DYADIC SYSTEMS IN DOUBLING METRIC SPACES AND APPLICATIONS TO POSITIVE INTEGRAL OPERATORS

ANNA KAIREMA

Academic dissertation

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Faculty of Science
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Supervisor
Tuomas Hytönen, University of Helsinki, Finland

Pre-examiners
David V. Cruz-Uribe, Trinity College, USA
Frédéric Bernicot, Université de Nantes, France

Opponent
Juha Kinnunen, Aalto University, Finland

Custos
Juha Oikkonen, University of Helsinki, Finland

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Dyadic systems in doubling metric spaces and applications to positive integral operators

Anna Kairema

Department of Mathematics and Statistics
P.O. Box 68 (Gustaf Hällströmin katu 2b)
FI-00014 University of Helsinki
anna.kairema@helsinki.fi
http://www.cs.helsinki.fi/u/kairema/

Abstract

This dissertation brings contribution to two interrelated topics. The first contribution concerns the so-called systems of dyadic cubes in the context of metric spaces. The second contribution consists of applications to one and two weight norm inequalities for linear and sublinear positive integral operators. Both of the topics are important in harmonic analysis and an ongoing area of study. The main novelties of the presented works consist of improving and extending existing results into more general situations. The work consists of four research articles and an introductory part.

The first two articles, written in collaboration with T. Hytönen, study systems of dyadic cubes in metric spaces. In the Euclidean space, dyadic cubes are well-known and define a convenient structure with useful covering and intersection properties. Such dyadic structures are central especially in the modern trend of harmonic analysis. In the first article extensions of these structures are constructed in general geometrically doubling metric spaces. These consist of a refinement of existing constructions and a completely new construction of finitely many adjacent dyadic systems which behave like "translates" of a fixed system but without requiring a group structure.

In this context, "cubes" are not properly cubes but rather more complicated sets that collectively have properties reminiscent of those in the Euclidean case. However, it is natural to ask what type of sets could or should be regarded cubes. In the second article, a complete answer is given to this question in the general framework of a geometrically doubling metric space by using the "plumpness" notion already appeared in the geometric measure theory.

From another side; the two latter articles study weighted norm inequalities. Via the new construction of adjacent dyadic systems, weighted estimates for positive integral operators are obtained in a general framework. In the third article, the two-weight problem is investigated for potential-type operators. Both strong and weak type estimates are characterized by "testing type" conditions: to show the full norm inequality it suffices to test the desired estimate on a specific class of simple test functions only. The results improve some previous results in the sense that the considered ambient space is more general (with more general measures and no additional geometric assumptions) and the testing is over a countable collection of test functions only (instead of a significantly larger collection appearing in the previous works on the topic). The main technical novelty of the proof is a decomposition of the operator, along the dyadic systems giving rise to certain finitely many "dyadic" versions of the original operator.

In the fourth article, the focus is on sharp constant estimates for generalized fractional integral operators. A positive answer and its sharpness are given in the context of a space of homogeneous type. The result is reduced to weak-type inequalities using the results from the third article. The sharpness of the result requires a construction of functions that locally behave similarly to the basic power functions on the Euclidean space, which seems to be a completely new discovery. The result extends a recent Euclidean result.

Keywords dyadic cube, adjacent dyadic systems, metric space, space of homogeneous type, potential-type operator, testing condition, weighted norm inequality, sharp bound

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Helsinki, May 2013 Anna Kairema

Contents

List of included articles

This thesis consists of an introductory part and the following four research papers:

[A] T. Hytönen and A. Kairema. Systems of dyadic cubes in a doubling metric space. *Colloq. Math.*, Volume 126, Issue 1, 1–33, (2012).

[B] T. Hytönen and A. Kairema. What is a cube? *Ann. Acad. Sci. Fenn. Math.*, Volume 38, 405–412, (2013).

[C] A. Kairema. Two-weight norm inequalities for potential type and maximal operators in a metric space. *Publ. Mat.*, Volume 57, 3–56, (2013).

[D] A. Kairema. Sharp weighted bounds for fractional integral operators in a space of homogeneous type. *Math. Scand.*, to appear. Preprint: arXiv:1202.6587, (2012).

In the introductory part, these papers are referred to as [A], [B], [C], and [D]. Other references will be numbered as [1],[2],...

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Author's contribution

Both authors had an equal role in the analysis and writing of the joint papers [A] and [B].

1. Overview

Various boundedness results for different integral operators have a central role in Harmonic Analysis, and they have a long-lasting history. The fundamental question in this context is a *two-weight problem*; the question of the boundedness of an operator from a space $L^p(d\sigma)$ to another space $L^q(d\omega)$. To be more precise, given an integral operator T, two measures σ and ω on X, and exponents p and q, we ask if there exists a constant C such that the norm inequality

$$\left(\int_X |Tf(x)|^q \, d\omega(x)\right)^{1/q} \le C \left(\int_X |f(x)|^p \, d\sigma(x)\right)^{1/p} \tag{1.1}$$

holds for every $f \in L^p(X, d\sigma)$.

A canonical framework for this problem is the Euclidean space $X = \mathbb{R}^n$ with two measures σ and ω that are generated by positive locally integrable functions u and w (called weights) and an underlying Lebesgue measure. The question is to identify and classify the pair of weights (u, w) for which there exists a constant C such that

$$\left(\int_{\mathbb{R}^n} |Tf(x)|^q \, w(x)dx\right)^{1/q} \le C \left(\int_{\mathbb{R}^n} |f(x)|^p \, u(x)dx\right)^{1/p} \tag{1.2}$$

holds for every $f \in L^p(\mathbb{R}^n, udx)$.

The study of weighted norm inequalities became an active area of research in the 1970s. At the basis of the developments related to these questions is the ground breaking work of B. Muckenhoupt, R. Wheeden, and their contemporaries. Originally, the question was studied on the unit circle of the complex plane, and in $\mathbb{R}^n$. Muckenhoupt [60] solved the *one-weight problem* with $u = w$ and $p = q$ for the Hardy–Littlewood maximal operator M in the Euclidean space by characterizing the weights w for which $M\colon L^p(\mathbb{R}^n, wdx) \to L^p(\mathbb{R}^n, wdx)$ is a bounded operator. This theory was later extended to include a variety of other operators in $\mathbb{R}^n$. Beginning with the work of A. P. Calderón [11], the Euclidean results have been extended to spaces of homogeneous type; quasi-metric spaces with doubling measures. It is now well-known that the one-weight problem is solved by Muckenhoupt's A_p-condition on the weight, which gives a qualitative characterization for the boundedness of many classical operators.

A subsequent question involves understanding the nature of the constant C in the norm inequality (1.2). In the past two decades, a general trend in the weighted theory has been the interest in obtaining *sharp* estimates by finding the precise dependence of operator norms on Muckenhoupt's weight constant which is given by

$$[w]_{A_p} := \sup_{B \text{ a ball}} \left(\frac{1}{|B|}\int_B w \, dx\right)\left(\frac{1}{|B|}\int_B w^{-\frac{1}{p-1}} \, dx\right)^{p-1} \quad \text{for } 1 < p < \infty.$$

To be more precise, given an operator T, the question is to find the smallest power $\alpha = \alpha(p)$ such that

$$\|Tf\|_{L^p(w)} \le C_{n,T}[w]_{A_p}^{\alpha}\|f\|_{L^p(w)}$$

holds for all $f \in L^p(\mathbb{R}^n, wdx)$.

The two-weight problem is, in general, much more complicated than the one-weight case, and there are several approaches to attack this. Important development in this direction is due to E. T. Sawyer [74] who solved the general two-weight problem for the maximal operator in the Euclidean space by the so-called *testing type conditions*: in order to have the full norm inequality, it is enough to test the desired inequality on a specific class of simple test functions only. This theory has been developed also in spaces of homogeneous type.

Weighted norm inequalities arise frequently in other areas of mathematics and in applications, and form an active area of on-going study. For example, weighted inequalities for the Riesz potential have a close connection to the positivity of the Schrödinger operators [12].

In this dissertation, we investigate one- and two-weight norm inequalities in the context of metric measure spaces and, in particular, in spaces of homogeneous type.

Our first results apply to a large class of potential-type operators with non-negative kernels $K(x,y)$ that satisfy certain monotonicity conditions in both the variables, and certain fractional maximal operators. We work in quite general metric measure spaces. Potential-type operators are investigated in a quasi-metric space with only the so-called geometrical doubling property, equipped with positive σ-finite Borel-measures. For fractional maximal operators, the set-up is a space of homogeneous type. In both cases, we solve the two-weight problem by characterizing (1.1) with Sawyer-type testing conditions (paper [C]).

