MISSPECIFICATION AND BOOTSTRAP TESTS IN MULTIVARIATE TIME SERIES MODELS WITH CONDITIONAL HETEROSEDASTICITY

PAUL CATANI
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Key words: Conditional heteroskedasticity, bootstrap, vector autoregressive model, misspecification test

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Paul Catani
Hanken School of Economics
Department of Finance and Statistics
P.O.Box 479, 00101 Helsinki, Finland

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THE ESSAYS


1 Introduction

This thesis has two main themes. The first is conditional heteroskedasticity. Heteroskedasticity is often encountered in economic and financial time series. If we consider a time series as a realisation of a stochastic process, then the time series is heteroskedastic if the variance of the process differs at different points in time. We may distinguish between two forms of heteroskedasticity: unconditional and conditional heteroskedasticity. Unconditional heteroskedasticity in a time series does not depend on past observations but may, for example, be caused by breaks in volatility. Heteroskedasticity causes, among other things, the least squares estimates of the standard errors in regression models estimated on time series data to be invalid, and therefore makes inference unreliable. Different ways of dealing with problems caused by heteroskedasticity have been developed. Heteroskedasticity-consistent covariance matrix estimators and weighted least squares offer valid inference under unconditional heteroskedasticity.

Conditional heteroskedasticity was introduced in econometrics by Engle (1982). Conditional heteroskedasticity occurs conditional on past information. Engle introduced the autoregressive conditional heteroskedasticity (ARCH) model and a test for ARCH effects in time series. In the ARCH model the variance of a time series process is dependent on past values of the time series. Let $\varepsilon_t$ be a serially uncorrelated random variable with variance $h_t$, conditional on all past information, that is $\text{Var}(\varepsilon_t|\mathcal{F}_{t-1}) = h_t$, where $\mathcal{F}_{t-1}$ is the sigma-field generated by all past observations. The first-order ARCH process is given by

$$\varepsilon_t = \sqrt{h_t} z_t, \quad h_t = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2, \quad z_t \sim \text{IID}(0,1),$$

where $\alpha_0 > 0$ and $0 \leq \alpha_1 < 1$. Since the introduction of the ARCH model, numerous variations and extensions have been proposed, and the literature on volatility modelling is still expanding rapidly. Many recent developments are dealing with modelling multivariate ARCH processes.

Conditional heteroskedasticity causes many asymptotic tests in time series models not to be valid. For example, tests for autocorrelation typically assume independent and identically distributed (IID) errors.

The second main theme of the thesis is the bootstrap. Improvements of small-sample properties of estimators and test statistics are topics under much research in econometrics. Bootstrap methods provide solutions to some of the problems. The bootstrap is a statistical technique for estimating the distributions of parameter estimates and test statistics that have unknown distributions. Thus we may draw inference based on the estimated distributions instead of asymptotic distributions which can be quite inaccurate in small samples. Early applications of the bootstrap are bootstrap standard errors and confidence intervals. Bootstrap methods
have been developed for a variety of different purposes, including unit root and cointegration tests.

Under conditional heteroskedasticity, standard residual based bootstrap methods that treat the error terms as IID cannot be used. The wild bootstrap is a method for dealing with conditional heteroskedasticity. Unlike standard IID bootstrap methods, the wild bootstrap replicates the pattern of heteroskedasticity in the original time series, and provides a solution to the problems with inference under conditional heteroskedasticity. The asymptotic validity of the wild bootstrap for autoregressive models under conditional heteroskedasticity of unknown form was established by Gonçalves and Kilian (2004). Recent research is dealing with issues such as volatility shifts and unstable volatility in non-stationary time series, and the application of wild bootstrap methods in such situations, e.g. Cavaliere and Taylor (2008), Cavaliere, Rahbek and Taylor (2010a, 2010b and 2012a).

The thesis consists of four separate papers dealing with conditional heteroskedasticity in multivariate time series models. The first paper studies wild bootstrap tests for autocorrelation in vector autoregressive (VAR) models with conditional heteroskedasticity. The effects of conditional heteroskedasticity on tests for autocorrelation are investigated through simulation and empirical examples. The second paper is an empirical study of tests for cointegration in the presence of conditional heteroskedasticity. It considers Chinese stock price data, in which the effects of conditional heteroskedasticity can be clearly observed. The third paper proposes and studies a new Lagrange multiplier test for the adequacy of a constant conditional correlation generalized autoregressive conditional heteroskedasticity (CCC-GARCH) model. The fourth paper studies tests for ARCH in VAR models. Tests considered are combined Monte Carlo (MC) tests introduced by Dufour, Khalaf and Beaulieu (2010), a multivariate Lagrange multiplier (LM) test described in Lütkepohl (2006) and asymptotic and bootstrap versions of the LM test proposed by Eklund and Teräsvirta (2007) for testing constancy of the error covariance matrix of a VAR model.

The contents of this introduction are the following. Sections 2–11 give an introduction to conditional heteroskedasticity in multivariate time series models, and the methods used in the essays. An overview of the essays is given in Section 12, and Section 13 concludes.

2 Multivariate Time Series Models

Let \( \{X_t\} \) be a time series variable observed at time \( t, t = 1, \ldots, T \). A widely used model, where \( X_t \) is a linear function of past values of itself, is the autoregressive (AR) model of order \( p \), given by
\[ X_t = \mu + \pi_1 X_{t-1} + \cdots + \pi_p X_{t-p} + \varepsilon_t, \quad (1) \]

where \( \mu \) is the constant and \( \{\varepsilon_t\} \) is an independent and identically distributed (IID) random variable with mean zero and variance \( \sigma^2 \). However, many economic time series not only depend on their own past values, but also on values of other economic variables. Therefore, many time series are best considered as components of a multivariate time series \( X_t = (X_{1t}, X_{2t}, \ldots, X_{Kt})' \), which not only allows for serial dependence within each component but also interdependence between the components. A multivariate generalization of the AR model is the vector autoregressive (VAR) model of order \( p \), given by

\[
X_t = \mu + \Pi_1 X_{t-1} + \cdots + \Pi_p X_{t-p} + \varepsilon_t
\]

\[
= \mu + \sum_{j=1}^{p} \Pi_j X_{t-j} + \varepsilon_t, \quad t = 1, \ldots, T,
\]

where \( \mu \) is a \( K \times 1 \) constant vector, \( \Pi_j, j = 1, \ldots, p \), are \( K \times K \) parameter matrices and \( \varepsilon_t = (\varepsilon_{1t}, \ldots, \varepsilon_{Kt})' \) is a \( K \)-dimensional IID vector such that

\[
\mathbb{E}(\varepsilon_t) = 0 \quad \text{and} \quad \mathbb{E}(\varepsilon_t \varepsilon_{t-h}') = \begin{cases} \Omega, & h = 0, \\ 0, & h \neq 0, \end{cases}
\]

where \( \Omega \) is a positive definite covariance matrix. Note that the VAR model does not contain current values of \( X_t \), i.e. the parameter matrix on \( X_t \) is \( \Pi_0 = I_K \).

The VAR process (2) can be written as

\[
(I_K - \Pi_1 L - \Pi_2 L^2 - \cdots - \Pi_p L^p) X_t = \mu + \varepsilon_t,
\]

where \( L \) is the lag operator: \( LX_t = X_{t-1} \). The process \( \{X_t\} \) is stable if it satisfies the condition

\[
\det(I_K - z \Pi_1 - z^2 \Pi_2 - \cdots - z^p \Pi_p) \neq 0 \quad \text{for} \quad |z| \leq 1.
\]

If \( X_t \) satisfies the condition (4), it is stationary and integrated of order 0, \( I(0) \). A nonstationary VAR process \( X_t \) is integrated of order 1, \( I(1) \), if the first difference \( \Delta X_t = X_t - X_{t-1} \) is stationary.

Suppose now that \( X_t \) is \( I(1) \). The first difference of \( X_t \) can then be written in error correction form as

\[
\Delta X_t = \mu + \Pi X_{t-1} + \Gamma_1 \Delta X_{t-1} + \cdots + \Gamma_{p-1} \Delta X_{t-p+1} + \varepsilon_t,
\]

where \( \Pi = \sum_{j=1}^{p} \Pi_j - I_K \) and \( \Gamma_j = \sum_{k=j+1}^{p} \Pi_k \). A nonstationary VAR process can have a stationary long run relation between the individual time series. If \( X_t \) is
nonstationary, it is cointegrated if there exists a nonzero vector $\beta$ such that $\beta'X_t$ is $I(0)$. The vector $\beta$ is the cointegrating vector.

3 Conditional Heteroskedasticity in Multivariate Time Series Models

All four essays deal with multivariate time series models with conditional heteroskedasticity. Conditional heteroskedasticity occurs conditional on past information. Since the introduction of autoregressive conditional heteroskedasticity (ARCH) by Engle (1982), modelling volatility has received much attention in financial econometrics. For example, in stock returns volatility clustering is often observed, where high volatility at time $t - 1$ is followed by high volatility at time $t$. Conditional heteroskedasticity causes many asymptotic tests in time series models not to be valid.

For example, tests for autocorrelation assume IID errors.

Let $\varepsilon_t$ be a serially uncorrelated random variable with variance $h_t$, conditional on all past information, that is $\text{Var}(\varepsilon_t | \mathcal{F}_{t-1}) = h_t$, where $\mathcal{F}_{t-1}$ is the sigma-field generated by all past observations. Engle (1982) defines the first order ARCH process

$$
\varepsilon_t = \sqrt{h_t}z_t, \quad h_t = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2, \quad z_t \sim \text{IID}(0, 1).
$$

The constraints $\alpha_0 > 0$ and $\alpha_1 \geq 0$ ensure that the process is positive. Furthermore, the process is stationary when $\alpha_1 < 1$. A generalisation of the ARCH model is the generalized ARCH (GARCH) model, introduced by Bollerslev (1986). The GARCH model augments the ARCH model by adding lags of $h_t$ to the model, so that $h_t$ depends on past values of itself in an autoregressive way. The GARCH(1,1) model of $h_t$ is given by

$$
h_t = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \beta_1 h_{t-1},
$$

where $\alpha_0 > 0$ and $\alpha_1$, $\beta_1 \geq 0$. The model is stationary if $\alpha_1 + \beta_1 < 1$. The GARCH(1,1) model has become one of the most widely used models for conditional heteroskedasticity in econometrics. The extensions proposed to the model are numerous, see e.g. Engle (1995) for a survey.

Multivariate GARCH (MGARCH) models have been proposed for modelling volatility in multivariate time series. Let $X_t$ be a $K$-dimensional time series process generated by

$$
X_t = \mu_t + \varepsilon_t,
$$

$$
\varepsilon_t = H_t^{1/2}z_t, \quad t = 1, \ldots, T,
$$

where $\mu_t$ is the conditional mean of $X_t$ and $\{z_t\}$ is a sequence of IID random variables
such that $E(z_tz_t') = I_K$. Then

$$E(\varepsilon_t|\mathcal{F}_{t-1}) = 0 \quad \text{and} \quad E(\varepsilon_t\varepsilon_t'|\mathcal{F}_{t-1}) = H_t,$$

where $H_t$ is the conditional covariance matrix given the sigma field $\mathcal{F}_{t-1}$ generated by the observed time series up to time $t-1$. There exists a large number of formulations for $H_t$, of which two families have become popular. The first are the Vector (VEC) and Baba-Engle-Kraft-Kroner (BEKK) GARCH models of Bollerslev, Engle and Wooldridge (1988), and Engle and Kroner (1995), respectively. These models are based on modelling $H_t$ directly. The BEKK-GARCH(1,1) model is given by the quadratic form

$$H_t = CC' + A'\varepsilon_{t-1}\varepsilon_{t-1}'A + B'H_tB,$$

where $A$, $B$ and $C$ are $K \times K$ parameter matrices and $C$ is lower triangular.

The second family are models of conditional variances and correlations including the constant conditional correlation GARCH (CCC-GARCH) model of Bollerslev (1990). The CCC-GARCH model is nested within other members of this family. The models of conditional variances and correlations are based on decomposing $H_t$ into conditional correlations and standard deviations, so that $H_t = D_tP_tD_t \cdot$ Here $D_t = \text{diag}(h_{1t}^{1/2}, \ldots, h_{Kt}^{1/2})$ is a diagonal matrix of conditional standard deviations and $P_t = [\rho_{ijt}], \ i, j = 1, \ldots, K,$ is a positive definite matrix of conditional correlations with ones on the main diagonal.

The CCC-GARCH model is one of the most widely used MGARCH models and also the one considered in Essays 1, 3 and 4. In the CCC-GARCH(1,1) model the correlation matrix $P_t = P$ is constant over time. Let $\varepsilon_t$ follow a CCC-GARCH model according to

$$\varepsilon_t = D_tz_t. \quad (8)$$

The sequence $\{z_t\}$ is a sequence of IID random variables with mean 0 and positive definite covariance matrix $P = [\rho_{ij}]$. The variance equations of $\{\varepsilon_t\}$ are defined as follows:

$$h_t = (h_{1t}, \ldots, h_{Kt}) = a_0 + A_1\varepsilon_{t-1}^{(2)} + B_1h_{t-1}, \quad (9)$$

where $\varepsilon_{t}^{(2)} = (\varepsilon_{1t}^{2}, \ldots, \varepsilon_{Kt}^{2})'$ is a $(K \times 1)$ vector, $a_0 = (\alpha_0, \ldots, \alpha_0K)'$ is a $(K \times 1)$ vector of positive constants, and $A_1$ and $B_1$ are $(K \times K)$ parameter matrices which are diagonal with non-negative diagonal elements $a_{ii}$ and $\beta_{ii}, i = 1, \ldots, K$, respectively. Setting $B = 0$ yield the CCC-ARCH model of Cecchetti, Cumby and Figlewski (1988).

Jeantheau (1998) proposes an Extended CCC (ECCC)-GARCH model where some of the off-diagonal elements have non-zero values. See also He and Teräsvirta (2004), who derive the fourth-moment structure for the ECCC-GARCH model.

Multivariate GARCH models can be estimated by maximum likelihood and the
adequacy of the models tested with Lagrange multiplier tests. For those purposes we need the log-likelihood function. Let \( \theta = (\omega', \rho')' \), where \( \omega \) contains the parameters in \( h_t \) and \( \rho = \text{vecl}(P) \). The operator \( \text{vecl} \) stacks the lower off-diagonal elements of \( P \) into a \( K(K-1)/2 \) vector. Using the Gaussian log-likelihood, the parameter estimates are obtained by maximising

\[
\frac{1}{T} \sum_{t=1}^{T} \ell_t(\theta) = \frac{1}{T} \sum_{t=1}^{T} \left( -\frac{1}{2} \ln(2\pi) - \ln |D_t| - \frac{1}{2} \ln |P| - \frac{1}{2} \varepsilon_t^\prime H_t^{-1} \varepsilon_t \right),
\]

conditional on \((X_0, \varepsilon_0)\), where \( H_t^{-1} = D_t^{-1}P^{-1}D_t^{-1} \). Since \( z_t \) are not assumed to be normal, the estimators are quasi-maximum-likelihood estimators (QMLE). Consistency and asymptotic normality of the QMLE is established under conditions given in Ling and McAleer (2003).

A number of other MGARCH models have also been proposed, some of which allow the correlations to change over time. See e.g. Bauwens, Laurent and Rombouts (2006) and Silvennoinen and Teräsvirta (2009) for extensive reviews of different MGARCH models.

4 The Bootstrap

Asymptotic theory may be a poor approximation to the distributions of estimators and test statistics in finite samples. One way of improving inference in small samples is bootstrapping. Bootstrapping is used to estimate the distributions of estimators and test statistics by resampling the data. The method was introduced by Efron (1979) and the name comes from the expression 'to pull oneself up one’s own bootstraps'. Essays 1, 2 and 4 consider bootstrap tests in multivariate time series models, where the bootstrap is used to reduce size distortions of tests.

Let \((X_1, X_2, \ldots, X_T)\) be a random sample of size \( T \) of IID random variables from a probability distribution with cumulative distribution function (CDF) \( F \) and parameter \( \theta \) characterizing the distribution. Let \( \tau = \tau(X_1, X_2, \ldots, X_T) \) be a test statistic upon which inference on \( \theta \) is based. Usually the exact distribution of \( \tau \) is not known. Instead of using asymptotic theory for inference, the finite sample distribution of \( \tau(X_1, X_2, \ldots, X_T) \) can be approximated using bootstrap.

One method of applying the bootstrap is a nonparametric bootstrap, which requires no assumptions about the distribution of the data. To simulate bootstrap samples, we sample independently from \((X_1, X_2, \ldots, X_T)\) with replacement. The bootstrap sample is denoted \((X_1^*, X_2^*, \ldots, X_T^*)\), where the star (*) notation is used to denote bootstrap quantities. From the bootstrap sample we calculate the statistic \( \tau^* = \tau(X_1^*, X_2^*, \ldots, X_T^*) \). This is repeated \( B \) times, where \( B \) is a sufficiently large number, to produce \( \tau_1^*, \ldots, \tau_B^* \). The empirical distribution of the bootstrap replications is used to approximate the distribution of \( \tau \) under the null hypothesis.
One way of performing bootstrap tests is by constructing bootstrap \( p \)-values. The \( p \)-values for a one-sided test, which rejects for large \( \tau \), can be calculated as

\[
\hat{p}^*(\tau) = \frac{1}{B} \sum_{b=1}^{B} I(\tau_b^* \geq \tau),
\]

where \( I(\cdot) \) is the indicator function.

Another method is a parametric bootstrap, which requires assumptions about the distribution of the data. The parameter \( \theta \) is estimated by \( \hat{\theta}_n \), and the unknown distribution function \( F(x|\theta) \) is replaced by \( F(x|\hat{\theta}_n) \). The bootstrap samples are then drawn from \( F(x|\hat{\theta}_n) \). The difference between the parametric and the nonparametric bootstrap is that in the parametric bootstrap the bootstrap samples are not re-samples of the original data, but new samples drawn from \( F(x|\hat{\theta}_n) \). The bootstrap \( p \)-value can again be calculated by (11).

In regression and time series models we resample the residuals. Consider, for example, the first order VAR model

\[
X_t = \Pi X_{t-1} + \epsilon_t, \quad \epsilon_t \sim \text{IID}(0, \sigma^2 I_K), \quad t = 1, \ldots, T,
\]

and suppose that we are interested in testing a hypothesis about the parameter \( \Pi \), e.g. \( H_0 : \Pi = \Pi_0 \) against \( H_1 : \Pi \neq \Pi_0 \). One possibility is to obtain the unrestricted residuals from

\[
\hat{\epsilon}_t = X_t - \hat{\Pi} X_{t-1},
\]

where \( \hat{\Pi} \) is the least squares estimate of \( \Pi \). We can then simulate bootstrap samples recursively from

\[
X^*_t = \hat{\Pi} X^*_{t-1} + \hat{\epsilon}^*_t,
\]

where \( \hat{\epsilon}^*_t \) is a random draw with replacement from \( \{\hat{\epsilon}_t\}, t = 1, \ldots, T \). Since \( \epsilon_t \) are assumed to be IID, this bootstrap procedure is referred to as the IID bootstrap.

5 Bootstrapping Autoregressions under Conditional Heteroskedasticity

If the errors are conditionally heteroskedastic, standard IID bootstrap methods are not valid.

Consider bootstrapping a test statistic to test some hypothesis about the parameters in the AR(1) model

\[
X_t = \mu + \pi_1 X_{t-1} + \epsilon_t, \quad \text{E}(\epsilon_t) = 0, \quad \text{E}(\epsilon_t^2) = \sigma^2_t,
\]

when there is heteroskedasticity of unknown form in the data. The test statistic
should be computed using a heteroskedasticity-consistent covariance matrix estimator (HCCME) (Davidson and MacKinnon (2007)). One of the best known HCCMEs is the one proposed by White (1980):

\[ \text{Var}(\hat{\pi}) = (X'X)^{-1}, \]

where \( \hat{\pi} = (\hat{\mu}, \hat{\mu}_1)' \), \( X = (\epsilon, X_{t-1}) \) is a \( T \times 2 \) matrix of regressors \( \epsilon \) is a vector of ones, \( X_{t-1} \) is a vector of lagged values and \( \hat{\Omega} \) is a \( T \times T \) diagonal matrix with squared residuals of the main diagonal. A number of possible adjustments to the HCCME are discussed and studied in Davidson and Flachaire (2008).

One way of implementing the bootstrap when the errors are conditionally heteroskedastic is the pairs bootstrap, proposed by Freedman (1981). The idea of the pairs bootstrap is to resample the original data in pairs of dependent and lagged variables \((X_t, X_{t-1})\). The procedure does not assume that the error terms are IID, but assumes instead that all the data are IID drawings from a multivariate distribution (Davidson and MacKinnon (2007)). Gonçalves and Kilian (2004) extend Freedman’s method to the context of autoregressive models and establish the asymptotic validity of the method.

An alternative to the pairs bootstrap is to use original residuals that are multiplied by random draws from a standard normal distribution, or some two-point distribution. That is, instead of using the bootstrap errors, \( \hat{\varepsilon}_t^* \), we use wild bootstrap errors \( \hat{e}_t^* \) given by

\[ \hat{e}_t^* = \hat{e}_t w_t, \]

where the scalar \( w_t \) is a random draw from a distribution with zero mean and unit variance, for example \( w_t \sim \text{NID}(0,1) \). This method is called the wild bootstrap (WB). It was developed by Liu (1988) and Mammen (1993) following suggestions by Wu (1986) and Beran (1986). Two-point distributions are discussed, among others, in Davidson, Monticini and Peel (2007). Davidson and Flachaire (2008) consider different possibilities for the distribution of \( w_t \), and find that the Rademacher distribution, where

\[ w_t = \begin{cases} 1, & \text{with probability } p = \frac{1}{2}, \\ -1, & \text{with probability } p = \frac{1}{2}, \end{cases} \]

(18)

gives the best result in many instances.

The wild bootstrap can be implemented in two ways. One is the recursive-design wild bootstrap, where the bootstrap sample is generated recursively as described in the previous section, but using the WB errors as in (17). That is, the bootstrap samples are generated by the recursion

\[ X_t^* = \hat{\mu} + \hat{\pi}_1 X_{t-1}^* + \hat{e}_t^*. \]
Following Gonçalves and Kilian (2004) the recursion is initalised at $X_0^* = 0$. The other method is the fixed-design WB where the bootstrap sample is generated according to

$$X_t^* = \hat{\mu} + \hat{\pi}_1 X_{t-1} + \hat{\epsilon}_t^*.$$  

(20)

The difference is that the fixed-design WB uses the lags of the original data to generate $X_t^*$.

The asymptotic validity of the three bootstrap methods under conditional heteroskedasticity of unknown form was established by Gonçalves and Kilian (2004). They assume the following conditions:

(i) $E(\epsilon_t|\mathcal{F}_{t-1}) = 0$, almost surely, where $\mathcal{F}_{t-1} = \sigma(\epsilon_{t-1}, \epsilon_{t-2}, \ldots)$ is the $\sigma$-field generated by $\{\epsilon_{t-1}, \epsilon_{t-2}, \ldots\}$.

(ii) $E(\epsilon_t^2) = \sigma^2 < \infty$.

(iii) $\lim_{n \to \infty} n^{-1} \sum_{t=1}^n E(\epsilon_t^2|\mathcal{F}_{t-1}) = \sigma^2 > 0$ in probability.

(iv) $\tau_{r,s} = \sigma^4 E(\epsilon_t^2 \epsilon_t \epsilon_{t-r} \epsilon_{t-s})$ is uniformly bounded for all $t, r \geq 1, s \geq 1; \tau_{r,s} > 0$ for all $r$.

(v) $\lim_{n \to \infty} n^{-1} \sum_{t=1}^n \epsilon_{t-r} \epsilon_{t-s} E(\epsilon_t^2|\mathcal{F}_{t-1}) = \sigma^4 \tau_{r,s}$, in probability for any $r \geq 1, s \geq 1$.

(vi) $E|\epsilon_t|^4$ is uniformly bounded, for some $r > 1$.

under which the pairs bootstrap and the fixed design WB are valid. Assumptions (i) and (ii) replace the IID assumption on the errors $\{\epsilon_t\}$ by the martingale difference (MD) sequence assumption. Assumption (iii) requires convergence of the conditional moments. Assumptions (iv) and (v) restrict the fourth-order cumulants of $\epsilon_t$. The assumptions require $\{\epsilon_t\}$ to be weakly stationary, but cover a wide range of conditionally heteroskedastic models, such as ARCH and GARCH, instead of imposing conditional homoskedasticity on $\{\epsilon_t\}$.

The recursive-design WB, however, requires a strengthening of the conditions. For the asymptotic validity of the recursive-design WB, conditions (iv) and (vi) are replaced by

(iv') $E(\epsilon_t^2 \epsilon_{t-r} \epsilon_{t-s}) = 0$ for all $r \neq s$, for all $t, r \geq 1, s \geq 1$.

(vi') $E|\epsilon_t|^4$ is uniformly bounded, for some $r \geq 2$ and for all $t$.

Assumption (vi') requires the existence of at least 8th moments for the MD sequence $\{\epsilon_t\}$. In particular, Gonçalves and Kilian show that Assumption (iv') ensures consistency of the recursive-design WB estimator and Assumption (vi') is required for the convergence of the recursive-design WB variance estimator to the correct limiting variance.

Gonzalves and Kilian find that the recursive-design WB is more accurate in finite samples compared to the fixed-design WB and the pairs bootstrap. The pairs bootstrap tends to be more accurate than the fixed design WB, but for large samples.
the differences vanish. They also find that the recursive design works well even when the assumptions (iv') and (vi') are not satisfied. Flachaire (2005) provides simulation evidence that suggests that the WB works better than the pairs bootstrap in regression models with unconditional heteroskedasticity.

The results of Gonçalves and Kilian (2004) apply to stationary time series, but the wild bootstrap can also be applied to time series that are $I(1)$. Cavaliere and Taylor (2008) consider bootstrap approaches to testing for unit roots when volatility shifts are present in the time series. While standard unit root tests become unreliable under such features, bootstrap methods based on the WB are found to perform well when the volatility process is nonstochastic. They find that wild bootstrap tests perform well even in small samples.

Essay 1 and 2 consider recursive-design WB tests for VAR models and show that inference based on them is more reliable than inference based on asymptotic tests when there is conditional heteroskedasticity in the data. The recursive design WB is generalized to the multivariate case, where the WB residuals are generated according to

$$\hat{e}_t^* = \hat{e}_t w_t. \quad (21)$$

That is, the residuals of all $K$ equations in the VAR model at time $t$ are multiplied by the same $w_t$. The bootstrap samples are then generated according to (14). The conditions from Gonçalves and Kilian for the asymptotic validity of the recursive design WB are adapted to the multivariate case of conditional heteroskedasticity.

6 Model Selection and Misspecification Tests

An underlying theme in the thesis is model selection using misspecification tests. When doing inference in time series models a crucial step is selecting the model. In the thesis VAR models and MARCH models are considered. A popular way of selecting a model is using information criteria such as the Akaike (AIC) or the Schwartz information criteria (SC). The criteria are calculated for a number of estimated models, and the model which minimizes the criteria is selected. Misspecification tests are a way of checking if the selected model satisfies the assumptions, e.g. no autocorrelation or heteroskedasticity. Misspecification tests have an important role in the evaluation of econometric models.

Essay 1 shows that under conditional heteroskedasticity commonly used tests for autocorrelation are oversized. A misspecification test for testing the adequacy of a CCC-GARCH model is proposed in Essay 3. Tests for conditional heteroskedasticity in VAR models are considered in Essay 4. Essay 2 is an empirical example of using wild bootstrap and misspecification tests in cointegrated VAR models.

Misspecification tests considered in the thesis are presented in the subsequent section. Tests for error autocorrelation are presented in section 7 and section 8
presents tests for ARCH. The LM test proposed in essay 3 for testing the adequacy of a CCC-GARCH model is presented in section 9.

7 Tests for Error Autocorrelation in VAR Models

Lagrange multiplier (LM) tests for error autocorrelation, originally proposed by Breusch (1978) and Godfrey (1978), are widely used in econometrics. These tests are commonly referred to as Breusch–Godfrey (BG) tests. In the tests a VAR($h$) model is considered for the error terms of the VAR given in (2):

$$\varepsilon_t = \Psi_1 \varepsilon_{t-1} + \cdots + \Psi_h \varepsilon_{t-h} + \varepsilon_t.$$  \hspace{1cm} (22)

The hypothesis being tested is

$$H_0 : \Psi_1 = \cdots = \Psi_h = 0 \quad \text{against} \quad H_1 : \Psi_j \neq 0 \text{ for at least one } j, 1 \leq j \leq h.$$

The test is implemented by first estimating a VAR model for $X_t$, and obtaining the residuals $\hat{\varepsilon}_t$. Then an auxiliary regression of the residuals is run on all the lags of $X_t$ in the VAR model and the lagged residuals up to order $h$:

$$\hat{\varepsilon}_t = \pi + \Pi_1 X_{t-1} + \cdots + \Pi_p X_{t-p} + \Psi_1 \hat{\varepsilon}_{t-1} + \cdots + \Psi_h \hat{\varepsilon}_{t-h} + \varepsilon_t$$

$$= \pi + (Z_t' \otimes I_K) \phi + \hat{\Phi}_t \psi + \varepsilon_t,$$

where $Z_t = (X_{t-1}', \ldots, X_{t-p}')$, $\phi = \text{vec}(\Pi_1, \ldots, \Pi_p)$, $\hat{\Phi}_t = (\hat{\varepsilon}_{t-1}', \ldots, \hat{\varepsilon}_{t-h}')$ and $\psi = \text{vec}(\Psi_1, \ldots, \Psi_h)$. Here $\otimes$ denotes the Kronecker product and the symbol $\text{vec}$ denotes the column vectorisation operator. The first $h$ values of the residuals are set to zero in the auxiliary model, so that the series length is equal to the series length in the original VAR model.

Bewley (1986) shows that the LM statistic can be written as

$$Q_{LM} = T \left( K - \text{tr}(\hat{\Omega}^{-1} \hat{\Omega}_e) \right),$$

where $\hat{\Omega} = T^{-1} \sum_{t=1}^T \hat{\varepsilon}_t \hat{\varepsilon}_t'$ is the estimator of the error covariance matrix from the VAR model and $\hat{\Omega}_e$ is the estimator of the error covariance matrix from the auxiliary regression. If the null hypothesis is correct, the two covariance matrix estimators have the same probability limit. The test statistic is asymptotically distributed as a $\chi^2$ random variable with $hK^2$ degrees of freedom under the null hypothesis. The test reduces to the single equation LM test when $K = 1$.

There are asymptotically equivalent likelihood ratio (LR) and Wald versions of the test statistic.

An alternative to the $\chi^2$ tests is the $F$ approximation to the LR test of Rao
(1973), given by

\[
Q_F = \left( \frac{\hat{\Omega}}{\hat{\Omega}_c} \right)^{1/s} \frac{Ns - \frac{1}{2}K^2h + 1}{K^2h},
\]

(25)

where

\[
s = \left( \frac{K^4h^2 - 4}{K^2 + K^2h^2 - 5} \right)^{1/2} \quad \text{and} \quad N = T - Kp - 1 - Kh - \frac{1}{2}(K - Kh + 1).
\]

The \( Q_F \) statistic is approximately distributed as \( F(hK^2, Ns - \frac{1}{2}K^2h + 1) \) under the null hypothesis. The formula (25) was derived in the multivariate linear regression model with fixed regressors by Rao (1951) and does not seem to have a theoretical justification in time series models. The \( F \) statistic was proposed in the context of testing for error autocorrelation by Doornik (1996).

The small sample properties of the tests are examined by Edgerton and Shukur (1999) in the stationary VAR model, and Brüggemann, Lütkepohl and Saikkonen (2006) in the cointegrated VAR model. Edgerton and Shukur find that the tests have satisfactory size properties only when \( K \) and \( h \) are small relative to the number of observations. The LM test is preferred to the LR and Wald versions of the test. The \( F \)-approximation to the LR test is found to have better size properties than the asymptotic \( \chi^2 \) tests when the dimensions are large. Brüggemann et al. obtain similar results when the tests are applied to the cointegrated VAR model. If the number of parameters is large relative to the number of observations, it is desirable to bootstrap the tests (see e.g. Davidson and MacKinnon (2004), Section 13.7).

In Essay 1 we consider Bootstrap and Wild Bootstrap tests for error autocorrelation in VAR models when the errors are conditionally heteroskedastic. The wild bootstrap tests are also used in Essay 2. Edgerton and Shukur (1999) show that Lagrange Multiplier tests have satisfactory size properties only when \( K \) and \( h \) are small relative to the number of observations, and that the tests are oversized in models with a large number of dimensions. The bootstrap can be applied to correct for the size distortions. In addition, if the errors are conditionally heteroskedastic the asymptotic and IID bootstrap tests are oversized, but the wild bootstrap performs well.

8 Tests for ARCH

The Lagrange multiplier (LM) test for ARCH of Engle (1982) in equation \( i = 1, \ldots, K \), of model (2) is a test of \( \alpha_1 = \alpha_2 = \cdots = \alpha_p = 0 \) in the auxiliary regression for equation \( i \)

\[
\varepsilon_{i,t}^2 = \alpha_0 + \alpha_1 \varepsilon_{i,t-1}^2 + \cdots + \alpha_p \varepsilon_{i,t-p}^2 + \varepsilon_{i,t}.
\]
The test statistic has the form

$$LM_i = TR_i^2,$$  \hspace{1cm} (26)

where $R^2_i$ is the coefficient of determination in the auxiliary regression. The LM statistic is asymptotically distributed as $\chi^2(p)$.

Multivariate tests for conditional heteroskedasticity in VAR models are studied in Essay 4. There exists a number of multivariate tests for conditional heteroskedasticity, however their use is not common and in statistical software like PcGive the tests are not available. The usual practice is to use (26) to test the errors separately. However, the results of Essay 4 show that multivariate tests may have higher power against MGARCH alternatives.

Dufour et al. (2010) consider a combined Monte Carlo (MC) test based on combining the outcomes of univariate tests. In this case one can use (26). Following Dufour et al. (2010), standardised versions of the test statistics are obtained by replacing $\tilde{e}_{it}$ by $\tilde{w}_{it}$, where $\tilde{w}_{it}$ are the elements in the $i$th column of the multivariate standardised residual matrix $\tilde{W} = \hat{E}(S_E)^{-1} = (\tilde{w}_1, \ldots, \tilde{w}_T)$. $\hat{E}$ is the associated residual covariance matrix from the estimated VAR model and $S_E$ the Cholesky factor of the error covariance matrix estimator $\hat{\Omega} = T^{-1}\tilde{E}'\tilde{E}$. The combined statistic is constructed as follows (Dufour et al. (2010)):

$$\hat{LM} = 1 - \min_{1 \leq i \leq K} [p(\hat{LM}_i)]$$  \hspace{1cm} (27)

where $p(\hat{LM}_i)$ is the individual $p$-value associated with $\hat{LM}_i$, and which may be derived from the asymptotic distribution $\hat{LM}_i \sim \chi^2(p)$. The idea of using the minimum of several $p$-values as a test statistic was originally proposed by Tippett (1931) for independent test statistics. In this case, however, the test statistics cannot be assumed to be independent. Dufour and Khalaf (2002) show that when the individual $p$-values, $p(\hat{LM}_i)$, are not independent, the size of the test in (27) can be controlled using Monte Carlo test techniques. The test is an exact test if the regressors are exogenous.

Observe that the combined test rejects if at least one of the individual tests is significant. The combined test is closely related to a Bonferroni-bound-based testing procedure, which requires to divide the significance level of the tests by the number of tests. If the significance level is $\alpha$, each of the individual tests is conducted at the level $\alpha_i = \alpha/K$ and the test rejects if the $p$-value of at least one of the tests is less than $\alpha_i$. The Boole-Bonferroni inequality (see e.g. Casella and Berger (2002), p 13) ensures that the probability of rejecting $H_0$ is not greater than $\alpha$. The testing procedure based on a Bonferroni bound can, however, lead to power losses if the number of equations, $K$, is large. Different from the Bonferroni bound the MC procedure delivers a simulated joint $p$-value and does not require the significance level to be divided by the number of equations. Dufour et al. (2010) show that
this leads to higher power of the test. Furthermore, they show that using the MC technique, exact \( p \)-values can be obtained in the case of exogenous regressors, with no need to evaluate the distribution of (27) analytically. The test is used in Essay 4 for testing for ARCH in a VAR model, where it is shown through simulation experiments that the test has good size properties even when the regressors are not exogenous. The \( p \)-value of the combined statistic is then calculated by a parametric bootstrap procedure.

