > 0$ is defined by

$$\mathrm{hom}_{\delta,\varepsilon,r}(\mu, x) = \sup\{\sharp\mathcal{B} : \mathcal{B} \text{ is a } (\delta r)\text{-packing of } B(x,r) \\ \text{with } \mu(B) > \varepsilon\mu(B(x,5r)) \text{ for all } B \in \mathcal{B}\}.$$

Here a *δ-packing* of a set $A \subset \mathbb{R}^d$ is a disjointed collection of balls of radius δ centred in A, see also Figure 2.1.

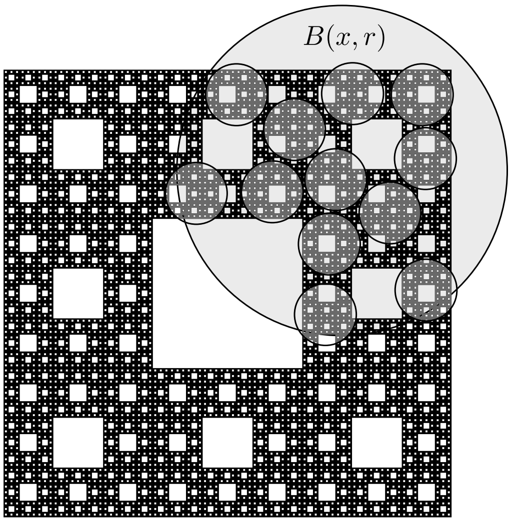

FIGURE 2.1. Our aim is to find an optimal (δr)-packing of $B(x,r)$ with large relative μ mass for each ball in the packing. The larger the size of the packing is, that is, the larger the local homogeneity is, the more uniformly distributed the measure μ is. In the picture the distribution of μ is given by the *Sierpiński carpet* and the small darker grey balls form an optimal homogeneous packing of $B(x,r)$.

The distribution of a measure μ is the most uniform near $x \in \mathbb{R}^d$ when for all small $\delta, \varepsilon, r > 0$ the local homogeneity

$$\mathrm{hom}_{\delta,\varepsilon,r}(\mu, x) \geq c\delta^{-d}$$

for some constant c independent of δ, ε, r. This happens for example in the case of Lebesgue measure $\mu = \mathcal{L}^d$. If the behavior of $\mathrm{hom}_{\delta,\varepsilon,r}(\mu, x)$ differs from this, that is, $\mathrm{hom}_{\delta,\varepsilon,r}(\mu, x) \leq c\delta^{-s}$ where $s < d$, the measure μ becomes less and less uniformly distributed the smaller the s is. The number s here is closely related to the dimension of the measure μ: Intuitively, if the dimension of μ is larger than s and if $\delta > 0$, then for many x and small $r > 0$ one expects to find at least δ^{-s} disjointed sub-balls of $B(x,r)$ of diameter δr with relatively large mass. This observation was made precise for upper local dimension in [13, Theorem 3.7], which is stated in the Euclidean case below:

Theorem 2.1. *Let $0 < m < s < d$. Then there exists a constant $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$, there exists $\varepsilon > 0$ with the following property: If μ is a measure on $\mathbb{R}^d$, then for μ*

almost every x with $\overline{\dim}_{\mathrm{loc}}(\mu, x) > s$, we have

$$\limsup_{r \searrow 0} \hom_{\delta,\varepsilon,r}(\mu, x) \geq \delta^{-m}.$$

However, Theorem 2.1 does not give us any extra information about the homogeneity if we replace the upper local dimension with lower local dimension. In the article [A] we studied this case, and we were also able to generalize Theorem 2.1 into a more quantitative form:

Theorem 2.2 (Theorem 1.1 of [A]). *Let $0 < m < s < d$. Then there exist constants $p > 0$ and $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$, there exists $\varepsilon > 0$ with the following property: If μ is a measure on $\mathbb{R}^d$, then*

(a) *for μ almost every x with $\underline{\dim}_{\mathrm{loc}}(\mu, x) > s$ and for all large enough $N \in \mathbb{N}$, we have*

$$\hom_{\delta,\varepsilon,r}(\mu, x) \geq \delta^{-m} \tag{2.1}$$

for at least pN dyadic scales $r \in \{2^{-1}, 2^{-2}, \dots, 2^{-N}\}$;