Our second results are of different nature. For this, the set-up is a space of homogeneous type. We solve the one-weight problem for the Hardy–Littlewood maximal operator (paper [A]) and fractional integral operators (paper [D]), and in both cases find the sharp dependence of the operator norms on the weight constants. In the Euclidean space, the result for the maximal operator is due to S. Buckley [10]. For fractional integrals, the result is relatively new even in the Euclidean case, and it is due to M. Lacey, K. Moen, C. Pérez and R. Torres [50].

One way to tackle these questions is a *dyadic discretization technique*: the operator is approximated by appropriate model operators, associated to systems of dyadic cubes. Consequently, the original theorem is governed by its dyadic analogues, which are often easier to obtain. Our main techniques involve this dyadic discretization, which depends on the construction of 'adjacent' dyadic systems.

Accordingly, the study of *dyadic cubes* forms a considerable large part of this thesis. We continue the seminal work of M. Christ [13] and study the construction of dyadic type families in general geometrically doubling metric spaces. The purpose of paper [A] is to develop the needed machinery for the analysis of papers [C, D] on one- and two-weight estimates discussed above. We have been pleased and surprised at how far the dyadic ideas and techniques can be taken and how well they work even in abstract metric spaces. We further discuss in detail the characterization of 'cubes' as geometric objects in the metric space context (paper [B]).

In all of our investigations presented in this dissertation, we have expended considerable effort to find as general a context as possible by dropping some common assumptions that appeared in the previous works on the subject (like the 'non-emptyness' of annuli and ruling out point masses). Also, much effort is put in providing a self-contained, careful, and comprehensive development.

2. Dyadic type families in metric spaces

2.1. **Background.** The system of dyadic cubes, and other nested structures, is an essential tool in mathematical analysis in the Euclidean space $\mathbb{R}^n$. The use of these structures comes forth in numerous techniques and conventions that have become standard in different areas of analysis, most notably in Harmonic Analysis. Dyadic cubes form a collection of certain special n-cubes of different sizes with the fundamental properties that any two cubes are either disjoint or one is contained in the other, and that the cubes of a given size partition the space; yet another useful property of these cubes is that every cube of a given size is a union of 2^n sub-cubes of the next smaller size. Since the space $\mathbb{R}^n$ has a strong geometrical structure, the system is easy to construct. In particular, the standard system of dyadic cubes is expressed by

$$\mathscr{D} = \{2^{-k}\big([0,1)^n + m\big) \colon k \in \mathbb{Z}, m \in \mathbb{Z}^n\}.$$

The Euclidean system of dyadic cubes has recently been generalized in two important directions. First, there is the *non-random choice of finitely many shifted dyadic systems*. For example, the systems

$$\mathscr{D}^t = \{2^{-k}\big([0,1)^n + m + (-1)^k t\big) \colon k \in \mathbb{Z}, m \in \mathbb{Z}^n\}, \quad t \in \{0, 1/3, 2/3\}^n,$$

provide a powerful tool for analysis, see e.g. [48, 54, 62, 63]. The novelty of working with these adjacent collections instead of just one system is that for any ball $B \subseteq \mathbb{R}^n$, there is a choice of some t and some $Q \in \mathscr{D}^t$ so that $B \subseteq \frac{9}{10}Q$ while $\operatorname{diam}(Q) \leq C \operatorname{diam}(B)$. As far as we know, the idea of using such adjacent dyadic systems goes back to Christ and Garnett–Jones, but we are not aware of the precise original occurrence of the systems.

Second, there is the *random choice of dyadic cubes*,

$$\mathscr{D}(\omega) = \Big\{2^{-k}\big([0,1)^n + m\big) + \sum_{j>k} 2^{-j}\omega_j \colon k \in \mathbb{Z}, m \in \mathbb{Z}^n\Big\},$$

where $\omega = (\omega_j)_{j\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$. These were introduced by Nazarov–Treil–Volberg [64, Section 4], and they are involved in several modern and recent techniques. The randomization method provides a powerful tool to avoid certain irregular situations by arguing that their occurrence only has a small probability. The random dyadic systems have been instrumental in the development of the non-doubling theory for singular integrals [64, 66] as well as the theory of sharp weighted estimates for singular integrals. Most notably, operators known as dyadic shifts, adapted to a random choice of dyadic cubes, have been of central importance in the study of sharp one-weight norm inequalities for singular integral operators [41, 70].

For the purposes of analysis in more general settings, the lack of a generalization of the Calderón–Zygmund decomposition in terms of dyadic cubes was a significant obstacle; cf. [78]. The first attempt to find a substitute for the system of dyadic cubes in more general metric spaces is due to Aimar–Macías [4] who constructed families of balls which had some, but not all, of the essential properties of dyadic cubes. The first truly satisfactory construction is due to Christ [13] and Sawyer–Wheeden [73] in the setting of a space of homogeneous type. In this setting, 'cubes' are not proper cubes but rather more complicated sets that collectively have the same essential properties that the standard dyadic cubes have in the Euclidean space. The first contribution of this dissertation is the continuation of these works and the development of the dyadic machinery in general quasi-metric spaces with a certain doubling property, as is explained in the following.

2.2. **Geometrically doubling quasi-metric spaces.** We consider a quasi-metric space (X,ρ) where $\rho\colon X\times X\to[0,\infty)$ satisfies the axioms of a metric except for the triangle inequality which is assumed in the weaker form

$$\rho(x,y)\leq A_0(\rho(x,z)+\rho(z,y))$$

with a constant $A_0\geq 1$ which is independent of the points $x,y,z\in X$. Important examples are provided by metric spaces with $A_0=1$, but the generality of quasi-metrics is needed in natural applications. This abstraction has a large and useful family of special cases, and it therefore deserves special attention.

We assume that the space has the well-known geometric property (sometimes referred to as *weak homogeneity*; cf. [15]) that every quasi-metric ball $B(x,r):=\{y\in X\colon \rho(x,y)<r\}$, where $x\in X$ and $r>0$, can be covered by at most A_1 (a fixed positive integer) balls $B(x_i,r/2)$ of half the radius. We call such quasi-metric spaces *geometrically doubling.*

2.3. **Dyadic systems and paper [A].** In a quasi-metric space (X,ρ), a family

$$\mathscr{D}=\bigcup_{k\in\mathbb{Z}}\mathscr{D}_k,\quad \mathscr{D}_k=\{Q^k_\alpha\colon \alpha\in\mathscr{A}_k\},$$

of Borel sets $Q^k_\alpha\subseteq X$ is called *a system of dyadic cubes with parameters* $\delta\in(0,1)$ and $0<c_1\leq C_1<\infty$ if it has the following properties:

$$\text{if } \ell\geq k,\text{ then either } Q^\ell_\beta\subseteq Q^k_\alpha \text{ or } Q^k_\alpha\cap Q^\ell_\beta=\emptyset; \tag{1}$$

$$X=\bigcup_{\alpha\in\mathscr{A}_k}Q^k_\alpha\quad\text{(disjoint union)}\quad\forall k\in\mathbb{Z}; \tag{2}$$

$$B(x^k_\alpha,c_1\delta^k)\subseteq Q^k_\alpha\subseteq B(x^k_\alpha,C_1\delta^k)=:B(Q^k_\alpha); \tag{3}$$

$$\text{if } \ell\geq k \text{ and } Q^\ell_\beta\subseteq Q^k_\alpha,\text{ then } B(Q^\ell_\beta)\subseteq B(Q^k_\alpha). \tag{4}$$

The set Q^k_α is called a *dyadic cube of generation* k with a center point $x^k_\alpha\in Q^k_\alpha$ and side length δ^k.

The most important example of a dyadic system is provided, of course, by the usual dyadic cubes in $\mathbb{R}^n$ of the form

$$Q_\alpha^k = 2^{-k}\big([0,1)^n + \alpha\big), \quad k \in \mathbb{Z}, \alpha \in \mathbb{Z}^n.$$

There are several constructions analogous to the Euclidean family of dyadic cubes constructed by several authors in more general metric or quasi-metric spaces with a suitable doubling structure. The various constructions have been motivated by different questions and tailored for special purposes. First results in this direction are due to Aimar–Macías [4] and G. David [22, Appendix A]; see also [23, Appendix I]. A more comprehensive construction, which has become the standard reference on the topic, was provided by M. Christ [13], in the full generality of Coifman–Weiss type spaces of homogeneous type and with the important metric control by balls property (3). A more elementary construction was provided in Sawyer–Wheeden [73]. Some further variants have been constructed by other authors [1, 42].

