Neutrino mass models
and leptonic CP violation at colliders

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27. joulukuuta 2013

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In this work the connection between neutrino mass mechanisms and leptonic CP violation at collider experiments is studied. These subjects are connected on a fundamental level: the neutrino mass models dictate the form of the neutrino mass matrix; leptonic CP violation is expressed in the neutrino mass matrix by the Dirac and Majorana phases.

The mass of neutrinos has been a subject of intensive study since the discovery of neutrino oscillation in the 1990s. This was the first, and still the only, observation that could not be explained by the standard model of particle physics. Neutrino mass mechanisms provide a way to extend the symmetry group of the standard model so that the neutrino masses are included. In addition to the tree-level seesaw mechanisms, four higher energy models, namely the minimal left-right symmetric model, the Littlest Higgs model, an SU(5) with an adjoint fermion, and the Altarelli-Feruglio model are reviewed. The first three extend the symmetry group of the standard model by a continuous symmetry, whereas the last extends it with a discrete flavor symmetry.

It is possible that the seesaw mediators responsible for neutrino masses are within the energy reach of the LHC. Then the parameters of the neutrino mass matrix could be probed at collider experiments. One could determine the existence of the Dirac and Majorana phases by studying their effects on observable quantities of the channels including the seesaw mediators. These effects are reviewed in this work. The non-zero values of the phases would still not determine the existence of leptonic CP violation: generally, a CP odd phase can lead into CP even processes. It is concluded that the discovery potential of the Majorana phases is quite promising at the studied processes. For the Dirac phase, the effects are more subtle and its value will probably be determined at other experiments.
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1 Introduction

The mass of neutrinos has been a subject of intensive study since the discovery of neutrino oscillation in the 1990s [77]. This was the first, and still the only, observation that could not be explained by the standard model of particle physics (SM). Although oscillation experiments have revealed that neutrinos are massive, the magnitude of the masses is still unknown. Moreover, even though experimental limits for the neutrino mixing parameters are constantly getting more exact, there is no information on whether neutrinos are Dirac particles like the other fermions, or if they are their own anti-particles (Majorana particles). Furthermore, the existence of neutrino mass opens the possibility for the CP symmetry to be violated also in the lepton sector, and not just for quarks.

Historically, in 1957-58 Pontecorvo made the first suggestions on the subject of neutrino oscillation [132,133]. In 1962, in an experiment suggested by Pontecorvo [134], the muon neutrino was observed [63]. In 1968, Pontecorvo suggested oscillations between electron and muon neutrinos [135] and in 1969 he and Gribov proposed the reason for oscillation to be the neutrino mixing. The standard theory of neutrino oscillations was built in the 1970s [40,68,73]. Later, experimental indications on this phenomenon were observed: an unexpected deficit in the flux of solar neutrinos was observed at the Homestake mine experiment [61], and later other experiments repeated the results. The conclusive evidence of the existence of the phenomenon was achieved in the late 1990s [77].

CP symmetry refers to a combination of two discrete symmetries: C for charge conjugation and P for parity transformation. Parity transformation flips the sign of the spacial coordinates of the particle and charge conjugation changes a particle into its antiparticle. In the 1950s it was observed that the parity symmetry (P) was maximally violated in weak interactions [146]. This was a shock for the physics community, since until then parity was thought to be conserved in all interactions. As also the charge conjugation (C) symmetry is maximally violated in weak interactions, it was soon suggested that it was the combined CP symmetry that should be conserved in all interactions, not parity.

However, in the 1960s it was observed that in the decays of the K meson also CP symmetry was violated [60]. The CP violation of quarks is manifested in the quark mixing matrix [49,111]. The phenomenon is possible because the quarks that participate in physical processes are not mass eigenstates but linear combinations of them. Later it has been discovered that CP is also violated in the decays of B [30] and D mesons [3]. The violation of CP symmetry is very slight; CP is an almost-symmetry in nature. The reason for this is unknown.
The observation of neutrino oscillation \[61,76\] meant that neutrinos are massive and thus leptonic CP violation could exist. It is, as all experiments with neutrinos, extremely difficult to observe, because neutrinos are so light and they interact only via weak interactions. Even though neutrinos cannot be seen, they can be observed indirectly. In the neutrino oscillation experiments one can study neutrinos originating from the Sun, from muon decay in the Earth’s atmosphere, from nuclear reactors, or from particle accelerators. These experiments are sensitive to the differences of neutrino masses and mixing angles of the lepton mixing matrix, as well as for the Dirac type of CP violation. However, the masses, the Majorana type of CP violating phases, or the nature of neutrinos, i.e., whether they are Dirac or Majorana particles, cannot be distinguished in oscillation experiments \[103\].

However, neutrino masses, Majorana phases, and the nature of neutrinos could be determined at collider experiments \[42\]. This is why studying neutrino properties at colliders is interesting: as neutrino mass generating mechanisms predict new particles with an energy of \(~1\text{ TeV}\), the Large Hadron Collider (LHC) could detect these particles that have been too heavy for previous colliders.

The SM is a relativistic gauge quantum field theory. That is, it describes particles as relativistic quantum fields satisfying gauge symmetries. The gauge group of the SM is \(SU(3)_c \otimes SU(2)_L \otimes U(1)_Y\): the first factor is the symmetry group of strong interactions; the second and third groups together describe the electroweak theory that combines weak and electromagnetic interactions. Thus, the SM describes weak, electromagnetic and strong interactions of all observed particles.

In the SM the elementary particles are divided into two categories, fermions with spin-1/2 and bosons with an integer spin. The former are further divided into quarks and leptons, the latter of which do not take part in strong interactions. The lepton sector is composed of three generations with one charged and one neutral lepton in each of them. The charged leptons are the electron \((e)\), the muon \((\mu)\), and the tau \((\tau)\). The corresponding neutrinos are \(\nu_e\), \(\nu_\mu\), and \(\nu_\tau\).

The SM has been a very successful theory: not only have its predictions matched the experimentally achieved results very accurately, but the theory has also predicted new particles. In 2012, the last theoretically predicted SM particle, the Higgs boson, was observed at CERN \[1,53\]. Given the success of the SM, the observation of neutrino oscillations was truly big news. Apart from neutrino oscillation, there is still no further direct evidence of physics beyond the SM, which makes the origin of neutrino mass a timely topic.

The SM describes neutrinos to be massless. Therefore, the origin of neutrino masses is one of the key topics of physics beyond the standard model. Even before
the experiments had proven the existence of the masses, different scenarios for the origin of neutrino masses were developed. If one wants to add a neutrino mass term into the particle content of the standard model, there is only a limited number of ways to do that.

At tree-level, there are three scenarios, called the seesaw mechanisms, that extend the standard model with either new scalars or fermions. Within the SM particle content the fermions get mass from the Yukawa interaction, which is not possible for neutrinos due to the lack of right-handed chiral particles. The new particles, called seesaw mediators, provide a way to write a mass term for neutrinos.

The seesaw models give an explanation on how a neutrino mass could be described. However, they do not give a physical reasoning for why the masses are generated in the way they are. To answer that question one has to look for theories of higher energies, many of which try to unify the three interactions included in the SM. Those models are called grand unified theories (GUT). In this work in addition to the tree-level seesaws a corresponding higher energy model for each of them are reviewed. They are the minimal left-right symmetric model (that includes the type I seesaw model), the Littlest Higgs model (type II seesaw), and an $SU(5)$ with an adjoint fermion (type III seesaw).

In addition to models extending the gauge group of the standard model, discrete flavor symmetry based models can also offer a viable way to explain neutrino mass properties. For long the data were in accordance with a neutrino mixing matrix pattern called tri-bi-maximal mixing, which has been used in a great number of flavor symmetry models. However, this pattern was excluded in 2012 when a mixing angle $\theta_{13}$ was found not to be zero [24,25], as was thought before. Attempts to save the approach have been made since. In this work, one of the most famous flavor symmetry models, the Altarelli-Feruglio model, is introduced.

The final focus of this work is to review the observable effects of CP violating Dirac and Majorana phases at collider experiments assuming the neutrino masses originating from interaction with seesaw mediators. Direct effects of leptonic CP violation are hard to measure, and unlike for quarks, there does not exist a lot of possible processes where CP violation could be directly detected. Therefore one has to look for weaker connections: to study effects that Dirac and Majorana phases would have on measurable quantities at collider experiments.

Leptonic CP violation is also a candidate for explaining the fundamental issue of baryon asymmetry in nature through a process called leptogenesis, in which the asymmetry is generated via decay of heavy sterile neutrinos, which leads to lepton asymmetry [65]. In fact, the attractiveness of the seesaw mechanisms stems also from
their ability to offer an explanation for the observed matter-antimatter asymmetry of the universe [89].

The work is organized as follows. In Sec. 2 the theoretical framework of neutrinos and in Sec. 3 the electroweak theory of the SM are reviewed. After that, in Sec. 4 the possible forms of the neutrino mass terms are shown. In Sec. 5 the aforementioned different neutrino mass mechanisms are introduced. The theoretical conditions for leptonic CP violation are studied in Sec. 6 before reviewing the observable effects of Dirac and Majorana phases at collider experiments in Sec. 7. Finally, conclusive remarks are made in Sec. 8.
2 Mathematical framework of spin-1/2 particles

Neutrinos are known to be massive, but in the SM they are described as massless. For massless particles the question of Dirac or Majorana nature is irrelevant, but for massive neutrinos it is of great importance.

In this section we go through the most important properties and restrictions of spinor fields of spin-1/2 particles stemming from their group theoretical properties. This is done in order to see the conceptual differences that will come necessary later on: to what extent is it profoundly different for a particle to have a small mass from to be massless; how the Majorana nature affects the Lagrangian of a particle; what kinds of mass terms are generally accepted for different kinds of spinors due to Lorentz invariance of the Lagrangian; and how the C, P, and CP symmetries are presented.

2.1 Dirac and Weyl spinors

The Lorentz group is of great importance in particle physics, since relativistic boosts, rotations, and inversions, which leave \( c^2 \tau^2 = x_0^2 - x^2 \) invariant, can be represented by it. Furthermore, one can relate different types of particles with different representations of the Poincaré group, which combines the Lorentz transformations and translations, and which is studied in detail in Appendix C.

The Lorentz group is marked as \( SO(3,1) \); its dimensionality can be regarded as having three space and one time dimensions. In Appendix B it is shown that the Lorentz symmetry can be broken into two disjoint \( SU(2) \) algebras. As the two disjoint \( SU(2) \) algebras commute, the group \( SO(3,1) \) can be represented as a direct product of them. One can label the representations of the restricted Lorentz group using the quantum numbers of the two \( SU(2) \)s, each of which has the dimension \( 2j + 1 \). The representations of the Lorentz group are denoted as \( (j_L,j_R) \), where subscripts \( L \) and \( R \) refer to left- and right-handedness. The dimension of a representation of the Lorentz group is then \( (2j_L + 1)(2j_R + 1) \). The lowest dimensional representations are known as spinor representations. They are denoted as \( (\frac{1}{2},0) \) and \( (0,\frac{1}{2}) \), and they are called the left- and right-handed Weyl representations, respectively. In the SM the massless neutrinos are described as Weyl spinor fields.

In Appendix B it is shown that the transformation properties for the \( (\frac{1}{2},0) \) and \( (0,\frac{1}{2}) \) spinors are the same under rotations, but opposite under Lorentz boosts. It is found that the representations are related to each other by a complex conjugation, so that one can write

$$ i\sigma_2 \phi^* = \chi, \quad i\sigma_2 \chi^* = \phi, \quad (1) $$
where \( \phi \) is a left-handed and \( \chi \) is a right-handed Weyl spinor, and where \( \sigma_2 \) is one of the Pauli spin matrices.

In the Weyl representations the spinors are two-component. However, in order to describe electrons, or any other Dirac-type particles, i.e., massive spin-1/2 particles, one needs to find a four-component representation. In fact, the Dirac representation is a reducible representation of Weyl representations, \( \sigma_2 \)

\[
\left( \frac{1}{2}, 0 \right) \oplus \left( 0, \frac{1}{2} \right),
\]

and the Dirac spinor can be represented as

\[
\psi = \begin{pmatrix} \phi_1 \\ \chi_2 \end{pmatrix},
\]

(2)

where the left and right chiral fields are numbered to emphasize them being independent spinors, i.e., a transformation \( i\sigma_2 \phi_1^* \neq \chi_1 \neq \chi_2 \). A Dirac spinor is four-dimensional, so the Weyl spinors are taken in this notation as

\[
\phi_1 = \begin{pmatrix} \phi_1 \\ 0 \end{pmatrix}, \quad \chi_2 = \begin{pmatrix} 0 \\ \chi_2 \end{pmatrix},
\]

(3)

where the zeros are two-component vectors. The representation is defined by

\[
L_{\mu\nu} = \frac{1}{2}\sigma_{\mu\nu} = \frac{1}{4}i [\gamma_\mu, \gamma_\nu],
\]

(4)

which fulfill the commutation relation of Eq. (43). Above \( \gamma^\mu \) are 4 \times 4 matrices that fulfill

\[
\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}1,
\]

(5)

where curly brackets denote the anti-commutator and \( g^{\mu\nu} \) is the Minkowski metric with signature \((+,-,−,−)\). The matrices \( \gamma^\mu \) are a matrix representation of a Clifford algebra \( Cl_{1,3}(R) \) and are called the Dirac matrices. Eq. (5) imposes the energy-momentum relation of a free particle Hamiltonian. It is a crucial condition in achieving the Klein-Gordon equation from Eq. (12). From Eq. (5), some useful properties follow directly:

\[
(\gamma^0)^2 = 1, \quad (\gamma^i)^2 = -1.
\]

(6)
We also require that
\[ \gamma_0 \gamma_\mu \gamma_0 = \gamma_\mu^\dagger, \]
which is to assure the hermiticity of the Dirac equation [124]. From Eq. (7) combined with Eq. (6) it can be noticed that \( \gamma_0 \) is Hermitian but \( \gamma_i \) are anti-Hermitian. One can have different representations for the \( \gamma_\mu \) matrices, which all are connected through a similarity transformation [102]
\[ \gamma_\mu' = U^\dagger \gamma_\mu U, \]
where matrix \( U \) is unitary and a prime on the left-hand side refers to a different representation. The lowest possible dimensionality for the Dirac matrices is four [84]. The Dirac matrices can be represented in the so called Weyl (or chiral) basis. The \( \gamma_\mu \) matrices are written as
\[ \gamma_0^W = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_W = \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix}, \]
where the subscript \( W \) stands for Weyl.

### 2.2 Dirac Lagrangian

The Lagrangian for a free Dirac particle is (in natural units)
\[ \mathcal{L}_D = \bar{\psi} (i \gamma_\mu \partial_\mu - m) \psi, \]
where \( \bar{\psi} \equiv \psi^\dagger \gamma^0 \) is the adjoint Dirac spinor. The canonical momentum is then \( \pi = i \psi^\dagger \), which leads to the Dirac Hamiltonian
\[ H = \int d^3x \left( \pi \partial_0 \psi - \mathcal{L}_D \right) = \int d^3x \psi^\dagger \left( -i \gamma^0 \gamma \cdot \nabla + \gamma^0 m \right) \psi. \]

The equation of motion is achieved from the Euler-Lagrange equations (see, e.g., [84])
\[ \partial_\mu \frac{\partial \mathcal{L}_D}{\partial (\partial_\mu \psi_r)} - \frac{\partial \mathcal{L}_D}{\partial \psi_r} = 0, \]
and reads
\[ (i \gamma_\mu \partial_\mu - m) \psi = 0. \]
This is the famous Dirac equation. If we decompose the spinor \( \psi \) into upper and lower parts as in Eq. (2), and define \( \sigma \equiv (1, \sigma) \) and \( \bar{\sigma} \equiv (1, -\sigma) \), the Dirac Lagrangian
takes the form

\[
\mathcal{L}_D = \left( \bar{\phi}_1 \tilde{\chi}_2 \right) \left( i\gamma^\mu \partial_\mu - m \right) \left( \begin{array}{c} \phi_1 \\ \chi_2 \end{array} \right)
\]

\[
= \left( \bar{\phi}_1 \tilde{\chi}_2 \right) \left( \begin{array}{cc} -m & i\sigma^\mu \partial_\mu \\ i\bar{\sigma}^\mu \partial_\mu & -m \end{array} \right) \left( \begin{array}{c} \phi_1 \\ \chi_2 \end{array} \right)
\]

\[
= i\phi_1^\dagger \bar{\sigma}^\mu \partial_\mu \phi_1 + i\chi_2^\dagger \sigma^\mu \partial_\mu \chi_2 + m \left( \chi_2^\dagger \phi_1 + \phi_1^\dagger \chi_2 \right).
\]

(13)

and the Dirac equation splits into a pair of equations

\[
\begin{cases}
i\sigma^\mu \partial_\mu \chi - m\phi = 0 \\
i\bar{\sigma}^\mu \partial_\mu \phi - m\chi = 0,
\end{cases}
\]

(14)

which states that in the case of massless particles, the equations decouple. This can also be seen from the Dirac Lagrangian in Eq. (13).

### 2.3 Chirality

In addition to the four Dirac matrices one can define a fifth gamma matrix,

\[ \gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3, \]

which is Hermitian, anticommutes with all the other \( \gamma^\mu \), and for which \((\gamma_5)^2 = 1\). It is called the chirality matrix. In the Weyl basis it reads

\[
\gamma^5_W = \begin{pmatrix} -\mathbb{1}_{2\times2} & 0 \\ 0 & \mathbb{1}_{2\times2} \end{pmatrix}.
\]

Using the chirality matrix \( \gamma^5 \) we can form projection operators, \( P_L \) and \( P_R \), which project the given Dirac spinor onto its \((\frac{1}{2}, 0)\) and \((0, \frac{1}{2})\) components, i.e.

\[
P_L \psi = \frac{1}{2} \left( \mathbb{1}_{4\times4} - \gamma_5 \right) \psi = \begin{pmatrix} \mathbb{1}_{2\times2} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} \phi_1 \\ 0 \end{pmatrix},
\]

\[
P_R \psi = \frac{1}{2} \left( \mathbb{1}_{4\times4} + \gamma_5 \right) \psi = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1}_{2\times2} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \chi_2 \end{pmatrix}.
\]

It is conventional to call the upper and lower components of Eq. (2) the left and right-handed spinors. This explains the subscripts \( L \) and \( R \): a spinor \((\frac{1}{2}, 0)\) is said to be of left-handed chirality; \((0, \frac{1}{2})\) spinors are called right chiral. The handedness,
or chirality, of a field is invariant under transformations of the restricted Lorentz group, which can be noted from the anticommutation of the chiral matrix $\gamma^5$ with the $\gamma^\mu$: both $P_L \psi$ and $P_R \psi$ commute with the commutators of the $SO^+(3,1)$ algebra of Eq. (4). One can also note that under rotations both the $\left(\frac{1}{2},0\right)$ and $\left(0,\frac{1}{2}\right)$ spinors transform in the same way, as is shown in Appendix B. However, the chirality of a massive particle is not conserved in time: as can be seen from Eq. (11), the mass term does not commute with $\gamma^5$. For massless particles chirality is conserved: it is the conserved quantity stemming from the invariance of the Lagrangian under a global $U(1)$ transformation $\exp(i\gamma^5 \alpha)$.

However, as shown in Appendix B, $\gamma^0$ acts as a parity operator: it leaves rotations invariant but flips the direction of a boost. A Dirac spinor transforms under parity as

$$\gamma_\mu^0 \psi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} \chi_2 \\ \phi_1 \end{pmatrix},$$

which means that parity transforms $\left(\frac{1}{2},0\right)$ spinors into $\left(0,\frac{1}{2}\right)$ spinors. In other words it changes the chirality of a Weyl spinor. It was noticed earlier that under boosts, a spinor $i\sigma_2 \phi^*$ transforms as a spinor $\chi$. A boost itself does not change a Weyl spinor into another representation, but it is a property of charge conjugation, which, like the parity transformation, does not belong to the restricted Lorentz group. They are discrete transformations, which we will come back to later.

Chirality is not to be confused with helicity, which also classifies fields into left- and right-handed parts. For massless particle the two concepts coincide, but for massive particles they are distinct. As shown in Appendix C, helicity is well defined only for massive particles, as it is the value of spin along the direction of the momentum.

### 2.4 Majorana equation

After the Dirac particle formulation it was proposed in 1937 by Majorana that one could describe a massive particle using only one two-component Weyl spinor $\chi_2$. Particles that could be represented this way are called Majorana particles. The idea is that the left and right-handed spinors do not need to be separate, but there can be a connection between them, so that the pair of equations of Eq. (2),

$$\begin{cases} i\sigma_\mu \partial_\mu \chi_2 - m\phi_1 = 0 \\ i\bar{\sigma}^\mu \partial_\mu \phi_1 - m\chi_2 = 0 \end{cases},$$

would be just two times the same equation. (Note that above $\chi_2$ and $\phi_1$ have four components in the sense of Eq. (3).) This can be done by taking a hermitian conjugate
of the lower equation and multiplying it with $\gamma^0$ from the right, giving

$$-i\partial_\mu \bar{\phi}_1 \gamma^\mu = m\bar{\chi}_2$$

$$i\gamma^\mu \partial_\mu C \bar{\phi}_1^T = mC\bar{\chi}_2^T,$$

which can be identified as the upper equation if

$$\eta_C C \bar{\phi}_1^T = \chi_2,$$

where $\eta_C$ is a phase factor which will be handled with more detail later when treating the charge conjugation symmetry. The matrix $C$ is defined through a relation,

$$C\gamma^\mu C^{-1} = -\gamma_\mu. \quad (15)$$

We then get the Majorana equation

$$i\gamma^\mu \partial_\mu \psi_L = \eta_C mC\bar{\psi}_L^T, \quad (16)$$

which makes it possible to write the complete field using only one chirality component,

$$\psi = \psi_L + \psi_R = \psi_L + C\bar{\psi}_L^T,$$

which differs from the Dirac fermion in that its left and right-handed chiral parts are connected. From this follows the Majorana condition,

$$\psi = C\bar{\psi}_L^T, \quad (17)$$

where the factor $\eta_C$ was rephased away. If one defines the charge conjugation (which will be treated later in detail) to be $\psi_C^T = C\bar{\psi}_L^T$, we get $\psi = \psi_L + \psi_C^T$, whence follows another form for the Majorana condition, $\psi = \psi_C$.

One notices that a charged particle cannot satisfy the condition; in the case of electromagnetism, for particles and antiparticles, having opposite charge $\pm e$, the expression of the covariant derivate becomes (from Eqs. [26] - [27]):

$$D_\mu \equiv \partial_\mu \pm ieA_\mu,$$

i.e., they differ by sign, which poses the requirement of zero charge for Majorana particles.
2.5 Majorana Lagrangian

If neutrinos are not Dirac particles, due to their neutrality, they could be Majorana particles, satisfying Eq. (17). When studying Weyl spinors, we noticed that a transformed spinor $-i\sigma^2\phi^*$ transforms as a spinor $\chi$ under Lorentz transformations. Thus, a Majorana spinor is in the Weyl basis

$$\psi_M = \begin{pmatrix} \phi \\ -i\sigma^2\phi^* \end{pmatrix},$$

where $\phi$ is a two component spinor. Using this, the Lagrangian for a Majorana field in a Weyl basis becomes

$$L_{M,W} = \frac{1}{2} \left( \chi^\dagger i\chi^T \sigma^2 \right) \gamma^0 \left( -m_M \ \ i\sigma^\mu\partial_\mu \\
\frac{i}{2} \sigma^\mu\partial_\mu - m_M \right) \left( \chi \ -i\sigma^2\chi^* \right)$$

$$= \frac{i}{2} \left( \chi^\dagger \sigma^\mu\partial_\mu\chi + \chi^T \sigma^\mu\sigma^\nu\partial_\mu\chi^* + \chi^\dagger m_M \sigma^2\chi^* - \chi^T \sigma^2 m_M \chi \right)$$

$$= \chi^\dagger \sigma^\mu\partial_\mu\chi + \frac{m_M}{2} \left( \chi^\dagger i\sigma^2\chi^* - \chi^T i\sigma^2\chi \right).$$

Note that the kinetic term is the same as for the left-handed part of the Dirac Lagrangian in the Weyl basis (Eq. (13)), but the mass term is different. Therefore in the case of massless neutrinos the question of whether neutrinos are Dirac or Majorana particles is irrelevant, but for massive neutrinos it becomes important.

2.6 Inversions

So far we have treated properties of continuous symmetries: rotations, boosts, and translations. In this section the discrete symmetries of charge conjugation $C$ and parity $P$, as well as the combined transformation $CP$ are reviewed. Discrete symmetries are profoundly different and have to be dealt with in a different way from continuous symmetries. For continuous symmetries, the Noether theorem states that there is a conserved quantity corresponding to every symmetry [131]. The quantity in question is the generator of a representation of a symmetry. In the case of $SO(3)$, generators are the angular momentum operators; angular momentum is conserved under rotations. However, there is no reason for a discrete symmetry to lead into a conserved quantity. Symmetries relevant for this work, charge conjugation and parity, are inversions, i.e., they are discrete transformations which applied twice give the initial state. For an inversion the transformation reads [141]

$$U^\dagger \phi_i(x)U = \eta S_{ij}\phi_j(x') \ \ \phi \rightarrow \phi' = \eta S\phi,$$
where $\eta$ is a phase factor and $S$ a matrix acting on a field $\phi$. Generally, the invariance of terms of the form $\phi^\dagger \phi$ under the inversion requires $|\eta| = 1$. For Majorana particles the condition is more strict: $\eta = \pm 1$.

For continuous symmetries the conservation law that follows from the Noether theorem is additive. For a case with transformation matrix $U = \exp(i\alpha G)$, and where an operator acts on a state as $G |\phi_1\rangle = g_1 |\phi_1\rangle$, we get

$$U |\phi_1 \phi_2\rangle = U |\phi_1\rangle |\phi_2\rangle = (U |\phi_1\rangle) (U |\phi_2\rangle) = e^{i\alpha (g_1 + g_2)} |\phi_1 \phi_2\rangle,$$

whereas for inversions we have (now the effect of an operator is $U_I |\phi_1\rangle = u_1 |\phi_1\rangle$),

$$U_I |\phi_1 \phi_2\rangle = (U_I |\phi_1\rangle) (U_I |\phi_2\rangle) = u_1 u_2 |\phi_1 \phi_2\rangle,$$

meaning that the conservation law is multiplicative. In the ensuing subsection the C, P, and CP symmetries are studied.

### 2.6.1 Charge conjugation

Under charge conjugation a spinor field transforms as

$$\psi \to \psi' \equiv \psi^C = \eta_C C \psi^T = -\eta_C \gamma^0 C \psi^*,$$

where $\eta_C$ is the intrinsic charge parity, which for a many particle system is the property that acts as in Eq. (21), i.e., the intrinsic charge parity of the system is the product of the parities of individual particles. As the charge conjugation is an inversion, there is a requirement that

$$|\eta_C|^2 = 1.$$
By performing the charge conjugation transformation to the quantized solution of the Dirac field, one notices that the charge conjugation converts particles into antiparticles, as was interpreted in the case of the Majorana condition. The transformations of vector and pseudovector bilinears differ by sign, i.e.,

\[
\bar{\psi}_a \gamma^\mu \psi_b \rightarrow (\bar{\psi}_a \gamma^\mu \psi_b)^C = \eta_{C,a} \eta_{C,b} \gamma^\mu \psi_a \eta_{C,b} \bar{\psi}_a \gamma^\mu \psi_b = \eta_{C,a} \eta_{C,b} \gamma^\mu \psi_a \eta_{C,b} \bar{\psi}_a \gamma^\mu \psi_b.
\]

Weak interactions have a coupling structure \( \gamma^\mu - \gamma^\mu \gamma^5 \), which is a combination of a vector and pseudovector. The above transformation properties then state that all weakly interacting processes violate the charge conjugation symmetry. On the contrary, for other interactions charge conjugation is a symmetry. The violation of \( C \) states that matter and antimatter are treated differently in nature. Since neutrinos interact in the SM only via weak interactions, for which \( C \) is not a symmetry, the intrinsic charge parities of neutrinos do not have a physical meaning and can be chosen arbitrarily \[84\].

