In this paper we define and study the Julia set and the Fatou set of an arbitrary polynomial \( f \), which is defined on the closed complex plane and whose degree is at least two. We are especially interested in the structure of these sets and in approximating the size of the Julia set.

First, we define the Julia and Fatou sets by using the concepts of normal families and equicontinuity. Then we move on to proving many of the essential facts concerning these sets, laying foundations for the main theorems of this paper presented in the fifth chapter. By the end of this chapter we achieve quite a good understanding of the basic structure of the Julia set and the Fatou set of an arbitrary polynomial \( f \).

In the fourth chapter we introduce the Hausdorff measure and dimension along with some theorems regarding them. In this chapter we also say more about fractals and self-similar sets, for example the Cantor set and the Koch curve. The main goal of this chapter is to prove a well-known result which allows to easily determine the Hausdorff dimension of any self-similar set that fulfills certain conditions. We end this chapter by calculating the Hausdorff dimension of the one-third Cantor set and the Koch-curve by using the result described earlier and notice, that their Hausdorff dimension is not integer-valued.

In the fifth chapter we study the structure of the Julia set further, concentrating on its connectedness, and introduce the Mandelbrot set. In this chapter we also prove the three main theorems of this paper. First we show a sufficient condition for the Julia set of a polynomial to be totally disconnected. This result, with some theorems proven in the third chapter, shows that in this case the Julia set is a Cantor-like set. The second result shows when the Julia set of a quadratic polynomial of the form \( f(z) = z^2 + c \) is a Jordan curve. The third and final result shows that given an arbitrary polynomial \( f \), there exists a lower bound for the Hausdorff dimension of the Julia set of the polynomial \( f \), which depends on the polynomial \( f \). This is the most important result of this paper.
On the Julia sets of polynomials and their Hausdorff dimension
Contents

1 Introduction 3
2 Basic tools from complex analysis 4
3 Basic properties concerning iteration of polynomials and the Julia set 9
4 Basic properties of self-similarity and geometric measure theory 24
5 The dynamics of the Julia set 37
6 Appendix 58
References 64
1 Introduction

In this paper we will mainly concentrate on the Julia sets of polynomials and try to estimate their Hausdorff dimension. We will start by introducing some basic tools from complex analysis that are needed through this paper in the second chapter. In the third chapter the theory of polynomial iteration on the complex sphere will be presented along with some basic definitions that will be used in this paper. We will then proceed to study how the points of the complex sphere behave under iterations of a polynomial and use this to define the Julia set and the Fatou set of a polynomial. We will then proceed to take a closer look at these sets and prove some of their properties.

In the fourth chapter we will give the definition of the Hausdorff measure and the Hausdorff dimension. We will also prove some of their properties, and in the end of the chapter give some ways to calculate the Hausdorff dimensions of some specific families of sets. In this chapter we will also discuss fractals and introduce an important concept of self-similarity concentrating mainly on its relation to the Hausdorff measure and dimension.

In the fifth chapter we will present the main results of this paper. We will start by looking at when the Julia set is totally disconnected and when it is a Jordan curve by studying what the behavior of the critical points tell us about the structure of the Julia set. In this chapter we will also introduce the Mandelbrot set, which will be related to a special subset of the Julia sets. We will then proceed to give an estimate for the Hausdorff dimension of the Julia set.

The study of the Julia set and the Fatou set started in early 20th century and was one of the starting points to the field of complex dynamics. The Julia set and the Fatou set are named after French mathematicians Gaston Julia and Pierre Fatou who were some of the first to study this field. Julia published a paper regarding the Julia set in 1918, and it is supposed that he was drawn to these studies by a paper from 1879 written by Arthur Cayley concerning the basins of attraction, which we will define later, of polynomial $f(z) = z^3 + c$. Julia was awarded largely due to this paper the Grand Prix de l’Académie des sciences. Fatou had published his work concerning complex iterations a year earlier in 1917. Also a great contribution to this field of study was due to Benoit Mandelbrot who continued Julia’s work. The Mandelbrot set, which is in a close relation to the Julia set, was named after him. Also the study of fractals was getting interest at the 20th century. Although there were many fractals studied before that, for example the Cantor set or the Weierstrass function, they were not studied as a group with some usual properties, but rather as individual examples of strange and pathological sets. The fact that these sets have some common properties and they should be studied as a group came on 20th century.

Nowadays complex dynamics is under active research, but there are still many important open questions left, for example, the structure of the Mandelbrot set is not yet completely understood.

The Hausdorff measure and the Hausdorff dimension were defined by Felix Hausdorff in 1918. They are the most used measure and definition for dimension when dealing with fractals as normal topological dimension is not sufficient. In some sense it could be said that the Hausdorff dimension is more refined than the usual topological dimension and can make a difference between sets that have the same topological dimension.
2 Basic tools from complex analysis

We will first briefly introduce some complex analytic results. We shall present these results in a
generality that is sufficient for our needs, and we will omit the proofs of these results. We recom-
end an interested reader to consult the literature of complex analysis, for example [AH] or [R]
for proofs of basic results in the field. For more involved results we will give a reference for the
proof.

**Theorem 2.1. Analytic bijection and local invertibility**

Let $h$ be analytic in a domain $D \subset \mathbb{C}$ and assume that $h'(z_0) \neq 0$ for some $z_0 \in D$. Then there
exists a neighborhood $V$ of $z_0$ such that $h$ is analytic bijection in $V$, and thus $h$ is conformal in $V$.
Additionally $h$ has a local inverse at the point $h(z_0)$. This means that there exists some neighbor-
hood $U$ of $h(z_0)$ such that we can define an analytic branch of $h^{-1}$ in it, such that it maps $U$ to
some neighborhood of $z_0$.

**Definition 2.1. Spherical metric**

Let us identify the complex plane with the plane $\mathbb{R}^2$ and embed it to the $\mathbb{R}^3$ in canonical way.
Now let there be a unit diameter ball, denoted by $S^2$, such that one point lies on the origin of the
complex plane and another point lies at the point $(0, 0, 1)$. We identify every point of the complex
plane with a point in $S^2$ using the stereographic projection, meaning that for every point $z \in \mathbb{C}$
we draw a straight line $L$ from the point $(0, 0, 1)$ to $z$ and identify $z$ with a point on $S^2$, defined as
$L \cap S^2 \setminus (0, 0, 1)$, and set that $\infty = (0, 0, 1)$. It is clear that by this definition we get a bijection from
$\mathbb{C} \cup \{\infty\}$ to $S^2$. We will refer to this as the closed complex plane and denote it by $\bar{\mathbb{C}}$. We define the
spherical distance of two points on the complex plane as the distance between the image points of
the stereographic projection on the $S^2$. Rigorously defined as follows, let $x, z \in \mathbb{C}$, then

$$d(x, z) = \frac{|z - x|}{\sqrt{(1 + |z|^2)(1 + |x|^2)}}$$

and if $x = \infty$, then $d(\infty, z) = \frac{1}{\sqrt{1 + |z|^2}}$. When we speak about equicontinuity in the definition of
the Fatou set and the Julia set we mean it with respect to the spherical metric.

**Definition 2.2. Normal family**

Let $\{h_n\}$ be a family of continuous functions which are mappings from a domain $D \subset \bar{\mathbb{C}}$ to the
closed complex plane. We say $\{h_n\}$ is normal if every sequence of members of $\{h_n\}$ contains a
subsequence $h_n$ that converges uniformly on compact subsets of the domain $D$ to some continuous
function from the domain $D$ to the closed complex plane.
Since we are interested in families of polynomial functions, which are analytic functions we addi-
tionally know that the locally uniform convergence on any domain $D' \subset \mathbb{C}$ happens in fact either
to an analytic function on $D'$ or towards the point infinity.

**Definition 2.3. Equicontinuity**
Family of continuous functions $h_n$ is equicontinuous at a point $z \in \mathbb{C}$, if for every $\varepsilon > 0$ there exists $\delta$ such that
$$|h_n(x) - h_n(z)| < \varepsilon$$
for every $n$ and for every $x \in B(z, \delta)$. The family $\{h_n\}$ is equicontinuous in some domain if it is equicontinuous in every point of that domain.

**Theorem 2.2.** Arzela–Ascoli theorem

The Arzela–Ascoli theorem states that a family of analytic functions is normal on a domain $D \subset \mathbb{C}$ if and only if it is equicontinuous on $D$ with respect to the spherical metric. For a proof see for example [AH] starting on the page 222.

**Theorem 2.3.**

Let $\{h_n\}$ be a family of analytic functions defined on a domain $D \subset \mathbb{C}$. Then if there are two distinct points $z, x \in \mathbb{C}$ such that all $h_n$ omit these values in $D$, then the family $\{h_n\}$ is normal on $D$. This result is referred to as the fundamental normality test and the proof can be found for example in [S] starting on the page 54.

By writing the above theorem in a slightly different way we obtain the following theorem which is a very useful tool for studying complex dynamics.

**Theorem 2.4.** Montel’s theorem

We will give two formulations for the Montel’s theorem. We shall proof the second one using the Arzela-Ascoli theorem 2.2.

First formulation: Assume that a family $\{h_n\}$ of analytic functions is not normal on a domain $D$ of complex plane. Then there exists a point $z \in D$ and for every neighborhood of that point $z$, denoted by $B_z$, the set $\bigcup_{n=1}^{\infty} h_n(B_z)$ is the whole complex plane except maybe for one single point.

Second formulation: Let $\{h_n\}$ be a family of analytic functions. If the family $\{h_n\}$ is locally bounded on a domain $D \subset \mathbb{C}$, meaning that for every $z \in D$ there is a real number $M$ such that $|h_n(x)| < M$ for every $n$ and for every $x$ in some neighborhood of $z$, then the family $\{h_n\}$ is normal on the domain $D$.

Proof: Due to the Arzela-Ascoli theorem 2.2, it is enough to show that a locally bounded family $\{h_n\}$ is equicontinuous at every point of the domain $D$ with respect to the spherical metric. But since for every point $z \in \mathbb{C}$ we can choose some neighborhood where the normal Euclidean metric is equivalent to the spherical metric it is enough to show equicontinuity with respect to the Euclidean metric. Choose an arbitrary $z \in D$ and an arbitrary $\varepsilon > 0$. Now as $\{h_n\}$ is locally bounded we have a neighborhood $B(z, 2r)$ of $z$ such that $B(z, 2r) \subset D$ and a bound $M > 0$ such that $|h_n(x)| \leq M$ for all $x \in B(z, 2r)$ and for all natural numbers $n$.

Next we use the Cauchy integral formula to obtain for all $h_n$ and all $x, y \in B(z, 2r)$, that
$$h_n(x) - h_n(y) = \frac{1}{2\pi i} \int_{\partial B(z, 2r)} \left( \frac{h_n(\theta)}{\theta - x} - \frac{h_n(\theta)}{\theta - y} \right) d\theta = \frac{1}{2\pi i} \int_{\partial B(z, 2r)} \left( \frac{(x - y) \cdot h_n(\theta)}{(\theta - x)(\theta - y)} \right) d\theta$$
Now if we restrict to \( x, y \in B(z, r) \), we have estimate \(|(\theta - y) \cdot (\theta - x)| > r^2\). From this we get, used in to the above integral, that
\[
|h_n(x) - h_n(y)| \leq \frac{2}{r} |x - y| \cdot \sup_{z \in \partial B(z, 2r)} |h_n(z)| \leq \frac{|x - y| \cdot 2M}{r}
\]
where \( M \) was the upper bound for functions \( h_n \) in a neighborhood \( B(z, 2r) \). Then we choose
\[
\delta = \min\{r, \varepsilon \cdot r / 4M\},
\]
and see that for every \( \varepsilon \) we can choose \( \delta \) as above, and equation
\[
|h_n(x) - h_n(z)| < \varepsilon
\]
will hold for all integers \( n \). Hence the family \( \{h_n\} \) is equicontinuous in a neighborhood \( B(z, \delta) \) and thus it is normal in the point \( z \) due to the Arzela-Ascoli theorem 2.2. As \( z \) was an arbitrary point in \( D \), we have that the family \( \{h_n\} \) is equicontinuous in the whole \( D \). Thus the proof is completed.

Next we will give another formulation for Montel’s theorem 2.4 using the spherical metric.

**Theorem 2.5.**
Let \( \{h_n\} \) be a family of analytic maps on a domain \( D \subset \mathbb{C} \). Assume that there exists a constant \( m \) and for each \( h_n \) three points \( a_n, b_n, c_n \) such that \( h_n \) does not obtain these values on \( D \) and that the minimal distance between these points is greater than \( m \) with respect to the spherical metric. Then the family \( \{h_n\} \) is normal on \( D \). For the proof see [B] theorem 3.3.5.

**Theorem 2.6.** Vitali’s theorem
Suppose that the family \( \{h_n\} \) of analytic functions is normal on a domain \( D \subset \mathbb{C} \) and assume that \( h_n \) converges pointwise to some function \( h \) on some non-empty open set \( U \subset D \). Then \( h \) can be extended to an analytic mapping from the whole \( D \) and \( h_n \to h \) locally uniformly on \( D \). For the proof see [B] theorem 3.3.3.

**Lemma 2.1.** Lipschitz condition
Let \( f \) be a polynomial. Then it satisfies the Lipschitz condition \( d(f(z), f(x)) \leq M \cdot d(z, x) \) with respect to the spherical metric, for some constant \( M > 0 \), and arbitrary \( z, x \) in the \( \mathbb{C} \). For the proof see [B] theorem 2.3.1.

**Definition 2.4.** Möbius mapping
A Möbius mapping is a rational mapping from the closed complex plane to itself of form
\[
f(z) = \frac{az + b}{cz + d}
\]
where \( a, b, c, d \) satisfy \( ad - bc \neq 0 \).
Definition 2.5. Valency

Let \( g \) be a function that is analytic at a point \( z \in \mathbb{C} \) and assume that it is non-constant at the neighborhood of \( z \). Then \( g \) can be written as a Taylor expansion in a neighborhood of the point \( z \). So we obtain that \( g(x) = a_0 + a_1(x-z) + a_2(x-z)^2 + \cdots \), where \( a_k \neq 0 \) and \( x \in B(z,r) \) with some positive radius \( r \). Its valency, denoted \( v_g(z) \) at the point \( z \) is the integer \( k \), which can also be defined as the unique integer such that the limit

\[
\lim_{x \to z} \frac{g(x) - g(z)}{(z-x)^k}
\]

exists, is finite and non-zero. We notice that for points \( z \) at which the derivative is not zero the valency is one. For critical points the valency is strictly greater than one.

Valency can also be defined as the number of solutions of the equation \( g(x) = g(z) \) at the point \( z \). For a polynomial \( f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 \) the valency can be defined at the point \( \infty \) using the Möbius mapping \( \frac{1}{z} \) as follows, \( v_f(\infty) = v_h(0) \) where \( h(z) = \frac{1}{f(1/z)} \). Calculating this for an arbitrary polynomial \( f \) with \( \deg f > 1 \) we obtain that

\[
\frac{1}{f(1/z)} = a_n z^n + \cdots + a_0 z^n
\]

Using the definition of valency it is clear that \( v_h(0) = n \), so \( v_f(\infty) = \deg f \). The valency satisfies the following equation for polynomials,

\[
\sum_{z \in \mathbb{C}} (v_g(z) - 1) = 2\deg(g) - 2
\]

see [B] page 43. We will next define the deficiency at a point \( z \) using the valency. We will denote deficiency by \( \delta \) and define \( \delta_g(z) = v_g(z) - 1 \).

Theorem 2.7. Argument principle

Let \( g \) be a meromorphic function on the closure of some simply connected domain, denoted as \( D \). This means that it is holomorphic in some neighborhood of the closure of \( D \) except for a set of isolated points, on which the \( g \) has a pole. Additionally we assume that it does not have any zeros or poles on the boundary of \( D \), denoted as \( C \) and that the set \( C \) is smooth enough so that the integral over it can be defined. Then we have that

\[
\int_C \frac{g'(x)}{g(x)} \, dx = 2\pi i \cdot (N - P)
\]

where \( N \) is the number of zeros on the interior of \( D \) and \( P \) is the number of poles, both counted by multiplicity.

Theorem 2.8. The Hurwitz theorem

Let \( D \) be an open subset of the complex plane and let \( \{h_n\} \) be a sequence of analytic functions on \( D \), such that \( \{h_n\} \) converges uniformly on compact subsets of \( D \) to an analytic function \( g \). Let \( B(z_0, r) \) be any disc such that its closure is contained in \( D \) and assume that the function \( g \) has no zeros on the boundary of our arbitrary disc \( B(z_0, r) \). Then there exists some integer \( k \) such that for any \( h_n \), such that \( n > k \), the functions \( h_n \) and \( g \) have the same number of zeros in \( B(z_0, r) \).
**Theorem 2.9.** The Lebesgue covering theorem

Let $X$ be a compact subset of a metric space and family $\{U_i : i \in I\}$ its open cover. Then there exists a positive number $\theta$ such that for all sets $U \subset X$ with a diameter smaller than $\theta$ there exist some $i \in I$ such that $U \subset U_i$.

**Theorem 2.10.** Monodromy theorem

Let $g$ be analytic on the disc $B$ and let there be a simply connected domain $U$ such that $B \subset U$. Then if it holds that for every point $z \in U$ the function $g$ can be analytically continued through a chain of discs from $B$ to some neighborhood of $z$, then there exists a unique analytic continuation of $g$ from $B$ to the whole set $U$. 
3 Basic properties concerning iteration of polynomials and the Julia set

In this section we will define the Julia set and the Fatou set and prove some of their basic properties. Mostly we follow Alan Beardon’s book [B], which considers rational mapping from closed complex plane to itself. As we restrict to the polynomials, there are some slight changes in some proofs. In addition to Beardon’s book this chapter has also been influenced by [AS], [CG], [F] and [R].

Let us first give some main definitions concerning polynomial maps, of form \( f(z) = a_nz^n + a_{n-1}z^{n-1} + \cdots + a_0 \), used in this paper. Let \( f \) be a polynomial from \( \overline{\mathbb{C}} \to \overline{\mathbb{C}} \) such that \( \deg f > 1 \). In this section \( f \) will always denote such polynomial. We will denote the \( n \)th iteration of \( f(z) \) by \( f^n(z) \), meaning \( f^n(z) = f(f^{n-1}(z)) \), and of course define \( f^0(z) = z \) for every \( z \in \overline{\mathbb{C}} \). We call a point \( z \) a fixed point if \( f(z) = z \) and a critical point if \( f'(z) = 0 \). Clearly there are \( \deg f \) finite fixed points, as equation \( f(z) = z \) has always \( \deg f \) solutions and \( \deg f - 1 \) finite critical points, as equation \( f'(z) = 0 \) has \( \deg f - 1 \) solutions, counting multiplicity in both cases. We classify the finite fixed points in the following way

1. If \( |f'(z)| = 0 \) we call the point \( z \) a super-attracting fixed point.
2. If \( |f'(z)| < 1 \) we call the point \( z \) an attracting fixed point.
3. If \( |f'(z)| = 1 \) we call the point \( z \) an indifferent fixed point.
4. If \( |f'(z)| > 1 \) we call the point \( z \) a repelling fixed point.

Additionally we classify the point infinity by evaluating the derivative of the function \( S : z \to \frac{1}{f(1/z)} \) at the origin. Computing this for polynomial \( f \) gives that \( S'(0) = 0 \) and hence the point \( \infty \) is a super attracting fixed point, since \( f(\infty) = \infty \) clearly holds.

We call a point \( z \) a periodic point if \( f^n(z) = z \) for some \( n \in \mathbb{N} \). The smallest natural number \( n \) satisfying this equation is called the period of \( z \). Notice that if \( z \) is a periodic point with the period \( n \), then also \( f^k(z) \) is a periodic point with the period of \( n \) for every positive integer \( k \). For a periodic point \( z \) with the period \( n \) we call the sequence \( z, f(z), \ldots, f^{n-1}(z) \) the cycle of \( z \). Clearly the cycle is the same for any starting point, and sometimes we will use shortened notation for the cycle \( z, z_1, \ldots, z_{n-2}, z_{n-1} \). Periodic point \( z \) of the period \( n \) can be thought as a fixed point of the function \( f^n \). We would like to classify the periodic points as we have classified the finite fixed points above, but we have to make sure that the derivative does not depend on a choice of a point in the cycle.

For proving this, we must first notice that using the chain rule gives us

\[
(f^n)'(z) = \prod_{k=0}^{k=n-1} f'(f^k(z))
\]

When we apply this to the case when \( z \) forms the cycle of the period \( n \) we get

\[
(f^n)'(f^i(z)) = \prod_{k=0}^{k=n-1} f'(f^k(z))
\]

showing that the derivative is independent of a choice of a representative in the cycle. Now we can classify cycles as we classified the fixed points.

Point \( z \) is called a pre-periodic point if it is not periodic under iterations of \( f \) but some iteration of it is. For example 1 is a pre-periodic point for the polynomial \( f(z) = z^2 - 1 \), as \( f(1) = 1^2 - 1 = 0 \), \( f^2(1) = f(f(1)) = f(0) = 0^2 - 1 = -1 \), and similarly \( f^3(1) = 0 \). From this we easily see that points \( 0, -1 \) form a cycle of a period 2 under iteration of \( f \).
**Definition 3.1. Conjugacy**

We say that two polynomials \( f \) and \( g \) are conjugate if there exists a Möbius mapping \( h \) such that \( f = h \circ g \circ h^{-1} \).

Conjugation respects fixed points, meaning that if \( f = h \circ g \circ h^{-1} \), then \( h(z) \) is a fixed point of \( f \), if and only if \( z \) is a fixed point of \( g \). It is also clear that \( f^n = h \circ g^n \circ h^{-1} \) as can easily be seen by induction, since \( f^{n+1} = h \circ g^{n+1} \circ h^{-1} \).

**Definition 3.2. Invariance**

Let \( g : \mathbb{C} \to \mathbb{C} \) be a function and \( D \) some subset of \( \mathbb{C} \). Set \( D \) is forward invariant under \( g \) if \( g(D) = D \), and backwards invariant if \( g^{-1}(D) = D \). Set \( D \) is completely invariant under \( g \) if it is both forward and backward invariant. We notice that if \( D \) is completely invariant set with respect to \( g \), then also \( D \) is completely invariant with respect to \( g^k \) for any integer \( k \).

**Definition 3.3. Attractors, basins of attraction and repellers**

Let \( G \) be a closed subset of \( \mathbb{C} \). We call the set \( G \) an attractor for \( f \), if \( G \) is forward invariant under \( f \) and there is some open set \( V \) such that \( G \subset V \) and that for every \( z \in V \) holds \( \lim_{k \to \infty} d(f^k(z), G) = 0 \). In addition we demand that \( G \) is minimal, meaning that there is no closed subset of \( G \) with the same properties.

The maximal \( V \) is called a basin of attraction of \( G \). We will denote the basin of attraction as \( A(G) \), and it consists of points converging to \( G \). When we speak of the immediate basin of attraction we mean the union of those components of \( A(G) \) that contain some point from the set \( G \).

We classified the point infinity as an attractive fixed point for polynomials, and hence we can next assume that \( G \subset \mathbb{C} \).

Let \( G \subset \mathbb{C} \) be a closed and forward invariant set. The set \( G \) is a repeller if there exist constants \( \varepsilon > 0 \), \( \delta > 0 \) such that all points \( x \notin G \) within the distance of \( \delta \) from the set \( G \) are repelled under some iterate of \( f \) to at least the distance of \( \varepsilon \) from \( G \). Note that after a point \( z \) has been repelled to the distance of \( \varepsilon \) from the set \( G \) it can return to the vicinity of the set \( G \) or even be mapped to the set \( G \) itself.

Here the distance between an arbitrary point \( v \) and the set \( G \) is defined in a usual way, the infimum between all distances of point \( v \) to all points \( w \in G \). In literature there are different definitions for repellers, but this one is used in this paper.

If we look at the function \( f : \mathbb{C} \to \mathbb{C} \), \( f(z) = z^2 \), we have the origin and the point infinity as attractors and for example the boundary of the unit disc centered at the origin as repeller. More generally attracting or super-attracting fixed points of a polynomial \( f \) are always attractors for \( f \) and repelling fixed points are repellers. This follows by studying the iterations of \( f \) in some small neighborhood of these points and remembering that the derivative is a continuous function.

Next we will first give some intuitive idea about the Fatou set and the Julia set and after this define them rigorously. We denote the Fatou set of a function \( f \) by \( F(f) \) and the Julia set by \( J(f) \). Sometimes we will use a shorter notation and denote \( F \) and \( J \) respectively.

The Fatou set is the set under which function \( f \) behaves in a stable manner. This means that for every \( z \in F(f) \) there exists a neighborhood of \( z \) in which all points act nearly in the same manner.
under iterations of \( f \). The Julia set is the complement of The Fatou set, meaning that for every point \( z \) in the Julia set and for any neighborhood of it there can be found points in that neighborhood which will not act in a same manner under iterations of \( f \).

For the simplest possible example of our intuition, let us again look at the polynomial \( f(z) = z^2 \).

Now the Fatou set consists of two components: of \( B(0, 1) \), where for every \( z \) we can find a neighborhood where all points in that neighborhood converge to 0, and of \( \mathbb{C} \setminus B(0, 1) \), where for every \( z \) we find a neighborhood where all points converge to infinity. The Julia set is the complement of the Fatou set. So we get that the Julia set is the boundary of \( B(0, 1) \), and for every point \( z \) in the Julia set and for every neighborhood \( V \) of that point we clearly can find two points such that one converges to 0 under iterations of \( f \) and other converges to infinity under iterations of \( f \).