In [A], we show the following result, which is a slight elaboration of seminal work by Christ [13]:

2.1. **Theorem.** (Theorem 2.2 of [A]) *Let (X, ρ) be a geometrically doubling quasi-metric space. Then there exists a system of dyadic cubes. The construction only depends on some fixed set of countably many center points x_α^k and a certain partial order among the index pairs (k, α).*

We remark that the construction requires the center points and a related partial order be fixed *a priori*. However, if either the system of points or the partial order is not given, their existence already follows from the assumptions. These lines will be discussed in more detail in [B].

The main novelty of our modification is that it only uses the condition of geometrical doubling and is independent of measure – no auxiliary doubling measure is needed for this, and since we do not assume our space to be complete, no such measure needs to even exist. Our construction also eliminates two minor drawbacks in the previous works: on the one hand, we obtain cubes on all length scales (rather than from a given level up), and we also obtain an exact partition of the space (rather than up to a set of measure zero).

While our geometric doubling property assumption is slightly more general than assuming the existence of a doubling measure (cf. Section 4.2), it should be mentioned that every complete metric space that has the geometric doubling property also supports a doubling measure [57]. Accordingly, some researchers refer to these two properties as "essentially equivalent". See Section 4.3 of this introductory part.

It should be noted that this dyadic system is not canonical, but on the contrary, a given space admits several different systems of dyadic cubes. In fact, a significant novelty of our construction is the fact that we use it to prove the existence of both random dyadic systems and a finite collection of adjacent dyadic systems that behaves like 'translates' of a fixed system but without an underlying group structure on the space; the latter structure is important for our study of weighted inequalities in [C, D]. These lines are discussed in more detail in the next sections.

2.4. **Adjacent dyadic systems.** In a quasi-metric space (X,ρ), a finite collection $\{\mathscr{D}^t\colon t=1,2,\dots,T\}$ of families $\mathscr{D}^t$ is called a *collection of adjacent systems of dyadic cubes with parameters* $\delta\in(0,1), 0<c_1\le C_1<\infty$ and $1\le C<\infty$ if it has the following properties: individually, each $\mathscr{D}^t$ is a system of dyadic cubes with parameters $\delta\in(0,1)$ and $0<c_1\le C_1<\infty$; collectively, the have the additional property that for any ball $B(x,r)\subseteq X$ there exist t and $Q\in\mathscr{D}^t$ such that

$$B(x,r)\subseteq Q\subseteq B(x,Cr). \tag{2.2}$$

The main contribution and a completely new result of [A] is the following metric space version of the non-random choice of boundedly many adjacent systems of dyadic cubes:

2.3. **Theorem.** (Theorem 4.1 and Proposition 4.3 of [A]) *Let (X,ρ) be a geometrically doubling quasi-metric space. Then there exists a collection of adjacent systems of dyadic cubes. Moreover, given a fixed point $x_0\in X$, this collection $\{\mathscr{D}^t\colon t=1,2,\dots,T\}$ can be constructed in such a way that for every $t=1,\dots,T$ and $k\in\mathbb{Z}$, there exists α such that $x_0=x^k_\alpha$, the center point of $Q^k_\alpha\in\mathscr{D}^t$.*

We remark that the fact that it is possible to insure a common point to become a center point at all levels in each family in our construction, as stated, is an extension of even the Euclidean standard cubes, and this property is crucial in applications; cf. [C, Remark 2.13 and Lemmata 2.11, 2.12].

For T (the number of the adjacent systems), the proof gives the upper bound

$$T=T(A_0,A_1,\delta)\le A_1^6(A_0^4/\delta)^{\log_2 A_1}. \tag{2.4}$$

There is, however, no reason to believe that (2.4) is even close to optimal. In particular, in the Euclidean space $\mathbb{R}^n$ with the usual structure we have $A_0=1, A_1\ge 2^n$ and $\delta=\frac{1}{2}$, so that (2.4) yields an upper bound of order 2^{7n}. However, T. Mei [59] has shown that the conclusion (2.2) can be obtained with just $n+1$ cleverly chosen systems $\mathscr{D}^t$. As for now, no better bound than (2.4) is known for general metric spaces.

2.5. **Random dyadic systems.** In addition to the mentioned non-random choice of boundedly many adjacent systems, we provide in [A] a streamlined and simplified metric space version of the randomized construction of dyadic systems originally due to Hytönen–Martikainen [39]. Our contribution includes an elaborate description of the underlying probability space; the details that turned out helpful in proving a vector valued Tb theorem for singular integrals by using the random cubes method in [58]. Moreover, we combine the two constructions, the random choice and the non-random choice, yielding a random family of adjacent dyadic systems $\mathscr{D}^t(\omega), t=1,2,\dots,T$. Given a positive σ-finite measure, this random family may, in particular, be chosen to have the additional property that for any cube, the measure of the boundary is zero. These lines, however, will not be pursued further in this thesis.

2.6. **Paper [B].** While the dyadic type families can be constructed even in abstract metric spaces, as stated, the notion of a 'cube' or even a 'dyadic cube' as an individual object barely makes sense anymore – it only becomes meaningful as a member of a dyadic system with properties reminiscent to those of the Euclidean standard system. Nevertheless, it is reasonable to ask what assumptions on a set should one impose so that it could or should be considered a member in some dyadic system?

Our motivation to understand this question arises from the recent developments in $\mathbb{R}^n$ where it has become standard to study singular integrals with the help of random dyadic cubes. This method was introduced in the seminal work of Nazarov–Treil–Volberg [66] and later generalized to the setting of an abstract metric space by Hytönen–Martikainen [39]. In $\mathbb{R}^n$, any cube can arise as a random dyadic cube whereas in abstract spaces (X, d), the question has been unclear. Another fact that motivated us is that in $\mathbb{R}^n$ a very common technique is to work with the dyadic sub-cubes of a given, *a priori* non-dyadic cube. Accordingly, our result allows this technique to be extended to other settings.

In [B], we give an intrinsic characterization of all subsets of a geometrically doubling metric space (X, ρ) that can arise as a member of some system of dyadic cubes by using the notion of 'plumpness', already appeared in the geometric measure theory: we say that a set $E \subseteq X$ is *plump* if for a small $r > 0$ and all $x \in E$, the ball $B(x, r)$ contains a sub-ball $B(z, cr)$ lying completely inside E, and where $c > 0$ is a uniform constant. According to our main result, a set E may become a 'cube' if and only if E is bounded and both E and its complement $X \setminus E$ are plump. We also provide a quantitative version of this result.

2.7. **Related developments.** The construction of dyadic type families depends on suitably chosen center points x_α^k and a certain partial order among their index pairs (k, α). The intuitive idea for adjacent dyadic systems would be to alter the choice of the center points. Accordingly, the basic idea behind our construction of several adjacent dyadic systems (random or non-random) in [A] was that new center points for the cubes of generation k are chosen among the old center points (from a reference system) of the previous (one level smaller) generation $k + 1$. This idea of randomizing the choice of center points seemed natural to us since it resembles the Euclidean approach of translating the standard dyadic lattice. However, it turns out that in abstract settings this approach is not the optimal one. Indeed, by keeping with a fixed set of center points and randomizing the partial order (the relations between center points of consecutive generations) instead, it is possible to improve some results related to random dyadic systems and yield powerful consequences. Most notably, by some new ideas [80], it is possible to give a more precise upper bound for the probability of a point ending up near the boundary of a cube of given size; cf. [A, Lemma 5.15] and [80, Theorem 5.19]. By considering finitely many adjacent dyadic systems a special case of the random dyadic systems, it is possible to use probabilistic arguments in the proofs which allows better results: for every ball, there is a cube that not only contains the ball but we can find such a cube that also the 'dyadic enlargement' of certain level contains the geometric enlargement of the ball; cf. [80, Theorem 5.27].

Dyadic type families on a measure space (X, μ) are basic examples of increasing sequences of σ-algebras that can be used to construct orthonormal bases for $L^2(X, \mu)$; see [35]. In [3], the authors consider the class of all dyadic systems in a space of homogeneous type and the associated Haar systems, and find sufficient conditions on two dyadic systems in order to obtain the equivalence of the corresponding Haar systems on Lebesgue spaces. For a more detailed discussion on the Haar bases associated to a dyadic family, see [2]. These lines are also discussed in [8] where the authors develop the theory of wavelets in spaces of homogeneous type.