An multivariate LM test of multivariate ARCH is considered in Lütkepohl (2006). The multivariate LM test is a generalization of (26) and is based on the auxiliary regression

\[
\text{vech}(\epsilon_t \epsilon'_t) = b_0 + B_1 \text{vech}(\epsilon_{t-1} \epsilon'_{t-1}) + \cdots + B_q \text{vech}(\epsilon_{t-q} \epsilon'_{t-q}) + e_t,
\]

where \( b_0 \) is a \( \frac{1}{2} K(K + 1) \)-dimensional parameter vector and \( B_1, \ldots, B_q \) are \( \frac{1}{2} K(K + 1) \times \frac{1}{2} K(K + 1) \) parameter matrices. The operator vech stacks the elements on and below the main diagonal of a \( K \times K \) matrix into a \( \frac{1}{2} K(K + 1) \)-dimensional vector.

The null hypothesis is \( B_1 = \cdots = B_q = 0 \). The test statistic is given by

\[
LM_{ARCH}(q) = \frac{1}{2} TK(K + 1) - T \text{tr}(\Sigma_{vech} \hat{\Sigma}_0^{-1}),
\]

where \( \Sigma_{vech} \) is the residual covariance matrix estimators of the auxiliary regression and \( \hat{\Sigma}_0 \) the corresponding matrix under \( H_0 \). Under regularity conditions, \( LM_{ARCH}(q) \) follows a \( \chi^2(qK^2(K + 1)^2/4) \) distribution. The major drawback of the test is that it requires the estimation of a large number of parameters even for moderate values of \( K \) and \( q \), and therefore performs poorly in small samples.

Another LM test for multivariate ARCH is the test of constant error covariances proposed by Eklund and Teräsvirta (2007). They propose a test for constant error covariance matrices which can be defined for different types of time varying covariance matrices under the alternative. When testing for ARCH, a suitable alternative is the CCC-ARCH process. Under the alternative the conditional variance \( h_t = (h_{1t}, \ldots, h_{Kt})' \) is given by

\[
h_t = a_0 + \sum_{i=1}^{p} A_i \epsilon_{i-1}^{(2)},
\]

where \( \epsilon_{i}^{(2)} = (\epsilon_{i1}^2, \ldots, \epsilon_{Kt}^2)' \). The null hypothesis is \( \text{diag}(A_1) = \cdots = \text{diag}(A_p) = 0 \).

Let \( \theta = (\omega_1', \ldots, \omega_n', \rho)' \), where \( \omega_j = (\alpha_{0j}, \alpha_{1j}, \ldots, \alpha_{p_j})' \), \( j = 1, \ldots, K \), and \( \rho = \text{vec}(P) \). The LM statistic has the form

\[
LM_{CCC} = TS_T(\hat{\theta})\hat{I}_T^{-1}(\hat{\theta})S_T(\hat{\theta}),
\]

where \( S_T(\hat{\theta}) \) and \( \hat{I}_T(\hat{\theta}) \) are the relevant blocks of the average score vector and the
information matrix, respectively, estimated under the null hypothesis (see Eklund and Teräsvirta (2007)). The $L_{M_{CCC}}$ statistic is asymptotically distributed as $\chi^2(np)$ under the null hypothesis.

9 A Misspecification Test for CCC-GARCH Models

Essay 3 proposes a Lagrange multiplier (LM) test for testing the adequacy of an estimated CCC-GARCH model. In order to construct the misspecification test, assume that in (8) $z_t = G_t u_t$, where

$$G_t = \text{diag}(g_{1t}^{1/2}, \ldots, g_{Kt}^{1/2})$$

with

$$g_{it} = 1 + \sum_{j=1}^{r} \zeta_{ij} z_{u-j}. \quad (31)$$

It follows that $u_t = (u_{1t}, \ldots, u_{Kt})' = (\varepsilon_{1t} h_{1t}^{-1/2}, \ldots, \varepsilon_{Kt} h_{Kt}^{-1/2})' \sim \text{IID}(0, \Sigma)$. Then (7) can be written as follows:

$$\varepsilon_t = D_t G_t u_t, \quad (32)$$

and (32) can be regarded as an 'ARCH nested in GARCH' model. For the univariate case, see Lundbergh and Teräsvirta (2002), and for another definition of $g_{it}$, in which $g_{it}$ is a deterministic positive definite function, see Amado and Teräsvirta (2013).

Let $\zeta = (\zeta_1', \ldots, \zeta_K')'$ be a $Kr \times 1$ matrix where $\zeta_i = (\zeta_{i1}, \ldots, \zeta_{ir})'$, $i = 1, \ldots, K$, is an $r \times 1$ vector. The null hypothesis is

$$H_0 : \zeta = 0 \text{ or } G_t \equiv I \quad (33)$$

in the model (31). Thus under $H_0$, $\{\varepsilon_t\}$ follows a CCC-GARCH model, and the alternative implies that there is dynamic structure unaccounted for in this model, because then none of the sequences $\{z_{it}\}$ is a sequence of independent random variables.

The LM test statistic is given by

$$LM_\zeta = T \bar{q}_\zeta(\tilde{\theta})' J^{-1}_{(\zeta, \zeta)}(\tilde{\theta}) \bar{q}_\zeta(\tilde{\theta}), \quad (34)$$

where

$$\bar{q}_\zeta(\tilde{\theta}) = -\frac{1}{T} \sum_{t=1}^{T} (\nabla G_t \text{vec}(I - \frac{1}{2} D_t P D_t^{-1} \varepsilon_t \varepsilon_t' - \frac{1}{2} \varepsilon_t \varepsilon_t' D_t P D_t^{-1}))$$
is the nonzero block of the average score, $\nabla G_t = \partial \text{vec}(G_t)' / \partial \zeta$, and

$$J(\zeta, \zeta)(\tilde{\theta}) = -\frac{1}{T} \sum_{t=1}^{T} \left\{ \frac{\partial^2 l_t(\tilde{\theta})}{\partial \zeta^t \partial \zeta} \right\}$$

$$- \left[ \frac{\partial^2 l_t(\tilde{\theta})}{\partial \omega' \partial \zeta} \quad \frac{\partial^2 l_t(\tilde{\theta})}{\partial \rho' \partial \zeta} \right] \left[ \begin{array}{cc} \frac{\partial^2 l_t(\tilde{\theta})}{\partial \omega' \partial \omega} & \frac{\partial^2 l_t(\tilde{\theta})}{\partial \rho' \partial \omega} \\ \frac{\partial^2 l_t(\tilde{\theta})}{\partial \omega' \partial \rho} & \frac{\partial^2 l_t(\tilde{\theta})}{\partial \rho' \partial \rho} \end{array} \right]^{-1} \left[ \begin{array}{c} \frac{\partial^2 l_t(\tilde{\theta})}{\partial \zeta' \partial \omega} \\ \frac{\partial^2 l_t(\tilde{\theta})}{\partial \zeta' \partial \rho} \end{array} \right]$$

is the corresponding block of the information matrix evaluated at $\tilde{\theta}$ under the null hypothesis. Under $H_0$, the test statistic in (34) has an asymptotic $\chi^2$ distribution with $Kr$ degrees of freedom.

If $D_t = \text{diag}(\alpha_0, \ldots, \alpha_{0K})$, $LM_\zeta$ becomes a special case of the test by Eklund and Teräsvirta (2007) of no conditional heteroskedasticity against CCC-ARCH.

## 10 Cointegration Analysis

Essay 2 is an empirical example of cointegration analysis when the data are conditionally heteroskedastic. The empirical examples in Essay 1 are also motivated by cointegration analysis.

Consider the $K$-dimensional VAR model written in VECM form in (5). If there exist cointegration relations in the data, the matrix $\Pi = \alpha \beta'$ has reduced rank, where $\alpha$ and $\beta$ are $K \times r$ matrices of rank $r < K$. The cointegration vectors are $\beta$ and the cointegrating relations are $\beta'X_t$, which are stationary and $I(0)$.

The cointegration rank is determined by a likelihood ratio test, testing the sequence of null hypotheses

$$H_0 : \text{rk}(\Pi) = r, \quad r = 0, 1, \ldots, K - 1. \quad (35)$$

The alternative hypotheses are $H_1 : r_0 < \text{rk}(\Pi) \leq K$, where $r_0$ is the value under the null hypothesis. The sequence is terminated when the null hypothesis cannot be rejected for the first time. If the test does not reject $H_0 : \text{rk}(\Pi) = 0$, we have no cointegration, but if the test rejects we proceed by testing $H_0 : \text{rk}(\Pi) = 1$. If the test does not reject, then the cointegration rank 1 is chosen, and so on. If the tests reject for all values of $r = 0, 1, \ldots, K - 1$, it indicates that the series are $I(0)$ and a VAR model in the levels can be applied.

The test statistic is constructed by first solving the eigenvalues $1 > \hat{\lambda}_1 > \ldots > \hat{\lambda}_K > 0$ from

$$|\lambda \Sigma_{22} - \Sigma_{21} \Sigma_{11}^{-1} \Sigma_{12}| = 0, \quad (36)$$

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where

$$\Sigma_{ij} = T^{-1} \sum_{t=1}^{T} \tilde{V}_{ti} \tilde{V}_{tj}, \quad i, j = 1, 2,$$

(37)

and $\tilde{V}_{t1}$ and $\tilde{V}_{t2}$ are the residuals from regressing $\Delta X_t$ and $X_{t-1}$ on the lagged differences and a constant. The likelihood ratio statistic is given by

$$Q_r = -T \sum_{i=r+1}^{K} \ln(1 - \hat{\lambda}_i).$$

(38)

The limiting distribution is nonstandard and the critical values based on simulations are given in Johansen (1996).

As shown by Cavaliere, Rahbek and Taylor (2010b), heteroskedasticity may cause the test to find too many cointegrating vectors. Cavaliere, Rahbek and Taylor (2010a) show that while $Q_r$ is valid under conditional heteroskedasticity, it is oversized in small samples and to correct for the size distortion they propose using the wild bootstrap. Cavaliere, Rahbek and Taylor (2012b) propose a bootstrap algorithm based on restricted residuals for bootstrap cointegration tests. Cavaliere, Rahbek and Taylor (2012a) show that the wild bootstrap implementation of the algorithm is asymptotically correctly sized in the presence of both conditional and unconditional heteroskedasticity.

The wild bootstrap algorithm used in Essay 2 for restricted residuals based on Cavaliere et al. (2012b) is given below. The algorithm can be used for robust cointegration analysis when there is heteroskedasticity in the errors.

**Algorithm 1** (Wild Bootstrap Cointegration Test)

1. Obtain the estimates $\hat{\alpha}^{(r)}$, $\hat{\beta}^{(r)}$, $\hat{\Gamma}_1^{(r)}$, $\ldots$, $\hat{\Gamma}_{k-1}^{(r)}$ and $\hat{\mu}^{(r)}$ together with the restricted residuals, $\hat{\varepsilon}_{r,t}$, by reduced rank regression under rank $r$.

2. Check if the characteristic polynomial $|\hat{\Pi}^{(r)}(z)| = 0$, with

$$\hat{\Pi}^{(r)}(z) = (1 - z)I_p - \hat{\alpha}^{(r)}\hat{\beta}^{(r)}z - \hat{\Gamma}_1^{(r)}(1 - z)z - \ldots - \hat{\Gamma}_{p-1}^{(r)}(1 - z)z^{p-1},$$

has $K - r$ roots equal to 1 and all other roots outside the unit circle. If so proceed to the next step.

3. Generate wild bootstrap errors as

$$\hat{\varepsilon}_{r,t}^* = \hat{\varepsilon}_{r,t}w_t,$$

where $w_t$ is a random draw from a distribution with mean 0 and variance 1.

4. Generate recursively the wild bootstrap sample $X_t^*$, $t = 1, \ldots, T$, where $X_t^* =$
\( X_t \) for \( t = 1, \ldots, k \), as
\[
\Delta X_t^* = \alpha^{(r)} \beta^{(r)} X_{t-1}^* + \sum_{i=1}^{p-1} \Gamma_i^{(r)} \Delta X_{t-i}^* + \mu^{(r)} + \varepsilon_{t,t}^*, \quad t = k + 1, \ldots, T.
\]

5. Obtain the wild bootstrap test statistic, \( Q_{r}^* = -T \sum_{t=r+1}^{T} \ln(1 - \hat{\lambda}_t^*) \), using the bootstrap sample.

6. Repeat steps 3-5 \( B \) times to obtain \( Q_{r1}^*, \ldots, Q_{rB}^* \).

7. Estimate the bootstrap \( p \)-value by
\[
\hat{p}^* = \frac{1}{B} \sum_{i=1}^{B} I(Q_{ri}^* > Q_r).
\]  

The null hypothesis is rejected at the significance level \( \alpha \) if \( \hat{p}^* < \alpha \). Step 2 is needed to explosive bootstrap samples. This may sometimes happen and may be an indication of model misspecification (see Swensen (2006)).

11 Empirical Applications

A number of empirical applications to financial time series are considered in the essays. In essays 1 and 4 the empirical examples deal with credit default swap (CDS) prices and Euribor interest rates. The empirical example in essay 2 deals with testing for cointegration between the Chinese stock markets and the Chinese markets and the US market. In these three examples it is shown that relying on asymptotic misspecification tests may erroneously lead to selecting models with many lags when there is conditional heteroskedasticity in the data. For example, in order to test the cointegration rank a model is first selected. Tests for the cointegration rank are known to be oversized and have low power in models with many lags. Taking conditional heteroskedasticity into account and using wild bootstrap methods lead to models with fewer lags and more accurate inference.

A CDS is a credit derivative which provides a bondholder with protection against the risk of default by the company. If a default occurs, the holder is compensated for the loss by an amount which equals the difference between the par value of the bond and its market value after the default. The CDS price is the annualised fee (expressed as a percentage of the principal) paid by the protection buyer. Duffie (1999) derived the equivalence of the CDS price and credit spread (defined as the bond yield minus the risk-free rate). We denote by \( p_t^{CDS} \) the CDS price and \( p_t^{CS} \) the credit spread on a risky bond over the risk-free rate. The basis is the difference
between the CDS price and the bond spread:

\[ s_t = p_t^{CDS} - p_t^{CS}. \]

If the two markets price credit risk equally in the long run, then the prices should be equal, so that the basis \( s = 0 \). Since both \( p_{CDS} \) and \( p_{CS} \) are \( I(1) \), the non-arbitrage relation can be tested as an equilibrium relation in the cointegrated VAR model, following Blanco, Brennan and Marsh (2005). The vector \( \mathbf{X}_t \) with the value 1 appended is \( \mathbf{X}_t = (p_t^{CDS}, p_t^{CS}, 1)' \). The financial theory posits that \( \mathbf{X}_t \) is cointegrated with cointegrating vector \( \mathbf{\beta} = (1, -1, c)' \), so that \( \mathbf{\beta} \mathbf{X}_t = p_{CDS} - p_{CS} + c \) is a cointegrating relation. Many papers have tested the equivalence of CDS prices and credit spreads for US and European investment-grade companies, and found that the parity relation holds for most companies, i.e. the bond and CDS markets price credit risk equally (see Blanco et al. (2005), Zhu (2006), Dötz (2007) and Forsbäck (2012)).

In the second empirical example we use Euribor interest rates. The problem of testing the expectations hypothesis of the term structure of interest rates is considered. The theory makes two predictions, which can be tested in the cointegrated VAR model (see Hall, Anderson and Granger (1992)). First, if there are \( K \) interest rate series in the system, then the cointegration rank is \( r = K - 1 \). Second, the spreads between the interest rates at different maturities span the cointegration space.

The third empirical example deals with testing for cointegration between the Chinese stock markets. The Chinese stock markets in Shanghai and Shenzhen are divided into A and B shares. In China only domestic investors were allowed to trade A shares, whereas B shares were only for foreign investors. The trading restrictions were relaxed in February 2001 when domestic investors were allowed to trade B shares. Further relaxations have been implemented since then.

Tian (2007) studies cointegration between the Chinese stock markets using Shanghai A and B shares indices and other Chinese, Taiwanese, Japanese and US stock indices. He finds evidence of cointegration in the period after the Asian crisis between the Shanghai A share market, the Hong Kong main market, the Taiwanese market and to some extent also the US market.

12 The Essays

This section gives a brief introduction to the four essays, the motivation behind them and their findings. All essays deal with multivariate time series models with conditional heteroskedasticity.
12.1 Essay 1

The first essay is written together with Niklas Ahlgren. The essay studies tests for autocorrelation in VAR models with conditional heteroskedasticity in the errors.

Standard asymptotic and residual-based bootstrap tests for error autocorrelation are unreliable in the presence of conditional heteroskedasticity. In this essay we propose wild bootstrap tests for autocorrelation in vector autoregressive models when the errors are conditionally heteroskedastic. In particular, we investigate the properties of Lagrange multiplier tests. Monte Carlo simulations show that the wild bootstrap tests have satisfactory size properties in models with CCC-GARCH errors, whereas the standard asymptotic and residual-based bootstrap tests are oversized.

The wild bootstrap tests for error autocorrelation perform well when the errors follow a CCC-GARCH process which satisfies the condition for the existence of the fourth moment matrix of the errors. The wild bootstrap tests perform reasonably well even if the condition is not satisfied. The wild bootstrap tests have the same power as the (size-corrected) asymptotic and bootstrap tests in the case of IID errors, but suffer a loss of power if the errors are conditionally heteroskedastic.

Empirical examples involving CDS prices and Euribor interest rates demonstrate that the asymptotic and bootstrap tests for error autocorrelation may falsely reject the null hypothesis of no error autocorrelation if the errors are conditionally heteroskedastic. In such cases the tests may erroneously lead to selecting a model with a long lag length. This potential pitfall can be avoided by using the wild bootstrap tests for error autocorrelation. We recommend the WB tests for use in practice if conditional heteroskedasticity is suspected.

12.2 Essay 2

The second essay studies the impact of conditional heteroskedasticity on cointegration analysis of the Chinese and US stock markets. Tests show that the Shanghai A and B share indices, the Hang Seng index of Hong Kong, the Hang Seng China Enterprises Index and Standard & Poor’s 500 index have significant conditional heteroskedasticity. Model selection based on standard asymptotic tests leads to models with a large number of lags. Wild bootstrap tests are robust to heteroskedasticity and using them we are able to select models with shorter lag lengths. Evidence for cointegration is weak in models selected using asymptotic tests. Using models selected by wild bootstrap tests leads to finding cointegration between the Chinese markets and the US market.

Asymptotic tests for cointegration find cointegrating relations in the Chinese stock markets. Using wild bootstrap tests we find no cointegration between the Chinese markets, but two cointegrating vectors are found between the Chinese markets and the US market in the period before the financial crisis 2007-2008. Further
analysis of the cointegrating vectors reveal volatility clustering in the cointegrating vectors. Therefore the results from the wild bootstrap tests should be more reliable.

12.3 Essay 3

Essay 3 is written together with Timo Teräsvirta. We propose a Lagrange multiplier test for testing the parametric structure of a CCC-GARCH model. The test is based on decomposing the CCC-GARCH model multiplicatively into two components, one of which represents the null model and the other the possible misspecification. The inspiration comes from the univariate 'no ARCH in GARCH' test in Lundbergh and Teräsvirta (2002). We derive the test statistic and study its performance through Monte Carlo simulations. We find that the test has good finite sample properties.

We compare the test with other tests for misspecification of multivariate GARCH models. A general portmanteau test for testing the adequacy of an MGARCH model proposed by Ling and Li (1997) and an LM test for time-varying correlations of Tse (2000) are considered. While the test is constructed for situations in which the GARCH equations may be misspecified, we also study whether the test has power against misspecification of the correlation structure. We find that our test has high power against alternatives where the misspecification is in the GARCH equations and is superior to the other two tests. In particular Tse’s test has very low power against misspecification in the conditional variance. Ling and Li’s test on the other hand has low power when the dimensions of the model are large. Our test is not greatly affected by misspecification in the conditional correlations and is generally outperformed by the test against time-varying conditional correlations of Tse (2000). Therefore it is well suited for considering misspecification of GARCH equations.

12.4 Essay 4

The fourth essay is written together with Niklas Ahlgren. In this essay we propose multivariate bootstrap tests for ARCH in VAR models, by following a suggestion in Dufour et al. (2010) of replacing an exact test by a bootstrap test when the model include lags. The tests generalise the univariate bootstrap LM test of Gel and Chen (2012) to the multivariate case. The tests are based on standardised multivariate least squares residuals and are therefore easy to calculate. The results show that the bootstrap tests perform better than the asymptotic tests. Our results also show that the test by Eklund and Teräsvirta (2007), with the alternative defined by a CCC-GARCH model, is more powerful than other multivariate LM tests such as combined univariate LM tests and multivariate LM tests which assume no particular alternative to the null hypothesis.

Simulation results indicate that the multivariate bootstrap LM tests have the correct size in small and moderate samples. Empirical application to CDS prices
and Euribor interest rates are provided as illustrations of the use of the tests with financial data.

13 Conclusions

The thesis makes contributions to the econometric literature on conditional heteroskedasticity in the analysis of multivariate time series. Essay 1 considers testing for autocorrelation in VAR models with conditional heteroskedasticity. The essay shows that wild bootstrap tests should be preferred when there is conditional heteroskedasticity in the data. Essay 2 studies cointegration between Chinese and US stock prices using wild bootstrap methods. Using wild bootstrap tests, models with fewer lags are selected. Cointegration is found between the Chinese and US stock markets before the financial crisis 2007-2008, but not between the Chinese markets. Essay 3 proposes an LM test for testing the adequacy of a CCC-GARCH model. The test statistic is derived and its properties are studied. Compared to other tests it performs well in testing the GARCH parametrisation. Essay 4 studies tests for ARCH in VAR models. The results show that bootstrap multivariate tests for conditional heteroskedasticity are correctly sized and have high power when the errors have conditional heteroskedasticity.

References


Wild Bootstrap Tests for Autocorrelation in Vector Autoregressive Models*

Niklas Ahlgren†  Paul Catani‡

October 9, 2013

Abstract

Many economic and financial time series exhibit conditional heteroskedasticity. Standard tests for autocorrelation are derived under the assumption of independent and identically distributed (IID) errors. These tests and residual-based bootstrap tests assuming IID errors are invalid in the presence of conditional heteroskedasticity. In this article we propose wild bootstrap tests for autocorrelation in vector autoregressive (VAR) models when the errors are conditionally heteroskedastic. In particular, we investigate the properties of Lagrange multiplier (LM) tests. Monte Carlo simulations show that the wild bootstrap LM tests have satisfactory size properties in models with constant conditional correlation generalised autoregressive conditional heteroskedastic (CCC-GARCH) errors. In contrast, the standard asymptotic and bootstrap LM tests are oversized. The tests are applied to VAR models for credit default swap (CDS) prices and Euribor interest rates.

1 Introduction

It is considered good practice to check the adequacy of an estimated time series model by testing for various types of misspecification such as error autocorrelation (AC) and autoregressive conditional heteroskedastic (ARCH) errors. Standard tests for error AC are derived under the assumption of independent and identically distributed (IID) errors. These tests and residual-based bootstrap tests assuming IID errors are invalid in the presence of conditional heteroskedasticity. Bera, Higgins and Lee (1992) show that the expression for the information matrix in Lagrange multiplier (LM) tests for error AC depends on the ARCH parameters. Consequently, standard LM tests for AC will be misleading if the presence of ARCH is neglected. They derive LM tests for AC based on transformed residuals obtained by dividing the residuals by an estimate of the conditional standard deviation. However, their approach requires the form of the conditional heteroskedasticity to be known. Typically, a GARCH process is assumed, but there is no guarantee that it provides an adequate description of the conditional heteroskedasticity. Secondly, it may be difficult to obtain reliable estimates of the GARCH parameters (Gonçalves and Kilian

*The authors want to thank Timo Teräsvirta for helpful comments.
†Hanken School of Economics, PO Box 479 (Arkadiagatan 22), 00101 Helsingfors, Finland. E-mail: niklas.ahlgren@hanken.fi.
‡Hanken School of Economics, PO Box 479 (Arkadiagatan 22), 00101 Helsingfors, Finland. E-mail: paul.catani@hanken.fi.
(2004)). Finally, further difficulties arise in the multivariate case. Estimating the parameters of multivariate GARCH models is substantially more complicated and it is often difficult to obtain reliable estimates of the parameters.

Conditional heteroskedasticity in the residuals of time series models estimated on macroeconomic and financial data at the monthly, weekly and daily frequencies is so pervasive that it is almost a stylised fact (see e.g. Gonçalves and Kilian (2004), and Sensier and van Dijk (2004)). Thus there is a need for tests for error AC that are robust to the presence of conditional heteroskedasticity of unknown form. The theoretical properties of wild bootstrap (WB) procedures for autoregressions with conditional heteroskedasticity are investigated by Gonçalves and Kilian. Godfrey and Tremayne (2005) report simulation results on WB tests for AC in dynamic regression models. A treatment of AC tests in vector autoregressive (VAR) models when the errors are conditionally heteroskedastic does not seem to be available, though.

In this article we propose WB tests for AC in vector autoregressive (VAR) models when the errors are conditionally heteroskedastic. Tests for error AC are frequently applied in the case when some of the variables are integrated and cointegrated. It is shown in Brüggemann, Lütkepohl and Saikkonen (2006) that LM tests for AC based on an auxiliary model are valid in integrated and cointegrated VAR models. In contrast, portmanteau tests for AC have to be modified to take into account the cointegration rank of the system, which is unknown in practice. For this reason, we focus on LM tests and do not consider portmanteau tests. We use a residual-based recursive WB procedure, which extends the results of Gonçalves and Kilian to the multivariate case. Monte Carlo simulations show that the WB LM tests have satisfactory size properties in models with constant conditional correlation generalised autoregressive conditional heteroskedastic (CCC-GARCH) errors. In contrast, the standard asymptotic and residual-based bootstrap LM tests are oversized. Some simulation evidence on the power of the WB tests is reported. The tests for error AC are applied to VAR models for credit default swap (CDS) prices and Euribor interest rates. The results show that there are significant ARCH effects in the residuals of the estimated models. The tests for AC are used to determine the lag length in the VAR models and it turns out that the WB tests lead to models with fewer lags than the asymptotic and bootstrap tests. Because it is important to avoid many lags in VAR models, this suggest that if conditional heteroskedasticity is suspected, the WB tests for AC should be used in place of the standard asymptotic and residual-based bootstrap tests.

The remainder of the paper is organised as follows. In Section 2 LM tests for error AC are reviewed. In Section 3 we describe the residual-based recursive WB algorithm. Section 4 presents simulation evidence on the size and power of the LM tests for AC. Section 5 contains empirical applications to CDS prices and Euribor interest rates. Section 6 concludes.

2 Tests for Error Autocorrelation

Let $X_t = (X_{1t}, \ldots, X_{kt})'$ be a $K$-dimensional vector of time series variables. It is assumed that $X_t$ is either stationary and $I(0)$, or $I(1)$ and cointegrated.
2.1 Stationary VAR Model

The $K$-dimensional vector of $I(0)$ time series variables $X_t$ is assumed to be generated by a vector autoregressive (VAR) model of order $p$:

$$X_t = \mu + \Pi_1 X_{t-1} + \ldots + \Pi_p X_{t-p} + \epsilon_t, \quad t = 1, \ldots, T.$$  \hfill (1)

Here $\Pi_1, \ldots, \Pi_p$ are $(K \times K)$ parameter matrices and $\mu$ is a $(K \times 1)$ vector of constants. The error process $\{\epsilon_t\}$ is assumed to be IID with mean zero and nonsingular and positive definite covariance matrix $\Omega$.

The Breusch–Godfrey Lagrange multiplier (LM) tests for error AC (Godfrey (1978), Godfrey (1991), and Breusch (1978)) assumes a VAR($h$) model for the error terms under the alternative:

$$\epsilon_t = \Psi_1 \epsilon_{t-1} + \ldots + \Psi_h \epsilon_{t-h} + e_t.$$  \hfill (2)

The hypothesis being tested is

$$H_0 : \Psi_1 = \cdots = \Psi_h = 0 \quad \text{against} \quad H_1 : \Psi_j \neq 0 \text{ for at least one } j, 1 \leq j \leq h.$$  

The test statistic is computed from an auxiliary model

$$\hat{\epsilon}_t = \mu + \Pi_1 X_{t-1} + \ldots + \Pi_p X_{t-p} + \hat{\Psi}_1 \hat{\epsilon}_{t-1} + \ldots + \hat{\Psi}_h \hat{\epsilon}_{t-h} + e_t$$

$$= \mu + (Z'_t \otimes I_K) \phi + (\hat{\Psi}'_t \otimes I_K) \psi + e_t,$$  \hfill (3)

where $Z'_t = (X'_{t-1}, \ldots, X'_{t-p})$, $\phi = \text{vec}(\Pi_1, \ldots, \Pi_p)$, $\hat{\Psi}'_t = (\hat{\epsilon}'_{t-1}, \ldots, \hat{\epsilon}'_{t-h})$ and $\psi = \text{vec}(\hat{\Psi}_1, \ldots, \hat{\Psi}_h)$. The symbol $\otimes$ denotes the Kronecker product and the symbol vec denotes the column vectorisation operator. The first $h$ values of the residuals $\hat{\epsilon}_t$ are set to zero in the auxiliary model, so that the series length is equal to the series length in the original VAR model.

The Breusch–Godfrey statistic is a standard LM statistic and is given by

$$Q_{LM} = T \hat{\psi}' (\hat{\Sigma}^{\psi\psi})^{-1} \hat{\psi},$$  \hfill (4)

where $\hat{\psi}$ is the generalised least squares (GLS) estimate of $\psi$ and $\hat{\Sigma}^{\psi\psi}$ is the block of

$$\hat{\Omega}^{-1}_e = \left( T^{-1} \sum_{t=1}^{T} \left[ \hat{E}_t \otimes I_K \right] \Omega^{-1} \left[ \hat{E}'_t \otimes I_K \ Z'_t \otimes I_K \right] \right)^{-1},$$

the estimator of the error covariance matrix from the auxiliary model (3), corresponding to $\psi$, and $\hat{\Omega} = T^{-1} \sum_{t=1}^{T} \hat{\epsilon}_t \hat{\epsilon}'_t$ is the estimator of the error covariance matrix from the VAR model (1).

Bewley (1986) showed that the LM statistic can alternatively be written as

$$Q_{LM} = T \left( K - \text{tr}(\hat{\Omega}^{-1} \hat{\Omega}_e) \right),$$  \hfill (5)

If the null hypothesis is true, the two covariance matrix estimators, $\hat{\Omega}$ and $\hat{\Omega}_e$, have the same probability limit. The test statistic $Q_{LM}$ is asymptotically distributed as a $\chi^2$ random variable with $hK^2$ degrees of freedom under the null hypothesis. The test reduces to the single equation LM test when $K = 1$.

There are asymptotically equivalent likelihood ratio (LR) and Wald versions of the test statistic, which are given by

$$Q_{LR} = T \left( \log \det \hat{\Omega} - \log \det \hat{\Omega}_e \right)$$  \hfill (6)
and

\[ Q_W = T \left( \text{tr}(\Omega^{-1}_e \tilde{\Omega}) - K \right), \]

respectively. An alternative to the \( \chi^2 \) tests is the \( F \) approximation to the LR test of Rao (1973), given by

\[ Q_F = \left[ \left( \frac{\det \tilde{\Omega}}{\det \Omega_e} \right)^{1/s} - 1 \right] \frac{Ns - \frac{1}{2}K^2h + 1}{K^2h}, \]

where

\[ s = \left( \frac{K^4h^2 - 4}{K^2 + K^2h^2 - 5} \right)^{1/2} \]

and \( N = T - Kp - 1 - Kh - \frac{1}{2}(K - Kh + 1) \).

The \( Q_F \) statistic is approximately distributed as \( F(hK^2, Ns - \frac{1}{2}K^2h + 1) \) under the null hypothesis. The formula (8) was derived in the multivariate linear regression model with fixed regressors and does not seem to have a theoretical justification in time series models. The \( F \) statistic was proposed in the context of testing for error AC by Doornik (1996).

The \( Q_{LM} \) and \( Q_F \) tests are the tests most commonly used in practice and are provided in e.g. PcGive and Eviews (in the latter only the \( Q_{LM} \) test is available).

The Breusch–Godfrey statistic \( Q_{LM} \) in (4) is not robust against conditional heteroskedasticity. Godfrey and Tremayne (2005) employ a heteroskedasticity-robust (HR) version of the LM statistic in dynamic regression models with conditionally heteroskedastic errors using the general formula of White (1980) for heteroskedasticity-consistent covariance matrix estimators (HCCME).

### 2.2 Vector Error Correction Model

We now turn to the case where \( X_t \) is \( I(1) \) and cointegrated. Following Brüggemann et al. (2006), we consider the cointegrated VAR model with \( r < K \) cointegrating relations. The model can be written in vector error correction model (VECM) form as

\[ \Delta X_t = \mu + \alpha \beta' X_{t-1} + \Gamma_1 \Delta X_{t-1} + \cdots + \Gamma_{p-1} \Delta X_{t-p+1} + \varepsilon_t, \]

where \( \alpha \) and \( \beta \) are \((K \times r)\) matrices with rank \( r \), \( \Gamma_1, \ldots, \Gamma_{p-1} \) are \((K \times K)\) parameter matrices and \( \mu \) is a \((K \times 1)\) vector of constants. The error process \{\( \varepsilon_t \)\} is assumed to be as before. Brüggemann et al. show that the LM statistic may be computed from the auxiliary model

\[ \tilde{\varepsilon}_t = \mu + \alpha \tilde{\beta}' X_{t-1} + \Gamma_1 \Delta X_{t-1} + \cdots + \Gamma_{p-1} \Delta X_{t-p+1} + \phi_1 + \Psi_1 \tilde{\varepsilon}_{t-1} + \cdots + \Psi_h \tilde{\varepsilon}_{t-h} + \varepsilon_t, \]

where \( \tilde{\beta}_\perp = (\beta_\perp', \Delta X_{t-1}', \ldots, \Delta X_{t-p+1}') \), \( \phi = \text{vec}(\alpha, \Gamma_1, \ldots, \Gamma_{p-1}) \) and \( \tilde{Z}_t = X_{t-1} \beta_{\perp} \otimes \tilde{\alpha} \). Here \( \beta_{\perp} \) denotes the orthogonal complement of \( \beta \) such that \( \beta' \beta_{\perp} = 0 \), and \( \beta_{\perp} \) is the estimator of \( \beta_{\perp} \) (see Johansen (1996), p. 95). The terms \( \mu \) and \( X_{t-1} \beta_{\perp} \otimes \tilde{\alpha} \) are related to the Gaussian scores of \( \mu \) and \( \beta \). Because no particular normalisation of \( \beta \) is employed, the term \( X_{t-1} \beta_{\perp} \otimes \tilde{\alpha} \) is used as an additional regressor in the auxiliary model (see Brüggemann et al. (2006), p. 587).
The limiting distribution of the LM statistic $Q_{LM}$ under the null hypothesis is the same $\chi^2$ distribution as in stationary VAR models.

The terms $X_{t-1}^\prime \beta_1 \otimes \hat{\alpha}$ may be deleted from the auxiliary model because the estimator of $\beta$ is asymptotically independent of the estimators of $\phi$ and $\psi$ (Brüggemann et al. 2006, Remark 1).

The tests for AC in the VECM (9) require the cointegration rank $r$ to be known. The value of $r$ is usually not known. In this case an unrestricted VAR model is estimated and the residuals are tested for AC before the cointegration rank is tested. The LM statistic $Q_{LM}$ in the unrestricted VAR model may then be computed from an auxiliary model that has the same form as the auxiliary model (3) for the stationary VAR model (Brüggemann et al. 2006), Remark 2).

### 3 Bootstrap and Wild Bootstrap Algorithms

We use the residual-based recursive WB procedure of Gonçalves and Kilian (2004). It is a modification of the recursive residual-based bootstrap method for autoregressions, which replaces the IID bootstrap by the WB when bootstrapping the residuals of the VAR model.