Usually the Julia set is not a standard geometrical object but a fractal, which will be discussed later in section 4.

Rigorous definitions are given next.

**Definition 3.4.** The Fatou set and the Julia set of a polynomial \( f \)

We define that a point \( z \in \mathbb{C} \) belongs to the Fatou set if \( z \) has a neighborhood where the family of iterate functions \( f^n \) forms a normal family. The Julia set is defined as the complement of the Fatou set. In other words it is the set of points which do not have a neighborhood where the family \( \{f^n\} \) is normal.

We will next make a few important remarks following from the definition. First we notice that due to the Arzela-Ascoli’s theorem 2.2 we could have equivalently defined that \( z \in F(f) \) if the family \( \{f^n\} \) is equicontinuous in some neighborhood of \( z \) with respect to the spherical metric of \( \mathbb{C} \), defined in definition 2.1.

Additionally from the definition it follows that the Fatou set is an open set, and that the Julia set is a closed set. As we have restricted to a case where \( f \) is a polynomial \( J(f) \) is also bounded, and thus \( J(f) \) is a compact set.

Boundedness can be confirmed from the fact that as \( f \) is a polynomial we can find a closed disc \( D \) centered at the origin such that \( f(\mathbb{C} \setminus D) \subset \mathbb{C} \setminus D \), and we can make the radius of the disc arbitrary big so that for all points \( x \in \mathbb{C} \setminus D \) iterations of \( f \) converge to infinity uniformly. Now it follows from the definition of the spherical metric 2.1, and from the uniform convergence that as we choose arbitrary points \( x, y \in \mathbb{C} \setminus D \), we have that for any \( \varepsilon \) there exists an integer \( N \) such that for all \( n \geq N \), it holds that

\[
d(f^n(x), f^n(y)) \leq d(f^n(x), \infty) + d(f^n(y), \infty) < \varepsilon
\]

And thus \( x \) has a neighborhood where the family of iterate functions \( \{f^n\}, n \geq N \) is equicontinuous. We know that all polynomials \( f^n \) are continuous and that we can add a finite family of continuous functions to an equicontinuous family and keep it equicontinuous by choosing minimal \( \delta \) that satisfy equation

\[
|f^n(x) - f^n(z)| < \varepsilon
\]

for all functions \( f^n \), and all points \( z \in B(x, \delta) \). Hence we have that \( x \) has a neighborhood where the family \( \{f^n\} \) is equicontinuous, and thus due to Arzela–Ascoli theorem 2.2 it is normal. Hence \( \mathbb{C} \setminus D \subset F(f) \), since \( x \) was an arbitrary point in \( \mathbb{C} \setminus D \). Therefore the Julia set is bounded.

If we compare our rigorous definition with our intuition we see that what we called behaving in a stable manner is in fact just the condition of equicontinuity. We will now prove some basic properties of the Fatou set and the Julia set.

**Theorem 3.1.**

The Julia set is nonempty.
Proof: Every point \( x \in \mathbb{C} \) will either converge towards infinity or stay bounded under iterations of a polynomial \( f \). It is clear that equation \( f(z) = z \) always has solutions in the complex plane, since \( \deg f > 1 \), and thus \( f \) always has fixed points. And clearly fixed points remain bounded under iterations of \( f \). On the other hand we know that for a polynomial we can always find a point \( z \) such that iteration of \( z \) under \( f \) will converge to infinity.

Thus we have two sets which are non-empty, disjoint and whose union is the whole complex plane. Hence they must have a non-empty boundary. In addition we know that the set of points that stay bounded under iterations of \( f \) must be a subset of some disc \( D \) with center in the origin and some constant radius \( c \), which was noted in the proof of boundedness of the Julia set in remarks on definition 3.4. Let \( y \) be a boundary point. From the definition of the boundary points it follows that in any neighborhood \( U \) of \( y \) there are points that converge to infinity and points that do not. And those points that do not converge to infinity will stay in some predefined disc \( D \). Therefore the family \( \{ f^n \} \) cannot be equicontinuous on \( U \) and thus it is not normal. Hence \( y \in J(f) \) by definition of the Julia set and the Julia set is non-empty.

The Julia set and the Fatou set are completely invariant sets.

Theorem 3.2.

The Julia set and the Fatou set are completely invariant sets.

Proof: It is enough to show that the Fatou set is completely invariant under \( f \) as the Julia set is its complement and \( f \) is mapping from \( \mathbb{C} \) onto itself. Because \( f \) is a polynomial it is surjective and thus it is enough to show that the Fatou set \( F \) is backwards invariant, meaning \( f^{-1}(F) = F \), since by surjectivity it holds that \( f(f^{-1}(F)) = F \). And from this and the backward invariance we get that \( f \) is also forward invariant.

To prove backward invariance we choose an arbitrary point \( z_0 \in f^{-1}(F) \) and \( w_0 = f(z_0) \). Hence \( w_0 \) belongs to \( F \).

We will first prove that \( z_0 \) has such a neighborhood that the family \( \{ f^n \} \) is equicontinuous in it. We know that this holds for \( w_0 \) since it belongs to the Fatou set. This means that functions \( f^{n+1} \) form an equicontinuous family for \( z_0 \), since \( f \) maps some neighborhood of \( z_0 \) to arbitrary neighborhood of \( w_0 \). Because \( f \) is continuous we can add it to the family \( f^{n+1} \) and obtain equicontinuous family \( \{ f^n \} \). Thus we have that functions \( f^n \) form normal family in \( z_0 \). Hence we have shown that if \( z \in f^{-1}(F) \) then also \( z \in F \). So we get that \( f^{-1}(F) \subset F \).

Now we have to prove that \( F \subset f^{-1}(F) \) to finish the proof. For this choose an arbitrary point \( z_0 \) from \( F \) and \( w_0 \) such that \( f(z_0) = w_0 \). Since \( z_0 \) is in \( F \), the family \( \{ f^n \} \) forms an equicontinuous family at a point \( z_0 \). We can take one function out of family and keep it equicontinuous, and we note that \( \{ f^{n+1} \} \) is equicontinuous family at the point \( z_0 \). Thus the family \( \{ f^n \} \) is equicontinuous at the point \( w_0 \). This can be seen as given any \( \varepsilon \) we have \( \delta_1 \) such that equation

\[
| f^{n+1}(z_0) - f^{n+1}(x) | < \varepsilon
\]

holds for all integers \( n \) and all \( x \in B(z_0, \delta_1) \). From this it follows as \( f(z_0) = w_0 \) and \( f \) is continuous that

\[
| f^n(w_0) - f^n(x) | < \varepsilon
\]

for all integers \( n \) and for all \( x \in f(B(z_0, \delta_1)) \). Hence we get that \( \{ f^n \} \) forms a equicontinuous family for \( w_0 \), as we can find some \( \delta_2 \) such that \( B(w_0, \delta_2) \subset f(B(z_0, \delta_1)) \), since \( f \) is analytic and thus it is an open mapping and hence \( f(B(z_0, \delta_1)) \) is open. Thus \( w_0 \in F \) and \( z_0 \in f^{-1}(F) \). This proves that the Fatou set is completely invariant, and hence the whole theorem.

We can easily see that the previous theorem holds for the example that we gave when \( f(z) = z^2 \). We remember that the Julia set was the boundary of the unit disc and indeed it is completely invariant as every point that is mapped on that set must have the modulus one and every point in that set gets mapped to another point with modulus exactly one. In same way we see that the Fatou set is also completely invariant, as it should be.
Theorem 3.3.
The Julia set has an empty interior.

Proof: Assume that \( J \) contains an open non-empty set \( U \). Because \( J \) is completely invariant \( f^k(U) \subset J \) for all integers \( k \). Since the set \( U \) is in the Julia set it is a neighborhood for some point \( z \) in the Julia set, and thus functions \( f^k \) do not form a normal family on \( U \). Due to Montel’s theorem 2.4, we know that

\[
\bigcup_{k=0}^{\infty} f^k(U)
\]

is the whole complex plane or the whole complex plane omitting a single point \( c \). Thus the Julia set would be unbounded, which is a contradiction to the Montel’s theorem. With this reasoning we can also show that there can be only one finite exceptional point. Since if there were more we knew from the beginning of the proof that there must be infinitely many. Hence there would be infinitely many disjoint completely

Definition 3.5. Exceptional points

Point \( z \) is exceptional if the completely invariant set generated by \( z \),

\[
\bigcup_{n=-\infty}^{\infty} f^n \{ z \}
\]

is finite. Here the notation \( f^{-j} \{ z \}, j > 0 \), means the set \( \{ x \in \mathbb{C} : f^j(x) = z \} \).

Theorem 3.4.

Let \( E \) denote the set of exceptional points in \( \hat{\mathbb{C}} \). Then there are at most two points in \( E \). If a point \( x \) is a finite exceptional point, \( \{ x \} \) is an arbitrary element of the Julia set and \( V \) its neighborhood, then \( W = \bigcup f^k(V) \) is the whole \( \mathbb{C} \) if \( x \in V \). And if \( x \notin V \), then \( W = \mathbb{C} \setminus \{ x \} \). Additionally we will show that this point \( x \) belongs to the Fatou set.

Proof: Assume first that the set \( E \) has \( k \) elements. Since \( E \) is finite it holds that \( f \) acts as a permutation on \( E \), since \( f \) can not map a point \( y \notin E \) to \( E \) and \( f : \mathbb{C} \rightarrow \hat{\mathbb{C}} \) is a surjection. And thus for some \( n \) the \( f^n \) fixes every point in \( E \). Suppose that \( f^n \) has the degree \( d \). It follows that for every \( e \in E \) the equation \( f^n(z) = e \) has \( d \) solutions which are all at \( e \) and hence due to the valency, definition 2.5, we have that

\[
k(d-1) = \sum_{e \in E} (v_{f^e}(e) - 1) \leq \sum_{z \in \mathbb{C}} (v_{f^e}(z) - 1) = 2d - 2
\]

from which we get that \( k \leq 2 \), since \( d \geq 2 \). It is easy to see that for polynomials \( \infty \) is always an exceptional point and as it belongs to the Fatou set we have that there can be only one finite exceptional point. Thus the first part is proven under an assumption that there exist only finitely many exceptional points. We also notice that \( f(\infty) = \infty \) and \( \{ f^{-1}(\infty) \} = \{ \infty \} \), and hence \( f(x) = x \) and \( \{ f^{-1}(x) \} = \{ x \} \).

Let \( z \) be an arbitrary element of the Julia set and \( V \) its neighborhood. From Montel’s theorem 2.4 it follows that \( W = \bigcup f^k(V) \) is the whole complex plane omitting at most one point \( y \). Clearly if \( x \notin V \), then \( x \notin \bigcup f^k(V) \) as \( \{ x \} \) is completely invariant, and hence \( y = x \). If \( x \in V \), then \( W \) must be the whole complex plane. Since if there were some point \( y \) that did not belong to \( W \) we could choose neighborhood \( V' = V \setminus x \) for a point \( z \), and notice that \( \bigcup f^k(V') = \mathbb{C} \setminus \{ x, y \} \) which is a contradiction to the Montel’s theorem. With this reasoning we can also show that there can be only one finite exceptional point. Since if there were more we knew from the beginning of the proof that there must be infinitely many. Hence there would be infinitely many disjoint completely
invariant sets that consist of finitely many points by the definition of exceptional points. Thus we can choose a point \( z \in J \) and finite exceptional points \( x, y \), such that \( z \notin \bigcup_{k=-\infty}^{\infty} f^k(x) \) and \( z \notin \bigcup_{k=-\infty}^{\infty} f^k(y) \). Now denote the set \( \bigcup_{k=-\infty}^{\infty} f^k(x) \cup \bigcup_{k=-\infty}^{\infty} f^k(y) \) by \( A \). Since \( A \) is finite we can find a neighborhood \( V \) of the point \( z \) that does not contain any points from the set \( A \). Hence we see that \( \bigcup_{k=0}^{\infty} f^k(V) \) does not contain any point from the set \( A \) since \( A \) is completely invariant. But now as the set \( A \) contains at least two points this leads to a contradiction with the Montel’s theorem.

Therefore all that is left to prove is that \( x \notin J(f) \). For proving this we first note that as \( x \) is completely invariant the only solution of \( f(z) - x = 0 \) must be \( x \). Thus we have that \( f(z) - x = c(z - x)^n \), where \( c \) is a non zero constant since \( f(z) - x \) is not identically zero. The derivative of \( f(z) \) is the same as the derivative of \( f(z) - x \), and hence \( f'(z) = nc(z - x)^{n-1} = 0 \) for \( z = x \), since \( n \geq 2 \). Thus we see that \( x \) is a super-attractive fixed point and it follows that \( x \) belongs to the Fatou set, which will be proved later in theorem 3.12. This will then conclude the proof.

If we look back at our example of function \( f(z) = z^2 \) we notice that 0 is the only finite exceptional point and it is indeed a super-attracting fixed point.

**Theorem 3.5.**

The Julia set contains infinitely many points.

Proof: We have shown that the Julia set is closed, nonempty and completely invariant, in remarks on the definition 3.4, theorem 3.1 and theorem 3.2 respectively. Thus if the Julia set consisted of finitely many points it would contain some exceptional point. But we have proven in theorem 3.4 that exceptional points do not belong to the Julia set, and thus the Julia set must contain infinitely many points. We will later note that in fact the Julia set must be an uncountable set.

**Theorem 3.6.**

The Julia set can be defined as the minimal closed and completely invariant set that contains at least two finite points.

Proof: We have shown in theorem 3.4 that there cannot be a completely invariant set that consists of finitely many points and contains at least two finite points. Now let set \( E \) be a closed infinite and totally invariant subset of the complex sphere that contains at least two finite points. We know that the complement of \( E \) which we will denote by \( H \) is also completely invariant and open. Thus each \( f^n \) maps \( H \) to itself. It is clear that we can find two points \( a, b \) which belong to the set \( E \) such that all \( f^n(H) \) omit these values, since \( f^n(H) \subset H \) for every \( n \). Therefore \( \{ f^n \}|_H \) forms a normal family and \( H \subset F \), and hence \( J \subset E \). This means that we can define the Julia set as the minimal closed subset which is completely invariant and has at least two finite points, since there can be only one finite exceptional point, as proven in theorem 3.4.

We will call this result the minimality property of the Julia set. For showing how convenient this can sometimes be we will calculate the Julia set of the polynomial \( f(z) = z^2 - 2 \). First we easily deduce that the inverse branches of \( f \) are \( g_1(z) = \sqrt{z+2} \) and \( g_2(z) = -\sqrt{z+2} \). We will denote the closed interval \([-2, 2] \) by \( L \), and notice that \( f(L) = L \) and \( g_1(L) \cup g_2(L) = L \). Thus \( L \) is completely invariant set with respect to the \( f \). As \( L \) is closed and clearly contains more than two points we conclude that the Julia set is contained in \( L \). We will come back to this example later in
this section and show that the Julia set is indeed the set $L$.
We could of course have obtained this result straight from definition of the Julia set and the Fatou set, but it would have required much more work and calculation even with this quite a simple polynomial.

**Proposition 3.6.**

The following holds for all points $z \in \mathbb{C}$ except for an exceptional point. If $V$ is an open set and it is intersecting $J$, then we can find a natural number $k$ such that $f^{-k}(z) \cap V$.

Proof: We have proven in theorem 3.4 that if $z$ is not an exceptional point then $z \in f^k(V)$ with some integer $k$. Thus at least one of the roots $f^{-k}(z)$.

We will now give the definitions for the orbits. We will denote the forward orbit of a point $z$ by $O^+(z)$ and define that $O^+(z) = \bigcup_{n=1}^{\infty} f^n(z)$. In a similar manner we will denote the backward orbit of a point $z$ by $O^-(z)$ and define $O^-(z) = \bigcup_{n=1}^{\infty} f^{-n}(z)$. If we speak about the orbit we mean the union of the forward and the backward orbit, and denote $O(z)$. We immediately notice that the orbit is non-trivial only for finite points, as $O(\infty) = \{\infty\}$, since the infinity is an exceptional point.

**Theorem 3.7.**

Let $f$ be as usual a polynomial with the degree at least two. Then the following two claims hold.

1. If $z$ is not an exceptional point, then $J$ is contained in the closure of $O^-(z)$
2. if $z \in J$ then $J$ is the closure of the $O^-(z)$

Proof: We will first prove the part one. We will prove that $J$ is contained in the closure of $O^-(z)$ by showing that any neighborhood of any point in the Julia set contains points from $O^-(z)$. Thus any point in the Julia set belongs to the closure of $O^-(z)$.

Let $x$ be any point in the Julia set and let $U$ be an arbitrary neighborhood of $x$. From the proposition 3.6 we know that the point $z$ lies in some $f^n(U)$ and thus its backward orbit does intersect $U$. Now as $x$ and $U$ were arbitrary this shows that $J$ is contained in the closure of $O^-(z)$.

To prove the second part we note that as the Julia set is completely invariant and closed the closure of $O^-(z)$ must lie in the Julia set, if $z \in J$. We know from theorem 3.4 that a point in the Julia set cannot be an exceptional point and thus we can use the first part of theorem and obtain that the Julia set must be contained in the closure of $O^-(z)$. And hence we have that the Julia set is the closure of $O^-(z)$.

**Proposition 3.7.**

The Julia set is the same for functions $f$ and $f^k$ for any given positive integer $k$.

Proof: We will show that the Fatou sets of $f$ and $f^k$ are the same, which means that their complements are also same.

The inclusion $F(f) \subseteq F(f^k)$ is clear, because every sequence of $\{f^{kn}\}$ is also a sequence of $\{f^n\}$ and thus if $z \in F(f)$, then also $z \in F(f^k)$.

For proving inclusion in the other direction we first notice that since the polynomials satisfy the
Lipschitz condition, lemma 2.1, we notice that \( \{ f^a(f^{mk}) \} \) forms equicontinuous family whenever \( \{ f^{mk} \} \) forms a equicontinuous family, where \( a \in \{ 1, 2, \ldots, k - 1 \} \). From this we get that families \( \{ f^{mk+a} \} \) for every integer \( a \) are equicontinuous and hence normal on the same set that the family \( \{ f^{mk} \} \) is normal. Now assume that \( \{ f^{kn} \} \) is normal family on a neighborhood of a point \( z \). Then as we noted above every family \( f^{kn+a} \) is normal on that neighborhood for every integer \( a \). Thus as we take a finite union of these normal families and add to this set the continuous functions \( \{ f, f^2, \ldots, f^{k-1} \} \) we will obtain the family \( \{ f^n \} \) which is normal on this neighborhood. This follows as the finite union of normal families is normal and as we can add a finite number of continuous functions to normal family and keep it normal. This shows that if \( z \in F(f^k) \) then also \( z \in F(f) \) and hence \( F(f^k) \subset F(f) \) which concludes the proof.

**Theorem 3.8.**

The Julia set has no isolated points.

**Proof:** Let \( z \in J \) be an arbitrary point and let \( V \) be an arbitrary neighborhood for such \( z \). We must prove that there is another point in \( V \) which belongs to \( J \). We will first study the situation in which \( z \) is not a fixed or a periodic point.

As \( z \) is not a fixed point we know that \( f^{-1}(z) \) cannot contain \( z \), and thus we can choose some \( z_1 \in f^{-1}(z) \) that satisfies \( z_1 \neq z \). It follows that \( V \) contains a point \( z_2 \in f^{-k}(z_1) \) for some natural number \( k \) due to the proposition 3.5, since we know that a point \( z_1 \) cannot be an exceptional point as it is in the Julia set, due to the theorem 3.4. Point \( z_2 \) belongs to the Julia set since the Julia set is completely invariant and it cannot be \( z \), since \( z \) was not periodic, and hence \( V \) contains another point of the Julia set in addition to \( z \).

Next we will consider that \( z \) is a fixed point. If \( f(z) = z \) and \( \{ f^{-1}(z) \} = \{ z \} \), we would have that \( z \) is an exceptional point and we have shown in theorem 3.4 that exceptional points belong to the Fatou set. Thus \( f(v) - z = 0 \) must have some solution \( x \) where \( x \neq z \). Therefore we obtain from proposition 3.6 that \( f^{-k}(x) \) intersects \( V \) for some integer \( k \), where \( V \) is again an arbitrary neighborhood of \( z \). This point belongs to the Julia set because of total invariance of the Julia set and it cannot be \( z \) as \( f^k(z) = z \neq x \).

To end the proof we will study a case when \( z \) is a periodic point. This can be reduced back to the previous case by noticing that \( z \) is a fixed point for the polynomial \( f^n \) where \( n \) is the period of \( z \). This can be done since \( J(f) \) is the same as \( J(f^k) \) for any integer \( k \) by the proposition 3.7. Thus we have covered all the cases for a point \( z \) and the proof is complete.

**Corollary 3.1.**

The Julia set is a perfect set.

**Proof:** We have proven that the Julia set is nonempty, closed and has no isolated points. Thus it is perfect. Due to the Baire’s category theorem we can say that the Julia set must be an uncountable set. Statement and proof of the Baire’s category theorem can be found for example in [R] starting on the page 97.

**Theorem 3.9.**

Let \( U \) be any open non-empty set that intersects the Julia set. Then we have that for any sufficiently large integer \( n \) it holds that \( J \subset f^n(U) \).
Proof: We can assume that the Julia set is not contained in $U$, since if it is the claim is trivial due to the complete invariance of the Julia set. We take three open subsets of the set $U$, denoted by $U_1, U_2, U_3$, each intersecting $J$ and at a positive distance apart from each other, using the spherical metric. This can be done since the Julia set is perfect. First we will show that for each $j \in \{1,2,3\}$, some forward image of $U_j$ covers some $U_k$. This means that for every $j$ there are some integers $k \in \{1,2,3\}$ and $n > 0$ such that $U_k \subset f^n(U_j)$.

Suppose this is not true. Then for some $j$ and every $n$ the function $f^n|_{U_j}$ fails to cover any of the sets $U_1, U_2, U_3$ and we can find three points, one from each set, such that $f^n|_{U_j}$ does not cover them. Therefore by theorem 2.5 we have that the family $\{f^n\}$ is normal on the set $U_j$, which is a contradiction as $U_j$ is a neighborhood of a point belonging to the Julia set. Thus $U_k \subset f^n(U_j)$ holds for every $j$, for some constants $k, n$.

We will denote any such $k$ by $\pi(j)$, and notice that $\pi$ maps set $\{1,2,3\}$ to itself and hence some iterate of $\pi$ has a fixed point. This means that for some $j$ and for some $n_1$ it holds that $U_j \subset f^{n_1}(U_j)$.

Now the sequence of $f^{m,n}(U_j)$ is increasing with $m$. We know from theorem 3.4 that $\cup_{m=0}^\infty f^{m,n}(U_j)$ is the whole complex plane omitting maybe a single point $x$ which does not belong to the Julia set of the polynomial $f^n$. And hence due to the proposition 3.7 it does not belong to the Julia set of the polynomial $f$ either. Thus the sets $f^{m,n}(U_j)$ form an increasing open cover for the Julia set, which is compact. From this it follows that a finite union of such sets covers the Julia set, and as the sets $f^{m,n}(U_j)$ were increasing it follows that in fact one of them covers $J$.

Let $n_2$ be the first integer for which $f^{n_1,n_2}(U_j)$ covers the Julia set. Now for every $n^* > n_2$ it holds that $f^{n}(U_j) = f^{n-n_2}(f^{n_2}(U_j))$ and since $f^{n_2}(U_j)$ contains the Julia set and the Julia set is completely invariant we notice that $J \subset f^{n}(U_j)$ for every $n^* > n_2$. And since we chose that $U_j \subset U$ this proves the claim, as $J \subset f^{n}(U_j) \subset f^{n}(U)$ with all integers $n^* > n_2$ for some integer $n_2$.

**Theorem 3.10.**

The Julia set contains the closure of the set of repelling periodic points.

Proof: Let $z \in \mathbb{C}$ be an arbitrary periodic repelling point. Then $f^k(z) = z$ for some $k$, where $k$ is the period of $z$. Let us study the polynomial $g = f^k$. As we have shown before in the proposition 3.7 the Julia set is the same for both $f$ and $f^k$, and clearly $z$ is a fixed point for $g$, and by assumptions $|g'(z)| > 1$.

Now assume that $\{g^n\}$ is normal family in a neighborhood of $z$. By definition of normality, definition 2.2, we would have some open and bounded neighborhood $V \subset \mathbb{C}$ of $z$ such that any subsequence of $\{g^n\}$ has a subsequence $\{g^m\}$ that converges locally uniformly towards some continuous function $h$ in it. Function $h$ can not be the constant function $\infty$ in $V$ since $g^n(z) = z$ for any $n$, and since all polynomials $g^n$ are analytic functions the function $h$ must be analytic. From the uniform convergence of $g^n$ on compact subsets of the set $V$ to the function $h$ and from the Cauchy integration theorem we know that the derivatives must also converge. Hence we get that $h'(z) = \lim_{i \to \infty} (g^{n_i})'(z)$. But using the chain rule we obtain that $| (g^{n_i})'(z) | = | (g'(z))^{n_i} | \to \infty$ as $i \to \infty$. This is a contradiction, since function $h$ was supposed to be analytic and proves that the point $z$ must belong to the Julia set. We have noticed before that the Julia set is closed, so as the set of repelling points belongs to the Julia set so does its closure, which completes the proof.