Finally, we wish to mention that our construction of non-random choice of finitely many adjacent dyadic systems was already found useful elsewhere [6, 17].

3. Characterization of two-weight estimates by testing conditions

3.1. Introduction.

We are interested in the question of finding necessary and sufficient conditions that guarantee the norm inequality

$$\|S(f\,d\sigma)\|_{L^q_\omega} \leq C\|f\|_{L^p_\sigma}, \quad 1 < p \leq q < \infty, \tag{3.1}$$

for all $f \in L^p_\sigma = L^p(X, d\sigma)$. In our investigations, S is a potential-type operator; a precise definition will be given below in Section 3.2 of this introductory part. Our set-up is a rather general metric measure space: a geometrically doubling quasimetric space (X, ρ) with positive Borel-measures σ and ω that are finite on balls.

We make two elementary observations concerning (3.1). First, an obviously necessary condition for (3.1) is that the desired estimate is true with $f = \chi_E$ where E is an arbitrary measurable set with finite measure. Thus, in order to have the full norm inequality, we in particular need to have that

$$\left(\int_E S(\chi_E\,d\sigma)^q\,d\omega\right)^{1/q} \leq C\sigma(E)^{1/p}. \tag{3.2}$$

The inequality (3.2) is called *a (local) testing condition.*

With a bounded linear operator $S\colon L^p_\sigma \to L^q_\omega$ on a normed space L^p_σ we can associate the adjoint operator $S^*\colon L^{q'}_\omega \to L^{p'}_\sigma$ of S which is defined under the pairing

$$\int_X S(f\,d\sigma)\cdot g\,d\omega = \int_X f\cdot S^*(g\,d\omega)\,d\sigma \quad \text{for all } f \text{ and } g.$$

Here p' is given by the equality $1/p + 1/p' = 1$. Thus, by Hölder's inequality,

$$\int_X f\cdot S^*(g\,d\omega)\,d\sigma = \int_X S(f\,d\sigma)\cdot g\,d\omega \leq \|S(f\,d\sigma)\|_{L^q_\omega}\cdot\|g\|_{L^{q'}_\omega}$$

for all $f \in L^p_\sigma$ and $g \in L^{q'}_\omega$. By using (3.1) and duality, we obtain

$$\|S^*(g\,d\omega)\|_{L^{p'}_\sigma} \leq C\|g\|_{L^{q'}_\omega}.$$

Thus, another necessary condition for (3.1) is that

$$\left(\int_E S^*(\chi_E\, d\omega)^{p'}\, d\sigma\right)^{1/p'} \le C\omega(E)^{1/q'}. \tag{3.3}$$

The inequality (3.3) is called *a (local) dual testing condition.*

A Sawyer-type theorem states that testing conditions are also sufficient for the full norm inequality when tested over a 'representative' collection of sets E: the operator is uniformly bounded if and only if it is bounded on a restricted class of functions, namely indicators of balls or cubes. These type of results go back to E. Sawyer [74, 75, 76] where the two-weight problem was solved for the Hardy–Littlewood maximal operator and other positive operators in the Euclidean space. Or course, for the maximal operator, testing conditions only involve the operator, and no condition for the adjoint operator (which is not well defined) is needed, but for linear operators the characterizations usually involve the adjoint operator.

The second contribution of this dissertation is an application of the adjacent dyadic systems discussed in Section 2.4 to characterizing two-weight strong- and weak-type norm inequalities for potential-type operators by means of Sawyer-type testing conditions, which will be discussed in the following.

3.2. **Potential-type operators and paper [C].** We consider a large class of potential-type operators. More precisely, we study integral operators T of the type

$$T(f\, d\sigma)(x) = \int_X K(x,y) f(y)\, d\sigma(y), \quad x \in X, \tag{3.4}$$

where the kernel $K\colon X \times X \to [0,\infty]$ is a non-negative function which satisfies the following monotonicity conditions: For every $k_2 > 1$ there exists $k_1 > 1$ such that

$$\begin{aligned} K(x,y) &\le k_1 K(x',y) \quad \text{whenever } \rho(x',y) \le k_2\rho(x,y), \\ K(x,y) &\le k_1 K(x,y') \quad \text{whenever } \rho(x,y') \le k_2\rho(x,y). \end{aligned} \tag{3.5}$$

We denote the formal adjoint of T by T^*, which is given by

$$T^*(g\, d\omega)(y) = \int_X K(x,y) g(x)\, d\omega(x), \quad y \in X.$$

Important examples are provided by fractional integrals, which we discuss in more detail in Section 4.4 of this introductory part.

We investigate the two-weight problem for T. Our main result reads as follows:

3.6. **Theorem.** (Theorem 1.12 of [C]) *Let (X,ρ) be a geometrically doubling quasi-metric space and $1 < p \le q < \infty$. Suppose σ and ω are positive Borel measures on X with the property that $\sigma(B) < \infty$ and $\omega(B) < \infty$ for all balls B. Then*

$$\|T\|_{L^p_\sigma \to L^q_\omega} \approx [\sigma,\omega]_{S_{p,q}} + [\omega,\sigma]^*_{S_{q',p'}}, \tag{3.7}$$

and the constants of equivalence only depend on the geometric structure of X, and p and q. Here

$$[\sigma,\omega]_{S_{p,q}} := \sup_Q \sigma(Q)^{-1/p} \|\chi_Q T(\chi_Q\, d\sigma)\|_{L^q_\omega}$$

and

$$[\omega,\sigma]^*_{S_{q',p'}} := \sup_Q \omega(Q)^{-1/q'} \left\| \chi_Q T^*(\chi_Q\, d\omega)\right\|_{L^{p'}_\sigma}$$

are the testing conditions where the supremum is over all dyadic cubes $Q \in \bigcup_{t=1}^T \mathscr{D}^t$*, and* $\infty \cdot 0$ *is interpreted as* 0*.*

We also give a characterization of the corresponding weak-type norm inequalities by dual testing conditions.

We emphasize the fact that no other assumptions, except the ones indicated in the Theorem, are imposed on the space. In particular, our measures are allowed to have atoms and our result applies to any measure space, as described, whether atom free or with atoms, or even to spaces consisting only of atoms, such as $\mathbb{Z}$.

Our result is a refinement of previous results [77, 81]; for related results, see also [73, 76, 82, 85]. Our contribution consists of weakening of the hypothesis as follows: First, our result does not require an underlying doubling measure, only the slightly weaker geometrical doubling property. Second, we do not assume any group structure on the space. Further, we have been able to drop the geometric non-empty annuli property that $B(x,R)\setminus B(x,r) \neq \emptyset$ for all $x \in X$ and $0 < r < R < \infty$ which appeared in the previous papers. Finally, we consider more general measures by allowing atoms.

The proof requires several steps and follows the general approach laid out in the earlier work on the topic. First, the problem is reduced to proving the desired estimate for appropriate model operators. This is done by constructing dyadic operators associated to T and each dyadic system $\mathscr{D}^t$, and the measures σ and ω, and showing that the original operator is pointwise equivalent to the sum of these models over the collection of adjacent dyadic systems. In the previous papers, the dyadic operators did not depend on the measures, but in our situation the presence of prospective atoms requires these extra details. The second step is to prove the testing result for these dyadic model operators. For this, the existing techniques with some technical modifications can be further pushed to yield the desired estimates.

We wish to also mention a related, and evidently quite important paper that was referred to us by the pre-examiner of this dissertation: The two-weight problem for potential-type operators was also studied by C. Pérez [69], where the author, though only working in the Euclidean space only, brings a significant contribution to the development of the technique of bounding potential-type operators by means of their dyadic counter-parts.

We make the important remark that testing over dyadic cubes from just one system $\mathscr{D}^t$ is not enough to obtain the full norm inequality; a counter example was provided in [77, Example 1.9]. Thus, a larger collection of cubes is needed. One of the novelty of our result is the specific collection of cubes involved in the testing conditions: in our result, it suffices to test over countably many dyadic cubes from the adjacent dyadic systems instead of all the translates of the dyadic lattice, which appeared in the previous result on the topic [77, 81]. We mention that the two-weight problem for potential-type operators was also considered in [82] where the authors provided characterizations in spaces of homogeneous type by testing conditions with indicators of balls rather than those of dyadic cubes.

In [C], we also extend the previous Euclidean characterization [74] of two-weight norm inequalities by testing conditions for fractional maximal operators into a space of homogeneous type.