2.6.2 Parity

The parity transformation flips the sign of the spatial coordinates, i.e., \( x^\mu \rightarrow x_\mu^P = (x^0, -x) \), so that the effect on a spinor reads

\[
\psi(x) \rightarrow \psi^P(x_P) = \eta_P \gamma^0 \psi(x_P),
\]

where \( \eta_P \) is the intrinsic parity, analogous to the intrinsic charge parity, for which there is a restriction stemming from parity being an inversion,

\[
\eta_P^2 = \pm 1.
\]

As stated before, in the case of fermions, single Dirac spinor fields are not physical, so that their phase can be freely chosen. This loosens the restriction for \( \eta_P \) slightly, for fermions we have then,

\[
|\eta_P| = 1.
\]

In Appendix \[3\] it is noted that \( \gamma^0 \) in the chiral (Weyl) basis acts as a parity operator for Weyl spinors. What it means is that \( \gamma^0 \) transformed a chiral Weyl field into another chirality. Weyl spinors describe a massless particle, and as for a massless particle chirality and helicity mean the same, also the sign of helicity was changed. Generally, for massive particles, it is helicity rather than chirality that is flipped by
parity. To see this, let’s study transformation properties of vector and pseudovector bilinears:

\[
\bar{\psi}_a \gamma^\mu \psi_b \to \left( \bar{\psi}_a \gamma^\mu \psi_b \right)^P = \eta^*_{P,a} \eta_{P,b} \bar{\psi}_a (x_P) \left( \gamma^\mu \right)^{1} \psi_b (x_P) = \eta^*_{P,a} \eta_{P,b} \bar{\psi}_a \gamma^\mu \psi_b \\
\bar{\psi}_a \gamma^5 \gamma^\mu \psi_b \to \left( \bar{\psi}_a \gamma^5 \gamma^\mu \psi_b \right)^P = -\eta^*_{P,a} \eta_{P,b} \bar{\psi}_a (x_P) \left( \gamma^\mu \right)^{1} \gamma^5 \psi_b (x_P) = -\eta^*_{P,a} \eta_{P,b} \bar{\psi}_a \gamma^\mu \gamma^5 \psi_b.
\]

They again differ by sign. As angular momentum is a vector, and spin is an axial vector, the transformation of helicity, defined as in Eq. (150), yields

\[
S \cdot P \to S^P \cdot P^P = -S \cdot P,
\]

stating that helicity is a pseudoscalar, and that it is reversed under parity transformation. Due to the V-A structure of weak interactions, also parity is then violated in weak interactions.

### 2.6.3 CP symmetry

CP transformation is a combined charge conjugation and parity transformation, and thus it reads

\[
\psi \to \psi^{CP} = -\eta_{C} \eta_{P} \gamma^0 \gamma^0 C \psi^* (x_P) = -\eta_{CP} C \psi^* (x_P),
\]

where \( \eta_{CP} \) is the intrinsic CP parity, \( \eta_{CP} = \eta_{C} \eta_{P} \), from which it follows that

\[
|\eta_{CP}|^2 = 1.
\]

As stated before the phase of a spinor does not have a physical meaning. Therefore, a CP transformation applied twice gives the original field, and is an inversion. As it is a combined charge conjugation and parity transformation, it transforms a particle into its antiparticle and flips the sign of the helicity. As both C and P are conserved under non-weak interactions, CP as their combination is clearly conserved. We are interested to see if the CP symmetry is violated in weak interactions. The vector and pseudovector bilinears transform as

\[
\bar{\psi}_a \gamma^\mu \psi_b \to \left( \bar{\psi}_a \gamma^\mu \psi_b \right)^{CP} = -\eta^*_{P,a} \eta_{P,b} \eta_{C,a} \eta_{C,b} \bar{\psi}_a \gamma^\mu \psi_b \\
\bar{\psi}_a \gamma^5 \gamma^\mu \psi_b \to \left( \bar{\psi}_a \gamma^5 \gamma^\mu \psi_b \right)^{CP} = -\eta^*_{P,a} \eta_{P,b} \eta_{C,a} \eta_{C,b} \bar{\psi}_a \gamma^\mu \gamma^5 \psi_b,
\]

which means that there is no direct reason to have CP violation for weak interactions; vectors and axial vectors transform in the same way. This is why it seemed to be so
strange that CP symmetry would be violated, and why the observation of its violation was such a shock. The violation of the CP symmetry is studied in more detail in Sec.6.
3  Gauge field theories

The Standard Model of particle physics is a non-abelian gauge theory of a symmetry group $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$. It has 12 generators and thus 12 gauge fields: $U(1)$ has one, $SU(2)$ has three, and $SU(3)$ has eight gauge fields. Generally, a gauge group is determined by what kinds of transformations are possible between gauges of the theory. For QED the gauge group is $U(1)$, for QCD it is $SU(3)$, and for isospin it is $SU(2)$ \[147\]. It turns out that the physical gauge fields arising from the localization of the gauge group $SU(2)_L \otimes U(1)_Y$ are combinations of the gauge fields of groups $SU(2)_L$ and $U(1)_Y$, whereas the gauge fields of $SU(3)_c$ are independent of the others. It follows that making the $SU(3)_c$ symmetry local leads to quantum chromodynamics, whereas the combined $SU(2)_L \otimes U(1)_Y$ local symmetry forms the electroweak theory \[85,137,142\].

3.1  $U(1)$ symmetry of QED

The Dirac Lagrangian $\mathcal{L}_D = \bar{\psi} (i\gamma^\mu \partial_\mu - m) \psi$ is invariant under a global $U(1)$ transformation, i.e. a complex phase shift,

$$\psi \rightarrow \psi' = \exp(i e \alpha) \psi,$$

where $e$ is the charge of an electron and $\alpha$ is an arbitrary constant, not dependent on $x$. In order to make it invariant under a local symmetry, i.e. one for which $\alpha = \alpha(x)$, one has to introduce a new vector field and the concept of a covariant derivative. Without them the Lagrangian is not invariant, but an extra term from taking the derivative appears: $-\bar{\psi} e \partial_\mu \alpha(x) \psi$. The new vector field, $A_\mu$, has to transform under a local $U(1)$ symmetry as

$$A_\mu \rightarrow A'_\mu = A_\mu + \partial_\mu \alpha(x),$$

and instead of the ordinary derivative we have to use a covariant derivative which transforms as the Dirac field,

$$D_\mu \equiv \partial_\mu - ie A_\mu.$$ \hspace{1cm} (26)

With these modifications we have a Lagrangian which is invariant under local $U(1)$ transformations,

$$\mathcal{L}'_D = \bar{\psi} (i\gamma^\mu D_\mu - m) \psi,$$

which is not anymore just a Lagrangian for a free Dirac particle, but instead it contains also an interaction term, $\mathcal{L}'_D = e \bar{\psi} \gamma^\mu A_\mu \psi$. As the field $\psi$ is taken to describe
electrons, and the vector field $A_\mu$ is just a photon, the term $\mathcal{L}'_D$ describes the interaction of an electron and a photon. It states that a possible vertex must consist of two electrons and one photon.

It is natural to assume that the Lagrangian has to be completed with a free Lagrangian for the vector field, which is the Lagrangian for Maxwell fields \[ \mathcal{L}_{em} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \]

The $U(1)$ invariant full Lagrangian of quantum electrodynamics (QED) is then

\[
\mathcal{L}_{QED} = \mathcal{L}'_D + \mathcal{L}_{em} = \bar{\psi} (i\gamma^\mu D_\mu - m) \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}.
\]

### 3.2 $SU(2)$ symmetry of isospin

Next we want to make the Dirac Lagrangian invariant under a local $SU(2)$ transformation, i.e. under a transformation

\[
\psi_a \rightarrow \psi'_a = U_{ab} \psi_b,
\]

where $U$ are constant unitary $2 \times 2$ matrices that depend on $x$. Elements of $SU(2)$ can be written in an exponential form, $U = \exp(i g T \cdot \alpha)$, where $T$ are the generators of the group (usage of the letter $T$ is conventional) and $\alpha$ depends on $x$. As we have seen before, in the fundamental $2 \times 2$ representation the generators are $T = \sigma/2$. As in the case of $U(1)$ symmetry, the Dirac Lagrangian has to be modified by adding new fields and the covariant derivative. The extra term due to derivation of Eq. (28) is now

\[-g \bar{\psi} \gamma^\mu T \cdot \partial_\mu \alpha \psi.\]

For each of the three generators one has to insert a corresponding gauge field, $W_\mu$, that would enable us to have an invariant Lagrangian. In general, there are as many gauge fields as there are generators in the symmetry group, and the gauge fields transform according to the symmetry group \[17\]. Analogously to the $U(1)$ symmetry, the covariant derivative takes a form

\[ D_\mu \equiv \partial_\mu - ig W_\mu \cdot T. \]
The Dirac Lagrangian now yields

\[ L'_{D} = \left( \bar{\psi}_1 \psi_1 \bar{\psi}_2 \right) U^{-1} \left[ i \gamma^\mu \left( \partial_\mu - igW'_\mu \cdot T \right) - \text{diag}(m_1, m_2) \right] U \left( \begin{array}{c} \psi'_1 \\ \psi'_2 \end{array} \right), \]

where for the covariant derivative we impose a condition

\[ U^{-1} D'_\mu U = D_\mu, \quad (29) \]

from which it follows that, by considering an infinitesimal transformation and using the known commutation relations for the generators \( T_i \), [102]

\[ W'_\mu = W_\mu + \partial_\mu \alpha - g \alpha \times W_\mu. \quad (30) \]

This is analogous to the transformation of the electromagnetic field \( A_\mu \). The cross product term stems from the commutation relation \([T_i, T_j] = i\epsilon_{ijk} T_k\).

We want to complete the Lagrangian with the free terms for the gauge fields \( W_\mu \). It turns out that it can be achieved by taking the commutator of the covariant derivative as in [131]

\[ [D_\mu, D_\nu] = [\partial_\mu - igW'_\mu T^i, \partial_\nu - igW'_\nu T^j] \equiv -igW^i_{\mu\nu} T^i, \]

where

\[ W^i_{\mu\nu} = \partial_\mu W^i_\nu - \partial_\nu W^i_\mu + g\epsilon_{ijk} W^j_\mu W^k_\nu = \partial_\mu W^i_\nu - \partial_\nu W^i_\mu + g W_\mu \times W_\nu. \]

Using this expression the full Lagrangian with \( SU(2) \) symmetry can be written as

\[ L_D = \bar{\psi} \left( i \gamma^\mu D_\mu - \text{diag}(m_1, m_2) \right) \psi - \frac{1}{4} \left( W^i_{\mu\nu} \right)^2. \quad (31) \]

### 3.3 \( SU(3) \) symmetry of QCD

The invariance of the Dirac Lagrangian under an \( SU(3) \) symmetry leads to the mathematical description of Quantum chromodynamics (QCD). The process of making the global symmetry local is analogous to the \( SU(2) \) symmetry. \( SU(3) \) has eight generators \( T_a \), of which the commutation relations are

\[ [T_a, T_b] = if_{ab}^c T_c. \]
The generators are in their defining representation $3 \times 3$ matrices $T_a = \lambda_a/2$ where

$$
\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},
$$

$$
\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},
$$

$$
\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},
$$

are known as the Gell-Mann matrices. As the rank of the SU(3) algebra is 2, two of the generators are diagonal in the defining representation and commute; generators labeled as $T_3$ and $T_8$ form the Cartan subalgebra of SU(3) $[102]$. The field $\psi$ must now have three components:

$$
\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix},
$$

and it is transformed under an SU(3) transformation as

$$
\psi \rightarrow \psi' = e^{i g a T_a \alpha} \psi.
$$

By promoting $\alpha$ to depend on $x$, we get an extra term from the derivative

$$
-g \bar{\psi} \gamma^\mu T_a \cdot (\partial_\mu \alpha_a) \psi,
$$

which can be gauged away with a covariant derivative of a form

$$
D_\mu = \partial_\mu - ig T_a G^a_\mu.
$$

The eight new vector fields $G^a_\mu$ are called gluons. The transformation of the gluon field is analogous to Eq. (30), namely

$$
G^a_\mu \rightarrow G'^a_\mu = G^a_\mu + \partial_\mu \alpha_a^a - gf_{bc}^a \alpha^b G^c_\mu.
$$

The self-interaction for the gluon fields are given from the commutator of the covariant
derivative as in the case of $SU(2)$. Only now the structure constant is not $\epsilon_{ijk}$, but $f_{abc}$:

$$G^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + ig f^{abc}_{\mu} G^b_\mu G^c_\nu,$$

with which the Lagrangian becomes

$$\mathcal{L}_D = \bar{\psi} (i\gamma^\mu D_\mu - \text{diag}(m_1, m_2, m_3)) \psi - \frac{1}{4} (G^a_{\mu\nu})^2.$$

The big difference with $SU(2)$ and $SU(3)$ compared to $U(1)$ symmetry is that the Lagrangians of the formers have terms with gluon and $W$ fields of third and forth order; they form vertices. Explicitly, for QCD,

$$\frac{1}{4} (G^a_{\mu\nu})^2 = \frac{1}{4} \left[ (\partial_\mu G^a_\nu - \partial_\nu G^a_\mu) (\partial^\mu G^{*a} - \partial^\nu G^{*a}) - g^2 f^{abc}_{\mu} f^{def}_{\nu} G^b_{\mu} G^c_{\nu} G^{*e}_{\rho} G^{*f}_{\sigma} + 2ig f^{abc}_{\mu} G^b_{\mu} G^c_{\nu} (\partial^\mu G^{*a} - \partial^\nu G^{*a}) \right]$$

where the first term is just like with electromagnetism, the second term describes a four gluon interaction, whereas the third one has a gluon-gluon-photon interaction. This is a fundamental difference between abelian and nonabelian gauge theories; in QED the gauge field (photon) does not have a self-interaction, but $W^n$ and gluon fields have. Next we look into the symmetry of weak interactions, for which the structure is $SU(2)_L \otimes U(1)_Y$. We will also achieve a physical meaning for the $W^n$ fields, when they are combined with a vector gauge field arising from the $U(1)$ symmetry.

### 3.4 $SU(2)_L \otimes U(1)_Y$ symmetry of electroweak theory

The $U(1)$ symmetry has one and $SU(2)$ has three generators, so the product group $SU(2)_L \otimes U(1)_Y$ has four generators, which, as we have noticed above, leads to four gauge fields. These are massive the $W^\pm$ and $Z^0$ bosons and the massless photon, $\gamma$. The group $SU(2)_L \otimes U(1)_Y$ does not mix with $SU(3)_C$ that describes interactions of quarks, so in order to study the interactions of neutrinos there is no need to treat QCD any further. As weak interactions treat left and right-handed particles differently: only left chiral particles interact through weak interactions. This can be put into a mathematical formulation: right-handed chiral fermion fields are singlets under weak interaction gauge group; left-handed leptons act as doublets under an $SU(2)$ symmetry, which leads to their coupling with $W$ bosons.

Let’s then combine the $SU(2)$ and $U(1)$ symmetries studied above. The $U(1)$ symmetry group for weak interactions is marked as $U(1)_Y$. Its generator is the weak
hypercharge $Y$, for which the Gell-Mann-Nishijima relation states \[ Q = T_3 + \frac{Y}{2}, \] (33)

where $Q$ is the charge operator and $T_3$ is the third generator of the $SU(2)_L$ symmetry. The generators $T_i$ satisfy the commutation relations of Eq. (131). The subscript $L$ stands for the chiral nature of weak interactions: the elements of the group act only on left-handed fields; the right-handed fields are singlets under their action.

In order to make the Lagrangian invariant under a local $SU(2)_L \otimes U(1)_Y$ symmetry, as in previous subsections, we need to add three vector fields, $A^a_\mu$, which correspond to each $T_i$, and one, $B_\mu$, for the generator $Y/2$, leading into a covariant derivative of the form

$$ D_\mu \equiv \partial_\mu - ig A_\mu \cdot T - i\frac{Y}{2} g' B_\mu, $$

(34)

where $g$ and $g'$ are the coupling constants for the two groups. We write the left-handed fermion weak isospin doublets as

$$ L'_L = \begin{pmatrix} \nu'_L \\ l'_L \end{pmatrix}, \quad Q'_L = \begin{pmatrix} u'_L \\ d'_L \end{pmatrix}, $$

(35)

where $l = e, \mu, \tau$, $u = u, c, t$, and $d = d, s, b$. The primes on all the fermion fields do not here mean that they are transformed; the fields are marked with a prime with far-sight as in Ref. [84]. The primes are there in anticipation for when we handle the mixing of the definite mass states of fermions. There are no right-handed neutrinos in the Standard Model. The right-handed fields are taken as singlets under weak interactions, i.e. there are

$$ l'_R, \quad u'_R, \quad d'_R. $$

(36)

The choice of the doublets determines the representation of the generators as $T_i = \tau_i/2$, where $\tau_i$ are the Pauli spin matrices. The operator $Y$ acts on the doublets according to Eq. (33):

$$ Y L'_L = (2Q - 2T_3) L'_L = - \begin{pmatrix} \nu'_L \\ l'_L \end{pmatrix}, $$

$$ Y Q'_L = \left( \begin{array}{cc} \frac{2}{3} & 0 \\ 0 & -\frac{1}{3} \end{array} \right) - \left( \begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array} \right) \left( \begin{array}{c} u'_L \\ d'_L \end{array} \right) = \frac{1}{3} \left( \begin{array}{c} u'_L \\ -d'_L \end{array} \right). $$

(37)
For right-handed fields we have (for all right-handed fields \( T_3 f_R = 0 \))

\[
Y l'_R = (2Q - 2T_3) l'_R = -2l'_R.
\]

For quarks we have \( Y u'_R = \frac{2}{3} u'_R, \) \( Y d'_R = -\frac{2}{3} d'_R. \) Let’s write the general \( SU(2)_L \otimes U(1)_Y \) transformation as

\[
U(\Theta, \eta) = e^{i (g T^\mu \Theta + g' Y^\mu \eta)}.
\]

The effect of such a transformation is then for the lepton doublet

\[
L'_L \rightarrow U(\Theta, \eta) L'_L = e^{i (g T^\mu \Theta + g' Y^\mu \eta)} \left( \begin{array}{c}
\nu'_{L,L} \\
\nu'_{L,R}
\end{array} \right) = e^{\sum_{L_R} (\gamma^\mu \eta)_{L,L}} \left( \begin{array}{c}
\nu'_{L,L} \\
\nu'_{L,R}
\end{array} \right),
\]

whereas for the singlet we have

\[
l'_R \rightarrow U(\Theta, \eta) l'\bar{R} = e^{i (g T^\mu \Theta + g' Y^\mu \eta)} l'_R = e^{-i \gamma^\mu \eta} l'_R.
\]

Because left and right-handed fields transform differently, we cannot insert mass terms of the Dirac equation, \( m \tilde{\psi} \psi = m \left( \psi^L \psi_R + \psi^R \psi_L \right). \) They would not be invariant under the transformation \( U \) of weak interactions. Instead, we now have for leptons (for quarks analogously)

\[
\mathcal{L}_D^l = \bar{l} i \gamma^\mu D^l_{\mu} l' = \bar{L'}_L i \gamma^\mu \left( \partial_\mu - ig A^\mu \right) L'_L + \bar{l'}_R i \gamma^\mu \left( \partial_\mu - \frac{i Y}{2} g' B^\mu \right) l'_R
\]

\[
= \bar{L'}_L i \gamma^\mu (\partial_\mu L'_L) + \bar{l'}_R i \gamma^\mu (\partial_\mu l'_R) + \frac{1}{2} \bar{L'}_L \gamma^\mu (g A^\mu \cdot \tau_i - g' B^\mu) L'_L - g' \bar{l'}_R \gamma^\mu B^\mu l'_R,
\]

where the weak hypercharge values were used and where \( l \) goes over all three lepton flavors. The first two terms are the kinetic terms. In order to achieve an expression for the interaction between physical gauge bosons, we need to focus on the last three terms,

\[
\mathcal{L}_D^{l\text{int}} = \frac{1}{2} \left( \bar{\nu}'_{L,L} - \bar{\nu}'_{L,R} \right) \gamma^\mu \left( -g' B^\mu + g A_{\mu,3} + g (A_{\mu,3} - i A_{\mu,2}) \right) \left( \begin{array}{c}
\nu'_{L,L} \\
\nu'_{L,R}
\end{array} \right) - g' \bar{l}'_R \gamma^\mu B^\mu l'_R,
\]

\[
= \frac{1}{2} \left( \bar{\nu}'_{L,L} \gamma^\mu ( -g' B^\mu + g A_{\mu,3}) \nu'_{L,L} + \bar{\nu}'_{L,L} \gamma^\mu g (A_{\mu,3} - i A_{\mu,2}) \nu'_{L,L} + \bar{\nu}'_{L,L} \gamma^\mu ( -g' B^\mu - g A_{\mu,3}) \nu'_{L,L} \right)
\]

\[
+ \bar{\nu}'_{L,L} \gamma^\mu g (A_{\mu,3} + i A_{\mu,2}) \nu'_{L,L} - g' \bar{l}'_R \gamma^\mu B^\mu l'_R,
\]

which can be split into charged and neutral current interactions. The former is

\[
\mathcal{L}_D^{CC} = \frac{g}{2} \left( \bar{\nu}'_{L,L} \gamma^\mu (A_{\mu,1} - i A_{\mu,2}) \nu'_{L,L} + \bar{\nu}'_{L,L} \gamma^\mu (A_{\mu,1} + i A_{\mu,2}) \nu'_{L,L} \right),
\]
where we can define the sums in brackets as the physical charged gauge bosons, \[ W_\mu^\pm \equiv \frac{1}{\sqrt{2}} (A_{\mu,1} \mp i A_{\mu,2}) , \] (39)

which leads to

\[ \mathcal{L}_D^{CC} = \frac{g}{\sqrt{2}} \left( \bar{\nu}_l \gamma^\mu W^+_\nu_l + \bar{\nu}_L \gamma^\mu W^-_\mu \nu_L \right) \]
\[ = \frac{g}{2\sqrt{2}} \bar{\nu}_l \gamma^\mu (1 - \gamma^5) l' W_\mu^+ + \text{h.c.} \]
\[ \equiv \frac{g}{2\sqrt{2}} j_{W,l}^\mu W_\mu^+ + \text{h.c.} , \]

where h.c. denotes Hermitian conjugate and where

\[ j_{W,l}^\mu \equiv \frac{1}{2} \bar{\nu}_l \gamma^\mu (1 - \gamma^5) l' , \] (40)

is the leptonic charged current. Above, the property \((1 - \gamma^5)(1 - \gamma^5) = 2(1 - \gamma^5)\) and anticommutation of \(\gamma^5\) with all the other \(\gamma^\mu\) were used. This generates vertices of lepton-neutrino-\(W\)-boson interactions. For the neutral current interaction we have

\[ \mathcal{L}_D^{NC} = \frac{1}{2} \left( \bar{\nu}_l \gamma^\mu (gA_{\mu,3} - g'B_\mu) \nu_l - \bar{\nu}_L \gamma^\mu (gA_{\mu,3} + g'B_\mu) \nu_L \right) - g' \bar{l}_L \gamma^\mu B_{\mu} l_R , \]

for which we can define as the physical gauge bosons

\[ Z_\mu^0 \equiv \frac{1}{\sqrt{g^2 + g'^2}} (gA_{\mu,3} - g'B_\mu) \]
\[ A_\mu \equiv \frac{1}{\sqrt{g^2 + g'^2}} (g'A_{\mu,3} + gB_\mu) , \]

which leads to

\[ \mathcal{L}_D^{NC} = \frac{1}{2} \left( \sqrt{g^2 + g'^2} \bar{\nu}_l \gamma^\mu Z_\mu^0 \nu_l - \sqrt{g^2 + g'^2} \bar{\nu}_L \gamma^\mu Z_\mu^0 \nu_L - \bar{\nu}_L \gamma^\mu 2g' \frac{1}{\sqrt{g^2 + g'^2}} (gA_\mu - g'Z_\mu^0) \nu_L \right) \\
- g' \bar{l}_L \gamma^\mu 2g' \frac{1}{\sqrt{g^2 + g'^2}} (gA_\mu - g'Z_\mu^0) \nu_L \\
= \frac{1}{\sqrt{g^2 + g'^2}} \left[ \frac{1}{2} \left( g^2 + g'^2 \right) \bar{\nu}_l \gamma^\mu Z_\mu^0 \nu_l - \left( g^2 + g'^2 \right) \bar{\nu}_L \gamma^\mu Z_\mu^0 \nu_L - \bar{\nu}_L \gamma^\mu 2g' (gA_\mu - g'Z_\mu^0) \nu_L \\
- g' \bar{l}_L \gamma^\mu (gA_\mu - g'Z_\mu^0) \nu_L \right] . \]
Let’s also define mixing angles in the \((A_\mu, Z_\mu^0)\)-plane and the electron charge \(e\) in the customary form as \[84\]

\[
\begin{align*}
\cos \theta_W &= \frac{g}{\sqrt{g^2 + g'^2}}, \\
\sin \theta_W &= \frac{g'}{\sqrt{g^2 + g'^2}}, \\
e &= \frac{gg'}{\sqrt{g^2 + g'^2}},
\end{align*}
\]

(43)

where \(\theta_W\) is the weak mixing angle \[85\]. The substitution of them gives

\[
\mathcal{L}_{D,(NC)} = \frac{1}{2\cos \theta_W} \left[ (\bar{\nu}'_L g \cos^2 \theta_w \gamma^\mu \nu'_L + \bar{\nu}'_L g' \cos \theta_w \sin \theta_w \gamma^\mu \nu'_L) \right. \\
+ \left. \cos \theta_w \sin \theta_w g' \bar{\nu}'_L \gamma^\mu \nu'_L + 2\bar{\nu}'_R g' \cos \theta_w \sin \theta_w \gamma^\mu \nu'_R \right] Z_\mu^0 - e \left( \bar{\nu}'_R \gamma^\mu \nu'_R + \bar{\nu}'_L \gamma^\mu \nu'_L \right) A_\mu \\
\equiv \frac{g}{2 \cos \theta_W} J^\mu_{Z_0} Z_\mu^0 + e J^\mu_{A_\mu},
\]

where \(g' \cos \theta_w / g = \sin \theta_w\) was used. Above the corresponding currents were

\[
\begin{align*}
J^\mu_{Z_0} &\equiv \bar{\nu}'_L \gamma^\mu \nu'_L - \left(1 - 2 \sin^2 \theta_w\right) \bar{\nu}'_L \gamma^\mu \nu'_L + 2 \sin^2 \theta_w \bar{\nu}'_R \gamma^\mu \nu'_R \\
&\equiv \bar{\nu}'_L \gamma^\mu \left( g'_V - g'_A \gamma^5 \right) \nu'_L + \bar{\nu}'_R \gamma^\mu \left( g'_V - g'_A \gamma^5 \right) \nu'_R \\
J^\mu_{A_\mu} &\equiv -\bar{\nu}' \gamma^\mu \nu',
\end{align*}
\]

(44)

where the leptonic weak interaction couplings were defined using the conventional V-A structure which describes the chiral nature of the interactions, where the vectorial and axial vectorial parts of the couplings are (for leptons): \(g'_V = \frac{1}{2}\), \(g'_A = \frac{1}{2}\), \(g'_V = -\frac{1}{2} + 2 \sin^2 \theta_w\), \(g'_A = -\frac{1}{2}\) as in \[84\]. Combining all the terms so far, namely the charged and neutral current interactions and the kinetic terms, all stemming from the Dirac Lagrangian, the leptonic part of the weak interaction Lagrangian reads

\[
\mathcal{L}_{\text{no Higgs}} = \sum_{l=e,\mu,\tau} i \bar{L}_L' \gamma^\mu \nu'_L D^\mu L'_L + \sum_{l=e,\mu,\tau} i \bar{\nu}'_R \gamma^\mu \nu'_R - \frac{1}{4} A_{\mu\nu} A^{\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu}.
\]

As one can notice, the mass terms for both gauge bosons and leptons are absent. In other words, the theory as it is describes them massless. As this is true only for photons, the model has to be augmented. The masses are generated through the Higgs mechanism.
3.4.1 The Higgs mechanism

In order to write mass terms for gauge bosons and fermions one adds a complex scalar Higgs doublet

\[ \Phi = \left( \begin{array}{c} \phi^+ \\ \phi^0 \end{array} \right), \]

where the superscripts denote the charge of the fields: ‘+’ for positively charged; ‘0’ for neutral. The weak hypercharge of the doublet is then 1, since

\[ Y_{\Phi} = (2Q - 2T_3) \Phi = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \left( \begin{array}{c} \phi^+ \\ \phi^0 \end{array} \right) = \left( \begin{array}{c} \phi^+ \\ \phi^0 \end{array} \right). \]