**Theorem 3.11.**

There is a periodic point in any neighborhood of any point in the Julia set.
Proof: Let $V$ be an arbitrary open set such that $V \cap J(f) \neq \emptyset$. Then choose a point $z \in V \cap J(f)$, such that $z$ is not a fixed point, and that it is not an image of a critical point. Because there is only a finite amount of fixed points or critical points and because the Julia set is perfect we can always find such $z$. Now, since $z$ is not an image of a critical point, we can find a branch $f^{-1}$, that is analytic function from $U \to \mathbb{C} \setminus U$, where $U$ is some small neighborhood of the point $z$. Thus we can define the family of analytic functions

$$g^k(x) = \frac{f^k(x) - x}{f^{-1}(x) - x}$$

for $x \in U$. Family $\{g^k\}$ cannot be normal, since $\{f^k\}$ is not normal by the definition of the Julia set. Hence by the Montel’s theorem $g^k$ must obtain value 0 or 1 for some integer $k$ and $x \in U$.

In the first case we have $f^k(x) = x$, and in the second $f^k(x) - x = f^{-1}(x) - x \iff f^k(x) = f^{-1}(x) \iff f^{k+1}(x) = x$. Thus we have shown that there is a periodic point in any neighborhood of $z \in J(f)$.

From this it clearly follows that there must be infinitely many periodic points, and this means that there must be infinitely many cycles.

**Corollary 3.2.**

The Julia set is the closure of the set of periodic repelling points.

**proof:** It is a known fact that there can be only a finite amount of attracting or indifferent cycles and we will later give a proof for the number of the attractive cycles. Thus because the Julia set is a perfect set and the above theorem holds we get that repelling periodic points are dense in the Julia set and hence the Julia set is the closure of repelling periodic points as they all belong in the Julia set. The proof for the statement considering number of indifferent cycles can be found in [B] theorem 9.6.1.

**Theorem 3.12.**

Attracting fixed points lie in the Fatou set.

**Proof:** We know that the point $\infty$ lies in the Fatou set, so we need to show the claim for attracting finite fixed points. First we will prove that there is a small disc around the attracting finite fixed point $z$, such that $f$ is forward invariant in that disc. From that we can deduce that the family $\{f^n\}$ is uniformly bounded on that disc.

Because the point $z$ is an attracting fixed point we know that $|f'(z)| < 1$. Thus there exists $\alpha < 1$ and a disc $D$ centered at $z$, such that $|f(x) - z| = |f(x) - f(z)| \leq \alpha |x - z|$, where $x \in D$ is an arbitrary point. Therefore it holds that $f^n(D) \subset D$ for any integer $n$. Hence Montel’s theorem 2.4 gives that $\{f^n\}$ forms a normal family in $D$, giving that $z$ must belong to the Fatou set.

Additionally continuing from the above proof by noticing that for every point $x \in D$ it holds that $\lim_{n \to \infty} f^n(x) = z$ and using the theorem 2.6 we obtain that the family $\{f^n\}$ converges locally uniformly towards $z$ on the set $F_z$, where $F_z$ is the component of the Fatou set that contains the point $z$. Thus we obtain that $F_z \subset A(z)$, where $A(z)$ is the basin of attraction of the point $z$.

**Corollary 3.3.**
Attracting periodic points belong to the Fatou set.

Let $z$ be a periodic point with the period $n$ and denote the cycle generated by $z$ as $C_z$. We have shown before in the proposition 3.7 that the Fatou set of polynomials $f$ and $f^k$ are the same for every integer $k$. Thus we can use the previous result as we study the function $g = f^n$ for which $z$ is an attractive fixed point.

Clearly this can be applied to every point of the cycle $C_z$ since the derivative is the same for every point in that cycle, with respect to the function $g$. So every attracting cycle has a basin of attraction $A(C_z)$. We defined the immediate basin of attraction denoted by $A^*(C_z)$ in the following way, $A^*(C_z) = A(C_z) \cap F_0^*(f)$, where $F_0^*(f)$ is the union of components of the Fatou set that contain a point from $C_z$. Clearly by definition the immediate basin of attraction is a subspace of the basin of attraction.

**Theorem 3.13.**

If there exists a periodic attracting point in any given component of the Fatou set, then there cannot be any other periodic points in that component.

Proof: Let $F_0$ be a component of the Fatou set and assume it contains an attracting periodic point $z$ and some other periodic point $x$, with periods $n,m$ respectively. Replacing $f$ with $f^{nm}$ both $z$ and $x$ become fixed points but the Fatou set does not change, $F(f) = F(f^{nm})$ by the proposition 3.7.

Then we notice that $f^{nmk}(x) \to z$ as $k \to \infty$, and thus only way that $x$ can be a fixed point is $x = z$. The convergence can be seen from the proof of the theorem 3.12, where we obtained that $F_0 \subset A(z)$, where $A(z)$ is the basin of attraction of the point $z$ with respect to the function $f^{nm}$. And since $x \in F_0$ it follows that $x \in A(z)$.

**Theorem 3.14.**

Let $F_0$ be a completely invariant component of $F(f)$. Then it holds that

1. $\partial (F_0) = J(f)$
2. $F_0$ is either a simply connected or infinitely connected
3. If there exists other components of $F(f)$ they are simply connected
4. $F_0$ is simply connected if and only if $J(f)$ is connected

Proof of 1: We will first show that as $F_0$ is completely invariant its closure $\overline{F}(f)$ is also completely invariant. To prove this we will first show forward invariance.

Since $f$ is continuous we know that for any $x \in F_0$ it holds that $f(x) = \lim_{z \to \infty} f(z)$, where $z \in F_0$ for all $k$ and $z_k \to x$. As $F_0$ is forward invariant we have that all $f(z_k) \in F_0$ and thus the $f(x)$ must belong to the closure of the $F_0$. We notice that for every point $z$ in the closure there is some point that is mapped on it by studying some inverse images $f^{-1}(z)$ and using the fact that if $z_j \to z$, then $f^{-1}(z_j) \to f^{-1}(z)$. The proof for the backward invariance is now identical to the proof for the forward invariance since for every $j$ the point $z_j \in F_0$, and hence $f^{-1}(z_j)$ lies in the component $F_0$ due to complete invariance.

We can use this with the minimality property of the Julia set, theorem 3.6, and obtain that $J(f) \subset \overline{F}(f)$, since $\overline{F}(f)$ is closed, completely invariant and has more than two points. As the Julia set is disjoint from $F_0$ and since all the boundary points of $F(f)$ belong to the Julia set we have that $J(f) = \partial (F_0)$.

Proof of 2: We assume that $F_0$ is a completely invariant component with some finite connectivity $m$, and denote the components of the complement of $F_0$ by $C_1, \ldots, C_m$. We will first prove the
result which states that there is an integer \( n \) such that each component \( C_j \) is completely invariant under \( f^n \).

As \( F_0 \) is completely invariant its complement is also completely invariant, so the \( f \) is mapping from the complement of \( F_0 \) to itself. As each \( C_i \) is connected and \( f \) is continuous we have that \( f(C_i) \) is connected for all \( i \). Thus \( f \) induces a map \( \tau \) of \( \{1, 2, ..., m\} \) into itself defined by \( \tau(i) = j \), where \( f(C_i) \subset C_j \). This is well defined since \( f \) is continuous; it must map all elements from one component to the same component as the components \( C_k \) are disjoint. Since \( f \) is surjective the mapping \( \tau \) is surjective and hence \( \tau \) is permutation of \( \{1, 2, ..., m\} \). Hence \( \tau \) has finite order, denoted by \( m \) and thus for all \( i \) it holds that \( f^m(C_i) = C_i \), so each \( C_i \) is completely invariant under \( f^m \).

Now as \( J(f) = J(f^n) \) is infinite we obtain that one of the components \( C_j \), denote as \( C_1 \) is infinite and that the minimality of \( J(f^n) \) implies that it lies in \( C_1 \). But from the 1 part we know that each \( C_1 \) meets \( J(f) = J(f^n) \) and thus \( m = 1 \).

Proof of 3: The set \( J(f) \cup F_0 \) is just the closure of \( F_0 \) and hence connected as \( F_0 \) is. Basic topological arguments concerning connectedness give that each component of the complement must be simply connected.

Proof of 4: From the point 1 we notice that \( J(f) = \partial(F_0) \) and now the simple topological argument gives that \( F_0 \) is simply connected if and only if its boundary is connected.

Corollary 3.4.

The Julia set is the boundary of the unbounded component of the Fatou set.

Proof: With the above theorem it is enough to show that the component of the Fatou set that contains the point \( \infty \), which we call the unbounded component of the Fatou set and denote by \( F_\infty \), is completely invariant. Due to the surjectivity it is enough to show that \( F_\infty \) is backward invariant. First note that since \( f(\infty) = \infty \) it is clear that \( f(F_\infty) \subset F_\infty \). Now let \( z \in f^{-1}(F_\infty) \) be arbitrary. Due to the complete invariance of the Fatou set it holds that \( z \in F_1 \), where \( F_1 \) is some component of the Fatou set and \( f(F_1) \subset F_\infty \). Assume that \( f(F_1) \neq F_\infty \). Then there exists some point \( x \in \partial F_1 \) such that \( f(x) \in F_\infty \). But this is a contradiction since \( x \in J \) and hence \( f(F_1) = F_\infty \) and we obtain that \( f(y) = \infty \) for some \( y \in F_1 \). But since \( f \) is polynomial we know that the only possibility is \( y = \infty \) and hence \( F_1 = F_\infty \), which completes the proof since the point \( z \) was arbitrary.

So we can define the Julia set of \( f \) as the boundary of the unbounded component of the Fatou set. In addition it gives that each bounded component of the Fatou set is simply connected and that the unbounded component is either simply or infinitely connected.

With this result we can return to study the Julia set of the polynomial \( f(z) = z^2 - 2 \). We had shown using the minimality property of the Julia set that \( J \subset L \), where \( L = [-2, 2] \). And now we note that the Fatou set must thus have only one component. This component must be the \( F_\infty \) and hence the Julia set is its boundary. But as \( L \) is completely invariant with respect to \( f \) we note that the boundary is the whole \( L \), since none of the points \( x \in L \) converge to infinity under iterations of \( f \). Thus the Julia set is the set \( L \).

As another remark to the previous theorem we obtain the following. Let us assume that the whole Fatou set is connected. Then either the Julia set is connected and the Fatou set is simply connected or the Julia consists of infinitely many components and the Fatou set is infinitely connected.

Theorem 3.15.
The boundary of the basin of attraction consist of the Julia set.

Proof: Let $C_z$ be an attractive cycle and $A(C_z)$ its basin of attraction. Since we have shown in the corollary 3.4 that the Julia set is the boundary of the component $F_0$ we can assume that both $C_z$ and $A(C_z)$ are bounded. We have proven that $A(C_z)$ is nonempty and we will now show that it contains some open discs around every point of the cycle $C_z$. Let $z$ be some point in the cycle and $n$ be the period of the cycle. Then $z$ is an attractive fixed point for $f^n$ and thus we can choose some disc $U$ around it such that all points in the closure of that disc converge uniformly towards the point $z$ as shown in the proof of the theorem 3.12. Hence we can choose this $U$ to be the disc for the point $z$ and for every other point in the cycle we choose some disc $U_k$ centered at $f^k(z)$ such that $U_k \subset f^{k}(U)$. Such disc $U_k$ exist for every integer $1 \leq k \leq n-1$ since $f^k$ is analytic and thus open map for every integer $k$. Now it is easy to see that for any point $x$ that lies in these discs it holds that $\lim_{k \to \infty} \text{dist}(C_z, f^k(x)) = 0$ since $\lim_{k \to \infty} \text{dist}(z, f^{kn}(x)) = 0$ and all mappings $f, f^2, \cdots, f^{n-1}$ are continuous.

Let us denote the smallest radius of these discs by $r$. We also know that $A(C_z)$ cannot be the whole complex plane since the Julia set is nonempty and does not belong to the basin of attraction. Hence there exists the boundary for $A(C_z)$. Let $c$ be an arbitrary point of the boundary and $V$ its arbitrary neighborhood. Then all points in $V \cap A(C_z) \neq \emptyset$ do converge to $C_z$ and all points in $V \setminus A(C_z) \neq \emptyset$ do not even enter in $A(C_z)$. Thus $c$ must belong to the Julia set, since it holds for the distance between iterations of two arbitrary points $z, x$ of which other converges to the cycle and other does not enter to the basin of attraction that $\lim_{n \to \infty} \text{dist}(f^n(z), f^n(x)) \geq r$. Therefore we have shown that $\partial A(C_z) \subset J(f)$.

To show inclusion to other direction we note that the boundary of $A(C_z)$ is closed and since $A(C_z)$ is completely invariant its boundary is also completely invariant, which follows from result in the proof of the theorem 3.14. From this we get by the minimality of the Julia set that $\partial A(C_z) = J(f)$, because clearly there are at least two points in the boundary of $A(C_z)$. If there would be only one boundary point we would get from a basic topology that $A(C_z)$ would be the whole complex plane omitting a single point, which is not possible.

If we take a look at our example of the Julia set for the polynomial $f(z) = z^2$ we see that the boundary of basins of attraction for both $0$ and $\infty$ is the boundary of the unit disc, and we note that the theorem clearly holds in this example.

We have defined valency $v_f(z)$ before in definition 2.5. We remind that we denoted deficiency of $f$ at $z$ by $\delta_f(z) = v_f(z) - 1$. Define deficiency over a set $A \subset \mathbb{C}$ as $\sum_{z \in A} \delta_f(z)$. Notice that $\delta_f(x) = 0$ for all but a finite set of points in arbitrary set $A$ and that for a polynomial $f$ the following equation holds $\delta_f(\mathbb{C}) = \deg f - 1$.

Additionally if $f$ is a continuous function from a domain $D \subset \mathbb{C}$ to a domain $\tilde{D} \subset \mathbb{C}$ we say it is $m$-fold if for every point $z_0 \in \tilde{D}$ the equation $f(z) = z_0$ has exactly $m$ solutions, counting multiplicities, in $D$. This in mind we proceed to the next theorem, for which we will give formulations that are suitable for our needs and omit the proof.

**Theorem 3.16.** The Riemann-Hurwitz relation

Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial with the degree at least two. We will give two formulations for the Riemann-Hurwitz relation.

First formulation: Let $f$ be as above and $F_0, F_1$ be components of the Fatou set. If it holds that $f$ maps the component $F_0$ to the component $F_1$, then there exists an integer $m$ such that $f$ is a $m$-fold map of $F_0$ into $F_1$ and

$$\chi(F_0) + \delta_f(F_0) = m \chi(F_1)$$

where $\chi$ denotes the Euler characteristic discussed in the appendix, definition 6.1 and definition
6.3.

Second formulation: Let $f$ be as before and assume that the following holds

1. $V$ is a domain in the closed complex plane which is bounded by a finite number of mutually disjoint Jordan curves.
2. $U$ is a component of the $f^{-1}(V)$
3. There are no critical values of $f$ on $\partial V$

Then it holds that there exists an integer $m$ such that $f$ is an $m$-fold map from $U$ to $V$ and

$$\chi(U) + \delta_f(U) = m\chi(V)$$

Proofs for these theorems can be found for example in [B], theorems 5.5.4 and 5.4.1 respectively.

**Theorem 3.17.**

The immediate basin of an attractive fixed point contains a critical point.

Proof: If an attractive fixed point is super attracting it is a critical point itself so we can concentrate on attractive but non-super-attracting points.

For a polynomial infinity is always a super attracting fixed point so we can assume that the fixed point $z$ is finite. Let $z$ be an attractive fixed point and denote the component of the Fatou set that contains $z$ by $F_z$. This component coincides with the immediate basin of attraction of the attractive fixed point and is forward invariant.

Assume now that the component $F_z$ does not contain a critical point. Then due to theorem 3.14 part 3 we notice that the component $F_z$ must be simply connected, since it does not contain the infinity as the infinity does not converge towards the point $z$. Using the Riemann-Hurwitz relation we obtain that

$$1 + 0 = \chi(F_z) + \delta(F_z) = m\chi(F_z) = m$$

and hence $m = 1$. From this we get that $f$ is a homeomorphism of the simply connected domain $F_z$ onto itself. Applying the Riemann mapping theorem we deduce that the restriction of the map $f$ to the component $F_z$ is analytically conjugate to an automorphism $S$ of the unit disc which preserves the origin. Thus $S$ must be a rotation of a unit disc. But from this we get that

$$|S'(0)| = |f'(z)| < 1$$

and this is a contradiction. Hence there must be a critical point in the immediate basin of attraction of an attractive fixed point.

**Corollary 3.5.**

The immediate basin of an attractive periodic point contains a critical point.

Proof: Let $(z_0, z_1, \ldots, z_{n-1})$ be a cycle of a period $n$. Above theorem says that for any $z_i$, $i \in \{0, 1, \ldots, n-1\}$ the $f^n$ has a critical point in $F_i$, which is the component of the Fatou set that contains the point $z_i$. In a critical point the derivative of a function is zero, so $(f^n)'(z) = 0$ for some $z$ in $F_i$. But we have shown that

$$(f^n)'(z) = \prod_{k=0}^{k=n-1} f'(f^k(z))$$

and thus $f$ must have a critical point in some $F_i$, as the product of elements is zero only if some of its element is.
From this we get an upper bound for the amount of different attractive cycles and for the basins of attractions. Since it is clear that the basins of attraction must be disjoint and every basin contains a critical point we conclude that there can be at most $\deg f - 1$ different finite attracting cycles for every polynomial $f$.

**Theorem 3.18.**

The Fatou set $F$ contains at most two completely invariant components. And if there are two such components they are both simply connected.

Proof: Assume that there exist more than one, but finitely many completely invariant components and denote them by $F_1, \ldots, F_k$, where $k \geq 2$. Additionally let $d$ denote the $\deg f$. We know from the third point of theorem 3.14 that if there is one completely invariant component of $F$, then all other components are simply connected. In this case this gives us that all components are simply connected as we can use this theorem to two different completely invariant components.

Next we will use the Riemann-Hurwitz relation, theorem 3.16, to each $F_j$. And as each $F_j$ is simply connected and completely invariant we obtain that $f$ is a $d$-fold mapping from $F_j \rightarrow F_j$, and thus

$$\delta_f(F_j) = (d - 1)\chi(F_j) = d - 1$$

We had defined deficiency in definition 2.5 and obtained formula which gives

$$k(d - 1) = \sum_{j=1}^{k} \delta_f(F_j) \leq \delta_f(\hat{C}) = 2d - 2$$

And thus $k \leq 2$ which completes the proof, since if there would be infinitely many completely invariant components we could choose three of them and obtain contradiction.

In fact it follows from the Sullivan’s no wandering domain theorem that if there are two completely invariant component in $F$ then these are the only components of $F$. We will not prove this result in this paper, but the proof can be found for example in [B] starting on the page 176.

**Theorem 3.19.**

The Fatou set of a polynomial $f$ can have $1, 2$ or $\infty$ many components.

Proof: Let us assume that $F$ consist of a finite number of components. Then we can show in the same way as in the proof of part 2 of theorem 3.14, that there exists such an integer $m$, that every component $F_i$ is completely invariant under $f^m$. Proposition 3.7 states that the Fatou set is the same for both $f$ and $f^m$ and thus the above theorem 3.18 states that there can be at most two components. To complete the proof we remember that the Julia set is bounded by theorem 3.4 and hence the Fatou set is non-empty and has at least one component.
4 Basic properties of self-similarity and geometric measure theory

In this section the definition, examples and some properties of self-similarity and the Hausdorff measure are given. We will also define the Hausdorff dimension using the Hausdorff measure and prove some of its basic properties. We shall also fractals and how to define them. We will then proceed to give a link between self-similar sets and the Hausdorff dimension. In this paper geometric measure theory, self-similarity and theory about fractals are used as tools to help to understand the properties of the Julia set better and therefore a few proofs that do not serve this goal are omitted in this paper. A closer look at this topic with proofs can be found for example in [FE] or [M]. Other sources that I have used for this chapter include [AIM], [F] and [I].

Definition 4.1. Similarity and contraction

Let \( g \) be a function from \( \mathbb{R}^n \) to itself. We call it a contraction if the condition \( |g(x) - g(y)| \leq c|x - y| \) holds for all \( x, y \in \mathbb{R}^n \), with some constant \( 0 < c < 1 \), where \( c \) is called the ratio of contraction. If the equality holds for every pair \( x, y \) we call \( g \) a similarity. This means that the function \( g \) maps set to a smaller copy of itself. The mapping \( g \) is then a composition of scaling with the constant \( c \), translation and a rotation.

Definition 4.2. Invariant sets and self-similarity

The non-empty compact subset \( X \) of the complex plane is called an invariant set with respect to the contractions \( g_k \), if there exists a finite family of contractions \( g_k \), from the complex plane to itself, such that

\[
X = \bigcup_{k \in K} g_k(X)
\]  

(1)

If all the functions \( g_k \) are similarities we call \( X \) a self-similar set with respect to these similarities. If \( X \) is self-similar then we notice that it consists of smaller copies of itself.

Self-similarity is best approached via examples. One of the most common examples of self-similarity is the one-third Cantor set. We start from the closed interval \( I = [0, 1] \) and take away the middle third of it, leaving endpoints \( 1/3 \) and \( 2/3 \) to the set, and obtain two closed intervals \([0, 1/3]\) and \([2/3, 1]\). Now we again take away the middle thirds of these intervals and obtain four closed intervals \([0, 1/9]\), \([2/9, 1/3]\), \([2/3, 7/9]\), and \([8/9, 1]\). We will obtain the Cantor set by recursively repeating this process infinitely. Next we will give a rigorous definition for this intuitive idea.

Examine functions \( g_1(x) = \frac{1}{3}x \) and \( g_2(x) = \frac{1}{3}x + \frac{2}{3} \), defined from the real-numbers to itself. Both are clearly similarities and \( g_1(I) = [0, 1/3] \) and \( g_2(I) = [2/3, 1] \). We can denote the first step in creating the Cantor set as \( C_1 = [0, 1] \), and we can construct the second step with \( C_2 = \bigcup_{i=1}^{2} g_i(I) \). Continuing like this we will construct the \( n \)th step with

\[
C_n = \bigcup_{i_1, \ldots, i_{n-1} = 1}^{2} g_{i_1} \circ \cdots \circ g_{i_{n-1}}(I)
\]

So we go through all compositions of \( g_1 \) and \( g_2 \), which consist of exactly \( n - 1 \) compositors. We obtain the Cantor set by defining \( C = \cap_{n=1}^{\infty} C_n \), that is as an intersection of all the steps. In the following picture we have illustrated seven first steps in creating the Cantor set. It can be checked that the Cantor set is the self-similar set with respect to the similarities \( g_1 \), \( g_2 \).
Many natural properties of the self-similar sets can be seen in the Cantor set. For example, taken any given composition $g = g_i \circ \cdots \circ g_n$ of $g_1$ and $g_2$ and taking the set $g(I) \cap C$ we obtain a set similar to the whole Cantor set. So we get the original Cantor set by scaling the part of the Cantor set obtained this way with an appropriate constant, which is $3^n$ where $n$ is the number of compositions in $g$ and then shifting if necessary. We also note that the Cantor set is obtained in a recursive fashion, and that it is very hard or impossible to give an exact geometric definition for the set without using recursion. It also has a fine structure, meaning that it resembles the original set on an arbitrary scale.

Another good example of self similarity is the Koch curve, which we will construct next and of which construction few first steps are given above. Start again from the closed unit interval $I = [0,1]$ and divide it into three parts, all of equal length. Leave first and last untouched, but take
out the middle one, just like in the construction of the Cantor set, and replace it with sides of an
upward pointing equilateral triangle. We denote the line segment on the left by \( I_1 \), the line segment
that builds left side of the triangle by \( I_2 \), the line segment that builds the right side of the triangle
by \( I_3 \) and the last one by \( I_4 \). In the next iteration we do the same thing for all lines in the curve
that we have got from the previous iterations. We obtain the Koch curve as a limit of these curves by
letting the number of iterations grow to infinity.

This time we define four similarities \( g_i, i \in \{1, 2, 3, 4\} \), which are defined by \( g_i(I) = I_i \). We omit
the exact calculation of these functions in this paper. The Koch curve is the self-similar set for
these similarities.

As we compare the Koch curve to the Cantor set we notice few properties they both share. Let
us denote the Koch curve by \( K \). As we take any composition \( g \) of similarities \( g_j \) and look at the
set \( g(K) \) we obtain a smaller self-similar copy of the whole set \( K \). Koch curve is also formed
recursively by iterating similarities, but as for the Cantor set it is very hard to give classical geo-
metric definition to it without recursion. For example it does not have a tangent at any point. It has
a fine structure, meaning that it resembles the original set in arbitrary scale, just like the Cantor
set.

For an arbitrary Julia set the condition of self similarity, condition (1), clearly does not hold with
similarities. However in some Julia sets there exists some kind of similarity. For studying this we
need some additional notions.

**Definition 4.3.** The hyperbolic Julia set

Let \( f \) be a polynomial and \( J \) its Julia set. We say that the Julia set is hyperbolic if forward or-
bits, defined just before theorem 3.7, of critical points converge either to an attractive cycle or to
infinity.