The WB errors are generated as $\varepsilon^*_t = w_t \hat{\varepsilon}_t$, where $\hat{\varepsilon}_t$ are the residuals from the estimated VAR model and $w_t$ is an IID sequence with mean zero, variance one and such that $\mathbb{E}|w_t|^4 \leq c < \infty$ (Gonçalves and Kilian (2004)). Several auxiliary distributions may be considered (see e.g. Davidson and Flachaire (2008) and Davidson, Monticini and Peel (2007)). We use the Rademacher distribution, which is given by the simple two-point distribution

$$ w_t = \begin{cases} 
1, & \text{with probability } \frac{1}{2} \\
-1, & \text{with probability } \frac{1}{2} 
\end{cases} \quad (11) $$

It is one of the most commonly used auxiliary distributions and is recommended by Davidson and Flachaire.

The asymptotic validity of the WB under conditional heteroskedasticity of unknown form was established by Gonçalves and Kilian. We assume the following conditions adapted from Gonçalves and Kilian to the case of multivariate conditional heteroskedasticity:

1. $\mathbb{E}(\varepsilon_t | \mathcal{F}_{t-1}) = 0$ almost surely, where $\mathcal{F}_{t-1} = \sigma(\varepsilon_{t-1}, \varepsilon_{t-2}, \ldots)$ is the $\sigma$-field generated by $\{\varepsilon_{t-1}, \varepsilon_{t-2}, \ldots\}$.
2. $\mathbb{E}(\varepsilon_t \varepsilon_t^\prime) = \Omega < \infty$ and positive definite.
3. $\lim_{n \to \infty} n^{-1} \sum_{t=1}^n \mathbb{E}(\varepsilon_t \varepsilon_t^\prime | \mathcal{F}_{t-1}) = \Omega > 0$ in probability.
4. $\mathbb{E}(\varepsilon_{it}^2 \varepsilon_{j,t-r} \varepsilon_{k,t-s}) = 0$ for all $r \neq s$, for all $i, j, k$, for all $t, r \geq 1, s \geq 1$.
5. $\lim_{n \to \infty} n^{-1} \sum_{t=1}^n \varepsilon_{it}^2 \varepsilon_{j,t-r} \varepsilon_{k,t-s} = \sigma_i^2 \sigma_j^{-1} \sigma_k^{-1} \mathbb{E}(\varepsilon_{it}^2)$. In probability for any $r \geq 1, s \geq 1$, where $\tau_{r,s,ijk} = \sigma_i^{-2} \sigma_j^{-1} \sigma_k^{-1} \mathbb{E}(\varepsilon_{it}^2 \varepsilon_{j,t-r} \varepsilon_{k,t-s})$.
6. $\mathbb{E}|\varepsilon_{it}|^{4r}$ is uniformly bounded for some $r \geq 2$ and for all $i$.

Assumptions (i) and (ii) replace the IID assumption on the errors $\{\varepsilon_t\}$ by the martingale difference (MD) sequence assumption. Assumption (iii) requires convergence of the conditional moments. Assumptions (iv) and (v) restrict the fourth-order cumulants of $\varepsilon_t$. Assumption (vi) requires the existence of at least 8th moments for the MD sequence $\{\varepsilon_t\}$. In particular, Gonçalves and Kilian show that Assumption (iv) ensures consistency of the recursive WB estimator and Assumption (vi) is required for the convergence of the recursive WB variance-covariance matrix estimator to the correct limiting variance-covariance matrix.
The bootstrap and WB algorithms for the $Q_{LM}$ test for error AC are detailed in Algorithms 1 and 2. The same algorithm is used with the $Q_{LR}$, $Q_{W}$ and $Q_{F}$ tests.

**Algorithm 1** *Bootstrap LM test for error AC.*

1. Compute the LM statistic $Q_{LM}$ from the data. Obtain the parameter estimates and the residuals $\hat{\varepsilon}_t$ from the VAR model.

2. Construct the bootstrap errors $\varepsilon^*_t$ by drawing independently with replacement a sample of size $T$ from $\{\hat{\varepsilon}_t\}_1^T$.

3. Generate recursively a bootstrap sample $X^*_t$ using the parameter estimates and the bootstrap errors from Steps 1 and 2.

4. Compute the bootstrap LM statistic $Q_{LM}^*$ from the bootstrap sample.

5. Repeat steps 2–4 $B$ times to obtain $Q_{LM}^*_1, \ldots, Q_{LM}^*_B$.

6. Estimate the bootstrap $p$-value by

$$\hat{p}^* = \frac{1}{B} \sum_{i=1}^{B} I(Q_{LM}^*_i > Q_{LM}),$$

(12)

where $I(\cdot)$ is the indicator function.

**Algorithm 2** *WB LM test for error AC.*

1. Same as in Algorithm 1.

2. Draw $w_t$, $t = 1, \ldots, T$, independently from a Rademacher distribution (12) and construct the WB errors as $\varepsilon^*_t = w_t \hat{\varepsilon}_t$.

3.–6. Same as in Algorithm 1.

The null hypothesis of no error AC is rejected at the significance level $\alpha$ if $\hat{p}^* < \alpha$.

## 4 Simulations

We investigate the properties of the tests for error AC when the errors are conditionally heteroskedastic by Monte Carlo simulation experiments. The small sample properties of the tests for error AC in stationary VAR models when the errors are IID are examined by Edgerton and Shukur (1999). They find that the tests have satisfactory size properties only when $K$ and $h$ are small relative to the number of observations. The LM test is preferable to the LR and Wald versions of the test. The $F$ approximation to the LR test is found to have better size properties than the asymptotic $\chi^2$ tests when the dimensions are large. The small sample properties of the tests for error AC in the context of the VECM when the errors are IID are investigated by Brüggemann et al. They find that the tests have similar properties in VECMs.

The tests considered are the four versions of the LM test, namely the $Q_{LM}$, $Q_{LR}$, $Q_{W}$ and $Q_{F}$ tests.
4.1 Monte Carlo Design

In the simulations we consider two data-generating processes (DGPs) for the conditional mean. The first DGP is a stationary VAR(1) model

\[ X_t = \mu + \Pi_1 X_{t-1} + \varepsilon_t. \]

The dimensions of the system are \( K = 2 \) and 5. The parameter values are contained in Table 1.

The second DGP is a VECM of the form used by Brüggemann et al. (2006)

\[ \Delta X_t = \mu + \alpha(\beta X_{t-1} - \tau(t - 1)) + \Gamma_1 \Delta X_{t-1} + \varepsilon_t. \]

The dimensions of the system are \( K = 2 \) and 5. The cointegration rank is \( r = 1 \) when \( K = 2 \) and \( r = 2 \) when \( K = 5 \). The parameter values are contained in Table 1.

For the errors we consider a constant conditional correlation generalised autoregressive conditional heteroskedasticity (CCC-GARCH) model (Bollerslev (1990))

\[ \varepsilon_t = D_t z_t, \]

where \( D_t = \text{diag}(h_{12}^{1/2}, \ldots, h_{K2}^{1/2}) \) is a diagonal matrix of conditional standard deviations of \( \varepsilon_t \) and \( z_t \sim \text{NID}(0, P) \) and \( P = (\rho_{ij}) \) is a positive definite covariance matrix with ones on the main diagonal. We focus on the CCC-GARCH(1,1) model with

\[ h_t = a_0 + A_1 \varepsilon_{t-1}^{(2)} + B_1 h_{t-1}, \]

where \( \varepsilon_t^{(2)} = (\varepsilon_{1t}^2, \ldots, \varepsilon_{Kt}^2)' \), \( h_t = (h_{1t}, \ldots, h_{Kt})' \) is a \((K \times 1)\) vector of conditional variances of \( \varepsilon_t \), \( a_0 \) is a \((K \times 1)\) vector of positive constants, and \( A_1 \) and \( B_1 \) are \((K \times K)\) parameter matrices which are diagonal with positive diagonal elements. Jeantheau (1998) proposes an extended CCC-GARCH model where some of the off-diagonal elements have non-zero values, but this extension is not considered here (see e.g. He and Teräsvirta (2004) and Nakatani and Teräsvirta (2009)). The parameter values are contained in Table 1. In DGP 1, \( A_1 = B_1 = 0 \), and then \( \varepsilon_t = z_t, z_t \sim \text{NID}(0, I_K) \). DGP 2 is characterised by low persistence in the volatility \( (a_{ii} = 0.5) \). DGP 3 is the multivariate generalisation of the DGP used by Godfrey and Tremayne (2005). It is characterised by moderate persistence in the volatility \( (a_{ii} = 0.8) \). DGP 4 is characterised by very high persistence in the volatility \( (a_{ii} + b_{ii} = 0.98) \). The values for \( \rho \) are the ones used by Nakatani and Teräsvirta (2009). Finally, when \( K = 5 \) the parameter \( \rho \) in \( P \) is the canonical correlation between the first component and the remaining \( K - 1 \) components.

The CCC-GARCH processes in DGPs 2–4 satisfy the conditions for weak and strict stationarity (He and Teräsvirta (2004) and Nakatani and Teräsvirta (2009)). Notice that the validity of the recursive residual-based WB procedure requires the existence of at least 8th moments. He and Teräsvirta (2004, p. 908) give a result concerning the existence of the fourth moment matrix of \( \varepsilon_t \). The condition is that the largest eigenvalue of a certain matrix is less than 1. DGPs 2 and 4 satisfy the condition (the largest eigenvalue is 0.75 and 0.9732, respectively). The condition is violated by DGP 3 (the largest eigenvalue is 1.92). The conditions for the existence of the 8th moment matrix of \( \varepsilon_t \) are not known. We therefore only know that the conditions are not satisfied by DGP 3.

The series lengths are \( T = 100, 200, 500 \) and 1000. The number of replications is 100000. The computations and simulations are performed in R, version 2.13.2
(R Development Core Team (2011)). We use the ccgarch package version 0.2.0 (Nakatani (2013)) for simulating the CCC-GARCH(1,1) models and checking the fourth moment condition. The size and power of the bootstrap and WB tests are simulated using the fast bootstrap method of Davidson and MacKinnon (2006).
# Table 1: Parameter values of the DGPs.

<table>
<thead>
<tr>
<th>Conditional mean</th>
<th>DGP 1. Stationary VAR(1): $X_t = \mu + \Pi_1 X_{t-1} + \varepsilon_t$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K = 2, 5$</td>
<td>$\Pi_1 = \phi_1 I_K, \phi_1 = 0.8$</td>
</tr>
<tr>
<td></td>
<td>$\mu = 0$</td>
</tr>
<tr>
<td>DGP 2. VECM:</td>
<td>$\Delta X_t = \mu + \alpha(\beta' X_{t-1} - \tau(t-1)) + \Gamma_1 \Delta X_{t-1} + \varepsilon_t$.</td>
</tr>
<tr>
<td>$K = 2, r = 1$</td>
<td>$\alpha = \begin{pmatrix} -0.2 \ 0 \end{pmatrix}$, $\beta = \begin{pmatrix} 1 \ -1 \end{pmatrix}$, $\mu = 0$, $\tau = 0.01$</td>
</tr>
<tr>
<td></td>
<td>$\Gamma_1 = \begin{pmatrix} 0.5 &amp; 0 \ -0.2 &amp; 0.5 \ -0.2 &amp; 0 \ 0 &amp; -0.2 \ 0 &amp; 0 \ 0 &amp; 0 \end{pmatrix}$</td>
</tr>
<tr>
<td>$K = 5, r = 2$</td>
<td>$\alpha = \begin{pmatrix} 0.5 &amp; 0 &amp; 0 &amp; 0 &amp; -0.2 \ -0.2 &amp; 0.5 &amp; 0 &amp; 0 &amp; 0 \ 0 &amp; -0.2 &amp; 0.5 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; -0.2 &amp; 0.5 &amp; 0 \ 0 &amp; 0 &amp; 0 &amp; -0.2 &amp; 0.5 \end{pmatrix}$, $\beta = \begin{pmatrix} 1 &amp; 0 \ -1 &amp; 1 \ 0 &amp; -1 \ 0 &amp; 0 \ 0 &amp; 0 \end{pmatrix}$, $\mu = 0$, $\tau = 0.01$</td>
</tr>
<tr>
<td></td>
<td>$\Gamma_1 = \begin{pmatrix} 0.5 \ 0.15 \ 0.15 \ 0.15 \ 0.15 \ 0.15 \end{pmatrix}$</td>
</tr>
<tr>
<td>$K = 2$</td>
<td>Errors: CCC-GARCH(1, 1) $a_0 = \begin{pmatrix} 0.15 \ 0.15 \end{pmatrix}$, $A_1 = \begin{pmatrix} 0.5 &amp; 0 \ 0 &amp; 0.5 \end{pmatrix}$, $B_1 = \begin{pmatrix} 0.9 \ 0 \end{pmatrix}$, $P = \begin{pmatrix} 1 &amp; \rho \ \rho &amp; 1 \end{pmatrix}$</td>
</tr>
<tr>
<td>DGP 1</td>
<td>$a_0 = \begin{pmatrix} 0.15 \ 0.15 \end{pmatrix}$, $A_1 = \begin{pmatrix} 0.5 &amp; 0 \ 0 &amp; 0.5 \end{pmatrix}$, $B_1 = \begin{pmatrix} 0.9 \ 0 \end{pmatrix}$, $P = \begin{pmatrix} 1 &amp; \rho \ \rho &amp; 1 \end{pmatrix}$</td>
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<tr>
<td>DGP 2</td>
<td>$a_0 = \begin{pmatrix} 0.15 \ 0.15 \end{pmatrix}$, $A_1 = \begin{pmatrix} 0.8 &amp; 0 \ 0 &amp; 0.8 \end{pmatrix}$, $B_1 = \begin{pmatrix} 0.9 \ 0 \end{pmatrix}$, $P = \begin{pmatrix} 1 &amp; \rho \ \rho &amp; 1 \end{pmatrix}$</td>
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<tr>
<td>DGP 3</td>
<td>$a_0 = \begin{pmatrix} 0.15 \ 0.15 \end{pmatrix}$, $A_1 = \begin{pmatrix} 0.8 &amp; 0 \ 0 &amp; 0.8 \end{pmatrix}$, $B_1 = \begin{pmatrix} 0.9 \ 0 \end{pmatrix}$, $P = \begin{pmatrix} 1 &amp; \rho \ \rho &amp; 1 \end{pmatrix}$</td>
</tr>
<tr>
<td>DGP 4</td>
<td>$a_0 = \begin{pmatrix} 0.15 \ 0.15 \end{pmatrix}$, $A_1 = \begin{pmatrix} 0.8 &amp; 0 \ 0 &amp; 0.8 \end{pmatrix}$, $B_1 = \begin{pmatrix} 0.9 \ 0 \end{pmatrix}$, $P = \begin{pmatrix} 1 &amp; \rho \ \rho &amp; 1 \end{pmatrix}$</td>
</tr>
<tr>
<td>$K = 5$</td>
<td>$a_0$ has same elements, $A_1$ and $B_1$ are diagonal with same diagonal elements, $P = (\rho)$, where $\rho$ the canonical correlation between the first component and the remaining $K - 1$ components.</td>
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</table>
4.2 Size

The simulated rejection probabilities of the asymptotic, bootstrap and WB tests for error AC of orders $h = 1, 4$ and 12 at the nominal significance level 5% are reported in Tables 2–5. The notation $Q_{LM}^{B}$, $Q_{LR}^{B}$, $Q_{W}^{B}$ and $Q_{F}^{B}$ is used with the bootstrap tests and the notation $Q_{LM}^{WB}$, $Q_{LR}^{WB}$, $Q_{W}^{WB}$ and $Q_{F}^{WB}$ is used with the WB tests.

The results show that the asymptotic $Q_{LM}$ and $Q_{F}$ tests should be preferred over the $Q_{LR}$ and $Q_{W}$ tests which are oversized in the stationary VAR model of DGP 1 with IID errors when $K = 2$. The size of the tests increase in the unrestricted VAR model when the data are generated by the VECM of DGP 2, as can be seen by comparing the rejection probabilities in Tables 2 and 4. Similar results were found by Edgerton and Shukur (1999) for stationary VAR models and Brüggemann et al. (2006) for VECMs. The $Q_{F}$ test is better at controlling the size in the unrestricted VAR model when the data are generated by the VECM of DGP 2. The asymptotic tests are all oversized when the dimensions are large ($K = 5$).

The size differences between the asymptotically equivalent tests disappear in finite samples when they are bootstrapped. This is expected, because the bootstrap provides a size-correction of the asymptotic tests (see Davidson and MacKinnon (2006)). To save space, we report only the rejection probabilities of the LM version of the bootstrap and WB tests. The $Q_{LM}^{B}$ test is oversized in DGPs 2–4 with conditionally heteroskedastic errors. It is not surprising that the rejection probabilities do not approach the nominal level as the number of observations increases, because the bootstrap test is not valid if the errors are not IID. The size of the $Q_{LM}^{B}$ test in the stationary VAR model of DGP 1 and DGP 2 for the errors with low persistence in volatility ($a_{ii} = 0.5$) is at least 35% for $h = 1$ and 4 and 23% for $h = 12$ (these are the rejection probabilities reported for $K = 2$ and $T = 1000$ in Table 2). The size of the test in DGP 4 for the errors with high persistence in volatility ($a_{ii} + b_{ii} = 0.98$) is at least 14% for $h = 1$, 24% for $h = 4$ and 36% for $h = 12$ (these are the rejection probabilities reported for $K = 2$ and $T = 1000$ in Table 2). The size distortion of the $Q_{LM}^{B}$ test at lag $h = 1$ is larger in DGPs 2 and 3 with ARCH errors than in DGP 4 with GARCH errors, which is caused by the interrelationship between AC and ARCH at lag $h = 1$ (see Bera et al. (1992)). It can be seen that the test attains its maximal size for $h = 1$ in DGP 2, whereas in DGP 4 the test attains its maximal size for $h = 12$. Similar results are obtained when the dimensions are $K = 5$ (see Table 2) and in the unrestricted VAR model when the data are generated by the VECM of DGP 2 (see Tables 4 and 5).

The size of the $Q_{LM}^{WB}$ test in the stationary VAR model of DGP 1 and DGP 4 for the errors with very high persistence in volatility when $K = 2$ is close to the nominal level for all $T$. There are size distortions in DGPs 2 and 3 for the errors when $K = 2$ and $T = 100$ and 200. The size of the $Q_{LM}^{WB}$ test approaches the nominal level as the number of observations increases. It can be noted that the convergence to the nominal level is slower in DGP 3 for which the fourth moment matrix of $\varepsilon_t$ does not exist than in DGPs 2 and 4 for which it does. The size of the $Q_{LM}^{WB}$ test appears to converge to 6% in DGP 3 rather than the nominal level 5%. However, longer series than $T = 1000$ observations may be needed for the size to converge to the nominal level. The simulation results show that failure of the sufficient condition of the existence of 8th moments to hold has little effect on the performance of the $Q_{LM}^{WB}$ test in finite samples. Gonçalves and Kilian (2004) obtained a similar result in the univariate case. The size of the $Q_{LM}^{WB}$ tests increases in higher-dimensional systems ($K = 5$) and in the unrestricted VAR model when the data are generated by the VECM of DGP 2, as can be seen by comparing the rejection probabilities in
4.3 Power

In the simulations for power the errors $\varepsilon_t$ have an autoregressive structure:

$$\varepsilon_t = \Psi_1 \varepsilon_{t-1} + \epsilon_t,$$

where $\Psi_1 = \varphi_1 I$, $\varphi_1 = 0, 0.01, \ldots, 0.99$ and $\epsilon_t$ is a $(K \times 1)$ vector of errors following DGP s 1–4 listed in Table 1. Because the different versions of the bootstrap and WB tests have identical power functions, we show the power functions only for the LM version of the tests. Figure 1 shows the simulated power functions of the level-adjusted asymptotic $Q_{LM}$ test, the bootstrap $Q_{LM}^B$ test and the WB $Q_{LM}^{WB}$ test for error AC in the stationary VAR model of DGP 1 when $K = 2$, $N = 100$ and $h = 1, 4$ and 12. The $Q_{LM}$, $Q_{LM}^B$ and $Q_{LM}^{WB}$ tests have identical power functions in DGP 1 with IID normal errors. Thus the $Q_{LM}^{WB}$ test does not incur a loss in power in comparison to the $Q_{LM}^B$ test when the errors are homoskedastic. The WB should therefore routinely replace the bootstrap based on IID errors in tests for AC if conditional heteroskedasticity in the errors is suspected. The $Q_{LM}^{WB}$ test has lower power when the errors are conditionally heteroskedastic compared to the case with IID errors. The power of the test decreases only slightly in DGP 4 for the errors, whereas the decrease in power is much larger in DGPs 2 and 3. It can be seen that the power against $h = 1$ approaches 1 in DGPs 1 and 4 when $\varphi_1 = 0.5$. In contrast, the power is below 0.8 in DGP 3. Figure 2 shows the power functions when $K = 5$, $N = 100$ and $h = 1, 4$ and 12. The decrease in power of the $Q_{LM}^{WB}$ test in DGPs 2–4 is smaller when the dimensions are large. Figures 3 and 4 show the power functions when $K = 2$ and 5 and $N = 500$. The decrease in power of the $Q_{LM}^{WB}$ tests when the errors follow DGP 3 is larger when $T$ increases. It can be seen that when $K = 5$ and $T = 500$, the power against $h = 1$ approaches 1 in DGPs 1, 2 and 4 when $\varphi_1 = 0.2$, but is below 0.5 in DGP 3.

5 Empirical Illustrations

In this section we illustrate the use of the WB LM tests for error AC with real data in two empirical applications to credit default swap (CDS) prices and Euribor interest rates.

The determination of the lag length in vector autoregressive models is an important area of application for tests for error AC (see e.g. Lütkepohl (2006)). The simulation results in the previous section strongly suggest that the asymptotic and bootstrap tests may falsely reject the null hypothesis of no error AC if the errors are conditionally heteroskedastic. The asymptotic and bootstrap tests for error AC may therefore erroneously indicate that a long lag length is required.

5.1 Credit Default Swap Prices

Our first empirical example deals with credit default swap (CDS) prices. A CDS is a credit derivative which provides a bondholder with protection against the risk of default by the company. If a default occurs, the holder is compensated for the loss by an amount which equals the difference between the par value of the bond and its market value after the default. The CDS price is the annualised fee (expressed as a
Table 2: Simulated size of the asymptotic, bootstrap and WB tests for error autocorrelation. The dimensions are $K = 2$. The nominal significance level is 5%.

\[
\begin{array}{ccccccccccc}
\hline
\rho & 0 & 0 & 0.3 & 0.9 & 0 & 0.3 & 0.9 & 0 & 0.3 & 0.9 \\
T = 100 & DGP 1 & DGP 2 & DGP 3 & DGP 4 \\
Q_{LM} & 0.053 & 0.157 & 0.161 & 0.201 & 0.280 & 0.288 & 0.368 & 0.070 & 0.071 & 0.079 \\
Q_F & 0.050 & 0.151 & 0.155 & 0.195 & 0.273 & 0.281 & 0.361 & 0.066 & 0.067 & 0.074 \\
Q_W & 0.071 & 0.189 & 0.193 & 0.234 & 0.316 & 0.324 & 0.404 & 0.092 & 0.092 & 0.102 \\
Q_{LR} & 0.062 & 0.173 & 0.177 & 0.218 & 0.298 & 0.306 & 0.387 & 0.081 & 0.081 & 0.090 \\
Q_{LM}^B & 0.050 & 0.152 & 0.154 & 0.197 & 0.277 & 0.282 & 0.360 & 0.066 & 0.066 & 0.074 \\
Q_{WB}^B & 0.050 & 0.063 & 0.063 & 0.070 & 0.070 & 0.076 & 0.085 & 0.055 & 0.052 & 0.054 \\
Q_{LM} & 0.048 & 0.110 & 0.111 & 0.141 & 0.229 & 0.235 & 0.317 & 0.070 & 0.075 & 0.088 \\
Q_F & 0.047 & 0.108 & 0.110 & 0.138 & 0.227 & 0.234 & 0.316 & 0.068 & 0.073 & 0.087 \\
Q_W & 0.128 & 0.225 & 0.227 & 0.262 & 0.363 & 0.370 & 0.452 & 0.169 & 0.175 & 0.195 \\
Q_{LR} & 0.084 & 0.165 & 0.167 & 0.199 & 0.297 & 0.304 & 0.386 & 0.115 & 0.121 & 0.138 \\
Q_{LM}^B & 0.050 & 0.113 & 0.114 & 0.145 & 0.233 & 0.235 & 0.312 & 0.072 & 0.079 & 0.088 \\
Q_{WB}^B & 0.052 & 0.058 & 0.059 & 0.061 & 0.067 & 0.069 & 0.074 & 0.052 & 0.054 & 0.054 \\
Q_{LM} & 0.030 & 0.047 & 0.049 & 0.061 & 0.104 & 0.108 & 0.157 & 0.049 & 0.050 & 0.064 \\
Q_F & 0.038 & 0.057 & 0.059 & 0.073 & 0.122 & 0.126 & 0.180 & 0.060 & 0.062 & 0.076 \\
Q_W & 0.498 & 0.538 & 0.538 & 0.553 & 0.597 & 0.600 & 0.636 & 0.563 & 0.568 & 0.599 \\
Q_{LR} & 0.218 & 0.260 & 0.263 & 0.280 & 0.342 & 0.347 & 0.402 & 0.276 & 0.280 & 0.309 \\
Q_{LM}^B & 0.052 & 0.075 & 0.078 & 0.088 & 0.140 & 0.145 & 0.194 & 0.079 & 0.079 & 0.096 \\
Q_{WB}^B & 0.052 & 0.054 & 0.055 & 0.057 & 0.063 & 0.064 & 0.068 & 0.053 & 0.053 & 0.054 \\
T = 200 & h = 4 & h = 1 & h = 12 & h = 1 & h = 4 & h = 12 & h = 12 \\
Q_{LM} & 0.052 & 0.186 & 0.192 & 0.247 & 0.371 & 0.382 & 0.486 & 0.078 & 0.078 & 0.093 \\
Q_F & 0.050 & 0.183 & 0.189 & 0.243 & 0.368 & 0.379 & 0.482 & 0.076 & 0.076 & 0.091 \\
Q_W & 0.060 & 0.202 & 0.209 & 0.264 & 0.388 & 0.399 & 0.503 & 0.089 & 0.089 & 0.104 \\
Q_{LR} & 0.056 & 0.194 & 0.200 & 0.256 & 0.380 & 0.391 & 0.494 & 0.084 & 0.083 & 0.098 \\
Q_{LM}^B & 0.052 & 0.186 & 0.190 & 0.245 & 0.369 & 0.381 & 0.481 & 0.077 & 0.076 & 0.093 \\
Q_{WB}^B & 0.052 & 0.060 & 0.060 & 0.062 & 0.068 & 0.069 & 0.076 & 0.053 & 0.052 & 0.054 \\
Q_{LM} & 0.048 & 0.146 & 0.151 & 0.202 & 0.364 & 0.372 & 0.484 & 0.093 & 0.096 & 0.123 \\
Q_F & 0.048 & 0.145 & 0.150 & 0.201 & 0.362 & 0.371 & 0.484 & 0.091 & 0.095 & 0.121 \\
Q_W & 0.082 & 0.202 & 0.209 & 0.265 & 0.431 & 0.441 & 0.548 & 0.143 & 0.145 & 0.178 \\
Q_{LR} & 0.064 & 0.173 & 0.180 & 0.234 & 0.398 & 0.408 & 0.517 & 0.117 & 0.120 & 0.150 \\
Q_{LM}^B & 0.049 & 0.148 & 0.153 & 0.202 & 0.363 & 0.370 & 0.478 & 0.095 & 0.097 & 0.123 \\
Q_{WB}^B & 0.050 & 0.058 & 0.059 & 0.060 & 0.066 & 0.067 & 0.070 & 0.052 & 0.052 & 0.052 \\
Q_{LM} & 0.039 & 0.080 & 0.082 & 0.110 & 0.214 & 0.224 & 0.314 & 0.091 & 0.095 & 0.135 \\
Q_F & 0.043 & 0.085 & 0.088 & 0.117 & 0.223 & 0.234 & 0.325 & 0.097 & 0.102 & 0.142 \\
Q_W & 0.208 & 0.288 & 0.291 & 0.326 & 0.445 & 0.457 & 0.538 & 0.324 & 0.331 & 0.388 \\
Q_{LR} & 0.106 & 0.171 & 0.175 & 0.206 & 0.327 & 0.337 & 0.427 & 0.195 & 0.201 & 0.253 \\
Q_{LM}^B & 0.050 & 0.096 & 0.101 & 0.129 & 0.234 & 0.243 & 0.327 & 0.111 & 0.116 & 0.155 \\
Q_{WB}^B & 0.051 & 0.054 & 0.055 & 0.056 & 0.064 & 0.063 & 0.066 & 0.051 & 0.054 & 0.055 \\
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\end{array}
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<td>0.667</td>
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Table 3: Simulated size of the asymptotic, bootstrap and WB tests for error autocorrelation. The dimensions are $K = 5$. The nominal significance level is 5%.

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<th>DGP 4</th>
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<td>0.213</td>
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<td>0.064</td>
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<p>| $T = 200$ | 0.061 | 0.215 | 0.215 | 0.251 | 0.481 | 0.486 | 0.547 | 0.086 | 0.088 | 0.094 |
|           | 0.052 | 0.196 | 0.196 | 0.232 | 0.461 | 0.466 | 0.528 | 0.074 | 0.075 | 0.082 |
|           | 0.107 | 0.297 | 0.298 | 0.336 | 0.564 | 0.569 | 0.625 | 0.143 | 0.145 | 0.153 |
|           | 0.082 | 0.255 | 0.256 | 0.293 | 0.524 | 0.528 | 0.587 | 0.113 | 0.115 | 0.122 |
|           | 0.050 | 0.193 | 0.193 | 0.227 | 0.452 | 0.459 | 0.520 | 0.071 | 0.073 | 0.079 |
|           | 0.050 | 0.069 | 0.066 | 0.071 | 0.089 | 0.089 | 0.096 | 0.053 | 0.053 | 0.056 |
|           | 0.066 | 0.169 | 0.173 | 0.201 | 0.447 | 0.454 | 0.518 | 0.113 | 0.115 | 0.127 |
|           | 0.052 | 0.143 | 0.147 | 0.174 | 0.417 | 0.423 | 0.491 | 0.091 | 0.092 | 0.104 |
|           | 0.370 | 0.549 | 0.551 | 0.581 | 0.776 | 0.777 | 0.813 | 0.472 | 0.475 | 0.492 |
|           | 0.189 | 0.350 | 0.353 | 0.389 | 0.629 | 0.635 | 0.687 | 0.274 | 0.278 | 0.294 |
|           | 0.051 | 0.138 | 0.143 | 0.167 | 0.399 | 0.406 | 0.471 | 0.089 | 0.091 | 0.099 |
|           | 0.050 | 0.063 | 0.064 | 0.065 | 0.084 | 0.084 | 0.084 | 0.054 | 0.056 | 0.054 |
|           | 0.062 | 0.101 | 0.103 | 0.114 | 0.250 | 0.258 | 0.304 | 0.117 | 0.118 | 0.134 |
|           | 0.050 | 0.086 | 0.088 | 0.099 | 0.238 | 0.247 | 0.296 | 0.099 | 0.100 | 0.114 |
|           | 0.999 | 0.999 | 0.999 | 0.999 | 0.999 | 0.999 | 0.999 | 0.999 | 1.000 | 1.000 |
|           | 0.878 | 0.907 | 0.906 | 0.909 | 0.934 | 0.936 | 0.942 | 0.923 | 0.925 | 0.931 |
|           | 0.051 | 0.085 | 0.089 | 0.100 | 0.223 | 0.233 | 0.277 | 0.100 | 0.101 | 0.118 |
|           | 0.051 | 0.059 | 0.057 | 0.058 | 0.071 | 0.075 | 0.074 | 0.054 | 0.055 | 0.056 |</p>
<table>
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<th>( \rho )</th>
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<th>DGP 2</th>
<th>DGP 3</th>
<th>DGP 4</th>
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<td>( h = 12 )</td>
<td>( h = 12 )</td>
<td>( h = 12 )</td>
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<tr>
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<td>0.254</td>
<td>0.261</td>
<td>0.312</td>
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<td>0.246</td>
<td>0.254</td>
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</tr>
<tr>
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Table 4: Simulated size of the asymptotic, bootstrap and WB tests for error autocorrelation in the cointegrated VAR model. The dimensions are $K = 2$. The nominal significance level is 5%.

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<tr>
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Table 5: Simulated size of the asymptotic, bootstrap and WB tests for error autocorrelation in the cointegrated VAR model. The dimensions are $K = 5$. The nominal significance level is 5%.

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<td>0.292</td>
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<td>( T = 1000 )</td>
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<td>( h = 4 )</td>
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<tr>
<td>( Q_{LM} )</td>
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<td>0.077, 0.196, 0.200, 0.235, 0.587, 0.596, 0.688, 0.167, 0.171, 0.193</td>
<td>0.087, 0.155, 0.158, 0.180, 0.429, 0.440, 0.518, 0.240, 0.250, 0.280</td>
<td>0.059, 0.113, 0.117, 0.139, 0.385, 0.394, 0.475, 0.181, 0.194, 0.223</td>
<td>0.059, 0.224, 0.229, 0.284, 0.685, 0.699, 0.780, 0.104, 0.109, 0.125</td>
<td>0.055, 0.216, 0.219, 0.272, 0.676, 0.690, 0.773, 0.097, 0.101, 0.119</td>
<td>0.063, 0.201, 0.202, 0.262, 0.743, 0.746, 0.835, 0.168, 0.178, 0.211</td>
<td>0.063, 0.150, 0.152, 0.183, 0.577, 0.581, 0.674, 0.259, 0.262, 0.320</td>
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<td>0.059, 0.113, 0.117, 0.139, 0.385, 0.394, 0.475, 0.181, 0.194, 0.223</td>
<td>0.066, 0.239, 0.245, 0.300, 0.697, 0.712, 0.791, 0.115, 0.120, 0.137</td>
<td>0.055, 0.216, 0.219, 0.272, 0.676, 0.690, 0.773, 0.097, 0.101, 0.119</td>
<td>0.062, 0.231, 0.237, 0.292, 0.691, 0.705, 0.786, 0.108, 0.114, 0.131</td>
<td>0.056, 0.215, 0.217, 0.270, 0.667, 0.685, 0.764, 0.095, 0.097, 0.116</td>
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<td>0.050, 0.057, 0.054, 0.061, 0.072, 0.070, 0.079, 0.056, 0.063, 0.058</td>
<td>0.049, 0.062, 0.063, 0.066, 0.071, 0.076, 0.079, 0.053, 0.057, 0.056</td>
<td>0.049, 0.062, 0.063, 0.066, 0.071, 0.076, 0.079, 0.053, 0.057, 0.056</td>
<td>0.054, 0.055, 0.058, 0.062, 0.068, 0.070, 0.079, 0.052, 0.052, 0.053</td>
<td>0.052, 0.059, 0.060, 0.062, 0.071, 0.069, 0.075, 0.051, 0.054, 0.056</td>
<td>0.048, 0.059, 0.057, 0.057, 0.070, 0.067, 0.071, 0.058, 0.054, 0.059</td>
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Figure 1: The power functions of the bootstrap and wild bootstrap tests for error autocorrelation when $K = 2$ and $N = 100$. 
Figure 2: The power functions of the bootstrap and wild bootstrap tests for error autocorrelation when $K = 5$ and $N = 100$. 
Figure 3: The power functions of the bootstrap and wild bootstrap tests for error autocorrelation when $K = 2$ and $N = 500$. 
Figure 4: The power functions of the bootstrap and wild bootstrap tests for error autocorrelation when $K = 5$ and $N = 500$. 
percentage of the principal) paid by the protection buyer. We denote by $p_t^{\text{CDS}}$ the CDS price and $p_t^{\text{CS}}$ the credit spread on a risky bond over the risk-free rate. The basis is the difference between the CDS price and the bond spread:

$$s_t = p_t^{\text{CDS}} - p_t^{\text{CS}}.$$ 

If the two markets price credit risk equally in the long run, then the prices should be equal, so that the basis $s_t = 0$. Because $p_t^{\text{CDS}}$ and $p_t^{\text{CS}}$ are $I(1)$ series, the non-arbitrage relation is tested as an equilibrium relation in the cointegrated VAR model (see Blanco, Brennan and Marsh (2005)). The vector $X_t$ with the value 1 appended is $X_t = (p_t^{\text{CDS}}, p_t^{\text{CS}}, 1)'$. The financial theory posits that $X_t$ is cointegrated with cointegrating vector $\beta = (1, -1, c)'$, so that $\beta'X_t = p_t^{\text{CDS}} - p_t^{\text{CS}} + c$ is a cointegrating relation. In theory $c = 0$, but in practice it may be different from zero (see Blanco et al. (2005) for details). Many researchers have tested the equivalence of CDS prices and credit spreads for US and European investment-grade companies and found that a parity relation holds for most companies, i.e. the bond and CDS markets price credit risk equally (see e.g. Blanco et al. (2005), Zhu (2006), Dötz (2007) and Forsbäck (2012)).