(b) *for μ almost every x with $\overline{\dim}_{\mathrm{loc}}(\mu, x) > s$ and for infinitely many $N \in \mathbb{N}$, the estimate* (2.1) *is satisfied for at least pN dyadic scales $r \in \{2^{-1}, 2^{-2}, \dots, 2^{-N}\}$.*

The main difference to Theorem 2.1 is that here we have a portion $p > 0$ of scales where we have an estimate for the local homogeneity. For the proof of Theorem 2.2 we have to introduce the notion of *dyadic homogeneity* where for a given dyadic cube we estimate the number of certain generation children of the cube with large relative μ mass. Dyadic homogeneity works really well with local entropy averages, and allows us immediately to have dyadic homogeneity estimates from local dimension estimates. Thus the main problem is to link dyadic homogeneity to the spherical homogeneity $\hom_{\delta,\varepsilon,r}(\mu, x)$.

2.2. **Conical densities.** As one can see from the results related to local homogeneity, if a dimension of a measure μ is large, μ should be locally fairly uniformly distributed. This would mean that if we "look" from many points x in the support of μ to some direction, we should see some of the mass of μ in arbitrarily small scales, see Figure 2.2. This observation can be made precise with *conical densities*.

The problem of relating the dimension of sets and measures to their distributions inside narrow cones has a long history beginning from the work of Besicovitch on the distribution of purely 1-unrectifiable sets. The conical density properties of Hausdorff measures have been extensively studied by J. Marstrand in [19], P. Mattila in [22], A. Salli in [34] and others and have been applied for example in unrectifiability [23] and removability problems [25, 18]. Analogous results for packing type measures were first obtained in [38, 15, 14]. Upper conical density theorems of arbitrary measures have been considered in [5, 13].

Let us introduce some notation. Let $m \in \{0, \dots, d-1\}$ and let $G(d, d-m)$ be the set of all $(d-m)$-dimensional linear subspaces of $\mathbb{R}^d$. For $x \in \mathbb{R}^d$, $r > 0$, $V \in G(d, d-m)$ and $0 \leq \alpha \leq 1$ we write

$$X(x, r, V, \alpha) = \{y \in \mathbb{R}^d : \mathrm{dist}(y - x, V) < \alpha|y - x|, y \in B(x, r)\}.$$

In the plane, this is a symmetric cone to the direction V around x with an opening angle $2 \arcsin \alpha$ restricted to the ball $B(x, r)$. When $\theta \in \mathbb{S}^{d-1}$ we also write

$$H(x, \theta, \alpha) = \{y \in \mathbb{R}^d : (y - x) \cdot \theta > \alpha|y - x|\}.$$

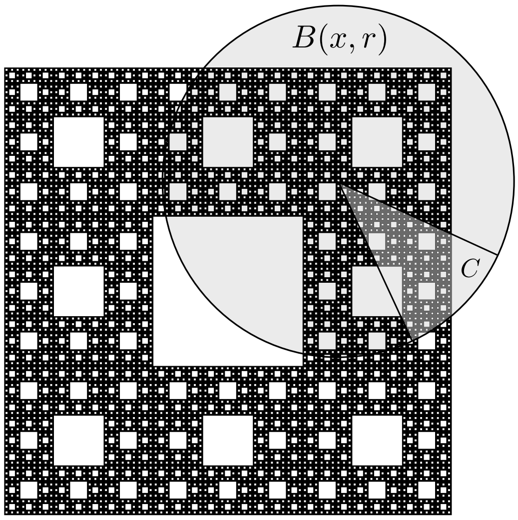

FIGURE 2.2. In the picture we are considering the mass of the Sierpiński carpet measure μ in a non-symmetric cone C formed around a line passing through x. If x is chosen wisely, we should see some mass of μ no matter to what direction we look from x.

If α here is close to 0, this cone is a near the half-space to the direction θ. We are interested in the following narrow cones

$$C(x, r, V, \theta, \alpha) = X(x, r, V, \alpha) \setminus H(x, \theta, \alpha)$$

and the distribution of measures in these cones, see also [Figure 1, A].