Let polynomial \( f \) be such that the Julia set \( J(f) \) is hyperbolic. From the definition of hyper-
bolic Julia sets it follows that the Julia set does not contain any critical points. Thus for any point
\( z \in J \) it holds that we can find some open ball centered at the point \( z \) such that \( f \) is conformal in
that ball, see theorem 2.1. The union of those balls provides an open cover for the Julia set and
since the Julia set is compact we can choose some finite union of these balls that will cover the
Julia set. Denote these balls by \( B_i \), where \( i \) goes from one to some finite number \( k \). Using the
Lebesgue’s covering theorem 2.9 we obtain that there exist some positive number \( \theta \) such that for
all points \( z \in J \) the ball \( B(z, \theta) \) is contained in some ball \( B_i \).

Next we will use the following theorem, proved in [AIM] theorem 2.10.9, which states that if \( f \) is
conformal in some disc \( B(x, r_1) \) we can choose any constant \( 0 < c < 1 \) and then it holds that

\[
\frac{|f(z_1) - f(z_2)|}{|f(z_1) - f(z_3)|} \leq \eta_c \frac{|z_1 - z_2|}{|z_1 - z_3|}
\]

(2)

where \( \eta_c \) is a constant that depends only on the constant \( c \), and \( z_1, z_2, z_3 \in B(x, cr_1) \). Here the
difference to similarity is that the constant \( \eta_c \) is greater than one and thus we can have some per-
turbation, and the amount of it depends on the constant \( c \).

With this theorem in mind we choose any positive \( r < \theta \). Now it holds for any \( z \in J \) that
\( B(z, r) \subset B(z, \theta) \subset B_i \) for some \( i \). Especially we obtain that \( f \) is conformal in the ball \( B(z, \theta) \).

And since we chose \( r < \theta \) we can choose a constant \( c = \frac{r}{\theta} < 1 \) and use the theorem presented
above. Additionally we notice that the choice of the constant \( c \) did not depend on the point \( z \in J \).
So we obtain that \( f \) fulfills the inequality (2) for points \( z_1, z_2, z_3 \in J \cap B(z, r) \). In this case we say
that \( J \cap B(z, r) \) is quasi self-similar with its image set.

We notice that this process can be iterated if it holds that the diameter of the set \( f(B(z, r)) \) is smaller
than \( \theta \). Since then it holds for some \( z_1 \in J \) that \( f(B(z, r)) \subset B(z_1, \theta) \) and additionally \( f(B(z, r)) \)
has a positive distance from the boundary of the set $B(z_1, \theta)$. Additionally it follows from the assumptions on $\theta$ that $f$ is conformal in the ball $B(z_1, \theta)$. So we can again use the inequality (2) with the constant $c_1 = \frac{\text{diam}(f(B(z,r)))}{\theta}$ on the ball $B(z_1, \theta)$ and obtain that

$$\frac{|f^2(z_1) - f^2(z_2)|}{|f^2(z_1) - f^2(z_3)|} \leq \eta_{c_1} \frac{|f(z_1) - f(z_2)|}{|f(z_1) - f(z_3)|} \leq \eta_{c_1, \theta} |z_1 - z_2| |z_1 - z_3|$$

for all $z_1, z_2, z_3 \in B(z,r)$. So we see that $J \cap B(z,r)$ is quasi self-similar with the set $f^2(B(z,r)) \cap J$. In the completely same manner we can iterate this process if the diameter of the set $f^2(B(z,r))$ is small enough. To study how many times the iteration can be done we notice that since the Julia set is compact and $f$ is a polynomial we have that the diameter of the set $f(B(z,r))$, where $r < \theta$ and $z \in J$, is smaller or equal to $mr$ where $m$ is the sup-norm of $f'$ on some compact disc $B$ that contains the Julia set and for which it holds that $\text{dist}(J, \partial B) > \theta$. Hence every sufficiently small subset of the Julia set is quasi self-similar with some other subset of the Julia set for which it holds that it is not a subset of any ball with radius smaller than $\frac{\theta}{2m}$ and centered in the point that belongs to the Julia set.

In other words any sufficiently small subset of the Julia set is quasi self-similar with some subset of the Julia set which is of some predefined magnitude. For illustration of this behavior we suggest reader to take a look at the pictures of the Julia sets on the pages 48 and 49.

Next we briefly discuss fractals.

Definition of the fractal is not straightforward and in fact there is no consensus for one definition that would rigorously define all sets that are considered fractals. So we will try to explain what kind of fractals there are and what are their usual properties, rather than giving a rigorous definition for them.

Let $F$ be a set we regard as a fractal. Then its usual properties include

1. At least a quasi self-similarity. With this we mean that for every point $z \in F$ and for every small enough ball $B(z,r)$, where $r < m$ and $m$ is some constant, there exists some continuous function $g$ such that $g(F \cap B(z,r)) \subset F$ and $F \cap B(z,r)$ is quasi self-similar with the set $g(F \cap B(z,r))$. Additionally we demand that $g(F \cap B(z,r))$ is of some predefined magnitude.

2. It has a fine structure. So it is as complex as the original set in an arbitrary scale

3. Its Hausdorff dimension, which will be defined shortly is greater than the topological dimension

4. It cannot be expressed using traditional geometrical objects

As said above this cannot be taken as a definition for the fractals, but more as an illustration of their usual behavior. So a set can be regarded as a fractal even if it does not fulfill all of these properties. For example both the Cantor set and the Koch curve are regarded as fractals since they fulfill all the requirements shown above, as we will later in this section prove that their Hausdorff dimension is indeed greater than topological dimension. The Julia sets are also considered as fractals for almost all polynomial $f$ since they cannot be expressed even locally with a conventional geometrical structures, have fine structure and are quasi-self-similar. We have already shown the quasi self-similarity for the hyperbolic Julia sets and we will later see some of the other properties for some Julia sets.

Next we will start studying the Hausdorff measure. For that we need the following notions. We will denote the diameter of the set $U \subset \mathbb{R}^d$ by $|U| = \text{sup}\{|x-z| \mid x, z \in U\}$. We will call the countable family of nonempty sets $U_i$ a $\delta$-cover of $U$ if $U \subset \bigcup U_i$ and $|U_i| < \delta$ for all $i$. Suppose $U$
is a subset of $\mathbb{R}^n$ and $s$ is an arbitrary non-negative number. Then for any $\delta > 0$ we define

$$H^s_\delta(U) = \inf\left\{ \sum_{i=1}^{\infty} |U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } U \right\}$$

Hence we try to cover $U$ with sets which diameter is at most $\delta$ and try to minimize the summation over their diameters $s$:th power. Clearly as $\delta$ decreases the number of acceptable covers is reduced, and thus the infimum taken over covers increases or stays same. Hence we obtain that $H^s_\delta(U)$ converges to a limit as $\delta \to 0$, if we allow one possible limit to be infinity. So we can give the following definition.

**Definition 4.4. Hausdorff measure**

Let $U$ be any set in $\mathbb{R}^n$. We define its $s$: dimensional (outer)Hausdorff measure as

$$H^s(U) = \lim_{\delta \to 0} H^s_\delta(U)$$

Limit exists for any set $U$ and any non-negative $s$ if we allow it to be infinity. In this paper we omit the proof that the Hausdorff measure is indeed an (outer)measure, for proof see [FE] page 171.

It can be shown that for the Borel sets the Hausdorff $s$:measure is just the Lebesgue $n$:measure multiplied with a constant, when $s$ is chosen to be a positive integer $n$. Proof for this result can be found for example in [M] starting on the page 45. Additionally it is easy to see that the Hausdorff measure for the value $s = 0$ is just the counting measure.

From the definition of the Hausdorff measure it is clear that the Hausdorff measure of a given set $U$ is non increasing with respect to $s$, since $|U_i| < \delta$ and $\delta \to 0$. So the bigger the $s$ is the smaller $|U_i|^s$ becomes, and thus the smaller the value of the series $\sum_{i=1}^{\infty} |U_i|^s$ will become.

In fact more can be said, if $t > s$ and $U_i$ is arbitrary $\delta$:cover of $U$, we have

$$\sum_{i=1}^{\infty} |U_i|^t \leq \delta^{t-s} \sum_{i=1}^{\infty} |U_i|^s$$

Since for every $|U_i|$ it holds that $|U_i|^t \leq \delta^{t-s} |U_i|^s$, we obtain that $|U_i|^t \leq \delta^{t-s} |U_i|^s$ which follows from assumption $|U_i| < \delta$ for all $i$. It follows from this that if $H^s(U) < \infty$ then $H^t(U) = 0$, since $\delta \to 0$. Therefore we obtain that there exists critical value $s_1$ for which it holds that $H^s(U) = 0$ when $s > s_1$ and $H^s(U) = \infty$ for all $0 \leq s < s_1$. And with this we give the following definition.

**Definition 4.5. Hausdorff dimension**

Let $U$ be an arbitrary set in $\mathbb{R}^n$. We define the Hausdorff dimension of the set $U$, denoted by $\dim_H(U)$ to be

$$\dim_H(U) = \inf\{s \geq 0 : H^s(U) = 0\}$$

In other words it is the smallest value of $s$ such that all the bigger values give zero for the Hausdorff measure of the set $U$. Thus we notice that the critical value discussed above is in fact defined to be the Hausdorff dimension of set $U$. At this critical value $H^s(U)$ can obtain any non negative value, including infinity.

We will next prove that bi-Lipschitz mappings preserve the Hausdorff dimension. For this we will first study how the Hölder condition affects the Hausdorff measure.
Proposition 4.6.
Let $U \subset \mathbb{R}^n$ and $g : U \rightarrow \mathbb{R}^m$ be a mapping such that $|g(x) - g(y)| \leq c|x - y|^\alpha$ for some constants $c > 0$, $\alpha > 0$. Then for each $s$ it holds that $\mathcal{H}^{s/\alpha}(g(U)) \leq c^{s/\alpha} \mathcal{H}^s(U)$. Notice that if $\alpha = 1$ we get the Lipschitz condition.

Proof: let $\{U_i\}$ be an arbitrary $\delta$-cover of the set $U$. From the assumptions we get that $|g(U \cap U_i)| \leq c|U_i|^\alpha$ for any $i$. From this it follows that $\{g(U \cap U_i)\}$ is $\varepsilon$-cover of $g(U)$, where $\varepsilon = c\delta^\alpha$. Then
\[
\sum_i |g(U \cap U_i)|^{s/\alpha} \leq \sum_i c^{s/\alpha} |U_i|^{s/\alpha}
\]
and thus we get
\[
\mathcal{H}^{s/\alpha}(g(U)) \leq c^{s/\alpha} \mathcal{H}^s(U)
\]
as we notice that $\varepsilon \rightarrow 0$ when $\delta \rightarrow 0$. Since $c$ and $\alpha$ were constants, we get the proposition.

Next we will show how the Hölder condition affects the Hausdorff dimension.

Corollary 4.1.
Let $U$ and $g$ be as above. Then $\dim_H(g(U)) \leq \frac{1}{\alpha} \dim_H(U)$.

Proof: From proposition 4.6 we get that if $s > \dim_H(U)$, then $\mathcal{H}^{s/\alpha}(g(U)) \leq c^{s/\alpha} \mathcal{H}^s(U) = 0$. And thus $\dim_H(g(U)) \leq \frac{s}{\alpha}$ for all $s > \dim_H(U)$, which proves the claim. We notice that if $\alpha = 1$, then $\dim_H(g(U)) \leq \dim_H(U)$.

As an another corollary to our proposition we obtain that a bi-Lipschitz mapping preserves the Hausdorff dimension.

Corollary 4.2.
Bi-Lipschitz mappings preserve the Hausdorff dimension.

Proof: As $g$ is a bi-Lipschitz mapping, the functions $g$ and $g^{-1}$ are both Lipschitz by the definition and above corollary gives that $\dim_H(g(U)) \leq \dim_H(U)$ and $\dim_H(U) \leq \dim_H(g(U))$. Thus $\dim_H(g(U)) = \dim_H(U)$, which proves the claim.

Actual calculation of the Hausdorff dimension for a given set $U$ is a non-trivial problem. In this section we will mainly concentrate on self-similar sets for which it is easier, using the method that we will prove later in the proposition 4.12. We will now give an example of calculating the Hausdorff dimension of some given set in order to make it better understandable.

Example 4.1. Calculating the Hausdorff dimension

We will construct a self-similar set by starting from the square that has the vertices at points $(0,0), (0,1), (1,0)$ and $(1,1)$. We denote this square by $U_1$. In the first step we then divide it to sixteen equal closed squares with side lengths of $\frac{1}{4}$. Then we choose from these four squares in such a manner that we have chosen a square from every column and remove the rest. We denote the union of these squares by $U_2$. In the $n$th step we will again divide every square in the set $U_n$
to sixteen equal squares and again choose one from every column in the same way as in the first step and remove the rest. Next we will define the set $U$ as

$$U = \bigcap_{n=1}^{\infty} U_n$$

and our goal is to show that $1 \leq \mathcal{H}^1(U) \leq \sqrt{2}$, and thus that $\dim_H(U) = 1$.

For the upper bound it is enough to find some covering $\{U_i\}$ of set $U$ such that the diameters $|U_i|$ can be made arbitrary small and $\sum_{i=1}^{\infty} |U_i| \leq \sqrt{2}$. We will construct covering by the squares themselves. On arbitrary step we get $4^k$ squares with $\frac{1}{4^k}$ of length of the side, and hence the diameter of such square is $\frac{\sqrt{2}}{4^k}$ which clearly converges to zero as $k \to \infty$. From this we get that

$$\sum_{i=1}^{4^k} |U_i| = \sqrt{2}$$

for an arbitrary integer $k$. In order to obtain the lower bound we will look at the orthogonal projection of the set $U$ onto $x-axis$. For the orthogonal projection to $x-axis$ it holds that

$$|P(x) - P(y)| = \sqrt{(x_1 - y_1)^2} \leq \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} = |x - y|$$

Hence it is a Lipschitz mapping with the Lipschitz constant one and thus by the corollary 4.1 it follows that the dimension of the projection is smaller or equal to the dimension of the original set. Because we chose a square from every column we obtain that the projection must be the whole interval $[0,1]$. Hence $1 = \mathcal{H}^1([0,1]) \leq \mathcal{H}^1(U)$. From this it follows that the dimension of the set $U$ must be 1.

Although we succeeded to calculate the Hausdorff dimension of this set the problem is generally non-trivial. So there is no universal way to calculate the Hausdorff dimension for some arbitrary given set.

Next our goal is to find a way to calculate the Hausdorff dimensions of self-similar sets that fulfill certain conditions. For this we first need some more results concerning self-similarity and the Hausdorff dimension.

**Definition 4.7.** Open set condition

We say that the finite family of similarities $g_i$, where $i \in \{1, 2, ..., m\}$ satisfy the open set condition if there exists an open bounded set $U \neq \emptyset$, such that

$$\bigcup_{i=1}^{m} g_i(U) \subset U$$

where the union is pairwise disjoint.

For example in the case of the similarities $g_1, g_2$ that define the Cantor set the set $U$ can be chosen to be an open interval $(0,1)$, and we notice that $g_1((0,1)) \cup g_2((0,1)) \subset (0,1)$ and that $g_1((0,1)) \cap g_2((0,1)) = \emptyset$. Thus the family $\{g_1, g_2\}$ satisfies the open set condition.

**Proposition 4.8.**
Let \( \{U_i\} \) be a collection of disjoint open sets in \( \mathbb{R}^n \) such that each set \( U_i \) contains a ball of radius \( a_1r \) and is contained in a ball of radius \( a_2r \). Then each ball \( B \) of radius \( r \) intersects at most \( (1 + 2a_2)^n a_1^n \) of the closures of sets \( U_i \) in the family \( \{U_i\} \).

Proof: If set \( U_i \) meets \( B \) then \( U_i \) is contained in a ball of radius \( (1 + 2a_2)r \), which is centered at some point of the set \( B \). Suppose that there are \( q \) sets in the family \( \{U_i\} \) that intersect \( B \). Then by summing up the volumes of interior balls of radius \( a_1r \) we get \( q(a_1r)^n \leq (1 + 2a_2)^n r^n \Rightarrow q \leq (1 + 2a_2)^n a_1^{-n} \), giving desired inequality.

We call \( \mu \) a mass distributor if it is a Borel measure on \( \mathbb{R}^n \) such that \( 0 < \mu(\mathbb{R}^n) < \infty \). We notice that it can have a bounded support and that it can be scaled to be a probability measure on the set \( \mathbb{R}^n \). With this definition we give the following proposition.

**Proposition 4.9.**

Let \( \mu \) be a mass distributor and \( U \subset \mathbb{R}^n \) be a set for which it holds that \( \mu(U) > 0 \). Now suppose that there exist constants \( c > 0 \) and \( \delta > 0 \) such that \( \mu(V) \leq c |V|^s \) for all sets \( V \) that fulfill condition \( |V| \leq \delta \). Then \( \mathcal{H}^s(U) \geq \frac{\mu(U)}{c} \) and \( s \leq \dim_H(U) \).

Proof: Let \( \{U_i\} \) be any \( \gamma \)-cover of \( U \), where \( \gamma < \delta \). Then

\[
0 < \mu(U) \leq \sum_i \mu(U_i) \leq c \sum_i |U_i|^s
\]

so we get \( \mathcal{H}^s(U) \geq \frac{\mu(U)}{c} \) as \( \gamma \) is smaller than \( \delta \). Hence as \( \gamma \to 0 \), we have \( \mathcal{H}^s(U) \geq \frac{\mu(U)}{c} \).

To prove the second part we denote \( s_1 = \dim_H(U) \), and assume \( s > s_1 \). Then we have from the definition of the Hausdorff dimension that \( \mathcal{H}^s(U) = 0 \). But we know that \( \mathcal{H}^s(U) \geq \frac{\mu(U)}{c} \), where \( \mu(U) > 0 \). This gives a contradiction, and the proof is complete.

This result is known as the mass distribution principle and will be referred with that name later on in this paper.

The claim is also valid if we restrict to the balls of radius at most \( \delta \), and this information will be used in the next chapter. In this case we claim that if \( \mu \) is a probability measure on the set \( U \subset \mathbb{C} \) and that for some positive constants \( c, s \) and \( r_0 \) it holds that

\[
\mu(B(z, r)) \leq cr^s
\]

for all \( 0 < r < r_0 \) and all \( z \in U \), then \( \dim_H(U) \geq s \).

Proof: First we notice that since \( s \) is some positive constant every point \( z \) must have a measure zero. Since otherwise we would have that any ball containing that point would have measure bigger than some constant regardless of its radius. This is a contradiction to the assumptions since \( cr^s \to 0 \) as \( r \to 0 \).

Then take any \( \delta < r_0/2 \) and let sets \( \{U_i\} \) form an arbitrary \( \delta \)-cover of the set \( U \). Then for each set \( U_i \) that meets the set \( U \) we choose some point \( \gamma_j \) in the intersection and a corresponding disc \( B(\gamma_j, p_j) \), where \( p_j = |U_i| < r_0 \). Then \( B(\gamma_j, p_j) \prec r_0 \). Now the set \( U_i \subset B(\gamma_j, p_j) \), and hence \( U \subset \bigcup_{j=1}^{\infty} B(\gamma_j, p_j) \). So we obtain

\[
1 = \mu(U) \leq \sum_{j=1}^{\infty} \mu(B(\gamma_j, p_j)) \leq c \sum_{j=1}^{\infty} p_j^s = c \sum_{k=1}^{\infty} |U_j|^s
\]
From the above equation it follows that \( H_\delta^c(U) \geq c^{-1} \) for all \( \delta < r_o/2 \). And thus as \( \delta \to 0 \), we obtain that \( H_\delta^c(U) \geq c^{-1} \), and therefore \( \dim_b(U) \geq s \). This proves the claim.

Now let us define a space \( \gamma \) to be the collection of non-empty compact sets of the space \( \mathbb{R}^n \). We will use \( \gamma \) to denote this set until the end of this chapter. We then define the \( \delta \)-parallel body of a set \( A \in \gamma \) to be the set of points within the distance \( \delta \) from the set \( A \), rigorously defined as \( A_\delta = \{ x \in \mathbb{R}^n : |x - a| \leq \delta \text{ for some } a \in A \} \). With these notions we can proceed to give definition for the distance of subsets.

**Definition 4.10.** Distance between subsets

Let \( A \in \gamma \) and \( B \in \gamma \). We define the distance between the sets \( A \) and \( B \), denoted by \( d(A, B) \), to be

\[
d(A, B) = \inf\{ \delta : B \subset A_\delta \text{ and } A \subset B_\delta \}
\]

In the other words it is the least \( \delta \) such that \( r \)-parallel body of the set \( A \) contains the set \( B \) and other way around, for every \( r > \delta \). We will next show that this defines a metric in the space \( \gamma \).

It is clear that \( d : \gamma \times \gamma \to \mathbb{R} \) is a non-negative function. From the definition of \( d \) it follows trivially that \( d(A, B) = 0 \) if and only if \( A = B \) and that \( d(A, B) = d(B, A) \), where \( A, B \in \gamma \). Thus the only thing left to prove is the triangle inequality. Let us assume that \( A, B, C \in \gamma \). We need to show that \( d(A, B) \leq d(A, C) + d(C, B) \). To do this fix arbitrary numbers \( r > d(A, C) \) and \( s > d(C, B) \). Then it holds that \( C \subset A_r, A \subset C_r, C \subset B_s \) and \( B \subset C_s \).

Let \( x \in A \) be arbitrary, then it holds that \( d[x, C] < r \) and hence we can choose some \( z \in C \) such that \( d[x, z] < r \), where \( d[\cdot, \cdot] \) is the normal Euclidean distance. With this and the fact that \( z \in C \subset B_s \) we obtain that

\[
d[x, B] = \inf_{v \in B} d[x, v] \leq \inf_{v \in B} d[x, z] + d[z, v] = d[x, z] + d[z, B] < r + s
\]

Since \( x \in A \) was arbitrary this shows that \( A \subset B_{r+s} \). In a completely similar way we obtain that \( B \subset A_{r+s} \). This shows that \( d(A, B) \leq r + s \), and since \( r > d(A, C) \) and \( s > d(C, B) \) were arbitrary this shows that \( d(A, B) \leq d(A, C) + d(C, B) \). Hence we obtain that \( d(\cdot, \cdot) \) is metric in the space \( \gamma \).

**Proposition 4.11.**

Let \( g_i, i \in \{1, 2, \ldots, m\} \) be contractions on \( \mathbb{R}^n \). Then there exists a unique non-empty compact set \( U \) that is invariant for \( g_i \), meaning that

\[
U = \bigcup_{i=1}^m g_i(U)
\]

In addition we can define a transformation \( G \) on the space \( \gamma \) of non-empty compact sets by

\[
G(F) = \bigcup_{i=1}^m g_i(F)
\]

and write \( G^n \) for the \( n \)-th iterate of \( G \) and get that

\[
U = \bigcap_{n=1}^\infty G^n(F)
\]

for any set \( F \in \gamma \) that fulfill condition \( g_i(F) \subset F \) for all \( i \).

**Proof:** First we note that the sets in \( \gamma \) are transformed to another set in \( \gamma \) by the transformation \( G \) as all the \( g_i \) are contractions, and thus continuous. And from the basic results in topology we
know that compact sets are mapped to compact sets under continuous mappings. Next we note that there indeed exists a compact set \( F \) such that \( g_i(F) \subset F \). For example the closure of a big enough ball centered at the origin fulfills the condition.

Next let \( F \) be any set in \( \gamma \) such that \( g_i(F) \subset F \), for all \( i \). Then we have that \( G^n(F) \subset G^{n-1}(F) \). Thus \( G^n(F) \) forms a decreasing sequence of non-empty compact sets and therefore has a non-empty compact intersection \( U = \bigcap_{i=1}^m G^n(F) \). From this it follows that \( G(U) = U \), so \( U \) is invariant.

Next we will show that \( U \) is unique. If \( A, B \in \gamma \), then

\[
d(G(A),G(B)) = d\left( \bigcup_{i=1}^m g_i(A), \bigcup_{i=1}^m g_i(B) \right) \leq \max_{1 \leq i \leq m} d(g_i(A), g_i(B))
\]

Since if \( \delta \) is such that the \( \delta \)-parallel body \((g_i(A))_{\delta}\) contains \( g_i(B) \) for all \( i \), then \((\bigcup_{i=1}^m g_i(A))_{\delta}\) contains \( \bigcup_{i=1}^m g_i(B) \).

From this we get that \( d(G(A),G(B)) \leq \left( \max_{1 \leq i \leq m} c_i \right) d(A,B) \), where \( c_i \) is the contraction ratio of the contraction \( g_i \). If \( A \) and \( B \) are both invariant then \( G(A) = A \) and \( G(B) = B \). Thus we have that \( d(A,B) \leq \left( \max_{1 \leq i \leq m} c_i \right) d(A,B) \) and since it holds that \( c_i < 1 \) for all \( c_i \), only possibility is that \( d(A,B) = 0 \), meaning that the sets are same.

Now we will check the case when \( F \) is an arbitrary element in the set \( \gamma \). Then we have that

\[
d(G(F),U) = d(G(F),G(U)) \leq c d(F,U)
\]

where \( c = \max_{1 \leq i \leq m} c_i < 1 \), and hence

\[
d(G^n(F),U) \leq c^n d(F,U) \to 0
\]

when \( n \to \infty \) since \( d(F,U) \) is some constant and \( c < 1 \). From this we obtain that an arbitrary set \( F \in \gamma \) converges to the set \( U \) in the sense that \( d(G^n(F), U) \to 0 \) as \( n \to \infty \). We also know that mapping \( G^n \) maps set \( F \) to a set in \( \gamma \), for any integer \( n \), and thus \( G^n(F) \) converges to \( U \) as an element of \( \gamma \).