The doublet transforms under an \( SU(2)_L \otimes U(1)_Y \) transformation as

\[ \Phi \rightarrow U(\theta, \eta) \Phi = e^{i(g^\tau \cdot \theta + g'^\eta \cdot \eta)} \left( \begin{array}{c} \phi^+ \\ \phi^0 \end{array} \right) = e^{i/2 (g^\tau \cdot \theta + g'^\eta)} \left( \begin{array}{c} \phi^+ \\ \phi^0 \end{array} \right). \]

As the Higgs field is scalar, it is described by the Lagrangian [131]

\[ \mathcal{L}_H = \left( D_\mu \Phi \right)^\dagger \left( D^\mu \Phi \right) - V(\Phi) = \left( D_\mu \Phi \right)^\dagger \left( D^\mu \Phi \right) - \mu^2 \Phi^+ \Phi - \lambda \left( \Phi^\dagger \Phi \right)^2, \]

where the renormalizability of the theory prevents terms of bigger order in \( \Phi \) [131]. As the covariant derivative transforms as \( D'_\mu = U(\theta, \eta) D_\mu U(\theta, \eta)^\dagger \), the scalar field Lagrangian is invariant under the symmetry. The choice of the coefficients \( \mu \) and \( \lambda \) is conventional. In order to introduce symmetry breaking, which requires a nonzero minimum of a bounded potential \( V(\Phi) \), we must have \( \lambda > 0 \) and \( \mu^2 < 0 \). The minimum of the potential is easily achieved, when the potential is written in a form

\[ V(\Phi) = \lambda \left( \Phi^\dagger \Phi + \frac{\mu^2}{2\lambda} \right)^2 \equiv \lambda \left( \Phi^\dagger \Phi - \frac{v^2}{2} \right)^2, \]

where \( v \equiv \sqrt{-\mu^2/\lambda} \) was defined and whence it is easy to see that the minimum of the potential occurs at \( \Phi^\dagger \Phi = v^2/2 \). The constant term is insignificant. The minimum of the potential is the potential of the vacuum, which means that now we have a nonzero vacuum expectation value (VEV). Since vacuum has to be electrically neutral, the charged field \( \phi^+ \) cannot contribute to the VEV. This leaves us with the VEV

\[ \langle \Phi \rangle = \frac{1}{\sqrt{2}} \left( \begin{array}{c} 0 \\ v \end{array} \right). \]
This breaks the $SU(2)_L \otimes U(1)_Y$ symmetry, since the weak hypercharge and weak isospin have a nonzero values for vacuum, which clearly cannot be:

\[
Y \langle \Phi \rangle = (2Q - 2T_3) \langle \Phi \rangle = \left( \begin{array}{cc}
2 & 1 \\
0 & 0
\end{array} \right) - \left( \begin{array}{ccc}
1 & 0 & -1 \\
0 & 0 & 0
\end{array} \right) \frac{1}{\sqrt{2}} \left( \begin{array}{c}
0 \\
v
\end{array} \right) = \langle \Phi \rangle,
\]

\[
T \langle \Phi \rangle = (T_1, T_2, T_3) \langle \Phi \rangle = \frac{1}{2\sqrt{2}} \left( \begin{array}{ccc}
v & 0 & 0 \\
0 & -i v & 0 \\
0 & 0 & -v
\end{array} \right).
\]

Instead, from these we notice that for the electric charge

\[
Q \langle \Phi \rangle = \left( T_3 + \frac{Y}{2} \right) \langle \Phi \rangle = \frac{1}{2\sqrt{2}} \left( \begin{array}{c}
0 \\
-v
\end{array} \right) + \frac{1}{2\sqrt{2}} \left( \begin{array}{c}
0 \\
v
\end{array} \right) = 0,
\]

which states that the electric charge is still a symmetry of the Lagrangian after the symmetry breaking. The remaining symmetry group is now $U(1)_Q$ instead of $SU(2)_L \otimes U(1)_Y$. The symmetry is said to be spontaneously broken, which means that the Lagrangian itself is still symmetric under $SU(2)_L \otimes U(1)_Y$, but the physical states, including the vacuum, are not. There is now only one massless gauge boson left corresponding to the generator of the $U(1)_Q$ symmetry, namely, the photon.

### 3.4.2 Gauge boson masses

The Goldstone theorem that states that for spontaneously broken continuous symmetry massless particles appear. One can write the Higgs doublet as \[84]^{139}

\[
\Phi(x) = \frac{1}{\sqrt{2}} \exp \left( \frac{i \xi(x) \cdot \tau}{v} \right) \left( \begin{array}{c}
0 \\
v + H(x)
\end{array} \right),
\]

where $H(x)$ is the physical Higgs field and $\xi(x)$ are three Goldstone bosons of the theory. Due to the gauge symmetries they can be rephased away with a transformation $U(\theta = -\xi(x)/v)$ of Eq. (38). This choice defines the unitary gauge \[84], where

\[
\Phi(x) = \frac{1}{\sqrt{2}} \left( \begin{array}{c}
0 \\
v + H(x)
\end{array} \right).
\]

The masses of the gauge bosons are found by evaluating the covariant derivative of the Higgs field Lagrangian of Eq. (45) in the unitarity gauge. The covariant derivative for the electroweak theory is given in Eq. (34). By using the gauge boson definitions
of Eqs. (39), (41), and (43) one can write the expression as
\[ D^\mu \Phi = \frac{1}{\sqrt{2}} \left( \frac{i}{\sqrt{2}} g W^\mu (v + H(x)) \right), \]
so that the whole Higgs Lagrangian, using Eq. (46) reads
\[ L_H = -\frac{1}{2} (\partial_\mu H(x))^2 + 2\lambda v^2 H^2(x) + \lambda v H^3(x) + \frac{\lambda}{4} H^4(x) + \frac{g^2 v^2}{4} W^\dagger_\mu W^\mu - \frac{g^2 v^2}{8 \cos^2 \theta_W} Z^\dagger_\mu Z^\mu \]
\[ + \frac{g^2 v}{2} W^\dagger_\mu W^\mu H(x) - \frac{g^2 v}{4 \cos^2 \theta_W} Z^\dagger_\mu Z^\mu H(x) - \frac{g^2 H^2(x)}{8 \cos^2 \theta_W} Z^\dagger_\mu Z^\mu + \frac{g^2}{4} W^\dagger_\mu W^\mu H^2(x), \]
where the constant terms have been omitted. The first term is the kinetic term of the Higgs boson. The second term is the mass term of Higgs particle, giving \( m_H = \sqrt{2\lambda v} = \sqrt{-2\mu^2} \). The mass of the Higgs boson is then separate from other parameters in the SM. The existence of the particle was finally observed in 2012 \[153\]. Then the value was measured to be \( m_H \approx 126 \text{ GeV} \). The third and fourth terms describe the self-couplings of the Higgs boson. The four last terms describe the possible vertices for the Higgs boson. The fifth and sixth terms give the masses of the gauge bosons, i.e.,
\[ m_W = \frac{gv}{2}, \quad m_Z = \frac{gv}{2 \cos \theta_W}, \]
from which one notices that \( m_W \cos \theta_W = m_Z \). From this relation one can write the \( \rho \) parameter,
\[ \rho = \frac{m_W \cos \theta_W}{m_Z}, \]
which for the SM is 1. If there are other scalar particles in addition to the Higgs particle, they also contribute to the gauge boson masses. The current value for the parameter is \( \rho = 0.9998^{+0.0008}_{-0.0005} \[36\] \) which is in a good accordance with the SM.

### 3.4.3 Lepton masses in the Standard Model

The addition of the Higgs field makes it possible to write mass terms also for fermions. It turns out that the fermion mass terms need both left and right chiral fields. As there are only left-handed neutrinos, neutrinos remain massless in the theory. This is why the observation of nonzero neutrino mass was so remarkable; it was a phenomenon that was not possible in the SM. The challenge with the lepton mass terms is that the left and right-handed fields do not transform similarly under the electroweak symmetry transformations. If we contract the VEV of \( \Phi \) with lepton fields, i.e.,
\[ L^\dagger_L \gamma^0 (\Phi) l_R \rightarrow L^\dagger_L e^{-\frac{i}{2}(\sigma_\tau \cdot \theta - \sigma_\eta \cdot \eta)} \gamma^0 e^{\frac{i}{2}(\sigma_\tau \cdot \theta + \sigma_\eta \cdot \eta)} \Phi e^{-ig'\eta} l_R' = L^\dagger_L \gamma^0 \Phi l_R', \]
the term stays invariant under an $SU(2)_L \otimes U(1)_Y$ transformation. This kind of a coupling is called a Yukawa interaction. As the gauge transformation properties are the same for leptons of different generations, there is no reason why the left-handed doublet $L'_L$ should be of the same generation as the singlet $l'_R$.

So the lepton mass terms in the Lagrangian are

$$\mathcal{L}_{\text{higgs}}^l = - \sum_{f_1, f_2 = e, \mu, \tau} Y_{f_1 f_2}^{l} L_{f_1}^\dagger L_{f_2} \frac{\gamma_0}{\sqrt{2}} f_{f_2, 0} + \text{h.c.}$$

If one wants to write a mass term for physical leptons, one should have terms where the doublet and singlet are both of the same generation. We can write the equation in matrix form,

$$\mathcal{L}_{\text{higgs}}^l = - \frac{1}{\sqrt{2}} (v + H) \bar{l}^L Y'^{l} v^R + \text{h.c.},$$

where $\bar{l}^L = (\bar{e}^L, \bar{\mu}^L, \bar{\tau}^L)$, $Y = (e^R, \mu^R, \tau^R)^T$, and $Y'^{l}$ is a $3 \times 3$ Yukawa coupling matrix, which can be diagonalized so that

$$U'^{l}_{L} Y'^{l} U_{L} = Y' = \text{diag}(Y_{ee}, Y_{\mu\mu}, Y_{\tau\tau}),$$

where $U_{R}, U_{L}$ are unitary matrices which transform the definite lepton mass states into the mixed states with primes, so that

$$U'^{l}_{R} Y'^{l} U^{l}_{R} = Y' = \text{diag}(Y_{ee}, Y_{\mu\mu}, Y_{\tau\tau}),$$

As neutrino fields are massless, we were able to transform them using the charged lepton transformation matrix. Now we have

$$\mathcal{L}_{\text{higgs}}^l = - \sum_{f_1 = e, \mu, \tau} m^l_{f_1} \bar{l}_{f_1} L_{f_1} \gamma_0 l_{f_1, 0} + \sum_{f_1 = e, \mu, \tau} m^l_{f_1} \frac{v}{\sqrt{2}} \bar{l}_{f_1} L_{f_1} l_{f_1, 0} + \text{h.c.},$$

where we defined the lepton mass,

$$m^l_{f_1} \equiv \frac{v Y^l_{f_1}}{\sqrt{2}}.$$
the definite mass fields,

\[ j_{W,l}^\mu = 2 \bar{\nu}_{l,L} U_L \gamma^\mu U_L^\dagger l_L = 2 \bar{\nu}_{l,L} \gamma^\mu l_L, \]

where \( U_L^\dagger \nu_{l,L} = \nu_{l,L} \) which states that there is no mixing of lepton mass states in the case of massless neutrinos.

Collecting all the terms the full Lagrangian for weak interaction then finally reads

\[
\mathcal{L}_{\text{weak}} = \sum_{l=e,\mu,\tau} i \bar{l}_L' \gamma_\mu D_\mu L_L' + \sum_{l=e,\mu,\tau} i \bar{l}_R' \gamma_\mu D_\mu l_R' - \frac{1}{4} A_{\mu\nu} A^{\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} \\
+ (D_\mu \Phi)^\dagger (D_\mu \Phi) - \mu^2 \Phi^\dagger \Phi - \lambda (\Phi^\dagger \Phi)^2 - \sum_{l=e,\mu,\tau} Y e (\bar{l}_L' \Phi l_R' + \bar{l}_R' \Phi^\dagger L'_L),
\]

where only leptonic terms are included.
4 Massive neutrinos

The simplest possibility to produce neutrino masses is through the Yukawa interaction just like for other fermions. For Dirac neutrinos, this requires the existence of right-handed neutrinos, but such particles have not been observed. However, this is not the only way to introduce mass for neutrinos. Since neutrinos are electrically neutral, they can also be Majorana particles. For Majorana neutrinos the Yukawa interaction is possible without right-handed neutrinos, but the minimal Majorana Yukawa interaction is not renormalizable. In this section the general mass matrix for left- and right-handed Dirac and Majorana neutrinos is derived.

4.1 Dirac mass

The simplest way of extending the SM is to add three sterile neutrinos, one for each generation, which are singlets under the SM gauge group (and their weak hypercharge is 0). It is conventional to call them right-handed in order to avoid misunderstandings, though one could as well use their left chiral representation, $\nu_{1,R}^C = \nu_{1,L}$ \[84\]. In order to have a mass term invariant under the electroweak symmetry, one also needs a Higgs doublet with hypercharge -1. This is achieved by introducing

$$\tilde{\Phi} = i\tau_2 \Phi^* = i\tau_2 \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & v + H \\ v + H & 0 \end{pmatrix}^* = \frac{1}{\sqrt{2}} \begin{pmatrix} v + H \\ 0 \end{pmatrix}. \quad (51)$$

Using this expression we can write gauge invariant Yukawa mass terms for neutrinos, since

$$\mathcal{L}_{\text{mass}}^\nu = \bar{L}_L i\tau_2 \langle \Phi \rangle^* \nu_R^I \quad (52)$$

stays invariant under the transformations of Eq. (38). The complete leptonic Yukawa mass terms are then

$$\mathcal{L}_{\text{mass}}^{\nu + l} = \sum_{f_1, f_2 = e, \mu, \tau} -Y_{f_1 f_2}^\nu \bar{L}_f \tilde{\Phi}_f \nu_{f_2,R}^I + \mathcal{L}_{\text{mass}}^{l} = -\frac{(v + H)}{\sqrt{2}} \left( \bar{\nu}_L Y^\nu \nu_R + \bar{\nu}_L Y^\nu Y_R \right) + \text{h.c.},$$

where the diagonalized Yukawa coupling matrix is

$$U_L^{\nu\nu} Y^\nu U_R^{\nu\nu} = Y^\nu = \text{diag}(Y_{11}, Y_{22}, Y_{33}),$$
and the definite neutrino mass states are

$$U_{R}^{\nu} \nu_{R} \equiv n_{R} = \left( \begin{array}{c} \nu_{1,R} \\ \nu_{2,R} \\ \nu_{3,R} \end{array} \right), \quad U_{L}^{\nu} \nu_{L} \equiv n_{L} = \left( \begin{array}{c} \nu_{1,L} \\ \nu_{2,L} \\ \nu_{3,L} \end{array} \right).$$  \hspace{1cm} (53)$$

Using these expressions the neutrino mass is generated analogously to the charged lepton mass, and it is

$$m_{i}^{\nu} \equiv \frac{vY_{i}^{\nu}}{\sqrt{2}}, \quad i = 1, 2, 3.$$  \hspace{1cm} (54)

In this expression the important problem with neutrino masses that emerge by just adding right-handed neutrinos is visible: the only difference between neutrino masses and charged lepton masses is the Yukawa coupling constant. The observed neutrino masses are of order $< 0.23 \text{eV}$ [8], whereas the charged lepton masses are $m_{e} \simeq 0.5 \text{MeV}, m_{\mu} \simeq 105, 7, \text{MeV}$, and $m_{\tau} \simeq 1777, 8 \text{MeV}$ [36]. The above expression does not give any insight on why neutrinos are so light; in other words, why the Yukawa couplings are so weak for neutrinos compared to the couplings of charged leptons.

### 4.1.1 The PMNS matrix for Dirac neutrinos

Within the SM, the leptonic charged current remained invariant when we changed to the mass definite basis. With massive neutrinos the situation is different, since left-handed neutrinos and charged leptons have different transformation matrices,

$$j_{W}^{\mu} = 2 \bar{\nu}_{L}^{\gamma_{\mu}} \gamma_{\mu}^{\nu} = 2 \bar{n}_{L} U_{PMNS}^{\dagger} \gamma_{\mu}^{\nu} = \sum_{f_{1}, f_{2} = e, \mu, \tau} 2 \bar{\nu}_{L,f_{1}} \gamma_{\mu}^{\nu} \nu_{L,f_{2}},$$  \hspace{1cm} (54)

where we defined the neutrino mixing matrix $U_{PMNS} \equiv U_{L}^{\dagger} U_{L}^{\nu}$, i.e.,

$$\nu_{L} \equiv U_{PMNS} n_{L} = \left( \begin{array}{ccc} U_{e,1} & U_{e,2} & U_{e,3} \\ U_{\mu,1} & U_{\mu,2} & U_{\mu,3} \\ U_{\tau,1} & U_{\tau,2} & U_{\tau,3} \end{array} \right) \left( \begin{array}{c} \nu_{1,L} \\ \nu_{2,L} \\ \nu_{3,L} \end{array} \right) = \sum_{i=1}^{3} \left( \begin{array}{c} U_{e,i} \nu_{i,L} \\ U_{\mu,i} \nu_{i,L} \\ U_{\tau,i} \nu_{i,L} \end{array} \right).$$  \hspace{1cm} (55)

The matrix $U_{PMNS}$ is known as the Pontecorvo–Maki–Nakagawa–Sakata (PMNS) matrix [121]. In the SM there is an analogous mixing matrix for quarks, which is called the Cabibbo-Kobayashi-Maskawa (CKM) matrix [49,111]. From Eq. (55) it is clear that it is not reasonable to speak about masses of neutrino flavor fields, since they are linear combinations of the three different mass states. It is also worth noting that the mixing occurs only between left-handed neutrinos. As the added three sterile right-handed neutrinos are SM singlets with a zero weak hypercharge, they do not
appear in the weak currents, but only in the Yukawa mass terms. Therefore there is no need to have a mixing matrix for them\cite{84}.

In the SM each particle has a lepton number $L = 1$ and all antiparticles have $L = -1$. One can classify lepton numbers further, so that every generation has its own lepton number: $L_e, L_\mu, L_\tau$. In the SM all the three lepton numbers are conserved, which is a consequence of invariance of the Lagrangian under a global $U(1)$ transformation of the form $\exp(i\phi_f)$, where $\phi_f$ stands for a different phase for each flavor.

The Lagrangian is thus invariant under a global transformation $\exp(i\phi)$, which acts on the mass fields, $l_L, l_R, \nu_{e,L}, \nu_{e,R}$. This enables the rephasing of the PMNS matrix phases. Generally, a unitary $N \times N$ matrix has $N(N+1)/2$ phases and $N(N-1)/2$ mixing angles. So, with three generations, there are six phases and three mixing angles. However, only one phase is physical and the others can be eliminated\cite{131}. Apart from the weak charged current part of the Lagrangian, the Lagrangian is invariant under the aforementioned global phase transformations $\exp(i\phi_f/i\phi)$, where there are three $\phi_f$ phases corresponding to the transformations of the charged fields, and one phase $\phi_i$ to each neutrino mass field.

The PMNS matrix arising from the addition of Dirac-type sterile neutrino fields can be parameterized as in\cite{36}

\[ U_{\text{PMNS}} = V K, \]  

where

\[ V = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}, \]

where $c_{ij} \equiv \cos \theta_{ij}$, $s_{ij} \equiv \sin \theta_{ij}$, and $\delta$ is the Dirac type CP violating phase that cannot be rephased away. The matrix $K$ becomes important for Majorana neutrinos. Here, as always for Dirac particles $K$ is just a unit matrix.

The current experimental values are listed in a later subsection. We will show that if the mixing matrix is real, there cannot be CP violation among neutrinos. This sounds reasonable: if the remaining physical phase $\delta \neq n\pi$, the weak charge current is not invariant under a CP transformation, as we will notice later on.
4.2 Majorana mass

The fact that only left-handed neutrinos have been observed raises the question of whether the observed neutrinos could be Majorana particles, satisfying the Majorana equation, Eq. (16). In Sec. 2.4 we noticed that a Majorana mass term is of the form

\[ \mathcal{L}_{\text{mass}}^{\text{Maj}} = \frac{m_{\text{Maj}}}{2} (\nu_L^T C^\dagger \nu_L + \text{h.c.}), \]  

(58)

where the notation is such that the field \( \chi \) has been replaced by \( \nu_L \) and the matrix \( i\sigma_2 \) by \( C \). The term is invariant under Lorentz transformations. Unlike in the case of the Dirac neutrino mass term, this form of a mass term is invariant under weak interactions, since the fields in Eq. (58) do not transform differently under weak interactions. Now there emerges a different kind of a problem with quantum numbers: the term in Eq. (58) has two left-handed fields, which means that the total Majorana mass term has \( Y = -2 \) and \( T_3 = 1 \). We would need a weak isospin triplet with weak hypercharge \( Y = -2 \) in order to make the corresponding Lagrangian term to have \( Y = 0 \). However, there is no such entity in the SM, so the Majorana mass term has to be built in a more complicated way.

It turns out that the lowest dimensional term possible to generate a Majorana mass term using the particles of the SM is the Weinberg operator \([143]\), which can be realized as \([84]\)

\[ \mathcal{L}_{\text{eff}}^{\text{Maj}} = \frac{1}{\mathcal{M}} \sum_{f_1, f_2 = e, \mu, \tau} g_{f_1 f_2} (L^T_{f_1 L} \tau_2 \Phi) C^\dagger (\Phi^T \tau_2 L_{f_2 L}) + \text{h.c.}, \]  

(59)

where \( g \) is a \( 3 \times 3 \) matrix of dimensionless coupling constants and \( \mathcal{M} \) is the scale of new physics having dimension of mass. The term in total has a mass dimension of five. This is problematic since only terms with mass dimension up to four are renormalizable \([131]\). After the symmetry breaking the term produces the Majorana wanted masses \([84]\).

\[ \mathcal{L}_{\text{mass}}^{\text{Maj}} = \frac{1}{2 \mathcal{M}} \sum_{f_1, f_2} g_{f_1 f_2} \nu_{f_1 L}^T \nu_{f_2 L}^\dagger C^\dagger \nu_{f_2 L}^\dagger + \text{h.c.}, \]  

(60)

where the neutrino mass is then

\[ (M_{\text{Maj}})_{f_1 f_2} = \frac{v^2}{\mathcal{M}} g_{f_1 f_2}. \]  

(61)

As Eq. (59) is of energy dimension five, it is not acceptable for the SM. This is not considered as such a bad setback, as it is generally believed that the SM is not the...
final theory, but just an effective theory, applicable at low energies. The tree-level realizations of the Weinberg operator correspond to the basic seesaw mechanisms \[57\]. In seesaw models the new heavy particles leave the effective dimension 5 operator after they are integrated out.

The higher the mass dimension, the more the corresponding terms are suppressed \[84\]. The dimension five term above, being of the lowest non-renormalizable dimension, is then the most accessible way to observe effects beyond the SM, and this is why the Majorana masses are so intensively studied.

### 4.2.1 The PMNS matrix for Majorana neutrinos

The mixing between definite mass states is possible for Majorana neutrinos as well. The mass term $\nu'^T L C \nu'_L$ can be expressed in mass states in the same way as the Dirac mass term, by using $\nu'_L = (\nu'_{eL}, \nu'_{\mu L}, \nu'_{\tau L})$ and $(U^{\nu,Maj}_L)^\dagger \nu'_L \equiv n_L = (\nu_{1L}, \nu_{2L}, \nu_{3L})$, which gives

$$L^{\text{mass}}_\text{Maj} = \frac{1}{2} \left( n^T_L \left( U^{\nu,Maj}_L \right)^T C^\dagger \tilde{M}^{\nu,Maj}_L U^{\nu,Maj}_L n_L + \text{h.c.} \right),$$

where the mass matrix is diagonalized,

$$\left( U^{\nu,Maj}_L \right)^T C^\dagger \tilde{M}^{\nu,Maj}_L U^{\nu,Maj}_L = M^{\text{Maj}}_{L,\text{diag}} = \text{diag} \left( m^{\text{Maj}}_1, m^{\text{Maj}}_2, m^{\text{Maj}}_3 \right).$$

From the anticommutation property of fermion fields combined with $C^T = -C$, it follows that the Majorana mass matrix $\tilde{M}^{\text{Maj}}_L$ has to be symmetric \[47\].

The charged lepton current becomes

$$j_{\mu \text{W,L,Maj}} = 2 \tilde{\nu}'_L \gamma^\mu Y_L = 2 \tilde{n}_L \left( U^{\nu,Maj}_L \right)^\dagger \gamma^\mu U^{\nu,Maj}_L \gamma^\mu 1_L \equiv 2 \tilde{n}_L \left( U^{\nu,\text{PMNS}}_L \right)^\dagger \gamma^\mu 1_L.$$

As Majorana neutrinos are massive, helicity is no longer a stable property. There are Majorana neutrinos with positive and negative helicity. It turns out that $j_{\mu \text{W,L,Maj}}^\mu$ creates mainly relativistic Majorana neutrinos with negative helicity; positive helicity neutrinos are strongly suppressed \[47\]. The hermitian conjugate $j_{\mu \text{W,L,Maj}}^\mu$ in turn creates mostly relativistic neutrinos with positive helicity \[84\]. Comparison to the Dirac neutrino type explains why it is common to call Majorana neutrinos with negative helicity as neutrinos, and particles with positive helicity as antineutrinos, although they are not antiparticles in the same sense as Dirac antiparticles.

As there are no antileptons present, Majorana type neutrinos violate the lepton number conservation by two units. This is possible, since the lepton number conservation is in the SM an accidental symmetry, i.e., there are no physical reasons for
it. The nonconservation of the lepton number means that Majorana neutrino fields are not invariant under a global $U(1)$ transformation. This can be seen in the mass term: given a transformation $\exp(i\phi_f)$, where $\phi_f$ is different for each flavor, we get

$$\mathcal{L}_{M,W} = \frac{m_M}{2} (\nu_L^T C \nu_L + \text{h.c.}) \rightarrow \frac{m_M}{2} (e^{2i\phi_f} \nu_L^T C \nu_L + \text{h.c.}).$$  \hspace{1cm} (62)

As stated above, as neutrinos are relativistic particles (neutrino masses are very small) we can approximate negative and positive helicity neutrinos to be effectively neutrinos and antineutrinos, so that neutrinos are considered massless left-chiral Weyl spinors and antineutrinos right-handed Weyl spinors. Using these, the effective charged current terms are invariant under the phase transformation, i.e., the effective lepton number is conserved. However, as can be seen from Eq. (62), this approximation does not help the mass terms: they break also the effective lepton number conservation. This is why the Majorana mass term can be viewed as a perturbation of the massless Lagrangian \cite{84}. There are experiments trying to detect signs of this kind of behavior, the neutrinoless double-beta decay being the most famous of them \cite{136}.

Due to the non-invariance under the rephasing of Eq. (62), the PMNS matrix for Majorana particles differs from the mixing matrix of Dirac neutrinos. The left-handed chiral neutrino fields cannot be rephased due to lepton number nonconservation. Therefore there are two more physical phases than for Dirac neutrinos. It is customary to write the PMNS matrix as in Eq. (55), so that $V$ is the same as for Dirac particles, but

$$K = \text{diag}\left(1, e^{i\Phi_1/2}, e^{i\Phi_2/2}\right),$$  \hspace{1cm} (63)

in which the phases $\Phi_i$ can lead into Majorana type CP violation. In the notation of Eq. (63) Majorana type CP violation then requires $\Phi_1, \Phi_2 \neq (-1)^n \pi$. If neutrinos are Dirac type particles, the Majorana phases vanish, whereas in the case of Majorana neutrinos the Dirac CP violating phase $\delta$ does not necessarily vanish; a Majorana neutrino can also have CP violation of the Dirac type \cite{42}.