3.3. **Related developments.** Sawyer-type characterizations for the two-weight problem have been verified for positive operators, but they also have been studied for singular integrals. First, in the Euclidean space and the (one-weight) Lebesgue measure situation, it is known that Sawyer-type testing conditions are enough to characterize norm estimates also for singular integral operators. In fact, the testing conditions of Sawyer and the testing conditions that are part of the celebrated $T1$ Theorem of David–Journé [24], which appeared at approximately the same time as Sawyer's results, are equivalent; cf. [79]. For a more detailed discussion of this topic, we refer to [68].

For the two-weight problem, Sawyer-type characterizations by testing conditions have been conjectured for singular integral operators [48]. Recently the result was achieved for the Hilbert transform by Lacey et. al. [49, 52]. For earlier results on the topic, we refer to the work of Nazarov–Treil–Volberg [65, 67, 68], and the papers by Lacey, Sawyer, Shen and Uriarte-Tuero [47, 48, 53].

We mention that there are several approaches to attack the two-weight problem. One active area of research today, and fundamentally different approach in comparison with the Sawyer-type results, involve the so-called "bump" conditions. For the history and recent developments on this approach, we refer to the book [21] and the paper [20].

4. Sharp one-weight estimates

4.1. **Background.** The theory of Muckenhoupt and his contemporaries [14, 37, 60] from the 1970's gives a full qualitative description of the weights w that are "good" in the sense that many classical operators – the Hardy–Littlewood maximal operator, the Hilbert transform, and the general classes of Calderón–Zygmund singular operators – act boundedly on the weighted space $L^p(\mathbb{R}^n, wdx)$. This class consists of weights that satisfy the condition

$$[w]_{A_p} := \sup_{B \text{ a ball}} \left(\frac{1}{|B|} \int_B w\, dx \right) \left(\frac{1}{|B|} \int_B w^{-\frac{1}{p-1}}\, dx \right)^{p-1} < \infty, \quad 1 < p < \infty,$$

nowadays known as the A_p-condition. This pioneering work also includes a similar result for fractional integral operators, or the Riesz potentials, which are defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \quad 0 < \alpha < n,$$

in the Euclidean space $\mathbb{R}^n$. In the non-weighted situation, it is well-known that I_α is a bounded operator from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ if and only if $p > 1$ and $1/p - 1/q =$

α/n. Muckenhoupt and Wheeden [61] solved the qualitative one-weight problem by characterizing the weights w for which the operator

$$I_\alpha\colon L^p(\mathbb{R}^n, w^p dx) \to L^q(\mathbb{R}^n, w^q dx)$$

is bounded: For $p > 1$ and $1/p - 1/q = \alpha/n$, this is the case if and only if

$$[w]_{A_{p,q}} := \sup_{B \text{ a ball}} \left(\frac{1}{|B|}\int_B w^q \, dx\right)\left(\frac{1}{|B|}\int_B w^{-\frac{p}{p-1}}\right)^{q(p-1)/p} < \infty.$$

Muckenhoupt's results led to an extensive study of weighted norm inequalities for other operators. The boundedness of many important operators on the weighted space $L^p(w) = L^p(\mathbb{R}^n, wdx)$ where $1 < p < \infty$ and $w \in A_p$, are now classical results, and the theory has been known for a long time; cf. [30], for example.

More recent developments involve understanding the relationship between operator norms and the weight constants. In the 1990's the attention of analysts shifted form the qualitative theory to the corresponding quantitative questions: Find the sharp dependence of operator norms on weight constants. This question was first investigated by S. Buckley [10]. He proved that for the Hardy–Littlewood maximal operator, the operator norm satisfies

$$\|M\|_{L^p(w)\to L^p(w)} \le C_{n,p}[w]_{A_p}^{\frac{1}{p-1}}, \quad 1 < p < \infty.$$

This result is *sharp* in the sense that $[w]_{A_p}^{1/(p-1)}$ cannot be replaced by $\varphi([w]_{A_p})$ for any positive non-decreasing function $\varphi\colon [0,\infty) \to [0,\infty)$ growing slower than the function $t \mapsto t^{1/(p-1)}$. Later A. Lerner [55] gave an extremely simple proof of Buckley's theorem using a centred version of the maximal operator together with the Besicovitch covering theorem.

In the same paper, Buckley also considered convolution type Calderón–Zygmund singular integral operators T and showed that

$$|T\|_{L^p(w)\to L^p(w)} \le C_{n,p}[w]_{A_p}^s, \tag{4.1}$$

where the best power s is at least $\max\{1, 1/(p-1)\}$ and at most $p/(p-1)$.

Subsequently, there has been increasing interest in finding the optimal powers. In particular, the investigations of Astala–Iwaniec–Saksman [7] showed that sharp estimates have important consequences. This attracted renewed attention to the topic, and the area has been a particularly active field of research for the past ten years. The central question was to solve the sharp constant result for Calderón–Zygmund singular integrals, but sharp constant estimates have also been of interest for other types of operators; the topic has been studied by a number of authors, for example [18, 19, 25, 26, 36, 51, 56, 70, 71, 83, 84].

Extrapolation is one of the most significant and powerful results in the weighted theory. It deals with the surprising phenomenon that, roughly speaking, if an operator satisfies sufficiently many weighted L^r inequalities for some exponent r, then structurally similar weighted L^p estimates follow for all exponents p. More precisely, if an operator $T\colon L^r(w) \to L^r(w)$ is bounded for a single exponent $r \ge 1$ and for all weights $w \in A_r$, then $T\colon L^p(w) \to L^p(w)$ is in fact bounded for all $1 < p < \infty$ and all weights $w \in A_p$. This is the content of the celebrated Rubio de Francia's extrapolation theorem [72], which is one of the highlights of the weighted theory.

At the time the extrapolation theorem was proven, the interest in quantitative estimates had not yet started. However, a careful examination of the original proof provides even these. Sharp versions of extrapolation theorem were achieved in [25], and they track down the dependence of operator norms on weight constants.

Extrapolation theorems have thus become a fundamental tool in the theory of weighted norm inequalities since they imply that in order to obtain (sharp) estimates for all p, it often suffices to obtain these for $p = 2$, or some other critical exponent specific to the particular operator.

The sharp weighted estimate for Calderón–Zygmund operators, or the A_2 conjecture, remained an open problem in this field for a long time. The first full solution was given by T. Hytönen [41] who established the linear dependence with the exponent $p = 2$ and the power $s = 1$ in (4.1). His solution required a probabilistic Dyadic Representation Theorem which states that any Calderón–Zygmund singular integral operator has a representation in terms of simpler dyadic shift operators associated to random dyadic cubes.

Sharp estimates for fractional integral operators were obtained somewhat earlier by Lacey–Moen–Pérez–Torres [50]. They proved that

$$\|I_\alpha\|_{L^p(w^p)\to L^q(w^q)} \le C_{p,q}[w]_{A_{p,q}}^{\left(1-\frac{\alpha}{n}\right)\max\{1,\frac{p'}{q}\}}, \quad p > 1,\ \tfrac{1}{p} - \tfrac{1}{q} = \tfrac{\alpha}{n},$$

and that this result is sharp. We mention that the authors also generalized Buckley's result by proving sharp bounds for fractional maximal operators M_α given by

$$M_\alpha f(x) = \sup_{B \text{ a ball}} \frac{\chi_B(x)}{|B|^{1-\alpha/n}} \int_B |f(y)|dy, \quad 0 \le \alpha < n.$$

The contribution of this dissertation to this area of research is the generalization the mentioned work of Buckley and that of Lacey et. al. to spaces of homogeneous type.

4.2. **A space of homogeneous type.** Our set-up is a space of homogeneous type in the sense of Coifman–Weiss [16]; a quasi-metric space (X, ρ) with a positive Borel-measure μ which satisfies the well-established (measure) doubling property that there exists a constant $C \ge 1$ such that

$$0 < \mu(B(x, 2r)) \le C\mu(B(x, r)) < \infty \quad \text{for all } x \in X \text{ and } r > 0.$$

Well-known properties of doubling measures that are relevant in our investigations are listed in the following lemma.

4.2. **Lemma.** *Suppose μ is a doubling measure on X. Then the following is true.*

(1) Let $x \in X$. Then $\mu(\{x\}) > 0$ if and only if there exists $\varepsilon > 0$ such that $\{x\} = B(x, \varepsilon)$.

(2) If $\mu(\{x\}) \ge \delta > 0$ for all $x \in X$, then X is countable.

(3) $\mu(X) < \infty$ if and only if $X = B(x, R)$ for some $x \in X$ and $R < \infty$.

In fact, the property (2) does not depend on the doubling property but is true for any σ-finite measure.