Now with these results we can proceed to calculate the Hausdorff dimension of self-similar sets that satisfy the open set condition.

**Proposition 4.12.**

Let \( g_i \) be a finite family of similarities with ratios \( c_i \), where \( 1 \leq i \leq m \), and assume that the open set condition holds. Let the set \( U \) be the unique non-empty compact and invariant set satisfying

\[
U = \bigcup_{i=1}^m g_i(U)
\]

and assume that \( s \) satisfies the condition

\[
\sum_{i=1}^m c_i^s = 1
\]

Then \( \dim_H(U) = s \) and for this value \( s \) it holds that \( 0 < \mathcal{H}^s(U) < \infty \).

**Proof:** Let \( s \) satisfy the assumptions above. For any set \( A \) we write \( A_{i_1,\ldots,i_k} = g_{i_1} \circ \cdots \circ g_{i_k}(A) \). Let \( J_k \) denote the set of all \( k \)-term sequences by \( 1 \leq i_j \leq m \). Then it follows from the assumptions that

\[
U = \bigcup_{J_k} U_{i_1,\ldots,i_k}
\]
We will now show that these covers of $U$ provide suitable upper bound for the Hausdorff measure. As the mapping $g_{i_1} \circ \cdots \circ g_{i_k}$ is a similarity of the ratio $c_{i_1} \cdots c_{i_k}$, we obtain that
\[
\sum_{k} |U_{i_1 \cdots i_k}|^q = \sum_{k} (c_{i_1} \cdots c_{i_k})^q |U|^q = \left( \sum_{i_1} c_{i_1}^q \right) \cdots \left( \sum_{i_k} c_{i_k}^q \right) |U|^q = |U|^q
\]
For any $\delta > 0$ we can choose $k$ such that $|U_{i_1 \cdots i_k}| \leq (\max, c_{i})^k \leq \delta$ and thus $H_\delta^q(U) \leq |U|^q$. From this we get $H_\delta^q(U) \leq |U|^q$, and we have found an upper bound. Notice that because the set $U$ was compact, it follows that $|U| < \infty$, and we have that for this $s$ it holds that $H_s^q(U) < \infty$.

For the lower bound let $I$ denote the set of all infinite sequences $I = \{(i_1, i_2, \ldots) : 1 \leq i_j \leq m\}$ and $I_{i_1 \cdots i_k} = \{(i_1, \ldots, i_k, q_{k+1}, \ldots) : 1 \leq q_j \leq m\}$ be the cylinder consisting of those sequences that have a starting sequence $(i_1, \ldots, i_k)$. We will now place a non-negative function $\mu$ on $U \cup \{\emptyset\}$ such that $\mu(I_{i_1 \cdots i_k}) = (c_{i_1} \cdots c_{i_k})^q$ and $\mu(\emptyset) = 0$. Since
\[
(c_{i_1} \cdots c_{i_k})^q = \sum_{i=1}^m (c_i \cdots c_k c_i)^q
\]
we obtain that
\[
\mu(I_{i_1 \cdots i_k}) = \sum_{i=1}^m \mu(I_{i_1 \cdots i_{k+1}})
\]
and we will extend $\mu$ to a measure on the $\sigma(I)$, which is the smallest $\sigma$-algebra that contains the set $I$, in the appendix after the theorem 6.6. Additionally it is easy to see that $\mu(I) = 1$. We will then transfer measure $\mu$ to a mass distribution $\tilde{\mu}$ on $U$ by defining
\[
\tilde{\mu}(A) = \mu\{(i_1, i_2, \ldots) : x_{i_1 i_2 \ldots} \in A\}
\]
for any Borel set $A$, where $x_{i_1 i_2 \ldots} = \cap_{k=1}^\infty U_{i_1 \cdots i_k}$, see discussion after the theorem 6.6 in the appendix. Clearly then from the definition of $\tilde{\mu}$ we get that $\tilde{\mu}(U) = \mu(I) = 1$.

Next we need to show that $\tilde{\mu}$ satisfies the conditions of the mass distribution principle, proposition 4.9. Let $V$ be an open set as in the definition 4.7. Since $\bigcup_{i=1}^m g_{i}(V) = G(V) \subset V$ the decreasing sequence of iterates $G^k(V)$ converges to $U$. Thus $U \subset V$ and $U_{i_1 \cdots i_k} \subset V_{i_1 \cdots i_k}$ for each finite sequence $(i_1, \ldots, i_k)$. Let $B$ be any ball of radius $r < 1$. We will estimate $\tilde{\mu}(B)$ with the sets $V_{i_1 \cdots i_k}$ that have diameters comparable with diameter of $B$ and with closures intersecting $U \cap B$.

We curtail each infinite sequence $(i_1, i_2, \ldots) \in I$ after the first term $i_k$ for which it holds that $(\min, c_i) r \leq c_i c_{i+1} \leq r$, and let $Q$ denote the finite set of all finite sequences obtained this way. Hence for all infinite sequences $(i_1, i_2, \ldots) \in I$ there is exactly one value of $k$ with $(i_1, \ldots, i_k) \in Q$. Since $V_{i_1}, V_m$ are disjoint it follows that $V_{i_1 \cdots i_k} \subset \bigcup_{i_1}^m V_{i_1 \cdots i_k}$. Using this in a nested way gives that the collection of open sets $\{V_{i_1 \cdots i_k} : (i_1, \ldots, i_k) \in Q\}$ is disjoint, and similarly $U \subset \bigcup_Q U_{i_1 \cdots i_k} \subset \bigcup_Q V_{i_1 \cdots i_k}$.

We choose $a_1$ and $a_2$ so that $V$ contains a ball of radius $a_1$ and is contained in a ball of radius $a_2$. Now for $(i_1, \ldots, i_k) \in Q$ the set $V_{i_1 \cdots i_k}$ contains a ball of radius $c_i \cdots c_k a_1$ and therefore one of radius $(\min, c_i) a_1 r$, and is contained in a ball of radius $c_i \cdots c_k a_2$ and hence in a ball of radius $a_2 r$. Let $Q_1$ denote those sequences $(i_1, \ldots, i_k)$ in $Q$ such that $B$ intersects $V_{i_1 \cdots i_k}$. From the proposition 4.8 giving covering result for balls we know that there are at most $q = (1 + 2a_2)^\gamma a_1^{-\gamma} (\min, c_i) - n$ sequences in $Q_1$. From this we get that
\[
\tilde{\mu}(B) = \tilde{\mu}(U \cap B) \leq \mu\{(i_1, i_2, \ldots) : x_{i_1 i_2 \ldots} \in U \cap B\} \leq \mu\{(i_1, \ldots, i_k) \in Q_1 : \}
\]

since if $x_{i_1 i_2 \ldots} \in U \cap B \subset \bigcup_Q V_{i_1 \cdots i_k}$, then there is an integer $k$ such that $(i_1, \ldots, i_k) \in Q_1$ and thus
\[
\tilde{\mu}(B) \leq \sum_{Q_1} \mu(I_{i_1 \cdots i_k}) = \sum_{Q_1} (c_i \cdots c_k)^q \leq \sum_{Q_1} r^q \leq r^q q
\]
since we have shown that \( c_1 \cdots c_k \leq r \). Since any set \( F \) is contained in a ball of radius \( |F| \) we have that \( \bar{\mu}(F) \leq |F|^q q \), so the mass distribution principle gives that \( \mathcal{H}^s(U) \geq q^{-1} > 0 \) and thus the \( \dim_{\mathcal{H}}(U) = s \). And we also note that for this value the condition \( 0 < \mathcal{H}^s(U) < \infty \) holds.

**Corollary 4.3.**

Let \( g_i \) be contractions for all \( i \), with the contraction ratios \( c_i \). Then it follows that \( \dim_{\mathcal{H}} U \leq s \), where \( s \) is given by the equation \( \sum_{i=1}^m c_i^s = 1 \).

This follows straight from the proof of proposition 4.12 as we note that instead of the equality \( d \)

Proof: Let \( d \) be a minimum distance between any pair of the compact disjoint sets \( g_1(U), \ldots, g_m(U) \), rigorously defined \( d = \min_{i \neq j} \inf \{|x-y|: x \in g_i(U), y \in g_j(U)\} \). Denote \( U_{i_1 \ldots i_k} = g_{i_1} \circ \cdots \circ g_{i_k}(U) \) and define \( \mu \) by \( \mu(U_{i_1 \ldots i_k}) = (b_{i_1} \cdots b_{i_k})^s \). Now since

\[
\sum_{i=1}^m \mu(U_{i_1 \ldots i_k}) = \sum_{i=1}^m (b_{i_1} \cdots b_{i_k} \cdot b_i)^s = (b_{i_1} \cdots b_{i_k})^s = \mu(U_{i_1 \ldots i_k}) = \mu \left( \bigcup_{i=1}^m U_{i_1 \ldots i_k} \right)
\]

we have that \( \mu \) defines mass distributor on \( U \) and \( \mu(U) = 1 \), as in the proof of proposition 4.12.

If \( x \in U \) there is unique infinite sequence \( i_1, i_2, \ldots \) such that \( x \in U_{i_1 \ldots i_k} \) for any \( k \). Let us choose numbers \( r \) and \( d \) such that \( 0 < r < d \) and let \( k \) be the least integer such that \( b_{i_1} \cdots b_{i_k} d \leq r < b_{i_1} \cdots b_{i_{k-1}} d \).

If \( i_1, \ldots, i_k \) is distinct from \( i_1, \ldots, i_k \), the sets \( U_{i_1 \ldots i_k} \) and \( U_{i_1 \ldots i_k} \) are disjoint and separated by a gap of width at least \( b_{i_1} \cdots b_{i_{k-1}} \cdot d > r \).

To see this we note that if \( j \) is the least integer such that \( i_j \neq i_j \), then \( U_{i_1 \ldots i_k} \subseteq U_{i_j} \) and \( U_{i_1 \ldots i_k} \subseteq U_{i_j} \) are separated by \( d \), so \( U_{i_1 \ldots i_k} \) and \( U_{i_1 \ldots i_k} \) are separated by at least \( b_{i_1} \cdots b_{i_{j-1}} d \). From this it follows that \( U \cap B_r(x) \subseteq U_{i_1 \ldots i_k} \) and thus

\[
\mu(U \cap B_r(x)) \leq \mu(U_{i_1 \ldots i_k}) = \mu(U_{i_j}) \leq d^{-s} r^s
\]
If $F$ intersects $U$ then $F \subset B_r(x)$ for some $x \in U$ with $r = |F|$. Therefore $\mu(F) \leq d^{-s} |F|^s$ and thus by the mass distribution principle we have that $\mathcal{H}^s(U) > 0$ and hence from the definition of the Hausdorff measure $\dim_H(U) \geq s$.

Now with the proposition 4.12 we can calculate the Hausdorff dimension of the Cantor set and the Koch curve.

**Example 4.2.**

We will start with the Cantor set, and remember that it is the unique self-similar set with respect to functions $g_1 = \frac{1}{3}x$ and $g_1 = \frac{1}{3}x + \frac{2}{3}$. Both are clearly similarities with the ratio $1/3$ and we know that the open set condition holds. Thus we can use the proposition 4.12. Hence we know that the Cantor set has the Hausdorff dimension $s$, where $s$ satisfies

$$\sum_{i=1}^{2} c_i^s = 1 \iff 2 \cdot \frac{1}{3^s} = 1$$

And hence we get that

$$s = \frac{\log 2}{\log 3} \approx 0.63$$

If we compare this to the usual topological dimension of the Cantor set, which is zero as it is totally disconnected, we notice that the Hausdorff dimension is strictly greater than the topological dimension, which was one of fractal’s usual properties.

For the Koch curve we have the four similarities discussed in examples of self-similarity 4.2, and we remember that the contraction ratio was $\frac{1}{3}$ for all of them. For using proposition 4.12 the open set condition must hold, and we check this by choosing the set $U$ which is the interior of the triangle with vertices on points $(0,0), (1,0)$ and $\left(\frac{1}{2}, \frac{1}{2\sqrt{3}}\right)$ and notice that the open set condition holds for the set $U$. Thus we can use proposition 4.12 and calculate in the similar manner as above that the Hausdorff dimension $s$ for the Koch curve is

$$\sum_{i=1}^{4} \frac{1}{3^s} = 1 \iff s = \frac{\log 4}{\log 3} \approx 1.26$$

As we again compare dimensions we notice that the Koch curve has also greater Hausdorff dimension than the topological dimension, just like the Cantor set.

In the same manner it is easy to calculate the Hausdorff dimension of any strictly self-similar set as long as we know the similarities that produce it and that it fulfills the open set condition.
5 The dynamics of the Julia set

In this section we will discuss more about the Julia sets as fractals and what the critical points of a polynomial $f$ can tell us about structure of the Julia set. We will also define the Mandelbrot set and examine its relation to the Julia sets of quadratic polynomials. Then we will proceed to give an estimate for the Hausdorff dimension of the Julia set. This chapter is based mostly on [AS] and [B], additionally I have used [CG] and [N]. Just like in the section three, in this section $f$ denotes always a polynomial such that $\deg f \geq 2$.

**Theorem 5.1.**

Let $f$ be a polynomial. Then the following things are equivalent.

1. $F_\infty$ is simply connected
2. $J(f)$ is connected
3. There are no finite critical points of $f$ in $F_\infty$

**Proof:** We have already shown that since $F_\infty$ is completely invariant 1 and 2 are equivalent by the theorem 3.14.

We will next show that 1 implies 3. Assume that $F_\infty$ is simply connected. Applying the Riemann-Hurwitz relation, theorem 3.16 to $f$ from $F_\infty$ to itself we have that $1 + \delta(F_\infty) = d$ and thus $\delta(F_\infty \setminus \{\infty\}) = 0$, where $\delta(F_\infty)$ is a deficiency, defined in the definition 2.5 and $d = \deg f$. This follows as we know that $f$ has a deficiency $d - 1$ at the $\infty$, and thus there cannot be any finite critical points in the component $F_\infty$.

To complete the proof we will show that 3 implies 1. We assume that there are no finite critical points in $F_\infty$. As $F_\infty$ is completely invariant with respect to $f$ it follows that $f^n$ has no finite critical points on $F_\infty$. This can be seen straight from the chain rule for the derivative and the fact that if $z$ is a critical point for $f^n$, then $(f^n)'(z) = 0$. Now we can find a disc $D$ centered at $\infty$, using the spherical metric 2.1, such that $f(D) \subset D \subset F_\infty$. We define $D_0 = D$ and $D_n = f^{-n}(D)$ and get that each $D_n$ is a domain containing $\infty$ and satisfying $D = D_0 \subset D_1 \subset ...$.

Applying the Riemann-Hurwitz relation 3.16 to the map $f^n$ of $D_n$ onto $D$ and noticing, that there are no finite critical points of $f^n$ in $F_\infty$, we get that

$$\chi(D_n) + (d^n - 1) = \chi(D_n) + \delta f^n(D_n) = d^n \chi(D)$$

As $D$ is simply connected we have that $\chi(D_n) = 1$ and hence $D_n$ is simply connected. As $F_\infty$ is the union of increasing sequence $D_n$ of simply connected sets that share a common point $\infty$, it must be simply connected.

As we remember that $F_\infty$ is completely invariant and consist of all points converging to infinity, we get that $F_\infty$ is simply connected if all critical points have bounded forward orbits.

**Corollary 5.1.**

For the quadratic polynomials of the form $f(z) = z^2 + c$, the Julia set is connected if and only if the forward orbit of the origin is bounded.

**Proof:** For a quadratic polynomial $f(z) = z^2 + c$ the origin is the only finite critical point. Thus by the above theorem the Julia set is connected if and only if the origin does not belong to the set $F_\infty$, which proves the claim.
Next we will briefly introduce and define the Mandelbrot set, which is in very close connection to the Julia sets of quadratic polynomials of form \( f(z) = z^2 + c \) and contains a lot of information about them. We are interested only in polynomials of form \( f(z) = z^2 + c \), because every quadratic polynomial \( g(z) = az^2 + 2bz + d \) can be conjugated, definition 3.1, to a polynomial of this form. This can be seen choosing \( c = ad + b - b^2 \) and using the Möbius mapping \( h(z) = az + b \) to obtain \( g(z) = h^{-1} \circ f \circ h(z) \), which proves that \( f \) and \( g \) are conjugated. We will now prove that for these polynomials, the Julia sets of \( f \) and \( g \) are connected in the following way

\[
J(g) = h^{-1}(J(f))
\]

Proof: Take an arbitrary \( z \in \mathbb{C} \) such that \( h(z) \in J(f) \). Then observe that \( g^n(z) = h^{-1} \circ f^n \circ h(z) = h^{-1} \circ f^n(x) \) where \( x = h(z) \in J(f) \). Hence \( f^n(x) \) stays bounded as \( n \to \infty \) and thus also \( g^n(z) \) stays bounded as \( n \to \infty \). Next we will show that in every neighborhood of such point there are points that converge to infinity. This is clear since \( h \) maps any neighborhood of \( z \) to some neighborhood of \( x \) and since the Julia set \( J(f) \) is the boundary of the \( F_{\infty}(f) \), we know that there are such points \( x' \) in any neighborhood of \( x \) that \( f^n(x') \) converge to infinity. And thus also \( h^{-1} \circ f^n(x') \) converges to infinity as \( n \to \infty \), hence \( h^{-1}(J(f)) \subset J(g) \). Then if we take any \( z_1 \in \mathbb{C} \) such that \( h(z_1) \notin J(f) \) we obtain that \( \{f^n(z)\} \) form equicontinuous family in some neighborhood of the point \( h(z_1) \). Hence we know that also the family \( \{h^{-1} \circ f^n(z)\} \) form equicontinuous family in this neighborhood since \( h^{-1} \) is a polynomial and hence satisfies the Lipschitz condition 2.1. Thus \( z_1 \in F(g) \), which proves the claim.

**Definition 5.1.** The Mandelbrot set

Let \( f \) be a quadratic polynomial of form \( f(z) = z^2 + c \). We define that those complex numbers \( c \) belong to the Mandelbrot set, for which the Julia set of \( f(z) = z^2 + c \) is connected. Due to the corollary 5.1, we can equivalently state that \( c \) belongs to the Mandelbrot set if \( \{f^n(0)\} \) forms a bounded sequence.

![Figure 3: The Mandelbrot set, picture courtesy of Wikipedia](image)

The Mandelbrot set has fractal like properties, as can be seen from the picture. We will next show that the Mandelbrot set lies inside the closed disc \( \overline{B}(0,2) \).
Let us assume \(|x| > 2\). We start by studying the iteration of the origin under a polynomial
\( f(z) = z^2 + x \), and we immediately notice that \( f(0) = x \).
We will show that for every \( n \geq 1 \) it holds that \( |f^{n+1}(0)| - |f^n(0)| \geq \varepsilon \), where \( \varepsilon = |x| - 2 > 0 \) is a constant. First we check that this holds for \( n = 1 \). In this case we have that
\[
|f^2(0)| - |f(0)| = |x^2 + x| - |x| \geq |x^2| - 2 |x| = (2 + \varepsilon) |x| - 2 |x| \geq \varepsilon
\]

So the claim holds for \( n = 1 \). Now assume that claim holds for all \( k < n \). Then we have that
\[
|f^n(0)| \geq p \geq |x| + \varepsilon > 2. \quad \text{Next we calculate}
\]
\[
|f^{n+1}(0)| - |f^n(0)| \geq |(f^n(0))^2 + x| - p \geq p^2 - |x| - p \geq 2p - p - |x| \geq \varepsilon
\]

Thus we have that the claim holds for an arbitrary \( n \). So in every iteration the modulus of \( f^n(0) \) increases at minimum with the constant \( \varepsilon > 0 \). Therefore it is clear that \( f^n(0) \to \infty \) as \( n \to \infty \) and \( x \) does not belong to the Mandelbrot set.

We presented earlier in the theorem 3.6 an example of the polynomial’s \( f(z) = z^2 - 2 \) Julia set and noticed that it was connected, and thus the above result is the best possible. We also could have noticed, that for the polynomial \( f(z) = z^2 - 2 \) the point 2 is in fact a fixed point, and hence the family \( \{f^n(0)\} \) is indeed bounded, since \( f^2(0) = 2 \). Hence the point \( c = -2 \) belongs to the Mandelbrot set.

We will next show that if all critical points converge to infinity under iterations of \( f \) we have that the Julia set is a Cantor-like set, also called the Cantor dust, meaning that it is totally disconnected. As we know the quadratic polynomials have only one critical point, so for them it holds, that either the Julia set is connected or it is totally disconnected.

**Theorem 5.2.**

Let \( f \) be a polynomial. If all critical points of \( f \) converge to infinity under iterations of \( f \) the Julia set is a Cantor dust.

Proof: Let us assume that all the critical points converge to infinity, so all critical points lie in the component \( F_\infty \). We have previously shown that \( F_\infty \) is connected and the Julia set is the boundary of \( F_\infty \).

We know that as \( J \) is a bounded set we can find a bounded disc \( D' \) such that \( J \subset D' \) and \( \partial D' \subset F_\infty \). Let us denote the set of critical points by \( C \), and the forward orbit of the set of critical points by \( C^+ \). By our assumptions every critical point converges to infinity and thus there are only finitely many points from the set \( C^+ \) in the disc \( D' \). Next we will construct a Jordan curve that separates the Julia set from the set \( C^+ \) such that the exterior part consist of points converging to infinity. We do this by connecting every point of the set \( C^+ \) that lies in \( D' \) to the boundary of \( D' \) with pairwise disjoint simple arcs \( \tau_i \) that lie in the \( F_\infty \). This can be done since \( F_\infty \) is connected and open, and thus path-connected. Now we have two disjoint compact sets, the Julia set and the union of the boundary of \( D' \) and all the arcs \( \tau_i \), which is path-connected. Additionally the Julia set lies inside the disc \( D' \). Hence we can separate these two sets with a Jordan curve \( \gamma \) and since the set \( \partial D' \cup ( \cup_{i=1}^n \tau_i ) \) is a compact subset of the open set \( F_\infty \), we can choose the \( \gamma \) such that the exterior points belong to the set \( F_\infty \). We will denote the set of points that are enclosed by \( \gamma \) by \( U \), the closure of the set \( U \) by \( M = U \cup \gamma \) and the exterior of the set \( U \) by \( W \). Clearly \( W \cup \gamma \) is a compact subset of the component \( F_\infty \) and thus for some integer \( n \) it holds that
\[
f^n(W \cup \gamma) \subset W
\]

and therefore also
\[
f^{-n}(M) \subset U
\]
We additionally notice that all the critical values of \( f^n \) are in the set \( C^+ \), and we see that \( f^n \) has no critical values in the closure of the set \( U \), where the critical value is just the value of a critical point under function \( f^n \). This follows since every critical point of \( f^n \) is in the set \( \{ C, f^{-1}(C), \ldots, f^{-(n-1)}(C) \} \), and thus \( f^n \) maps such a point to the set \( C^+ \).

We have proven that the Julia set is the same for \( f \) and \( f^n \) with any integer \( n \) in the proposition 3.7. And hence we can rename \( f^n \) as \( f \), and set that \( d = \deg f \). With this we obtain

1. \( f(W \cup \gamma) \subset W \)
2. \( f^{-1}(M) \subset U \)
3. \( J \subset U \)

Now \( f \) is an analytic map of each component of \( f^{-1}(U) \) onto \( U \). This follows as \( f^{-1}(U) \) is defined locally analytically on every point \( z \in U \), since there are no critical points in the set \( U \) and as set \( U \) is simply connected, the monodromy theorem 2.10 gives that there exist a unique analytic continuation of \( f^{-1} \) on the set \( U \). And as \( U \) is simply connected, the restriction of \( f \) to each such component is a homeomorphism of that component onto \( U \). Hence we can define branches \( S_1, \ldots, S_d \) of \( f^{-1} \) on \( M \) and the sets \( (S_1(M), \ldots, S_d(M)) \) are pairwise disjoint compact subsets of \( M \).

We will use these sets to show that \( J \) is a Cantor dust.

For each sequence \( (i_1, \ldots, i_n) \), where each \( i_j \in \{1, 2, \ldots, d\} \), we will define that

\[ M(i_1, \ldots, i_n) = S_{i_1} \circ \cdots \circ S_{i_n}(M) \]

We notice that \( M(i_1, \ldots, i_n, i_{n+1}) \subset M(i_1, \ldots, i_n) \) and that for every fixed \( n \) sets \( M(i_1, \ldots, i_n) \) are pairwise disjoint, and that there are \( d^n \) sets in such a family. The sets \( M(i_1, \ldots, i_n) \) also fulfill the condition

\[ \bigcup_{i_1, \ldots, i_n} M(i_1, \ldots, i_n) = f^{-n}(M) \]

Now we will write

\[ M_\infty = \cap_{n=0}^{\infty} f^{-n}(M) \]

We notice that \( M_\infty \) is compact, as it is an intersection of compact sets, and non-empty, as it is an intersection of nested non-empty sets.

Next we will show that in fact \( J \subset M_\infty \) and that \( M_\infty \) is totally disconnected, which will complete the proof.

By the definition of the set \( M \) the Julia set belongs to it. Hence as the Julia set is completely invariant by the theorem 3.2, the Julia set must belong to \( M_\infty \), since all points that do not belong to \( M_\infty \) are mapped outside \( M \) with some iteration of the function \( f \).