### 4.3 The general Dirac-Majorana mass term

Let’s form the most general expression for the neutrino mass matrix. If neutrinos are Dirac particles, we need the sterile right-handed neutrinos, as was shown before. But the added right-handed neutrinos do not have to be Dirac neutrinos, they could as well be Majorana neutrinos. On the other hand, it is possible that the left-handed observed neutrinos are Majorana particles. Also, it is possible that there are right and left-handed neutrinos that are all Majorana particles, but which together form a
Dirac mass term. All these possibilities can be written in a compact form,

\[ L_{\text{D}+\text{M}} = L_{\text{mass},L} + L_{\nu,\text{mass}} + L_{\text{mass},R} \]  

(64)

\[
\begin{aligned}
&= \frac{1}{2} \nu_L^T C^\dagger M_{\text{Maj}}^L \nu'_L - \frac{1}{2} \nu_R^T m_{\nu} \nu'_L - \frac{1}{2} \nu_R^T m_{\nu}^T \nu'_L + \frac{1}{2} \nu_R^T C M_{\text{Maj}}^R \nu' + \text{h.c.} \\
&= \frac{1}{2} \left( \nu_L^T \left( \nu_R^C \right)^T \right) C^\dagger \left( \begin{array}{cc} M_{\text{Maj}}^L & m_{\nu}^T \\ m_{\nu} & M_{\text{Maj}}^R \end{array} \right) \left( \begin{array}{c} \nu'_L \\ \nu'^C \end{array} \right) + \text{h.c.} \\
&= \frac{1}{2} N_{\nu}^T C^\dagger \left( \begin{array}{cc} M_{\text{Maj}}^L & m_{\nu}^T \\ m_{\nu} & M_{\text{Maj}}^R \end{array} \right) N'_L + \text{h.c.},
\end{aligned}
\]  

(65)

(66)

(67)

where properties \( \nu_{L/R}^C = C \nu_{L/R}^T \) and \( C^T = -C \) were used. Above, the vectors \( \nu'_L \) and \( \nu'_R \) are \( \nu'_L \equiv (\nu'_{eL}, \nu'_{\mu L}, \nu'_{\tau L}) \) and \( \nu'_R \equiv (\nu'_{C1 R}, \ldots, \nu'_{C N_s R}) \), where \( N_s \) is the number of sterile neutrinos. There is no reason why the number of right-handed neutrinos should be the same as for the left-handed observed neutrinos. However, for simplicity we study the situation for \( N_s = 3 \). Note that there are no primes on right chiral neutrino fields, since they are sterile and they do not take part in weak interactions. In the mass equation there is a symmetric mass matrix,

\[ M_{\text{D}+\text{M}} = \left( \begin{array}{cc} M_{\text{Maj}}^L & m_{\nu}^T \\ m_{\nu} & M_{\text{Maj}}^R \end{array} \right), \]  

(68)

in which \( M_{\text{Maj}}^L \) is a \( 3 \times 3 \) and \( M_{\text{Maj}}^R \) is a \( 3 \times 3 \) matrix. It then follows that \( m_{\nu} \) is a \( 3 \times 3 \) matrix. All three matrices are complex. The Majorana mass matrices are symmetric, as noted before. The matrix \( M_{\text{D}+\text{M}} \) is non-diagonal, and hence it does not give definite masses for the fields. As a symmetric matrix it can be diagonalized by a unitary matrix \( U^M \) so that

\[ (U^M)^T M U^M = M^{\text{diag}} = \text{diag}(m_{\nu 1}, \ldots, m_{\nu 6}), \]  

(69)

where submatrices of \( U^M \) are real, diagonal, and non-negative \([45]\). The interacting neutrino fields can be written in terms of the definite mass states \( n_L \), i.e.,

\[ N'_L = \left( \begin{array}{c} \nu'_L \\ \nu'_R^C \end{array} \right) = U^M n_L = U^M \left( \begin{array}{c} \nu_{1L} \\ \vdots \\ \nu_{6L} \end{array} \right) = \left( \begin{array}{c} U^M_{11} \nu_{1L} + \ldots + U^M_{16} \nu_{NL} \\ \vdots \\ U^M_{21} \nu_{1L} + \ldots + U^M_{26} \nu_{NL} \end{array} \right), \]  

(70)

36
so that the mass term becomes

\[ \mathcal{L}_{\nu; \text{mass}}^{D+M} = \frac{1}{2} \sum_{i=1}^{6} m_\nu^i n_L^i C^i n_L + \text{h.c.,} \]

which has the structure of a Majorana mass term. This means that there are 6 massive Majorana fields that together produce mass terms corresponding to the combination of Dirac and Majorana mass terms. From Eq. (70) we see that in this form of the mass term, in addition to the observed oscillation between the observed flavor states, the interacting neutrino fields \( \nu'_L \) and the sterile fields \( \nu^C_R \) can oscillate into each others, because they consist of the same definite mass fields.

### 4.4 Experimental status of neutrino mass parameters

The neutrino mass parameters are extremely difficult to determine because neutrinos interact only through weak interactions and they are very light. The experiments studying neutrino oscillations can only determine the squared mass differences between neutrinos, i.e., they do not determine whether neutrinos are Dirac or Majorana particles [39]. The current neutrino oscillation data require there to be at least two massive neutrinos.

By studying the energy of an electron in the tritium \( \beta \)-decay \( ^3\text{H} \rightarrow ^3\text{He} + e^- + \bar{\nu}_e \) one can determine the absolute mass of an electron neutrino. There have been two major experiments using this method: in Troitzk [115] and in Mainz [113]. A new, more accurate experiment, KATRIN, with sensitivity 0.2 eV, is under construction [67]. The current, most stringent upper bounds obtained in these experiments are

\[ m_\nu < 2.5 \text{ eV, \ with } 95\% \text{ CL.} \]

The most recent cosmological bounds are from the Planck satellite, [8]

\[ m_\nu < 2.3 \text{ eV.} \]

As stated before, the Majorana mass terms violate the lepton charge conservation. This is possible, since although within the SM the lepton number is conserved, there is no physical requirement for the conservation. The Majorana nature of neutrinos could manifest itself in processes that violate lepton number conservation by two units. The most established test is the neutrinoless double-beta decay [136], proposed for the first time already in 1939 [78]. The interest in this experiment comes from the
fact that it can be used not only to determine the Majorana nature of neutrinos, but also to measure the Majorana phases in Eq. 56 and the scale of neutrino masses [36].

As stated before, quarks have a CKM mixing matrix, which is analogous to the PMNS matrix of leptons. However, there are important differences between the cases of quarks and leptons. First, it is still unknown why neutrinos are so light; second, the mixing parameters are different. For quarks the current values of CKM matrix are [36]

$$|V_{CKM}| \simeq \begin{pmatrix} 0.97 & 0.23 & 0.0035 \\ 0.23 & 0.97 & 0.041 \\ 0.0087 & 0.04 & 1.0 \end{pmatrix}, \quad (71)$$

so that the CKM matrix is close to the unit matrix. The elements of the CKM matrix are determined by tree-level processes but also by studying flavor changing neutral currents [41]. The current experimental values are [16, 36]

$$\sin^2(2\theta_{13}) = 0.089 \pm 0.010\text{(stat.)} \pm 0.005\text{(syst.)},$$

$$\sin^2(\theta_{23}) = 0.42^{+0.08}_{-0.03},$$

$$\sin^2(\theta_{12}) = 0.306^{+0.018}_{-0.015},$$

$$\Delta m^2_{21} \equiv \Delta m^2_{\text{sol}} = 7.58^{+0.22}_{-0.26} \times 10^{-5} \text{eV}^2,$$

$$\Delta m^2_{31} \equiv \Delta m^2_{\text{A}} = 2.35^{+0.12}_{-0.09} \times 10^{-3} \text{eV}^2,$$

where the value of $\theta_{13}$ is at $7.7\sigma$, all the others are determined at a 1$\sigma$ statistical significance, and where $\Delta m^2_{\text{sol}}$ refers to the mass difference observed from oscillations of solar neutrinos, and $\Delta m^2_{\text{A}}$ to that of atmospheric neutrinos. There are no data about the magnitudes of the CP violating Dirac or Majorana phases. Together, the numerical values for the PMNS matrix are [86]

$$|V_{PMNS}| = \begin{pmatrix} 0.81 & 0.56 & < 0.22 \\ 0.39 & 0.59 & 0.68 \\ 0.38 & 0.55 & 0.70 \end{pmatrix},$$

so that the structure of the PMNS matrix is not even near the unit matrix; its structure is very different from the CKM matrix. The reason for this is still unknown. In degrees the current values for the mixing angles are [5]

$$\theta_{12} \simeq 34^\circ, \quad \theta_{23} \simeq 45^\circ, \quad \theta_{13} \simeq 9^\circ.$$
5 Neutrino mass mechanisms

In this section the mechanisms which lead to neutrino mass terms described in the previous section are reviewed. The basic idea behind all the mechanisms is the same: we introduce new particles that couple left-handed neutrinos to the Higgs field. Often the new particles theorized are so heavy that they are out of existing detectors’ reach, but mediators of the so-called seesaw mechanisms can be within the energy scale of the LHC. The neutrino mass generating mechanisms predict mass matrix parameters and patterns that can determine or exclude leptonic CP violation.

Listing all the different models is out of the scope of this work. Some extensions add new particles within the SM symmetry group. These include the seesaw models, in which one adds extra fermions or scalars to the theory. The tree-level seesaw models are reviewed in the first subsection.

One can also augment the symmetry group of the SM either by a discrete or a continuous symmetry. These are fundamentally different situations: for a continuous symmetry, in the same way as in Sec.3 with $U(1)$, $SU(2)$, and $SU(3)$ symmetries, gauge fields appear when the symmetry is made local; for discrete symmetries this is not needed. As an example of discrete symmetry extensions the Altarelli-Feruglio model is described.

In order to show how extensions of the gauge groups can include the seesaw mechanisms at low energies, one higher energy model corresponding to each tree-level seesaw model is introduced: The minimal left-right symmetric model (that includes the type I seesaw model), the Littlest Higgs model (type II seesaw), and an $SU(5)$ with an adjoint fermion (type III seesaw).

5.1 Seesaw mechanisms

There are three tree-level realizations of generating the dimension 5 operator of Eq. (59) that generates the neutrino masses [11]. These are called seesaw mechanisms (see, e.g., [10, 15, 57, 138] for details). Two of these are achieved by extending the fermion sector. By denoting the left-handed lepton isospin doublet $L_L$ with $Y = -1$ (from Eq. (37)) as $(2, -1)$ and the $Y = 1$ Higgs isospin doublet as $(2, 1)$ makes it possible to write [84]

$$\mathcal{L} \propto \bar{L}_L \tilde{\Phi} \sim (2, 1) \otimes (2, -1) = (3, 0) \oplus (1, 0),$$

where $\tilde{\Phi}$ was introduced in Eq. (51) and where the form of the equation was implied by Eq. (52). Therefore, in order to have a mass term that stays invariant under an
SU(2)_L \otimes U(1)_Y$ transformation, and knowing that for $SU(2)$ $3 \otimes 3 = 5 \oplus 3 \oplus 1$, one has two options: either to add a fermion singlet $(1,0)$ or a fermion isospin triplet with $Y = 0$, i.e., $(3,0)$. These are called type I and III seesaws, respectively.

One can also extend the scalar sector. As no right-handed neutrinos are added, the left-handed neutrinos have to be Majorana particles. The total mass term stays invariant under the SM symmetries if one adds a triplet with $Y = 2$ [125]. The extension of a scalar triplet with $Y = 2$ is called the type II seesaw. The three tree-level seesaw mechanisms are pictured in Fig.1. There are also other more complex seesaw schemes that are not studied here in detail. These include inverse and linear seesaws [96,126].

![Figure 1: Diagrams of the three most common seesaw mechanisms: the type I seesaw with a heavy Majorana neutrino $N$ (left); the type II seesaw is mediated by a heavy scalar triplet $\Delta$ (middle); in the type III seesaw a fermion triplet $\Sigma$ is added to the SM particle content. Figure from Ref. [11].](image)

The name seesaw comes from inserting very massive particles as a counterweight to the light observed neutrinos. The mass of the observed light neutrinos is then suppressed in the same way as in Eq. (61), where the Majorana mass is suppressed by $M$. The theoretical motivation for the seesaw mechanisms is that they offer a natural explanation for why the scale of neutrino masses is many orders of magnitude smaller than that of other fermions.

5.1.1 Seesaw I and III

Consider first a single right-handed sterile neutrino added to the SM with only one generation. In this case the eigenvalues of the mass matrix in Eq. (68) are

$$m_\pm = \frac{1}{2} \left[ m_{R}^{\text{Maj}} + m_{L}^{\text{Maj}} \pm \sqrt{ (m_{R}^{\text{Maj}} - m_{L}^{\text{Maj}})^2 + 4m_D^2} \right],$$

40
In the case of $m^{\text{Maj}}_L = 0$ and $m^{\text{Maj}}_R \gg m_D$, the eigenvalues are

$$m_+ = m^{\text{Maj}}_R + \frac{m_D^2}{m^{\text{Maj}}_R} \approx m^{\text{Maj}}_R$$

$$m_- = \frac{1}{2} \left[ m^{\text{Maj}}_R - m^{\text{Maj}}_R \left( 1 + \frac{4m_D^2}{2m^{\text{Maj}}_R} \right) \right] \approx m^{\text{Maj}}_R.$$ 

This is called the seesaw mechanism, because it gives a reasonable explanation for the small masses of neutrinos; it predicts one light and one heavy neutrino.

**Seesaw I**

For three generations the process is not as straightforward. If we set $M^{\text{Maj}}_L = 0$, and approximate that elements of $M^{\text{Maj}}_R$ are much larger than $m^\nu$, we can diagonalize the mass matrix by blocks, i.e., \(104,138\)

$$U^T_M M U_M \simeq \text{diag}(m^\nu, M^{\text{Maj}}_R). \quad (72)$$

If we divide the matrix to four parts as in \(43\), i.e.,

$$U_M = \begin{pmatrix} V_L & V_{LR} \\ V_{RL} & V_R \end{pmatrix},$$

we get from Eq. (72) a set of equations,

$$V^\dagger_L m^\nu V^*_L + V^\dagger_{RL} m^\nu V^*_L + V^\dagger_{RL} M^{\text{Maj}}_R V^*_RL = m^\nu$$

$$V^\dagger_{LR} m^\nu V^*_RL + V^\dagger_R m^\nu V^*_R + V^\dagger_R M^{\text{Maj}}_R V^*_R = 0$$

$$V^\dagger_{LR} m^\nu V^*_RL + V^\dagger_R m^\nu V^*_R + V^\dagger_R M^{\text{Maj}}_R V^*_R = M^\nu,$$

whence we get, after straight-forward manipulation using $M_D \ll M_R$ and the fact that $M_R$ is non-singular that the effective light neutrino mass is

$$m_{\nu,\text{effective}} = -m^\nu \frac{1}{M^{\text{Maj}}_R} m^{\nu T}_R.$$ 

By writing the Dirac mass term as $m^\nu = \frac{Y^\nu v}{\sqrt{2}}$, we find

$$m_{\nu,\text{effective}} = -\frac{v^2}{2} Y^\nu \frac{1}{M^{\text{Maj}}_R} Y^{\nu T}.$$ 

The effective mass is suppressed by the large mass of right-handed neutrinos.
Seesaw III

Neutrino masses can also be generated by extending the fermion sector with three SU(2) leptonic triplet $\Sigma_i$ with weak hypercharge $Y = 0$, where the components are Weyl spinors $\{4, 13, 72, 114\}$. The triplets can be represented as

$$\Sigma_i = \begin{pmatrix} \Sigma_i^0 \\ \sqrt{2} \Sigma_i^- + \Sigma_i^0 \\ \sqrt{2} \Sigma_i^- - \Sigma_i^0 \end{pmatrix},$$

where $\Sigma_i^0 = \Sigma_i^3$, $\Sigma_i^+_i = 1/\sqrt{2} (\Sigma_i^1 - i \Sigma_i^2)$, and $\Sigma_i^- = 1/\sqrt{2} (\Sigma_i^1 + i \Sigma_i^2)$. The calculation of the neutrino mass term is analogous to that in the seesaw I. Since the Higgs sector is not extended, $M_{M,L} = 0$ in Eq. (68). The physical particles of $\Sigma_i$ are combinations of $\Sigma_i^0$, $\Sigma_i^+$, and $\Sigma_i^-$, so that there are charged Dirac fermions $l_{\Sigma,i}$ and Majorana neutrinos $\nu_{\Sigma,i}$

$$l_{\Sigma,i} = \Sigma_i^- + (\Sigma_i^+)^C, \quad \nu_{\Sigma,i} = \Sigma_i^0 + (\Sigma_i^0)^C,$$

for which one can choose $l_{\Sigma,i,L} = \Sigma_i^-, l_{\Sigma,i,R} = \Sigma_i^+ - (\Sigma_i^0)^C$, and $\nu_{\Sigma,i,R} = \Sigma_i^0$. The relevant parts of the Lagrangian in the seesaw III are

$$L_{\text{seesaw III}} = -\sum_{i,j} \left( Y_{\nu i}^T \bar{L}_i L \Sigma_j \tilde{\phi} + \frac{1}{2} M_{R,ij} \Sigma_i \cdot \Sigma_j \right) + \text{h.c.},$$

from which one gets the neutrino mass terms after the spontaneous symmetry breaking,

$$L_{\nu,\text{mass}} = -\frac{1}{2} (\bar{\nu}_L^T \tilde{N}_L) \begin{pmatrix} 0 & m_{\nu}^T \\ m_{\nu} & M_{R}^{\text{Maj}} \end{pmatrix} \begin{pmatrix} \nu_R^T \\ N_R^T \end{pmatrix} + \text{h.c.}$$

$$= -\frac{1}{2} \bar{\nu}_L^T m_{\nu}^T N_R^T - \frac{1}{2} \tilde{N}_L m_{\nu}^T \nu_R^T - \frac{1}{2} \tilde{N}_L M_{R}^{\text{Maj}} N_R^T,$$

from which, by integrating out the heavy Majorana mass fields and writing $m_{\nu}^T = \frac{Y_{\nu \nu}}{\sqrt{2}}$, one gets

$$m_{\nu,\text{effective,III}} = -\frac{v^2}{2} Y_{\nu \nu} \frac{1}{M_R^{\text{Maj}}} Y^{\nu \nu T}. \quad (74)$$

Sequential dominance

Sequential dominance $^{[\text{106}]}$ is a mechanism that explains the neutrino mass hierarchy. It is based on the seesaw I, so that it assumes the existence of at least two new heavy Majorana singlet neutrinos, of which one is heavier than the others. Consider a single heavy Majorana neutrino $\nu_{R,3}$ with a mass $M_3$. In a basis with diagonal charged lepton
and heavy neutrino mass matrices, the Yukawa term becomes (using the notation of Ref. [106]),

\[ H (dL_e + eL_\mu + f L_\tau) \nu_{R,3}, \]

where \(d, e, f\) are the Yukawa couplings. The couplings are assumed to be such that \(d \ll e, f\). The seesaw mechanism gives a light neutrino mass

\[ m_3 \simeq (e^2 + f^2) \frac{v^2}{M_3}. \]

By identifying the neutrino with a mass \(m_3\) to be the atmospheric neutrino, one can write the atmospheric mixing angle as [109]

\[ \tan \theta_{23} \simeq \frac{e}{f}, \tag{75} \]

whereas the reactor angle becomes [109]

\[ \theta_{13} \simeq \frac{d}{\sqrt{e^2 + f^2}}. \tag{76} \]

Next, one introduces another heavy neutrino, \(\nu_{R,2}\), with a mass \(M_2\), for which, analogously to \(\nu_{R,3}\) one writes

\[ H (aL_e + bL_\mu + c L_\tau) \nu_{R,2}, \]

where this time the Yukawa couplings are conventionally marked as \(a, b, c\). Then, due to an assumption that \((a^2 + b^2 + c^2)/M_2 \ll (e^2 + f^2)/M_3\), one can write the second light neutrino as [106,109]

\[ m_2 \simeq (a^2 + (b \cos \theta_{23} - c \sin \theta_{23})^2) \frac{v^2}{M_2}, \tag{77} \]

from which one gets the second mixing angle,

\[ \tan \theta_{12} \simeq \frac{a}{b \cos \theta_{23} - c \sin \theta_{23}}. \tag{78} \]

Hence, this method gives a way to write the mixing angles and light neutrino masses using only Yukawa couplings.
5.1.2 Seesaw II

If we assume $M_{L}^{Maj} \neq 0$ in Eq. (68), the Higgs sector has to be extended. The small neutrino masses can also be generated by adding a scalar triplet $\Delta$ with a mass $M_{\Delta}$ and a hypercharge $Y = 2$, which in the $2 \times 2$ representation reads

$$\Delta = \begin{pmatrix} \Delta^+/\sqrt{2} & \Delta^{++} \\ \Delta^0 & -\Delta^+/\sqrt{2} \end{pmatrix},$$

where the superscripts $0$, $+$, and $++$ denote the charge of the scalar field. As we do not enlarge the fermion sector with right-handed neutrinos, the light left-handed SM neutrinos have to be Majorana particles [124].

The neutrino mass matrix in this scenario is achieved from the Yukawa interaction of the scalar triplet and left-handed light neutrinos, as in [13,19,79]

$$L_{\Delta}^Y = Y_\nu^\Delta L_L^T Ci\tau_2 \Delta L_L + h.c.,$$

which, after a non-zero VEV for the neutral component of $\Delta$, $\langle \Delta^0 \rangle \equiv v_\Delta/\sqrt{2}$ [79], gives the the mass matrix, so that

$$m_\nu = v_\Delta Y^\Delta.$$  \hfill (79)

The triplet Higgs mass terms couple with the Higgs doublet $\phi$. The most general Higgs potential contains an interaction term of the SM Higgs and the new triplet, and the relevant terms are [19,42]

$$V = M_{\Delta}^2 \text{Tr}(\Delta^\dagger \Delta) + \frac{1}{\sqrt{2}} \mu(\Phi^T i\tau_2 \Delta^\dagger \Phi + h.c.) + \ldots,$$

where $\mu$ is a dimensionless coupling. The potential can be minimized so that the triplet VEV can be written as [19]

$$v_\Delta \simeq \frac{\mu v^2}{2M_{\Delta}^2}. \hfill (80)$$

As the triplet is assumed to be heavy, $v_\Delta$ is much lighter than $v$. Thus the corrections to the $Z$ and $W$ boson masses caused by the Higgs triplet are negligible [124], and the $\rho$ parameter is not much affected.
5.1.3 Other seesaws

Other seesaw scenarios are also studied in the literature. In the so-called double and inverse seesaw schemes (the latter being a special case of the former) one adds two sets of heavy SM singlet fermions, the right-handed \( N \) and the left-handed \( S \) to the SM, so that the Yukawa parts of the Lagrangian become \[126\]

\[
-\mathcal{L}_Y = Y^\nu \bar{L}_L \Phi N + \frac{1}{2} \bar{N} \mu_R N + \frac{1}{2} \bar{S} \mu_S S^\dagger + \text{h.c.},
\]

from which it follows that the neutrino mass matrix is of the form \[57\]

\[
M_\nu = \begin{pmatrix}
0 & M_D & 0 \\
M_D^T & \mu_N & M_N \\
0 & M_N^T & \mu_S
\end{pmatrix},
\]

where \( M_D \) refers to the Dirac mass term. The standard assumption is to consider the limit \( \mu_N \ll v \lesssim M_N \), which gives the lightest neutrino mass \[57\],

\[
m_\nu \simeq (M_D M_N^{-1}) \mu_S (M_D M_N^{-1})^T.
\]

The advantage of this model is that the TeV energy scale is technically natural for its seesaw mediators \[126\]. Therefore it is testable at the LHC.

5.2 Gauge symmetry extensions

The seesaw mechanisms give an explanation for the light neutrino masses. However, there are still unanswered questions that the mechanisms do not explain, such as why do the SM Higgs boson, the scalar triplets, or the extra fermions happen to have the masses they have.

There are different theories with higher energies that try to make connections between different properties of particles. One can do this, e.g., by assuming a symmetry group that contains the symmetry group of the SM. The larger symmetry is then broken, leading to the observed properties. One main reason to the failure of the minimal SU(5) model by Georgi and Glashow was because the neutrinos remained massless in the theory like they do in the SM \[81\]. The addition of a dimension five Weinberg operator could not fix the problem because the neutrino mass is still restricted to be extremely small due to the \( SU(5) \) symmetry \[126\].

The Littlest Higgs model is based on an \( SU(5) \) symmetry, like the original Georgi-Glashow model, but it is more complex. It predicts the existence of a scalar triplet \( \Delta \)
(along with other particles) which can generate the neutrino masses as in the seesaw II mechanism.

The minimal left-right symmetric model extends the symmetry group of the SM so that left- and right-handed particles are described symmetrically. This is to be contrasted with electroweak theory, where only left-handed chiral particles participate in weak interactions. The model adds heavy right-handed neutrinos to the particle content of the SM, and therefore contains a type I seesaw model.

In the last subsection a model with an adjoint fermion added to the minimal non-supersymmetric $SU(5)$ is introduced. At 1 TeV energy level the model manifests a fermion triplet, and thus includes the type III seesaw model.

5.2.1 The Littlest Higgs model

The LtH models were introduced in 2001 [27,28], although the key idea behind the models, the Higgs boson being a pseudo-Goldstone boson, was introduced already in the 1970s [82]. The LtH models have been studied extensively during recent years: the first LtH articles have been cited already almost a thousand times. In the models the SM gauge group is augmented. The name comes from the models’ capability of explaining the lightness of the Higgs mass. There are different versions to do that but here the so-called “the Littlest Higgs” [28] is reviewed. This is because within this model neutrino mass parameters and possible CP violating effects could be detected in the LHC [91,95,103].

The model consists of a global $SU(5)$ symmetry that contains two sets of $SU(3) \otimes SU(2) \otimes U(1)$ subgroups. The $SU(5)$ is broken to $SO(5)$ by making a subgroup of $(SU(2) \otimes U(1))^2$ local. As an $SU(N)$ group has $N^2 - 1$ and an $SO(N)$ has $N(N-1)/2$ generators [80], the symmetry breaking $SU(5) \to SO(5)$ leaves 14 broken generators.