We take a subsample of the companies in Table 1 of Blanco et al. The companies in our subsample are Bank of America, Citigroup, Goldman Sachs, Barclays Bank and Vodafone, the first three of which are US and the remaining two European companies. We use 5-year maturity CDS prices and credit spreads from Datastream. The data are daily observations from 1 January 2009 to 31 January 2012, and the number of daily observations is $T = 804$. Blanco et al. contain a discussion of issues related to the construction of the series. Figure 5 plots the CDS prices and credit spreads for the companies in our subsample. The differences between the two series are small for all companies.

We use information criteria to determine the lag length in the unrestricted VAR models for $p_t^{\text{CDS}}$ and $p_t^{\text{CS}}$. The Akaike information criterion (AIC) selects $p = 18$ for Bank of America, whereas the Schwarz (SC) and Hannan–Quinn (HQ) information criteria select $p = 2$. The AIC selects $p = 10$ for Barclays Bank, whereas the SC and HQ select $p = 4$. Similar results are obtained for the other companies. The AIC tends to select a model with a long lag length, which is what we would expect since it is inconsistent. Because the sample size is large, we rely on the SC and HQ information criteria, and select the lag length $p = 2$ for Bank of America, $p = 3$ for Citigroup, Goldman Sachs and Vodafone, and $p = 4$ for Barclays Bank.

Table 6 shows the $p$-values of the asymptotic $Q_{L,M}, Q_{L,R}, Q_{W}$, and $Q_F$ tests, bootstrap $Q_{L,M}^B, Q_{L,R}^B, Q_{W}^B$ and $Q_F^B$ tests and WB $Q_{L,M}^{WB}, Q_{L,R}^{WB}, Q_{W}^{WB}$ and $Q_F^{WB}$ tests for error AC of orders $h = 1, 4$ and 12. Not surprisingly, the $p$-values of the four versions of the LM test are almost identical because the dimensions ($K = 2$) are small relative to the number of observations ($T = 804$). We first discuss the overall picture and then the results for Barclays Bank in detail. The asymptotic tests for error AC of orders $h = 1$ and 4 are significant at the 5% level for Goldman Sachs, Barclays Bank and Vodafone. The asymptotic tests for error AC of order $h = 12$ are significant for all companies. The results for the bootstrap tests are more mixed, but the tests for error AC of order $h = 12$ are significant for all companies. The overall picture is encouraging for the WB tests. The WB tests for error AC of orders $h = 1, 4$ and 12 are insignificant for all companies. Based on the WB test for error AC we conclude that the VAR models selected by the SC and HQ information criteria provide good descriptions of the data.

We now turn to a detailed analysis of the results for Barclays Bank and the
Figure 5: The CDS prices and bond spreads for Bank of America, Citigroup, Goldman Sachs, Barclays Bank and Vodafone. The sample period is 1 January 2009 to 31 January 2012.
$Q_{LM}$ test. Similar results are obtained for the other companies and versions of the LM test, as is readily seen in Table 6. The asymptotic $Q_{LM}$ test rejects the null hypothesis of no error AC at the 5% level. The $p$-value is 2.6% for $h = 1$, 0.1% for $h = 4$ and 0 for $h = 12$. The corresponding $p$-values of the bootstrap $Q_{LM}^{BM}$ test are 7.2%, 2.7% and 2.8%. The WB $Q_{LM}^{WB}$ test does not reject the null hypothesis of no error AC of orders $h = 1$, 4 and 12. The $p$-value is 32% for $h = 1$, 44% for $h = 4$ and 61% for $h = 12$. We therefore have a situation where the asymptotic test rejects but the WB test does not reject. Based on the WB test for error AC we conclude that the VAR(4) model provides a good description of the data. Finally, we mention that all VAR models with lag lengths up to $p = 25$ are rejected by the asymptotic test (see Table 7). The bootstrap test leads to a VAR(5) model (see Table 7).
Table 6: Tests for error AC in the estimated VAR models for the CDS prices and bond spreads. The table reports the $p$-values of the tests.

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<tr>
<th>$h$</th>
<th>$Q_{LM}$</th>
<th>$Q_F$</th>
<th>$Q_W$</th>
<th>$Q_{LR}$</th>
<th>$Q_{LM}^R$</th>
<th>$Q_F^R$</th>
<th>$Q_W^R$</th>
<th>$Q_{LR}^R$</th>
<th>$Q_{LM}^{WB}$</th>
<th>$Q_F^{WB}$</th>
<th>$Q_W^{WB}$</th>
<th>$Q_{LR}^{WB}$</th>
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VAR(2) model for Bank of America

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<th>$Q_W^R$</th>
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<th>$Q_F^{WB}$</th>
<th>$Q_W^{WB}$</th>
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VAR(3) model for Citigroup

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<th>$Q_W$</th>
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<th>$Q_F^{WB}$</th>
<th>$Q_W^{WB}$</th>
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VAR(4) model for Goldman Sachs

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<th>$Q_F^R$</th>
<th>$Q_W^R$</th>
<th>$Q_{LR}^R$</th>
<th>$Q_{LM}^{WB}$</th>
<th>$Q_F^{WB}$</th>
<th>$Q_W^{WB}$</th>
<th>$Q_{LR}^{WB}$</th>
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<tbody>
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<td>0.001</td>
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<td>0.027</td>
<td>0.028</td>
<td>0.027</td>
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</tr>
<tr>
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<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
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VAR(3) model for Barclays Bank

<table>
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<tr>
<th>$h$</th>
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<th>$Q_F$</th>
<th>$Q_W$</th>
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<th>$Q_{LM}^R$</th>
<th>$Q_F^R$</th>
<th>$Q_W^R$</th>
<th>$Q_{LR}^R$</th>
<th>$Q_{LM}^{WB}$</th>
<th>$Q_F^{WB}$</th>
<th>$Q_W^{WB}$</th>
<th>$Q_{LR}^{WB}$</th>
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<td>0.002</td>
<td>0.002</td>
<td>0.799</td>
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<td>0.802</td>
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<td>0.533</td>
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<td>0.533</td>
<td>0.533</td>
</tr>
<tr>
<td>4</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
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<td>0.245</td>
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<td>0.245</td>
<td>0.504</td>
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<td>0.505</td>
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</tr>
<tr>
<td>12</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.019</td>
<td>0.020</td>
<td>0.021</td>
<td>0.020</td>
<td>0.504</td>
<td>0.504</td>
<td>0.504</td>
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</tr>
</tbody>
</table>
Table 7: Lag length of the VAR models determined by the bootstrap and WB LM tests for error AC and the cointegration rank determined by the LR test for cointegration rank in the VAR models.

<table>
<thead>
<tr>
<th></th>
<th>$Q^{\text{H}}_{\text{LM}}$</th>
<th>$Q^{\text{W}}_{\text{LM}}$</th>
<th>$Q^{\text{H}}_{\text{LM}}$</th>
<th>$Q^{\text{W}}_{\text{LM}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lag length</td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Bank of America</td>
<td>7</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Citigroup</td>
<td>10</td>
<td>3</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Goldman Sachs</td>
<td>4</td>
<td>3</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Barclays Bank</td>
<td>5</td>
<td>4</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Vodafone Bank</td>
<td>5</td>
<td>3</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

The residuals from the estimated VAR models are plotted in Figure 6. Notice the volatility clustering in the residuals. This is confirmed by Table 8, which reports asymptotic and bootstrap LM tests for ARCH up to order 2. The bootstrap $Q^{\text{B}}_{\text{LM}}$ test for ARCH is presented in Catani and Ahlgren (2013). The asymptotic and bootstrap tests for ARCH find a strong ARCH effect in the residuals for all companies, except the equation for $p_{t}^{\text{CS}}$ for Citigroup. The multivariate ARCH tests are significant at the 1% level, with the exception of Vodafone. Because there is very strong evidence of ARCH in the errors, the WB tests for error AC should be preferred over the asymptotic and bootstrap tests which assume IID errors.

Finally, we mention that Barclays Bank is the only company for which the LR test of cointegration rank (Johansen (1996)) accepts the cointegration rank $r = 1$ (see Table 7). The restrictions on $\beta$ implied by the theory are, however, not accepted by the LR test (see Johansen (1996), Chapter 7).

In order to further investigate the size properties of the tests for error AC, we fit CCC-GARCH(1, 1) models to the residuals from the unrestricted VAR models. Table 9 summarises the parameter estimates. The parameter estimates imply severe ARCH effects. The stationarity condition for the CCC-GARCH(1, 1) model is $\lambda(\Gamma_{C}) < 1$, where $\lambda$ is the modulus of the largest eigenvalue of a certain matrix $\Gamma_{C}$ (see He and Ter"{a}svirta (2004)). Notice that the stationarity condition is satisfied only for the models for Goldman Sachs and Barclays Bank, indicating that the CCC-GARCH(1, 1) model does not provide a good fit to the residuals of the other companies. The fourth moment condition is satisfied when $\lambda(\Gamma_{C \otimes C}) < 1$, where $\lambda$ is the modulus of the largest eigenvalue of a certain matrix $\Gamma_{C \otimes C}$ (see He and Ter"{a}svirta (2004)). The fourth moment condition is not satisfied by any of the models for the errors. We investigate by simulation the rejection probabilities of the tests for error AC for Barclays Bank. In each simulation we use the estimated parameters from the VAR(4)-CCC-GARCH(1, 1) model to define the DGP. The errors $z_{t}$ are drawn from a multivariate normal distribution with covariance matrix equal to the estimated covariance matrix. We simulate 100000 time series of length $T = 804$. We test for error AC in the unrestricted VAR model and the cointegrated VAR model with cointegration rank $r = 1$. The simulated size of the tests are reported in Table 10. The asymptotic and bootstrap tests are severely oversized, whereas the WB tests have size close to the nominal level 5%. Focusing on the $Q_{\text{LM}}$ test, the size of the asymptotic test in the unrestricted VAR model is 13% for $h = 1$, 22% for $h = 4$ and 30% for $h = 12$. The bootstrap $Q^{\text{B}}_{\text{LM}}$ test is oversized, which is what we would expect since the errors are not IID. The size of the WB test $Q^{\text{WB}}_{\text{LM}}$ is 5.3% for $h = 1$, 5.5% for $h = 4$ and 5.7% for $h = 12$. The size of the asymptotic $Q_{\text{LM}}$ test in the
Figure 6: The residuals from the estimated VAR models for the CDS prices and bond spreads.
Table 8: LM tests for ARCH in the estimated VAR models for the CDS prices and bond spreads. The table reports the p-values of the tests.

<table>
<thead>
<tr>
<th></th>
<th>Q_{LM}</th>
<th>Q_{LM}^B</th>
<th>Q_{F}</th>
<th>Q_{F}^B</th>
</tr>
</thead>
<tbody>
<tr>
<td>VAR(2) model for Bank of America</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>pCDS</td>
<td>0.000</td>
<td>0.007</td>
<td>0.000</td>
<td>0.007</td>
</tr>
<tr>
<td>pCS</td>
<td>0.003</td>
<td>0.024</td>
<td>0.003</td>
<td>0.024</td>
</tr>
<tr>
<td>Multivariate</td>
<td>0.000</td>
<td>0.000</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>VAR(3) model for Citigroup</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>pCDS</td>
<td>0.000</td>
<td>0.002</td>
<td>0.000</td>
<td>0.002</td>
</tr>
<tr>
<td>pCS</td>
<td>0.956</td>
<td>0.564</td>
<td>0.957</td>
<td>0.564</td>
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<tr>
<td>Multivariate</td>
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<td>0.028</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>VAR(3) model for Goldman Sachs</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>pCDS</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>pCS</td>
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<td>0.024</td>
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<tr>
<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>VAR(4) model for Barclays Bank</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>pCDS</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>pCS</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>Multivariate</td>
<td>0.000</td>
<td>0.000</td>
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<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>VAR(3) model for Vodafone</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>pCDS</td>
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<td>0.000</td>
<td>0.002</td>
</tr>
<tr>
<td>pCS</td>
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<td>0.018</td>
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</tr>
<tr>
<td>Multivariate</td>
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<td>0.093</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

cointegrated VAR model with \( r = 1 \) is 19% for \( h = 1 \), 25% for \( h = 4 \) and 31% for \( h = 12 \). The bootstrap \( Q_{LM}^B \) test is oversized. The size of the WB test \( Q_{LM}^{WB} \) is 5.5% for \( h = 1 \), 7.0% for \( h = 4 \) and 4.7% for \( h = 12 \). Comparing the results for the unrestricted VAR model and the cointegrated VAR model with cointegration rank \( r = 1 \) shows small differences in size only for the test of \( h = 1 \). The size of the \( Q_{LM}^{WB} \) test is 0.053 in the unrestricted VAR model and 0.055 in the cointegrated VAR model. The differences in size are larger for the tests of \( h = 4 \) and 12, but different from the simulation results reported in Brüggemann et al (2006) who find that the size of the tests increases in the unrestricted VAR model compared to the cointegrated VAR model, no clear overall picture emerges. The size of the \( Q_{LM}^{WB} \) test for \( h = 4 \) increases from 0.055 in the unrestricted VAR model to 0.070 in the cointegrated VAR model. The opposite happens with the size of the test for \( h = 12 \), which decreases from 0.057 in the unrestricted VAR model to 0.047 in the cointegrated VAR model. The size of the WB test \( Q_{LM}^{WB} \) is in both models much closer to the nominal level than the size of the asymptotic and bootstrap tests.
Table 9: The parameter estimates of the CCC-GARCH(1, 1) models fitted to the residuals from the VAR models for the CDS prices and bond spreads. Standard errors are reported in parentheses below the parameter estimates.

<table>
<thead>
<tr>
<th></th>
<th>Bank of America</th>
<th>Citigroup</th>
<th>Goldman Sachs</th>
<th>Barclays Banks</th>
<th>Vodafone</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_{CDS}$</td>
<td>$a_{01}$</td>
<td>1.153</td>
<td>(1.049)</td>
<td>0.114</td>
<td>(0.155)</td>
</tr>
<tr>
<td></td>
<td>$a_{11}$</td>
<td>0.191</td>
<td>(0.124)</td>
<td>0.138</td>
<td>(0.096)</td>
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<tr>
<td></td>
<td>$b_{11}$</td>
<td>0.832</td>
<td>(0.085)</td>
<td>0.888</td>
<td>(0.058)</td>
</tr>
<tr>
<td>$p_{CS}$</td>
<td>$a_{02}$</td>
<td>2.265</td>
<td>(1.775)</td>
<td>0.386</td>
<td>(1.223)</td>
</tr>
<tr>
<td></td>
<td>$a_{22}$</td>
<td>0.075</td>
<td>(0.046)</td>
<td>0.071</td>
<td>(0.037)</td>
</tr>
<tr>
<td></td>
<td>$b_{22}$</td>
<td>0.936</td>
<td>(0.025)</td>
<td>0.951</td>
<td>(0.006)</td>
</tr>
<tr>
<td></td>
<td>$\rho$</td>
<td>0.082</td>
<td>(0.057)</td>
<td>0.046</td>
<td>(0.041)</td>
</tr>
<tr>
<td></td>
<td>$\lambda(\Gamma_C)$</td>
<td>1.023</td>
<td>(0.057)</td>
<td>1.026</td>
<td>(0.041)</td>
</tr>
<tr>
<td></td>
<td>$\lambda(\Gamma_{C\otimes C})$</td>
<td>1.119</td>
<td>(0.094)</td>
<td>1.091</td>
<td>(0.044)</td>
</tr>
</tbody>
</table>
Table 10: Simulated size of the asymptotic, bootstrap and WB tests for error AC in the estimated VAR model for the CDS price and bond spread for Barclays Bank.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$Q_{LM}$</th>
<th>$Q_{LR}$</th>
<th>$Q_{W}$</th>
<th>$Q_{F}$</th>
<th>$Q^B_{LM}$</th>
<th>$Q^B_{LR}$</th>
<th>$Q^B_{W}$</th>
<th>$Q^B_{F}$</th>
<th>$Q^{WB}_{LM}$</th>
<th>$Q^{WB}_{LR}$</th>
<th>$Q^{WB}_{W}$</th>
<th>$Q^{WB}_{F}$</th>
</tr>
</thead>
<tbody>
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<td>0.135</td>
<td>0.137</td>
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<td>0.130</td>
<td>0.130</td>
<td>0.130</td>
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<td>0.053</td>
<td>0.053</td>
<td>0.053</td>
<td>0.053</td>
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<tr>
<td>4</td>
<td>0.221</td>
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<td>0.237</td>
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<td>0.208</td>
<td>0.055</td>
<td>0.055</td>
<td>0.055</td>
<td>0.055</td>
<td>0.055</td>
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<tr>
<td>12</td>
<td>0.304</td>
<td>0.334</td>
<td>0.365</td>
<td>0.295</td>
<td>0.276</td>
<td>0.275</td>
<td>0.275</td>
<td>0.057</td>
<td>0.057</td>
<td>0.058</td>
<td>0.058</td>
<td>0.057</td>
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Unrestricted VAR(4) model

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<tr>
<th>$h$</th>
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<th>$Q_{W}$</th>
<th>$Q_{F}$</th>
<th>$Q^B_{LM}$</th>
<th>$Q^B_{LR}$</th>
<th>$Q^B_{W}$</th>
<th>$Q^B_{F}$</th>
<th>$Q^{WB}_{LM}$</th>
<th>$Q^{WB}_{LR}$</th>
<th>$Q^{WB}_{W}$</th>
<th>$Q^{WB}_{F}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>0.189</td>
<td>0.193</td>
<td>0.183</td>
<td>0.073</td>
<td>0.073</td>
<td>0.074</td>
<td>0.073</td>
<td>0.055</td>
<td>0.056</td>
<td>0.058</td>
<td>0.056</td>
</tr>
<tr>
<td>4</td>
<td>0.249</td>
<td>0.264</td>
<td>0.276</td>
<td>0.240</td>
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<td>0.187</td>
<td>0.189</td>
<td>0.187</td>
<td>0.070</td>
<td>0.067</td>
<td>0.067</td>
<td>0.067</td>
</tr>
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<td>0.343</td>
<td>0.373</td>
<td>0.306</td>
<td>0.245</td>
<td>0.242</td>
<td>0.241</td>
<td>0.242</td>
<td>0.047</td>
<td>0.046</td>
<td>0.046</td>
<td>0.046</td>
</tr>
</tbody>
</table>

Cointegrated VAR(4) model with $r = 1$
5.2 Euribor Interest Rates

In the second empirical example we consider the problem of testing the expectations hypothesis of the term structure of interest rates. The theory makes two predictions, which can be tested in the cointegrated VAR model (Hall, Anderson and Granger (1992)). First, if there are $K$ interest rate series in the system, then the cointegration rank is $r = K - 1$. Second, the spreads between the interest rates at different maturities span the cointegration space.

We use monthly data from December 1998 to September 2008 on the 1, 3, 6, 9 and 12 month Euribor interest rates. All interest rates are nominal and annualised. The data were retrieved from www.euribor.org. The SC and HQ information criteria select the lag length $p = 3$. The estimated VAR model includes a constant and dummy variables to account for interest rate shocks in April 2001 and September 2001.

Table 11 shows the $p$-values of the asymptotic $Q_{LM}$, $Q_{LR}$, $Q_{W}$, and $Q_{F}$ tests, bootstrap $Q_{LM}^B$, $Q_{LR}^B$, $Q_{W}^B$ and $Q_{F}^B$ tests and WB $Q_{LM}^{WB}$, $Q_{LR}^{WB}$, $Q_{W}^{WB}$ and $Q_{F}^{WB}$ tests for error AC of orders $h = 1, 4$ and 12. We first discuss the overall picture and then the results for the LM test in detail. The $p$-values of the four versions of the LM test are close to zero, with the exception of the $Q_{F}$ test for AC of order $h = 1$, which is 0.059. The asymptotic $Q_{LM}$ test rejects the null hypothesis of no error AC at the 5% level. The $p$-value is 1.6% for $h = 1$, and 0 for $h = 4$ and 12. The corresponding $p$-values of the bootstrap $Q_{LM}^B$ test are 10.6%, 2.6% and 0.2%. The WB $Q_{LM}^{WB}$ test does not reject the null hypothesis of no error AC of orders $h = 1, 4$ and 12. The $p$-value of is 36% for $h = 1$, 38% for $h = 4$ and 19% for $h = 12$. We therefore have a situation where the asymptotic test rejects but the WB test does not reject. Based on the WB test for error AC we conclude that the VAR(3) model provides a good description of the data. Finally, we mention that the VAR model with the smallest value of the lag length $p$ which is not rejected by the asymptotic $Q_{LM}$ test is a VAR(10) model. The bootstrap $Q_{LM}^B$ test leads to a VAR(6) model.
Table 11: Tests for error AC in the estimated VAR(3) model for the Euribor interest rates. The table reports the $p$-values of the tests.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$Q_{LM}$</th>
<th>$Q_{LR}$</th>
<th>$Q_{W}$</th>
<th>$Q_{F}$</th>
<th>$Q_{LM}^{R}$</th>
<th>$Q_{LR}^{R}$</th>
<th>$Q_{W}^{R}$</th>
<th>$Q_{F}^{R}$</th>
<th>$Q_{LM}^{WB}$</th>
<th>$Q_{LR}^{WB}$</th>
<th>$Q_{W}^{WB}$</th>
<th>$Q_{F}^{WB}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.016</td>
<td>0.006</td>
<td>0.002</td>
<td>0.059</td>
<td>0.106</td>
<td>0.100</td>
<td>0.095</td>
<td>0.100</td>
<td>0.364</td>
<td>0.354</td>
<td>0.344</td>
<td>0.354</td>
</tr>
<tr>
<td>4</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.014</td>
<td>0.026</td>
<td>0.036</td>
<td>0.049</td>
<td>0.036</td>
<td>0.376</td>
<td>0.440</td>
<td>0.500</td>
<td>0.440</td>
</tr>
<tr>
<td>12</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.003</td>
<td>0.002</td>
<td>0.006</td>
<td>0.002</td>
<td>0.191</td>
<td>0.260</td>
<td>0.382</td>
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</table>
Table 12: Asymptotic and bootstrap tests for ARCH in the estimated VAR(3) model for the Euribor interest rates. The table reports the p-values of the tests.

<table>
<thead>
<tr>
<th></th>
<th>$Q_{LM}$</th>
<th>$Q_{LM}^{B}$</th>
<th>$Q_{F}$</th>
<th>$Q_{F}^{B}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 month</td>
<td>0.000</td>
<td>0.002</td>
<td>0.001</td>
<td>0.002</td>
</tr>
<tr>
<td>3 months</td>
<td>0.559</td>
<td>0.466</td>
<td>0.623</td>
<td>0.466</td>
</tr>
<tr>
<td>6 months</td>
<td>0.675</td>
<td>0.669</td>
<td>0.727</td>
<td>0.669</td>
</tr>
<tr>
<td>9 months</td>
<td>0.342</td>
<td>0.322</td>
<td>0.417</td>
<td>0.322</td>
</tr>
<tr>
<td>12 months</td>
<td>0.329</td>
<td>0.306</td>
<td>0.403</td>
<td>0.306</td>
</tr>
<tr>
<td>Multivariate</td>
<td>0.000</td>
<td>0.000</td>
<td>–</td>
<td>–</td>
</tr>
</tbody>
</table>

LM tests for ARCH up to order 2 are reported in Table 12. The asymptotic and bootstrap tests find a strong ARCH effect in the residuals from the equation for the 1 month interest rate. The multivariate ARCH tests are significant at the 1% level. Because there is very strong evidence of ARCH in the errors, the WB tests for error AC should be preferred over the asymptotic and bootstrap tests which assume IID errors.

Finally, we mention that the outcome of the sequence of LR ratio tests of cointegration rank (Johansen (1996)) is $r = 2$, so the financial theory is not supported by the data.

6 Conclusions

In this article we study the performance of LM tests for error AC in stationary and cointegrated vector autoregressive models with conditionally heteroskedastic errors. The main results are that the asymptotic and residual-based bootstrap tests based on the IID error assumption are oversized when the errors are conditionally heteroskedastic, whereas the WB test performs well whether the errors are IID or conditionally heteroskedastic. Based on simulation results, the WB test for error AC perform well when the errors follow a CCC-GARCH process which satisfies the condition for the existence of the fourth moment matrix of the errors. The WB test performs reasonably well even if the condition is not satisfied. The WB test has the same power as the level-adjusted asymptotic and bootstrap tests in the case of IID errors, but suffers a loss of power if the errors are conditionally heteroskedastic.

The empirical examples demonstrate that the asymptotic and bootstrap tests for error AC may falsely reject the null hypothesis of no error AC if the errors are conditionally heteroskedastic. In such cases the tests may erroneously lead to selecting a model with a long lag length. This potential pitfall is avoided by the WB test. We therefore recommend the WB test for error AC for use in practice if conditional heteroskedasticity is suspected.

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Cointegration Analysis of Chinese Stock Markets
— An Example of Wild Bootstrap Testing

Paul Catani*

October 9, 2013

Abstract

Conditional heteroskedasticity is common in financial time series. This article studies the impact of conditional heteroskedasticity on cointegration analysis between Chinese stock markets and the US stock market. Tests show that the Shanghai A and B share indices, the Hang Seng index of Hong Kong, the Hang Seng China Enterprices Index and and Standar & Poor’s 500 index have significant conditional heteroskedasticity and that model selection based on standard asymptotic tests leads to models with a large number of lags. Wild bootstrap tests are robust to conditional heteroskedasticity and using them we are able to select models with shorter lag lengths. Asymptotic tests for cointegration find cointegrating relations between the Chinese stock markets. Using wild bootstrap tests of cointegration we find no cointegration in the Chinese markets. When the Standard & Poor’s 500 index is included in the system two cointegrating relations are found in a period before the financial crisis 2007-2008.

1 Introduction

Conditional heteroskedasticity is a problem often encountered in economic and financial time series. Inference based on asymptotic tests may be misleading if conditional heteroskedasticity is not taken into account. A method for dealing with conditional heteroskedasticity is the wild bootstrap. An example of inference under conditional heteroskedasticity considered in this paper is cointegration analysis of Chinese stock prices.

The Chinese stock markets in Shanghai and Shenzhen have since their opening in 1993 been divided into A and B shares. In China only domestic investors were allowed to trade A shares, whereas B shares were reserved for foreign investors. The trading restrictions were relaxed for the first time in February 2001, when domestic investors were allowed to trade B shares. Further relaxations have been implemented since. In 2002 Qualified Foreign Institutional Investors were allowed to trade A shares and in April 2006 access to foreign security markets was granted to Qualified Domestic Institutional Investors. Other notable relaxations are the gradual abolishment of foreign exchange restrictions, the most important being the currency policy change in July 2005, which allowed minimal daily appreciation of the yuan (Ahlgren, sjö and Zhang (2009)).

*Hanken School of Economics, PO Box 479 (Arkadiagatan 22), 00101 Helsinki, Finland. E-mail: paul.catani@hanken.fi.
A majority of the companies listed on the Shanghai and Shenzhen stock exchanges issue only A shares, while some issue both A and B shares. Furthermore, H shares are issued for Hong Kong residents and listed on the Hong Kong Stock Exchange.

Cointegration of A and B shares on the Chinese markets have been studied, among others, by Sjöö and Zhang (2000), Shen, Chen and Chen (2007), Tian (2007) and Ahlgren et al. (2009). Sjöö and Zhang (2000) study information flows between the A and B shares markets, and find no cointegration between the A and B shares. Tian (2007) finds no evidence of cointegration between the A and B share markets in Shanghai. Shen et al. (2007) find that the Shenzhen A and B shares have an asymmetric cointegration relationship after the B share market were opened to domestic investors in February 2001, and that the A shares on both markets are asymmetrically cointegrated in the period from October 1992 to September 2002. Ahlgren et al. (2009) examine information flows on China’s stock exchanges by panel cointegration tests on the Shanghai and Schenzen on A and B shares. They find that the A and B shares are cointegrated after January 2001, when domestic investors were allowed to trade B shares, but not in the time period before the event.

An extensive literature on the integration of the Chinese, neighboring markets and the US market is available. Huang, Yang and Hu (2000) find no cointegration between the markets in the South China region, Japan and the US. However, Huang et al. (2000), and Groenewold, Tang and Wu (2004) find evidence of cointegration in the A-share markets between Shanghai and Shenzhen before the Asian crisis, which occurred in 1997–1998. However, Groenewold et al. (2004) no longer find cointegration after the crisis. Tian (2007) studies cointegration in the Chinese stock markets using Shanghai A and B shares indices, the Hang Seng China enterprises index, the Hang Seng index of Hong Kong, the Taiwanese value weighted index, the Tokio Stock Exchange index and Standard & Poor’s 500 (SP 500) index. They find evidence of cointegration in the period after the Asian crisis between the Shanghai A shares, Hong Kong main market, the Taiwanese market and to some extent also the US market.

Studying the relations between the Shanghai, Shenzhen and Hong Kong markets, Chan and Lo (2000) and Huang et al. (2000) find evidence of the Hong Kong market leading the Shanghai B-share market. Groenewold et al. (2004) find evidence that Hong Kong has weak predictive power for the Shanghai and Shenzhen markets, but no cointegration between the three markets is found after the Asian crisis.

Following Tian (2007) and Ahlgren et al. (2009), I test the integration of the Chinese stock markets. I study whether there exists cointegration between the Shanghai A and B shares indices, the Hang Seng index of Hong Kong and the Hang Seng China enterprises index.

The indices are characterised by conditional heteroskedasticity, which may cause problems in cointegration analysis, and the likelihood ratio test of cointegration rank of Johansen (1996) may be unreliable under such conditions in finite samples (Cavaliere, Rahbek and Taylor (2010a)). Cavaliere et al. (2010a) and Cavaliere, Rahbek and Taylor (2012) propose a wild bootstrap method for cointegration rank tests and show that it produces more reliable inferences when there is heteroskedasticity in the data.

In this paper I study cointegration relations in the Chinese stock prices using both standard asymptotic tests and wild bootstrap tests. I show that neglecting conditional heteroskedasticity leads to selecting models with many lags. The likelihood ratio test of cointegration rank finds cointegration in the Chinese stock markets.
Using wild bootstrap methods leads to the selection of a model with fewer lags and no cointegration is found in the Chinese stock markets. When the SP 500 index is included in the system asymptotic tests favour models with many lags where no cointegration is found. However, using wild bootstrap tests for model selection, models with fewer lags are selected and cointegration is found in the period before the financial crisis of 2007-2008.

The outline of the paper is the following. The second section introduces the data and the methods used to test for cointegration. The results are presented and discussed in the third section and the fourth section concludes.

2 Data and Methods

2.1 Data

The data consist of four time series of logarithms of the Shanghai A (SHA) and B (SHB) share indices, the Hang Seng index of Hong Kong (HKSI), the Hang Seng China enterprises index (HKH) and SP 500 index (SP500). Following Tian (2007), the Shenzhen market is not considered. I use daily data obtained from Datastream from 1 January 2002 to 24 September 2012. The total number of daily observations is $T = 2799$. Based on tests for structural breaks in the VAR models performed when modeling the data, the data are further divided into two subperiods: one before (1 January 2002 to 31 July 2007, $T = 1978$) and one after (1 September 2008 to 24 September 2012, $T = 1038$) the beginning of the financial crisis of 2007-2008. The data are presented in Figure 1.

2.2 Likelihood Ratio Test of Cointegration Rank

For testing cointegration I use the likelihood ratio test of Johansen (1996). Two systems are considered in the analysis, one containing only the Chinese market and one with all the Chinese indices and the SP 500 index. The four-dimensional time series of Chinese stock prices can be represented by $X_t = (SHA_t, SHB_t, HKSI_t, HKH_t)'$. I consider the $K$-dimensional VAR model

$$X_t = \Pi_1 X_{t-1} + \Pi_2 X_{t-2} + \ldots + \Pi_p X_{t-p} + \mu + \epsilon_t, \quad t = 1, \ldots, T, \quad (1)$$

where $\mu$ is an unrestricted constant vector. The model can be written in vector error correction (VECM) form

$$\Delta X_t = \Pi X_{t-1} + \sum_{i=1}^{p-1} \Gamma_i \Delta X_{t-i} + \mu + \epsilon_t, \quad (2)$$

where $\Pi = \sum_{i=1}^{p} \Pi_i - I$ and $\Gamma_i = -\sum_{j=i+1}^{p} \Pi_j$. If there exist cointegration relations in the data, the matrix $\Pi = \alpha \beta'$ has reduced rank, where $\alpha$ and $\beta$ are $K \times r$ matrices of rank $r < 4$. The cointegration vectors are $\beta$ and the cointegrating relations are $\beta'X_t$, which are stationary and $I(0)$.

The cointegration rank is determined by the likelihood ratio test, testing the sequence of null hypotheses

$$H_0 : \text{rk}(\Pi) = r, \quad r = 0, 1, \ldots, K - 1. \quad (3)$$

The alternative hypotheses are $H_1 : r_0 < \text{rk}(\Pi) \leq K$, where $r_0$ is the value under the null hypothesis. The sequence is terminated when the null hypothesis cannot
Figure 1: The logarithms of the Shanghai A and B shares indices, the Hang Seng index of Hong Kong, the Hang Seng China enterprises index and the S&P 500 index. The sample period is 1 January 2002 to 24 September 2012.
be rejected for the first time. Here \( K = 4 \) and if the test does not reject \( H_0 : \text{rk}(\Pi) = 0 \), we have no cointegration, but if the test rejects we proceed by testing \( H_0 : \text{rk}(\Pi) = 1 \). If the test does not reject, then the cointegration rank 1 is chosen, and so on. If the tests reject for all values of \( r = 0, 1, \ldots, K - 1 \), it indicates that the series are \( I(0) \) and a VAR model for levels can be applied.

The test statistic is constructed by first solving the eigenvalues \( 1 > \hat{\lambda}_1 > \ldots > \hat{\lambda}_K > 0 \) from

\[
|\lambda \Sigma_{22} - \Sigma_{21} \Sigma_{11}^{-1} \Sigma_{12}| = 0,
\]

where

\[
\Sigma_{ij} = T^{-1} \sum_{t=1}^{T} \hat{V}_{t-i} \hat{V}_{t-j}, \quad i, j = 1, 2,
\]

and \( \hat{V}_{t1} \) and \( \hat{V}_{t2} \) are the residuals from regressing \( \Delta X_t \) and \( X_{t-1} \) on the lagged differences and a constant. The likelihood ratio statistic is given by

\[
Q_r = -T \sum_{i=r+1}^{K} \ln(1 - \hat{\lambda}_i).
\]

The limiting distribution is nonstandard and the critical values based on simulations are given in Johansen (1996).

### 2.3 Wild Bootstrap Likelihood Ratio Test of Cointegration Rank

The motivation for using the wild bootstrap is that it is robust against heteroskedasticity. As shown by Cavaliere, Rahbek and Taylor (2010b), the likelihood ratio test tends to overreject when the data are subject to unconditional heteroskedasticity. This can lead to findings of false cointegrating relations between the stock market indices. Cavaliere et al. (2010a) show that the likelihood ratio test is valid under conditional heteroskedasticity, but can be misleading in finite samples.