When considering the upper conical density theorems for measures, the most general result before the one in [A] is the following result [13, Theorem 5.1] by Käenmäki, Rajala, and Suomala:

Theorem 2.3. *Let $0 < m < s < d$, $m \in \mathbb{N}$ and $0 < \alpha < 1$. Then there exists a constant $c > 0$ such that the following holds: If μ is a measure on $\mathbb{R}^d$, then for μ almost every x with $\overline{\dim}_{\mathrm{loc}}(\mu, x) > s$ we have*

$$\limsup_{r \searrow 0} \inf_{\substack{\theta \in \mathbb{S}^{d-1} \\ V \in G(d, d-m)}} \frac{\mu(C(x, r, V, \theta, \alpha))}{\mu(B(x, r))} > c.$$

In the article [A], we extended this result in a similar way we extended the local homogeneity result in Theorem 2.2:

Theorem 2.4 (Theorem 1.2 of [A]). *Let $0 < m < s < d$, $m \in \mathbb{N}$ and $0 < \alpha < 1$. Then there exist constants $p > 0$ and $c > 0$ such that the following holds: If μ is a measure on $\mathbb{R}^d$, then*

(a) *for μ almost every x with $\underline{\dim}_{\mathrm{loc}}(\mu, x) > s$ and for all large enough $N \in \mathbb{N}$ we have*

$$\inf_{\substack{\theta \in \mathbb{S}^{d-1} \\ V \in G(d, d-m)}} \frac{\mu(C(x, r, V, \theta, \alpha))}{\mu(B(x, r))} > c \tag{2.2}$$

for at least pN dyadic scales $r \in \{2^{-1}, 2^{-2}, \ldots, 2^{-N}\}$.

(b) *for μ almost every x with $\overline{\dim}_{\mathrm{loc}}(\mu, x) > s$ and for infinitely many $N \in \mathbb{N}$, the estimate* (2.2) *is satisfied for at least pN dyadic scales $r \in \{2^{-1}, 2^{-2}, \dots, 2^{-N}\}$.*

For the statement (a), a weaker form of the result was already available. Namely, the results of M. Csörnyei, Käenmäki, Rajala, and Suomala in [5] (see Remark 4.7 in [5]) yield that for *infinitely many* $N \in \mathbb{N}$ the estimate (2.2) holds at least pN dyadic scales $r \leq 2^{-N}$. Our result strengthens this to *all* large enough N. In the proof we once again invoke local entropy averages and solve a discrete version of the conical density theorem, see [A, Lemma 3.2].

2.3. **Porosity.** Porosity is a degree of singularity of sets and measures which describes the size of "holes" in arbitrarily small scales. The ideas surrounding porosity were already present in the work of A. Denjoy in [6] and the notion was rigorously considered by E. P. Dolženko in [7] when he studied certain exceptional sets for complex functions.

The relationship between dimension and porosity is of much later origin. J. Sarvas proved in [35] that when a set has uniformly positive porosity at every small scale, then its Hausdorff dimension is less than the dimension of the space. O. Martio and M. Vuorinen [21] then extended this result to give sharper bounds for the dimension. In [22] Mattila studied the asymptotics of this phenomenon when porosity increases, and using techniques from conical densities he managed to prove that when the porosity of a set in $\mathbb{R}^d$ increases to its maximal value $1/2$, then the Hausdorff dimension of the set decreases to $d-1$. The reasoning for dimension $d-1$ here is that $(d-1)$-dimensional hyperplanes are already maximally porous. In [34] Salli found the correct rate of asymptotics for this phenomenon, and generalized it to packing dimension. Moreover, in [16] P. Koskela and S. Rohde introduced *mean porosity* in which we assume that holes of the set appear in average at some portion of small scales rather than all small scales. For dimension theory, relaxation to mean porosity was not too radical since previous results could be generalized to mean porous sets. For example, the work by Salli was generalized by D. Beliaev and S. Smirnov in [3].