Let’s assume that the $SU(5)$ symmetry is broken due to a non-zero VEV of a scalar field $\Sigma(x)$, i.e., [28]

$$\langle \Sigma \rangle \equiv \Sigma_0 = \begin{pmatrix} 0_{2\times2} & 0 & 1_{2\times2} \\ 0 & 1 & 0 \\ 1_{2\times2} & 0 & 0_{2\times2} \end{pmatrix},$$

Under a general $SU(5)$ transformation $U = \exp(i\theta_a T_a)$, where $a = 1 \ldots 24$, $\Sigma(x)$ transforms as $\Sigma(x) \to U \Sigma(x) U^T$ [28]. There are different conditions for the 14 broken $(T_b)$ and 10 unbroken $SU(5)$ generators $(T_u)$ [139], i.e.,

$$T_u \langle \Sigma \rangle + \langle \Sigma \rangle T_u = 0, \quad T_b \langle \Sigma \rangle - \langle \Sigma \rangle T_b = 0,$$  

(81)
using which, one can write $\Sigma(x)$, analogously to Eq.\,(48), in a form \[139\]

$$
\Sigma(x) = \exp(2i\pi_b T_b / f)\Sigma_0,
$$

where the VEV $f$ states the cut-off scale of the symmetry breaking to be $\Lambda \simeq 4\pi f \sim 10\text{ TeV}$. By identifying the SM Higgs doublet $\Phi = (\phi^+, \phi^0)$ and an electroweak scalar triplet $\Delta$, in addition to a real SM singlet $\eta$ and a triplet $\chi = \chi_a \sigma_a$, the two latter of which are Goldstone bosons that will vanish, one can explicitly write $\pi_b T_b$ as \[139\]

$$
\pi_b T_b = \Pi = \begin{pmatrix}
(\chi + \eta)/2\sqrt{5} & \Phi^\dagger/\sqrt{2} & \Delta^\dagger \\
\Phi/\sqrt{2} & -2\eta/\sqrt{5} & \Phi^*/\sqrt{2} \\
\Delta & \Phi^T/\sqrt{2} & (\chi^T + \eta)/2\sqrt{5}
\end{pmatrix}.
$$

Let’s denote the two $SU(2) \otimes U(1)$s as $G_1$ and $G_2$. The generators of the subgroups, $Q_i$ and $Y_i$, resemble the generators of the electroweak theory, with the distinction that here they are $5 \times 5$ matrices, so that \[91\]

$$
Q^a_1 = \begin{pmatrix}
\frac{\sigma^a}{2} & 0_{2 \times 3} \\
0_{3 \times 2} & 0_{3 \times 3}
\end{pmatrix}, 
Q^a_2 = \begin{pmatrix}
0_{3 \times 3} & 0_{3 \times 2} \\
0_{2 \times 3} & \frac{\sigma^a}{2}
\end{pmatrix}, 
(82)
$$

$$
Y_1 = \frac{1}{10} \begin{pmatrix}
-3 \cdot 1_{2 \times 2} & 0_{2 \times 3} \\
0_{3 \times 2} & 2 \cdot 1_{3 \times 3}
\end{pmatrix}, 
Y_2 = \frac{1}{10} \begin{pmatrix}
-2 \cdot 1_{3 \times 3} & 0_{3 \times 2} \\
0_{2 \times 3} & 3 \cdot 1_{2 \times 2}
\end{pmatrix},
$$

By writing the covariant derivative of $\Sigma(x)$, i.e., \[91\]

$$
D_\mu \Sigma(x) = \partial_\mu \Sigma - i \sum_{i=1,2} (g_i W^a_{\mu i}(Q^a_i \Sigma + \Sigma Q^a_i^T) + g'_i B_{\mu i}(Y_i \Sigma + \Sigma Y_i^T)),
$$

which, using Eq.\,(81), breaks the global $SU(5)$ symmetry. The symmetry $(SU(2) \otimes U(1))^2$ breaks down to its subgroup $SU(2)_L \otimes U(1)_Y$, where the generators of the electroweak symmetry are $Y = Y_1 + Y_2$ and $T_i = Q_{1i} + Q_{2i}$, as can be noticed from Eq.\,(82). The above expression of the covariant derivative prevents the SM Higgs boson from having mass if either $g_i$s or $g'_i$s are zero. This is because the Higgs field transforms non-linearly under the remaining $SU(3)$ symmetries. The scalar triplet $\Delta$ is not protected by symmetries, and therefore it can have a mass larger than the mass of the Higgs boson, which can in this model achieve mass via electroweak symmetry breaking at lower energies. There are estimates that the mass of the scalar could be of order $1\text{ TeV}$. In addition to the scalar triplet, the model predicts the existence of new gauge bosons $W_{LH}$ and $Z_{LH}$, and a vector-like quark $T$ with charge $2/3$. All of these particles could be within the energy range of the LHC.
5.2.2 The minimal left-right symmetric model

The type I seesaw can be originated from minimal left-right symmetric model. The model connects the lightness of observed neutrinos to the suppression of the interactions of right-handed neutrinos at low energies by a seesaw mechanism, which is called the L-R seesaw \[58\]. The L-R symmetric model is said to give an explanation on why the seesaw scale is what it is in addition to the obvious parity symmetry restoration \[57\]. The model predicted neutrino masses long before they were found \[140\].

In the minimal left-right symmetric model the SM gauge group is extended with the symmetry $SU(2)_R$, so that the electroweak gauge group becomes $SU(2)_L \otimes SU(2)_R \otimes U(1)_Y \[119\]$. In the theory one adds right-handed neutrinos in order for the fermion sector becoming left-right symmetric; there are both left and right-handed lepton doublets (wrt. Eqs. (35)-(36)),

$$L_L = \left( \begin{array}{c} \nu_l \\ l \end{array} \right)_L, \quad L_R = \left( \begin{array}{c} N_l \\ l \end{array} \right)_R.$$

For the electromagnetic charge, instead of Eq. (33), now holds

$$Q = I_{3,L} + I_{3,L} + \frac{B - L}{2},$$

where $B$ and $L$ are the baryon and lepton numbers. In addition to right-handed neutrinos, one adds a bi-doublet, i.e., a doublet under both $SU(2)_L$ and $SU(2)_R$ symmetries, $\Phi$ with zero $B - L$ charge, triplet $\Delta_L$ with $B - L = 2$ which acts as a singlet under $SU(2)_R$, and a triplet $\Delta_R$, a singlet under $SU(2)_L$ with $B - L = 2$, i.e.,

$$\Phi = \left( \begin{array}{c} \phi_1^0 \\ \phi_1^+ \\ \phi_1^- \end{array} \right), \quad \Delta_{L,R} = \left( \begin{array}{cc} \Delta^+/\sqrt{2} & \Delta^{++} \\ \Delta^0 & -\Delta^+ / \sqrt{2} \end{array} \right).$$

The Yukawa couplings for the theory are \[140\]

$$\mathcal{L}_\Delta = \frac{1}{2} \left( L_L^T C i \tau_2 Y_{\Delta L} \Delta_L L_L + L_R^T C i \tau_2 Y_{\Delta R} \Delta_R L_R \right) + \text{h.c.} \quad (83)$$

In order to break the L-R symmetry into $SU(2)_L \otimes U(1)_Y$, the triplets first get the VEVs \[119\]

$$\langle \Delta_L \rangle = 0, \quad \langle \Delta_R \rangle = \left( \begin{array}{cc} 0 & \nu_R \\ 0 & 0 \end{array} \right).$$
Next, the bi-doublet gets a non-zero VEV,

\[ \langle \Phi \rangle = \begin{pmatrix} v_1 & 0 \\ 0 & v_2 e^{i\beta} \end{pmatrix}, \]

where \( v_1^2 + v_2^2 = v \). This makes \( \Delta_L \) to develop a small VEV, \( \langle \Delta_L \rangle \sim v^2/v_R \). There are new bosons, \( W_R \) and \( Z_R \). There exists a relation for the masses of these new particles, \( M_{Z_R} \approx 1.7 M_{W_R} \) [140]. In this theoretical setup the lower limit for the mass of the boson \( W_R \) is 2.5 TeV [119], and it is currently being searched at the LHC [56].

The heavy right-handed neutrinos obtain mass terms from Eq. (83), so that

\[ \mathcal{L}_{M_N} = Y_{\Delta} v_R \left( N^T_i C N_i + N^*_i C^T N_i \right), \]

from which it follows that \( M_N = Y_{\Delta} v_R \). Assuming the Majorana nature of light and heavy neutrinos one can write the Dirac Yukawa couplings of the triplets as [140]

\[ \mathcal{L} = \bar{L}_L \left( Y_{\Phi} \Phi + \tilde{Y}_{\Phi} \tilde{\Phi} \right) L_R + h.c., \]

which gives after the symmetry breaking the neutrino Dirac mass matrix,

\[ M_D = v_1 Y_{\Phi} + v_2 \tilde{Y}_{\Phi} e^{-i\beta}, \]

so that the neutrino mass terms become

\[ M_D \bar{L}_L L_R + M_N L_R^T C L_R + h.c., \]

which leads into type I seesaw mechanism.

5.2.3 SU(5) with an adjoint fermion

Also the type III seesaw model can be extended so that the origin of the fermionic triplet is explained. This can be done by building on the SU(5) symmetry. The problems of the minimal SU(5) are that the neutrinos in the theory are massless and the gauge coupling constants do not unify. The way to overcome this is to add an adjoint fermionic multiplet 24_ to the theory. The multiplet is decomposed under the SM gauge group so that it contains both a fermionic triplet and a singlet. In addition to the adjoint fermion the model contains three generations of fermionic 10_ and 5_, and scalars 5_H and 24_H, the latter of which is responsible for the SU(5) breaking with a VEV proportional to \( Y_1 \) of Eq. (82), i.e, \( \langle 24_H \rangle = \text{diag}(2, 2, 2, -3, -3) \). After
the symmetry breaking the Yukawa couplings read \([26]\)

\[
\mathcal{L}_Y = Y_N^f L_{L,f} \Phi S + Y_\Sigma^f L_{L,f} \Phi \Sigma - \frac{M_S}{2} SS - \frac{M_\Sigma}{2} \Sigma \Sigma + \text{h.c.},
\]

where \(\Sigma\) is the fermion triplet of type III seesaw, and \(S\) is a heavy neutrino singlet. This gives the mass term for the light neutrinos after the symmetry breaking of the SM Higgs (Eq. (47)), so that integrating out the heavy triplet and singlet fields gives \([26]\)

\[
m_{\nu}^{ij} = -v^2 \left( \frac{Y_i^j \Sigma}{M_\Sigma} - \frac{Y_i^j S}{M_S} \right).
\]

In order to get the mass matrix of the type III seesaw, Eq. (74), one assumes \(M_S\) to be very large. The advantage of the theory compared to many other models is that it can be directly tested at the LHC: it predicts the type III seesaw with a lepton triplet with \(M_\Sigma \lesssim 1\text{ TeV}\) \([33,34]\).

### 5.3 Discrete flavor symmetry extensions

One way to overcome the hierarchy problem of neutrino masses is to augment the SM symmetry group with a flavor symmetry, i.e.,

\[
G_{SM} = SU(3)_c \otimes SU(2)_L \otimes U(1)_Y \otimes G_{\text{flavor}}.
\]

The idea is to have an additional flavor symmetry that restricts the form of the mass matrices. This would give an explanation for the mass scale of neutrinos as well as for the mixing pattern of the PMNS matrix.

The motivation comes from the fact that the majority of the free parameters of the SM concerns fermion masses \([75]\). Also, it is not known why there are three different families. In the SM masses and three families are considered as given parameters, without trying to explain the structure. One way to explain this is to address a new symmetry.

The motivation for flavor groups to be discrete comes from the fact that the spontaneous breaking of a discrete symmetry does not lead to Goldstone bosons \([109]\). The idea of an additional family symmetry to the SM is not new: in 1979 it was suggested that there could be a \(U(1)\) symmetry under which the different families would have different charges \([74]\). This would cause the mass ratios of quarks to be so large. Due to the current information on the mixing pattern, one needs to have a non-abelian flavor symmetry \([109]\). There are models describing both the quark and the lepton sectors; here we only study properties of models with leptons.
As the symmetry should describe the behavior of lepton flavors, it is natural to require the group to have an irreducible triplet representation. Being the easiest groups having this property, the alternating symmetry $A_4$ and the permutation symmetry $S_4$ have been the most used symmetries (see Appendix D for group properties of $A_4$ in detail).

The first model using $A_4$ as a flavor symmetry was published in 2001 [117,118]. After that, a large amount of different neutrino mass predicting mechanisms using $A_4$ and $S_4$ have been suggested. Many of the models were based on the tribimaximal (TBM) mixing pattern [94], which was suggested in 2002. The pattern was in accordance with the neutrino oscillation data until recently. The TBM model assumes the mixing angles to be

$$\sin \theta_{12} = \frac{1}{\sqrt{3}}, \quad \sin \theta_{23} = \frac{1}{\sqrt{2}}, \quad \sin \theta_{13} = 0,$$

which leads to the mixing matrix of the form

$$U_{\text{PMNS}}^{TBM} = \begin{pmatrix}
\sqrt{2/3} & 1/\sqrt{3} & 0 \\
-1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\
-1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2}
\end{pmatrix} \quad (84)$$

This was excluded in 2012, as the reactor angle $\theta_{13}$ was found to be non-zero [24].

The procedure of a flavor symmetry extension is the following. First, one decides on a particular family symmetry. Next, one assigns all the fields in the Lagrangian with certain representation of the new flavor symmetry group so that the Lagrangian density stays invariant under the new symmetry. In Sec.3 gauge fields were added to secure the invariance under continuous local symmetries. Here, instead of gauge fields one introduces flavor fields, flavons.

5.3.1 Altarelli-Feruglio model

Although the Altarelli-Feruglio (A-F) model with $A_4$ family symmetry has been ruled out, we study it here as an example of a discrete flavor symmetry model. This is because the model plays a central role in flavor physics. There are many different versions and extensions of the A-F model that agree with the data, but the basic theory is the following.

$A_4$ has two subgroups that are generated by the generators $S$ and $T$ (see Appendix B for the notation). First, having order 2, the generator $S$ spans an $Z_2$ which is a cyclic group of two elements: it contains $S$ itself and a unit element. It is denoted
as $G_S$. The generator $T$ is of order 3 and it spans a cyclic group $Z_3$, which contains itself, it applied twice, and a unit element. This group is denoted as $G_T$.

In the A-F model the flavor symmetry $A_4$ is broken by the VEV of scalar triplet fields of the form $\phi = (\phi_1, \phi_2, \phi_3)$. The VEV has two possible patterns. A VEV of the form

$$\langle \phi_S \rangle = (v_S, v_S, v_S),$$

breaks the $A_4$ symmetry to $G_S$, whereas a VEV

$$\langle \phi_T \rangle = (v_T, 0, 0),$$

breaks it into $G_T$. The breaking occurs in the same way as in the Higgs mechanism described in Sec. 3.4.1. As mentioned above, all the fields of the theory must transform as the representations of $A_4$. The lepton doublets as assigned to be triplets, and right-handed charged leptons $e_R, \mu_R, \tau_R$ transform as $1, 1'', 1'$, respectively \[109\].

One adds three scalar fields that are singlets under the SM gauge group. The first two are triplets $\phi_S$ and $\phi_T$. The former has the VEV $\langle \phi_S \rangle$ of Eq. (85) and the latter the VEV $\langle \phi_T \rangle$ of Eq. (86). The third scalar is a singlet $\xi$ with a VEV $\langle \xi \rangle = u$. In addition to these, $h_1$ and $h_2$, two SM gauge group doublet scalar fields, singlets under the $A_4$ symmetry, are added. The corresponding VEVs are $\langle h_1 \rangle = v_1$ and $\langle h_2 \rangle = v_2$. The model assumes that there is a cut-off scale that is much larger than the VEVs of the scalar doublets \[109\]. Using these the Yukawa terms of the lepton sector Lagrangian density becomes \[22\]

$$L_l = Y_{e} e_R \langle \phi_T \rangle L \frac{h_3}{\Lambda} + Y_{\mu} \mu_R \langle \phi_T \rangle L \frac{h_2}{\Lambda} + Y_{\tau} \tau_R \langle \phi_T \rangle L \frac{h_1}{\Lambda} + h.c.,$$

where the representations of the product of two triplets were produced according to the Kronecker products studied in Appendix B. The subscript of the brackets refers to in which representation the product is. Next one lets the scalar fields to develop VEVs of Eqs. (85) - (86). The Yukawa terms become

$$L_l = v_2 \frac{v_T}{\Lambda} (Y_{e} e_R e_L + Y_{\mu} \mu_R \mu_L + Y_{\tau} \tau_R \tau_L) + X_a \frac{v^2}{\Lambda} (\nu_e \nu_e + 2 \nu_\mu \nu_\mu)$$

$$+ X_b \frac{2v_S}{3\Lambda^2} (\nu_e \nu_e + \nu_\mu \nu_\mu + \nu_\tau \nu_\tau - \nu_e \nu_\mu - \nu_\mu \nu_\tau - \nu_\tau \nu_e) + h.c.,$$

so that the charged leptons remain invariant under a $Z_3$ transformations, but neutrinos have a $Z_2$ symmetry, which means that the $A_4$ symmetry is broken \[22\]. By
defining
\[ a \equiv 2X_a \frac{u}{\Lambda}, \quad b \equiv 2X_b \frac{v_S}{\Lambda}, \quad m_l = Y_l \frac{v_T}{\Lambda}, \quad l = e, \mu, \tau, \]
one can write the mass matrices as
\[
\begin{pmatrix}
  m_e & 0 & 0 \\
  0 & m_\mu & 0 \\
  0 & 0 & m_\tau
\end{pmatrix}, \quad m_\nu = \frac{v_1^2}{\Lambda} \begin{pmatrix}
  a + 2b/3 & -b/3 & -b/3 \\
  -b/3 & 2b/3 & a - b/3 \\
  -b/3 & a - b/3 & 2b/3
\end{pmatrix}.
\]

The neutrino mass matrix is diagonalized in this model by the TBM mixing matrix of Eq. (84), and it leads to neutrino masses [22]
\[ m_1 = a + b, \quad m_2 = a, \quad m_3 = b - a. \]

Although the original A-F model was excluded, there are extensions that allow a non-zero \( \theta_{13} \) and thus a Dirac CP violating phase \( \delta \). The motivation for not abandoning the model is that although the TBM predicts a wrong \( \theta_{13} \), other two mixing angles fit well with the observations. The needed tuning can be done, e.g., by introducing two more flavon fields that transform as the representations \( 1'' \) and \( 1' \), and stating that \( A_4 \) is the symmetry that remains after an \( S_4 \) symmetry is broken [108].

There are also many supersymmetric models combining \( A_4 \) with other symmetries, such as models based on a symmetry \( A_4 \otimes SU(5) \) [62].

**Tribimaximal-Cabibbo mixing** As most of the flavor symmetry models, like A-F model, were based on a zero reactor angle, there has been a lot of research on how to save the models after the confirmation of non-zero \( \theta_{13} \). It has been noticed that a certain modification of the TBM, named tribimaximal-Cabibbo mixing (TBC) leads to values of the mixing parameters within the experimental error limits [107].

The idea of somehow linking the Cabibbo angle, or other quark sector parameters to the lepton sector is not new [64]. It has even been suggested to use it in the TBM before, although in a different way [123]. The TBC mixing pattern is based on the fact that the numerical value of the reactor angle written as
\[ \sin \theta_{13} = \frac{\sin \theta_C}{\sqrt{2}} = \frac{\lambda}{\sqrt{2}}, \]
is in good agreement with the experimental results, giving \( \theta_{13} \approx 9.2^\circ \). Above \( \lambda \approx 0.2253 \pm 0.0007 \) [36] is the Wolfenstein parameter used in the Wolfenstein parameterization of the CKM matrix [145] and \( \theta_C \approx 13^\circ \) is the Cabibbo angle [49]. The
mixing angles in the TBC mixing are written as

\[ \sin \theta_{12} = \frac{1}{\sqrt{3}}(1 + s), \quad \sin \theta_{23}(1 + a) = \frac{1}{\sqrt{2}}, \quad \sin \theta_{13} = \frac{r}{\sqrt{2}}, \]

where \( r = \lambda \). The values for \( s \) and \( a \) are not determined. The labels refer to reactor (\( r \)), atmospheric (\( a \)), and solar (\( s \)) angles. The origin of the mixing pattern is not clear, but one way to explain is is to assume the existence of at least two heavy neutrinos (the seesaw I), and the sequential dominance structure \([107]\). Consider the Eqs.\([75]-[78]\) If, for an unknown reason, there is a restriction for the Yukawa couplings so that \([107]\)

\[
\begin{pmatrix}
d \\
e \\
f
\end{pmatrix} = a_3 \begin{pmatrix}
\lambda \\
1 \\
1
\end{pmatrix}, \quad \begin{pmatrix}
a \\
b \\
c
\end{pmatrix} = a_2 \begin{pmatrix}
1 \\
1 \\
-1
\end{pmatrix},
\]

one gets for the mixing angles the values \( \theta_{12} \simeq 1/\sqrt{2}, \theta_{13} \simeq \lambda/\sqrt{2} \) and \( \theta_{23} \simeq 1/\sqrt{3} \), which are in accordance with TBC.

There have been attempts to make connections between the CP violation effects that could be detected at colliders, and flavor symmetries. Such attempts include extending a seesaw mechanism with a flavor symmetry, like the \( A_4 \) symmetry \([96]\).

In Ref. \([50]\) a model that combines the discrete \( T_7 \) symmetry and \( U(1)_{B-L} \) is introduced. It contains heavy neutrinos, and the light neutrino model is also generated by type I seesaw. The SM singlet scalars are well beyond the energy scale of the LHC, but it has been suggested that one could detect the two extra Higgs doublets that the model requires. As many other models, also this assumes \( \theta_{13} \) to be zero. The discovery of the non-zero value of \( \theta_{13} \) messed up the field. Because of that, and because most of the new physics occurs at energy scales too high for the LHC, the phenomenology of flavor symmetry models at low energies is not as established as it is for tree-level seesaw mediators \([97]\). Moreover, the connection between CP violating phases is even less studied, although it has been noted that heavy right-handed neutrinos added in certain models could have complex phases in the mass matrix \([97]\). In Sec.\([7]\) the phenomenology of the tree-level seesaw mediators is studied, but their possible origin in flavor symmetry models is left without attention in this work.
6 Conditions for leptonic CP violation

After the observation of neutrino oscillation, the structure of the neutrino mixing matrix became a timely topic. The mixing, together with the discovery of non-zero $\theta_{13}$ in 2012, makes it possible also for leptons to violate the CP symmetry.

In this section the theoretical conditions for CP invariance are studied. It is noticed that the conditions for Dirac and Majorana particles differ. It is also noted that neutrino oscillations are the only feasible way to measure direct CP odd effects, although it is not impossible for other processes to manifest CP violation. For processes for which CP odd effects are not observable, one can still try to probe CP violation by studying effects that Majorana and Dirac phases have on some observables, although the connection would be weaker.

6.1 CP invariance of neutrino mass terms

Let’s first study the conditions on which the Lagrangian is invariant under the CP transformation. As noted in Sec.2.6.3, the weak coupling does not violate the CP symmetry, even though both C and P are maximally violated. The possible sources of CP violation are the PMNS matrix and the neutrino mass terms. In this subsection we derive the conditions for the Majorana and the Dirac type CP violation.

6.1.1 CP invariance of the charged current

The charged current interactions,

$$\mathcal{L}_{CC} = \frac{g}{2\sqrt{2}} 2\bar{n}_{L}U_{PMNS}^{\dagger} \gamma^{\mu} l_{L} W_{\mu}^{+} = \frac{g}{2\sqrt{2}} \bar{\nu}_{l}^{\prime} \gamma^{\mu} (1 - \gamma^{5}) l_{l} W_{\mu}^{+} + \text{h.c.},$$

have terms with $l_{l}'$ and $\bar{\nu}_{l}'$ which transform under CP as

$$l_{l}' \rightarrow -\eta_{CP}^{l} \gamma^{0} C l_{l}^{T}(x_{P}),$$

$$\bar{\nu}_{l}' \rightarrow (\eta_{CP}^{\nu l})^{*} \nu_{l}^{T}(x_{P}) C \gamma^{0},$$

$$W_{\mu} \rightarrow \eta_{CP}^{W} W_{\mu}.$$
The charged current transforms under CP transformation as

\[ j_{CC,l}^\mu \rightarrow \left( 2 \sum_{l=e,\mu,\tau} \bar{n}_{l,L} U_{PMNS}^\dagger \gamma_\mu l_{L} + \text{h.c.} \right)^{CP} \]

\[ = -2 \sum_{l=e,\mu,\tau} \left( \eta_{CP}^{l,D} \right)^* \eta_{CP}^{l,D} \gamma_{\mu} l_{L} U_{PMNS}^\dagger U_{L} + \text{h.c.,} \quad (88) \]

where the property \( \gamma_{\mu}^\dagger = \gamma_{\mu} \) was used. As \( W_\mu \) transforms so that the Lorentz indices are raised, the lowering of the indices of the charged current is allowed. The hermitian conjugate of the charged current term used above is

\[ -2 \bar{l}_{L}^{'\dagger} \gamma_{\mu} U_{PMNS} n_{l,L} \].

It is the same as in Eq.(88) if the phases satisfy \( \left( \eta_{CP}^{\nu,l,D} \right)^* \eta_{CP}^{\nu,l,D} \eta_{W}^{CP} \eta_{CP} = 1 \), and if

\[ U_{PMNS} = U_{PMNS}^s, \]

i.e., the mixing matrix has to be real in order to have CP invariance for weak interactions. This is satisfied if the phase \( \delta = n\pi \) or if \( \theta_{13} = n\pi \). As the reactor angle does not satisfy this, the value of the phase \( \delta \) determines the CP symmetry of weak interactions.

### 6.1.2 CP invariance of the mass terms

Let’s study the CP transformation properties of the mass terms of charged leptons, Dirac, and Majorana neutrinos. From Eqs.(64) and (49), we have

\[ \mathcal{L}_{\text{mass}}^{l+D+M} = -\bar{L}_{L} M^{l} L_{R} - \bar{R}_{R} M^{D} R_{L} - \bar{\nu}_{L} M^{\nu} \nu_{R} - \bar{\nu}_{R} M^{\nu} \nu_{L} \]

\[ + \frac{1}{2} \nu_{L}^T C_{l}^{l} M_{l}^{Maj} \nu_{L}^{*} + \frac{1}{2} \nu_{R}^T C_{R}^{R} M_{R}^{Maj} \nu_{R}^{*} \]

where the fact that Majorana mass matrices are symmetric was used. The fields transform according to the CP transformations in Sec.2.6.3, i.e.,

\[ \begin{align*}
I_{L} &\rightarrow W_{L} \gamma^{0} C I_{L}^T, & \bar{I}_{L} &\rightarrow I_{L}^T C^\dagger (\gamma^0)^T W_{L}^\dagger \\
\nu_{L} &\rightarrow W_{L} \gamma^{0} C \nu_{L}^T, & \bar{\nu}_{L} &\rightarrow \nu_{L}^T C^\dagger (\gamma^0)^T W_{L}^\dagger \\
I_{R} &\rightarrow W_{R} \gamma^{0} C I_{R}^T, & \bar{I}_{R} &\rightarrow I_{R}^T C^\dagger (\gamma^0)^T W_{R}^\dagger \\
\nu_{R} &\rightarrow W_{R} \gamma^{0} C \nu_{R}^T, & \bar{\nu}_{R} &\rightarrow \nu_{R}^T C^\dagger (\gamma^0)^T W_{R}^\dagger
\end{align*} \]
so that the Lagrangian mass terms transform as

$$\mathcal{L}_{\text{mass}} \rightarrow -\bar{l}_R^l \left[ W_R^T (M^l)^T W_L^l \right] l_L - \bar{l}_L^l \left[ W_L^T M^l \nu^R \right] \nu_R^l$$

$$- \bar{\nu}_R^\nu \left[ W_R^\nu (m^\nu)^T W_L^\nu \right] \nu_L^\nu - \bar{\nu}_L^\nu \left[ W_L^\nu m^\nu \nu_R^\nu \right] \nu_R^\nu$$

$$- \frac{1}{2} \bar{\nu}_L^\nu \left[ W_L^T M^l_{\text{Majorana}} W_L^l \right] C \nu_L^\nu - \frac{1}{2} \bar{\nu}_R^\nu \left[ W_R^\nu M^l_{\text{Majorana}} W_R^\nu \right] \nu_R^\nu$$

$$- \frac{1}{2} \bar{\nu}_R^\nu \left[ W_R^\nu T M^l_{\text{Majorana}} W_R^\nu \right] C \nu_R^\nu - \frac{1}{2} \bar{\nu}_L^\nu \left[ W_L^T M^l_{\text{Majorana}} W_L^l \right] \nu_R^\nu,$$

where the property $C C^\dagger = -\gamma^0$ was used. The Lagrangian is then invariant under CP transformation if there exists such matrices $W^l_R$, $W^\nu_R$, and $W^l_L$ that satisfy

$$W^l_R^l M^l W^l_R = M^l$$

$$W^\nu_R^\nu m^\nu W^l_R = m^\nu$$

$$W_L^T M^l_{\text{Majorana}} W_L^l = -M^l_{\text{Majorana}}$$

$$W_R^\nu T M^l_{\text{Majorana}} W_R^\nu = -M^l_{\text{Majorana}}$$

where matrices $W_R$ and $W_L$ have to be symmetric, as can be noticed by solving the first condition for $M^l$ in the left hand side, and comparing this to the complex conjugate of the original equation. It turns out that in a theory with three left-handed Majorana fields, if one chooses $M^l$ to be real, diagonal, and positive, (as it clearly should be in a weak interaction basis for charged leptons) the elements of the mass matrix $M^l_{\text{Majorana}}$ are restricted to be either real or imaginary [42].

### 6.2 Condition for Dirac CP violation

In the following, the famous Jarlskog invariant is derived. Usually in the literature the expression for the invariant is taken as known, but the derivation contains physically interesting features. It defines the condition for Dirac type CP violation. It is especially interesting, since it consists of rephasing invariant quartets, i.e., quantities that have the same values in any parameterization of the PMNS matrix.