So all that remains is to show that \( M_\infty \) is totally disconnected. For this, we will study an arbitrary component \( M' \) of \( M_\infty \). We remember that the family of sets \( M(i_1, \ldots, i_n) \) is pairwise disjoint and thus \( M' \) can belong to only one set in that family. Hence there is a sequence \( j_n \) such that

\[ M' = \cap_{n=0}^{\infty} M(j_1, \ldots, j_n) \]

Now only thing that remains to be proven is that \( | M(j_1, \ldots, j_n) | \to 0 \) as \( n \to \infty \), showing that diameter of every component converges to zero. For proving this we need a few results.

**Lemma 5.1.**

If a family \( \{ S_n \} \) is such that each \( S_n \) is a single-valued analytic branch of some \( f^{-m} \) in a domain \( D \), then \( \{ S_n \} \) is normal in \( D \).

Proof: Let \( A \) and \( B \) be disjoint cycles of \( f \), each containing at least three points. Existence of such cycles follows easily from the facts that there is only finitely many cycles with less that three elements, that the Julia set is perfect, corollary 3.1 and that there is a periodic point in every neighborhood of any point from the Julia set, theorem 3.11. Now \( \{ S_n \} \) cannot map any point \( z \in D \setminus A \)
to $A$, since then for some $m$, $z = f^m \circ S_n(z) \in A$. So the family $\{S_n\}$ must be normal in $D \setminus A$. The same argument holds for $B$, and thus $\{S_n\}$ is normal in $D \setminus B$, and hence it is normal in their union, which is $D$, as $A$ and $B$ were disjoint.

Therefore we have obtained that in the proof of main theorem the family $\{S_n\}$ is normal in $U$ since every $S_m$ is some branch of $f^{-m}$.

**Lemma 5.2.**

Let the domain $D$ and the family $\{S_n\}$ be as in the previous lemma. Then if $D$ meets the Julia set any locally uniform limit $\phi$ of a subsequence of $\{S_n\}$ is a constant.

Proof: For each $n$, let $m(n)$ be such that $S_n$ is a branch of $(f^{m(n)})^{-1}$ in $D$. Then suppose that $S_n \to \phi$ locally uniformly in $D$ as $n \to \infty$ in some infinite set $N_1$ of positive integers. We will aim to show a contradiction if the analytic function $\phi$ is not a constant.

Assume $\phi$ is not a constant. Then each $S_n$ is univalent in $D$, since $S_n$ has a left inverse $f^{m(n)}$. As $\phi$ is not a constant, the Hurwitz’s theorem 2.8 implies that $\phi$ is univalent in $D$.

Next take any $z \in D \cap J$ and draw a small circle $\gamma$ about $z$, which with its interior lies in $D$. As $S_n \to \phi$ uniformly on $\gamma$ it follows from the argument principle, theorem 2.7 that for all sufficiently large $n \in N_1$, $S_n(D)$ contains some neighborhood $W$ of $\phi(z)$. Now we can use theorem 3.9. Since as $S_n(z) \to \phi(z)$, $\phi(z)$ is also in $J$, since every $S_n(z)$ is in $J$ due to the complete invariance of the Julia set. Thus there exists an integer $k_0$ such that for all $k \geq k_0$, we have $J \subset f^k(W)$.

Hence we can take $n$ in $N_2$ such that $m(n) \geq k_0$ and obtain that $J \subset f^{m(n)}(W) \subset f^{m(n)}(S_n(D)) = D$

Finally, apply the above to any subdomain $D_0 \subset D$ that does intersect, but does not contain the Julia set. We would have $J \subset D_0$, which is false by our assumption on $D_0$. Thus we see that $\phi$ is constant in $D_0$, and hence also in $D$, since $\phi$ is analytic function. This proves the lemma 5.2.

With these results we can move forward proving the original statement of theorem 5.2. From the above lemmas it follows that any locally uniform limit of a subsequence of $S_{j_1}, S_{j_1} \circ S_{j_2}, ...$ is a constant, and the limit exists by the normality of $\{S_n\}$. So we have obtained that this sequence converges locally uniformly to some point $x$ on $M$, and that for a sufficiently large $n$, the sets $M(j_1, ..., j_n)$ lie in some preassigned arbitrary small neighborhood of $x$. This shows that the diameter of an arbitrary component of the set $M_n$ converges to zero, and thus completes the proof.

In this case we have proven that the Julia set is regarded as a fractal, since it does satisfy points 1, 2 and 4 in our list of typical properties for fractals, given on the page 26. For the point 1, note that since all critical points converge to infinity, the Julia set is hyperbolic. We will later in the theorem 5.7 prove that its Hausdorff dimension is greater than zero, and thus it also fulfills the point 3 on the list of typical properties for fractals.

**Corollary 5.2.**

The quadratic polynomials have only one finite critical point and thus it either converges to infinity under iterations of $f$ or stays bounded. Therefore the Julia set of such a polynomial is always either connected or totally disconnected.

Next we will study when the Julia set of a polynomial $f$ is a Jordan curve. For this we will
need some results.

First result we need is the topological result known as the converse of the Jordan curve theorem, which we will state in a generality sufficient to our needs and omit the proof. For interested readers the proof and a closer look at this theorem can be found for example in [N], page 166.

We will also need the notion of accessibility, which is present in this theorem. We say that every point of the set $A$ being accessible from the set $B$ means that for every point $z \in A$ there exists a simple curve $\gamma$ with one end point being $z$ and all the other points belonging to $B$.

**Theorem 5.3.**

Let $A$ be a bounded and closed set such that $\mathbb{C} \setminus A$ consist of two open components $B$ and $C$ from each of which it is at every point accessible. Then the set $A$ is a Jordan curve.

This result is known as the converse of the Jordan curve theorem.

Now we will apply this to the Julia sets. Let $f$ have two completely invariant components of the Fatou set, denoted by $F_1$ and $F_2$, both containing either an attractive or a super-attractive fixed point. In this case they are both simply connected, by the same reasoning as in proof of the theorem 3.18.

As the $F_1$ and $F_2$ are both completely invariant, the Julia set is their boundary, which is proven in the theorem 3.14. These are the only components of the Fatou set, which follows from the Sullivan’s theorem as noted in the end of the theorem 3.18.

Thus we can apply the converse of a Jordan curve theorem if we know that the Julia set is accessible from both components $F_1$ and $F_2$, since then the conditions of converse of the Jordan curve theorem are met with respect to the Julia set. Hence we have that the Julia set is a Jordan curve. We will next study when this holds for a quadratic polynomial of form $f(z) = z^2 + c$.

For this we will not use the no wandering domain theorem, but we require some additional results.

First we notice that the Julia set of a quadratic polynomial of form $f(z) = z^2 + c$ is centrally symmetric. This follows as the points $z$ and $-z$ are both mapped to a same point by a polynomial $f(z) = z^2 + c$ and thus they either both converge to infinity or stay bounded under iterations of a polynomial $f$. As the Julia set is the boundary of the component $F_\infty$ we see from this that the Julia set is indeed centrally symmetrical.

Additionally we will need the notion of expanding maps.

**Definition 5.2.** Expanding maps

We say that $f$ is expanding on the Julia set if there exists an integer $m$ such that

$$| (f^m)'(z) | > 1$$

holds for every point $z \in J$.

Next we will prove the following lemma that will give a connection between the expansive maps and the Julia sets for some polynomials.

**Lemma 5.3.**
Let $f$ be a polynomial and $C^+$ be the forward orbit of the set of critical points. If it holds that $J \cap C^+ = \emptyset$, then $f$ is expanding on the Julia set.

Proof: Since $J \cap C^+ = \emptyset$ we know that we can find $\delta_z > 0$ for every point $z \in J$ such that $B(z, \delta_z) \cap C^+ = \emptyset$. Thus in this disc we know that for every integer $n$ the $f^{-n}$ has well defined analytic branches $S_n$ in $B(z, \delta_z)$, due to the theorems 2.1 and 2.10. Additionally we notice that $B(z, \delta_z)$ intersects the Julia set, since $z \in J$. Hence we can use the lemmas 5.1 and 5.2 to obtain that the family $\{S_n\}$ is normal and $S_n|_{B(z,\delta_z)} \to a$ when $n \to \infty$, where $a$ is some constant and $\{S_n\}$ any converging subsequence of the family $\{S_n\}$.

Now we claim that for the constant $\frac{\delta_z}{2}$ the following is true. For every $0 < \varepsilon < 1$ it holds that $|S_n'(x)| < \varepsilon$ for all $x \in B(z, \frac{\delta_z}{2})$ and for every choice of $S_n$, where $n > n_0(z)$, and $n_0(z)$ is an integer that depends on $z$.

To prove this, let $\{S_k\}$ be an arbitrary sequence from the family $\{S_n\}$. Due to normality we would have some subsequence $\{S_{k_i}\}$ that converges in the disc $B(z, \delta_{z_i})$, and we know that the convergence must be towards a constant function and it must be locally uniform. Thus the derivatives of $\{S_{k_i}\}$ should converge locally uniformly towards the derivative of the limit function due to the Cauchy integration theorem. But since the derivative of the constant function is zero in the whole closed disc $\tilde{B}(z, \frac{\delta_z}{2})$, we see that $|S_{k_i}'(x)| < \varepsilon$ for all sufficiently big $k_i$. This now gives the claim since if it did not hold, we could choose a sequence $\{S_{m_i}\}$ where $|S_{m_i}'(x_i)| \geq \varepsilon$ for every positive integer $i$ and for some $x_i \in B(z, \frac{\delta_z}{2})$. But this would be a contradiction to the result that every sequence has a subsequence that converges uniformly to a constant function, as sequence $\{S_{m_i}\}$ clearly does not have a subsequence that would converge locally uniformly to a constant function on the disc $\tilde{B}(z, \frac{\delta_z}{2})$.

Next choose any $\varepsilon < 1$. We notice that $f^n \circ S_n(x)$ is identical mapping for every choice of $S_n$. Hence using the chain rule and assuming $x \in B(z, \frac{\delta_z}{2})$ and $n > n_0(z)$, we have that

$$1 = |(f^n)'(S_n(x))| \cdot |S_n'(x)| \leq |(f^n)'(S_n(x))| \cdot \varepsilon$$

From this we obtain that $|(f^n)'(w)| \geq \frac{1}{\varepsilon}$ for every $w \in f^{-n}(B(z, \frac{\delta_z}{2}))$ and all $n > n_0(z)$. And since we chose $\varepsilon < 1$ we have that $|(f^n)'(w)| > 1$. The discs $B(z, \frac{\delta_z}{2})$ provide an open cover for the Julia set, and since the Julia set is compact, we know that there exists a finite subcover. So we obtain that $J \subset \bigcup_{i=1}^{k} B(z_i, \frac{\delta_{z_i}}{2})$. Then, due to the complete invariance of the Julia set, we know that $J \subset \bigcup_{i=1}^{k} f^{-m}(B(z_i, \frac{\delta_{z_i}}{2}))$ for every integer $m$.

From this it follows that $|(f^n)'(z)| > 1$ for every $z \in J$ and for every integer $n > \max_{1 \leq i \leq k \leq 1} n_0(z_i)$, which completes the proof.

**Theorem 5.4.**

For quadratic polynomials of the form $f(z) = z^2 + c$ the Julia set is a Jordan curve, if there exist a finite attracting fixed point.

Proof: For the polynomials of form $f(z) = z^2 + c$ the point $\infty$ is always a super attractive point
and the component $F_\infty$ is always completely invariant. We will next try to find another completely
invariant component with attracting or super-attracting fixed point.

Let us denote the derivative in a fixed point by $\lambda$. So $\lambda = f'(z) = 2z$, where $z$ is a fixed point. 
We will now find points in the Mandelbrot set for which it holds that there exist a finite fixed attracting point.
For fixed points it holds that 
\[ c = \frac{2z}{2} \cdot (1 - \frac{2z}{2}) = z(1-z) = z - z^2 \iff z^2 + c = z \]
As we are interested in finding points $c$, for which fixed points are attractive, we have the condition 
$|\lambda| < 1$. And thus we have that the set of points that satisfy this condition is 
\[ C_M = \left\{ \frac{\lambda}{2} \cdot (1 - \frac{\lambda}{2}) : |\lambda| < 1 \right\} \]
This set is a subset of the Mandelbrot set, since every polynomial $f(z) = z^2 + c$, where $c$ is from 
this set, has a finite attractive fixed point, and hence the Fatou set must have a finite simply 
connected component that contains the finite attractive fixed point. Let us denote this component by $F_c$. 
Since the Julia set contains the boundary of the component $F_c$, the Julia set cannot be totally 
disconnected. Thus we obtain from the corollary 5.2 that the Julia set must be connected. The set $C_M$ is 
called the main cardioid of the Mandelbrot set.

Since the Julia set is connected for polynomials $f(z) = z^2 + c$, where $c$ is from the main cardioid, the component $F_\infty$ is simply connected and as always completely invariant. We notice that the 
component $F_c$ coincides with the immediate basin of attraction of the finite attractive fixed point.
We will next show that this component is completely invariant with respect to $f$.
In the theorem 3.17 we proved that the immediate basin of attraction of the attractive point must hold a critical point. In addition we know that it is a forward invariant set, since it holds a fixed point.
We can connect the finite attractive fixed point to the critical point, which for this $f$ is the 
origin, by a curve $\gamma$ for which it holds that $\gamma \in F_c$, since every component of the Fatou set is open and connected, and thus path-connected. For the backward invariance we first note that if $F_c$ is not a backward invariant set, then the branch of $f^{-1}$ that does not fix the fixed point, let us denote it by $S_2$, would map the component $F_c$ to some other component of the Fatou set. Thus it is enough to find one point from the component $F_c$ such that $S_2$ maps it to the component $F_c$ for proving that $F_c$ is backwards invariant.

Now we will use the central symmetry of the Julia set and notice that the component $F_c$ must be 
central symmetric, since it contains the origin and its boundary consists of points that belong to 
the Julia set. Additionally we calculate that $f^{-1}(0) = \{ \pm \sqrt{-c} \}$ and see that branches $S_1$ and $S_2$ map the origin to a central symmetric points, where $S_1$ is the branch that fixes the finite attractive 
fixed point. It follows from the definition of $S_1$ that $S_1(F_c) \subset F_c$, so especially $S_1(\gamma) \subset F_c$. 
Additionally $S_1(\gamma)$ is a curve that connects the finite fixed point to the point $S_1(0)$. Thus the curve $\gamma' = \gamma \cup S_1(\gamma)$ is contained in the component $F_c$ and connects the origin to the point $S_1(0)$. Taking 
the mirrored curve of $\gamma'$ with respect to the origin we obtain the curve $\gamma''$ which is contained 
in the component $F_c$ due the central symmetry of $F_c$ and connects the origin to the point $S_2(0)$ 
due to the central symmetry of the points $S_1(0)$ and $S_2(0)$. Now if we take the curve $\gamma' \cup \gamma''$ we obtain a curve that is contained in the component $F_c$ and connects the points $S_1(0)$ and $S_2(0)$. 
This shows that the branch $S_2$ maps the origin to a point in the component $F_c$, which proves the claim.

Next we need to show that the Julia set is accessible from the both components $F_c$ and $F_\infty$. For a 
polynomial of form $f(z) = z^2 + c$, where $c$ is from the main cardioid, we know that the origin is the 
only finite critical point, and that it converges towards the finite attracting fixed point. Hence 
$J \cap \overline{C_T} = \emptyset$ and the conditions of the lemma 5.3 are satisfied. So we know that there exists such 
integer $m$ that $|(f^m)'(z)| > 1$ for every $z \in J$. And since the Julia set is compact, we obtain that
\[ |(f^m)'(z)| \geq k > 1 \] for all \( z \in J \). Since the Julia set is same for the polynomials \( f \) and \( f^m \), we can simplify the notation and denote \( f^m = f \) and assume that \( |f'(z)| \geq k > 1 \) for all \( z \in J \). From this it follows that we can find a \( \delta > 0 \) such that the condition \( |f'(x)| \geq k > 1 \) holds for all \( x \in J_\delta \), where \( J_\delta = \{ x \in \mathbb{C} : |x - z| < \delta \) for some \( z \in J \}. Clearly \( J_\delta \) is a neighborhood of the Julia set, and next we will show that the preimages of every point in it contract at some specific rate towards the Julia set. First note that for an arbitrary point \( y \in J_\delta \) we can find a point \( z \in J \) such that \( y \in B(z, \delta) \). Additionally, since the Julia set is completely invariant, it holds that \( f^{-1}(z_i) \subset J \) for every \( z_i \in J \). Then since the condition \( |f'(x)| \geq k > 1 \) holds for every point in the set \( J_\delta \) it follows that every branch of \( f^{-1} \) contracts any ball \( B(z_i, r) \subset J_\delta \), \( z_i \in J \) in the following way, \( S_i(B(z_i, r)) \subset B(S_i(z_i), cr) \) where \( c = \frac{1}{k} < 1 \) is the ratio of contraction and a constant. Thus the ball \( B(z, \delta) \) gets mapped inside a ball \( B(S_n(z), c^n \delta) \) for every \( n \) and for every branch \( S_n \in f^{-n} \), which shows that the preimages of any point \( y \in J_\delta \) converges towards the Julia set with some specific rate. From the previous computation it also follows that \( f^{-1}(J_\delta) \subset J_\delta \).

We will now show that every point in the Julia set is accessible from the component \( F_m \), and the accessibility from the component \( F \), can be verified in a similar manner.

Choose an arbitrary point \( x \in F_m \cap J_\delta \), and notice that then \( f^{-n}(x) \to J \) uniformly. Next denote the length of an arbitrary curve \( \theta \subset F_m \cap J_\delta \) with \( |\theta| \). Choose a constant \( A \), such that we can connect the point \( x \) to any of its preimages \( f^{-1}(x) \) with a simple curve which length is at most \( A \) and which is contained in the set \( F_m \cap J_\delta \). Let us then define a family of curves \( \sigma_n \), with \( \sigma_n = \{ S_{i_n} \circ \cdots \circ S_{i_1}(\rho_j) \} \), where \( j \in \{ 1, 2 \} \), \( S_1, S_2 \) are the branches of \( f^{-1} \) and \( \rho_1, \rho_2 \) are simple curves between the points \( x, S_1(x) \) and \( x, S_2(x) \) respectively such that they both lie in the set \( F_m \cap J_\delta \) and \( A \geq |\rho_j| \). Then we claim that the length of an arbitrary representative \( \sigma_n \) from the family \( \sigma_n \) is uniformly bounded. Now since all branches of \( f^{-1} \) are contractions in the set \( J_\delta \) we know that
\[
|S_n(\theta)| \leq c |\theta|,
\]
where \( c < 1 \) is a ratio of contraction and is a constant. Then it follows that
\[
|\sigma_n| = \sum_{n=0}^{\infty} |S_{i_n} \circ \cdots \circ S_{i_1}(\rho_j)| \leq A \sum_{n=0}^{\infty} c^n \leq C \cdot A
\]

Where the final step comes from the fact that the geometric series converge to some constant \( C \) when \( c < 1 \). Hence the length of an arbitrary representative \( \sigma_n \) of the family \( \sigma_n \) is uniformly bounded.

Next we will show that for every \( z \in J \) there exists some sequence of branches of \( f^{-1} \) such that
\[
\lim_{n \to \infty} S_{i_n} \circ \cdots \circ S_{i_1}(x) = z.
\]
Choose an arbitrary \( z \in J \) and a sequence \( \{ \epsilon_n \}_{n=0}^{\infty} \), defined with \( \epsilon_0 = 1 \) and \( \epsilon_{n+1} = \frac{\epsilon_n}{2} \). We know that \( x \) is not an exceptional point, since \( f'(x) \neq 0 \). So using the proposition 3.6 we obtain that there exists an integer \( k \) such that \( f^{-k}(x) \) intersects \( B(z, \epsilon_1) \), meaning that \( x_1 = S_{i_k} \cdots \circ S_{i_1}(x) \in B(z, \epsilon_1) \). Now if \( x_1 \in B(z, \epsilon_1) \), we choose \( x_2 = x_1 \) and study if \( x_2 \in B(z, \epsilon_2) \). And if \( x_1 \notin B(z, \epsilon_1) \) we obtain again using the proposition 3.6 that \( f^{-k_l}(x_1) \) intersects \( B(z, \epsilon_1) \) for some integer \( k_l \). Then we obtain that \( x_2 = S_{i_{k+1}} \circ \cdots \circ S_{i_1} \circ S_{i_1} \circ \cdots \circ S_{i_1}(x) \in B(z, \epsilon_1) \). Continuing like this we obtain a sequence of points \( \{ x_j \}_{j=1}^{\infty} \) such that \( x_j \to z \), since \( \epsilon_n \to 0 \). Thus we have branches \( S_i \) of \( f^{-1} \) such that \( S_i \circ \cdots \circ S_{i_1}(x) \) accumulates to the point \( z \) as \( n \to \infty \).

But since an arbitrary curve from the family \( \sigma_n \) had a finite length we know that any such curve must converge towards some point. Thus the curve \( \sigma_n \), that accumulates towards the point \( z \) must also converge towards it, since if a curve converges towards some point it can accumulate only to that point. Hence we can construct a curve \( \sigma_n \), which has a starting point \( x \) and which converges towards a pre-defined \( z \in J \). From the construction it follows that \( \sigma_n \subset F_m \), since the component \( F_m \) is completely invariant. So as we take the closure of the \( \sigma_n \), we obtain a curve with a finite length, which one endpoint is the point \( z \) and all the other points belong to the component \( F_m \). This curve might intersect itself and hence not be simple, but we can make a simple curve out of it by cutting out all loops. This can be done with the following construction.

First we notice that if we have a simple curve \( \phi \in F_m \cap J_\delta \), then also the curve \( S_i(\phi) \) where
$i \in \{1, 2\}$ must be a simple curve, since $S_i$ is a homeomorphism. Then with this result we start the actual construction of a simple curve. Start from the simple curve $\rho_i$ and construct the curve $\rho_i \cup S_i(\rho_i)$. If this curve is simple we proceed to study the curve $\rho_i \cup S_i(\rho_i) \cup S_i(\rho_i)$. If the curve $\rho_i \cup S_i(\rho_i)$ is not simple the curve $S_i(\rho_i)$ must intersect the curve $\rho_i$ at least two times. This follows since if the simple curves $\rho_i$ and $S_i(\rho_i)$ would intersect only at the point $S_i(x)$ the curve $\rho_i \cup S_i(\rho_i)$ would be simple. Next choose the last point at which the curve $S_i(\rho_i)$ intersects the curve $\rho_i$ and denote it by $b$. Denote the part of the curve $\rho_i$ that starts from the point $x$ and ends to the point $b$ by $\bar{\gamma}_i$ and the part of the curve $S_i(\rho_i)$ that starts from the point $b$ and ends at the endpoint of the curve $S_i(\rho_i)$ by $\bar{S}_i(\rho_i)$. Then construct a curve $\rho_i \cup \bar{S}_i(\rho_i)$. The curve we obtain this way is simple, since it is a union of two simple curves that intersect only at the point $b$, which is the end point of the curve $\rho_i$ and the starting point of the curve $\bar{S}_i(\rho_i)$. Additionally it is shorter than the curve $\rho_i \cup S_i(\rho_i)$ and it has the same starting point and ending point as the curve $\rho_i \cup S_i(\rho_i)$. Next we proceed to study the curve $\rho_i \cup \bar{S}_i(\rho_i) \cup S_i(\rho_i)$.

By iterating this construction we obtain a simple curve with finite length that converges towards the point $z$ and lies in the set $F_w$. So taking the closure of this curve we obtain a simple curve which one endpoint is $z$ and all the other points lie in the set $F_w$. Thus the Julia set is accessible from the component $F_w$ and the proof for the component $F_c$ is similar.

Therefore we can apply the converse of the Jordan curve theorem, theorem 5.3 and obtain that the Julia set must be a Jordan curve as soon as we have proven that there are no other components in the Fatou set. For this we will use the following lemma.

**Lemma 5.4.**

Let us assume that $C^+ \cap J = \emptyset$. Then it holds that

$$F(f) = \cup_{\gamma} A(\gamma)$$

Where $\gamma$ is the set of attractive periodic cycles and $A(\gamma)$ are their basins of attraction.

**Proof:** Let $F_i$ be a component of the Fatou set. Then it follows that $\partial F_i \subset J$. Additionally we know that there exists a subsequence $\{n_k\}$ such that $f^{n_k}|_{F_i} \to g$ uniformly on compact subsets of $F_i$ due to normality and $g$ is analytic since every $f^{n_k}$ is analytic. Assume first that $g$ is not a constant function. Now $g(F_i)$ is a domain since $g$ is analytic and thus an open mapping. Additionally we claim that $\partial g(F_i) \subset J$. This follows since if $z \in \partial g(F_i)$, then $f^{n_k}(F_i) \cap B(z, \varepsilon) \neq \emptyset$ for every $n_k > n(\varepsilon)$. Now if we assume that $z \in F$ then there exist some $\varepsilon > 0$ such that $B(z, \varepsilon) \subset F$, and thus $B(z, \varepsilon) \subset f^{n_k}(F_i)$ since $B(z, \varepsilon)$ cannot belong to more than one component of the Fatou set. But from this it follows that $B(z, \varepsilon) \subset g(F_i)$, which is a contradiction and our claim holds.