In the wild bootstrap procedure the residuals are not drawn randomly, but instead they are multiplied with an independent scalar sequence \( \{w_t\}_{t=1}^{T} \), which follows a distribution with mean 0 and variance 1. One alternative is the Rademacher distribution

\[
w_t = \begin{cases} 
1 \text{ with probability } 1/2 \\
-1 \text{ with probability } 1/2.
\end{cases}
\]

Other distributions for \( w_t \) have also been suggested. Cavaliere et al. (2010a) use the standard normal distribution, but according to Davidson and Flachaire (2008) the Rademacher distribution is the best choice.

The wild bootstrap algorithm using restricted residuals, based on Cavaliere et al. (2012), is given below.

**Algorithm 1** *(Wild Bootstrap Cointegration Test)*

1. Obtain the estimates \( \hat{\alpha}^{(r)}, \hat{\beta}^{(r)}, \hat{\Gamma}_1^{(r)}, \ldots, \hat{\Gamma}_{k-1}^{(r)} \) and \( \hat{\mu}^{(r)} \) together with the restricted residuals, \( \hat{e}_{r,t} \), by reduced rank regression.

2. Check if the characteristic polynomial \( |\tilde{\Pi}^{(r)}(z)| = 0 \), with

\[
\tilde{\Pi}^{(r)}(z) = (1 - z)I_p - \hat{\alpha}^{(r)} \hat{\beta}^{(r)} z - \hat{\Gamma}_1^{(r)} (1 - z) z - \ldots - \hat{\Gamma}_{p-1}^{(r)} (1 - z) z^{p-1}.
\]
has $K - r$ roots equal to 1 and all other roots outside the unit circle. If so proceed to the next step.

3. Generate wild bootstrap errors as

$$\hat{\varepsilon}_{r,t}^* = \hat{\varepsilon}_{r,t} w_t,$$

where $w_t$ is a random draw from a distribution with mean 0 and variance 1.

4. Generate recursively the wild bootstrap sample $X_t^*, t = 1, ..., T$, where $X_t^* = X_t$ for $t = 1, \ldots, k$, as

$$\Delta X_t^* = \tilde{\alpha}^{(r)} \beta^{(r)} X_{t-1} + \sum_{i=1}^{p-1} \tilde{I}_{i}^{(r)} \Delta X_{t-i}^* + \tilde{\mu}^{(r)} + \hat{\varepsilon}_{r,t}^*, \quad t = k + 1, ..., T.$$ 

5. Obtain the wild bootstrap test statistic, $Q_{r}^* = -T \sum_{i=r+1}^{p} \ln(1 - \hat{\lambda}_{i}^*)$, using the bootstrap sample.

6. Repeat steps 3-5 and obtain $Q_{r1}^*, \ldots, Q_{rB}^*$.

7. Estimate the bootstrap $p$-value by

$$\hat{p}^* = \frac{1}{B} \sum_{i=1}^{B} I(Q_{r,i}^* > Q_r),$$

where $I(\cdot)$ is the indicator function.

The null hypothesis is rejected at the significance level $\alpha$ if $\hat{p}^* < \alpha$.

3 Results

The results are presented in this section. The labels 'System 1' and 'System 2' are used to separate the model with the four Chinese stock indices from the model including also the SP 500 index.

In order to test for cointegration a VAR model is selected. This can be done based on information criteria, such as the Akaike information criterion (AIC), the Schwartz information criterion (SC) or the Hannan-Quinn information criterion (HQ). Models with lag length between 1 and 20 are estimated, and for each the information criteria are calculated. The lag length which minimizes the information criteria are presented in table 1. The AIC selects a model with more lags compared to the SC and HQ in all sample periods. For the full sample and the subsample before the financial crisis, the SC and HQ are minimized for $p = 2$, while AIC is minimized for $p = 7$. For the subsample after the financial crisis, the SC and HQ are minimized for $p = 1$, while AIC is minimized for $p = 2$.

In System 2 the models suggested by the SC, HQ and AIC for the full sample are of order 2, 3 and 7 respectively. For the first subsample all criteria suggest a VAR model of order 2 and for the second subsample the SC, HQ and AIC select models of order 2, 3 and 9 respectively.

Another way of selecting the lag length of the VAR model is to use likelihood ratio tests for sequential testing of $H_0 : \Pi_p = 0$ against $\Pi_p \neq 0$. The sequence is started from a large number $p$ and moves to smaller values until the test rejects for
Table 1: VAR lag length determined by information criteria.

<table>
<thead>
<tr>
<th>System</th>
<th>Sample</th>
<th>SC</th>
<th>HQ</th>
<th>AIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Full</td>
<td>2</td>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>Before</td>
<td>2</td>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>After</td>
<td>1</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>Full</td>
<td>2</td>
<td>3</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>Before</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>After</td>
<td>2</td>
<td>3</td>
<td>9</td>
</tr>
</tbody>
</table>

Table 2: Likelihood ratio tests for the lag length in the VAR model.

<table>
<thead>
<tr>
<th>System</th>
<th>Sample</th>
<th>Test</th>
<th>p = 6</th>
<th>p = 5</th>
<th>p = 4</th>
<th>p = 3</th>
<th>p = 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Full</td>
<td>Asymptotic</td>
<td>0.151</td>
<td>0.019</td>
<td>0.006</td>
<td>0.001</td>
<td>0.000</td>
</tr>
<tr>
<td></td>
<td>WB</td>
<td>Asymptotic</td>
<td>0.419</td>
<td>0.199</td>
<td>0.201</td>
<td>0.089</td>
<td>0.000</td>
</tr>
<tr>
<td></td>
<td>Before</td>
<td>Asymptotic</td>
<td>0.085</td>
<td>0.633</td>
<td>0.521</td>
<td>0.007</td>
<td>0.000</td>
</tr>
<tr>
<td></td>
<td>WB</td>
<td>Asymptotic</td>
<td>0.213</td>
<td>0.838</td>
<td>0.787</td>
<td>0.080</td>
<td>0.000</td>
</tr>
<tr>
<td></td>
<td>After</td>
<td>Asymptotic</td>
<td>0.056</td>
<td>0.112</td>
<td>0.157</td>
<td>0.043</td>
<td>0.000</td>
</tr>
<tr>
<td></td>
<td>WB</td>
<td>Asymptotic</td>
<td>0.270</td>
<td>0.490</td>
<td>0.696</td>
<td>0.422</td>
<td>0.014</td>
</tr>
<tr>
<td>2</td>
<td>Full</td>
<td>Asymptotic</td>
<td>0.033</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td></td>
<td>WB</td>
<td>Asymptotic</td>
<td>0.395</td>
<td>0.582</td>
<td>0.015</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td></td>
<td>Before</td>
<td>Asymptotic</td>
<td>0.041</td>
<td>0.469</td>
<td>0.133</td>
<td>0.009</td>
<td>0.000</td>
</tr>
<tr>
<td></td>
<td>WB</td>
<td>Asymptotic</td>
<td>0.141</td>
<td>0.739</td>
<td>0.445</td>
<td>0.110</td>
<td>0.000</td>
</tr>
<tr>
<td></td>
<td>After</td>
<td>Asymptotic</td>
<td>0.004</td>
<td>0.079</td>
<td>0.001</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td></td>
<td>WB</td>
<td>Asymptotic</td>
<td>0.189</td>
<td>0.626</td>
<td>0.291</td>
<td>0.006</td>
<td>0.000</td>
</tr>
</tbody>
</table>

Note: Bold indicates the first time the test sequence rejects the null hypothesis. The number of wild bootstrap replications is $B = 9999$.

the first time. I use the $F$-approximation of Rao (1973) to the likelihood ratio test. Since there is evidence of heteroskedasticity in the data, both asymptotic and wild bootstrap $p$-values are calculated for the tests. The wild bootstrap is implemented in a similar way as described in Algorithm 1 for the test of cointegration rank. The difference being that in this case the bootstrap samples are constructed recursively using the unrestricted estimates according to:

$$X_t^* = \hat{\Pi}_1 X_{t-1}^* + \hat{\Pi}_2 X_{t-2}^* + \ldots + \hat{\Pi}_p X_{t-p}^* + \hat{\mu} + \hat{\varepsilon}_t^*, \quad (8)$$

where $\hat{\varepsilon}_t^* = \hat{\varepsilon}_t w_t$ and $w_t$ is a random draw from the Rademacher distribution.

Table 2 reports the $p$-values of the likelihood ratio tests starting from $p = 6$. For the full sample the asymptotic test rejects the null hypothesis for the first time when $p = 5$, and the wild bootstrap test when $p = 2$. For the two subsamples the asymptotic tests reject for $p = 3$, and the wild bootstrap tests reject for $p = 2$. That is, $p = 2$ seems appropriate for all sample periods in the first system using the wild bootstrap test. When the SP 500 index is included in the model the results are similar in that the wild bootstrap test suggests models with less lags compared to the asymptotic test. The asymptotic test selects a VAR(6) model for all sample periods. The wild bootstrap test selects a VAR(4) model for the full sample, a VAR(2) for the first subsample and a VAR(3) model for the second subsample.

Tests for autocorrelation in the VAR models are calculated using the LM principle of Breusch (1978) and Godfrey (1978). Again, both asymptotic and wild bootstrap $p$-values are calculated and reported in Table 3. Following Ahlgren and Catani
(2012), a recursive design wild bootstrap is implemented, where the recursion in (8) is used. In all sample periods the asymptotic tests find significant autocorrelation for all values of $h$. Here $h$ is the order of autocorrelation tested against. The wild bootstrap test does not reject for any value of $h$ in the subsamples, but for the full sample the wild bootstrap tests find evidence for autocorrelation. Using a VAR(3) model for the full sample, the wild bootstrap tests no longer find autocorrelation. The results for VAR models with order $p > 3$ are not reported here. The asymptotic tests find significant autocorrelation in all considered VAR models up to $p = 20$.

<table>
<thead>
<tr>
<th>System</th>
<th>Test</th>
<th>VAR(2)</th>
<th></th>
<th></th>
<th>VAR(3)</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$h = 1$</td>
<td>$h = 5$</td>
<td>$h = 10$</td>
<td>$h = 1$</td>
<td>$h = 5$</td>
<td>$h = 10$</td>
</tr>
<tr>
<td>1</td>
<td>Full</td>
<td>Asymptotic</td>
<td>0.002</td>
<td>0.000</td>
<td>0.000</td>
<td>0.001</td>
<td>0.000</td>
</tr>
<tr>
<td></td>
<td></td>
<td>WB</td>
<td>0.104</td>
<td>0.048</td>
<td>0.019</td>
<td>0.382</td>
<td>0.143</td>
</tr>
<tr>
<td></td>
<td>Before</td>
<td>Asymptotic</td>
<td>0.280</td>
<td>0.002</td>
<td>0.000</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>WB</td>
<td>0.773</td>
<td>0.590</td>
<td>0.163</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>After</td>
<td>Asymptotic</td>
<td>0.061</td>
<td>0.014</td>
<td>0.000</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>WB</td>
<td>0.222</td>
<td>0.278</td>
<td>0.063</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

|        |          |        |       |       |        |       |       |

**Note:** * indicates significance at the 5% level. The number of wild bootstrap replications is $B = 9999$.

Tests for autoregressive conditional heteroskedasticity (ARCH) in the VAR models selected according to the wild bootstrap tests for autocorrelation in Table 3 are reported in Table 4. In testing the system, I use the Lagrange multiplier (LM) test of Eklund and Teräsvirta (2007), which assumes a constant conditional correlation ARCH (CCC-ARCH) of second order under the alternative hypothesis. The LM test, proposed by Engle (1982) is used to test the residuals from the individual equations. In addition to asymptotic $p$-values, Table 4 also reports bootstrap $p$-values. Following Catani and Ahlgren (2013) the bootstrap $p$-values have been calculated using a fixed design bootstrap. Different from the recursive wild bootstrap implemented in the sequential likelihood ratio tests and the tests for autocorrelation the bootstrap sample is constructed using the original lags of the VAR-model and resampled standardized residuals as bootstrap errors (See Catani and Ahlgren (2013) for details). All tests reject the null hypothesis of no ARCH, in all series and for all sample periods. The evidence for conditional heteroskedasticity is strong. Both systems and all series were also tested using White’s test for heteroskedasticity (White (1980)). In all cases the $p$-values were 0.000, indicating strong evidence for unconditional heteroskedasticity.

Ahlgren and Catani (2012) show that the asymptotic $p$-values are unreliable when testing for autocorrelation under conditional heteroskedasticity. Since the evidence of conditional heteroskedasticity is strong, the wild bootstrap $p$-values should be preferred. Based on Table 3 and 4 this indicates that a VAR(3) model fits the data for the full sample and VAR(2) models are appropriate for the subsamples in system
Table 4: LM tests for ARCH(2) errors in the VAR models.

<table>
<thead>
<tr>
<th>Sample</th>
<th>Test</th>
<th>S1</th>
<th>S2</th>
<th>SHA</th>
<th>SHB</th>
<th>HKH</th>
<th>HKSI</th>
<th>SP500</th>
</tr>
</thead>
<tbody>
<tr>
<td>Full</td>
<td>Asymptotic</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td></td>
<td>Bootstrap</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>Before</td>
<td>Asymptotic</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td></td>
<td>Bootstrap</td>
<td>0.000</td>
<td>0.000</td>
<td>0.001</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td>After</td>
<td>Asymptotic</td>
<td>0.000</td>
<td>0.000</td>
<td>0.001</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
<td>0.000</td>
</tr>
<tr>
<td></td>
<td>Bootstrap</td>
<td>0.000</td>
<td>0.000</td>
<td>0.002</td>
<td>0.000</td>
<td>0.000</td>
<td>0.001</td>
<td>0.000</td>
</tr>
</tbody>
</table>

Note: * indicates significance at the 5% level. The number of bootstrap replications is $B = 9999$.

1. For System 2 a VAR(2) is appropriate for the first subsample, a VAR(4) for the second subsample and a VAR(7) for the full sample.

Table 5 reports asymptotic and wild bootstrap $p$-values from trace tests of cointegration rank in the three sample periods. The model is selected based on the wild bootstrap likelihood ratio tests in Table 2 and wild bootstrap autocorrelation tests in Table 3. In System 1 the tests find no evidence for cointegration in the full sample. In both subsamples the asymptotic tests find one cointegrating vector. The wild bootstrap tests, however, find no evidence of cointegration in the subsamples. Wild bootstrap $p$-values are in all cases larger than the corresponding asymptotic $p$-values. Following Cavaliere et al. (2010a) and Cavaliere et al. (2010b), it can be concluded that the wild bootstrap $p$-values are more reliable than the asymptotic ones. Therefore no evidence of cointegration in found between the four stock indices. When the SP 500 index is included in the system both asymptotic and bootstrap tests find two cointegrating vectors in the period before the financial crisis. The cointegration however disappears after the financial crisis.

Table 6 reports the results when model selection is based on the asymptotic likelihood ratio tests in Table 2. The results for System 1 do not differ much from the results in Table 5. For System 2, the results are different. The asymptotic tests only find one cointegrating vector in the first subsample, while wild bootstrap tests find no cointegrating vectors.

Table 5 also reports the 95% quantiles of the LR test. The asymptotic quantiles are from Johansen (1996, Table 15.3), and the wild bootstrap quantiles were calculated for the different sample periods from the data. The wild bootstrap quantiles are larger than the asymptotic quantiles. The difference can best be seen when the null hypothesis is $r = 0$. For the second subsample, the difference to the asymptotic quantiles is substantial. The difference is observed in the $p$-values, where the asymptotic test rejects $r = 0$ with $p$-value 0.002, while the wild bootstrap $p$-value is 0.239. The results for the first subsample are fairly similar for both asymptotic and wild bootstrap tests in all cases. The difference between the 95% asymptotic and wild bootstrap quantiles are also smallest in the first subsample.

In order to examine the results of the asymptotic tests further, the cointegrating relations found by the asymptotic tests between the four Chinese series are estimated and reported in Table 7. The estimated cointegrating relations $\hat{\beta}' \mathbf{X}_t$ and their first differences $\Delta \hat{\beta}' \mathbf{X}_t$ are plotted in Figure 2. The cointegrating relations do not look stationary and we can see volatility clustering in the first differences of the relations. This indicates the presence of conditional heteroskedasticity in the cointegrating relations.
Table 5: LR tests of Cointegration Rank when the VAR model selection is based on wild bootstrap LR tests.

<table>
<thead>
<tr>
<th>Sample</th>
<th>$H_0 : r = 0$</th>
<th>$r \leq 1$</th>
<th>$r \leq 2$</th>
<th>$r \leq 3$</th>
<th>$r = 0$</th>
<th>$r \leq 1$</th>
<th>$r \leq 2$</th>
<th>$r \leq 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test</td>
<td>System 1</td>
<td></td>
<td></td>
<td></td>
<td>System 2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Full</td>
<td>$Q_r$</td>
<td>0.157</td>
<td>0.234</td>
<td>0.265</td>
<td>0.066</td>
<td>0.521</td>
<td>0.606</td>
<td>0.667</td>
</tr>
<tr>
<td></td>
<td>$Q_r^*$</td>
<td>0.307</td>
<td>0.365</td>
<td>0.513</td>
<td>0.470</td>
<td>0.740</td>
<td>0.724</td>
<td>0.761</td>
</tr>
<tr>
<td>Before</td>
<td>$Q_r$</td>
<td>0.020</td>
<td>0.208</td>
<td>0.280</td>
<td>0.947</td>
<td>0.006</td>
<td>0.048</td>
<td>0.146</td>
</tr>
<tr>
<td></td>
<td>$Q_r^*$</td>
<td>0.087</td>
<td>0.223</td>
<td>0.477</td>
<td>0.956</td>
<td>0.035</td>
<td>0.042</td>
<td>0.202</td>
</tr>
<tr>
<td>After</td>
<td>$Q_r$</td>
<td>0.002</td>
<td>0.077</td>
<td>0.252</td>
<td>0.129</td>
<td>0.064</td>
<td>0.157</td>
<td>0.225</td>
</tr>
<tr>
<td></td>
<td>$Q_r^*$</td>
<td>0.239</td>
<td>0.298</td>
<td>0.540</td>
<td>0.490</td>
<td>0.762</td>
<td>0.573</td>
<td>0.378</td>
</tr>
</tbody>
</table>

95% Quantiles

<table>
<thead>
<tr>
<th>Sample</th>
<th>Asymptotic</th>
<th>Asymptotic</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Full</td>
<td>47.21</td>
<td>29.38</td>
</tr>
<tr>
<td></td>
<td>15.34</td>
<td>3.84</td>
</tr>
<tr>
<td>Before</td>
<td>68.68</td>
<td>47.21</td>
</tr>
<tr>
<td></td>
<td>29.38</td>
<td>15.34</td>
</tr>
<tr>
<td>After</td>
<td>123.36</td>
<td>60.49</td>
</tr>
<tr>
<td></td>
<td>33.67</td>
<td>17.68</td>
</tr>
</tbody>
</table>

Note: * indicates significance at the 5% level. The number of wild bootstrap replications is $B = 9999$.

Table 6: LR tests of Cointegration Rank when the VAR model selection is based on asymptotic LR tests.

<table>
<thead>
<tr>
<th>Sample</th>
<th>$H_0 : r = 0$</th>
<th>$r \leq 1$</th>
<th>$r \leq 2$</th>
<th>$r \leq 3$</th>
<th>$r = 0$</th>
<th>$r \leq 1$</th>
<th>$r \leq 2$</th>
<th>$r \leq 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test</td>
<td>System 1</td>
<td></td>
<td></td>
<td></td>
<td>System 2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Full</td>
<td>$Q_r$</td>
<td>0.203</td>
<td>0.331</td>
<td>0.262</td>
<td>0.063</td>
<td>0.531</td>
<td>0.616</td>
<td>0.678</td>
</tr>
<tr>
<td></td>
<td>$Q_r^*$</td>
<td>0.272</td>
<td>0.397</td>
<td>0.510</td>
<td>0.456</td>
<td>0.701</td>
<td>0.658</td>
<td>0.792</td>
</tr>
<tr>
<td>Before</td>
<td>$Q_r$</td>
<td>0.031</td>
<td>0.336</td>
<td>0.374</td>
<td>0.990</td>
<td>0.016</td>
<td>0.225</td>
<td>0.328</td>
</tr>
<tr>
<td></td>
<td>$Q_r^*$</td>
<td>0.086</td>
<td>0.317</td>
<td>0.537</td>
<td>0.994</td>
<td>0.074</td>
<td>0.222</td>
<td>0.363</td>
</tr>
<tr>
<td>After</td>
<td>$Q_r$</td>
<td>0.006</td>
<td>0.085</td>
<td>0.251</td>
<td>0.136</td>
<td>0.194</td>
<td>0.177</td>
<td>0.314</td>
</tr>
<tr>
<td></td>
<td>$Q_r^*$</td>
<td>0.289</td>
<td>0.346</td>
<td>0.425</td>
<td>0.467</td>
<td>0.913</td>
<td>0.713</td>
<td>0.448</td>
</tr>
</tbody>
</table>

Note: * indicates significance at the 5% level. The number of wild bootstrap replications is $B = 9999$.  

10
Table 7: System 1. The restricted estimates of the cointegrating vectors $\beta$ and the adjustment parameters $\alpha$ for $r = 1$.

<table>
<thead>
<tr>
<th>Sample</th>
<th>$\hat{\beta}$</th>
<th>$\hat{\alpha}$</th>
<th>$\hat{\beta}$</th>
<th>$\hat{\alpha}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>SHA</td>
<td>1.000</td>
<td>0.011</td>
<td>1.000</td>
<td>-0.014</td>
</tr>
<tr>
<td>SHB</td>
<td>-0.777</td>
<td>0.024</td>
<td>0.109</td>
<td>-0.020</td>
</tr>
<tr>
<td>HKH</td>
<td>-0.231</td>
<td>0.005</td>
<td>-2.996</td>
<td>0.002</td>
</tr>
<tr>
<td>HKSI</td>
<td>0.149</td>
<td>0.007</td>
<td>2.281</td>
<td>-0.002</td>
</tr>
</tbody>
</table>

The cointegrating relation for the first subsample when SP 500 is included in the system are reported in Table 8. The estimated relations $\hat{\beta}'X_t$ are plotted in Figure 3. The relations look more stationary compared to the implied relations estimated for the four Chinese stock indices. Again the first differences reveal volatility clustering in the cointegrating relations.

Table 8: System 2. The normalized estimates of the cointegrating vectors $\beta$ and the adjustment parameters $\alpha$ for $r = 2$.

<table>
<thead>
<tr>
<th>Sample</th>
<th>$\hat{\beta}$</th>
<th>$\hat{\alpha}$</th>
<th>$\hat{\beta}$</th>
<th>$\hat{\alpha}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>SHA</td>
<td>1.000</td>
<td>-1.411</td>
<td>0.007</td>
<td>-0.001</td>
</tr>
<tr>
<td>SHB</td>
<td>-0.887</td>
<td>1.000</td>
<td>0.013</td>
<td>-0.008</td>
</tr>
<tr>
<td>HKH</td>
<td>0.400</td>
<td>1.009</td>
<td>-0.005</td>
<td>-0.011</td>
</tr>
<tr>
<td>HKSI</td>
<td>0.886</td>
<td>0.539</td>
<td>-0.001</td>
<td>-0.006</td>
</tr>
<tr>
<td>SP500</td>
<td>-2.490</td>
<td>-2.369</td>
<td>0.004</td>
<td>0.0002</td>
</tr>
</tbody>
</table>

4 Conclusions

In this article I have studied cointegrating relations of the Chinese A and B share indices in Shanghai, the Hang Seng index of Hong Kong, the Hang Seng China enterprises index and Standard & Poor’s 500 index. The analysis is carried out by likelihood ratio test for cointegration and bootstrapping the test statistic using the wild bootstrap.

Asymptotic tests find evidence for cointegration between the four Chinese markets in the two subsamples before and after the 2007–2008 financial crisis. However, using the wild bootstrap no cointegration is found on the Chinese stock markets. In the sample before the financial crisis two cointegrating vectors are found between the Chinese and the US market.

The motivation for using wild bootstrap tests is that diagnostic tests reveal that heteroskedasticity is a problem in the data. Asymptotic tests for model selection favour models with many lags, while wild bootstrap tests in general select models with less lags. The study on Chinese stock markets indicates that there is less evidence of heteroskedasticity when using models with more lags. The wild bootstrap also finds less evidence compared to asymptotic tests. Further study of the cointegrating vectors reveal volatility clustering in the cointegrating vectors. Therefore results from the wild bootstrap tests should be more reliable.
Figure 2: System 1. Estimated cointegration relations and their first differences for the two subsamples.
Figure 3: System 2. Estimated cointegration relations and their first differences for the subsample before the financial crisis 2007-2008.
References


A Lagrange Multiplier Test for Testing the Adequacy of the Constant Conditional Correlation GARCH Model

Paul Catani* Timo Teräsvirta†

October 9, 2013

Abstract

A Lagrange multiplier test for testing the parametric structure of a constant conditional correlation generalized autoregressive conditional heteroskedasticity (CCC-GARCH) model is proposed. The test is based on decomposing the CCC-GARCH model multiplicatively into two components, one of which represents the null model, whereas the other one describes the misspecification. A simulation study shows that the test has good finite sample properties. We compare the test with other tests for misspecification of multivariate GARCH models. The test has high power against alternatives where the misspecification is in the GARCH parameters and is superior to other tests. The test is not greatly affected by misspecification in the conditional correlations and is therefore well suited for considering misspecification of GARCH equations.

1 Introduction

Multiple GARCH models have become an important tool in forecasting volatility of portfolios. There are several classes of multivariate GARCH models, beginning with the general Vector GARCH model of Bollerslev, Engle & Wooldridge (1988). This model is even ‘too general’ in the sense that conditional covariance matrices generated by this model are positive definite with probability less than one. Following this first attempt at joint modelling of conditional variances and covariances using the GARCH approach, the main goal of econometricians has been to develop models whose parametric structure would guarantee positive definiteness of the conditional covariance matrix. Two classes of such models have become quite popular. The first one is the so-called BEKK-GARCH model discussed by Engle & Kroner (1995), and the second one is the family of conditional correlation models. The basic model nested in the other members of this family is the Constant Conditional Correlation GARCH (CCC-GARCH) model by Bollerslev (1990). For information about these and other multivariate GARCH models, see Bauwens, Laurent & Rombouts (2006) and Silvennoinen & Teräsvirta (2009).

In this paper the focus is on conditional correlation GARCH models. While they are frequently fitted to financial time series, testing the parametric structure

*Hanken School of Economics, email: paul.catani@hanken.fi
†CREATEES, Aarhus University, email: tterasvirta@creates.au.dk
of the GARCH equations in them has not been very common. Our aim is to derive a portmanteau test for testing misspecification of the GARCH structure of these models. The predecessor of our test is the portmanteau test of Ling & Li (1997) who generalised the univariate test of Li & Mak (1994) to a multivariate situation. Their test is not restricted to conditional correlation GARCH models, but by a suitable choice of the conditional covariance matrix it becomes a misspecification test of the GARCH equations in the CCC-GARCH model.

Nakatani & Teräsvirta (2009) derived a test of the CCC-GARCH model against the Extended CCC-GARCH model of Jeantheau (1998). In their Lagrange multiplier (LM-) test the alternative to the GARCH equations is the model with GARCH equations that contains lags of squared errors and conditional variances from other GARCH equations. Our aim is to derive a general portmanteau test in the spirit of Ling & Li (1997) such that the alternative to the GARCH equations is more general than in the test of Nakatani & Teräsvirta (2009). It is based on decomposing the conditional variance equations in the CCC-GARCH model multiplicatively into two components, one of which represents the null model, whereas the other one describes the misspecification. The inspiration comes from the univariate 'no ARCH in GARCH' test in Lundbergh & Teräsvirta (2002). This leads to a portmanteau test that is more general than that of Ling & Li (1997).

A practical question in applying tests of the GARCH structure of the CCC-GARCH model is whether these tests also have power against misspecification of the correlation structure. This will be investigated by simulation. There are also tests of the correlation structure of the CCC-GARCH model. Tse (2000) derived a portmanteau-type test against the alternative that the conditional correlations are not constant over time. Silvennoinen & Teräsvirta (2009a) constructed an LM test against the Smooth Transition Conditional Correlation GARCH (STCC-GARCH) model. The question then is whether tests of constant conditional correlations in turn have power against misspecification in the GARCH equations. In this paper this problem is investigated by simulating the test of Tse (2000). His test can be viewed as a portmanteau-type test without a specific alternative to constant correlations.

It would be useful to test the adequacy of GARCH equations when the estimated model is a time-varying conditional correlation model such as the DCC-GARCH model of Engle (2002), the STCC-GARCH model, or the Markov-switching CC-GARCH model of Pelletier (2006). The difficulty is, however, that asymptotic normality of the maximum likelihood estimators of the parameters of these models has not been proven. The corresponding proof exists for the CCC-GARCH model, see Ling & McAleer (2003), which is why that model constitutes the null hypothesis for the test derived in this paper.

The plan of the paper is as follows. In section 2 the CCC-GARCH process is defined and we present the decomposition of the conditional variance equations which our test is based upon. In section 3 we give the first and second order partial derivatives of the quasi-log-likelihood function of the decomposed CCC-GARCH model. The LM test is derived in section 4 and section 5 contains a bivariate illustration of the test. The finite sample properties of the test are studied by Monte Carlo simulations in section 6. Section 7 concludes. Mathematical proofs can be found in the Appendix.
2 Model

Consider the following multivariate GARCH model with a zero unconditional mean:

\[ y_t = \varepsilon_t \quad t = 1, 2, \ldots, T \]  

where \( y_t = (y_{1t}, \ldots, y_{mt})' \) is an \((m \times 1)\) vector. The \(m\)-dimensional error term \( \varepsilon_t \) is decomposed as follows:

\[ \varepsilon_t = D_t \zeta_t \]  

where

\[ D_t = \text{diag}(h_{1t}^{1/2}, \ldots, h_{mt}^{1/2}) \]

is a diagonal matrix of conditional standard deviations of the elements of \( \varepsilon_t \). Assume that \( h_{it} \) follows a GARCH(1,1) process

\[ h_{it} = \alpha_{i0} + \alpha_{i1} \varepsilon_{i,t-1}^2 + \beta_{i1} h_{i,t-1}, \]

where \( \alpha_{i1} \) and \( \beta_{i1} \) are nonnegative, \( i = 1, \ldots, m \). Furthermore, \( \zeta_t \sim \text{IID}(0, \mathbf{P}) \), where

\[ \mathbf{P} = [ \rho_{ij} ] \]

is a positive definite correlation matrix, i.e. \( \rho_{ii} = 1 \), \( i = 1, \ldots, m \).

Equation (4) may be generalised to contain asymmetric or higher-order terms. From (2) we have

\[ \zeta_t = (z_{1t}, \ldots, z_{mt})' = D_t^{-1} \varepsilon_t = (\varepsilon_{1t} h_{1t}^{-1/2}, \ldots, \varepsilon_{mt} h_{mt}^{-1/2})' \]

and (2) with (5) defines a Constant Conditional Correlation (CCC-) GARCH model. The model can be written as

\[ \mathbf{h}_t = \mathbf{a}_0 + \mathbf{A}_1 \varepsilon_{t-1}^{(2)} + \mathbf{B}_1 \mathbf{h}_{t-1}, \]

where \( \varepsilon_{t}^{(2)} = (\varepsilon_{1t}^2, \ldots, \varepsilon_{mt}^2)' \), \( \mathbf{h}_t = (h_{1t}, \ldots, h_{mt})' \) and \( \mathbf{a}_0 = (\alpha_{10}, \ldots, \alpha_{m0})' \) are \((m \times 1)\) vectors and \( \mathbf{A}_1 \) and \( \mathbf{B}_1 \) are diagonal \((m \times m)\) parameter matrices with positive diagonal elements \( \alpha_{i1} \) and \( \beta_{i1} \), \( i = 1, \ldots, m \), respectively.

In order to construct a misspecification test for the CCC-GARCH model (2), we assume that \( \zeta_t = \mathbf{G}_t \zeta_t \), where

\[ \mathbf{G}_t = \text{diag}(g_{1t}^{1/2}, \ldots, g_{mt}^{1/2}) \]

with

\[ g_{it} = 1 + \sum_{j=1}^{r} \zeta_{ij}^2 z_{i,t-j}. \]

It follows that

\[ \mathbf{u}_t = (u_{1t}, \ldots, u_{mt})' = (\varepsilon_{1t} h_{1t}^{-1/2} g_{1t}^{-1/2}, \ldots, \varepsilon_{mt} h_{mt}^{-1/2} g_{mt}^{-1/2})' \sim \text{IID}(0, \mathbf{P}). \]

Then (2) can be written as follows:

\[ \varepsilon_t = \mathbf{D}_t \mathbf{G}_t \mathbf{u}_t \]

and (9) can be regarded as an 'ARCH nested in GARCH' model. For the univariate case, see Lundbergh & Teräsvirta (2002) and for another definition of \( g_{it} \), in which \( g_{it} \) is a deterministic positive definite function, see Amado & Teräsvirta (2013).

Let \( \zeta = (\zeta_1', \ldots, \zeta_m')' \) be an \( mr \times 1 \) matrix where \( \zeta_i = (\zeta_{i1}, \ldots, \zeta_{im})', i = 1, \ldots, m, \) is an \( r \times 1 \) vector. Our misspecification test consists of testing

\[ H_0 : \zeta = 0 \quad \text{or} \quad \mathbf{G}_t \equiv \mathbf{I} \]

in the model (8). Thus under \( H_0 \), \( \{ \varepsilon_t \} \) follows a CCC-GARCH model, and the alternative implies that there is dynamic structure unaccounted for in this model, because none of the sequences \( \{ z_{i,t} \} \) is a sequence of independent random variables.
3 The log-likelihood function and its partial derivatives

3.1 The log-likelihood function

First, we introduce some notation. Let \( \mathbf{0}_m \) be an \( m \times 1 \) null vector, \( \mathbf{0}_{mn} \) an \( mn \times 1 \) null vector, \( \mathbf{1}_m \) an \( m \times 1 \) vector of ones, \( \mathbf{I}_m \) an \( m \times m \) identity matrix, and \( \text{diag}(\mathbf{a}) \) a diagonal matrix whose diagonal elements are the elements of vector \( \mathbf{a} \). In order to derive the Lagrange Multiplier statistic for testing the null hypothesis (10), we need the log-likelihood function of the model and its first two partial derivatives. Under the null hypothesis, we assume that \( \{ \varepsilon_t \} \) is a sequence of vector white noise with \( \mathbb{E}\varepsilon_t = 0 \) and the conditional covariance matrix \( \Sigma_t = \mathbf{D}_t \mathbf{P}_t \mathbf{D}_t \). Let \( \mathbf{w} = (\mathbf{w}_1', ..., \mathbf{w}_m')' \) be a \( 3m \)-dimensional vector where \( \mathbf{w}_i = (\alpha_{i0}, \alpha_{i1}, \beta_{i1})' \), \( i = 1, ..., m \), and \( \rho = \text{vec}(\mathbf{P}) = (\rho_{12}, ..., \rho_{1m}, \rho_{23}, ..., \rho_{2m}, ..., \rho_{m-1,m})' \) be an \( m(m-1)/2 \)-dimensional vector. Furthermore, let \( \zeta = (\zeta_1', ..., \zeta_m')' \) be an \( mr \)-dimensional vector such that \( \zeta_i = (\zeta_{i1}, ..., \zeta_{ir})' \), \( i = 1, ..., m \), is an \( r \times 1 \) vector, and finally, set \( \theta = (\mathbf{w}', \rho', \zeta')' \). Thus, the quasi-log-likelihood of the CCC-GARCH model for observation \( t \) takes the form of the Gaussian log-likelihood:

\[
\begin{align*}
    l_t(\theta) &= -(1/2) \sum_{i=1}^{m} \ln h_{it} - (1/2) \sum_{i=1}^{m} \ln g_{it} - (1/2) \ln |\mathbf{P}| - (1/2) \mathbf{u}_t' \mathbf{P}^{-1} \mathbf{u}_t \\
    &= - \ln |\mathbf{D}_t| - \ln |\mathbf{G}_t| - (1/2) \ln |\mathbf{P}| - (1/2) \mathbf{u}_t' \mathbf{P}^{-1} \mathbf{u}_t \\
    \text{(11)}
\end{align*}
\]

and maximising

\[
L_T(\theta) = \sum_{t=1}^{T} l_t(\theta)
\]

with respect to \( \theta \) yields the quasi maximum likelihood estimator (QMLE) \( \hat{\theta} \).