Porosity was historically just set theoretical until the work by J.-P. Eckmann and Järvenpääs in [8] where they introduced an analogous notion for measures. For measures "holes" are determined by the areas where the measure has relatively small amount of mass compared to its surrounding area, see Figure 2.3. Most dimension theorems for porous sets could be deduced also in the setting of porous measures, see the papers [8, 3, 14, 2, 13, 37].

In the article [A] we are able to obtain a new generality to some previous results in this field. Let us first introduce some precise notion we need.

Definition 2.2. Fix $\ell = 1, \dots, d$ and a measure μ on $\mathbb{R}^d$. The *ℓ-porosity* of μ at $x \in \mathbb{R}^d$ with parameters $r, \varepsilon > 0$ is the number

$$\begin{aligned}\mathrm{por}_\ell(\mu, x, r, \varepsilon) = \sup\{\varrho > 0 : \text{there are } y_1, \dots, y_\ell \in \mathbb{R}^d, (y_i - x) \cdot (y_j - x) = 0, \text{ with } \\ B(y_i, \varrho r) \subset B(x, r) \text{ and } \mu(B(y_i, \varrho r)) \leq \varepsilon \mu(B(x, r))\}.\end{aligned}$$

Let $0 < \alpha < 1/2$, $0 < p \leq 1$, and $x \in \mathbb{R}^d$. The measure μ is

(a) *lower mean (ℓ, α, p)-porous at x* if, for any $\varepsilon > 0$ and for all large enough $N \in \mathbb{N}$, we have

$$\mathrm{por}_\ell(\mu, x, r, \varepsilon) \geq \alpha \tag{2.3}$$

for at least pN dyadic scales $r \in \{2^{-1}, 2^{-2}, \ldots, 2^{-N}\}$;

(b) *upper mean (ℓ, α, p)-porous at x* if, for any $\varepsilon > 0$ and for infinitely many $N \in \mathbb{N}$, the estimate (2.3) is satisfied for at least pN dyadic scales $r \in \{2^{-1}, 2^{-2}, \ldots, 2^{-N}\}$.

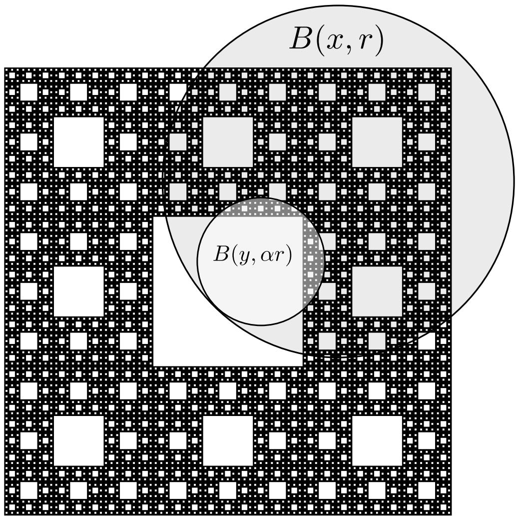

FIGURE 2.3. In the picture we have $\mathrm{por}_1(\mu, x, r, \varepsilon) > \alpha$ for the Sierpiński carpet measure μ. Thus there exists a "hole" $B(y, \alpha r)$ for μ in $B(x, r)$ with relative size α. Notice that the hole can have some μ mass in it, but we have control over it. The threshold ε is an upper bound for the relative μ mass of $B(y, \alpha r)$ in the ball $B(x, r)$.

The concept ℓ-porosity was introduced by Käenmäki and Suomala in [15] for sets and by Käenmäki, Rajala and Suomala for measures in [13]. This generalizes the usual notion of porosity in the sense that now one requires ℓ holes in ℓ orthogonal directions near a point rather than just one hole. When assuming such a condition, dimension should drop more drastically than in the case of usual porosity. This observation was made precise for measures by Käenmäki, Rajala and Suomala in [13, Theorem 5.2], which we also state below.