In our investigations it is further assumed that X is sufficiently non-trivial in that X contains infinitely many points. We mention the following result from [D], which we find interesting and which is relevant in our investigations. It states that such a homogeneous space X is one of the three types listed in the lemma:

4.3. **Lemma.** (Lemma 2.1 of [D]) *Suppose that (X, μ) is a space of homogeneous type and $\#X = \infty$. Then precisely one of the following is satisfied:*

1) $\mu(X) < \infty$;
2) X is countably infinite and $\mu(\{x\}) \geq \delta > 0$ for all $x \in X$;
3) $\mu(B)$ can have arbitrarily small and large values with balls B.

4.3. **Different notions of doubling.** We wish to clarify by the following remarks the basis of the research presented in this dissertation. First, it is well-known that measure doubling implies geometrical doubling [15]. Conversely, every complete metric space that has the geometric doubling property also supports a doubling measure [57] which, in many cases, is the "natural" measure associated to the space. A large part of the research presented in this dissertation (papers [B] and [C], and the main results in [A]) only require the geometric doubling property. Even if often referred as "essentially equivalent", the two properties are, however, not "strictly equivalent". Indeed, there are easy examples of non-complete spaces that have the geometric doubling property but do not support any doubling measures: consider, for example, the set of rational numbers; it is easy to see that $\mathbb{Q}$ equipped with the usual distance satisfies the geometric doubling property whereas there are no doubling measures (to check this, we refer to Lemmata 4.2 and 4.3 above). Another, perhaps more philosophical, motivation for working in geometrically doubling spaces only is the following. Here we are, in some sense, forced to find the most fundamental arguments since only purely geometric considerations are available. This constraint sometimes leads to better results, too. For example, in the original Christ construction of dyadic cubes, depending on a doubling measure, the cubes on each dyadic scale cover the space up to a measure zero while our construction gives an exact partition. These two properties are again "essentially equivalent", but in some situations the slight difference might give trouble.

To another direction, there is the *reverse doubling condition* that for some $C \geq 1$ and $\varepsilon > 0$,

$$\mu(B(x, Cr)) \geq (1 + \varepsilon)\mu(B(x, r)) \quad \text{for all } x \in X \text{ and } 0 < r < \operatorname{diam}(X)/C,$$

which is a common assumption in metric analysis allowing a control from below for the growth of balls. We do not assume this in any of our investigations. The reverse doubling condition relates to the common geometric assumption of the non-emptyness of the annuli that $B(x, R) \setminus B(x, r) \neq \emptyset$ for $x \in X$ and $0 < r < R < \infty$, which appeared in the previous works related to our investigations on potential-type operators. In a space of homogeneous type, these two conditions are equivalent

[78, Lemma 20, p. 11]. The reverse doubling property excludes in particular the presence of point masses; yet another common assumption, even if empty annuli are otherwise allowed, which we also have been able to drop in the research presented in this dissertation.

4.4. **Fractional integral operators and paper [D].** Fractional integrals over quasi-metric measure spaces (X, ρ, μ), which generalize the usual Riesz potential defined in the Euclidean space, have been considered in different forms. One common and widely studied notation; see for example [28, 29, 31, 32, 33, 43, 45, 46], is given by the formula

$$I^s f(x) := \int_X \frac{f(y)\,d\mu(y)}{\rho(x,y)^s}, \quad s > 0.$$

These operators are better suited for non-doubling measure spaces (X, μ) with the *upper Ahlfors regularity condition* that for some $n > 0$,

$$\mu(B(x,r)) \leq C_1 r^n \tag{4.4}$$

where $C_1 > 0$ does not depend on $x \in X$ and $r > 0$. Indeed, I^s is a bounded operator from $L^p(X, \mu)$ to $L^q(X, \mu)$ for $1 < p < q < \infty$ if and only if μ satisfies the condition (4.4) for some $n > 0$, $s = n - \alpha$ with $0 < \alpha < n$, and $1/p - 1/q = \alpha/n$; for this result, see e.g. [31, Theorem 1]. Other types of fractional integrals [5, 34] are given by

$$\mathfrak{T}^\alpha f(x) := \int_X \frac{\rho(x,y)^\alpha}{\mu(B(x,\rho(x,y)))} f(y)\,d\mu(y), \quad \alpha > 0.$$

These operators are better adjusted for and commonly studied in measure spaces (X, μ) with the *lower Ahlfors regularity condition* that for some $n > 0$,

$$\mu(B(x,r)) \geq C_2 r^n, \tag{4.5}$$

where $C_2 > 0$ does not depend on $x \in X$ and $r > 0$. As an easy calculation shows, for example all doubling measures satisfy the lower Ahlfors regularity condition for all $x \in \Omega$ and $0 < r \leq \ell$ where $\ell < \infty$ is a fixed number and $\Omega \subseteq X$ is any open set with the property that $c := \inf_{x \in \Omega} \mu(B(x,\ell)) > 0$. The constant n in (4.5) only depends on the doubling constant and we may further take $C_2 := c(2\ell)^{-n}$.

We further mention that also some further types of fractional integrals have been considered elsewhere; see [28, Chapter 6].

In [D], we study fractional integrals of the type

$$T_\gamma f(x) := \int_X \frac{f(y)\,d\mu(y)}{\mu(B(x,\rho(x,y)))^{1-\gamma}}, \quad 0 < \gamma < 1, \tag{4.6}$$

in a space of homogeneous type. These operators have been studied e.g. in [9, 28, 33, 44]. Obviously, the upper Ahlfors regularity condition (4.4) implies that

$$I^s f(x) = I^{n-\alpha} f(x) \leq C_1 \begin{cases} T_\gamma f(x) \\ \mathfrak{T}^\alpha f(x) \end{cases} \quad \text{and} \quad T_\gamma f(x) \leq C_1 \mathfrak{T}^\alpha f(x)$$

for $f \geq 0$ and $\gamma := \alpha/n$. Similarly, the lower Ahlfors regularity condition (4.5) implies

$$\mathfrak{T}^\alpha f(x) \leq \frac{1}{C_2} \begin{cases} I^{n-\alpha} f(x) \\ T_\gamma f(x) \end{cases} \quad \text{and} \quad T_\gamma f(x) \leq \frac{1}{C_2} I^{n-\alpha} f(x)$$

for $f \geq 0$ and $\gamma := \alpha/n$. If X has a constant dimension in the sense that μ satisfies both the regularity conditions (4.4) and (4.5), then all the three variants of fractional integrals mentioned are equivalent. Accordingly, our results apply to all of them in such spaces. In particular, in the usual Euclidean space $\mathbb{R}^n$ with the Lebesgue measure, all the three operators reduce to the classical Riesz potentials.

The main result in [D] reads as follows:

4.7. **Theorem.** (Theorem 3.3 of [D]) *Let (X, ρ, μ) be a space of homogeneous type. Let $0 < \gamma < 1$ and suppose $1 < p \leq q < \infty$ satisfy $1/p - 1/q = \gamma$. Then*

$$\|T_\gamma\|_{L^p(w^p) \to L^q(w^q)} \lesssim [w]_{A_{p,q}}^{(1-\gamma)\max\left\{1, \frac{p'}{q}\right\}}.$$

The estimate is sharp in any space X with infinitely many points.

The broad outlines of our proof follow the Euclidean proof [50] and involve several reductions. First, it suffices to prove the corresponding sharp weak-type estimate; the strong-type estimates then follow by the Sawyer-type results discussed in Section 3. Second, it is enough to prove the case $p = 1$ and $q = 1/(1-\gamma)$ for the weak-type estimate; the other values of exponents follow from a weak-type extrapolation theorem with sharp constants. Finally, the desired inequality is proved by showing a slightly more general estimate. For the sharpness of the result, we show that any space of homogeneous type with infinitely many points supports functions which, at least locally, behave sufficiently similarly to the basic power functions $|x|^{-\alpha}$ on the Euclidean space, which seems to be a completely new discovery.

4.5. **Related developments.** There is some recent development in the study of extrapolation that the author was not aware of when writing this dissertation; cf. [27]. In particular, there are now better sharp extrapolation results for the "off-diagonal" case $T\colon L^p(w^p) \to L^q(w^q)$ with $w \in A_{p,q}$; see [21, p. 24]. Consequently, the details of the proof of Theorem 4.7 discussed above might be unnecessarily complicated, and the result may, in fact, extend to a still more general setting.