To ensure asymptotic normality of the QMLE, we make the following assumptions:

**Assumption 1 (Stationarity).** All the roots of \( \det(\mathbf{I}_m - \mathbf{A}_1 \mathbf{x} - \mathbf{B}_1 \mathbf{x}) \) lie outside the unit circle.

**Assumption 2.** The parameter space \( \Theta \) is a compact subspace of Euclidean space; the matrix \( \mathbf{P} \) is a finite and positive definite symmetric matrix, with the elements on the diagonal being 1 and the eigenvalue of the matrix \( \mathbf{P} \) with largest absolute value having a positive lower bound over \( \Theta \); each \( \alpha_{i1} \) and \( \beta_{i1} \) are nonnegative, \( i = 1, ..., m \) and each element of \( \{ \alpha_{i0}, i = 1, ..., m \} \) has positive lower and upper bounds over \( \Theta \). Furthermore, if \( \beta_{i1} > 0 \), then \( \alpha_{i1} > 0, i = 1, ..., m \).

**Assumption 3 (Identifiability).** The formulation at the true parameter value \( \theta_0 \) of the CCC-GARCH-model is minimal.

**Assumption 4.** \( \mathbb{E}|\varepsilon_{it}| < \infty, i = 1, ..., m \).

Under assumption 1 the CCC-GARCH(1,1) model has a weakly stationary solution, which is unique. Furthermore it is also strictly stationary and ergodic (see Jeantheau (1998) and Ling & McAleer (2003)).

Jeantheau (1998) shows that under assumption 3 the model is identifiable. Define \( \mathbf{B}(\mathbf{L}) = \mathbf{I}_m - \mathbf{B}_1 \mathbf{L} \) and \( \mathbf{A}(\mathbf{L}) = \mathbf{A}_1 \mathbf{L} \) where \( \mathbf{L} \) is the lag operator. Sufficient conditions for assumption 3 to hold are:

- \( \det(\mathbf{A}(\mathbf{L})) \neq 0 \) and \( \det(\mathbf{B}(\mathbf{L})) \neq 0 \).
- \( \mathbf{A}(\mathbf{L}) \) and \( \mathbf{B}(\mathbf{L}) \) are left coprime.
• A(L) or B(L) is column reduced.

A(L) and B(L) are left coprime if any of the greatest common left divisors, D, of A(L) and B(L) are unimodular. D is unimodular if det(D) is not equal to zero and independent of the lag operator L. Furthermore, the polynomial matrix A(L) or B(L) is column reduced if det(A_1) ≠ 0 or det(B_1) ≠ 0 respectively. See Jeantheau (1998) for details and proof.

Assumptions 2 and 4 are crucial for the proof of asymptotic normality of the QMLE, see Ling & McAleer (2003).

### 3.2 The score and the information matrix of the log-likelihood function

In this section, we define the first and second partial derivatives of (11). Let \( q_t(\theta) = \partial l_t(\theta)/\partial \theta \) be the score vector for observation t, and let

\[
\bar{q}(\theta) = (1/T) \sum_{t=1}^{T} q_t(\theta) = (1/T)q(\theta)
\]

be the average score. We use notation \( q(\hat{\theta}) \) for the score evaluated at \( \theta = \hat{\theta} \). The \( 3m + m(m-1)/2 + mr \)–dimensional score vector for the observation t of (11) has the following form

\[
q_t(\theta) = \begin{bmatrix}
\frac{\partial l_t(\theta)}{\partial \omega} \\
\frac{\partial l_t(\theta)}{\partial \rho}
\end{bmatrix},
\]

where

\[
\frac{\partial l_t(\theta)}{\partial \omega} = -\nabla D_t \text{vec} \left( D_t^{-1} - \frac{1}{2} D_t^{-1} G_t^{-1} E_t E_t' G_t^{-1} M_t^{-1} - \frac{1}{2} M_t^{-1} G_t^{-1} E_t E_t' G_t^{-1} D_t^{-1} \right)
\]

and

\[
\frac{\partial l_t(\theta)}{\partial \rho} = -\frac{1}{2} \nabla P \text{vec} \left( P^{-1} - P^{-1} D_t^{-1} G_t^{-1} E_t E_t' G_t^{-1} D_t^{-1} P^{-1} \right)
\]

where \( M_t = D_t P D_t, H_t = G_t M_t G_t, \nabla D_t = \frac{\partial \text{vec}(D_t)'}{\partial \omega} \) and \( \nabla P = \frac{\partial \text{vec}(P)'}{\partial \rho} \). These two vectors of partial derivatives are given in Nakatani & Teräsvirta (2009).

The following lemma gives the first-order partial derivative of the log-likelihood function with respect to \( \zeta \).

**Lemma 1** The third element of the score vector has the following form

\[
\frac{\partial l_t(\theta)}{\partial \zeta} = -\nabla G_t \text{vec} \left( G_t^{-1} - \frac{1}{2} H_t^{-1} E_t E_t' G_t^{-1} - \frac{1}{2} G_t^{-1} E_t E_t' H_t^{-1} \right)
\]

where \( \nabla G_t = \frac{\partial \text{vec}(G_t)'}{\partial \zeta} \). Under \( H_0 \)

\[
\frac{\partial l_t(\theta)}{\partial \zeta} = -\nabla G_t \text{vec} \left( I - \frac{1}{2} M_t^{-1} E_t E_t' - \frac{1}{2} E_t E_t' M_t^{-1} \right).
\]

**Proof.** See Appendix A.1.
Under the null hypothesis, the average score vector thus has the form

$$
\bar{q}(\theta) |_{H_0} = \frac{1}{T} \sum_{t=1}^{T} \left[ \frac{\partial l_t(\theta)}{\partial \omega'}, \frac{\partial l_t(\theta)}{\partial \rho'}, \frac{\partial l_t(\theta)}{\partial \zeta'} \right]'
$$

$$
= \frac{1}{T} \sum_{t=1}^{T} \left[ \nabla D_{t, \text{vec}}(D_t^{-1} - \frac{1}{2} D_t^{-1} \varepsilon_t' \varepsilon_t'M_t^{-1} - \frac{1}{2} M_t^{-1} \varepsilon_t' \varepsilon_t'D_t^{-1}) \\
- \nabla P_{t, \text{vec}}(P^{-1} - P^{-1} D_t^{-1} \varepsilon_t' \varepsilon_t'D_t^{-1} P^{-1}) \\
- \nabla G_{t, \text{vec}}(I - \frac{1}{2} M_t^{-1} \varepsilon_t' \varepsilon_t' - \frac{1}{2} \varepsilon_t' \varepsilon_t'M_t^{-1}) \right]. \tag{15}
$$

The population information matrix is

$$
\mathcal{I}(\theta_0) = (1/T) E(q(\theta_0)'q(\theta_0)) = E(q(\theta_0)'q(\theta_0))' \tag{16}
$$

where $\theta_0$ is the true parameter. The negative of the expected Hessian evaluated at $\theta_0$ equals

$$
\mathcal{J}(\theta_0) = -(1/T) E \sum_{t=1}^{T} \frac{\partial^2 l_t(\theta)}{\partial \theta \partial \theta'} |_{\theta = \theta_0}. \tag{17}
$$

The Hessian for observation $t$ has the form

$$
\mathcal{H}_t(\theta) = \frac{\partial^2 l_t(\theta)}{\partial \theta \partial \theta'} = \begin{bmatrix}
\frac{\partial^2 l_t(\theta)}{\partial \omega' \partial \omega} & \frac{\partial^2 l_t(\theta)}{\partial \omega' \partial \rho} & \frac{\partial^2 l_t(\theta)}{\partial \omega' \partial \zeta} \\
\frac{\partial^2 l_t(\theta)}{\partial \rho' \partial \omega} & \frac{\partial^2 l_t(\theta)}{\partial \rho' \partial \rho} & \frac{\partial^2 l_t(\theta)}{\partial \rho' \partial \zeta} \\
\frac{\partial^2 l_t(\theta)}{\partial \zeta' \partial \omega} & \frac{\partial^2 l_t(\theta)}{\partial \zeta' \partial \rho} & \frac{\partial^2 l_t(\theta)}{\partial \zeta' \partial \zeta}
\end{bmatrix}, \tag{18}
$$

where the $2 \times 2$ upper left block is given in Nakatani & Teräsvirta (2009). The information matrix for observation $t$ under the null hypothesis is given by

$$
\mathcal{J}(\theta_0) = -E[\mathcal{H}_t(\theta)] |_{\theta = \theta_0} = \begin{bmatrix}
J_{11} & J_{12} & J_{13} \\
J_{12} & J_{22} & J_{23} \\
J_{13} & J_{23} & J_{33}
\end{bmatrix}, \tag{19}
$$

where $J_{11}, J_{12}$ and $J_{22}$ are given in Nakatani & Teräsvirta (2009). The following lemma gives the second partial derivatives of the log-likelihood function with respect to $\zeta$.

**Lemma 2** The second partial derivatives of the log-likelihood function with respect to $\zeta$ are given by

$$
\frac{\partial^2 l(\theta)}{\partial \omega' \partial \zeta} = -\frac{1}{2} \frac{\partial \text{vec}(G_t)'}{\partial \zeta} (D_t u_t u_t' P^{-1} D_t^{-1} \otimes G_t^{-1} D_t^{-1} + D_t u_t u_t' \otimes G_t^{-1} M_t^{-1}) \tag{20}
$$

$$
+ G_t^{-1} M_t^{-1} \otimes D_t u_t u_t' + G_t^{-1} D_t^{-1} \otimes D_t u_t u_t' P^{-1} D_t^{-1}) \frac{\partial \text{vec}(D_t)}{\partial \omega'},
$$

$$
\frac{\partial^2 l(\theta)}{\partial \rho' \partial \zeta} = \frac{1}{2} \frac{\partial \text{vec}(G_t)'}{\partial \zeta} \left\{ D_t u_t u_t' P^{-1} \otimes G_t^{-1} D_t^{-1} P^{-1} + G_t^{-1} D_t^{-1} P^{-1} \otimes D_t u_t u_t' P^{-1} \right\} \frac{\partial \text{vec}(P)}{\partial \rho'} \tag{21}
$$

$$
+ G_t^{-1} D_t^{-1} P^{-1} \otimes D_t u_t u_t' P^{-1} \right\} \frac{\partial \text{vec}(P)}{\partial \rho'}
$$

$$
+ G_t^{-1} D_t^{-1} P^{-1} \otimes D_t u_t u_t' P^{-1} \right\} \frac{\partial \text{vec}(P)}{\partial \rho'}
$$

$$
+ G_t^{-1} D_t^{-1} P^{-1} \otimes D_t u_t u_t' P^{-1} \right\} \frac{\partial \text{vec}(P)}{\partial \rho'}
$$
\[\frac{\partial^2 l(\theta)}{\partial \zeta' \partial \zeta} = - \left\{ \begin{bmatrix} \text{vec}(G_t^{-1})' \otimes I \end{bmatrix} - \frac{1}{2} \begin{bmatrix} \text{vec}(G_t^{-1} M_t^{-1} D_t u_t' D_t) & \otimes I \end{bmatrix} \right\} \frac{\partial^2 \text{vec}(G_t)'}{\partial \zeta' \partial \zeta} + \frac{1}{2} \frac{\partial \text{vec}(G_t)'}{\partial \zeta} \begin{bmatrix} 2(G_t^{-1} \otimes G_t^{-1}) - D_t u_t' D_t \otimes H_t^{-1} \\ -G_t^{-1} \otimes G_t^{-1} M_t^{-1} D_t u_t' D_t - D_t u_t' D_t M_t^{-1} G_t^{-1} & \otimes G_t^{-1} \\ -G_t^{-1} \otimes D_t u_t' D_t M_t^{-1} G_t^{-1} - G_t^{-1} M_t^{-1} D_t u_t' D_t \otimes G_t^{-1} \\ -H_t^{-1} \otimes D_t u_t' D_t \end{bmatrix} \frac{\partial \text{vec}(G_t)}{\partial \zeta'}.
\]

Taking conditional expectations with the relation \(E[u_t u_t'] = P\) and setting \(G_t = I\) under \(H_0\) in (20)-(22) yields

\[J_{13t} = \frac{1}{2} \nabla D_t \left\{ I \otimes D_t^{-1} + PD_t \otimes M_t^{-1} + M_t^{-1} \otimes PD_t + D_t^{-1} \otimes I \right\} \nabla G_t', \quad (23)
\]

\[J_{23t} = \frac{1}{2} \nabla P \left\{ D_t \otimes P^{-1} D_t^{-1} + P^{-1} D_t^{-1} \otimes D_t \right\} \nabla G_t', \quad (24)
\]

\[J_{33t} = \frac{1}{2} \nabla G_t \left\{ 2(I \otimes I) + M_t \otimes M_t^{-1} + M_t^{-1} \otimes M_t \right\} \nabla G_t'. \quad (25)
\]

Proof. See Appendix A.2.

4 The LM test statistic

When assumptions 1-4 hold the asymptotic null distribution of the maximum likelihood estimator \(\hat{\theta}\) is given by

\[\sqrt{T}(\hat{\theta} - \theta_0) \xrightarrow{D} N(0, \mathcal{I}^{-1}(\theta_0) \mathcal{I}(\theta_0) \mathcal{I}^{-1}(\theta_0)) \quad (26)
\]

Under the assumption \(z_t \sim \text{NID}(0, P)\), the information matrix \(\mathcal{I}(\theta_0) = -\mathcal{J}(\theta_0)\) and the asymptotic covariance matrix reduces to \(\mathcal{I}^{-1}(\theta_0)\). Ling & McAleer (2003) show that \(\mathcal{I}(\theta_0)\) and \(\mathcal{J}(\theta_0)\) can be consistently estimated by

\[\mathcal{I}(\theta) = \frac{1}{T} \sum_{t=1}^{T} q_t(\hat{\theta}) q_t(\hat{\theta})' \quad (27)
\]

or by

\[\mathcal{J}(\hat{\theta}) = - \frac{1}{T} \sum_{t=1}^{T} \frac{\partial^2 l(\hat{\theta})}{\partial \hat{\theta} \partial \hat{\theta}'}, \quad (28)
\]

respectively. See also Nakatani & Teräsvirta (2009).

Let \(\hat{\theta} = (\hat{\omega}', \hat{\rho}', \hat{\zeta}')'\) be the QML estimator of \(\theta_0\) under the null hypothesis. The average score evaluated at \(\hat{\theta}\) equals

\[\hat{q}(\hat{\theta}) = \frac{1}{T} \sum_{t=1}^{T} \begin{bmatrix} \frac{\partial l(\hat{\theta})}{\partial \omega} \\ \frac{\partial l(\hat{\theta})}{\partial \rho} \\ \frac{\partial l(\hat{\theta})}{\partial \zeta} \end{bmatrix} = \begin{bmatrix} \hat{q}_\omega(\hat{\theta}) \\ \hat{q}_\rho(\hat{\theta}) \\ \hat{q}_\zeta(\hat{\theta}) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \hat{\alpha}_\zeta(\hat{\theta}) \end{bmatrix} \quad (29)
\]
Then
\[
\tilde{q}_\zeta(\tilde{\theta}) = -\frac{1}{T} \sum_{t=1}^{T} (\nabla G_t, \text{vec}(I - \frac{1}{2} M_t^{-1} \varepsilon_t \varepsilon_t' - \frac{1}{2} \varepsilon_t \varepsilon_t' M_t^{-1}))
\] (30)

is the relevant (nonzero) block in the LM test statistic. The corresponding block of the information matrix in (19) evaluated at \( \theta \) under the null equals
\[
\mathcal{J}_{(\zeta, \zeta)}(\tilde{\theta}) = \tilde{J}_{33} - \left[ \begin{array}{c} \tilde{J}_{13} \\ \tilde{J}_{23} \end{array} \right] \left[ \begin{array}{c} \tilde{J}_{11} \\ \tilde{J}_{12} \\ \tilde{J}_{22} \end{array} \right]^{-1} \left[ \begin{array}{c} \tilde{J}_{13} \\ \tilde{J}_{23} \end{array} \right]
\] (31)

The main result is in the following theorem:

**Theorem 1** (the LM test statistic) Under \( H_0: \zeta = 0 \) or \( G_t = I \), the LM statistic
\[
LM_\zeta = T \tilde{q}_\zeta(\tilde{\theta})' \mathcal{J}^{-1}_{(\zeta, \zeta)}(\tilde{\theta}) \tilde{q}_\zeta(\tilde{\theta})
\] (32)

has an asymptotic \( \chi^2 \) distribution with \( mr \) degrees of freedom.

Obviously, if \( D_t = \text{diag}(\alpha_{01}, \ldots, \alpha_{qm}) \), \( LM_\zeta \) becomes a test of no conditional heteroskedasticity against CCC-ARCH. This test statistic is a special case of the constant error covariance matrix test derived by Eklund & Teräsvirta (2007).

**5 Bivariate illustration**

In this section we discuss the bivariate case. Consider the case where the number of dimensions \( m = 2 \), then \( \omega = (\omega_1', \omega_2')' \), where \( \omega = (\alpha_{i0}, \alpha_{i1}, \beta_{i1})' \) for \( i = 1, 2 \) and \( \zeta = (\zeta_1', \zeta_2')' \) and
\[
h_{it} = \alpha_{i0} + \alpha_{i1} \varepsilon_{i,t-1}^2 + \beta_{i1} h_{i,t-1}, i = 1, 2.
\] (33)

The block of the score vector corresponding to the parameter \( \zeta \) in Lemma 1 becomes
\[
\tilde{q}_\zeta(\tilde{\theta}) = -\frac{1}{T} \sum_{t=1}^{T} \left[ \begin{array}{c} \bar{v}^{(0)}_{1t} \\ \bar{v}^{(0)}_{2t} \end{array} \right] \left( 1 - \frac{1}{1 - \rho^2} \right) \varepsilon_{1t}^2 h_{2t} - \frac{1}{1 - \rho^2} \varepsilon_{1t} \varepsilon_{2t} \sqrt{h_{1t} h_{2t}}
\] (34)

where \( \bar{v}^{(0)}_{ijt} = \partial \bar{v}_{ijt} / \partial \zeta_j = \frac{1}{2} \bar{z}^{(2)}_{jt} \) estimated under \( H_0 \) and \( \bar{z}^{(2)}_{jt} = (\bar{z}_{j,t-1}^2, \ldots, \bar{z}_{j,t-r}^2)' \) for \( i, j = 1, 2 \), and \( \rho \) is the conditional correlation between \( \varepsilon_{1t} \) and \( \varepsilon_{2t} \).

The block of the maximum likelihood estimated information matrix corresponding to \( \zeta \) in Theorem 1 equals
\[
\mathcal{J}_{(\zeta, \zeta)}(\tilde{\theta}) = \tilde{J}_{33} - \left[ \begin{array}{c} \tilde{J}_{13} \\ \tilde{J}_{23} \end{array} \right]_{2r \times 7} \left[ \begin{array}{c} \tilde{J}_{11} \\ \tilde{J}_{12} \\ \tilde{J}_{22} \end{array} \right]_{7 \times 7}^{-1} \left[ \begin{array}{c} \tilde{J}_{13} \\ \tilde{J}_{23} \end{array} \right]_{7 \times 2r}
\] (35)

where
\[
\left[ \begin{array}{c} \tilde{J}_{11} \\ \tilde{J}_{12} \\ \tilde{J}_{12} \\ \tilde{J}_{22} \end{array} \right] = \frac{1}{4T} \sum_{t=1}^{T} \left[ \begin{array}{c} (1 + \frac{1}{1 - \rho^2}) \tilde{k}_{11t} \tilde{k}_{11t}' \\ -\frac{\rho^2}{1 - \rho^2} \tilde{k}_{22t} \tilde{k}_{11t}' \\ -2 \frac{\rho^2}{1 - \rho^2} \tilde{k}_{11t}' \\ -\frac{\rho^2}{1 - \rho^2} \tilde{k}_{22t} \end{array} \right]
\] (36)
\[
\begin{align*}
\begin{bmatrix}
\tilde{J}_{13} \\
\tilde{J}_{23}
\end{bmatrix}
&= \frac{1}{2T} \sum_{t=1}^{T} \begin{bmatrix}
\left(1 + \frac{1}{1-\rho^2} \right) \tilde{k}_{11t}\tilde{\nu}_{11t}^{0y} \\
-\frac{\rho^2}{1-\rho^2} \tilde{k}_{22t}\tilde{\nu}_{11t}^{0y} \\
\end{bmatrix} \\
&\quad + \frac{1}{2T} \sum_{t=1}^{T} \begin{bmatrix}
-\frac{\rho^2}{1-\rho^2} \tilde{k}_{22t}\tilde{\nu}_{22t}^{0y} \\
\left(1 + \frac{1}{1-\rho^2} \right) \tilde{\nu}_{22t}^{0y} \\
\end{bmatrix} \\
\end{align*}
\]  

(37)

and

\[
\tilde{J}_{33} = \frac{1}{T} \sum_{t=1}^{T} \begin{bmatrix}
\left(1 + \frac{1}{1-\rho^2} \right) \tilde{\nu}_{11t}^{0y} \\
-\frac{\rho^2}{1-\rho^2} \tilde{\nu}_{22t}^{0y} \\
\end{bmatrix} \\
\begin{bmatrix}
\left(1 + \frac{1}{1-\rho^2} \right) \tilde{\nu}_{22t}^{0y} \\
-\frac{\rho^2}{1-\rho^2} \tilde{\nu}_{22t}^{0y} \\
\end{bmatrix}
\]  

(38)

In (36) and (37) \( \tilde{k}_{ijt} = \tilde{h}_{it}^{-1}\partial \tilde{h}_{it}/\partial \omega_j \) estimated under \( H_0 \), where \( \partial \tilde{h}_{it}/\partial \omega_j = \tilde{x}_{jt} + \tilde{\beta}_i \partial \tilde{h}_{it-1}/\partial \omega_j \), and \( \tilde{x}_{jt} = (1, \varepsilon_{jt-1}, \tilde{h}_{jt-1})' \) for \( i, j = 1, 2 \). Furthermore \( \tilde{h}_{it} \) is \( h_{it} \) estimated under \( H_0 \). Following the suggestion by Fiorentini, Calzolari & Panattoni (1996), we can set the initial values of the recursions as

\[
\tilde{x}_{j0} = (1, \frac{1}{T} \sum_{t=1}^{T} \varepsilon_{jt}^2, \frac{1}{T} \sum_{t=1}^{T} \varepsilon_{jt}^2)' \\
\]  

(39)

\[
\partial \tilde{h}_{i0}/\partial \omega_j = 0.
\]

The LM test statistic (32) in the bivariate case has an asymptotic \( \chi^2 \) distribution with \( 2r \) degrees of freedom.

## 6 Monte Carlo simulations

We study the size and power properties of the test statistic \( LM_\zeta \), using Monte Carlo simulations. The power of \( LM_\zeta \) is studied in situations in which the GARCH equations are misspecified and in situations in which the alternative is a model with time-varying correlations. Our test is constructed for situations in which the GARCH equations may be misspecified. Nevertheless, it is interesting to know whether it can also reveal misspecification in the conditional correlation structure.

We compare the power of the test to the power of the portmanteau test of Ling & Li (1997) and the LM-test of constant conditional correlations of Tse (2000).

Ling & Li (1997) introduced a general portmanteau test for testing the adequacy of a multivariate GARCH(p, q) model. Let \( \varepsilon_t = \mathbf{H}_t^{1/2} \eta_t \), where \( \mathbf{H}_t \) is the conditional covariance matrix of \( \varepsilon_t \) and \( \eta \sim \text{IID}(0, \mathbf{I}) \) under the null hypothesis. The test statistic is given by

\[
Q(r) = T\tilde{\mathbf{R}}\tilde{\Omega}^{-1}\tilde{\mathbf{R}},
\]  

(40)

where \( \tilde{\mathbf{R}} = (\tilde{R}_1, \ldots, \tilde{R}_r)' \) with

\[
\tilde{R}_j = \sum_{t=j+1}^{T}(\varepsilon_t'\mathbf{H}_t^{-1}\varepsilon_t - m)(\varepsilon_t'\mathbf{H}_t^{-1}\varepsilon_t - m) \bigg/ \sum_{t=1}^{T}(\varepsilon_t'\mathbf{H}_t^{-1}\varepsilon_t - m)^2 
\]  

(41)

and \( \tilde{\Omega} \) is the covariance matrix estimator of \( \tilde{\mathbf{R}} \). Under the null hypothesis \( \mathbb{E}\varepsilon_t'\mathbf{H}_t^{-1}\varepsilon_t = m \) and (41) is the \( j \)th order autocorrelation between \( \varepsilon_t'\mathbf{H}_t^{-1}\varepsilon_t \) and \( \varepsilon_{t-j}'\mathbf{H}_{t-j}^{-1}\varepsilon_{t-j} \). The test statistic has an asymptotic \( \chi^2(r) \)-distribution under \( H_0 \). The test can be viewed as a multivariate extension of the portmanteau test of Li & Mak (1994) for testing the adequacy of a univariate GARCH model. When \( m = 1 \) (40) collapses into the test of Li and Mak.

Tse & Tsui (1999) study the power of Ling & Li’s test in testing the adequacy of a multivariate model for conditional heteroskedasticity. They find that the test
has low power in most cases where the conditional correlation structure of the true model differs from the estimated one.

The LM test of constant conditional correlations by Tse (2000), denoted $LMC$ following the original article, is based on allowing for time-varying correlations given by

$$
\rho_{ijt} = \rho_{ij} + \delta_{ij} \varepsilon_{i,t-1} \varepsilon_{j,t-1}, \quad 1 \leq i < j \leq m,
$$

where $\delta_{ij}$ are additional parameters under the alternative hypothesis. The null hypothesis is $H_0 : \delta_{ij} = 0$, for $1 \leq i < j \leq m$, and the test statistic is given by

$$
LMC = l' \tilde{S} (\tilde{S}' \tilde{S})^{-1} \tilde{S}' l,
$$

where $l$ is the $T \times 1$ vector of ones, $\tilde{S}$ is the $T \times m$ matrix of partial derivatives $\partial l_t / \partial \theta'$ evaluated under $H_0$ and $\theta$ is the vector of parameters in the model under the alternative hypothesis. Under $H_0$ $LMC$ is asymptotically distributed as $\chi^2(M)$, where $M = m(m-1)/2$.

### 6.1 Size

The size of $LMC$ is simulated for five different CCC-GARCH(1, 1) models at sample sizes $T = 1000, 2500, 5000$ and $10000$ and dimensions $m = 2$ and $5$. The nominal size of the tests is 5% in all cases. The data are generated from the five bivariate CCC-GARCH(1, 1) models used in Nakatani & Teräsvirta (2009). DGP 1 has moderate persistence in volatility, while DGPs 2 and 3 represent models with high persistence and DGPs 4 and 5 models with low persistence in volatility. The correlation is low ($\rho = 0.3$) in DGPs 1, 3 and 5 and high ($\rho = 0.9$) in DGPs 2 and 4. All simulations have been performed in R (R Core Team (2013)) using the ccgarch package by Nakatani (2013).

We simulate both two- and five-dimensional models. When simulating the latter models the DGPs are extensions of the former models. For example, in the two-dimensional case DGP 1 $A = \text{diag}(0.1, 0.2)$ on the main diagonal, whereas in the five-dimensional model $A = \text{diag}(0.1, 0.2, 0.1, 0.2, 0.1)$. The five-dimensional conditional correlation matrices are of the form

$$
P = \begin{bmatrix}
1 & \rho & \rho^2 & \rho^3 & \rho^4 \\
\rho & 1 & \rho & \rho^2 & \rho^3 \\
\rho^2 & \rho & 1 & \rho & \rho^2 \\
\rho^3 & \rho^2 & \rho & 1 & \rho \\
\rho^4 & \rho^3 & \rho^2 & \rho & 1
\end{bmatrix}, \quad (42)
$$

which is selected simply because it depends on a single parameter. There is no statistical theory behind this choice.

Table 1 summarises the results when $m = 2$. The test has a reasonable size already when $T = 1000$ for all CCC-GARCH processes considered and for both lag lengths $r = 1$ and $4$. The only exception is DGP 5 with $T = 1000$ and $r = 4$. Table 2 summarises the results for $m = 5$. The test has good size properties even in this case.

### 6.2 Power

We begin by considering the power of the test when a CCC-GARCH(1, 1) model is fitted to the data while the data are generated by a CCC-ARCH(2) or a CCC-GARCH(2, 1) process. We continue by studying the situation in which a CCC-GARCH(1, 1) model is fitted to the data, but the true process is an MGARCH
Table 1: Simulated size of the LM test for testing the adequacy of the estimated CCC-GARCH model when $m = 2$ and $r = 1, 4$. The nominal significance level is 0.05.

<table>
<thead>
<tr>
<th>$T$</th>
<th>DGP 1</th>
<th>DGP 2</th>
<th>DGP 3</th>
<th>DGP 4</th>
<th>DGP 5</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$r = 1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1000</td>
<td>0.045</td>
<td>0.050</td>
<td>0.043</td>
<td>0.052</td>
<td>0.047</td>
</tr>
<tr>
<td>2500</td>
<td>0.050</td>
<td>0.049</td>
<td>0.048</td>
<td>0.051</td>
<td>0.049</td>
</tr>
<tr>
<td>5000</td>
<td>0.052</td>
<td>0.049</td>
<td>0.051</td>
<td>0.052</td>
<td>0.047</td>
</tr>
<tr>
<td>10000</td>
<td>0.050</td>
<td>0.047</td>
<td>0.051</td>
<td>0.051</td>
<td>0.051</td>
</tr>
<tr>
<td>$r = 4$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1000</td>
<td>0.049</td>
<td>0.051</td>
<td>0.052</td>
<td>0.051</td>
<td>0.082</td>
</tr>
<tr>
<td>2500</td>
<td>0.048</td>
<td>0.052</td>
<td>0.048</td>
<td>0.051</td>
<td>0.050</td>
</tr>
<tr>
<td>5000</td>
<td>0.048</td>
<td>0.050</td>
<td>0.049</td>
<td>0.053</td>
<td>0.047</td>
</tr>
<tr>
<td>10000</td>
<td>0.052</td>
<td>0.052</td>
<td>0.053</td>
<td>0.052</td>
<td>0.049</td>
</tr>
</tbody>
</table>

**Note:** The number of replications equals 10000. The nominal significance level is 5%.

Table 2: Simulated size of the LM test for testing the adequacy of the estimated CCC-GARCH model when $m = 5$ and $r = 1$. The nominal significance level is 0.05.

<table>
<thead>
<tr>
<th>$T$</th>
<th>DGP 1</th>
<th>DGP 2</th>
<th>DGP 3</th>
<th>DGP 4</th>
<th>DGP 5</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$r = 1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1000</td>
<td>0.047</td>
<td>0.055</td>
<td>0.046</td>
<td>0.050</td>
<td>0.047</td>
</tr>
<tr>
<td>2500</td>
<td>0.049</td>
<td>0.052</td>
<td>0.040</td>
<td>0.045</td>
<td>0.051</td>
</tr>
<tr>
<td>5000</td>
<td>0.051</td>
<td>0.050</td>
<td>0.047</td>
<td>0.053</td>
<td>0.045</td>
</tr>
<tr>
<td>10000</td>
<td>0.049</td>
<td>0.052</td>
<td>0.048</td>
<td>0.051</td>
<td>0.043</td>
</tr>
<tr>
<td>$r = 4$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1000</td>
<td>0.045</td>
<td>0.054</td>
<td>0.052</td>
<td>0.061</td>
<td>0.115</td>
</tr>
<tr>
<td>2500</td>
<td>0.052</td>
<td>0.058</td>
<td>0.050</td>
<td>0.050</td>
<td>0.052</td>
</tr>
<tr>
<td>5000</td>
<td>0.051</td>
<td>0.058</td>
<td>0.048</td>
<td>0.053</td>
<td>0.043</td>
</tr>
<tr>
<td>10000</td>
<td>0.053</td>
<td>0.055</td>
<td>0.055</td>
<td>0.049</td>
<td>0.047</td>
</tr>
</tbody>
</table>

**Note:** The number of replications equals 5000. The nominal significance level is 5%.

process with time-varying conditional correlations. We consider cases where the correlations follow the Dynamic Conditional Correlation (DCC) GARCH model of Engle (2002), the Smooth Transition Conditional Correlation (STCC) GARCH model of Silvennoinen & Teräsvirta (2009a) and the Baba-Engle-Kraft-Kroner (BEKK) GARCH model, defined in Engle & Kroner (1995).

All simulations are again performed in R. As the empirical size of the our statistic is very close to the nominal 5% size and the simulations require plenty of CPU time we have used the asymptotic null distribution in calculating the power. All estimates of the power of the test statistics are rejection rates under the alternative.

Three different parametrisations are considered for the CCC-GARCH, one for the DCC- and the STCC-GARCH process and two for the BEKK-GARCH processes. The parameters of these models appear in Table 3.
<table>
<thead>
<tr>
<th>DGP 1</th>
<th>DGP 2</th>
<th>DGP 3</th>
<th>DGP 4-5</th>
<th>DGP 6-8</th>
<th>DGP 9</th>
<th>DGP 10</th>
</tr>
</thead>
<tbody>
<tr>
<td>A_1</td>
<td>0.3, I_2</td>
<td>0.3, I_2</td>
<td>0.3, I_2</td>
<td>0.3, I_2</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.07</td>
<td>0.07</td>
<td>0.07</td>
<td>0.07</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>A_2</td>
<td>0.1, I_2</td>
<td>0.1, I_2</td>
<td>0.1, I_2</td>
<td>0.1, I_2</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.08</td>
<td>0.08</td>
<td>0.08</td>
<td>0.08</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>B_1</td>
<td>0.8, 0, I_2</td>
<td>0.8, 0, I_2</td>
<td>0.8, 0, I_2</td>
<td>0.8, 0, I_2</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
<td>0.25</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>C_1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.2, 0.1, I_2</td>
<td>0.2, 0.1, I_2</td>
<td>0.2, 0.1, I_2</td>
<td>0.2, 0.1, I_2</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.2, 0.2</td>
<td>0.2, 0.2</td>
<td>0.2, 0.2</td>
<td>0.2, 0.2</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.1, 0.1</td>
<td>0.1, 0.1</td>
<td>0.1, 0.1</td>
<td>0.1, 0.1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note: a_0 = (0.1, 0.2)' in DGPs 1-9.
Table 4 presents the results when \( m = 2 \). In DGPs 1-3 the constant conditional correlation matrix

\[
P = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix},
\]

with \( \rho = 0.3 \) (DGP 1a-3a) and \( \rho = 0.9 \) (DGP 1b-3b). The power of \( LM_\zeta \) is higher than the power of \( Q(r) \) or \( LMC \) in all six cases. In addition \( Q(r) \) outperforms \( LMC \) in most cases. \( Q(r) \) has good power for DGP 3 and in large samples also for DGPs 1 and 2. \( LMC \) has rather low if any power at all sample sizes when \( \rho = 0.3 \). For \( LM_\zeta \) and \( LMC \) there is an increase in power when the correlation changes from 0.3 to 0.9, while the power of \( Q(r) \) in that case slightly decreases. In particular, when the conditional correlation is large, also \( LMC \) which is not designed to detect misspecification in GARCH equations, can have considerable power when the time series are sufficiently long.

In the DCC-GARCH model (DGPs 4-5) the conditional correlation is generated by the following process:

\[
Q_t = (1 - a - b)P + az_{t-1}z'_{t-1} + bQ_{t-1},
\]

where \( a \) and \( b \) are the DCC-parameters and \( P \) is now the unconditional correlation matrix \( P = \{ \rho_{ij} \} \). Furthermore, to produce valid correlation matrices \( Q_t \) is rescaled as follows:

\[
P_t = (I \otimes Q_t)^{-1/2}Q_t(I \otimes Q_t)^{-1/2},
\]

where \( \otimes \) is the Hadamard product. The values for the DCC-parameters are

\[
\begin{align*}
\text{DGP 4} & : \quad a = 0.09, b = 0.9 & \text{and} \\
\text{DGP 5} & : \quad a = 0.05, b = 0.9.
\end{align*}
\]

In DGP 4 the persistence in the conditional correlation is very high, i.e. the conditional correlation can deviate substantially from its mean for long periods, whereas in DGP 5 the attraction towards the mean is stronger than in DGP 4. We consider two values for the unconditional correlation: \( \rho = 0.3 \) (DGP 4a and 5a) and \( \rho = 0.9 \) (DGP 4b and 5b). From Table 4 we can see that the power of \( LM_\zeta \) more or less equals its size for all four DGPs at all sample sizes. Interestingly, the power of \( Q(r) \) considerably increases when the correlation increases from 0.3 to 0.9. It can be quite high when the persistence of the correlation is high as in DGP 4. As may be expected, \( LMC \) is the best performer, displaying strong power against both DGPs at all sample sizes.