The conditions of Eqs. (91) depend on a specific weak interaction basis. One can define Hermitian matrices $H^l$, $H^{\nu,D}$, and $H^{\nu,M}$ in order to eliminate $W_L$, $W_R^l$, and $W_R^\nu$, i.e.,

$$H^l \equiv M^l M^{l\dagger} = W_L^T M^l \nu^{\dagger T} W^l_L = W_L^T (H^l)^T W^l_L = W_L H^{l\dagger} W^l_L.$$  

For neutrinos we define

$$H^{\nu} \equiv M^{\nu} M^{\nu\dagger},$$
where $M^\nu$ can be $M_L^{\text{Maj}}$, $M_R^{\text{Maj}}$, or $m^\nu$. In the case of Dirac neutrinos, $H^\nu \equiv H^\nu_D$, the matrix has the same transformation properties as $H^l$ in Eq. (92) [37],

$$W_L^\dagger H^\nu W_L = H^{\nu*} = H^{\nu T}.$$  

(93)

This means that the existence of the matrices $W_R^l$ and $W_R^{\nu}$ follows directly from $W_L$, i.e. the existence of $W_L$ is sufficient condition to have a CP invariant theory without Majorana fields. For Majorana fields we have

$$H^\nu_L \equiv M_L^{\text{Maj}} M_L^{\text{Maj} T} W_L$$

$$H^\nu_R \equiv M_R^{\text{Maj}} M_R^{\text{Maj} T} W_R^{\nu}$$

and for them we have

$$W_L H^{\nu T}_L W_L = H^{\nu*}_L = (H^\nu_L)^T,$$  

(94)

and analogously to the right-handed Majorana fields with the substitution $W_L \to W_R$. The properties of $H^{\nu/l}$ are independent of the particular basis of $H^{\nu/l}$ [37]. This can be proved as follows: let’s first consider $H^\nu_D$ and $H^{\nu T}_D$ that are defined in different bases. The case of Majorana fields is studied later. Matrices $H^\nu_D$ and $H^{\nu T}_D$ are related by [37]

$$H^\nu_D = V_L^\dagger H^{\nu T}_D V_L,$$  

(95)

where $V_L$ is a unitary matrix. If $W_L$ satisfies Eq. (93), then we have

$$H^\nu_D^{T} = V_L^\dagger H^\nu_D^{T} V_L = V_L^\dagger W_L^\dagger H^\nu_D W_L V_L^\dagger = X^\dagger H^\nu_D X,$$

where $X \equiv V_L W_L V_L^T$ was defined [37]. Thus $H^\nu_D$ has the same properties as $H^\nu_D$ in Eq. (93). It is easy to see that

$$W_L^\dagger [H^\nu_D, H^l_D] W_L = W_L^\dagger [H^\nu_D H^l_D - H^l_D H^\nu_D] W_L = - [H^\nu_D, H^l_D]^T,$$  

(96)

but from Eq. (92) it follows that $(H^\nu_D)^2 = W_L H^{\nu T}_D H^{\nu T}_D W_L = U_L (H^{\nu T}_D)^2 U_L^\dagger$, so that

$$(H^\nu_D)^p = W_L (H^\nu_D)^T W_L^\dagger, \quad p = 1, 2, \ldots,$$  

(97)

Using Eqs. (96) - (97) we get a result

$$U_L^\dagger [(H^\nu_D)^p, (H^l)^T] U_L = - [(H^\nu_D)^p, (H^l)^T]^T.$$  

(98)
From Eq. (98) one notices a general rule,

\[
\left( [(H_D^\nu)^p, (H_l^q)]^T \right)^n = U_L^\dagger [H^\nu_p, H^l_q]^n U_L,
\]

from which the famous commutator identity follows \([46, 88]\),

\[
\text{Tr} \left( [(H_D^\nu)^p, (H_l^q)]^n \right) = \text{Tr} \left( U_L^\dagger [(H_D^\nu)^p, (H_l^q)]^n U_L \right)
\]  \( (99) \)

First, for \( n = 1 \) this leads to \(-\text{Tr} [(H_D^\nu)^p, (H_l^q)] = \text{Tr} [(H_D^\nu)^p, (H_l^q)]\), which means that the commutator vanishes identically. For \( n = 2 \) we have \( \text{Tr} \left( [(H_D^\nu)^p, (H_l^q)]^2 \right) = \text{Tr} \left( [(H_D^\nu)^p, (H_l^q)]^2 \right) \). This is an identity, which is the result for all even numbers, due to the minus sign in Eq. (98). Instead, for odd numbers \( n \) we get from Eq. (99) the condition for CP invariance,

\[
\text{Tr} \left( [(H_D^\nu)^p, (H_l^q)]^n \right) = 0, \quad n = 1, 3, ...
\]

In the case of three generations the mass matrices are of order \( 3 \times 3 \). As the trace of a commutator vanishes, the case \( n = 1 \) is trivial. The lowest value for \( n \) that gives contribution for the CP invariance condition is \( n = 3, p = q = 1 \) \([37]\),

\[
\text{Tr} \left( [H_D^\nu, H_l^q]^3 \right) = 0.
\]  \( (100) \)

Let’s evaluate this expression. After some algebra one finds that

\[
\text{Tr} \left( [H_D^\nu, H_l^q]^3 \right) = -3 \text{Tr} \left( (H_D^\nu)^2 (H_l^q) (H_l^q)^* (H_l^q) \right) + 3 \text{Tr} \left( (H_D^\nu)^2 (H_l^q)^* (H_l^q) (H_l^q)^* \right) = 6i \text{Im} \left[ \text{Tr} \left( (H_D^\nu)^2 (H_l^q)^* (H_l^q) \right) \right].
\]

By evaluating the expression using the elements of the matrices one finds,

\[
\text{Tr} \left( [H_D^\nu, H_l^q]^3 \right) = 6i \sum_{i,j=1}^3 \sum_{\alpha,\beta=e,\mu,\tau} m_i^4 m_\alpha^4 m_j^2 m_\beta^2 \text{Im} Q^*_{i\alpha j\beta}
\]

where the Jarlskog invariant \([101]\) was defined as

\[
J_{\alpha i \beta j} \equiv \text{Im} Q_{\alpha i \beta j} = (U_{\alpha i} U_{\beta j} U_{\alpha j}^* U_{\beta i}^*),
\]  \( (101) \)

where \([45]\)

\[
Q_{\alpha i \beta j} = Q_{\beta j \alpha i} = Q^*_{\alpha j \beta i} = Q^*_{\beta i \alpha j},
\]  \( (102) \)
The sums can be evaluated so that

$$\text{Tr} \left( \left[ H_D^\nu, H^l \right]^3 \right) = -6i \Delta m^2_{21} \Delta m^2_{31} \Delta m^2_{32} (m^2_\tau - m^2_\mu) (m^2_\mu - m^2_e) (m^2_\tau - m^2_e) J_{2211},$$

where the customary notation $\Delta m^2_{ij} \equiv (m^2_i - m^2_j)$ was used. One sees that if neutrinos are non-degenerate, the only possible source for the Dirac type CP violation is a nonzero Jarlskog invariant of Eq. (101), which in the standard parametrization (using Eq. (57)) reads

$$J_{2211} = \frac{1}{8} \sin(2\theta_{12}) \sin(2\theta_{13}) \sin(2\theta_{23}) \sin \delta,$$

(103)

which agrees with our previous findings that nonzero phase and mixing angles are the necessary and sufficient cause for Dirac type CP violation. The parameters $Q_{\alpha i\beta j}$ are called rephasing invariants, since their values do not alter under rephasing of lepton fields. Using them one can write the minimal set of independent CP violating parameters as

$$\text{Im}(Q_{\alpha i\beta j}), \quad \alpha \leq i, \quad \alpha \neq 1, \quad i \neq N,$$

(104)

where $N$ is the number of generations. For the three generation case the non-trivial invariants are then $\text{Im}(Q_{2231}), \text{Im}(Q_{2213}), \text{Im}(Q_{2233}),$ and $\text{Im}(Q_{2211}),$ which was used above.

### 6.3 Condition for Majorana CP violation

The Jarlskog invariant is not sensitive to Majorana phases. For the condition of Majorana type CP violation one then has to derive its own expression.

The CP invariance conditions of the mass matrices in Eqs. (90) - (91) imply the matrices $W_L$ and $W_R^\nu$ to be, in the case of three generations, of the form

$$W_L = \text{diag}(e^{i\alpha_1}, e^{i\alpha_2}, e^{i\alpha_3}), \quad W_R^\nu = \text{diag}(e^{i\beta_1}, e^{i\beta_2}, e^{i\beta_3}),$$

where $\alpha_i = (2n_i + 1)\pi/2, \beta_i = (2m_i + 1)\pi/2,$ where $n_i, m_i \in \mathbb{Z}.$ These with Eq. (89) constrain the phase of the Dirac and charged lepton mass matrices so that

$$\text{phase}(m^\nu)_{ij} = (n_i - m_j)\pi/2, \quad \text{phase}(H^l)_{ij} = (n_i - n_j)\pi/2,$$

(105)

are required in order to maintain CP invariance. Through a similar derivation as for Dirac neutrinos one achieves a condition for non-degenerate Majorana neutrinos for
three generations [46],

$$\text{Im} \left[ \text{Tr} \left( H^l M_{\text{Maj}}^* M_{\text{Maj}}^* H^{l*} M_{\text{Maj}} \right) \right] = 0,$$

(106)

which restricts $H^l$ to have [46] $\text{Im}(H^l_{12}) = \text{Im}(H^l_{13}) = \text{Im}(H^l_{23}) = 0$, which is in accordance with the CP invariance condition of Eq. (105). Therefore, any non-zero value of the expression of Eq. (106) is due to neutrino mass matrices.

For degenerate Majorana neutrinos the above conditions do not hold, since Eq. (106) is trivially satisfied. However, unlike for Dirac type particles, for which the CP invariance is assured by the degeneracy, for Majorana neutrinos there can be CP violation even in the case of degenerate masses [46]. For degenerate neutrinos the CP invariance condition reads [45]

$$\text{Tr} \left[ (M_{\text{Maj}} H^l M_{\text{Maj}}^* H^{l*})^3 \right] = 0.$$

As was noted before, the rephasing invariant quantities $Q_{\alpha i \beta j}$ are not sensitive for Majorana phases. Instead, one can construct rephasing invariant bilinears which do take the Majorana phases into account, namely [127,128]

$$S_{\alpha ij} = V_{\alpha i} V_{\alpha j}^* K_i^* K_j = U_{\alpha i} U_{\alpha j}^*,$$

which can be written using $Q_{\alpha i \beta j}$ as

$$Q_{\alpha i \beta j} = S_{\alpha ij} S_{\beta ij}^*,$$

(107)

and for which the minimal set of CP violating parameters reads [128]

$$\text{Im}(S_{\alpha ij}), \quad \alpha \leq i, \quad i \neq N, \quad j = N,$$

where $N$ is the number of generations. For three generations the independent non-trivial bilinears are then $\text{Im}(S_{113})$, $\text{Im}(S_{123})$, and $\text{Im}(S_{223})$. In Ref. [44] it is shown that, using unitarity of the PMNS matrix, the whole matrix can be written using six “Majorana type phases”. These phases are arguments of the rephasing invariant bilinears, i.e., $\arg(S_{\alpha ij})$. Because of that, they do not depend on a certain parametrization of the matrix. These six phases are $\gamma_{jk} \equiv \arg(S_{ijk})$.

### 6.4 Leptonic CP violation in neutrino oscillations

As will become clear in the next subsection, it is difficult to find processes that would explicitly violate CP symmetry in the lepton sector. As neutrino oscillations do manifest the violation is that exists, the process is studied in detail in this subsection.
When neutrinos are created in physical processes, they are in a quantum state that is an eigenstate of weak interaction. That is, they represent a flavor: electron, muon or tau. It has been observed that a neutrino originally of one flavor can be transformed into another flavor, or more technically: the probability of measuring a neutrino of a certain flavor is time-dependent. This phenomenon, neutrino oscillation, could not be possible if neutrinos were massless.

The flavor states we observe are not mass eigenstates, as was shown in Sec. 4.1.1, but instead linear superpositions of three mass fields which are referred to as $\nu_1$, $\nu_2$, and $\nu_3$. The mass states are eigenstates for the time evolution of the field, i.e.,

$$|\nu_f(t)\rangle = \sum_{i=1}^{3} e^{-iE_i t} |\nu_i\rangle,$$  \hspace{1cm} \text{(108)}$$

where the subscript $f$ refers to a flavor. As neutrinos travel almost at the speed of light, in natural units we can write $t \simeq x$, where $x$ is the distance the particle travels in time $t$. This assumption is justified as neutrinos are described as localized wave packets rather than plane waves [105], traveling at a group velocity close to the speed of light. The relativistic energy-momentum relation reads $E_i = \sqrt{p_i^2 + m_i^2}$, which can be simplified by assuming $p_i$s to be the same for all three neutrino mass fields, denoted by $p$, as in [124]. Observable neutrinos have to be ultrarelativistic: the scale of neutrino masses is under 1 eV [36], but their energy has to be of the order of 100 keV in order to be detected [84]. In that case one can write $p \gg m_i$, and the energy relation can be expanded as $E_i \simeq p + \frac{1}{2} m_i^2 / p$. With these Eq. (108) can be written as

$$|\nu_f(x)\rangle = \sum_{i=1}^{3} e^{-ipx} e^{-\frac{im_i^2}{2p}x^2} |\nu_i\rangle.$$  \hspace{1cm} \text{(109)}$$

This means that if the mass state neutrinos have different masses, they also evolve in time (in distance) differently. This is the mechanism for the oscillation. On the other hand, this is also why the observation of oscillations implies neutrinos to be massive and the masses of $\nu_1$, $\nu_2$, and $\nu_3$ to be distinct.

The amplitude of finding a neutrino of flavor $f$ to be of different flavor $f'$ after a distance $x$ is

$$\langle \nu_{f'} | \nu_f(x) \rangle = \sum_{i,j=1}^{3} \langle \nu_{f'} | U_{f',j} U_{i,j} | \nu_i(x) \rangle = \sum_{i=1}^{3} e^{-ipx} e^{-\frac{im_i^2}{2p}x^2} U_{f',i}^* U_{i,j}.$$ 

The factor $\exp(-ipx)$ can be left without attention, because it cancels in the calcu-
lation of the probability of the process, i.e.,
\[ P_{f \rightarrow f'} = \left| \langle \nu_f | \nu_f(x) \rangle \right|^2 = \sum_{i,j=1}^{3} |Q_{ff'ij}| \cos \left( \frac{x}{2p} \Delta m^2_{ij} - \arg (Q_{ff'ij}) \right), \tag{110} \]

where the notation \( \Delta m^2_{ij} \equiv m^2_i - m^2_j \) and \( Q_{ff'ij} \equiv U_{f,i}^* U_{f',j}^* U_{f,j} U_{f',i} \) was used. Let us then calculate the probability of the CP conjugated process \( \bar{f} \rightarrow \bar{f}' \). Notice that as the leptons in this case are antineutrinos, they can be regarded as neutrinos travelling backwards in time, i.e., \( \bar{\nu}_i(t) = e^{iE_i t} \bar{\nu}_i \), from which it follows that
\[ P_{\bar{f} \rightarrow \bar{f}'} = \left| \langle \bar{\nu}_{\bar{f}'} | \nu_f(x) \rangle \right|^2 = \sum_{i,j=1}^{3} |Q_{ff'ij}| \cos \left( \frac{x}{2p} \Delta m^2_{ij} + \arg (Q_{ff'ij}) \right). \]

The invariance of the CP symmetry requires that a process and its CP conjugated process are as likely to occur. Using Eq. (102) one finds that
\[ P_{f \rightarrow f'} - P_{\bar{f} \rightarrow \bar{f}'} = 4 \sum_{i=1<j}^{3} \text{Im} (Q_{ff'ij}) \sin \left( \frac{i\pi}{2p} \Delta m^2_{ij} \right), \tag{111} \]

whence one gets the condition for the CP invariance,
\[ \text{Im} (Q_{ff'ij}) = 0. \tag{112} \]

A non-zero \( J_{ff'ij} = \text{Im} (Q_{ff'ij}) \) is then a sufficient condition for the CP violation also in neutrino oscillations. It does not depend on whether neutrinos are Dirac or Majorana particles. The Majorana type CP violation cannot be detected this way, since the Majorana phases cancel in the expression. Eq. (112) depends only on the Dirac phase, and can therefore only manifest Dirac type CP violation. The same condition holds for quarks \[101\].

6.5 Conditions for CP odd effects

Although non-zero values for CP violating phases are a necessary condition for CP violation, their existence would still not conclude the existence of leptonic CP violation. For that, one has to show that the effect of the phases is CP odd, i.e., they affect CP transformed processes differently from the original processes. It is possible that the phases contribute equally to both processes and their CP mirrored images, which would not result in leptonic CP violation, as will be shown in this subsection.

For the Dirac type phase \( \delta \) it is generally acknowledged that it can lead to CP violating effects, as was the case in neutrino oscillation. An observation of non-zero
In any experiment would then state the existence of CP violation also in neutrino oscillation. However, it is unclear if non-zero Majorana phases would have similar implications. It is unclear if direct CP odd effects can be measured at collider experiments and the research on the subject is limited. In this subsection, the general conditions for the phenomenon are derived and the possibility of CP violation in some processes is studied.

Generally, for a process to be CP violating, one has to have an amplitude $A$ with at least two contributions, $A_1$ and $A_2$, for which the amplitudes are different and can be written as

$$A_i = |A_i| e^{i\phi_i} \equiv a_i e^{i\phi_i},$$

so that the total amplitude reads

$$A = a_1 e^{i\phi_1} e^{i\delta_1} + a_2 e^{i\phi_2} e^{i\delta_2},$$

where $\delta_i$ are the CP odd phases which take the opposite sign under CP transformation. For the CP transformed process the corresponding amplitude is then

$$\bar{A} = a_1 e^{i\phi_1} e^{-i\delta_1} + a_2 e^{i\phi_2} e^{-i\delta_2},$$

and the total difference between the processes is

$$\Delta_{CP} = 4a_1 a_2 \sin(\delta_1 - \delta_2) \sin(\phi_1 - \phi_2),$$

whence one can conclude the three necessary conditions for CP odd effects [47]: the amplitude of the physical process must have at least two contributions; the CP odd phases must be distinct for different contributions so that $\delta_i - \delta_j \neq n\pi$; also the CP even phases $\phi_i$ must hold $\phi_i - \phi_j \neq n\pi$.

It is for the third condition why neutrinoless double beta decays cannot be used for detecting CP violation but only to probe the magnitudes of Majorana phases [87]. This is because the amplitude for the process can be written as

$$A_{0\nu\beta\beta} = \sum_i (\lambda_i U_{ei}^2) m_i K,$$

where $\lambda_i$ are phase factors, $m_i$ the masses of the neutrino mass states, $U_{ei}$ are the elements of the first row of the PMNS matrix and $K$ is a kinematical factor, which is the same for all values of $i$. As the CP even phases in this case are $\phi_i = \arg(K)$ [87], the phases are the same for all contributions and the third condition is not satisfied.
On the contrary, as studied before, for neutrino oscillation a CP odd effect is possible. This is in accordance with the three conditions presented above: there are distinct contributions from different mass states; the CP odd phase results from $\delta$ in the mixing matrix; the phase factor $\exp(-im^2 x/2p)$ acts as the even CP phase.

In theory, the direct signs of leptonic CP violation are also possible through certain meson decays. K mesons can be produced at the LHC, and they decay semileptonically as

$$K^+ \rightarrow \pi^- \mu^+ \mu^+.$$ 

The dominant tree-level and first order correction decay channels for this are pictured in Fig. 2.

![Figure 2: Diagrams of the dominant tree-level (a) and a first order correction processes for the $K^+ \rightarrow \pi^- \mu^+ \mu^+$ decay. Figure from Ref. [87].](image)

The amplitude for the tree-level process (Fig. 2a) can be written as [87]

$$A_1 = \sum_i (\lambda_i U^2_{\mu})^* m_i K \equiv |A_1| e^{i\delta_1},$$

which by itself cannot lead to CP odd effects for the same reason as the neutrinoless double beta decay. Majorana phases will enter the picture if one can take into account the one-loop corrections, such as pictured in Fig. 2b, where the process $K^+ \rightarrow \mu^+ \nu \rightarrow \mu^+ \mu^+ \pi^-$ contains intermediate states that violate lepton number by two units. For this process CP even phases appear to the corresponding amplitude. This is because of the physical intermediate state, from the absorptive part of the amplitude [87]. The amplitude reads

$$A_2 = |A_2| e^{i\delta_2} e^{i\phi_2},$$

where $A_2$ is the magnitude of the amplitude, $\delta_2$ is the CP odd, and $\phi_2$ is the CP even phase. In total, the amplitude for the whole process and its CP conjugate process,
assuming that there are only these two contributions, are

\[ A(K^+ \to \mu^+ \mu^+ \pi^-) = |A_1| e^{i\delta_1} + |A_2| e^{i\delta_2} e^{i\phi_2} \]
\[ A(K^- \to \mu^- \mu^- \pi^+) = |A_1| e^{-i\delta_1} + |A_2| e^{-i\delta_2} e^{i\phi_2}, \]

which, according to Eq. (113), gives the difference for the rates of the processes,

\[ \Delta_{CP}(K^\pm \to \mu^\pm \mu^\pm \pi^\mp) = 4 |A_1| a_2 \sin(\delta_1 - \delta_2) \sin(-\phi_2), \]

(114)

and therefore, there can be CP odd effects from this kind of decay. Unfortunately, Eq. (114) requires the CP odd phases be distinct, \( \delta_1 \neq \delta_2 \), which is not the case if the only source for CP odd phases is the PMNS matrix. Therefore, in order to observe leptonic CP violation in K meson decays, one should have new physics sources, as suggested in Ref. [87]. Additionally, even if this process would manifest leptonic CP violation, it could not be observed: the upper limit for the branching ratio \( K^\pm \to \mu^\pm \mu^\pm \pi^\mp \) is \( 10^{-8} \) [36], which makes it unobservable.

In Ref. [128] the dependence of the squared amplitudes on observable effects for different leptonic processes is studied. They emphasize that unlike it is commonly stated, processes with lepton number violation are not the only processes that can be used to probe Majorana phases. Instead, the phases are present if there is lepton flavor violation, even if the total lepton number is conserved [128]. Because the charged current interactions can be the only source of lepton flavor violation in the SM, only processes with subprocesses pictured in Fig. 3 can be taken into consideration.

![Figure 3: Diagrams that are allowed in the processes studied in Refs. [127–129]. The diagram (a) conserves total lepton number although it violates two flavors whereas (b) violates both the total lepton number and the two flavors \( \alpha, \beta \). Figure from Ref. [128].](image)

For these processes the amplitudes (neglecting the kinematical parts) are [128]

\[ A_{\beta\alpha i} = V_{\beta i} V_{\alpha i}^*, \quad B_{\beta\alpha i} = V_{\beta i} V_{\alpha i} (K_i^*)^2, \]

where the former is for the process of Fig. 3a, and the latter for Fig. 3b. The dependence of the Majorana phases in the latter comes from the fact that in this approach
one only includes processes with no external neutrinos, i.e., the amplitudes must be invariant under a rephasing of the neutrino field \([128]\). In accordance with the above, in Ref. \([38]\) observables in the process \(e^+e^- \rightarrow \tau^+\tau^-\) are derived. In Ref. \([129]\) it is shown that by this approach CP odd effects can be manifested in lepton-lepton scattering processes, such as \(e + \mu \rightarrow e + \tau\), even without new physics contribution. The CP odd observable arising from the process is a function of rephasing invariant quartets and bilinears presented before. However, the possible magnitude and the detectability of the quantity were not discussed. Although theoretically observable effects can be found, they are often experimentally unaccessible \([59]\). Still, the inclusion of only processes with subprocesses as in Fig. 3 makes this approach of little use for collider processes with seesaw mediators.

Overall, the literature on possible measurable effects of leptonic CP violation is very limited. For CP odd effects in processes with seesaw mediators there does not seem to be any research. Instead, there is an abundance of research trying to connect the CP violating phases and neutrino mass mechanisms by studying effects that they cause to branching ratios of related processes. Although this is a weaker link than a measurement of direct CP odd effects would be, an observation of non-zero Majorana phases would certainly shed some light on the subject, not to mention that it would mean that neutrinos are Majorana particles.
7 Determination of Dirac and Majorana phases at colliders

Despite of all the efforts, leptonic CP violation has never been observed. Motivation for believing in its existence comes from the quark sector: the CP symmetry has been observed to be violated in meson decays. It has been a subject of theoretical research for decades, but only recently has there been experimental evidence for the mixing angle $\theta_{13}$ to be nonzero \cite{24}. This has opened the possibility for determination of the Dirac phase $\delta$, which always appears in conjunction with $\theta_{13}$. In Ref. \cite{98} it is estimated that the determination of $\delta$ could be possible in the 2020s.

The Dirac phase has been searched for at neutrino oscillation experiments for years. As can be seen from Eq. \eqref{111}, oscillation experiments are not sensitive to Majorana phases, and therefore Majorana type CP violation must be probed in other kinds of experiments. The Dirac phase is hoped to be measurable in future long baseline experiments \cite{42}. Although one can determine the squared mass differences, mixing angles, and Dirac type CP violation through oscillation experiments, the masses and the Majorana phases have to be studied elsewhere.

Majorana phases have been studied in neutrinoless double beta decay experiments, but no signs of them have yet been found. An observation of the neutrinoless double beta decay, $(A, Z) \rightarrow (A, Z + 2) + 2e$, would directly imply neutrinos to be Majorana particles. In those decays and rare lepton flavor violating decays the predictions of the event rates depend on Majorana phases \cite{136}.

As the Dirac and Majorana phases are part of the neutrino mixing matrix, one can verify or exclude the existence of leptonic CP violation by studying the PMNS matrix. The properties of the PMNS matrix are studied currently for other reasons as well: if light and heavy neutrinos mix, the PMNS matrix cannot be unitary. This is why a deviation of the PMNS matrix from the unitarity would imply extra particles \cite{58}.

The neutrino mass scale and the Majorana phases can also be studied at collider experiments. If neutrino masses are generated through mass mechanisms with new physics at low energies, the CP violating phases can be probed at the LHC.

As there are no similar established observable quantities as there are in neutrino oscillation experiments, one cannot use the same strategy as with oscillations. Instead, as stressed in the previous section, one has to study the existence of CP violation by studying the light neutrino mass parameters, which contain the CP violating phases. Due to the tiny mass of neutrinos, leptonic CP violation is more difficult to observe than the CP violation of the quark sector. Where as there are many possible observables, e.g., processes involving K and B mesons \cite{30,60}, for the
lepton sector these observables are not accessible by experiments, as was seen in the previous section \[42\].

The measurement of the neutrino masses and their scale are important for the possible observation of the CP violating phases, because the phases are visible in the branching ratios which depend on the mass ordering \[13\]. It is still unknown whether the neutrino masses are ordered as \(\nu_1 < \nu_2 < \nu_3\) (normal hierarchy) or as \(\nu_3 < \nu_1 < \nu_2\) (inverted hierarchy). If the lightest neutrino turns out to be of order 0.1 eV, the observed mass differences are so small that the masses can be approximated as degenerate. In this case neutrinos are called quasi-degenerate (QD).

As pointed out earlier, in the case of mass degeneracy, there cannot be Dirac type CP violation for degenerate neutrinos. This however, does not apply to Majorana type CP violation, for which there can be CP violation even in the case of degenerate masses \[46\], as was noted before.

The phenomenology of different seesaw mechanisms at the LHC has been studied extensively (see, e.g., \[9,10,12–15,18–21,26,32,33,122\]). However, in many approaches the focus is in finding new particles rather than Dirac and Majorana phases. The CP violating phases are often either ignored or the studied decay channels are chosen so that the effect of the phases on branching ratios is minimal. In this section possible processes with effects from Dirac and Majorana phases and their current experimental limits are reviewed.

7.1 Experimental effects of the scalar triplet

There are many models containing an \(SU(2)_L\) triplet scalar with \(Y = 2\), such as the type II seesaw studied in Sec.\[5.1.2\]. This is because the Yukawa couplings of the added scalar triplet to neutrinos are arbitrary, whereas the couplings between heavy and light neutrinos are strongly restricted \[103\]. The triplet \(\Delta\) couples to other particles as described in Sec.\[5.1.2\] which leads to the neutrino mass term of Eq.\[79\],

\[
(m^\nu)_{f_1f_2} = v_\Delta Y^\Delta_{f_1f_2}.
\]

The reason for the popularity of the scalar triplet model is visible above: the neutrino mass matrix is proportional to the leptonic Yukawa coupling constants. As no right-handed neutrinos are added to the SM particle content, the light neutrinos are Majorana particles. Therefore, by studying the scalar triplet decay one could observe both non-zero Dirac and Majorana phases. It turns out that the decays of the doubly charged scalar are especially interesting from the point of view of neutrino physics. This is because the branching ratios of the decays could describe not only the Dirac
and Majorana phases but also the neutrino mass hierarchy and the smallest light neutrino mass \([69,79,103]\).