We assumed that $C^+ \cap J = \emptyset$ and hence if $z \in \partial g(F_i)$ we can find a disc $D$ centered at the point $z$ such that $D \cap C^+ = \emptyset$ and $D \cap J \neq \emptyset$. Now as we assumed that $g$ is not a constant function and it holds that $f^{n_k}|_{F_i} \to g$, we have that $f^{-n_k} \circ g(z) \to h(z)$ where $h(z)$ is identical mapping in compact subsets of $g^{-1}(D) \cap F_i$. This gives that $f^{-n_k}|_{D}$ does not converge towards constant, which is a contradiction to the lemma 5.2. So we know that $g$ must be a constant function.

Assume then that $g$ is a constant function $g \equiv a$, where $a \in J$. Due to the assumptions we can choose an $\varepsilon > 0$ such that $B(a, \varepsilon) \cap C^+ = \emptyset$. Thus by the lemma 5.2 we have that $|f^{-n}(B(a, \varepsilon))| \to 0$. Now let $K \subset F_i$ be compact. Then it holds for all sufficiently big $n_k$ that $f^{n_k}(K) \subset B(a, \varepsilon) \Rightarrow K \subset f^{-n_k}(B(a, \varepsilon))$, which is a contradiction since $f^{-n_k}(B(a, \varepsilon))$ converges towards a point in the Julia set and $K$ is a compact subset of the Fatou set.

The above gives that $g \equiv a$ and $a \in F$. Let $W$ be the component of the $F$ that contains the point $a$. Then $f^{n_k}(F_i) = W$ for all $k \geq k_0$ since they are both components of the Fatou set and their
intersection is nonempty when $k_0$ is large enough.

Denote $m = n_{k_0+1} - n_{k_0}$. Then it holds that $f^m(W) = f^{n_{k_0+1} - n_{k_0}}(F_i) = f^{n_{k_0+1}}(F_i) = W$ and $f^{n_{k_0} - n_{k_0}}|_W \to a$ uniformly on compact subsets of $W$. Next we denote $n_k - n_{k_0} = m_j k - s$ where $0 \leq s < m$, and notice that then $f^{m_j}|_W \to a$ uniformly on a compact subset of $W$. Thus $f^m(a) = f^{m_j}(\lim_{k \to \infty} f^{m_j}(a)) = \lim_{k \to \infty} f^{m_j}(f^m(a)) = a$, since $f^m(a) \in W$. Hence $a$ is $m$-periodic.

We can choose $\varepsilon > 0$ such that $B(a, \varepsilon) \subseteq W$. Hence $f^{m_j}|_{B(a, \varepsilon)} \to a$, and we note that $| (f^{m_j})'(a) | = | (f^m)'(a) |^{|h|} \to 0$. This gives that $| (f^m)'(a) | < 1$ and shows that $a$ is $m$-periodic attractive point. Now it holds that $f^m|_{F_i} \to a$ and thus $F_i \subset A(\gamma)$ where $\gamma = \{a, f(a), \ldots, f^{m-1}(a)\}$, which concludes the proof of the lemma.

As our polynomial $f(z) = z^2 + c$ where $c$ is from the main cardioid satisfies the conditions of the previous lemma we know that the Fatou set consist of attractive basins, $F(f) = \bigcup_{\gamma} A(\gamma)$. On the other hand from the corollary 3.5 we get an upper bound $\deg f - 1$ for the number of finite attractive cycles. So in this case we can have only one finite attractive cycle and we already know that it is just the attractive fixed point, which we will denote by $z_c$. Additionally we have shown that the component $F_c$ containing the point $z_c$ is completely invariant and thus the component $F_c$ coincides with the basin of attraction of the attracting fixed point $z_c$. The same holds for the component $F_{\infty}$ and the attracting fixed point $\infty$. Hence there can be no other components in the Fatou set.

Therefore we can apply the converse of the Jordan curve theorem, theorem 5.3, since the complement of the Julia set consist of two components $F_{\infty}$ and $F_c$ which both contain an attractive fixed point, and the Julia set is accessible from both of them. Thus the Julia set is a Jordan curve for a polynomial $f(z) = z^2 + c$ when $c$ is from the main cardioid.

We have obtained that there are three possibilities for the Julia set of a quadratic polynomial.

1. The Julia set is a Jordan curve
2. The Julia set is connected, but not a Jordan curve
3. The Julia set is a Cantor dust

Our very first example of the Julia set was of form 1, since the Julia set of the polynomial $f(z) = z^2$ is just the boundary of the unit disc and it is clearly a Jordan curve. But clearly this Julia set is not a fractal, so next we show a picture of the Julia set that is a Jordan curve and a fractal, in a sense that we have described in this paper. In the picture the interior of the Julia set and the Julia set itself are both colored with black to make the picture clearer, so the Julia set is just the boundary of the set colored with black. This set is called the filled-in-Julia set.
Figure 4: Filled-in-Julia set of $f(z) = z^2 - 0.6$, which I generated using the following program www.easyfractalgenerator.com

We will illustrate the second possibility next. When the Julia set is connected but not a Jordan curve it can be either intersecting itself or forking. We will give an illustration on the both cases.

Figure 5: The Julia set of the polynomial $f(z) = z^2 + i$, picture taken from http://personal.maths.surrey.ac.uk/st/H.Bruin/res.html.

This is the forking case. On the next page we have the intersecting case.
Figure 6: The Julia set of the polynomial \( f(z) \approx z^2 - 0.1226 + 0.7449i \), called the Douady rabbit. Picture taken from http://en.wikipedia.org/wiki/File: Douady rabbit.png

The last possibility is that the Julia set is a Cantor dust, example given next. In this case the picture is not as clear as in the previous cases as it is hard to illustrate a Cantor set. We stress the fact that this set is totally disconnected, even if it would look like there are some small connected discs is the picture.

Figure 7: Julia set of the polynomial \( f(z) = z^2 + 0.285 + 0.01i \), picture taken from http://en.wikipedia.org/wiki/File:Julia0.2850.01.png

We have covered every possible form of the Julia set of quadratic polynomials and it is one of the types illustrated above. For higher degree polynomials there are more possibilities as they have
more finite critical points and therefore we can have situations where some critical point stays bounded under iterations and some other does not. We will now give an example of such behavior.

**Example 5.1.**

We will study the structure of the Julia set of the polynomial $f(z) = z^2 - \frac{z^3}{9}$. First we will note that $f(z)$ has critical points at 0, 6 and $\infty$. Additionally it has super-attracting fixed points at 0 and $\infty$, which are lying in the components $F_0$ and $F_∞$. These components $F_0$ and $F_∞$ must be separate, since no component can contain two attractive fixed points. We will prove that

1. $F_∞$ contains the domain $A = \{z : |z| > 10\}$
2. $F_0$ contains the disc $D = \{z : |z| < \frac{9}{10}\}$
3. $F_∞$ contains the circle $C = \{z : |z| = 6\}$

For the first part, if $|z| > 10$, then

$$\left|\frac{f(z)}{z}\right| = |z| \cdot \left(\frac{z}{9}\right)^{-1} \geq \frac{|z|}{9} \geq \frac{10}{9}$$

And thus the first part holds, since every point $z$, satisfying condition $|z| > 10$ converges to infinity under iterations of $f$.

For the second part we notice in a similar manner that if $|z| \leq \frac{9}{10}$, we obtain that

$$\left|\frac{f(z)}{z}\right| \leq |z| \cdot \left(1 + \frac{|z|}{9}\right) \leq \frac{99}{100}$$

and thus all such points converge to 0 and belong to the component $F_0$.

For the third point we notice that if $|z| = 6$, then we have that

$$|f(z)| \geq |z|^2 \cdot \left(1 - \frac{|z|}{9}\right) = 12$$

And hence we know from the part one that $C$ lies in the completely invariant component $F_∞$.

Next we notice that as 9 is mapped onto 0 with $f$, we have that $9 \notin F_∞$. And thus as $F_∞$ contains $C$ but not the point 9 it cannot be simply connected. Therefore by the theorem 3.14 it must be infinitely connected. We also know that the Fatou set has at least three components as the component containing 9 is separated from $F_0$ by the set $C$ which belongs to $F_∞$. And thus by the theorem 3.19 we see that it has infinitely many components which are all simply connected by the theorem 3.14, and all components, except $F_∞$ are bounded.

Hence the Julia set must have infinitely many components but it is not a Cantor set, as for example the boundary of the simply connected component $F_0$ belongs to it.

Next we will prove some more results about the Julia and the Fatou sets, aiming to the theorem which would let us approximate the Hausdorff dimension of the Julia sets.
Theorem 5.5.

Let $f$ be a polynomial and $E$ be a compact subset of the complex plane such that for every $z \in F(f)$ the sequence $f^n(z)$ does not accumulate at any point in $E$. Furthermore let $U$ be any open set containing $J$. Then we have that $f^{-n}(E) \subset U$ for all sufficiently large integers $n$.

Proof: We will prove the claim by a contradiction, so assume for the moment that the claim is false. Thus we would have some open set $U$ which contains $J$, and additionally we have some sequence $n_1, n_2, \ldots$ such that $z_{n_i}$ belongs to $f^{-n_i}(E)$ but not to $U$. Without the loss of generality the points $z_{n_i}$ converge to some point $w$, since for all positive integers $n$ it holds that $f^{-n}(E) \subset D$ where $D$ is some closed disc centered at the origin. From this it follows that all points $z_{n_i}$ belong to the closed and bounded set $D \setminus U$ and thus they have a converging subsequence, which we can choose to replace $n_1, n_2, \ldots$ if needed. Hence as $U$ is open $w \notin U$, and clearly $w \in F$ as the Julia set was contained in the set $U$.

Now as the family $\{f^n\}$ is equicontinuous on every point $z \in F$ we can choose an arbitrary $\varepsilon > 0$ and there exists a positive $\delta$ such that for all points $z$ satisfying $d(z, w) < \delta$ it holds that $d(f^n(z), f^n(w)) < \varepsilon$, for all integers $n$. Thus for the large enough $n_i$ it holds that

$$d(f^{n_i}(z_{n_i}), f^{n_i}(w)) < \varepsilon$$

and that $f^{n_i}(z_{n_i}) \in E$. We claim that the sequence $f^{n_i}(w)$ accumulates to some point in $E$, which would be a contradiction, and thus the claim would hold. This can be seen in the following way. We know that only finitely many points $f^{n_i}(w)$ do not belong to the set $\{x : d(x, E) \leq 1\}$, which we can obtain by choosing $\varepsilon = 1$. This set is closed and bounded subset of the complex plane, so there exists an accumulation point. Next we will show that this accumulation point must belong to the set $E$. Let us assume that a point $k$ is an accumulation point and $k \notin E$. Then $d(E, k) = s$ must be positive since the sets $E$ and $k$ are compact and disjoint. But now we can choose $\varepsilon = \frac{s}{2}$ and we see that $f^{n_i}(w) \in \{x : d(x, E) \leq \frac{s}{2}\}$ for all $n_i > m$ where $m$ is some constant. Hence there is at most finite number of points from the sequence $f^{n_i}(w)$ in the disc $B(k, \frac{s}{2})$ which is a contradiction. This shows that there exists an accumulation point of the sequence $\{f^{n_i}(w)\}$ that belongs to the set $E$, which is a contradiction to assumptions of the theorem and thus proves the claim.

Theorem 5.6.

There exists a point $z \in J$ such that $|f'(z)| > 1$ if every point in the Fatou set converges towards an attractive cycle.

Proof: Let us assume that the claim is false. Then for every $z \in J$ it holds that $|f'(z)| \leq 1$. We will first check the case when there exists a point $z_1 \in J$ such that $|f'(z_1)| < 1$. Now there exists some neighborhood $U$ of $z_1$ such that $|f'(x)| < 1$ for all $x \in U$. By making $U$ smaller we can assume that there are no points from the attractive cycles in the set $U$, since there is only finitely many attractive cycles as noted after the proof of the corollary 3.5 and all attractive cycles lie in the Fatou set, due to the corollary 3.3. We know that in this neighborhood there exists a periodic point, by the theorem 3.11. This periodic point can not belong to the Fatou set since by our assumption every point in the Fatou set converges towards some attractive cycle. Thus this periodic point is in the Julia set. Let us denote it by $z_2$ and its period by $n$. But now as we calculate the derivative of $(f^n)'(z_2)$ and remember that by our assumption $|f'(z)| \leq 1$ for all $z \in J$ we obtain

$$| (f^n)'(z_2) | = | \prod_{k=0}^{k=n-1} f'(f^k(z_2)) | < 1$$
This is a contradiction since as mentioned above all attractive cycles must lie in the Fatou set. Then assume that | f'(z) | = 1 for every z ∈ J. Then especially there are no critical points in the Julia set. And since every point in the Fatou set converges towards some attractive cycle we know that J \cap \mathbb{C}^+ = \emptyset. Thus f is expanding on the Julia set and the claim holds.

Next we notice that as f is a polynomial the | f'(z) | attains its finite maximum on the Julia set, as the Julia set is compact and f'(z) is continuous. Therefore we can define

\[ K_0 = \max \{|f'(z)| : z \in J\} \]

From the theorem 5.6 we see that K_0 > 1 when every point in the Fatou set converges to attractive cycle. To show that K_0 > 1 for an arbitrary polynomial f we refer to the corollary 3.2 from which it follows that the Julia set contains a repelling cycle. Now with this definition we will proceed to the next theorem where we will give an estimate for the Hausdorff dimension of the Julia set. In the proof we will use the following notation. We will denote the normal Euclidean distance of given sets A, B by d[A,B] and the cardinality of a set A by ||A||.

**Theorem 5.7.**

Let f be as usual a polynomial of degree d, where d is at least two. Then we have that

\[ \dim_H(J) \geq \frac{\log d}{\log K_0} \]

Proof: We will denote the \( \delta \)-parallel body, defined just before the definition 4.10, of J by \( J_\delta \), and denote

\[ K(\delta) = \sup \{|f'(z)| : z \in J_\delta\} \]

We see immediately that 1 < K_0 < K(\delta) < \infty, and K(\delta) → K_0 as \( \delta \to 0 \). We will prove that

\[ \dim_H(J) \geq \frac{\log d}{\log K(\delta)} \]

From which the claim follows as \( \delta \to 0 \). To simplify notation we will denote \( K(\delta) \) by K as \( \delta \) will not vary while we prove this inequality. Additionally we can assume that \( \delta < 1 \).

We will first find a point a such that for every n the set \( f^{-n}(a) \) has exactly \( d^n \) elements and that as \( n \to \infty \) the set \( f^{-n}(a) \) converges to the Julia set.

To find such a point we first note that as there are only finitely many critical points the set \( \bigcup_{n=1}^{\infty} f^n(C) \), where C is the set of critical points, is countable. Thus we can choose some finite point a in the set \( F_\infty \), which does not have any critical points in its backward orbit. Hence for this point there are exactly \( d^n \) elements in the set \( f^{-n}(a) \) for every integer n. It follows from the theorem 5.5 that the backward orbit of such point does converge towards the Julia set since we know that in the set \( F_\infty \setminus \{\infty\} \) every point converges to \( \infty \), and thus the forward orbit of any point in the Fatou set does not accumulate to a. Hence we have found a point a with the desired properties.

So we obtain that there is such an integer m that \( f^{-m}(a) \subset J_\delta \) when \( n > m \). We may now replace a with some point in \( f^{-m}(a) \) for some integer \( n > m \) and relabeling the chosen point as a we get

\[ \bigcup_{n=0}^{\infty} f^{-n}(a) \subset J_\delta \]

Next we will introduce some notations used in the proof.
1. \( f^{-n}(a) = \{ a_n(i) : i = 1, \ldots, d^n \} \)

2. \( d_n = d[J, f^{-n}(a)] \) and thus \( d_0 = d[J, a] \)

3. \( \mu_n \) is a uniform probability measure on \( f^{-n}(a) \). So the \( \mu_n \) places a measure of \( d^{-n} \) on every \( a_n(i) \), hence for every set \( E \) the following holds \( \mu_n(E) = d^{-n} \cdot || \{ i : a_n(i) \in E \} || \)

We know that \( d_n < \delta \) and theorem 5.5 implies that \( d_n \to 0 \) as \( n \to \infty \).

Next we will prove a lemma needed for proving the theorem.

**Lemma 5.5.**

With the notation of the theorem 5.7 it holds that \( d_n \leq K d_{n+1} \), and thus \( d_0 \leq K d_1 \leq K^2 d_2 \leq \ldots \)

Proof: Let \( z \in J \) and \( a_{n+1}(i) \) be arbitrary. Then one of the following holds

1. \( \delta \leq |z - a_{n+1}(i)| \)
2. \( |z - a_{n+1}(i)| < \delta \)

If 1 holds, we remember that \( K > 1 \) and \( d_n < \delta \), and obtain that

\[
d_n < \delta < K\delta \leq K |z - a_{n+1}(i)|
\]

If 2 holds, then the linear segment from \( z \) to \( a_{n+1}(i) \) lies entirely in the \( J_\delta \). Hence \( |f'(x)| \leq K \) on this segment. As \( f(z) \) and \( f(a_{n+1}(i)) \) are in \( J \) and \( f^{-n}(a) \) respectively we get

\[
d_n \leq |f(z) - f(a_{n+1}(i))| \leq K |z - a_{n+1}(i)|
\]

Thus we have in both cases that \( d_n \leq K |z - a_{n+1}(i)| \). By letting \( z \) and \( a_{n+1}(i) \) vary we obtain the claim, and thus the lemma is proven.

With this lemma we proceed with the proof of the main theorem. We will first prove it in the case that \( J \) does not contain critical points and afterwards take a look at what changes need to be done in order to get the universal case proved. Everything done this far remains valid without this assumption.

This assumption gives us that for each \( z \in J \) there exists a positive \( r_z \) such that \( f \) is univalent in the disc \( B(z, r_z) \), since \( f \) is analytic and its derivative is non-zero on the point \( z \). Now the discs \( B(z, r_z) \) form an open cover for the compact set \( J \). So by the Lebesgue’s covering theorem, theorem 2.9, there exists a positive \( \theta \) such that \( f \) is univalent on each disc \( B(z, \theta) \), where \( z \in J \). We may degrease the \( \theta \) if we wish, so we can obtain a \( \theta \) that is smaller than \( \delta \). Additionally we can choose such a positive integer \( q \) that the following holds

\[
\frac{d_0}{K^q} < \theta < \delta
\]

Next we will prove another lemma needed for the proof, which proves that points \( f^{-n}(a) \) are distributed in some sense evenly around the Julia set.

**Lemma 5.6.**
For all \( z \in J \), and all integers \( n, m \) satisfying \( n \geq 1, m \geq q \), there are at most \( d^{n+q-m} \) points from the set \( f^{-n}(a) \) in a disc \( B\left(z, \frac{d_0}{K^m}\right) \). Thus the following equation holds

\[
\| B\left(z, \frac{d_0}{K^m}\right) \cap f^{-n}(a) \| \leq d^{n+q-m}
\]

Proof: We shall prove this for all \( z \in J \), and for all \( n \geq 1 \) by an induction over \( m \), where \( m \geq q \) as in the assumptions. First we check that claim holds for all \( z \in J \) and all integers \( n \geq 1 \) in the case that \( m = q \). In this case the right side of equation is \( d^n \). But we know from the assumptions regarding \( a \) that in the set \( f^{-n}(a) \) there are \( d^n \) elements. Thus the inequality holds, as there can be at most as many elements in the intersection as there is in one of the sets in intersection.

Now we will assume that claim holds for all integers \( m \), where \( q \leq m \leq M \) and prove that for all \( z \in J \) and for all \( n \geq 1 \) it holds that

\[
\| B\left(z, \frac{d_0}{K^{m+1}}\right) \cap f^{-n}(a) \| \leq d^{n+q-(M+1)}
\]

We will first suppose that \( n = 1 \). Then we obtain from the lemma 5.5, and from the fact \( K > 1 \), that the following holds

\[
\frac{d_0}{K^{m+1}} < \frac{d_0}{K} < d_1 = d[J, f^{-1}(a)]
\]

This means that in this case none of the points \( a_1(i) \) are within the distance of \( \frac{d_0}{K^{m+1}} \) of \( J \). Thus the left-hand side of inequality in the claim is zero and hence the claim holds.

Next we will assume that \( n \geq 2 \), and especially \( n-1 \geq 1 \). Then it holds that

\[
\frac{d_0}{K^{m+1}} < \frac{d_0}{K^q} < \theta < \delta
\]

Thus \( f \) is univalent on the disc \( B\left(z, \frac{d_0}{K^{m+1}}\right) \) and \( |f'(x)| \leq K \) for all \( x \in B\left(z, \frac{d_0}{K^{m+1}}\right) \). From this we obtain that \( f \) is univalent map of \( B\left(z, \frac{d_0}{K^{m+1}}\right) \) into the disc \( B\left(f(z), \frac{d_0}{K^q}\right) \). And hence the distinct points of \( f^{-n}(a) \) in \( B\left(z, \frac{d_0}{K^{m+1}}\right) \) are mapped to distinct points of \( f^{-(n-1)}(a) \) in the disc \( B\left(f(z), \frac{d_0}{K^q}\right) \). From this we get using the induction step, the fact that \( n-1 \geq 1 \) and the complete invariance of the Julia set the following inequality

\[
\| B\left(z, \frac{d_0}{K^{m+1}}\right) \cap f^{-n}(a) \| \leq \| B\left(f(z), \frac{d_0}{K^q}\right) \cap f^{-(n-1)}(a) \| \leq d^{n-1+q-M} = d^{n+q-(M+1)}
\]

Hence we have proven our lemma for every \( n \) and \( m \).

Now we will continue with the proof of our main theorem. Rewriting the lemma 5.6 in terms of the probability measures \( \mu_n \) defined earlier in the proof we get

\[
\mu_n\left(B\left(z, \frac{d_0}{K^m}\right)\right) \leq \frac{d^n}{d^n}
\]

We will next aim at modifying this inequality in such a way that we can use the mass distribution principle, proposition 4.9. For that we need to be able to give an upper bound for the Hausdorff
measure on small discs.

If \(0 < r < \frac{d_0}{K}\), then there is some positive integer \(m\) such that \(\frac{d_0}{K^{m+1}} < r \leq \frac{d_0}{K^m}\). Now denote \(t = \frac{\log d}{\log K}\) and notice that for this choice of \(t\) it holds that \(K^t = d\). With this we can proceed and obtain

\[
\mu_n(B(z, r)) \leq \mu_n(B\left(z, \frac{d_0}{K^m}\right)) \leq \frac{d^n}{K^m} \leq d^n \left(\frac{rK}{d_0}\right)^t = \left(\frac{d^{q+1}}{d_0}\right)^t r^t.
\]

We will then use the theorem 6.4, which states that we have some subsequence of the \(\{\mu_n\}\) that converges weakly to some probability measure \(\mu\) supported on \(J\), in such a way that the above condition is inherited by \(\mu\).

With this result we obtain that for all \(z\) in \(J\) and all \(r\) in \((0, \frac{d_0}{K})\), we have the following

\[
\mu(B(z, r)) \leq \left(\frac{d^{q+1}}{d_0}\right)^t r^t.
\]

And this with the proposition 4.9 shows that \(\dim_H(J) \geq t = \frac{\log d}{\log K}\), since \(\left(\frac{d^{q+1}}{d_0}\right)\) is just a constant, and the above inequality holds for all discs that have a positive, but small enough radius and which are centered at any point \(z \in J\).

This proves the claim in the case that there are no critical points in the Julia set. Next we will try to get rid of this assumption.

Now assume that \(f\) has critical points in the Julia set. Let us denote the set of critical points that lie in the Julia set by \(C_J = \{c_1, \ldots, c_k\}\). As these are critical points of the function \(f\), its derivative is zero at these points. Thus any iterate \(f^n(c_i)\) has also derivative zero in these points. Hence no iterate of \(f\) can fix any of these points, since if it would, then that point would be a super-attracting fixed point and thus lie in the Fatou set by the theorem 3.12.

Therefore for every \(i\) it holds that there exists an integer \(n_1\) such that \(f^n(c_i)\) lies outside \(C_J\) when \(n > n_1\). This can be seen from the fact that as set \(C_J\) is finite and none of the iterates of a function \(f\) can fix any point in it, the iteration of any \(c_i\) can return to \(C_J\) only finitely many times. And from this it follows that we can choose \(n_1\) to be that \(n\) for which the iteration \(f^n(c_i)\) returns to the set \(C_J\) for the last time, and thus for all \(n > n_1\) the \(f^n(c_i)\) lies outside the set \(C_J\).

This can be done for all finitely many points in the set \(C_J\), and hence we can choose the maximum of such \(n_1\) and obtain a constant \(p\) such that \(f^p(C_J)\) is disjoint from \(C_J\) when \(n \geq p\). We will now fix \(p\) to be any integer satisfying this condition.

Next we will choose any positive \(\rho\) satisfying conditions

1. The discs \(B(c_j, 2\rho)\) are disjoint
2. \(|(f^p)'(y)| \leq 1\) on each disc \(B(c_j, 2\rho)\)
3. \(4\rho < d[f^p(C_J), C_J]\)

Next we will define

\[
H = J \setminus \bigcup_{j=1}^k B(c_j, \rho)
\]

From the definition of the set \(H\) it follows that it is a compact subset of \(J\) and it does not intersect \(C_J\). Now we can obtain in a same manner that we used in an earlier part of the proof, using the
Lebesgue’s covering theorem 2.9, that there is a positive number \( \theta \) such that \( f \) is univalent on a disc \( B(z, \theta) \) when \( z \in H \).