In the STCC-GARCH model (DGPs 6-8) the time-varying correlations are defined as follows:

\[
P_t = (1 - G_t)P_{(1)} + G_tP_{(2)},
\]

where \( P_t \) fluctuates between two positive definite correlation matrices \( P_{(1)} \) and \( P_{(2)} \) according to a transition function \( G_t \) which takes values between 0 and 1 depending on a continuous transition variable \( s_t \). In our simulations \( G_t \) is a logistic function:

\[
G_t(c, \gamma, s_t) = \left(1 + e^{-\gamma(s_t-c)}\right)^{-1}, \quad \gamma > 0
\]  

(43)

where \( \gamma \) is the speed and \( c \) the location of transition. In DGPs 6 and 7, \( s_t = \varepsilon_{1,t-1} \) in (43) whereas in DGP 8, the transition variable \( s_t \) follows a first-order autoregressive process whose innovation is \( \varepsilon_{1,t-1} \):

\[
s_t = 0.99s_{t-1} + \varepsilon_{1,t-1}.
\]
In this case, the transition variable is quite persistent. The difference between DGP 6 and DGP 7 is that in the former the transition is fairly smooth, $\gamma = 5$, whereas it is rapid in the latter as $\gamma = 100$. In both DGPs, $c = 3$, which means that $P_t$ on average stays closer to $P_{(1)}$ than $P_{(2)}$. In DGP 8, $\gamma = 5$ and $c = 0$, so the transition is smooth and due to persistent $\{s_t\}$ the correlations change slowly over time.

In all these DGPs the two correlation matrices are

$$P_{(1)} = \begin{bmatrix} 1 & 0.3 \\ 0.3 & 1 \end{bmatrix}, \quad P_{(2)} = \begin{bmatrix} 1 & 0.9 \\ 0.9 & 1 \end{bmatrix}. $$

Again, $LMC$ has the highest power of the three tests, but contrary to the DCC-GARCH alternative, $LM_\xi$ also has power against DGPs 6 and 7 where $s_t = \varepsilon_{1,t-1}$. It has very little power against DGP 8. It seems that if the correlation fluctuates sufficiently slowly, $LM_\xi$ does not respond such time-variation. The performance of $Q(1)$ lies between that of $LMC$ and $LM_\xi$. This test has power against all three DGPs but the power is clearly weaker than that of $LMC$, in small samples in particular. Tse and Tsui (1999) found $Q(1)$ have low power against a diagonal BEKK-GARCH model, but they used a different specification from ours.

Finally we consider two diagonal BEKK-GARCH alternatives, where the model of the conditional covariances is given by

$$H_t = CC' + A_1'\varepsilon_{t-1}\varepsilon'_{t-1}A_1 + B_1'H_{t-1}B_1. $$

The results in Table 4 show that, as in the case of DCC-GARCH, $LM_\xi$ only has trivial power against the BEKK-GARCH models considered. $Q(1)$ has some power against the simplest diagonal BEKK-GARCH alternative (DGP 10) but trivial power against DGP 9. As can be expected, $LMC$ has the highest power of the three tests.

The power of the tests is also simulated for $m = 5$. The results reported in Table 5 are similar to the ones obtained when $m = 2$. The $LM_\xi$ test has in general higher power when $m = 5$ and the difference in power between the tests in favor of $LM_\xi$ is even larger than in the bivariate case. The portmanteau test has slightly less power when $m = 5$ than when $m = 2$ when the alternative is a CCC-GARCH(2, 1) process. When the alternative is an STCC-GARCH model, the power of $LM_\xi$ marginally increases with the dimension of the model.

The test results seem to suggest following guidelines as to what to do in practice after estimating a CCC-GARCH model. First carry out the three tests. If both $LM_\xi$ and $Q(r)$ reject the null hypothesis of no ARCH in GARCH whereas $LMC$ does not or does so only weakly, conclude that at least some of the GARCH equations have to be respecified. If all tests strongly reject, no conclusions can be drawn at this stage. If $LMC$ rejects the null hypothesis of constant correlations whereas $LM_\xi$ does not, tentatively assume that the correlations are not constant and fit either a DCC-GARCH or BEKK-GARCH model to the data. If both tests reject but $LMC$ provides the strongest rejection, consider again giving up the assumption of constant conditional correlations but also consider the STCC-GARCH model as an alternative. If all three tests reject very strongly, reconsidering both the GARCH equations and the CCC-assumption could be useful. Note, however, that these guidelines are based on a rather limited number of simulation designs and are rather tentative. Finally, if one has reason to suspect spillover effects, these tests can be completed by the GARCH misspecification test in Nakatani and Teräsvirta (2009).
Table 4: Simulated power of three test statistics for testing the adequacy of the estimated CCC-GARCH model when \( m = 2 \) and \( r = 1 \).

<table>
<thead>
<tr>
<th>DGP</th>
<th>CCC-GARCH</th>
<th>DCC-GARCH</th>
<th>STCC-GARCH</th>
<th>BEKK-GARCH</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho )</td>
<td>1a</td>
<td>2a</td>
<td>3a</td>
<td>1b</td>
</tr>
<tr>
<td>0.3</td>
<td>0.048</td>
<td>0.043</td>
<td>0.043</td>
<td>0.040</td>
</tr>
<tr>
<td>0.9</td>
<td>0.056</td>
<td>0.051</td>
<td>0.274</td>
<td>0.112</td>
</tr>
</tbody>
</table>

\( T = 1000 \)

| \( LM_\zeta \) | 0.752 | 0.660 | 1.000 | 0.816 | 0.805 | 1.000 | 0.048 | 0.043 | 0.043 | 0.040 | 0.142 | 0.145 | 0.063 | 0.047 | 0.044 |
| \( Q(r) \) | 0.260 | 0.338 | 0.764 | 0.283 | 0.314 | 0.740 | 0.056 | 0.051 | 0.274 | 0.112 | 0.176 | 0.180 | 0.439 | 0.048 | 0.129 |
| \( LMC \) | 0.060 | 0.063 | 0.074 | 0.197 | 0.253 | 0.616 | 0.883 | 0.433 | 0.868 | 0.502 | 0.559 | 0.543 | 0.662 | 0.989 | 0.944 |

\( T = 2500 \)

| \( LM_\zeta \) | 0.988 | 0.973 | 1.000 | 0.995 | 0.995 | 1.000 | 0.049 | 0.049 | 0.049 | 0.046 | 0.265 | 0.280 | 0.063 | 0.051 | 0.055 |
| \( Q(r) \) | 0.535 | 0.709 | 0.983 | 0.587 | 0.687 | 0.980 | 0.062 | 0.053 | 0.559 | 0.173 | 0.350 | 0.365 | 0.713 | 0.059 | 0.265 |
| \( LMC \) | 0.059 | 0.072 | 0.103 | 0.285 | 0.386 | 0.899 | 0.998 | 0.762 | 0.993 | 0.794 | 0.869 | 0.862 | 0.937 | 1.000 | 1.000 |

\( T = 5000 \)

| \( LM_\zeta \) | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.048 | 0.049 | 0.052 | 0.047 | 0.511 | 0.516 | 0.068 | 0.049 | 0.052 |
| \( Q(r) \) | 0.825 | 0.947 | 1.000 | 0.869 | 0.931 | 1.000 | 0.085 | 0.058 | 0.833 | 0.279 | 0.617 | 0.603 | 0.917 | 0.052 | 0.445 |
| \( LMC \) | 0.070 | 0.079 | 0.151 | 0.415 | 0.547 | 0.989 | 1.000 | 0.957 | 1.000 | 0.963 | 0.985 | 0.988 | 0.995 | 1.000 | 1.000 |

\( T = 10000 \)

| \( LM_\zeta \) | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 0.053 | 0.054 | 0.064 | 0.051 | 0.799 | 0.804 | 0.067 | 0.047 | 0.047 |
| \( Q(r) \) | 0.984 | 0.999 | 1.000 | 0.992 | 0.998 | 1.000 | 0.111 | 0.056 | 0.980 | 0.507 | 0.881 | 0.884 | 0.995 | 0.046 | 0.735 |
| \( LMC \) | 0.087 | 0.104 | 0.244 | 0.584 | 0.715 | 0.999 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 | 1.000 |

**Note:** The number of replications equals 5000. The nominal significance level is 5%.
Table 5: Simulated power of three test statistics for testing the adequacy of the estimated CCC-GARCH model when \( m = 5 \) and \( r = 1 \).

<table>
<thead>
<tr>
<th>DGP</th>
<th>CCC-GARCH</th>
<th>DCC-GARCH</th>
<th>STCC-GARCH</th>
<th>BEKK-GARCH</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho )</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>0.3</td>
<td>0.3</td>
<td>0.9</td>
<td>0.3</td>
<td>0.9</td>
</tr>
</tbody>
</table>

\( T = 1000 \)

\( LM_\xi \) & 0.969 & 0.934 & 1.000 & 0.994 & 0.994 & 1.000 & 0.042 & 0.043 & 0.053 & 0.047 & 0.154 & 0.162 & 0.088 & 0.052 & 0.064 \\
\( Q(r) \) & 0.234 & 0.334 & 0.672 & 0.387 & 0.304 & 0.791 & 0.060 & 0.059 & 0.513 & 0.159 & 0.375 & 0.371 & 0.959 & 0.059 & 0.214 \\
\( LMC \) & 0.083 & 0.083 & 0.104 & 0.361 & 0.466 & 0.907 & 1.000 & 0.949 & 1.000 & 0.896 & 0.300 & 0.290 & 0.858 & 1.000 & 1.000 \\

\( T = 2500 \)

\( LM_\xi \) & 1.000 & 1.000 & 1.000 & 1.000 & 1.000 & 1.000 & 0.048 & 0.050 & 0.059 & 0.052 & 0.329 & 0.340 & 0.098 & 0.053 & 0.068 \\
\( Q(r) \) & 0.474 & 0.674 & 0.959 & 0.714 & 0.641 & 0.987 & 0.076 & 0.052 & 0.881 & 0.330 & 0.736 & 0.738 & 0.999 & 0.065 & 0.485 \\
\( LMC \) & 0.083 & 0.076 & 0.115 & 0.512 & 0.663 & 0.998 & 1.000 & 1.000 & 1.000 & 0.998 & 0.522 & 0.521 & 0.991 & 1.000 & 1.000 \\

\( T = 5000 \)

\( LM_\xi \) & 1.000 & 1.000 & 1.000 & 1.000 & 1.000 & 1.000 & 0.056 & 0.048 & 0.071 & 0.057 & 0.602 & 0.632 & 0.102 & 0.047 & 0.073 \\
\( Q(r) \) & 0.740 & 0.927 & 0.999 & 0.944 & 0.909 & 1.000 & 0.109 & 0.067 & 0.994 & 0.563 & 0.961 & 0.950 & 1.000 & 0.063 & 0.796 \\
\( LMC \) & 0.080 & 0.089 & 0.173 & 0.686 & 0.837 & 1.000 & 1.000 & 1.000 & 1.000 & 1.000 & 0.815 & 0.806 & 1.000 & 1.000 & 1.000 \\

\( T = 10000 \)

\( LM_\xi \) & 1.000 & 1.000 & 1.000 & 1.000 & 1.000 & 1.000 & 0.047 & 0.048 & 0.096 & 0.063 & 0.906 & 0.911 & 0.104 & 0.058 & 0.073 \\
\( Q(r) \) & 0.957 & 0.998 & 1.000 & 0.999 & 0.998 & 1.000 & 0.161 & 0.075 & 1.000 & 0.852 & 1.000 & 0.999 & 1.000 & 0.070 & 0.975 \\
\( LMC \) & 0.083 & 0.121 & 0.316 & 0.890 & 0.955 & 1.000 & 1.000 & 1.000 & 1.000 & 1.000 & 0.984 & 0.985 & 1.000 & 1.000 & 1.000 \\

Note: The number of replications equals 5000. The nominal significance level is 5%.
7 Conclusion

We have derived a LM test for testing the adequacy of a fitted CCC-GARCH model. Monte Carlo simulations show that the test has good size properties. The test has good power when the GARCH equations are misspecified and we find that the power of the test increases with the dimensions of the model. We also study the power of the test against misspecification in the conditional correlation structure.

Comparing with other tests, our test has higher power than the portmanteau test of Ling & Li (1997) when the GARCH equations are misspecified. On the other hand, the test is not greatly affected by misspecification in the conditional correlations, the STCC-GARCH alternative being an exception. Therefore it is well suited for considering misspecification of GARCH equations. Furthermore, we find that the test for time varying correlations of Tse (2000), while having very low power when the misspecification is in the conditional covariances, performs remarkably well when the conditional correlation structure is misspecified. The Portmanteau test of Ling & Li (1997) has some power against misspecification in both the GARCH equations and in the conditional correlations structure, but is in both cases outperformed by either our test or the LMC test of Tse (2000). It therefore seems a good idea to perform both tests and, if either one or both reject the CCC-GARCH model, then decide how to proceed from there.

Appendix A

The matrix derivations in this part are based on Lütkepohl (1996), see also Nakatani & Teräsvirta (2009).

Appendix A.1 Proof of Lemma 1

\[
\frac{\partial l(\theta)}{\partial \zeta} = -\frac{\partial \ln |G_t|}{\partial \zeta} - 2 \frac{\partial \varepsilon_t' H_t^{-1} \varepsilon_t}{\partial \zeta} - \frac{1}{2} \frac{\partial \varepsilon_t' H_t^{-1} \varepsilon_t}{\partial \zeta},
\]

\[
- \frac{1}{2} \frac{\partial \varepsilon_t' H_t^{-1} \varepsilon_t}{\partial \zeta} = \frac{1}{2} \frac{\partial \text{vec}(G_t)'}{\partial \zeta} \text{vec}(H_t^{-1} \varepsilon_t' \varepsilon_t + G_t^{-1} \varepsilon_t' \varepsilon_t' H_t^{-1}),
\]

\[
\frac{\partial l(\theta)}{\partial \zeta} = - \frac{\partial \text{vec}(G_t)'}{\partial \zeta} \text{vec}(G_t^{-1} - \frac{1}{2} H_t^{-1} \varepsilon_t' \varepsilon_t G_t^{-1} - \frac{1}{2} G_t^{-1} \varepsilon_t' \varepsilon_t' H_t^{-1}),
\]

where \( H_t = G_t D_t P D_t G_t \).

\[
\left. \frac{\partial l(\theta)}{\partial \zeta} \right|_{H_0} = - \frac{\partial \text{vec}(G_t)'}{\partial \zeta} \text{vec}(I - \frac{1}{2} M_t^{-1} \varepsilon_t' \varepsilon_t - \frac{1}{2} \varepsilon_t' \varepsilon_t M_t^{-1}),
\]

where \( \theta = (\omega', \rho', \zeta')', \omega = (\omega_1', ..., \omega_m')', \omega_i = (\alpha_{i0}, \alpha_{i1}, \beta_{i1})', \rho = \text{vec}(P), \zeta = (\zeta_1', ..., \zeta_m'), \zeta_i = (\zeta_{i1}, ..., \zeta_{ir})'. \)
Appendix A.2 Proof of Lemma 2

The second partial derivatives of the log-likelihood function w.r.t. \( \theta \), the Hessian, is given by:

\[
\mathcal{H}_t(\theta) = \begin{bmatrix}
\frac{\partial^2 l(\theta)}{\partial \omega' \partial \omega} & \frac{\partial^2 l(\theta)}{\partial \omega' \partial \rho} & \frac{\partial^2 l(\theta)}{\partial \omega' \partial \xi} \\
\frac{\partial^2 l(\theta)}{\partial \rho' \partial \omega} & \frac{\partial^2 l(\theta)}{\partial \rho' \partial \rho} & \frac{\partial^2 l(\theta)}{\partial \rho' \partial \xi} \\
\frac{\partial^2 l(\theta)}{\partial \xi' \partial \omega} & \frac{\partial^2 l(\theta)}{\partial \xi' \partial \rho} & \frac{\partial^2 l(\theta)}{\partial \xi' \partial \xi}
\end{bmatrix}.
\]  

(46)

Appendix A.2.1 The elements on the third row of the Hessian

Appendix A.2.1.1 The lower left element of the Hessian

The left element of the third row is given by:

\[
\frac{\partial^2 l(\theta)}{\partial \omega' \partial \xi} = -\frac{\partial}{\partial \omega'} \left( \frac{\partial \text{vec}(G_i')}{\partial \xi} \text{vec}(G_t^{-1}) \right)
\]

+ \frac{1}{2} \frac{\partial}{\partial \omega'} \left( \frac{\partial \text{vec}(G_i')}{\partial \xi} \text{vec}(G_t^{-1}D_t^{-1}P^{-1}D_t^{-1}G_t^{-1}G_t^{-1}) \right)

+ \frac{1}{2} \frac{\partial}{\partial \omega'} \left( \frac{\partial \text{vec}(G_i')}{\partial \xi} \text{vec}(G_t^{-1}G_t^{-1}D_t^{-1}P^{-1}D_t^{-1}G_t^{-1}) \right)

1st:

\[
-\frac{\partial}{\partial \omega'} \left( \frac{\partial \text{vec}(G_i')}{\partial \xi} \text{vec}(G_t^{-1}) \right) = 0
\]

2nd:

\[
\frac{1}{2} \frac{\partial}{\partial \omega'} \left( \frac{\partial \text{vec}(G_i')}{\partial \xi} \text{vec}(G_t^{-1}D_t^{-1}P^{-1}D_t^{-1}G_t^{-1}G_t^{-1}) \right)
\]

\[
= \frac{1}{2} \frac{\partial}{\partial \xi'} \left( \frac{\partial \text{vec}(G_i')}{\partial \xi} \right) \text{vec}(G_t^{-1}D_t^{-1}P^{-1}D_t^{-1}G_t^{-1}G_t^{-1}) \\
+ G_t^{-1}G_t^{-1}D_t^{-1}P^{-1}D_t^{-1}G_t^{-1}G_t^{-1} \frac{\partial}{\partial \omega'} \text{vec}(D_t)
\]

\[
= -\frac{1}{2} \frac{\partial}{\partial \xi'} \left( \frac{\partial \text{vec}(G_i')}{\partial \xi} \right) \text{vec}(G_t^{-1}G_t^{-1}D_t^{-1}P^{-1}D_t^{-1}G_t^{-1}G_t^{-1}) \\
+ G_t^{-1}G_t^{-1}D_t^{-1}P^{-1}D_t^{-1}G_t^{-1}G_t^{-1} \frac{\partial}{\partial \omega'} \text{vec}(D_t)
\]

Similarly the 3rd:

\[
\frac{1}{2} \frac{\partial}{\partial \omega'} \left( \frac{\partial \text{vec}(G_i')}{\partial \xi} \text{vec}(G_t^{-1}G_t^{-1}D_t^{-1}P^{-1}D_t^{-1}G_t^{-1}) \right)
\]

\[
= -\frac{1}{2} \frac{\partial}{\partial \xi'} \left( \frac{\partial \text{vec}(G_i')}{\partial \xi} \right) \text{vec}(G_t^{-1}D_t^{-1}P^{-1}D_t^{-1} \otimes G_t^{-1}G_t^{-1}D_t^{-1}) \\
+ G_t^{-1}D_t^{-1} \otimes G_t^{-1}G_t^{-1}D_t^{-1}P^{-1}D_t^{-1} \frac{\partial}{\partial \omega'} \text{vec}(D_t)
\]
Putting all together and using \( \epsilon_t = G_t D_t u_t \) yields

\[
\frac{\partial^2 l(\theta)}{\partial \omega' \partial \zeta} = \frac{-1}{2} \frac{\partial \text{vec}(G_t)^T}{\partial \zeta} \left( D_t u_t u_t' \right)^{-1} \frac{D_t^{-1} G_t^{-1} D_t^{-1} + D_t u_t u_t' \otimes G_t^{-1} M_t^{-1}}{D_t u_t u_t' + G_t^{-1} D_t^{-1} \otimes D_t u_t u_t'} \frac{\partial \text{vec}(D_t)}{\partial \omega'}
\]

(47)

\[
+ G_t^{-1} M_t^{-1} \otimes D_t u_t u_t' + G_t^{-1} D_t^{-1} \otimes D_t u_t u_t' \left( P^{-1} D_t^{-1} D_t^{-1} \right) \frac{\partial \text{vec}(D_t)}{\partial \omega'}
\]

Appendix A.2.1.2 The lower middle element of the Hessian

The middle element of the third row:

\[
\frac{\partial^2 l(\theta)}{\partial \rho' \partial \zeta} = -\frac{\partial}{\partial \rho'} \left( \frac{\partial \text{vec}(G_t)^T}{\partial \zeta} \frac{\text{vec}(G_t^{-1})}{\partial \zeta} \right)
\]

1st:

\[
-\frac{\partial}{\partial \rho'} \left( \frac{\partial \text{vec}(G_t)^T}{\partial \zeta} \frac{\text{vec}(G_t^{-1})}{\partial \zeta} \right) = 0
\]

2nd:

\[
\frac{1}{2} \frac{\partial}{\partial \rho'} \left( \frac{\partial \text{vec}(G_t)^T}{\partial \zeta} \frac{\text{vec}(G_t^{-1} D_t^{-1} P^{-1} D_t^{-1} G_t^{-1} \epsilon_t \epsilon_t' G_t^{-1})}{\partial \zeta} \frac{\text{vec}(P^{-1})}{\partial \rho'} \right)
\]

\[
= \frac{1}{2} \left( \frac{\partial \text{vec}(G_t)^T}{\partial \zeta} \frac{\partial \text{vec}(G_t^{-1} D_t^{-1} P^{-1} D_t^{-1} G_t^{-1} \epsilon_t \epsilon_t' G_t^{-1})}{\partial \zeta} \frac{\text{vec}(P^{-1})}{\partial \rho'} \right) \frac{\partial \text{vec}(P)}{\partial \rho'}
\]

\[
\left( G_t^{-1} \epsilon_t \epsilon_t' G_t^{-1} D_t^{-1} \otimes G_t^{-1} D_t^{-1} \right) \left( P^{-1} \otimes P^{-1} \right) \frac{\partial \text{vec}(P)}{\partial \rho'}
\]

\[
= -\frac{1}{2} \frac{\partial \text{vec}(G_t)^T}{\partial \zeta} \left( G_t^{-1} \epsilon_t \epsilon_t' G_t^{-1} D_t^{-1} \otimes G_t^{-1} D_t^{-1} \right) \frac{\partial \text{vec}(P)}{\partial \rho'}
\]

3rd:

\[
\frac{1}{2} \frac{\partial}{\partial \rho'} \left( \frac{\partial \text{vec}(G_t)^T}{\partial \zeta} \frac{\text{vec}(G_t^{-1} \epsilon_t \epsilon_t' G_t^{-1} D_t^{-1} P^{-1} D_t^{-1} G_t^{-1})}{\partial \zeta} \frac{\text{vec}(P)}{\partial \rho'} \right)
\]

\[
= -\frac{1}{2} \frac{\partial \text{vec}(G_t)^T}{\partial \zeta} \left( G_t^{-1} D_t^{-1} P^{-1} \otimes G_t^{-1} \epsilon_t \epsilon_t' G_t^{-1} D_t^{-1} P^{-1} \right) \frac{\partial \text{vec}(P)}{\partial \rho'}
\]

Putting these together yields

\[
\frac{\partial^2 l(\theta)}{\partial \rho' \partial \zeta} = -\frac{1}{2} \frac{\partial \text{vec}(G_t)^T}{\partial \zeta} \left\{ D_t u_t u_t' P^{-1} \otimes G_t^{-1} D_t^{-1} P^{-1} + G_t^{-1} D_t^{-1} P^{-1} \otimes D_t u_t u_t' P^{-1} \right\} \frac{\partial \text{vec}(P)}{\partial \rho'}
\]

(48)
Appendix A.2.1.3 The lower right element of the Hessian

The lower right element is given by

\[
\frac{\partial^2 l(\theta)}{\partial \zeta' \partial \zeta} = - \frac{\partial}{\partial \zeta'} \left( \frac{\partial \text{vec}(G_i)}{\partial \zeta} \text{vec}(G_i^{-1}) \right) \frac{\partial \text{vec}(G_i)}{\partial \zeta'} \\
+ \frac{1}{2} \frac{\partial}{\partial \zeta'} \left( \frac{\partial \text{vec}(G_i)}{\partial \zeta} \text{vec}(G_i^{-1}D_i^{-1}P_i^{-1}D_i^{-1}G_i^{-1} \epsilon_i' \epsilon_i' G_i^{-1}) \right) \frac{\partial \text{vec}(G_i)}{\partial \zeta'} \\
+ \frac{1}{2} \frac{\partial}{\partial \zeta'} \left( \frac{\partial \text{vec}(G_i)}{\partial \zeta} \text{vec}(G_i^{-1} \epsilon_i' \epsilon_i' G_i^{-1}) \right) \frac{\partial \text{vec}(G_i)}{\partial \zeta'}.
\]

1st:

\[
- \frac{\partial}{\partial \zeta'} \left( \frac{\partial \text{vec}(G_i)}{\partial \zeta} \text{vec}(G_i^{-1}) \right) = \frac{\partial \text{vec}(G_i)}{\partial \zeta} (G_i^{-1} \otimes G_i^{-1}) \frac{\partial \text{vec}(G_i)}{\partial \zeta'} \\
- \left[ \text{vec}(G_i^{-1}) \otimes I \right] \frac{\partial^2 \text{vec}(G_i)}{\partial \zeta' \partial \zeta}.
\]

2nd:

\[
\frac{1}{2} \frac{\partial}{\partial \zeta'} \left( \frac{\partial \text{vec}(G_i)}{\partial \zeta} \text{vec}(G_i^{-1}M_i^{-1}G_i^{-1} \epsilon_i' \epsilon_i' G_i^{-1}) \right) \\
= \frac{1}{2} \left[ \text{vec}(G_i^{-1}M_i^{-1}G_i^{-1} \epsilon_i' \epsilon_i' G_i^{-1}) \otimes I \right] \frac{\partial^2 \text{vec}(G_i)}{\partial \zeta' \partial \zeta} \\
+ \frac{1}{2} \frac{\partial \text{vec}(G_i)}{\partial \zeta'} \frac{\partial \text{vec}(G_i^{-1}M_i^{-1}G_i^{-1} \epsilon_i' \epsilon_i' G_i^{-1})}{\partial \zeta'} \frac{\partial \text{vec}(G_i)}{\partial \zeta'} \\
= \frac{1}{2} \left[ \text{vec}(G_i^{-1}M_i^{-1}G_i^{-1} \epsilon_i' \epsilon_i' G_i^{-1}) \otimes I \right] \frac{\partial^2 \text{vec}(G_i)}{\partial \zeta' \partial \zeta} \\
- \frac{1}{2} \frac{\partial \text{vec}(G_i)}{\partial \zeta'} \left( G_i^{-1} \epsilon_i' \epsilon_i' G_i^{-1} \otimes G_i^{-1}M_i^{-1}G_i^{-1} + G_i^{-1} \otimes G_i^{-1}M_i^{-1}G_i^{-1} \epsilon_i' \epsilon_i' G_i^{-1} \\
+ G_i^{-1} \epsilon_i' \epsilon_i' G_i^{-1}M_i^{-1}G_i^{-1} \otimes G_i^{-1} \right) \frac{\partial \text{vec}(G_i)}{\partial \zeta'} \\
= \frac{1}{2} \left[ \text{vec}(G_i^{-1}D_i^{-1}P_i^{-1}D_i) \otimes I \right] \frac{\partial^2 \text{vec}(G_i)}{\partial \zeta' \partial \zeta} \\
- \frac{1}{2} \frac{\partial \text{vec}(G_i)}{\partial \zeta'} \left( D_i \epsilon_i' \epsilon_i' D_i \otimes H_i^{-1} + G_i^{-1} \otimes G_i^{-1}M_i^{-1}D_i \epsilon_i' \epsilon_i' D_i \\
+ D_i \epsilon_i' \epsilon_i' D_i \otimes G_i^{-1} \right) \frac{\partial \text{vec}(G_i)}{\partial \zeta'}.
\]

3rd:
\[
\frac{1}{2} \frac{\partial}{\partial \zeta} \left( \frac{\partial \text{vec}(G_t)'}{\partial \zeta} \text{vec}(G_t^{-1} \varepsilon_t \varepsilon_t' G_t^{-1} M_t^{-1} G_t^{-1}) \right)
\]

\[
= \frac{1}{2} \left[ \text{vec}(G_t^{-1} \varepsilon_t \varepsilon_t' G_t^{-1} M_t^{-1} G_t^{-1})' \otimes I \right] \frac{\partial^2 \text{vec}(G_t)'}{\partial \zeta' \partial \zeta} + \frac{1}{2} \frac{\partial \text{vec}(G_t)'}{\partial \zeta} \frac{\partial (G_t^{-1} \otimes \text{vec}(G_t)')}{\partial \zeta} \frac{\partial \text{vec}(G_t)}{\partial \zeta} \frac{\partial \text{vec}(G_t)}{\partial \zeta'}
\]

\[
+ \frac{1}{2} \frac{\partial \text{vec}(G_t)'}{\partial \zeta} \left( G_t^{-1} \otimes D_t u_t u_t' D_t M_t^{-1} G_t^{-1} + G_t^{-1} M_t^{-1} D_t u_t u_t' D_t \otimes G_t^{-1} \right)
\]

\[
+ H_t^{-1} \otimes D_t u_t u_t' D_t \frac{\partial \text{vec}(G_t)}{\partial \zeta'}
\]

Putting the elements together yields

\[
\frac{\partial^2 l(\theta)}{\partial \zeta' \partial \zeta} = - \left\{ \left( \text{vec}(G_t^{-1})' \otimes I \right) - \frac{1}{2} \left( \text{vec}(G_t^{-1} M_t^{-1} D_t u_t u_t' D_t)') \otimes I \right) \right\} \frac{\partial^2 \text{vec}(G_t)'}{\partial \zeta' \partial \zeta}
\]

\[
+ \frac{1}{2} \frac{\partial \text{vec}(G_t)'}{\partial \zeta} \left\{ 2(G_t^{-1} \otimes G_t^{-1}) - D_t u_t u_t' D_t \otimes H_t^{-1} \right. \left. \right. 
\]

\[
- G_t^{-1} \otimes G_t^{-1} M_t^{-1} D_t u_t u_t' D_t - D_t u_t u_t' D_t M_t^{-1} G_t^{-1} \otimes G_t^{-1} 
\]

\[
- G_t^{-1} \otimes D_t u_t u_t' D_t M_t^{-1} G_t^{-1} - G_t^{-1} M_t^{-1} D_t u_t u_t' D_t \otimes G_t^{-1} 
\]

\[
- H_t^{-1} \otimes D_t u_t u_t' D_t \right\} \frac{\partial \text{vec}(G_t)}{\partial \zeta'}.
\]

References


**URL:** [http://CRAN.r-project/package=ccgarch](http://CRAN.r-project/package=ccgarch)


**URL:** [http://www.R-project.org](http://www.R-project.org)


Finite-Sample Multivariate Tests for ARCH in Vector Autoregressive Models

Paul Catani*   Niklas Ahlgren†

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Abstract

In this paper we propose finite-sample multivariate tests for ARCH effects in the errors of vector autoregressive (VAR) models using Monte Carlo testing techniques and the bootstrap. The tests under consideration are combined equation-by-equation LM tests, multivariate LM tests and LM tests of constant error covariance matrix. The tests are based on standardised multivariate residuals. We use a parametric bootstrap to circumvent the problem that the test statistics in VAR models are not free of nuisance parameters under the null hypothesis. The tests are evaluated in simulation experiments and the finite-sample bootstrap tests are found to have excellent size and power properties. The LM tests of constant error covariance matrix outperform the combined LM tests and multivariate LM tests in terms of power. The tests are applied to VAR models estimated on credit default swap (CDS) prices data and Euribor interest rates data.

1 Introduction

The Lagrange multiplier (LM) test for ARCH of Engle (1982) is widely used as a diagnostic test in regression and time series models. In multivariate models, testing for ARCH is often done equation-by-equation by applying univariate LM tests to the residuals from the individual equations. However, as emphasised by Dufour, Khalaf and Beaulieu (2010), univariate statistics from individual equations are not independent if the errors are contemporaneously correlated. The application of univariate tests in multivariate models leads to problems with combining the outcomes of the tests. Multivariate tests for ARCH are available in the literature (see e.g. Lütkepohl (2006)), but they have not been much used. General multivariate GARCH models have $O(n^4)$ parameters, where $n$ is the dimension of the process (Engle and Kroner (1995), p. 126). The multivariate generalisation of the LM test for ARCH may therefore perform poorly for small and moderate sample sizes, particularly when $n$ is large. Eklund and Teräsvirta (2007) propose a test of constancy of the error covariance matrix. The idea behind the test is that the error variances are time-varying, whereas the correlations are constant over time. As Eklund and Teräsvirta point out, their test may be viewed as a multivariate diagnostic test for ARCH.

*Hanken School of Economics, PO Box 479 (Arkadiagatan 22), 00101 Helsingfors, Finland. E-mail: paul.catani@hanken.fi.
†Hanken School of Economics, PO Box 479 (Arkadiagatan 22), 00101 Helsingfors, Finland. E-mail: niklas.ahlgren@hanken.fi.
Monte Carlo (MC) tests were introduced in econometrics by Dufour (2006) and applied to specification testing by Dufour et al. (2010). Dufour et al. propose multivariate specification tests for ARCH, which are based on combining equation-by-equation univariate tests. MC testing techniques deliver exact finite-sample tests in regression models when the regressors are exogenous. In dynamic regression models and time series models which contain lags of the dependent variable, the distributions of test statistics are not free of nuisance parameters. The bootstrap may then be used to obtain finite-sample tests which are asymptotically exact.

In this paper we propose multivariate bootstrap tests for ARCH in vector autoregressive (VAR) models, by following a suggestion in Dufour et al. of replacing an exact test by a bootstrap test when the model includes lags. The tests generalise the univariate bootstrap LM test of Gel and Chen (2012) to the multivariate case. Simulation results indicate that the multivariate bootstrap LM tests have the correct size in small and moderate samples. Empirical application to CDS prices and Euribor interest rates are provided as illustrations of the use of the tests with financial data.

2 The Model Setup and Standardised Multivariate Residuals

The observations on the $n \times 1$ vector $y_t = (y_{1t}, \ldots, y_{nt})'$ are assumed to be generated by an $n$-variate vector autoregressive (VAR) model

$$y_t = \Pi_1 y_{t-1} + \ldots + \Pi_p y_{t-p} + u_t, \quad (1)$$

where $\Pi_1, \ldots, \Pi_p$ are $n \times n$ parameter matrices. The errors $u_t$ are assumed to be independent and identically distributed (IID) with mean zero, and nonsingular and positive definite covariance matrix $\Omega$.

Dufour et al. (2010) develop a framework for Monte Carlo (MC) tests which employs Cholesky-standardised multivariate residuals from the multivariate linear regression model

$$Y = XB + U, \quad (2)$$

where $Y = (y_1, \ldots, y_n)$ is a $T \times n$ matrix, $X$ is a $T \times k$ matrix of full column rank, $B$ is a $k \times n$ parameter matrix and $U = (u_1, \ldots, u_n)$ is a $T \times n$ matrix of errors. The regressors are assumed to be exogenous, i.e. $X$ is taken as fixed for statistical analysis.