Many 'sharp' results of recent interest can, in fact, be further improved. This was investigated by Hytönen–Pérez [38] where they replace a part of the A_p bounds by weaker A_∞ estimates involving an A_∞ weight constant given by

$$[w]_{A_\infty} := \sup_{B \text{ a ball}} \frac{1}{w(B)} \int_B M(w\chi_B)\, dx.$$

By using appropriate mixed $A_p - A_\infty$ bounds, also the quantitative extrapolation can be made even slightly more precise, as shown in [38]. These lines are also discussed in [40] where the authors provide a slightly more precise version of Buckley's result for the sharp weighted bound for the Hardy–Littlewood maximal function involving these A_∞ constants.

References

[1] Hugo Aimar, Ana Bernardis, and Bibiana Iaffei. Comparison of Hardy-Littlewood and dyadic maximal functions on spaces of homogeneous type. *J. Math. Anal. Appl.*, 312(1):105–120, 2005.

[2] Hugo Aimar, Ana Bernardis, and Bibiana Iaffei. Multiresolution approximations and unconditional bases on weighted Lebesgue spaces on spaces of homogeneous type. *J. Approx. Theory*, 148(1):12–34, 2007.

[3] Hugo Aimar, Ana Bernardis, and Luis Nowak. Equivalence of Haar bases associated with different dyadic systems. *J. Geom. Anal.*, 21(2):288–304, 2011.

[4] Hugo Aimar and Roberto A. Macías. Weighted norm inequalities for the Hardy-Littlewood maximal operator on spaces of homogeneous type. *Proc. Amer. Math. Soc.*, 91(2):213–216, 1984.

[5] Alexandre Almeida and Stefan Samko. Fractional and hypersingular operators in variable exponent spaces on metric measure spaces. *Mediterr. J. Math.*, 6(2):215–232, 2009.

[6] Theresa C. Anderson and Armen Vagharshakyan. A simple proof of the sharp weighted estimate for Calderón–Zygmund operators on homogeneous spaces. Preprint, arXiv:1206.2489, 2012.

[7] Kari Astala, Tadeusz Iwaniec, and Eero Saksman. Beltrami operators in the plane. *Duke Math. J.*, 107(1):27–56, 2001.

[8] Pascal Auscher and Tuomas Hytönen. Orthonormal bases of regular wavelets in spaces of homogeneous type. *Appl. Comput. Harmon. Anal.*, 34(2):266–296, 2013.

[9] Marco Bramanti and M. Cristina Cerutti. Commutators of singular integrals and fractional integrals on homogeneous spaces. In *Harmonic analysis and operator theory (Caracas, 1994)*, volume 189 of *Contemp. Math.*, pages 81–94. Amer. Math. Soc., Providence, RI, 1995.

[10] Stephen M. Buckley. Estimates for operator norms on weighted spaces and reverse Jensen inequalities. *Trans. Amer. Math. Soc.*, 340(1):253–272, 1993.

[11] A.-P. Calderón. Inequalities for the maximal function relative to a metric. *Studia Math.*, 57(3):297–306, 1976.

[12] S.-Y. A. Chang, J. M. Wilson, and T. H. Wolff. Some weighted norm inequalities concerning the Schrödinger operators. *Comment. Math. Helv.*, 60(2):217–246, 1985.

[13] Michael Christ. A $T(b)$ theorem with remarks on analytic capacity and the Cauchy integral. *Colloq. Math.*, 60/61(2):601–628, 1990.

[14] R. R. Coifman and C. Fefferman. Weighted norm inequalities for maximal functions and singular integrals. *Studia Math.*, 51:241–250, 1974.

[15] Ronald R. Coifman and Guido Weiss. *Analyse harmonique non-commutative sur certains espaces homogènes*. Lecture Notes in Mathematics, Vol. 242. Springer-Verlag, Berlin, 1971. Étude de certaines intégrales singulières.

[16] Ronald R. Coifman and Guido Weiss. Extensions of Hardy spaces and their use in analysis. *Bull. Amer. Math. Soc.*, 83(4):569–645, 1977.

[17] Jose M. Conde, José García-Cuerva, and Javier Parcet. Sharp dyadic coverings and nondoubling Calderón–Zygmund theory. Preprint, arXiv:1201.3513, 2012.

[18] David Cruz-Uribe, José María Martell, and Carlos Pérez. Sharp weighted estimates for approximating dyadic operators. *Electron. Res. Announc. Math. Sci.*, 17:12–19, 2010.

[19] David Cruz-Uribe, José María Martell, and Carlos Pérez. Sharp weighted estimates for classical operators. *Adv. Math.*, 229(1):408–441, 2012.

[20] David Cruz-Uribe, Alexander Reznikov, and Alexander Volberg. Logarithmic bump conditions and the two weight boundedness of Calderón–Zygmund operators. Preprint, arXiv:1112.0676, 2012.

[21] David V. Cruz-Uribe, José Maria Martell, and Carlos Pérez. *Weights, extrapolation and the theory of Rubio de Francia*, volume 215 of *Operator Theory: Advances and Applications*. Birkhäuser/Springer Basel AG, Basel, 2011.

[22] Guy David. Morceaux de graphes lipschitziens et intégrales singulières sur une surface. *Rev. Mat. Iberoamericana*, 4(1):73–114, 1988.

[23] Guy David. *Wavelets and singular integrals on curves and surfaces*, volume 1465 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1991.

[24] Guy David and Jean-Lin Journé. A boundedness criterion for generalized Calderón-Zygmund operators. *Ann. of Math. (2)*, 120(2):371–397, 1984.

[25] Oliver Dragičević, Loukas Grafakos, María Cristina Pereyra, and Stefanie Petermichl. Extrapolation and sharp norm estimates for classical operators on weighted Lebesgue spaces. *Publ. Mat.*, 49(1):73–91, 2005.

[26] O. Dragičević and A. Volberg. Sharp estimate of the Ahlfors–Beurling operator via averaging martingale transforms. *Michigan Math. J.*, 51(2):415–435, 2003.

[27] Javier Duoandikoetxea. Extrapolation of weights revisited: new proofs and sharp bounds. *J. Funct. Anal.*, 260(6):1886–1901, 2011.

[28] David E. Edmunds, Vakhtang Kokilashvili, and Alexander Meskhi. *Bounded and compact integral operators*, volume 543 of *Mathematics and its Applications*. Kluwer Academic Publishers, Dordrecht, 2002.

[29] José García-Cuerva and A. Eduardo Gatto. Boundedness properties of fractional integral operators associated to non-doubling measures. *Studia Math.*, 162(3):245–261, 2004.

[30] José García-Cuerva and José L. Rubio de Francia. *Weighted norm inequalities and related topics*, volume 116 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1985. Notas de Matemática [Mathematical Notes], 104.

[31] A. Eduardo Gatto. On fractional calculus associated to doubling and non-doubling measures. In *Harmonic analysis*, volume 411 of *Contemp. Math.*, pages 15–37. Amer. Math. Soc., Providence, RI, 2006.

[32] A. Eduardo Gatto, Carlos Segovia, and Stephen Vági. On fractional differentiation and integration on spaces of homogeneous type. *Rev. Mat. Iberoamericana*, 12(1):111–145, 1996.

[33] A. Eduardo Gatto and Stephen Vági. Fractional integrals on spaces of homogeneous type. In *Analysis and partial differential equations*, volume 122 of *Lecture Notes in Pure and Appl. Math.*, pages 171–216. Dekker, New York, 1990.

[34] Ioseb Genebashvili, Amiran Gogatishvili, Vakhtang Kokilashvili, and Miroslav Krbec. *Weight theory for integral transforms on spaces of homogeneous type*, volume 92 of *Pitman Monographs and Surveys in Pure and Applied Mathematics*. Longman, Harlow, 1998.

[35] Maria Girardi and Wim Sweldens. A new class of unbalanced Haar wavelets that form an unconditional basis for L_p on general measure spaces. *J. Fourier Anal. Appl.*, 3(4):457–474, 1997.

[36] S. Hukovic, S. Treil, and A. Volberg. The Bellman functions and sharp weighted inequalities for square functions. in: Complex Analysis, Operators, and Related Topics, Operator Theory Advances and Applications, vol. 113, Birkhäuser, Basel, 2000, pp. 97–113.

[37] Richard Hunt, Benjamin Muckenhoupt, and Richard Wheeden. Weighted norm inequalities for the conjugate function and Hilbert transform. *Trans. Amer. Math. Soc.*, 176:227–251, 1973.

[38] T. Hytönen and Carlos Pérez. Sharp weighted bounds involving A_∞. Preprint, arXiv:1103.5562, 2011.