Next we will define \( q \) modifying it from what it was in a previous part of the proof. We will choose \( q \) that satisfy \( q > p \) and

\[
\frac{d_0}{K^q} < \min\{\theta, \rho, \delta\}
\]

We will next show that with these modifications the lemma 5.6 remains valid and the rest of the proof follows as in the previous case. Thus the whole proof will hold.

Proof for lemma 5.6 with modifications: We will again use an induction on \( m \) for all \( m \geq q \), and as before the inequality which we are proving is

\[
\| B\left(z, \frac{d_0}{K^m}\right) \cap f^{-n}(a) \| \leq d^{m+q-m}
\]

and it holds when \( m = q \) for the same reason as in the previous proof of this result.

Next we assume that the inequality holds for all \( z \in J \), for all \( n \geq 1 \), and for all \( m \) that satisfy \( q \leq m \leq M \). Then we prove that the inequality holds for \( M + 1 \), thus we have to show that

\[
\| B\left(z, \frac{d_0}{K^{M+1}}\right) \cap f^{-n}(a) \| \leq d^{M+q-(M+1)}
\]

First we suppose that \( 1 \leq n \leq M + 1 \). Hence by the lemma 5.5 we have that

\[
\frac{d_0}{K^{M+1}} \leq \frac{d_0}{K^n} \leq d_n
\]

And from this we get that

\[
\| B\left(z, \frac{d_0}{K^{M+1}}\right) \cap f^{-n}(a) \| \leq \| B(z, d_n) \cap f^{-n}(a) \| = 0
\]

Where the last equality comes from the definition of \( d_n \). So the claim holds for these integers \( n \).

Hence we may suppose that \( n > M + 1 > q > p \). Now if \( z \in H \) the inequality follows from the induction hypothesis just as in the previous proof of this lemma.

This means that we must concentrate on the points \( z \in \bigcup_{j=1}^k B(c_j, \rho) \). As these discs are pairwise disjoint the point \( z \) belongs only to one of them, and we denote that disc by \( B(c_1, \rho) \). Next we notice that

\[
\frac{d_0}{K^{M+1}} \leq \frac{d_0}{K^q} < \rho
\]

and remember that by the assumptions on \( p \) we have that \( | (f^p)'(y) | \leq 1 \) on each disc \( B(c_j, 2\rho) \).

Thus the function \( f^p \) maps the disc \( B\left(z, \frac{d_0}{K^{M+1}}\right) \) inside the disc \( B\left(f^p(z), \frac{d_0}{K^{M+1}}\right) \).

As \( d \) is the degree of \( f \) it is clear that there are at most \( d \) points that can be mapped to any given point \( z \). Thus we have that at most \( d^p \) points in \( B\left(z, \frac{d_0}{K^{M+1}}\right) \) can be mapped to any point \( z \) in \( B\left(f^p(z), \frac{d_0}{K^{M+1}}\right) \) by the mapping \( f^p \). And from this we get that

\[
\| B\left(z, \frac{d_0}{K^{M+1}}\right) \cap f^{-n}(a) \| \leq d^p \cdot \| B\left(f^p(z), \frac{d_0}{K^{M+1}}\right) \cap f^{-(n-p)}(a) \|
\]
Now from the third assumption regarding $\rho$ we know that the point $f^p(z)$ lies in the set $H$, since the distance of $f^p(c_i)$ from the set $C_J$ must be at least $4\rho$ and the point $f^p(z)$ lies in the disc $B(f^p(c_i), \rho)$. As we already have confirmed the claim for points in the set $H$ we get by continuing the previous inequalities that

$$\| B\left(z, \frac{d_0}{K^{M+1}} \right) \cap \mathcal{f}^{-n}(a) \| \leq d^p \cdot d^{(n-p)+q-(M+1)} = d^{n+q-(M+1)}$$

And therefore the lemma 5.6 holds, completing the proof for the whole theorem.

So we have proven that

$$\dim_H(J) \geq \frac{\log d}{\log K_0}$$

holds for the Julia set of every polynomial with degree at least two. We had an example of the Julia sets for the polynomial $f(z) = z^2$, and as we use the above theorem we get that

$$\dim_H(J) \geq \frac{\log 2}{\log 2} = 1$$

We see that in this case the lower bound is in fact the Hausdorff dimension of $J$, so the bound is sharp. This is not always the case though, in fact our approximation can be quite rough in some situations. For example we had studied the Julia set of the polynomial $f(z) = z^2 - 2$ in an example regarding theorem 3.6. For this polynomial the Julia set is the interval $[-2, 2]$. If we approximate the Hausdorff dimension of the Julia set for this polynomial using the above theorem we get

$$\dim_H(J) \geq \frac{\log 2}{\log 4} = \frac{1}{2}$$

But it is easy to see that the Hausdorff dimension of the interval $[-2, 2]$ is 1.

Most important consequence of this theorem is that all the Julia sets of polynomials with degree at least two have a positive Hausdorff dimension, since this theorem provides a positive lower bound. As we remember from the discussion in the end of an example 4.1 the calculation of the Hausdorff measure and thus determining the Hausdorff dimension or even the lower bound for it is a non-trivial problem so theorem 5.7 is quite strong result.
6 Appendix

**Definition 6.1.** Euler characteristic for domains bounded with Jordan curves

We will discuss the Euler characteristics with some informality and suggest interested readers to consult for example [K] chapters 4 and 5 for more details and formality.

Let $S$ be a compact or a bordered surface, for our needs a sphere or a plane domain with its boundary, and assume that the boundary consist of a finite number of simple closed curves. A triangulation $T$ of $S$ is a partition of $S$ into a finite number of mutually disjoint subsets called vertices, edges, and faces with the following properties

1. Each vertex is a point of $S$
2. For each edge $e$ there is homeomorphism $\phi$ of a closed interval $[a, b]$ in $\mathbb{R}$ into $S$ which maps open interval $(a, b)$ onto $e$, and the endpoints $a$ and $b$ to the vertices of $T$
3. For each face $f$ there is homeomorphism $\phi$ of a closed triangle $Q$ in $\mathbb{C}$ into $S$ which maps the edges and vertices of $Q$ to the edges and vertices of $T$, and such that $f$ is the $\phi$-image of the interior of $Q$

So $T$ partitions $S$ into a disjoint subsets of $S$. Each such subset is either a vertex, edge or a face and we call these the simplexes of $T$, and they have dimension 0, 1, 2 respectively. We define Euler characteristic for every simplex as $\chi(s) = (-1)^m$, where $m$ is a dimension of a simplex. More generally if $S_0$ is a subset of $S$ consisting of a union of simplex $s_1, \ldots, s_n$, we define

$$\chi(S_0) = \sum_{i=1}^{n} \chi(s_i) = \sum_{i=1}^{n} (-1)^m,$$

If triangulation contains $F$ faces, $E$ edges and $V$ vertices, we have that $\chi(S) = F - E + V$. It is known that $\chi(S)$ is topologically invariant and does not depend on used triangulation, see theorem 5.13 in [K]. Let $S_1$ and $S_2$ be a subsets of a set $S$, each comprising a union of simplexes in the set $S$. Then it holds that $\chi(S_1 \cup S_2) + \chi(S_1 \cap S_2) = \chi(S_1) + \chi(S_2)$, and thus $\chi(S) = \chi(S_0) + \chi(\partial S)$, where $S_0$ is the interior of the set $S$. But since the boundary of $S$ consist of disjoint simple close curves there must equally many vertices and edges on the boundary. And since there can be no faces we have that $\chi(\partial S) = 0$ and we obtain that the Euler characteristic is same for the set $S$ and its interior.

Next by a straight calculation it can be obtained that $\chi(\bar{C}) = 2$, and $\chi(B(0, 1)) = 1$. Now assume that $D \subset \bar{C}$ is the complement of the $k$ mutually disjoint topological closed discs $B_1, B_2, \ldots, B_k$ such that the boundary of each disc is a Jordan curve. So $D$ is of connectivity $k$. Then we can use a triangulation such that each of $D, B_1, \ldots, B_k$ is a union of simplexes and obtain that

$$2 = \chi(\bar{C}) = \chi(D) + \sum_{i=1}^{k} \chi(B_i).$$

Since the Euler characteristic is topologically invariant we know that $\chi(B_i) = 1$ for every $i$ and hence we obtain that $\chi(D) = 2 - k$.

Next we must define Euler characteristic for an arbitrary domain $D$ of the complex sphere.

**Definition 6.2.** Regular subdomains

Let $D$ be any domain on the sphere. We say that its subdomain $\Omega$ is regular if the following two conditions hold.

1. $\Omega$ is bounded by a finite amount of mutually disjoint Jordan curves, let us denote them by $\gamma_1, \ldots, \gamma_n$, which all lie in $D$. 

2. The complement of $\Omega$ consist of $n$ topological discs $B_1, \cdots B_n$, which are bounded by $\gamma_1, \cdots \gamma_n$ respectively and each $B_k$ meets the complement of $D$.

We notice that $\chi(\Omega)$ is well defined for every regular subdomain of $D$. Next we will give few properties of regular subdomains.

**Theorem 6.1.**

Let $D$ be any proper subdomain of the complex sphere. Then the following two claims hold

1. Any compact subset of $D$ lies in some regular subdomain of $D$.
2. If $\Omega_1$ and $\Omega_2$ are regular subdomains of $D$ then there exists some regular subdomain $\Omega$ of $D$ such that $\Omega_1 \subset \Omega$ and $\Omega_2 \subset \Omega$.

Proof of part 1: Without loss of generality we can assume that the point $\infty$ lies in the set $D$. Let us cover the plane with a grid of closed squares of diameter $\frac{1}{n}$ where $n$ is a positive number. Then let $K_n$ be union of those squares that contain some point from the complement of the domain $D$, and let $D_n$ be that component of the complement of $K_n$ which contains the point $\infty$. Next we notice that the sets $D_n$ form an increasing sequence of regular domains of the domain $D$ and that $\bigcup_{n=1}^{\infty} D_n = D$.

Then let $K$ be any compact subset of $D$. We know that sets $D_n$ form open cover for it and hence there exists some finite subcover consisting of sets from the family $\{D_n\}$. But since sets $D_n$ were increasing there exists some set $D_{n_1}$ which is a regular subdomain of the domain $D$ and it holds that $K \subset D_{n_1}$.

Proof of part 2: Let $\Omega_1$ and $\Omega_2$ be regular subdomains of the domain $D$. Then it holds that $\bar{\Omega}_1 \cup \bar{\Omega}_2$ is a compact subset of the domain $D$ and hence the claim follows from the part one.

Next we will show that $\chi(\Omega)$ is a monotonic function on the class of all regular subdomains of $D$, and so tends to a limit as $\Omega$ increases to $D$ through class of regular subdomains.

**Theorem 6.2.**

Let $\Omega_1$ and $\Omega_2$ be regular subdomains of the domain $D$. If it holds that $\Omega_1 \subset \Omega_2$ then $\chi(\Omega_2) \leq \chi(\Omega_1)$.

Proof: Let $W_1, \cdots, W_n$ be the components of the complement of $\Omega_1$ and $V_1, \cdots, V_m$ be the components of the complement of $\Omega_2$. Then since we assumed that $\Omega_1 \subset \Omega_2$ it holds that $V_1 \cup \cdots \cup V_m \subset W_1 \cup \cdots \cup W_n$. Then for each $j \in \{1, 2, \cdots, n\}$ choose a point $z_j \in W_j$ such that $z_j \notin D$. As $z_j \notin D$ it lies in some $V_k$. then since every $V_k$ is connected it holds that $V_k \subset W_j$. Hence it follows that every $W_j$ contains some $V_k$ and thus $n \leq m$. But then we obtain that $\chi(\Omega_2) = 2 - m \leq 2 - n = \chi(\Omega_1)$, which completes the proof.

Now we are ready to extend the definition of Euler characteristics for an arbitrary domain $D$ of the complex sphere.

**Definition 6.3.** Euler characteristics for an arbitrary domains

Let $D$ be an arbitrary domain of the complex sphere, then we define that

$$\chi(D) = \inf \{ \chi(\Omega) : \Omega \text{ is a regular subdomain of } D \}$$
where we accept the $-\infty$ as one possible limit.

So either there exists some sequence \( \{\Omega_n\} \) of regular subdomains of \( D \) such that \( \chi(\Omega_n) \to -\infty \), or there is some regular subdomain \( \Omega_0 \) of \( D \) such that

\[
\chi(\Omega_0) = \inf \{ \chi(\Omega) : \Omega \text{ is a regular subdomain of } D \} > -\infty
\]

and then we define \( \chi(D) = \chi(\Omega_0) \).

Now if \( D \) is simply connected domain then \( \partial D \) is connected and every regular subdomain of \( D \) can have only one component in its complement. Hence \( \chi(D) = 1 \) for every simple connected domain. More generally if a domain \( D \) is of \( k \)-connectivity, then in similar manner we deduce that \( \chi(\Omega) = 2 - k \) for every large enough regular subdomain of \( D \). And hence \( \chi(D) = 2 - k \) for all such domains \( D \).

**Definition 6.4.** Radon measure

Let \( \mu \) be a measure on the Borel sets of \( \mathbb{C} \). We say it is a Radon measure if it fulfills the following two conditions.

1. It is locally finite, meaning that for every point \( z \in \mathbb{C} \) there exists a neighborhood \( U \) for \( z \) such that \( \mu(U) < \infty \).

2. It is inner regular, meaning that for every set \( V \) it holds that \( \mu(V) = \sup_{K \subset V} \mu(K) \) where \( K \) is compact.

In [H], theorem 1.28 it is proven that we can equivalently state that a Borel measure \( \mu \) on \( \mathbb{R}^n \) is a Radon measure if and only if \( \mu \) is locally finite.

As an corollary from the above theorem we obtain that every probability measure defined on the Borel sets of \( \mathbb{C} \) is a Radon measure.

**Definition 6.5.** Weak convergence

Let \( \{\mu_n\} \) be a family of Radon measures. We say that \( \{\mu_n\} \) converges weakly to the Radon measure \( \mu \) if

\[
\lim_{n \to \infty} \int \phi d\mu_n = \int \phi d\mu
\]

holds for every continuous and compactly supported function \( \phi : \mathbb{C} \to \mathbb{C} \).

**Theorem 6.3.**

Let \( \{\mu_n\} \) be a family of Radon measures in \( \mathbb{C} \) for which it holds that \( \sup_{n \in \{1, 2, \ldots\}} \mu_n(K) < \infty \) for every compact set \( K \subset \mathbb{C} \). Then it holds that there exists a subsequence \( \{\mu_{n_i}\} \) and a Radon measure \( \mu \) such that \( \{\mu_{n_i}\} \) converges weakly to the measure \( \mu \).

Proof: We omit the proof in this paper, but it can be found for example in [H] starting on the page 50.

For the next theorem let \( \mu \) be a Borel measure on the set \( \mathbb{C} \). Then we denote its support by \( \text{supp}(\mu) \) and define it as follows, \( \text{supp}(\mu) = \mathbb{C} \setminus \bigcup \{ V : V \text{ is open and } \mu(V) = 0 \} \).

Theorem 6.4.
Let \( \{ \mu_n \} \) be a sequence of probability measures defined on the Borel sets of \( \mathbb{C} \) such that their supports converge uniformly to the Julia set of the polynomial \( f \) when \( n \to \infty \). This means that given any \( \varepsilon > 0 \) we can find an integer \( n_\varepsilon \) such that \( \text{supp}(\mu_n) \subseteq J_f \) for every \( n > n_\varepsilon \). Additionally assume that the following inequality holds, \( \mu_n(B(z, r)) \leq cr^n \) for all \( n, 0 < r < r_z, z \in J \) and some constant \( c \). Then we have some subsequence \( \{ \mu_{n_i} \} \) that converges weakly to the probability measure \( \mu \) on the Julia set and the above inequality is inherited by \( \mu \).

Proof: Since every measure in the sequence \( \{ \mu_n \} \) is a probability measure the condition of the theorem 6.3 clearly holds. Thus we obtain that there exists some \( \mu \) such that some subsequence of \( \{ \mu_{n_i} \} \) converges to it. Additionally we notice that the support of \( \mu \) must be contained in the Julia set since the supports of \( \mu_n \) converges uniformly to \( J \). Now we have to check that the condition \( \mu(B(z, r)) \leq cr^n \) is inherited by \( \mu \).

Since probability measures on the complex plane are Radon measures we can use the following result. For a sequence \( \{ \mu_n \} \) of radon measures that converge weakly towards the radon measure \( \mu \) it holds that \( \liminf_{n \to \infty} \mu_n(U) \geq \mu(U) \) when \( U \) is an arbitrary open set in the complex plane. Proof for this claim can be found for example in [H] on the page 50.

Using the above result we have that \( \mu(B(z, r)) \leq \liminf_{n \to \infty} \mu_n(B(z, r)) \leq cr^n \) for every sufficiently small but positive \( r \) and \( z \in J \). Hence we see that the condition is inherited by the measure \( \mu \) and the claim holds.

Definition 6.6. Semialgebra

Let \( X \) be a set. Then the class \( S \subseteq \mathcal{P}(X) \) is a semialgebra if the following conditions hold.

1. \( \emptyset \in S \)
2. \( A, B \in S \Rightarrow A \cap B \in S \)
3. \( A \in S \Rightarrow X \setminus A = \bigcup_{i=1}^n A_i \) where every set \( A_i \in S \) and sets \( A_i \) are disjoint.

Next we will show that the class \( I \cup \emptyset \) on which we have defined the non-negative function \( \mu \) in the proposition 4.12 is a semialgebra. The condition 1 is trivial so let us check conditions 2 and 3.

To obtain condition 2 choose an arbitrary sets \( A, B \in I \cup \emptyset \). We know from the definition of the set \( I \) that either \( A \cap B = \emptyset \) or the sets \( A \) and \( B \) are nested. Thus the condition 2 holds.

To show condition 3 choose an arbitrary set \( A \in I \cup \emptyset \). If \( A = \emptyset \) the claim is trivial so we can assume that \( A \neq \emptyset \). Hence the set \( A \) is of form \( I_{i_1, \ldots, i_n} \) where \( n \) is some natural number and every \( i_k \in \{1, 2, \ldots, m\} \). Then choose all sequences \( j_1, \ldots, j_n \), where \( j_k \in \{1, 2, \ldots, m\} \), such that \( j_k \neq i_k \) for some \( k \in \{1, 2, \ldots, n\} \). Next we choose all sets \( I_{j_1, \ldots, j_n} \) corresponding these sequences and notice that each \( I_{j_1, \ldots, j_n} \in I \), they are disjoint and \( (I \cup \emptyset) \setminus A = \bigcup I_{j_1, \ldots, j_n} \cup \emptyset \). This shows that \( I \cup \emptyset \) is a semialgebra.

Definition 6.7. Algebra

Let \( X \) be a set. Then the class \( \mathcal{A} \subseteq \mathcal{P}(X) \) is an algebra if the following conditions hold.

1. \( \emptyset \in \mathcal{A} \)
2. \( A, B \in \mathcal{A} \Rightarrow A \cap B \in \mathcal{A} \)
3. \( A \in \mathcal{A} \Rightarrow X \setminus A \in \mathcal{A} \)
Theorem 6.5.
Let $S$ be a semialgebra, then
\[ \bar{S} = \{ \bigcup_{i=1}^{n} A_i : A_i \in S, n \in \mathbb{N} \text{ where sets } A_i \text{ are disjoint} \} \]
is an algebra.

Proof: Proof can be found for example in [H] lemma 1.39.

Definition 6.8. Measure on algebra

We say that $\nu$ is a measure on algebra $\mathcal{A}$ if $\nu : \mathcal{A} \to [0, +\infty]$ is $\sigma$-additive and $\nu(\emptyset) = 0$

Theorem 6.6. Caratheodory extension theorem

Let $\bar{S}$ be algebra and $\nu$ a measure on it. Then there exists a measure $\bar{\nu} : \sigma(\bar{S}) \to [0, +\infty]$ such that $\bar{\nu}(A) = \nu(A)$ for every $A \in \bar{S}$. Additionally if $\nu(\bar{S}) < \infty$ the measure $\bar{\nu}$ is unique.

Proof: Proof can be found for example in [H] theorem 1.44.

Next we will show that the non-negative function $\mu$ defined on the set $I \cup \emptyset$ can be extended to the whole $\sigma(I)$, where $\sigma(I)$ is the smallest $\sigma$-algebra that contains the set $I$. We have already noticed that $I \cup \emptyset$ is a semialgebra and hence we know that it can be extended to algebra $\bar{I}$ by taking all finite unions of disjoint sets belonging to $I \cup \emptyset$. And since the function $\mu$ is clearly finitely additive it can be extended to the non-negative function $\mu_*$ on the algebra $\bar{I}$. Now all that we need is to show that $\mu_*$ is $\sigma$-additive on the $I$ and use the Caratheodory extension theorem to obtain the result. The proof for the fact that $\mu_*$ is $\sigma$-additive is a bit technical so we will not give it here, but the proof can be found for example in [W] starting on the page 214.

In the following we use a notation from the proposition 4.12.

We will try to produce a mass distribution $\bar{\mu}$ on the set $U$ from the measure $\mu_*$ defined on $\sigma(I)$. For this we define that $\bar{\mu}(A) = \mu((i_1, i_2, \ldots) : x_{i_1, i_2, \ldots} \in A)$ for any Borel set $A$, where $x_{i_1, i_2, \ldots} = \bigcap_{k=1}^{\infty} U_{i_1, \ldots, i_k}$ and we will show that every Borel set is indeed measurable with this definition. We first notice that clearly $\bar{\mu}$ is indeed a measure, since for any sets $U_{i_1, \ldots, i_k}$ and $U_{i_1, \ldots, i_{(k+1)}}^*$ either $U_{i_1, \ldots, i_{(k+1)}}^* \subset U_{i_1, \ldots, i_k}$ or the intersection of these two sets has Haussdorf $s$-measure zero for every integer $m \geq 0$ and every choice of the set $U_{i_1, \ldots, i_{(k+1)}}^*$. Additionally it holds that $\bar{\mu}(U) = \mu(I) = 1$, and similarly any set $U_{i_1, \ldots, i_k}$ is trivially $\bar{\mu}$-measurable and $\bar{\mu}(U_{i_1, \ldots, i_k}) = \mu(U_{i_1, \ldots, i_k}) = (c_{i_1} \cdots c_{i_k})^s$. Next we will aim to show that any Borel set $A$ is measurable.

First we notice that the diameter of an arbitrary set $U_{i_1, \ldots, i_k}$ converges uniformly to 0 as $k \to \infty$ and that $U = \bigcup_{J_k} U_{i_1, \ldots, i_k}$ where $J_k$ denotes the set of all $k$-term sequences with $1 \leq i_j \leq m$. To show that every Borel set is $\bar{\mu}$-measurable it is enough to show that every open set $V$ is $\bar{\mu}$-measurable.

We remind our selves that given two sets $U_{i_1, \ldots, i_k}$ and $U_{i_1, \ldots, i_{(k+1)}}$ either $U_{i_1, \ldots, i_{(k+1)}} \subset U_{i_1, \ldots, i_k}$ or the intersection of these two sets has Haussdorf $s$-measure zero for every integer $m \geq 0$ and every choice of the set $U_{i_1, \ldots, i_{(k+1)}}$.

Then we show that $V = \bigcup_{k=0}^{\infty} U_{i_1, \ldots, i_k}$ where every set $U_{i_1, \ldots, i_k}$ must fulfill the following two conditions.
1. \( U'_{i_1,\ldots,i_k} \subset V \)

2. \( \mathcal{H}^s(U'_{i_1,\ldots,i_k} \cap U'_{i_1,\ldots,i_d}) = 0 \) where \( d \leq k \) and \( U'_{i_1,\ldots,i_d} \) has been chosen to the union

This follows since for an arbitrary point \( x \in V \cap U \) it holds that there exist such a disc \( B(x,r) \) that \( B(x,r) \cap U \subset V \) as \( V \) is open. Then since the diameter of every set \( U_{i_1,\ldots,i_k} \) converges uniformly to zero and additionally \( U = \bigcup_{k} U_{i_1,\ldots,i_k} \) for every integer \( k \) there exists some integer \( k^* \) and some sequence \( i_1,\ldots,i_k^* \) such that \( x \in U_{i_1,\ldots,i_k^*} \subset B(x,r) \). Hence \( U_{i_1,\ldots,i_k^*} \subset V \) for this sequence \( i_1,\ldots,i_k^* \in J_k^* \), and thus the set \( U_{i_1,\ldots,i_k^*} \) or some set that contains the set \( U_{i_1,\ldots,i_k^*} \) belongs to the union \( \bigcup_{k=0}^\infty U'_{i_1,\ldots,i_k} \).

Since the point \( x \in V \) was arbitrary this gives the claim.

Let us denote the family of those sets that belong to the union \( \bigcup_{k=0}^\infty U'_{i_1,\ldots,i_k} \) by \( L \). Then we obtain that

\[
\bar{\mu}(V) = \bar{\mu}(\bigcup_{k=0}^\infty U'_{i_1,\ldots,i_k}) = \sum_{k=0}^\infty \mu(\bigcup_{i \in L} U_{i_1,\ldots,i_k})
\]

where every set \( U_{i_1,\ldots,i_k} \) is \( \bar{\mu} \)-measurable and last equality holds since every intersection of the different sets in the family \( L \) has measure zero. Thus every open set \( V \) is \( \bar{\mu} \)-measurable.
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