Estimate the model by LS and obtain the LS residuals

$$\hat{U} = (\hat{u}_1, \ldots, \hat{u}_n), \quad \hat{u}_i = (\hat{u}_{i1}, \ldots, \hat{u}_{iT})', \quad i = 1, \ldots, n. \quad (3)$$

Compute the multivariate standardised residual matrix

$$\tilde{W} = \hat{U} S_{\hat{U}}^{-1}, \quad (4)$$

where $S_{\hat{U}}$ is the Cholesky factor of $T^{-1}\hat{U}'\hat{U}$, i.e.

$$\hat{\Omega} = S_{\hat{U}}' S_{\hat{U}}, \quad \hat{\Omega}^{-1} = (T^{-1}\hat{U}'\hat{U})^{-1} = S_{\hat{U}}^{-1} (S_{\hat{U}}^{-1})'. \quad (5)$$

The following notation is used: $\tilde{W} = (\tilde{w}_1, \ldots, \tilde{w}_n)$ and $\tilde{w}_i = (\tilde{w}_{i1}, \ldots, \tilde{w}_{iT})', \quad i = 1, \ldots, n$. The framework delivers exact diagnostic tests in finite samples.
The VAR model in (1) defines a pure autoregression, which can be written in the linear regression form (2) with \( X_t = (y_{t-1}, \ldots, y_{t-p})' \) a typical row of \( X \), and \( B \) is an \( np \times n \) parameter matrix. The distribution of test statistics in the VAR model based on Cholesky-standardised multivariate residuals are not free of nuisance parameters. To circumvent the problem, we use a parametric bootstrap, which sets \( B \) in (2) equal to the LS estimator \( B \).

### 3 Multivariate Test for ARCH

We consider multivariate tests for conditionally heteroskedastic (ARCH) errors in the \( n \)-variate vector autoregressive (VAR) model (1). The null hypothesis is that the errors \( u_i \) are IID(0, \( \Omega \)) against the alternative hypothesis that they are conditionally heteroskedastic:

\[
u_i = H_i^{1/2} \varepsilon_i,
\]

where \( H_i = E(u_i u_i'|F_{t-1}) \) is the conditional covariance matrix of the errors \( u_i \), \( F_{t-1} = \sigma(\varepsilon_{t-1}, \varepsilon_{t-2}, \ldots) \) is the \( \sigma \)-field generated by \( \{\varepsilon_{t-1}, \varepsilon_{t-2}, \ldots\} \) and \( \{\varepsilon_t\} \) is a sequence of IID(0, \( \Omega \)) random variables.

We focus on Lagrange multiplier (LM) tests based on LS residuals. Hence, we do not consider multivariate Portmanteau tests and kernel-based tests.

#### 3.1 Combined Univariate Tests

The Lagrange multiplier (LM) test for ARCH (Engle (1982)) in equation \( i \) is a test of \( b_1 = \cdots = b_n = 0 \) in the auxiliary regression

\[
\hat{\sigma}^2_u = b_0 + b_1 \sigma^2_{t-1} + \cdots + b_p \sigma^2_{t-h} + e_{it}.
\]

The test statistic has the form

\[
LM_i = TR_i^2,
\]

where \( R_i^2 \) is the coefficient of determination in the auxiliary regression for equation \( i \). The LM statistic is asymptotically distributed as \( \chi^2(h) \) under the null hypothesis.

Following Dufour et al. (2010), standardised versions of the test statistics are obtained by replacing \( \hat{\sigma}^2_u \) by \( \hat{\sigma}^2_{it} \), where \( \hat{\sigma}^2_{it} \) are the elements in the \( i \)th column of the multivariate standardised residual matrix \( \hat{W} \) in (4).

The combined statistic is constructed as follows (Dufour et al. (2010)):

\[
\overline{LM} = 1 - \min_{1 \leq i \leq n} (p(LM_i)),
\]

where \( p(LM_i) \) are the individual \( p \)-values associated with the standardised LM statistics \( LM_i \). The \( p \)-values may be derived from the asymptotic distribution of \( \overline{LM}_i \), which is the \( \chi^2(h) \) distribution.

Observe that the combined test rejects if at least one of the individual tests is significant. The combined test is closely related to a Bonferroni-type testing procedure, but different from the Bonferroni bound the MC procedure delivers a simulated joint \( p \)-value (Dufour et al. (2010)).
3.2 Multivariate LM Tests

The multivariate LM test for ARCH is a generalisation of the univariate LM test, and is based on the auxiliary regression

\[
\text{vech}(\hat{\mathbf{u}}, \hat{\mathbf{u}}') = \mathbf{b}_0 + \mathbf{B}_1 \text{vech}(\hat{\mathbf{u}}_{t-1}, \hat{\mathbf{u}}'_{t-1}) + \cdots + \mathbf{B}_p \text{vech}(\hat{\mathbf{u}}_{t-h}, \hat{\mathbf{u}}'_{t-h}) + \mathbf{e}_t, \tag{9}
\]

where \( \mathbf{b}_0 \) is a \( \frac{1}{2}n(n+1) \)-dimensional parameter vector and \( \mathbf{B}_1, \ldots, \mathbf{B}_h \) are \( \frac{1}{2}n(n+1) \times \frac{1}{2}n(n+1) \) parameter matrices. The operator \( \text{vech} \) stacks the elements on and below the main diagonal of an \( n \times n \) matrix into a \( \frac{1}{2}n(n+1) \)-dimensional vector. The null hypothesis is that \( \mathbf{B}_1 = \cdots = \mathbf{B}_h = \mathbf{0} \), and then there is no ARCH in the errors \( \mathbf{u}_t \). The multivariate LM statistic can be shown to be of the form

\[
\text{MLM} = \frac{1}{2}T(n+1) - T \text{tr}(\hat{\Omega}_{\text{vech}}\hat{\Omega}^{-1}), \tag{10}
\]

where \( \hat{\Omega}_{\text{vech}} \) is the estimator of the error covariance matrix from the auxiliary model (9) and \( \hat{\Omega} = T^{-1} \sum_{t=1}^{T} \hat{\mathbf{e}}_t \hat{\mathbf{e}}'_t \) is the estimator of the error covariance matrix from the VAR model (1) (Lütkepohl (2006)). Following Dufour et al. (2010), a standardised version of the test statistic is obtained by replacing \( \hat{\mathbf{u}}_t \) by \( \bar{\mathbf{w}}_t \), where \( \bar{\mathbf{w}}_t \) are the elements in the \( t \)th row of the multivariate standardised residual matrix \( \bar{\mathbf{W}} \) in (4). The MLM statistic is asymptotically distributed as \( \chi^2(hn^2(n+1)^2/4) \) under the null hypothesis.

Eklund and Teräsvirta (2007) propose a test for constant error covariance matrices. The test can be defined for different types of time varying covariances under the alternative. When testing for ARCH, a suitable alternative is the constant conditional correlation autoregressive conditional heteroskedastic (CCC-ARCH) process of order \( h \). Then in (6)

\[
\mathbf{H}_t = \mathbf{D}_t \mathbf{P} \mathbf{D}_t',
\]

where

\[
\mathbf{D}_t = \text{diag}(h_{1t}^{1/2}, \ldots, h_{nt}^{1/2}) \tag{11}
\]

is a diagonal matrix of conditional standard deviations of the errors \( \mathbf{u}_t \). Further, \( \mathbf{D}_t^{-1} \mathbf{u}_t = \mathbf{e}_t \), where \( \mathbf{e}_t \sim \text{IID}(\mathbf{0}, \mathbf{P}) \) and \( \mathbf{P} = (\rho_{ij}) \), \( i, j = 1, \ldots, n \), is a positive definite matrix of conditional correlations, i.e. \( \rho_{ij} = 1 \) for \( i = j \). The conditional variance \( \mathbf{h}_t = (h_{1t}, \ldots, h_{ht})' \) follows a CCC-ARCH(\( h \)) process

\[
\mathbf{h}_t = \mathbf{a}_0 + \sum_{k=1}^{h} \mathbf{A}_k \mathbf{u}_{t-k}^{(2)}, \tag{12}
\]

where \( \mathbf{a}_0 = (a_{01}, \ldots, a_{0n})' \) is an \( n \)-dimensional vector of positive constants, \( \mathbf{A}_1, \ldots, \mathbf{A}_h \) are \( n \times n \) diagonal matrices and \( \mathbf{u}_t^{(2)} = (u_{1t}^2, \ldots, u_{nt}^2)' \).

The null hypothesis is \( \text{diag}(\mathbf{A}_1) = \cdots = \text{diag}(\mathbf{A}_h) = \mathbf{0} \). Let \( \mathbf{\theta} = (\omega'_1, \ldots, \omega'_h, \rho)' \), where \( \omega_i = (a_{0i}, a_{1i}, \ldots, a_{hi})' \) and \( \rho = \text{vecl}(\mathbf{P}) \). The operator \( \text{vecl} \) stacks the elements below the main diagonal of a \( n \times n \) matrix into a \( \frac{1}{2}n(n-1) \)-dimensional vector. The LM statistic has the form

\[
\text{LM}_{\text{CCC}} = T\bar{\mathbf{s}}_T(\tilde{\mathbf{\theta}})\tilde{\mathbf{I}}_T^{-1}(\tilde{\mathbf{\theta}})\bar{\mathbf{s}}_T(\tilde{\mathbf{\theta}}), \tag{13}
\]

where \( \bar{\mathbf{s}}_T(\tilde{\mathbf{\theta}}) \) and \( \tilde{\mathbf{I}}_T(\tilde{\mathbf{\theta}}) \) are the relevant blocks of the average score vector and information matrix, respectively, estimated under the null hypothesis (see Eklund and Teräsvirta 2007). The \( \text{LM}_{\text{CCC}} \) statistic is asymptotically distributed as \( \chi^2(nh) \) under the null hypothesis.
4 Bootstrap Monte Carlo Tests for ARCH

In this section we present the Monte Carlo testing technique and the bootstrap algorithm. For a general treatment and proofs, see Dufour (2006); for a treatment of specification tests, see Dufour et al. (2010).

The combined Monte Carlo (MC) test is implemented using the following algorithm. The algorithm is a modification of the algorithm in Dufour et al. (2010, p. 270) to autoregressions. Because the VAR model contains lagged dependent variables, the LM statistics are not free of nuisance parameters under the null hypothesis. Following a suggestion in Dufour et al., the LS estimator  \( \hat{B} \) of \( B \) under the null hypothesis is used in the parametric bootstrap in step 3. The tests based on the parametric bootstrap are not exact in finite samples. They are only exact as the sample size tends to infinity.

Algorithm (Bootstrap Monte Carlo tests for ARCH)

1. From the observed data, compute \( \hat{L}M \) in (8) and denote it \( \hat{L}M^{(0)} \). 
2. Obtain \( N \) draws from

   \[ \mathbf{W}_1, \ldots, \mathbf{W}_T \sim \text{NID}(0, \mathbf{I}_n) \]

   and denote the drawn variates \( \mathbf{W}^{(j)} \), \( j = 1, \ldots, N \).
3. For each draw \( j \), conditional on the observed regressor matrix \( \mathbf{X} \), the Cholesky factor \( \mathbf{S}_\mathbf{U} \) of the residuals \( \mathbf{U} \) and the LS estimator \( \mathbf{B} \) of \( B \), construct a bootstrap replication

   \[ \mathbf{Y}^{(j)} = \mathbf{X}\hat{\mathbf{B}} + \mathbf{W}^{(j)}\mathbf{S}_\mathbf{U}, \quad j = 1, \ldots, N. \]

   Regress \( \mathbf{Y}^{(j)} \) on \( \mathbf{X} \) and obtain the associated residual matrix \( \hat{\mathbf{U}}^{(j)} \), covariance matrix \( \hat{\mathbf{\Omega}}^{(j)} = \mathbf{T}^{-1}\hat{\mathbf{U}}^{(j)\prime}\hat{\mathbf{U}}^{(j)} \) and its Cholesky factor \( \mathbf{S}^{(j)}_\mathbf{U} \). Obtain the simulated standardised residuals

   \[ \hat{\mathbf{W}}^{(j)} = \hat{\mathbf{U}}^{(j)}(\mathbf{S}^{(j)}_\mathbf{U})^{-1} = (\hat{\mathbf{w}}_1^{(j)}, \ldots, \hat{\mathbf{w}}_n^{(j)}), \]

   where \( \hat{\mathbf{w}}_i^{(j)} = (\hat{w}_{i1}^{(j)}, \ldots, \hat{w}_{iT}^{(j)})' \), \( i = 1, \ldots, n \).
4. Compute the LM statistic for equation \( i \) and MC draw \( j \) using (7), denoting it \( \hat{L}M_i^{(j)} \). Compute \( \hat{L}M^{(j)} = 1 - \min_{1 \leq i \leq n}(p(\hat{L}M_i^{(j)})) \) using (8) as in step 1.
5. Given \( \hat{L}M^{(j)} \), \( j = 1, \ldots, N \), compute the number of simulated values greater than or equal to \( \hat{L}M^{(0)} \) (denoted \( N\hat{G}_N(\hat{L}M^{(0)}) \)). The MC \( p \)-values is

   \[ \hat{p}_N(\hat{L}M) = [N\hat{G}_N(\hat{L}M^{(0)}) + 1]/(N + 1). \]

Dufour et al. (2010) use \( N = 999 \) to implement the MC tests. The null hypothesis is rejected at the significance level \( \alpha \) if \( \hat{p}_N(\hat{L}M) \leq \alpha \).

The same algorithm is used with the multivariate LM tests MLM in (10) and the LM-CCC test in (13).

The asymptotic validity of the bootstrap tests follows from Theorem 1 of Dufour et al. (2010) and the consistency of the LS estimator \( \hat{B} \) of \( B \).
5 Simulations

We conduct Monte Carlo simulations for size and power of the multivariate tests for ARCH in finite samples with \( n = 2 \) and \( n = 5 \). The sample sizes are \( T = 100, 200 \) and 400. The number of Monte Carlo replications is 5000 for \( T = 100 \) and 200, and 2000 for \( T = 400 \). In the bootstrap Monte Carlo tests the number of replications is \( N = 499.1 \).

The model for the conditional mean is a stationary VAR(2) model:

\[
y_t = 0.5 \cdot I_n y_{t-1} + 0.3 \cdot I_n y_{t-2} + u_t.
\]

Four different data generating processes (DGP) are considered for the errors \( u_t \). In DGP 1, \( u_t \sim \text{NID}(0, I_n) \). In DGPs 2 and 3, \( u_t \) follow CCC-GARCH(1, 1) processes:

\[
u_t = H_t^{1/2} \varepsilon_t, \quad H_t = D_t P D_t, \quad D_t = \text{diag}(h_{1t}^{1/2}, \ldots, h_{nt}^{1/2}), \quad h_t = a_0 + A_1 u_{t-1}^{(2)} + B_1 h_{t-1},
\]

where \( h_t = (h_{1t}, \ldots, h_{nt})' \) and \( u_t^{(2)} = (u_{1t}^2, \ldots, u_{nt}^2)' \). Furthermore, \( \varepsilon_t \sim \text{IID}(0, P) \), where \( P = (\rho_{ij}) \). The parameter values for \( A_1 \) and \( B_1 \) in DGP 2 are

\[
A_1 = 0.08 \cdot I_n \quad \text{and} \quad B_1 = 0.90 \cdot I_n,
\]

and in DGP 3

\[
A_1 = 0.50 \cdot I_n \quad \text{and} \quad B_1 = 0.
\]

The constant vector \( a_0 \) has all its elements \( a_{0i} = 0.02, \ i = 1, \ldots, n \). The five-dimensional conditional correlation matrices are defined by

\[
P = \begin{pmatrix}
1 & \rho & \rho^2 & \rho^3 & \rho^4 \\
\rho & 1 & \rho & \rho^2 & \rho^3 \\
\rho^2 & \rho & 1 & \rho & \rho^2 \\
\rho^3 & \rho^2 & \rho & 1 & \rho \\
\rho^4 & \rho^3 & \rho^2 & \rho & 1
\end{pmatrix},
\]

and in the bivariate case \( P \) is the upper-left block of the above matrix. There is no statistical theory behind the choice of \( P \) and it is chosen simply because it depends on a single parameter. The constant correlation parameter is \( \rho = 0.5 \). In DGP 4 when \( n = 2 \), one series follows a univariate ARCH(1) process with parameters \( a_0 = 0.02 \) and \( a_1 = 0.50 \), and the other series is \( \text{NID}(0, 1) \). In DGP 4 when \( n = 5 \), two series follows univariate ARCH(1) processes with parameters \( a_0 = 0.02 \) and \( a_1 = 0.50 \), and the remaining three series are \( \text{NID}(0, I_3) \).

We consider the following five competing tests: the combined LM test (denoted \( \widetilde{LM} \)), asymptotic multivariate LM test (denoted \( MLM \)), bootstrap multivariate LM test (denoted \( MLM^* \)), asymptotic LM-CCC test (denoted \( LM_{\text{CCC}} \)) and bootstrap LM-CCC test (denoted \( LM_{\text{CCC}}^* \)).

Table 1 presents the results for testing against ARCH of orders \( h = 2, 5 \) and 10 in bivariate models. In addition to the multivariate tests, the table shows the results for the individual LM tests (denoted \( LM_1 \) and \( LM_2 \), respectively). The multivariate tests for ARCH tend to be slightly undersized, with the exception of the multivariate LM test \( MLM \). In particular, the LM-CCC test \( LM_{\text{CCC}} \) is undersized, with size against \( h = 2 \) of 2.4% when \( N = 100 \), 3.8% when \( N = 200 \) and 4.2% when \( N = 400 \). Bootstrapping the test brings its size closer to the nominal level. Turning to power,
we see that the LM-CCC test $LM_{CCC}$ is the most powerful test in all DGPs. Despite being slightly undersized in small samples, the asymptotic LM-CCC test $LM_{CCC}$ performs well in terms of power. If the errors are CCC-ARCH (DGP 3), then $LM_{CCC}$ is outperformed only by its bootstrap version $LM^*_CCC$. The combined LM test $LM$ has lower power. The multivariate LM tests $MLM$ and $MLM^*$ have lower power than the other multivariate tests.

Table 2 presents the results for testing against ARCH of orders $h = 2, 5$ and $10$ in models with $n = 5$. The performance of the multivariate LM tests $MLM$ and $MLM^*$ deteriorates further in comparison with the combined LM test $LM$ and the LM-CCC tests $LM_{CCC}$ and $LM^*_{CCC}$. 
Table 1: Simulated size and power of tests for ARCH when \( n = 2 \). The nominal significance level is 5%.

<table>
<thead>
<tr>
<th>DGP</th>
<th>Test</th>
<th>( h = 2 )</th>
<th>( h = 3 )</th>
<th>( h = 4 )</th>
<th>( h = 5 )</th>
<th>( h = 10 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( L_M )</td>
<td>( L_M )</td>
<td>0.05</td>
<td>0.20</td>
<td>0.45</td>
<td>0.60</td>
<td>0.75</td>
</tr>
<tr>
<td>( L_{M*} )</td>
<td>( L_{M*} )</td>
<td>0.10</td>
<td>0.23</td>
<td>0.40</td>
<td>0.55</td>
<td>0.73</td>
</tr>
<tr>
<td>( L_M )</td>
<td>( L_M )</td>
<td>0.05</td>
<td>0.20</td>
<td>0.45</td>
<td>0.60</td>
<td>0.75</td>
</tr>
<tr>
<td>( L_{M*} )</td>
<td>( L_{M*} )</td>
<td>0.10</td>
<td>0.23</td>
<td>0.40</td>
<td>0.55</td>
<td>0.73</td>
</tr>
<tr>
<td>( L_{M_{CC}} )</td>
<td>( L_{M_{CC}} )</td>
<td>0.05</td>
<td>0.20</td>
<td>0.45</td>
<td>0.60</td>
<td>0.75</td>
</tr>
<tr>
<td>( L_{M_{CC}} )</td>
<td>( L_{M_{CC}} )</td>
<td>0.10</td>
<td>0.23</td>
<td>0.40</td>
<td>0.55</td>
<td>0.73</td>
</tr>
</tbody>
</table>

\( T = 100 \)
Table 2: Simulated Size and power of tests for ARCH when \( n = 5 \). The nominal significance level is 5%.

<table>
<thead>
<tr>
<th>DGP</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>( h = 2 )</td>
<td></td>
<td></td>
<td>( h = 5 )</td>
<td></td>
<td></td>
<td>( h = 10 )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \hat{L}M )</td>
<td>0.036</td>
<td>0.188</td>
<td>0.829</td>
<td>0.620</td>
<td>0.048</td>
<td>0.233</td>
<td>0.730</td>
<td>0.519</td>
<td>0.048</td>
<td>0.250</td>
<td>0.596</td>
<td>0.411</td>
</tr>
<tr>
<td>( MLM )</td>
<td>0.035</td>
<td>0.145</td>
<td>0.703</td>
<td>0.215</td>
<td>0.001</td>
<td>0.032</td>
<td>0.106</td>
<td>0.012</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>( MLM^* )</td>
<td>0.024</td>
<td>0.114</td>
<td>0.642</td>
<td>0.168</td>
<td>0.032</td>
<td>0.138</td>
<td>0.309</td>
<td>0.101</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>( LM_{CCC} )</td>
<td>0.024</td>
<td>0.175</td>
<td>0.915</td>
<td>0.656</td>
<td>0.023</td>
<td>0.211</td>
<td>0.843</td>
<td>0.536</td>
<td>0.015</td>
<td>0.182</td>
<td>0.752</td>
<td>0.416</td>
</tr>
<tr>
<td>( LM_{CCC}^* )</td>
<td>0.042</td>
<td>0.208</td>
<td>0.929</td>
<td>0.701</td>
<td>0.045</td>
<td>0.286</td>
<td>0.885</td>
<td>0.614</td>
<td>0.047</td>
<td>0.295</td>
<td>0.832</td>
<td>0.530</td>
</tr>
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<td></td>
<td></td>
<td></td>
<td></td>
<td>( T = 200 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \hat{L}M )</td>
<td>0.044</td>
<td>0.553</td>
<td>0.998</td>
<td>0.960</td>
<td>0.050</td>
<td>0.685</td>
<td>0.989</td>
<td>0.901</td>
<td>0.048</td>
<td>0.726</td>
<td>0.968</td>
<td>0.853</td>
</tr>
<tr>
<td>( MLM )</td>
<td>0.055</td>
<td>0.502</td>
<td>0.992</td>
<td>0.656</td>
<td>0.031</td>
<td>0.539</td>
<td>0.880</td>
<td>0.339</td>
<td>0.003</td>
<td>0.263</td>
<td>0.280</td>
<td>0.033</td>
</tr>
<tr>
<td>( MLM^* )</td>
<td>0.032</td>
<td>0.420</td>
<td>0.987</td>
<td>0.580</td>
<td>0.033</td>
<td>0.539</td>
<td>0.881</td>
<td>0.346</td>
<td>0.048</td>
<td>0.482</td>
<td>0.564</td>
<td>0.189</td>
</tr>
<tr>
<td>( LM_{CCC} )</td>
<td>0.037</td>
<td>0.647</td>
<td>1.000</td>
<td>0.974</td>
<td>0.035</td>
<td>0.774</td>
<td>0.999</td>
<td>0.940</td>
<td>0.028</td>
<td>0.797</td>
<td>0.997</td>
<td>0.890</td>
</tr>
<tr>
<td>( LM_{CCC}^* )</td>
<td>0.047</td>
<td>0.669</td>
<td>1.000</td>
<td>0.976</td>
<td>0.047</td>
<td>0.797</td>
<td>0.999</td>
<td>0.949</td>
<td>0.048</td>
<td>0.829</td>
<td>0.999</td>
<td>0.918</td>
</tr>
<tr>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td>( T = 400 )</td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>( \hat{L}M )</td>
<td>0.039</td>
<td>0.928</td>
<td>1.000</td>
<td>1.000</td>
<td>0.044</td>
<td>0.983</td>
<td>1.000</td>
<td>1.000</td>
<td>0.051</td>
<td>0.986</td>
<td>1.000</td>
<td>0.997</td>
</tr>
<tr>
<td>( MLM )</td>
<td>0.066</td>
<td>0.892</td>
<td>1.000</td>
<td>0.955</td>
<td>0.049</td>
<td>0.956</td>
<td>1.000</td>
<td>0.842</td>
<td>0.031</td>
<td>0.940</td>
<td>0.979</td>
<td>0.525</td>
</tr>
<tr>
<td>( MLM^* )</td>
<td>0.040</td>
<td>0.854</td>
<td>1.000</td>
<td>0.938</td>
<td>0.035</td>
<td>0.945</td>
<td>1.000</td>
<td>0.820</td>
<td>0.046</td>
<td>0.946</td>
<td>0.979</td>
<td>0.563</td>
</tr>
<tr>
<td>( LM_{CCC} )</td>
<td>0.038</td>
<td>0.973</td>
<td>1.000</td>
<td>1.000</td>
<td>0.039</td>
<td>0.995</td>
<td>1.000</td>
<td>1.000</td>
<td>0.036</td>
<td>0.996</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td>( LM_{CCC}^* )</td>
<td>0.043</td>
<td>0.975</td>
<td>1.000</td>
<td>1.000</td>
<td>0.046</td>
<td>0.995</td>
<td>1.000</td>
<td>1.000</td>
<td>0.047</td>
<td>0.997</td>
<td>1.000</td>
<td>1.000</td>
</tr>
</tbody>
</table>
6 Empirical Examples

In this section the tests for ARCH are applied to CDS prices data and Euribor interest rates data.

6.1 Credit Default Swap Prices

Our first empirical example deals with credit default swap (CDS) prices. A CDS is a credit derivative which provides a bondholder with protection against the risk of default by the company. If a default occurs, the holder is compensated for the loss by an amount which equals the difference between the par value of the bond and its market value after the default. The CDS price is the annualised fee expressed as a percentage of the principal paid by the protection buyer. The equivalence of CDS prices and credit spreads of US and European investment-grade firms has been tested in the cointegrated VAR model and it has been found that a parity relation holds for most companies, i.e. the bond and CDS markets price credit risk equally (Blanco, Brennan and Marsh (2005), Zhu (2006), Dötz (2007) and Forsbäck (2012), inter alia).

We take a subsample of the companies in Table 1 of Blanco et al. The companies in our subsample are Bank of America, Citigroup, Goldman Sachs, Barclays Bank and Vodafone, the first three of which are US and the remaining two European companies. We use 5-year maturity CDS prices and credit spreads from Datastream. The data are daily observations from 1 January 2009 to 31 January 2012, and the number of daily observations for each company is $T = 804$.

Figure 1 displays the standardised residuals from VAR models for the CDS prices data. There is some evidence that the bond and CDS markets share periods of high volatility; for all companies large movements in one series is matched by large movements in the other series. This suggests that multivariate tests for ARCH effects will be more powerful than either univariate tests or combined tests. Finally, large outliers is a major feature of the CDS prices data, and there is some evidence that large outliers coincide for the two series. For all companies and both series there are some extremely large standardised residuals in the beginning of the sample period (corresponding to the first half of 2009). We have included dummy variables in the estimated VAR models to account for outliers which are deemed not to belong to high-volatility clusters. The dummy variables are listed in the notes to Table 3. The main reason for including dummy variables in the estimated VAR models is that the univariate LM tests are found to be sensitive to large outliers. The impact of a large outlier (typically larger than 8 standard deviations) would be that the $p$-values of the univariate LM tests shoot up to 0.999 or 1.000 in some cases. The multivariate tests are less sensitive to outliers, which is another reason for preferring multivariate tests over univariate tests.

The results of tests for ARCH of orders $h = 2$, 5 and 10 are reported in Table 3. In addition to the whole sample period, we divide the data into 2 sub-periods of $T = 402$ observations and 4 sub-periods of $T = 201$ observations. We report the $p$-values of the asymptotic univariate LM tests (denoted $L_{\text{CDS}}$ and $L_{\text{CS}}$), combined LM test (denoted $\tilde{L}$), asymptotic multivariate LM test (denoted $\text{MLM}$), bootstrap multivariate LM test (denoted $\text{MLM}^*$), asymptotic LM-CCC test (denoted $L_{\text{CCC}}$) and bootstrap LM-CCC test (denoted $L_{\text{CCC}}^*$). In the resampling scheme of the combined LM test and bootstrap tests we use $N = 999$. Hence the smallest possible $p$-value of the MC and bootstrap tests equals 0.001.
Figure 1: The standardised residuals $\tilde{w}_t$ from the estimated VAR models for the CDS prices data.
For the full sample period of $T = 804$ observations, all tests are significant at the 5%, level and all tests are significant at the 1% level, except the univariate LM test for $h = 5$ in the equation for $p_{CS}$ for Bank of America. In fact, most $p$-values are either 0.000 or 0.001. The $p$-values reported in Table 3 therefore do not reveal any differences between the tests.

For the sub-period of $T = 402$ observations the multivariate tests $MLM$, $MLM^*$, $LM_{CCC}$ and $LM^*_{CCC}$ are almost all significant at the 5% and 1% levels. An exception is Vodafone in sub-period 2; it is only the $LM_{CCC}$ and $LM^*_{CCC}$ tests that reject the null hypothesis of no ARCH effect of orders $h = 2$ and 5. An instance where the univariate tests do not reject, but the multivariate tests reject, is Bank of America in sub-period 2 when testing for ARCH of order $h = 2$. The $p$-values of the univariate tests are 0.168 and 0.112, and the $p$-value of the combined test is 0.196. The $p$-values of the multivariate tests are 0.000 and 0.001.

The results for the sub-periods of $T = 201$ observations are more interesting from the point of view of being able to detect differences between the tests. In view of our simulation results in Section 5, we would expect the $LM_{CCC}$ and $LM^*_{CCC}$ tests to be more powerful than the combined test $\bar{LM}$, which in turn is more powerful than the $MLM$ and $MLM^*$ tests. We observe that the $p$-values of the asymptotic test $MLM$ are larger than the $p$-values of the bootstrap test $MLM^*$, whereas the opposite holds for the asymptotic test $LM_{CCC}$ and the bootstrap test $LM^*_{CCC}$, which is in agreement with the findings in the simulations that the $MLM$ test is slightly oversized, whereas the $LM_{CCC}$ test is conservative. On balance, the univariate tests do not detect ARCH effects and the combined test only in about half of the cases in Table 3. The multivariate tests find more evidence of ARCH. More rejections are recorded for the $LM_{CCC}$ and $LM^*_{CCC}$ tests than for the $MLM$ and $MLM^*$ tests, and the $p$-values of the former tests are smaller than the $p$-values of the latter. The bootstrap LM-CCC test $LM^*_{CCC}$ finds the most evidence of ARCH.

The combined test $\bar{LM}$ tends to reject the null hypothesis of no ARCH effect when at least one of the univariate LM tests reject with a $p$-value close to 0. The multivariate tests $MLM$ and $MLM^*$, and $LM_{CCC}$ and $LM^*_{CCC}$ tend to reject also when both $p$-values of the univariate tests are moderate. For example, for Citigroup in sub-period 2, neither of the individual LM tests rejects the null hypothesis of no ARCH for any value of $h$, with the $p$-values varying between 0.103 and 0.333 for the equation for $p_{CDS}$, and between 0.737 and 0.980 for the equation for $p_{CS}$. The $p$-values of the combined tests $\bar{LM}$ lie between the $p$-values of the univariate LM tests, and the combined test does not reject for any value of $h$. The multivariate tests $MLM$, $MLM^*$, $LM_{CCC}$ and $LM^*_{CCC}$ reject for all values of $h$.

In a few cases the tests lead to conflicting decisions. For example, for Goldman Sachs in sub-period 3, the univariate LM tests, combined test $\bar{LM}$, and $LM_{CCC}$ and $LM^*_{CCC}$ tests do not reject for $h = 2$, but the multivariate tests $MLM$ and $MLM^*$ reject the null hypothesis of no ARCH effect. In contrast, for $h = 10$ the combined test $\bar{LM}$, and the $LM_{CCC}$ and $LM^*_{CCC}$ tests reject, but the $MLM$ and $MLM^*$ tests do not reject.
Table 3: Tests for ARCH in the estimated VAR models for CDS prices. The table reports the p-values of the tests. Notes: The lag length of the VAR model is $p = 2$ for Bank of America, $p = 3$ for Citigroup, $p = 3$ for Goldman Sachs, $p = 4$ for Barclays Bank and $p = 3$ for Vodafone. The estimated VAR models contain dummy variables taking the value 1 for the date in question and 0 otherwise: 25 February 2009, 10 April 2009 and 8 June 2009 for Citigroup, 9 April 2009 for Goldman Sachs, 6 February 2009 and 4 June 2009 for Barcalys Bank, and 8 June 2009 for Vodafone.

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6.2 Euribor Interest Rates

Our second empirical example uses data from the first two decades of Euribor interest rates. The common European currency, or the euro, was launched on 1 January 1999 and became legal tender on 1 January 2002 in 12 European Union countries. With the introduction of the euro in 1999, a new inter-bank reference rate was created within the European Monetary Union. Euribor (Euro Inter-bank Offered Rate) became the benchmark for short fixed-term inter-bank deposits within the euro zone. This makes Euribor one of the most important financial benchmarks in the world.

The data consist of $T = 172$ monthly observations from December 1998 to March 2013 on the 1, 3, 6, 9 and 12 month Euribor interest rates. All interest rates are nominal and annualised. The Data were retrieved from www.euribor-ebf.eu.

We fit a VAR model with lag length $p = 3$ to the interest rates data and test for ARCH effects in the errors. Figure 2 graphs correlograms of the squares and cross products of the standardised residuals. The correlograms show significant correlations in the squares and cross products. The significant cross correlations suggest that multivariate tests for ARCH effects will be more powerful than either univariate tests or combined tests.

The results of tests for ARCH of orders $h = 2$ and 12 are reported in Table 4. We report the $p$-values of the asymptotic univariate LM tests (denoted $LM$), combined LM test (denoted $\widetilde{LM}$), asymptotic multivariate LM test (denoted $MLM$), bootstrap multivariate LM test (denoted $MLM^*$), asymptotic LM-CCC test (denoted $LM_{CCC}$) and bootstrap LM-CCC test (denoted $LM_{CCC}^*$). In the resampling scheme of the combined LM test and bootstrap tests we use $N = 999$. Hence the smallest possible $p$-value of the MC and bootstrap tests equals 0.001. The $p$-values of the multivariate tests are all 0.000 or 0.001, which suggests significant ARCH effects. The results for the individual LM tests show that for $h = 2$, 3 out of 5 tests are significant at the 5% and 1% levels, and for $h = 12$, 3 out of 5 tests are significant at the 5%, and 2 out of 5 tests at the 1% level.
Figure 2: Correlograms of the squares $\hat{\omega}_t^2$ and cross products $\hat{\omega}_t \hat{\omega}_j$ of the standardised residuals from the VAR model for the Euribor interest rates data.
Table 4: Tests for ARCH in the estimated VAR model for Euribor interest rates. The table reports the *p*-values of the tests. Note: The lag length of the VAR model is *p* = 3. For *h* = 12 the number of parameters in the multivariate LM tests exceeds the number of observations.

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7 Conclusions

In this paper we have introduced and evaluated multivariate bootstrap tests for ARCH in vector autoregressive models. The tests are based on standardised multivariate least squares residuals and are therefore easy to calculate. The results show that the bootstrap tests perform better than the asymptotic tests in terms of both size and power. Our results also show that a less frequently used test against constant conditional correlation GARCH is more powerful than other multivariate LM tests such as combined univariate LM tests and multivariate LM tests which assume no particular alternative to the null hypothesis. The tests are applied to credit default swap (CDS) prices and Euribor interest rates. The multivariate tests find significant ARCH effects in almost all series. Our results indicate that ARCH effects should be taken into account by using wild bootstrap tests e.g. when testing for cointegration between CDS prices and credit spreads, and the Euribor interest rates at different maturities.

References


Conditional heteroskedasticity is often encountered in economic and financial time series. Since the introduction of autoregressive conditional heteroskedasticity (ARCH) by Engle in 1982, modelling volatility has received much attention in financial econometrics. Conditional heteroskedasticity also causes many asymptotic tests in time series models not to be valid. For example, tests for autocorrelation typically assume independent and identically distributed errors. The wild bootstrap provides a solution to the problem with inference under conditional heteroskedasticity.

This thesis consists of an introduction and four papers dealing with conditional heteroskedasticity in multivariate time series models. The first paper studies wild bootstrap tests for autocorrelation in vector autoregressive (VAR) models with conditional heteroskedasticity. The second paper is an empirical study of tests for cointegration in Chinese stock price data in the presence of conditional heteroskedasticity. The third paper proposes and studies a new Lagrange multiplier test for testing the adequacy of an estimated constant conditional correlation generalized ARCH model. The fourth paper studies tests for ARCH in VAR models.