[39] Tuomas Hytönen and Henri Martikainen. Non-homogeneous Tb Theorem and Random Dyadic Cubes on Metric Measure Spaces. *J. Geom. Anal.*, 22(4):1071–1107, 2012.

[40] Tuomas Hytönen, Carlos Pérez, and Ezequiel Rela. Sharp Reverse Hölder property for A_∞ weights on spaces of homogeneous type. *J. Funct. Anal.*, 263(12):3883–3899, 2012.

[41] Tuomas P. Hytönen. The sharp weighted bound for general Calderón-Zygmund operators. *Ann. of Math. (2)*, 175(3):1473–1506, 2012.
[42] Antti Käenmäki, Tapio Rajala, and Ville Suomala. Existence of doubling measures via generalised nested cubes. *Proc. Amer. Math. Soc.*, 140(9):3275–3281, 2012.
[43] V. Kokilashvili and A. Meskhi. On some weighted inequalities for fractional integrals on nonhomogeneous spaces. *Z. Anal. Anwendungen*, 24(4):871–885, 2005.
[44] V. M. Kokilashvili and S. G. Samko. Operators of harmonic analysis in weighted spaces with non-standard growth. *J. Math. Anal. Appl.*, 352(1):15–34, 2009.
[45] Vachtang M. Kokilashvili and Alois Kufner. Fractional integrals on spaces of homogeneous type. *Comment. Math. Univ. Carolin.*, 30(3):511–523, 1989.
[46] Vakhtang Kokilashvili and Alexander Meskhi. Fractional integrals on measure spaces. *Fract. Calc. Appl. Anal.*, 4(1):1–24, 2001.
[47] M. T. Lacey, E. T. Sawyer, and I. Uriarte-Tuero. Two weight inequalities for discrete positive operators. Preprint, arXiv:0911.3437, 2009.
[48] Michael Lacey, Eric T. Sawyer, and Ignacio Uriarte-Tuero. A characterization of two weight norm inequalities for maximal singular integrals with one doubling measure. *Anal. PDE*, 5(1):1–60, 2012.
[49] Michael T. Lacey. On the two weight Hilbert transform inequality. Preprint, arXiv:1301.4663, 2013.
[50] Michael T. Lacey, Kabe Moen, Carlos Pérez, and Rodolfo H. Torres. Sharp weighted bounds for fractional integral operators. *J. Funct. Anal.*, 259(5):1073–1097, 2010.
[51] Michael T. Lacey, Stefanie Petermichl, and Maria Carmen Reguera. Sharp A_2 inequality for Haar shift operators. *Math. Ann.*, 348(1):127–141, 2010.
[52] Michael T. Lacey, Eric T. Sawyer, Chun-Yen Shen, and Ignacio Uriarte-Tuero. Two weight inequality for the Hilbert transform: a real variable characterization. Preprint, arXiv:1201.4319, 2012.
[53] Michael T. Lacey, Eric T. Sawyer, and Ignacio Uriarte-Tuero. A two weight inequality for the Hilbert transform assuming an energy hypothesis. *J. Funct. Anal.*, 263(2):305–363, 2012.
[54] A. Lerner. A simple proof of the A_2 conjecture. *Int. Math. Res. Notices*, (2012). DOI: 10.1093/imrn/rns145.
[55] Andrei K. Lerner. An elementary approach to several results on the Hardy-Littlewood maximal operator. *Proc. Amer. Math. Soc.*, 136(8):2829–2833, 2008.
[56] Andrei K. Lerner. Sharp weighted norm inequalities for Littlewood-Paley operators and singular integrals. *Adv. Math.*, 226(5):3912–3926, 2011.
[57] Jouni Luukkainen and Eero Saksman. Every complete doubling metric space carries a doubling measure. *Proc. Amer. Math. Soc.*, 126(2):531–534, 1998.
[58] Henri Martikainen. Vector-valued non-homogeneous Tb theorem on metric measure spaces. *Rev. Mat. Iberoam.*, 28(4):961–998, 2012.
[59] Tao Mei. BMO is the intersection of two translates of dyadic BMO. *C. R. Math. Acad. Sci. Paris*, 336(12):1003–1006, 2003.
[60] Benjamin Muckenhoupt. Weighted norm inequalities for the Hardy maximal function. *Trans. Amer. Math. Soc.*, 165:207–226, 1972.
[61] Benjamin Muckenhoupt and Richard Wheeden. Weighted norm inequalities for fractional integrals. *Trans. Amer. Math. Soc.*, 192:261–274, 1974.
[62] C. Muscalu, T. Tao, and C. Thiele. The bi-Carleson operator. *Geom. Funct. Anal.*, 16(1):230–277, 2006.
[63] Camil Muscalu, Terence Tao, and Christoph Thiele. Multi-linear operators given by singular multipliers. *J. Amer. Math. Soc.*, 15(2):469–496 (electronic), 2002.
[64] F. Nazarov, S. Treil, and A. Volberg. Cauchy integral and Calderón-Zygmund operators on nonhomogeneous spaces. *Internat. Math. Res. Notices*, (15):703–726, 1997.

[65] F. Nazarov, S. Treil, and A. Volberg. The Bellman functions and two-weight inequalities for Haar multipliers. *J. Amer. Math. Soc.*, 12(4):909–928, 1999.

[66] F. Nazarov, S. Treil, and A. Volberg. The Tb-theorem on non-homogeneous spaces. *Acta Math.*, 190(2):151–239, 2003.

[67] F. Nazarov, S. Treil, and A. Volberg. Two weight inequalities for individual Haar multipliers and other well localized operators. *Math. Res. Lett.*, 15(3):583–597, 2008.

[68] Fedor Nazarov, Sergei Treil, and Alexander Volberg. Two weight estimate for the Hilbert transform and corona decomposition for non-doubling measures. Preprint, arXiv:1003.1596, 2005/2010.

[69] Carlos Pérez. Two weighted inequalities for potential and fractional type maximal operators. *Indiana Univ. Math. J.*, 43(2):663–683, 1994.

[70] S. Petermichl. The sharp bound for the Hilbert transform on weighted Lebesgue spaces in terms of the classical A_p characteristic. *Amer. J. Math.*, 129(5):1355–1375, 2007.

[71] Stefanie Petermichl. The sharp weighted bound for the Riesz transforms. *Proc. Amer. Math. Soc.*, 136(4):1237–1249, 2008.

[72] José L. Rubio de Francia. Factorization theory and A_p weights. *Amer. J. Math.*, 106(3):533–547, 1984.

[73] E. Sawyer and R. L. Wheeden. Weighted inequalities for fractional integrals on Euclidean and homogeneous spaces. *Amer. J. Math.*, 114(4):813–874, 1992.

[74] Eric T. Sawyer. A characterization of a two-weight norm inequality for maximal operators. *Studia Math.*, 75(1):1–11, 1982.

[75] Eric T. Sawyer. A two weight weak type inequality for fractional integrals. *Trans. Amer. Math. Soc.*, 281(1):339–345, 1984.

[76] Eric T. Sawyer. A characterization of two weight norm inequalities for fractional and Poisson integrals. *Trans. Amer. Math. Soc.*, 308(2):533–545, 1988.

[77] Eric T. Sawyer, Richard L. Wheeden, and Shiying Zhao. Weighted norm inequalities for operators of potential type and fractional maximal functions. *Potential Anal.*, 5(6):523–580, 1996.

[78] Jan-Olov Strömberg and Alberto Torchinsky. *Weighted Hardy spaces*, volume 1381 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1989.

[79] Terence Tao. The T(b) theorem and its variants. In *International Conference on Harmonic Analysis and Related Topics*, volume 41 of *Proc. Cent. Math. Appl.*, pages 143–160. Aust. Natl. Univ., 2003.

[80] Olli Tapiola. Random and non-random dyadic systems in doubling metric spaces. *Master's thesis, University of Helsinki*, 2012.

[81] Igor E. Verbitsky and Richard L. Wheeden. Weighted norm inequalities for integral operators. *Trans. Amer. Math. Soc.*, 350(8):3371–3391, 1998.

[82] Richard L. Wheeden and Shiying Zhao. Weak type estimates for operators of potential type. *Studia Math.*, 119(2):149–160, 1996.

[83] J. Wittwer. A sharp estimate on the norm of the martingale transform. *Math. Res. Lett.*, 7(1):1–12, 2000.

[84] J. Wittwer. A sharp estimate on the norm of the continuous square function. *Proc. Amer. Math. Soc.*, 130(8):2335–2342, 2002.

[85] Shi Ying Zhao. On weighted inequalities for operators of potential type. *Colloq. Math.*, 69(1):95–115, 1995.