Doubly charged scalars, if they exist, can be generated in a proton-proton collision at the LHC. Their most important decay channels are \([13,18]\)

\[
\begin{align*}
pp \rightarrow & \Delta^{++}\Delta^{--} \rightarrow l^+l^-l^-l^-, \\
pp \rightarrow & \Delta^{\pm\pm}\Delta^\mp \rightarrow l^\pm l^\mp l^\mp\nu,
\end{align*}
\]

which are widely studied in the literature (see, e.g., \([95,100]\)) and which are pictured in Fig. 4. Especially the first one is considered promising because it has almost no SM background \([79,100]\). However, for that channel there are technical challenges due to the fact that it is difficult to distinguish electrons and muons that are a product of tau decay from electrons and muons decaying directly from \(\Delta^{\pm\pm}\) \([79]\). This is why the decays \(\Delta^{\pm\pm} \rightarrow l^\pm l^\pm\), where \(l = e, \mu\), have gotten more attention in the literature than the tau channels.

In various studies final states with differing charged lepton flavors are considered best: on the one hand, it has been estimated that for normal hierarchy the scalar triplet’s decay channels containing taus dominate \([103]\); on the other hand, in many studies the same final states with taus are left without attention \([10,13]\), as the mass reconstruction for taus is more difficult due to the missing transverse energy from neutrinos \([79]\). The inclusion of the final states with taus is especially important when trying to distinguish the mass scale or hierarchy of neutrinos \([69]\).

![Figure 4: Feynman diagrams for the two dominant production and decay processes of the doubly charged scalar at the LHC. Figure from Ref. \[52\].](image)

The production of the doubly-charged Higgs has been previously studied experimentally at LEP \([6]\) and at Tevatron \([7]\). It is currently being searched for at the LHC. Nothing beyond the SM predictions has yet been observed. The latest lower limits (from 2012) for the doubly charged scalar are determined for the final states \(e^\pm e^\pm, \mu^\pm\mu^\pm\), and \(e^\pm\mu^\pm\) assuming a branching ratio of 100% for each case, and they are 409 GeV, 398 GeV, and 395 GeV, respectively \([2,52]\).

There is an upper limit for the triplet VEV from precision data, \(v_\Delta < 2\text{ GeV} \,[13]\), and therefore it can be chosen to be small. This bound lets the Yukawa couplings to be
of the same magnitude as for other particles. The plain type II seesaw model does not predict the mass of the triplet, but more complex models do. For example Littlest Higgs model, from Sec. 5.2.1, predicts the mass of the triplet to be \( m_\Delta \simeq 1 \text{ TeV} \) [91,95].

The doubly charged scalar has several decay channels: it can decay into a lepton pair, but also into \( W^\pm W^\pm, \Delta^\pm W^\pm, \) or \( \Delta^\pm \Delta^\pm, \) of which the first one is the most significant [79]. Also for the singly charged scalar the lepton channel is dominant, when \( v_\Delta < 10^{-4} \text{ GeV} \) [69]. The decay widths for the dominant channels are [13,103]

\[
\Gamma(\Delta^{\pm\pm} \rightarrow l_i^\pm l_j^\pm) = \frac{M_\Delta}{4\pi(1 + \delta_{ij})} |Y_{ij}|^2,
\]
\[
\Gamma(\Delta^{\pm\pm} \rightarrow W^\pm W^\pm) = kv_\Delta^2,
\]
\[
\Gamma(\Delta^{\pm} \rightarrow l_i^\pm \nu_j) = \frac{M_\Delta}{8\pi} |Y_{ij}|^2,
\]

where \( i, j = e, \mu, \tau \) and in the bosonic channel \( k \) is a function of masses \( m_{W^\pm} \) and \( m_\Delta. \)

The masses of the triplet are assumed degenerate, although there is mass splitting due to radiative corrections [126]. The ratio between the branching ratios of the leptonic and bosonic decay channels can be approximated as in Refs. [69,70], namely

\[
\frac{\Gamma(\Delta^{++} \rightarrow l_i^+ l_j^+)}{\Gamma(\Delta^{++} \rightarrow W^+ W^+)} \simeq \frac{|Y_{ij}|^2 v^4}{M_{\Delta}^2 v_\Delta^2} \simeq \left( \frac{M_{\text{Maj}}}{M_\Delta} \right)^2 \left( \frac{v}{v_\Delta} \right)^4,
\]

whence one can conclude that for \( m_\nu \sim 1 \text{ eV}, M_\Delta \sim 1 \text{ TeV} \) the values of \( v_\Delta < 10^{-4} \text{ GeV} \) imply the domination of leptonic modes [126]. Then, ignoring the bosonic decay channel is a good approximation [69].

When the contribution of the bosonic decay is assumed to be negligible, one gets the branching ratios [79]

\[
\text{BR}_{ij} = \frac{\Gamma(\Delta^{\pm\pm} \rightarrow l_i^\pm l_j^\pm)}{\sum_{k,l} \Gamma(\Delta^{\pm\pm} \rightarrow l_k^\pm l_l^\pm)} = \frac{2}{1 + \delta_{ij}} \frac{|(M_{\text{Maj}})_{ij}|^2}{\sum_{k,l} |(M_{\text{Maj}})_{kl}|^2},
\]

(117)

where

\[
\sum_{k,l} |(M_{\text{Maj}})_{kl}|^2 = \sum_{i=1}^{3} m_i^2 = \begin{cases} 3m_0^2 + \Delta m_{21}^2 + \Delta m_{31}^2 & (\text{NH}) \\ 3m_0^2 + \Delta m_{21}^2 + 2|\Delta m_{31}^2| & (\text{IH}) \end{cases},
\]

where NH refers to normal hierarchy and IH to inverted hierarchy. By assuming normal hierarchy, \( \sin^2 \theta_{13} < 0.05 \), and a zero mass for the lightest neutrino, and by
defining \( r \equiv \Delta m^2_{21} / |\Delta m^2_{31}| \simeq 0.03 \), Eq. (117) gives \[79\]

\[
\begin{align*}
\text{BR}^{\text{NH}}_{\nu e} &\simeq r \sin^4 \theta_{12} + 2 \sqrt{r} \sin^2 \theta_{12} \sin^2 \theta_{13} \cos(\Phi_2 - 2\delta), \\
\text{BR}^{\text{NH}}_{\nu e\mu} &\simeq 2 \left[ r \sin^2 \theta_{12} \cos^2 \theta_{12} \cos^2 \theta_{23} + \sin^2 \theta_{23} \sin^2 \theta_{13} \\
&\quad + 2 \sqrt{r} \sin \theta_{12} \cos \theta_{12} \sin \theta_{23} \cos \theta_{23} \sin \theta_{13} \cos(\Phi_2 - 2\delta) \right], \\
\text{BR}^{\text{NH}}_{\mu\mu} &\simeq \sin^4 \theta_{23} + 2 \sin^2 \theta_{23} \cos^2 \theta_{23} \cos^2 \theta_{12} \sqrt{r} \cos \Phi_2 + r \cos^4 \theta_{23} \cos^4 \theta_{12} \\
&\quad - 4 \sin^3 \theta_{23} \cos \theta_{23} \sin \theta_{12} \cos \theta_{12} \sin \theta_{13} \sqrt{r} \cos(\Phi_2 - \delta),
\end{align*}
\]

whence one notices that the branching ratios depend slightly on CP phases and that a nonzero value of \( \theta_{13} \) has an impact on the ratios. In fact, a zero value for \( \theta_{13} \) would imply that the branching ratios for channels containing electrons would not have CP phases \[79\]. For inverted hierarchy the situation is different: even in the case of \( \theta_{13} = 0 \) the channels with electrons depend on Majorana phases \[79\]. In Fig. 5 the branching ratios are pictured as a function of lightest neutrino masses with two values of \( \sin \theta_{13} \) and for both hierarchical mass orderings. One notices that for inverted hierarchy the lines corresponding to \( \theta_{13} = 0 \) and \( \theta_{13} = 0, 1 \simeq 5, 7^\circ \) are almost overlapping. For normal hierarchy the lines differ significantly, especially for channels containing \( e \), as discussed above. Moreover, there is a great difference between different channels: for the final state \( ee \) the difference between IH and NH is very clear for small neutrino masses whereas it diminishes for bigger masses; instead, for the \( e\tau \) and \( e\mu \) channels even for small masses the NH and IH ranges are completely overlapping.
Figure 5: Branching ratios of the doubly charged scalar into a lepton pair as a function of lightest neutrino mass with 2σ significance. NH refers to normal hierarchy and IH to inverted hierarchy. The thin red and blue lines correspond to the BRs with fixed values for the Majorana phases when $\theta_{13} = 0$: $\Phi_2 = \pi$ and $\Phi_1 = 0, \pi/4, \pi/2, 3\pi/4, \pi$. Figure from Ref. [79]

 neutrino mass parameters could then be determined by studying branching ratios of the decay of the doubly charged scalar. Using $\chi^2$-distribution analysis it is found [79] that the Majorana phases have completely different discovery potentials in different mass orderings. For the quasi-degenerate case the possible accuracy of the determination of the Majorana phases depends on their values: for $\Phi_1 = \Phi_2 = \pi$ the accurate determination is more difficult, although unambiguous; for $\Phi_1 = \Phi_2 = \pi/2$ the determination is more accurate but there is a four-fold ambiguity in the values of the Majorana phases, as shown in Fig. 6.
Figure 6: Determination of the Majorana phases for quasi-degenerate neutrinos with $m = 0, 15\, \text{eV}$ with 1000 doubly-charged scalar pair decays. Figure from Ref. [79]

However, the task of analyzing the leptonic branching ratios becomes more difficult if one does not ignore the $\Delta^{\pm\pm} \rightarrow W^\pm W^\pm$ channel, since this channel is difficult to measure [103]. It has been suggested that this could be overcome by studying the six different leptonic decay channels assuming fixed values for squared neutrino mass differences and mixing angles from oscillation experiments. With these assumptions, with small $\theta_{13}$, one can derive a parameter $C'_1$ [103]

$$C'_1 \equiv \frac{2\text{BR}_{\mu\mu} + 2\text{BR}_{\tau\tau} + 2\text{BR}_{\mu\tau} - 2\text{BR}_{e\tau}}{2\text{BR}_{ee} + \text{BR}_{e\mu} + \text{BR}_{e\tau}} = \frac{-m_1^2 + m_2^2 + 3m_3^2}{2m_1^2 + m_2^2} + \mathcal{O}(\sin^2(\theta_{13})), \quad (119)$$

where the branching ratios are as in Eq. (117). The mass hierarchy is manifested in the magnitude of the parameter $C_1$: if $C_1 < 1$, the mass hierarchy is inverted; if $C_1 > 1$, it is normal; if $C_1 \simeq 1$, masses are degenerate. From the Eq. (7) one can solve the lightest neutrino mass, so that, e.g., for the NH case one has [103]

$$m_1^2 = \frac{\Delta m_{\text{sol}}^2 (4 - C'_1) + 3\Delta m_{\text{atm}}^2}{3(C'_1 - 1)}, \quad (120)$$

stating that the absolute neutrino masses could be determined at collider experiments. For the IH case there exists a similar relation. In Fig. 7 the branching ratios of different di-lepton channels are pictured as a function of both for the $\theta_{13} = 0$ and $\theta_{13} = 0.22 \simeq 12, 7^\circ$ with maximal Dirac CP phase $\delta = \pi/2$ cases, assuming a zero value for the lightest neutrino. In the NH case the ratios are presented as a function of the remaining Majorana phase (the other is rephased away), whereas in the IH case they are presented as functions of the difference of the two possible Majorana phases. For both cases the existence of Dirac CP violating phase causes asymmetry.
to the distributions and the non-zero theta makes it possible for the $e^\pm\mu^\pm$ and $e^\pm\tau^\pm$ final states to become observable in normal hierarchy [103]. One notices that the effect of Majorana phases on the branching ratios is considerable for IH, where as for NH the dependence is only moderate. Similar results are derived in Ref. [69].

Figure 7: Estimated branching ratios of the decay $\Delta^{\pm\pm} \rightarrow l^\pm l^\pm$, $l^\pm = e, \mu, \tau$. The upper (lower) figures correspond to the NH (IH) case. For the former the branching ratios are shown as a function of a Majorana phase, whereas for the latter as a function of the difference of Majorana phases. The lightest neutrino mass is assumed to be zero. Figures from Ref. [103]

In the case of degenerate neutrinos, the branching ratios of Eq. (117) do not anymore depend on the neutrino mass. The absolute mass scale cannot then be probed at collider experiments.

In the case zero values for both Majorana phases and $\theta_{13}$ the only significant non-zero branching ratios are $\text{BR}_{\mu\mu} = \text{BR}_{\tau\tau} = \text{BR}_{ee} = 1/3$ [103]. Non-zero contribution of the other states or deviations from 1/3 for contributing final states would then imply either a non-zero value for at least one of the Majorana phases or non-zero $\theta_{13}$. In Fig. 8 are pictured the effects of non-zero Majorana phases on different branching ratios.

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Figure 8: Impact of non-zero Majorana phases on different flavored branching ratios of the process $\Delta^{\pm \pm} \rightarrow l^\pm l^\pm$, $l^\pm = e, \mu, \tau$ for degenerate light neutrino masses. The mixing angle $\theta_{13}$ and Dirac CP violating phase $\delta$ are assumed to be negligible. Figures from Ref. [103].

The dependance of the branching ratios for different flavored di-lepton final states of the doubly charged scalar decays on Majorana phases is shown in Fig.9. The existence of the Majorana phases makes the spread of the predicted values of the BRs larger, but some characteristics can still be noticed: different flavored di-lepton channels dominate for different mass hierarchies and some information on the Majorana phases could be achieved. The impact of Majorana phases on the different branching ratios is most remarkable on different flavored lepton final state with IH (Fig.9 lower right diagram), where the dominant branching ratios changes as a function of the Majorana phase.
Overall, the effect of CP phases exists for models with a scalar triplet, but the connection is weak. Only in specific cases some information on the Majorana phases could be determined directly from the branching ratios of the doubly charged scalar. For Dirac CP violating phase the situation is even more difficult: its effects are weak and mix easily with the signs of Majorana phases as in Eq. (118).

7.2 Experimental effects of heavy neutrinos

Heavy singlet Majorana neutrinos are present in many extensions, from the simplest seesaw mechanism to complex GUT models. For heavy neutrinos, unlike for scalar triplet, the SM background is different for distinctly flavored final states: this requires
the searches to separate different flavor states from each others; the final states including taus are interfered by the background too much to give valid information \cite{12}. Most of the studies on heavy neutrinos assume there to be something more than just plain heavy right-handed neutrinos \cite{9,12,29,35,57,58,71,92}. This is both because for the plain SM extended with right-handed neutrinos there does not seem to be a physical reason for heavy neutrinos to be within the energy reach of the LHC, but also because the gauge couplings to $N$ in processes described in Eq. (121) are strongly suppressed within the SM content, as they are of the order $O(m_\nu/M_N)$ \cite{71}.

If there are SM singlet neutrinos, which generate small neutrino masses for observed neutrinos via the seesaw mechanism, as presented in Sec. 5.1, they could, in theory, be observed at the LHC. However, if their masses are of order 1 TeV, the Yukawa coupling constants might become too small to be detectable \cite{103}. In order to have the observed light neutrino masses and Yukawa couplings of the same size as for rest of the SM particles while having an energy for singlet neutrinos of 1 TeV, the models have to be more complex than the basic seesaw mechanism. This could be done by introducing new gauge interactions as in left-right symmetric (L-R) models. Then the heavy neutrinos could be observable up to 2 TeV \cite{126}.

Within the SM gauge group, heavy neutrino singlets $N$ can be produced in processes \cite{71}

\begin{equation}
pp \rightarrow W^{\pm*} \rightarrow t^\pm N,
\end{equation}

\begin{equation}
pp \rightarrow Z^* \rightarrow NN.
\end{equation}

The singlet $N$ has four main decay channels,

\begin{equation}
N \rightarrow l^\pm W^\mp, \quad N \rightarrow \nu Z, \quad N \rightarrow \nu H,
\end{equation}

which further decay into leptons. The channel with $W$ boson is presented in Fig. 11a. There are other channels to produce $N$, but the background effects are so large that the channels are unobservable \cite{12}. The decay channels of Eq. (123) are possible only when the heavy neutrinos are heavier than the gauge bosons. The lightest of the bosons are $W$ ($\sim 80$ GeV), so in the case of $m_N < 80$ GeV, instead of the bosonic decays the heavy neutrinos decay into three leptons via off-shell bosons \cite{12}.

The most studied final state is \cite{21,93,130}

\begin{equation}
pp \rightarrow l^\pm N \rightarrow l^\pm l^\mp + 2 \text{ jets}
\end{equation}

with no missing energy being a characteristic signature of heavy singlets \cite{66}. This
is because the heavy Majorana neutrinos are considered heavier than the $W$ boson, which means that the first $W$ in the process is off-shell, but the second $W$ in turn is on-shell \[66\]. This channel has been probed at the CMS experiment at $\sqrt{s} = 7$ TeV corresponding to $5.0 \text{ fb}^{-1}$, but nothing over the SM background has yet been observed \[55\].

In Ref. \[10\] it is pointed out that the tri-lepton final state via a process

$$pp \rightarrow l^{\pm} N \rightarrow l^{\pm} l^{\mp} l^{\mp} \nu$$

could perform as a better channel for detecting effects of heavy neutrinos than the two-lepton final state. This is because the like-sign di-lepton final state requires neutrinos to be Majorana particles. Not all the seesaw models contain heavy Majorana states and therefore do not contain lepton number violation. Instead, inverse seesaws I and III contain heavy quasi-Dirac neutrinos that do not produce the same-sign di-lepton final states \[26\], but there are final states with three charged leptons even in this case.

In a recent study it has been stated that the process

$$pp \rightarrow W^* \gamma^* jj \rightarrow l^{\pm} N + 2 \text{jets}, \quad (124)$$

with four different channels pictured in Fig. 10 is dominating over the $pp \rightarrow Nl^{\pm}$ channel for the heavy neutrino masses $m_N > 300 \text{ GeV} \[66\]$. This is because the process of Eq. 124 gets contribution from both hadronic and electroweak channels, as pictured in Fig. 11; it turns out that the dependence on heavy neutrino mass of the electroweak channels of the process is not as strong as it is for the $pp \rightarrow Nl^{\pm}$ process, and its rate drops slower \[66\]. This far heavy neutrino masses that high have not been experimentally probed, but the experiments at the LHC are aiming to eventually reach $m_N = 500 \text{ GeV}$.

![Figure 10: Different production channels of heavy Majorana neutrinos of the process $pp \rightarrow l^{\pm} N + 2 \text{jets}$ from Ref. \[66\].](image)

If the heavy neutrinos are Dirac particles, their decay must conserve lepton num-
ber, and the same-sign final states with two charged leptons are not allowed. For the same-sign channel it is estimated that the LHC can exclude or confirm the heavy Majorana neutrinos up to 200 GeV at a statistical significance of $5\sigma$ for the process $pp \rightarrow \mu^\pm \mu^\pm jj$. For processes $pp \rightarrow e^\pm e^\pm jj$ and $pp \rightarrow e^\pm \mu^\pm jj$ the background effects are larger due to detector properties \[12\]. For the channel $pp \rightarrow e^\pm e^\pm jj$ at $5\sigma$ the lower limit of the neutrino mass is 145 GeV \[55\].

For heavy Dirac singlets the channel $pp \rightarrow e^\pm \mu^\pm jj$ is studied. The SM background is larger for opposite charged lepton final states. In fact, it is considered unlikely that the LHC could make any conclusions of heavy Dirac neutrinos due to large background effects \[12\]. They could better be found at $e^+e^-$ colliders (CLIC and ILC) through a process $e^+e^- \rightarrow N\nu \rightarrow lW\nu$ \[12\].

If there are new gauge interactions, the neutrino singlets could be produced in pairs in a process \[71\]

$$pp \rightarrow Z' \rightarrow NN,$$

where $Z'$ is a generic new neutral gauge boson, e.g., of the $U(1)_{B-L}$ gauge symmetry. The heavy neutrinos $N$ decay as in Eq. \[123\]. They can decay conserving lepton number, i.e., $NN \rightarrow l^\pm W^\mp l^\pm W^\mp$, or violating it, i.e., $NN \rightarrow l^\pm W^\mp l^\pm W^\mp$. The gauge bosons further decay into four jets in both cases \[71\].

The mass of the heavy neutrino, $M_N$, is not constrained by any symmetry or new physics, but in a recent study it is stated that a TeV scale L-R symmetric theory could be detectable at the LHC \[58\], with the final state of two same-sign charged leptons and two jets signal with no missing energy being in accordance with the L-R models.

![Figure 11: Feynman diagrams for four possible processes within a L-R model](image)

The signs of the minimal L-R extension are easier to observe than plain heavy right-handed neutrinos. In Fig. 11 the most significant production and decay pro-
cesses of the heavy neutrinos within the minimal left-right symmetric model are presented, namely

\[ pp \rightarrow W_L^*/W_R \rightarrow l^\pm N, \]

and depending on the Majorana or Dirac nature the decay channel is lepton number violating or conserving. Without \( W_R \) the decay channels of \( N \) are presented in Eq. (123). Within an L-R extension \( N \) can also decay into three body final states: \( N \rightarrow lW_R^* \rightarrow ljj \), which is suppressed by the mass of \( W_R \) and not by the small mixing value \[57\]. The decay mode \( N \rightarrow lW_R^* \rightarrow ll\nu \) is strongly suppressed by both mixing and the mass of the \( W_R \) boson, and it is therefore neglected \[57\]. In addition, in order to be able to determine the lepton number violating or conserving nature of the process, one needs to study processes without neutrinos in the final states \[29\].

Fig. 11a corresponds to the traditional type I seesaw scheme; the mixing between heavy and light neutrinos is \( V_{lN} \simeq \sqrt{m_\nu/M_N} \), which is too small to be detected assuming the scale of \( m_\nu \) to be \( \leq 0.5 \text{eV} \) and a TeV-scale \( M_N \). Only for certain textures of \( m_\nu \) the mixing is not negligible \[90\]. For the Fig. 11b case, the LHC has excluded \( M_{W_R} \) up to 2.5 TeV for a TeV scale \( M_N \). In the case of non-negligible \( V_{lN} \) the process of Fig. 11c would give a dominant contribution to the \( l^\pm l^\pm jj \) signal, whereas the case of Fig. 11d is strongly suppressed. The strong contribution of the Fig. 11c case is due to the facts that production of the heavy neutrino through the channel \( pp \rightarrow W_R \rightarrow Nl^\pm \) is suppressed only by the ratio \( (M_W/M_{W_R})^4 \) (and not dependent on mixing \( V_{lN} \)) and that the heavy neutrino can decay through an on-shell \( W \) boson, i.e., \( N \rightarrow l^\pm W \rightarrow l^\pm jj \).

Ref. \[58\] states that due to vacuum stability heavy neutrinos should be lighter than the new gauge bosons in minimal L-R models. It is estimated that if the new gauge bosons have masses below 4-5 TeV, there should be enough information by combining the oscillation data and the LHC results to determine the Dirac mass matrix. The signs of the channel of Figs. 11a and 11b have been looked for at the CMS and ATLAS experiments. In Ref. \[58\] it is emphasized that the signal of Fig. 11c could be dominant and should be searched for at the experiments as well. The current lower limit for \( M_{W_R} \) is \( \simeq 2.5 \text{TeV} \), which excludes also right-handed heavy neutrinos \( M_{N_\mu} < M_{W_R} \) for L-R models with exact L-R symmetry \[56\].
For the bosonic decays the partial widths are \[13, 71, 72\]

\[\Gamma(N \to l^\pm W^\mp) = \frac{g^2}{64\pi} |V_{lN}|^2 \frac{m_N^3}{m_W^2} \left(1 - \frac{m_W^2}{m_N^2}\right) \left(1 + \frac{m_W^2}{m_N^2} - \frac{2m_W^4}{m_N^4}\right)\]

\[\Gamma(N \to \nu Z) = \frac{g^2}{64\pi \cos \theta_w} |V_{lN}|^2 \frac{m_N^3}{m_Z^2} \left(1 - \frac{m_Z^2}{m_N^2}\right) \left(1 + \frac{m_Z^2}{m_N^2} - \frac{2m_Z^4}{m_N^4}\right)\]

\[\Gamma(N \to \nu H) = \frac{g^2}{64\pi} |V_{lN}|^2 \frac{m_N^3}{m_W^2} \left(1 - \frac{m_H^2}{m_N^2}\right)^2,\]

where \(\theta_w\) is the weak mixing angle and \(g\) is the coupling constant. The branching ratios \(\Gamma_i/\sum_j \Gamma_j\) then depend only on the masses of the particles, as couplings and the mixing matrix elements are reduced. Compared to the scalar triplet BR of Eq. (117) this situation is less attractive for light neutrino mass parameters and hence for CP phases.

In Ref. [14] it is noted that if one assumes extra gauge interactions, the discovery potential of the heavy neutrino singlets is larger. The addition of \(Z'\) into the seesaw I scenario is widely studied in the literature [9,14,35,71,99,126]. The insertion of the new gauge boson causes the possible mass range of the heavy neutrinos to increase up to 800 GeV [9,21,126], whereas without the extension the heavy Majorana neutrino mass can be tested at 2\(\sigma\) (5\(\sigma\)) accuracy only up to 375 (250) GeV [29]. However, the branching ratios of to \(NN\) decays do not depend on Majorana phases [71].

For the inverse type I seesaw model the same-sign dilepton channel does not contribute, which is a great difference compared to the models studied above. The non-contribution of this channel is due to the pseudo-Dirac nature of the two sets of new particles. However, the lepton number violating three-lepton final state is possible [57]. The difference of the decays in case of Dirac and Majorana heavy neutrinos is pictured in Fig. [12]

![Feynman diagrams](image)

Figure 12: Feynman diagrams of the most promising channels for Majorana (left) and Dirac (right) heavy neutrino signals at the LHC assuming existence of the left-right symmetry. Figure from Ref. [57].

For Dirac type heavy neutrinos the final state with three charged leptons and one light neutrino is the most promising, whereas for the detection of heavy Majorana
neutrinos the best final state has two same sign charged leptons and two jets with no missing energy \cite{29}. As the different branching ratios are not connected to the neutrino mass matrix, the connection to CP violating phases is not as direct as it is with the scalar triplet. However, some information on the Majorana phases could be determined.

One can learn about the light neutrino mass matrix properties by studying the mixing between light and heavy neutrinos. In Refs. \cite{71,114} an interesting relation is derived:

\[ V_{lN}^* M_N^{\text{diag}} V_{lN}^\dagger = -V_{PMNS}^* m_\nu^{\text{diag}} V_{PMNS}^\dagger, \]

(125)

where the mixing between heavy and light neutrinos can be written in the Casas-Ibarra parameterization (using Eq. (129))

\[ V_{lN} = m_\nu M_R^{-1} = iU_{PMNS} \sqrt{\text{diag}(m_1, m_2, m_3)} \Omega \frac{1}{\sqrt{\text{diag}(M_{R,1}, M_{R,2})}}, \]

and where the matrix \( \Omega \) is parameterized using rotational angles \( w_{ij} \), so that \( \Omega(w_{21}, w_{31}, w_{32}) = R_{12}(w_{21}) R_{13}(w_{31}) R_{23}(w_{32}) \), where, e.g.,

\[ R_{12} = \begin{pmatrix} \cos w_{21} & -\sin w_{21} & 0 \\ \sin w_{21} & \cos w_{21} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \]

With this notation, one gets, e.g., for \( V_{lN}^{e1} \), \cite{114}

\[ -iV_{lN}^{e1} \sqrt{M_1} = \sqrt{m_2 c \theta_{13} s \theta_{12 c} w_{21} c w_{31} + \sqrt{m_1 c \theta_{12 c} c \theta_{13 c} w_{31} e^{i \Phi_1/2}} + \sqrt{m_3 s \theta_{13 c} s w_{31} e^{i (\Phi_2/2 - \delta)}}}, \]

whence one sees that the determination of the Majorana and Dirac phases \( \Phi_1, \Phi_2, \) and \( \delta \) is far from unambiguous: their values depend not only on heavy-light mixing elements but also on the absolute neutrino masses, mixing angles, and the rotational angles of \( R \), which are completely unknown \cite{71}. Even if the values of other parameters would be known, the magnitude of \( V_{lN} \sim \sqrt{m_i/M_1} \) is very small which means that their dependance on the CP phases is so subtle that it is possible to be undetectable.