Theorem 2.5. *Let $\ell = 1, 2, \ldots, d$, $0 < \alpha < \frac{1}{2}$ and μ be a measure on $\mathbb{R}^d$. Then at μ almost every x where*

$$\lim_{\varepsilon \searrow 0} \liminf_{r \searrow 0} \mathrm{por}_\ell(\mu, x, r, \varepsilon) \geq \alpha$$

we have

$$\overline{\dim}_{\mathrm{loc}}(\mu, x) \leq d - \ell + \frac{c}{\log \frac{1}{1-2\alpha}},$$

where $c > 0$ depends only on the dimension d.

Motivated by Theorem 2.5 one can ask whether an analogous result holds for measures that satisfy a mean ℓ-porosity condition rather than the strong porosity condition of Theorem 2.5. The case $\ell = 1$ was already established in [2, Theorem 3.1]:

Theorem 2.6. *Let $0 < \alpha < \frac{1}{2}$, $0 < p \leq 1$, and μ a measure on $\mathbb{R}^d$. Then at μ almost every x where μ is lower mean $(1, \alpha, p)$-porous, we have*

$$\overline{\dim}_{\text{loc}}(\mu, x) \leq d - p + \frac{c}{\log \frac{1}{1-2\alpha}},$$

where $c > 0$ depends only on the dimension d.

The method used in [2] relies heavily on the co-dimension being one and cannot be used when $\ell > 1$. In the paper [A] we managed to generalize Theorem 2.6 for any $\ell = 1, 2, \ldots, d$. Moreover, we were able to say something about the case when the measure is upper mean porous. Before the work in [A], upper mean porosity was not previously considered.

Theorem 2.7 (Theorem 1.3 of [A])**.** *Let $\ell = 1, 2, \ldots, d$, $0 < \alpha < \frac{1}{2}$, and $0 < p \leq 1$. If μ is a measure on $\mathbb{R}^d$, then*

(a) *for μ almost all x where μ is lower mean (ℓ, α, p)-porous, we have*

$$\overline{\dim}_{\text{loc}}(\mu, x) \leq d - p\ell + \frac{c}{\log \frac{1}{1-2\alpha}}, \tag{2.4}$$

(b) *for μ almost all x where μ is upper mean (ℓ, α, p)-porous, we have*

$$\underline{\dim}_{\text{loc}}(\mu, x) \leq d - p\ell + \frac{c}{\log \frac{1}{1-2\alpha}}. \tag{2.5}$$

Here $c > 0$ is a constant that depends only on the dimension d.

The estimates in Theorem 2.7 are asymptotically sharp when $\alpha \to 1/2$. However, in the paper [A] we do not provide any examples verifying this, but modifications of the example for general ℓ in [2, Example 3.9] works in our setting as well. The proof of Theorem 2.7 is once more based on local entropy averages and a discrete version of the problem, which is given in [A, Lemma 3.6].

3. Tangent measures

Tangent measures could considered to be analogous concepts to measures as what derivatives are to a function. Both are defined as blowing-up the object in study at some point in small scales, and taking a suitable limit of the blow-ups when we increase resolution. Information about the fine structure of the measure can then be deduced from tangent measures. Ideas surrounding tangent measures were already present in the work [20] by Marstrand and they were first defined by Preiss in [33] as an effective tool to solve an old conjecture on the characterization of rectifiability with respect to densities.

Tangent measures are still playing a central role in the study of rectifiability. For example, symmetricity of tangent measures is useful in the characterization of rectifiability with respect to principal values for singular integral, see [26]. Furthermore, they are also present in the work related to rectifiability in the Heisenberg group, see for example [24, 27]. Nowadays, tangent measures are also well represented in the study of scenery flows and tangent measure distributions, which in turn have turned out to be essential in the study of dimensions of fractals, see for example the paper [10] by Hochman and Shmerkin.

Let us now introduce the basic notation related to tangent measures. Recall that a *measure* is always a Radon-measure on $\mathbb{R}^d$. Let $\mathcal{M}$ be the space of all measures on $\mathbb{R}^d$. Given $\mu \in \mathcal{M}$, $x \in \mathbb{R}^d$, and a scale $r > 0$, the *blow-up* of μ at $x \in \mathbb{R}^d$ is the image measure $T_{x,r\sharp}\mu$, defined by

$$T_{x,r\sharp}\mu(A) = \mu(rA + x), \quad \text{for every Borel set } A \subset \mathbb{R}^d.$$

Definition 3.1. Let $\mu \in \mathcal{M}$ and $x \in \mathbb{R}^d$. A measure $\nu \in \mathcal{M} \setminus \{0\}$ is a *tangent measure* of μ at x if there exists a sequence of radii $(r_i)_{i\in\mathbb{N}}$, $r_i > 0$, $r_i \searrow 0$, and normalization constants $(c_i)_{i\in\mathbb{N}}$, $c_i > 0$, such that the blow-ups

$$c_i T_{x,r_i\sharp}\mu \rightharpoonup \nu, \quad \text{as } i \to \infty,$$

where $\rightharpoonup$ denotes the weak convergence of measures in $\mathcal{M}$. The set of tangent measures of μ at x is denoted by $\mathrm{Tan}(\mu, x)$.

3.1. **Non-doubling measures.** The doubling condition for measures rises from the research related to analysis on homogeneous metric spaces. Homogeneous metric spaces are finite dimensional in a certain sense, and this can be characterized by the existence of *doubling measures*, that is, measures satisfying a certain uniform growth condition for the measures of balls in the space. We are interested in a local version of the doubling condition and how this behavior could be observed from tangent measures.

Definition 3.2. Let $\mu \in \mathcal{M}$ and $x \in \operatorname{spt}\mu$, where $\operatorname{spt}\mu$ is the support of μ. The *doubling constant* of μ at x is defined by

$$D(\mu, x) = \limsup_{r\searrow 0} \frac{\mu(B(x, 2r))}{\mu(B(x, r))}.$$

We say that μ satisfies the *doubling condition* at x if $D(\mu, x) < \infty$. A measure μ in $\mathbb{R}^d$ is *non-doubling* if

$$D(\mu, x) = \infty$$

at μ almost every $x \in \mathbb{R}^d$.

Tangent measures are not blind to the doubling condition of a measure. A compactness property of tangent measures immediately gives that $\mathrm{Tan}(\mu, x)$ is non-empty at those x where μ satisfies the doubling condition, see [23, Theorem 14.3]. Moreover, Preiss proved in [33, Proposition 2.2 & Corollary 2.7] the following characterization:

Theorem 3.1. *Let $\mu \in \mathcal{M}$ and $x \in \mathbb{R}^d$ with $\mathrm{Tan}(\mu, x) \neq \varnothing$. Then*

$$D(\mu, x) = \infty \quad \Longleftrightarrow \quad \sup_{\nu \in \mathrm{Tan}(\mu,x)} \frac{\nu(B(0,R))}{\nu(B(0,1))} = \infty \quad \textit{for every } R > 1.$$

Can anything more be said about the behavior of the tangent measures of non-doubling measures? For example, is non-doubling always preserved to them, or do they even have some form of singularity? Non-doubling of a measure is a particularly strong form of singularity, but singularity itself is too weak to pose any impact on the tangent measures. This can be seen from the following example by Preiss [33, Example 5.9]:

Theorem 3.2. *There exists a singular measure μ on $\mathbb{R}$ such that every tangent measure ν of μ is a constant multiple of $\mathcal{L}^1$.*

However, as Theorem 3.1 indicates, non-doubling measures cannot have the same property as the measure constructed in Theorem 3.2. Nonetheless, in the article [B] we exhibited an analogous result for non-doubling measures:

Theorem 3.3 (Theorem 1.1 of [B]). *There exists a non-doubling measure μ on $\mathbb{R}$ such that every tangent measure ν of μ is equivalent to Lebesgue measure.*

Here "ν is equivalent to Lebesgue measure" means that ν and Lebesgue measure are mutually absolutely continuous to each other. The construction of the measure μ in Theorem 3.3 is based on finding a continuous map $\varphi : [-1, 1] \to [0, \infty)$ which vanishes very rapidly when approaching to -1 or 1. Then μ is constructed using φ as a rule to distribute mass over rapidly nesting grids of dyadic cubes.

Theorem 3.3 answers to the questions we posed in the beginning of this section: No form of singularity is in general preserved to the tangent measures of non-doubling measures. Moreover, Theorem 3.3 has some implications to the theory of porosity. In [28, Lemma 5] M. E. Mera and M. Morán characterized positive *upper porosity*

$$\overline{\mathrm{por}}(\mu, x) = \lim_{\varepsilon \searrow 0} \limsup_{r \searrow 0} \mathrm{por}_1(\mu, x, r, \varepsilon),$$

recall Definition 2.2, with respect to tangent measures in the following way:

Theorem 3.4. *Let $\mu \in \mathcal{M}$. Then*

$$\overline{\mathrm{por}}(\mu, x) > 0 \qquad \Longleftrightarrow \qquad \textit{there exists } \nu \in \mathrm{Tan}(\mu, x) \textit{ such that } \operatorname{spt} \nu \neq \mathbb{R}^d$$

at μ almost every $x \in \mathbb{R}^d$ where μ satisfies the doubling condition.

V. Suomala asked in a personal communication from 2009 that could one generalize Theorem 3.4 for non-doubling points. This question is reasonable since Preiss proved in [33, Theorem 2.5] that tangent measures do exist at μ almost every point for any measure μ. However, the example given in Theorem 3.3 provides a counterexample. First of all, it was proven in [29, Proposition 3.3] that the condition $D(\mu, x) = \infty$ occurs precisely when $\overline{\mathrm{por}}(\mu, x) = 1$. Moreover, if a measure ν is equivalent to Lebesgue measure, we immediately have $\operatorname{spt} \nu = \mathbb{R}^d$. Hence we have

Corollary 3.1 (Corollary 1.1 of [B]). *There exists a measure μ on $\mathbb{R}$ such that $\overline{\mathrm{por}}(\mu, x) = 1$ at μ almost every $x \in \mathbb{R}^d$, yet every tangent measure ν of μ has $\operatorname{spt}\nu = \mathbb{R}^d$.*

3.2. **Typical measures.** During recent years, understanding the properties of objects that are generic in the sense of Baire category have gained a lot of attention. Let X be a complete metric space. A set $T \subset X$ is *thin* if for any $x \in T$ there exists a sequence of balls $B_i \subset X \setminus T$, $i \in \mathbb{N}$, such that $B_i \searrow x$. A countable union of thin sets is called *meagre*. A property P of points $x \in X$ is *typical* if the set

$$\{x \in X : x \text{ does not satisfy P}\}$$

is meagre. The reason why completeness of X is assumed is that then we have the Baire category theorem at our disposal: In particular, the Baire category theorem yields that X itself cannot be meagre in X. In our setting we are interested in typical properties of measures in $\mathcal{M}$. The set $\mathcal{M}$ can be equipped with a metric d that makes it a complete metric space.

In [32] T. O'Neil constructed a Radon measure μ in $\mathbb{R}^d$ with a curious property: for μ almost every $x \in \mathbb{R}^d$ the set of tangent measures $\operatorname{Tan}(\mu, x) = \mathcal{M} \setminus \{0\}$. Moreover, in his PhD thesis [31] O'Neil managed to extend this result by showing that such a property of measures is typical:

Theorem 3.5. *Typical $\mu \in \mathcal{M}$ satisfies $\operatorname{Tan}(\mu, x) = \mathcal{M} \setminus \{0\}$ at μ almost every $x \in \mathbb{R}^d$.*

In [C, Theorem 1.1] we provide a different self-contained proof for Theorem 3.5. The main difference between the proofs is that O'Neil's original proof relied on a special property of the measure μ constructed in [32], but we do not require O'Neil's measure in our approach.

Recalling non-doubling measures from the previous section, we also noticed in [C, Corollary 4.1] that Theorem 3.5 has the following immediate consequence if we invoke Theorem 3.1:

Corollary 3.2. *Typical $\mu \in \mathcal{M}$ is non-doubling.*

Furthermore, D. Bate and G. Speight proved in [1] that when a measure μ on a metric space admits a differentiable structure in the sense of Cheeger, then μ satisfies the doubling condition μ almost everywhere. Hence Corollary 3.2 also says that with respect to Euclidean metric a typical $\mu \in \mathcal{M}$ does not admit a differentiable structure. However, Corollary 3.2 is proven in $\mathbb{R}^d$, so motivated by this it would be interesting to see if Corollary 3.2 could be generalized for measures in more general metric spaces.

We also show that Theorem 3.5 is sharp: it cannot be extended to hold at *every* point $x \in \mathbb{R}^d$. More precisely, we obtain the following

Proposition 3.1 (Proposition 5.1 of [C]). *For any $\mu \in \mathcal{M} \setminus \{0\}$ there exists $x \in \operatorname{spt}\mu$ with*

$$\text{either } \mathcal{L}^d \notin \operatorname{Tan}(\mu, x) \qquad \text{or} \qquad \mathcal{L}^d \llcorner [0,\infty)^d \notin \operatorname{Tan}(\mu, x).$$

In the paper [C] we also studied micromeasures – an analogous concept to tangent measures on trees, and their typical properties. Let I be any finite set, and $I^{\mathbb{N}}$ be the corresponding infinite tree. Let $\mathcal{P}$ be the set of all Borel probability measures on $I^{\mathbb{N}}$. If $\mu \in \mathcal{P}$ and y is a finite word with $\mu[y] > 0$, we define the *normalized blow-up* of μ with respect to y as follows:

$$\mu_y[z] = \frac{\mu[yz]}{\mu[y]}, \quad \text{for any finite word } z.$$

In other words, μ_y is the normalized restriction of μ to $[y]$ shifted back to $I^{\mathbb{N}}$. Here $[y] \subset I^{\mathbb{N}}$ is the cylinder generated by y. This notion defines a Borel probability measure on $I^{\mathbb{N}}$.

Definition 3.3. A measure $\nu \in \mathcal{P}$ is a *micromeasure* of $\mu \in \mathcal{P}$ at $x \in I^{\mathbb{N}}$ if there exist generations $(n_i)_{i\in\mathbb{N}}$, $n_i \nearrow \infty$, such that the normalized blow-ups

$$\mu_{x|n_i} \rightharpoonup \nu, \quad \text{as } i \to \infty,$$

where $\rightharpoonup$ also denotes the weak convergence of measures in $\mathcal{P}$. The set of micromeasures of μ at x is denoted by $\mathrm{micro}(\mu, x)$.

The weak topology of $\mathcal{P}$ is metrizable and compact, so asking for typical properties of measures in $\mathcal{P}$ makes sense. In [C] we obtain the following analogous result to Theorem 3.5:

Theorem 3.6 (Theorem 6.1 of [C]). *Typical $\mu \in \mathcal{P}$ satisfies $\mathrm{micro}(\mu, x) = \mathcal{P}$ at every $x \in I^{\mathbb{N}}$.*

The proof of Theorem 3.6 is much simpler than the proofs of Theorem 3.5 even though the ideas involved are similar. The main reason for this is that we are now working in a purely symbolic environment. This is also the reason why Theorem 3.6 has "at every x" rather than "at μ almost every x" as it was in Theorem 3.5.

Further problems. (1) Theorems 3.5 and 3.6 do not rule out that on average over small scales, some measures might appear more often than others as local blow-ups when we zoom in to a point. *Micro-* and *tangent measure distributions* can be used here to describe the statistical behavior of the blow-ups of a measure and the distribution of micro- and tangent measures. These notions have risen naturally when studying dimensions of measures, as it was in the study of projections of measures in [10]. It would be interesting to see what kind of distribution of micro- and tangent measures typical measures have.

(2) J. Christensen introduced in [4], and B. Hunt, T. Sauer and J. York later developed in [11] the notion *prevalence*, a type of genericity that has been considered to be a generalization of "Lebesgue almost everywhere" in topological vector spaces. Prevalence can also be studied in $\mathcal{M}$ as it was already done by L. Olsen in [30], where the L^q-dimensions of prevalent measures were studied. Dimensions of prevalent measures have been observed to be drastically different than typical properties, so it would be interesting to see if the sets of micro- and tangent measures for prevalent measures are different than the ones obtained in Theorems 3.5 and 3.6.

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