The branching ratios of the decay \( N \rightarrow l^\pm W^\mp \), where \( l = e, \mu, \tau \), as a function of the Majorana phases for NH and IH are shown in Fig.13 where the Casas-Ibarra parameterization with random values of \( \Omega \) is used. As discussed above, no accurate information can be extracted from the impact of Majorana phases, but only some overall tendencies that they cause on the branching ratios.
Figure 13: Branching ratios of the process $N \to l^\pm W^\mp$, where $N$ is the lightest of the heavy neutrinos. The left (right) figure corresponds to the NH (IH) case, where the lightest neutrino is assumed massless. The assumed masses for the heavy neutrino and Higgs boson are $m_N = 300$ GeV and $m_H = 120$ GeV, and the values of the matrix $\Omega$ are random. Figure from Ref. [71]

Non-zero values of the Majorana phases could also be determined at collider experiments by studying observed deviations on the rate of the process $pp \to l^\pm l^\pm + 2$ jets $+ X$, where $X$ denotes two quark jets, with $\alpha_1 = \alpha_2 = 0$ [112]. The estimated values for the cross sections are proportional to the squares of the mixings, $|V_{lN}|^2$.

From neutrinoless double beta decay experiments there are stringent conditions for the heavy-light Majorana mixing (assuming three heavy neutrinos), i.e.,

$$\sum_{i=1}^{3} \frac{|V_{eN_i}|^2 e^{i\phi_{N_i}}}{M_N} + \sum_{i=1}^{3} \frac{|V_{\mu N_i}|^2 e^{i\phi_{\nu_i}} m_{\nu_i}}{q_0^2} \leq 5 \times 10^{-8} \text{ GeV}^{-1}, \quad (126)$$

where for the neutrino momentum $q_0 \simeq 105$ MeV. Ref. [112] uses the lower limit on the half-life of the neutrinoless double-beta decay from 2000, namely $T_{1/2}^{0\nu\beta\beta}(^{76}\text{Ge}) \geq 1.9 \times 10^{25} \text{ yrs}$ [110], whereas the most recent lower limit, from 2012, reads $T_{1/2}^{0\nu\beta\beta}(^{136}\text{Xe}) \geq 1.6 \times 10^{25} \text{ yrs}$ [31], but as there is no significant change, their results are still plausible. The limits for the heavy-light neutrino mixing with the 2000 the neutrinoless double-beta decay limits are [112]

$$\frac{|V_{eN}|^2}{M_N} \leq 5 \times 10^{-8} \text{ GeV}^{-1}, \quad |V_{\mu N}|^2 \leq 10^{-4}, \quad |V_{\tau N}|^2 \leq 10^{-2}. \quad (127)$$
If the Majorana phases are non-zero, they can affect the rate of the like-sign di-lepton final state. In the case of observing more events of the process \( pp \rightarrow l^\pm l^\pm + 2\text{ jets} + X \) than are expected from the bounds from neutrinoless double beta decay experiments, one could conclude that this is due to the Majorana phases. This can be seen from Eq. (126): the magnitudes of the mixings can be larger if the Majorana phases have suitable values than zero values. In Fig. 14 from Ref. [112] the estimated maximal cross sections of the processes \( pp \rightarrow e^\pm e^\pm jj + X \) and \( pp \rightarrow e^\pm e^\pm W^\mp X \) with no CP violation are pictured along cross sections of the same processes with different mixings.

\[ \text{Figure 14: The estimated maximal cross sections of the processes } pp \rightarrow e^\pm e^\pm jj + X \text{ (left)} \text{ and } pp \rightarrow e^\pm e^\pm W^\mp X \text{ (right) at the LHC. The lower lines correspond to the values of the mixing as in Eq. (127) and assume the Majorana phases to be zero; for the upper lines the values are assumed to be } |V_{eN}|^2 \simeq |V_{\mu N}|^2 \simeq |V_{\tau N}|^2 \simeq 10^{-3}. \text{ Figures from Ref. [112].} \]

In summary, weak indirect connections are possible to construct between Majorana phases and heavy neutrino singlets at collider experiments. For the Dirac CP phase the situation does not look as promising. Instead, if heavy neutrinos exist, the value of \( \delta \) should be determined at neutrino oscillation experiments.

### 7.3 Experimental effects of the heavy fermion triplet

The phenomenology of triplet fermions is more promising than the case of fermion singlets, because the triplet components can couple to gauge bosons as well [57]. However, there is no physical reason for the mass of the fermion triplet in type III seesaw mechanism to be within the reach of the LHC [72]. Because of this, there are models that contain a fermion triplet are more complex than the plain seesaw
For example, Refs. [14,15] study the inverse type III seesaw, where two quasi-degenerate Majorana fermion triplets form a Dirac fermion triplet. The physical new particles are a heavy Dirac neutrino $N$ and two heavy charged leptons $E_{-1}$ and $E_{+2}$. The production processes for these particles are the same as for the ones of a Majorana triplet, but the decays are different, since Dirac particles do not violate lepton number conservation, but Majorana particles do.

Phenomenology of minimal non-supersymmetric $SU(5)$ theory of Sec.5.2.3 is studied in Refs. [26,33,34]. In the theory one adds an adjoint fermionic multiplet, $24_F$, which is decomposed under the SM gauge group so that it contains both a fermionic triplet and a singlet. The underlying $SU(5)$ structure requires one light neutrino to be massless [26]. This is not against the oscillation data, since the data imply the existence of at least two massive neutrinos. If only two neutrinos are massive, there can be only one Majorana phase in the theory.

In the basic type III seesaw an insertion of a fermion triplet $\Sigma_i$ leads to charged leptons $E_i$ and Majorana neutrinos $N_i$. At proton-proton collisions an $E_iE_i$ or $E_iN_i$ pair can be produce, i.e.,

$$pp \to E^+E^-, \quad pp \to E^\pm N.$$  

For $N_i$ the decay channels are the same as in the previous subsection, whereas the heavy charged leptons decay into $E^\pm \to \bar{\nu}W^\pm/l^\pm Z/l^\pm H$. There are plenty of different final states for the fermion triplet model. The mass splitting between the components is only of order $10^2$ MeV, and can thus be neglected [114]. If it is taken into account, one has more possible decay channels, which are not considered here [72]. The most studied final states are the same as for the fermion singlet, namely the final states with two same-sign charged leptons or three charged leptons [13,34,72],

$$pp \to E^\pm N \to l^\pm l^\mp l^\pm X,$$

$$pp \to E^\pm N \to l^\pm l^\mp l^\pm X,$$

$$pp \to E^\pm N \to l^\pm l^\pm X,$$

where $X$ denotes other final state particles, i.e., quarks and neutrinos. For all of these channels the SM background is quite small. For the last two the discovery potential is found to be equal [13].

It would be easiest to distinguish the fermion triplet scenario from others by studying final states that do not exist for fermion singlets or scalar triplets. Such
states include a four-lepton final state with total charge $Q = \pm 2$,

$$pp \rightarrow E^\pm N \rightarrow l^\pm Z l^\pm W^\mp \rightarrow l^\pm l^\pm l^\pm l^- X,$$

which offers a good possibility to measure the heavy triplet mass: it could be found with luminosity $\sim 15\, fb^{-1}$ [13]. In Ref. [72] it is suggested to study the lepton flavor violating channel $pp \rightarrow l^+l^- + 4\, \text{jets}$. The channel has a large SM background, but its advantage is that it can be used for model discrimination: it is a signature of the type III seesaw, whereas di-lepton channels are possible for all the basic seesaw models.

The corresponding partial widths for the $E^\pm$ decays are [13,72]

$$\Gamma(E^\pm \rightarrow \bar{\nu}W^\pm) = \frac{g^2}{32\pi} |V_{lN}|^2 \frac{m_E^3}{m_W^2} \left( 1 - \frac{m_W^2}{m_E^2} \right) \left( 1 + \frac{m_W^2}{m_E^2} - 2 \frac{m_W^4}{m_E^4} \right),$$

$$\Gamma(E^\pm \rightarrow l^\pm Z) = \frac{g^2}{32\pi \cos^2 \theta_W} |V_{lN}|^2 \frac{m_E^3}{m_Z^2} \left( 1 - \frac{m_Z^2}{m_E^2} \right) \left( 1 + \frac{m_Z^2}{m_E^2} - 2 \frac{m_Z^4}{m_E^4} \right),$$

$$\Gamma(E^\pm \rightarrow l^\pm H) = \frac{g^2}{64\pi} |V_{lN}|^2 \frac{m_E^3}{m_W^2} \left( 1 - \frac{m_H^2}{m_E^2} \right)^2,$$

for $N$ the corresponding values are given in the previous subsection. As with fermion singlets, the branching ratios do not depend on the mixing parameters or couplings, but only on particle masses, and thus the connection to CP phases is not straightforward. In Fig.15 the branching ratios of the heavy triplet components are shown.

![Figure 15](image)

Figure 15: The branching ratios of $E^\pm$ and $N$ as function of the triplet mass. The mass of the Higgs boson is estimated to be $m_H = 120\,\text{GeV}$ (the study is from 2009). Figure from Ref. [114].

For a charged lepton mass $m_E = 300\,\text{GeV}$ the four-lepton signal could give results with luminosity $5\, fb^{-1}$, and the three-lepton signal (pictured in Fig.) with $3\, fb^{-1}$ at $5\sigma$.
confidence level [13]. The pair production of $E^\pm N$ has been searched at the LHC, but no evidence of the process has been discovered. The current collected data from the CMS experiment at the LHC correspond to an integrated luminosity of 4.9 fb$^{-1}$. The lower limit for the heavy fermion triplet masses is 180 GeV [54]. It is interesting that, as can be seen in Fig. 16, the decay $N \rightarrow l^\pm W^\mp$ is dominant only up to $\simeq 200$ GeV, after which it is equally likely with the channel $N \rightarrow \nu Z$. The three lepton final state is also possible for the latter decay mode of $N$, with $Z \rightarrow l^+ l^-$ decay, but it is less likely: the probability of the $Z$ boson decay into a lepton pair is $\simeq 3.37\%$ for each flavor, whereas for the decay $W^\pm \rightarrow l^\pm \nu$ the corresponding value is $\simeq 10.8\%$ [36]. It is estimated that the LHC should be able to probe the heavy fermion triplet mass up to 1.5 TeV [72,114].

For the fermion triplet the branching ratio for $l^\pm l^\pm l^\mp$ is smaller than for the $l^\pm l^\pm$ final state, where $l^\pm = e^\pm, \mu^\pm$ [10]. Still in these kinds of processes the background is larger in the latter and it needs more binding cuts, which removes the gain of a larger BR [10]. This is shown in Table (1). The three lepton final state seems to be explicitly a more promising final state in both the seesaw and the inverse seesaw cases.

<table>
<thead>
<tr>
<th>Signal</th>
<th>$l^\pm l^\pm$</th>
<th>$l^\pm l^\pm l^\mp$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E^+ E^- (\Sigma_{\text{Maj}})$</td>
<td>1.6</td>
<td>26.3</td>
</tr>
<tr>
<td>$E^\pm N (\Sigma_{\text{Maj}})$</td>
<td>240.0</td>
<td>192.2</td>
</tr>
<tr>
<td>$E^+_i E^-<em>i (\Sigma</em>{\text{D}})$</td>
<td>4.2</td>
<td>80.9</td>
</tr>
<tr>
<td>$E^+<em>i N (\Sigma</em>{\text{D}})$</td>
<td>12.3</td>
<td>398.3</td>
</tr>
</tbody>
</table>

Table 1: Comparison of the estimated number of events of the two leading final states for the fermion triplet scenario at a luminosity of 30 fb$^{-1}$. $\Sigma_D$ $(\Sigma_{\text{Maj}})$ refers to the triplet being of Dirac (Majorana) nature. The data are from Ref. [10].

For the $SU(5)$ based fermion triplet [26,33,34] the Majorana phases do have an
impact. From the restrictions from the proton decay and unification constraints, it is concluded that the mass of the fermion triplet has to be between 100 GeV and 1 TeV \cite{33}. This is interesting since, e.g., in the seesaw I models there is no similar mechanism that would keep the heavy neutrino masses low enough for the energy range of the LHC. In the model the effect of the Majorana phase is visible: by assuming the lowest neutrino mass be zero and thus be able to set $\Phi_1 = 0$ for the normal hierarchy the branching ratios are $\text{BR}(V\mu) \simeq \text{BR}(V\tau) \gg \text{BR}(V\bar{e})$, where $V = W, Z, H$, and for inverted hierarchy $\text{BR}(V\mu) \simeq \text{BR}(V\tau) < \text{BR}(V\bar{e})$. In Fig. 17 the effect of a non-zero Majorana phase $\alpha_1$ is shown. The difference compared to the zero approximation is largest as the absolute value of $\Phi_1$ is $\pm \pi/2$. Although the existence of the Majorana phase alters the branching ratios, the predictivity of the model is weakened by the experimental uncertainty of the Majorana phase.

![Figure 17](image)

Figure 17: Different branching ratios of the components of the fermion triplet, $T \rightarrow Vl^\pm$, where $V = W, Z, H$, as a function of the Majorana phase for NH (left) and IH (right). The value of $\text{Im}(z)$ (see text) is $\geq 2$ (upper) and 1 (lower). Figure from Ref. \cite{26}.

In Fig. 17 the Casas-Ibarra parameterization \cite{51} of the neutrino parameters is
used. Here $z$ is a complex parameter of the matrix $\Omega$, which defined in an equation [89]

$$vY_\Delta^{\ast} = m_\nu^D = iU_{PMNS} \sqrt{\text{diag}(m_1, m_2, m_3)} \Omega \sqrt{\text{diag}(M_{R,1}, M_{R,2})},$$

(129)

with a different form in normal and inverted hierarchies. The defining condition for $\Omega$ is $\Omega^T \Omega = 1$. For this case, assuming normal hierarchy, the matrix $\Omega$ can be written as

$$\Omega = \begin{pmatrix} 0 & 0 \\ \cos z & -\sin z \\ \pm \sin z & \pm \cos z \end{pmatrix}.$$ 

(130)

By combining Eqs. (129)-(130) the Yukawa couplings can be written as

$$vY_\Delta^{\ast} = \begin{cases} 
  i\sqrt{M_\Delta} \left( U_{i2} \sqrt{m_2^\nu} \cos z + U_{i3} \sqrt{m_3^\nu} \sin z \right) & \text{(NH)} \\
  i\sqrt{M_\Delta} \left( U_{i1} \sqrt{m_1^\nu} \cos z + U_{i2} \sqrt{m_2^\nu} \sin z \right) & \text{(IH)}
\end{cases},$$

For inverted hierarchy the matrix is of similar form: the third row is then zero and the first row nonzero. The most promising final states in this triplet model are the states with two same-sign charged leptons with four jets [26]. The branching ratios of different flavored charged leptons would follow the pattern of Fig. 17 for normal hierarchy they would be $\text{BR}(\mu^\pm \mu^\mp / \mu^\pm \tau^\pm / \tau^\pm \tau^\pm + \text{jets}) \gg \text{BR}(e^\pm e^\pm + \text{jets})$, whereas for inverted hierarchy they would have $\text{BR}(\mu^\pm \mu^\pm / \mu^\pm \tau^\pm / \tau^\pm \tau^\pm + \text{jets}) < \text{BR}(e^\pm e^\pm + \text{jets})$. The Majorana phase would modify the ratios so that with the maximal $\alpha = \pm \pi/2$ for every $\mu^\pm$ ($\tau^\pm$) in the final state the branching ratio would be halved (doubled) in the case of NH; the maximal Majorana phase would suppress all the other channels but the ones with $e^\pm$ in the case of IH [26]. It is estimated that the LHC could find the triplet of up to 450 GeV (700 GeV) with luminosity of 10 fb$^{-1}$ (100 fb$^{-1}$) [26,33].

The predicted distribution of the branching ratios of different charged dilepton channels offers a way to distinguish the model from other scenarios or to exclude it. As in the case of fermion singlets, the presence or absence of lepton number violating processes (the most notable being the same-sign di-lepton signal) can make it possible to determine the nature of neutrinos. Overall, although the discovery potential of fermion triplet is better than for the heavy neutrino singlets, the connection to CP violation is weak: in some cases the branching ratios depend on the Majorana phases, but the spread in possible values due to experimental inaccuracy might prevent finding conclusive evidence. The possibilities of finding signs of Dirac CP violating phase $\delta$ are even weaker. However, due to the observation of large $\theta_{13}$ the measurement of $\delta$ should be in principal possible at oscillation experiments [26].
8 Conclusions

In this work the restrictions for the mathematical formulation of neutrino mass terms, specific mass generating models, general conditions for leptonic CP violation, and the manifestation of Dirac and Majorana phases at collider experiments were reviewed. These subjects are connected on a fundamental level: the neutrino mass models dictate the form of the neutrino mass matrix; leptonic CP violation is expressed in the neutrino mass matrix by the Dirac and Majorana phases. In addition the mathematical framework of neutrinos and the formulation of the electroweak theory were studied.

The neutrino mass mechanisms considered in this work included the three tree-level seesaw mechanisms. A higher energy model that explains how a seesaw scheme could originate was introduced corresponding to each seesaw model: for the type I seesaw, the minimal left-right symmetric model was studied; for the type II seesaw, the Littlest Higgs model was reviewed; and finally for the type III seesaw an $SU(5)$ with an adjoint fermion was introduced. Finally, as an example of flavor symmetry extensions, the structure of the Altarelli-Feruglio model was reviewed. It offers a way to understand the mixing pattern of neutrino mass matrices.

The different decay channels of the heavy seesaw mediators were reviewed. There has not been any indication as of yet of the seesaw mediators. Still, their discovery is possible for all three tree-level seesaw mechanisms at the LHC.

For the doubly charged scalar (which is a component of the scalar triplet) the most promising final state seems to be the state of two same-sign leptons. The background effects are small and the dependency of the branching ratios of the decays on CP violating phases is clear.

For the heavy neutrino case the discovery potential of new physics is not considered to be as promising, as there are more background processes and there is no direct dependance on the CP violating phases, although some weak connections were shown. The incomplete nature of the topic is emphasized by the fact that there does not seem to be consensus on which decay channel would be the best or even on which is the dominant production channel for heavy neutrinos [10,66]. The most studied decay channels also for the heavy neutrino singlet are the states with two like-sign charged leptons or three charged leptons.

For the triplet fermion case that consists of new heavy neutrinos and charged leptons, the best channels to observe new physics are the same as for the heavy neutrino singlet, in addition to four-lepton final states that are impossible for other scenarios. A connection between CP violating phases and decays of heavy neutrinos
has been proposed \cite{71,114} and was shown, but the dependance is unlikely to be
detectable because of other unknown parameters.

The final state with three charged leptons is considered to be a good channel for
all three different seesaw mediators \cite{10}, and therefore it could be the most promising
channel to study. By testing the branching ratios of the tri- and di-lepton final
states one can determine the Dirac or Majorana nature of the particles, as they have
different final states.

Overall, the determination of Dirac and Majorana phases is very challenging.
Not only is this because their effects are subtle; current inaccuracy of other neutrino
mixing parameters makes it even more difficult to draw any conclusions for the phases.
The impact of the Dirac CP violating phase $\delta$ has not even been taken into account
in most of the studies because it appears in the mixing matrix only in terms with
$\theta_{13}$, which was until 2012 thought to be zero. In general, it seems to be that while
the discovery potential of the Majorana phases is, although depending on the model
in question, quite promising at the studied processes, the same cannot be said about
the Dirac phase: for that, the $e^+e^-$ collisions could serve better \cite{12}.

The non-zero values of the Majorana phases would not necessarily imply leptonic
CP violation, as CP odd phases can lead into CP even processes, as was shown in
Sec.6.5. Instead, a non-zero $\delta$ would imply violation of the CP symmetry, as was
shown in Sec.6.4.

In the case of a positive signal the challenge of distinguishing the right model
becomes relevant. For example, the $SU(5)$ with an adjoint fermion has a very similar
phenomenology as an addition of an extra lepton doublet to the SM \cite{26}.

The recent discoveries of both the SM Higgs and the non-zero value for the $\theta_{13}$
mixing angle in 2012 have had a great impact on the field. Thanks to the former
discovery one does not have extra uncertainty in the simulations due to an unknown
Higgs mass as in Refs. \cite{71,72,93}; due to the latter observation there is a need
for updating many approximations and models which relied on a zero value for $\theta_{13}$
as in Ref. \cite{22}. It is not anymore a good approximation to neglect the Dirac CP
violating phase as has been common to do, as in Refs. \cite{69,71}. In light of these
recent developments and the continually evolving precision of neutrino experiments,
more insight on the subject will certainly be achieved during the upcoming years.
A  $SU(2)$ and $SO(3)$ symmetries

$SU(2)$ is a non-Abelian group of $2 \times 2$ unitary matrices with a unit determinant. The commutation relations for the generators are

$$[J_i, J_j] = i\epsilon_{ijk}J_k.$$  \hfill (131)

We cannot choose a basis where all of them would be diagonal, i.e., they would have the same eigenstates. Instead, we can construct linear combinations of the generators, which do commute and thus have common eigenstates. Generally, one chooses the maximal set of commuting generators of the group, which form the basis of the Cartan subalgebra. In this case, the generators do not commute, so one can pick one of them to form the Cartan subalgebra. It is common to choose the third component as the Cartan subalgebra. The commutation relations of Eq. (131) are fulfilled by generators $J_i = \frac{1}{2}\sigma_i$ \hfill (102):

$$J_1 = \frac{1}{2}\sigma_1 = \frac{1}{2}\left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right), \quad J_2 = \frac{1}{2}\sigma_2 = \frac{1}{2}\left(\begin{array}{cc} 0 & -i \\ i & 0 \end{array}\right), \quad J_3 = \frac{1}{2}\sigma_3 = \frac{1}{2}\left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right),$$  \hfill (132)

where $\sigma_i$ are the Pauli spin matrices and where the diagonality of $J_3$ refers to the choice of direction: we have chosen the rotation described by $SU(2)$ to be around the $z$-axis. The raising and lowering operators of $SU(2)$ are linear combinations of $J_1$ and $J_2$, namely:

$$J_+ \equiv J_1 + ij_2 = \left(\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right), \quad J_- = J_1 - ij_2 = \left(\begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array}\right),$$  \hfill (133)

for which the commutation relations become

$$[J_3, J_+] = J_+$$
$$[J_3, J_-] = -J_-$$
$$[J_+, J_-] = 2J_3.$$  \hfill (134)

They show that $J_3$ does not commute with $J_\pm$. Thus $J_3$ and $J_\pm$ do not have common eigenstates. Still, there is a connection for them. Let us mark an eigenstate of $J_3$ as $|m_j\rangle$, i.e.,

$$J_3 |m_j\rangle = m_j |m_j\rangle.$$
Using the commutation relations of Eq. (134) one gets

\[ J_3 J_\pm |m_j\rangle = ([J_3, J_\pm] + J_\pm J_3) |m_j\rangle = (\pm J_\pm + J_\pm m_j) |m_j\rangle = (m \pm 1) J_\pm |m_j\rangle, \]

which means that an eigenstate of \( J_3 \), on which \( J_\pm \) has operated, is also an eigenstate of \( J_3 \) with an eigenvalue raised or lowered by one. This is the reason behind the names 'raising' and 'lowering' operators \( J_\pm \). The total angular momentum operator is

\[ \mathbf{J}^2 = J_1^2 + J_2^2 + J_3^2, \]

which commutes with all the \( J_i \), and thus it is called a Casimir operator of the \( SU(2) \) algebra. The common eigenstates of \( \mathbf{J}^2 \) and \( J_3 \) are characterized by \( |j, m_j\rangle \). The value \( j \) corresponds to the highest possible value of \( m_j \), i.e., the state for which \( J_+ |m_j = j\rangle = 0 \). The eigenvalues can be calculated using the commutation relations

\[ \mathbf{J}^2 |j, m_j = j\rangle = \left\{ (J_1 + iJ_2) (J_1 + iJ_2) - i [J_1, J_2] + J_3^2 \right\} |j, m_j = j\rangle \\
= [J_- J_+ + J_3 (J_3 + 1)] |j, m_j = j\rangle \\
= j (j + 1) |j, m_j = j\rangle. \quad (135) \]

The effect of \( J_\pm \) on eigenstates of \( J_3 \) and \( \mathbf{J}^2 \) is achieved from the norm \( \| J_\pm |j, m_j\rangle \| \).

For \( J_+ \)

\[ \| J_+ |j, m_j\rangle \|^2 = \langle j, m_j | J_- J_+ |j, m_j\rangle = \langle j, m_j | \left[ \mathbf{J}^2 - J_3 (J_3 + 1) \right] |j, m_j\rangle \\
= j (j + 1) - m_j (m_j + 1), \]

where the orthogonality of states \( |j, m_j\rangle \) and hermiticity of \( J_i \) were used. From this it follows that (taking the conventional choice of the phase being real and positive)

\[ J_+ |j, m_j\rangle = \sqrt{j (j + 1) - m_j (m_j + 1)} |j, m_j + 1\rangle = \sqrt{(j - m_j) (j + m_j + 1)} |j, m_j, + 1\rangle, \]

which is equal to 0 in the case \( m_j = j \), as was required before. For \( J_- \) the derivation is similar: by writing \( \mathbf{J}^2 \) as \( J_+ J_- - i [J_2, J_1] + J_3^2 \) and acting with this expression on the lowest state, correspondingly, for which \( J_- |m_j\rangle = 0 \), one gets \( \mathbf{J}^2 |j, m_j\rangle = m_j (m_j - 1) |j, m_j\rangle \). Comparing this to Eq. (135) one sees that \( m_j = -j \). Thus the values of \( m_j \) range between \( m_j = 2j + 1 \). On the other hand the amount of possible values of \( m_j \) must be an integer, so \( j \) can have also half-integer values. The value \( m_j \) gives the dimensionality of an irreducible representation of \( SU(2) \), i.e., the matrices of a representation of \( SU(2) \) are of a size \((2j + 1) \times (2j + 1)\). The Clebsch-Gordan
series can be written using the dimensionalities as

\[(2j_1 + 1) \otimes (2j_2 + 1) = \sum_{j = |j_1 - j_2|}^{j_1 + j_2} \oplus (2j + 1) . \tag{136}\]

The difference between the \(SU(2)\) and the group of three-dimensional rotations, \(SO(3)\), is quite subtle. A general \(2 \times 2\) matrix of \(SU(2)\) can be expressed in an exponential form

\[U_{SU(2)} = e^{-iJ \cdot n \theta} = e^{-\frac{1}{2}i\sigma \cdot n \theta}, \tag{137}\]

where the infinitesimal generator is expressed as \(n \theta\), where \(n\) is a unit vector and \(\theta\) is an angle. By using the property \((\sigma \cdot n)^2 = 1\), one can write

\[U_{SU(2)} = 1 \cos \left(\frac{1}{2} \theta\right) - i (\sigma \cdot n) \sin \left(\frac{1}{2} \theta\right), \tag{138}\]

whereas the general element of \(SO(3)\) is similarly

\[U_{SO(3)} = e^{-iX \cdot n \theta} = 1 \cos \theta - i (X \cdot n) \sin \theta,\]

inserting values \(\theta = 2\pi\) and \(\theta = 4\pi\) yields

\[U_{SO(3)}(\theta = 2\pi) = 1; \quad U_{SU(2)}(\theta = 2\pi) = -1,\]

\[U_{SO(3)}(\theta = 4\pi) = 1; \quad U_{SU(2)}(\theta = 4\pi) = 1,\]

which means that the parameter space of \(SO(3)\) is half of that of \(SU(2)\); instead of a 1:1 mapping \(SO(3)\) can be regarded as a quotient group of \(SU(2)\), namely

\[SO(3) \cong SU(2)/\mathbb{Z}_2.\]

This is physically reasonable: \(SO(3)\) describes a physical rotation, for which rotations must be of periodicity \(2\pi\). The different periodicity is visible also in the allowed values of \(j\). As mentioned, requirement of \(2j + 1\) being an integer allows \(j\) to be integer or half-integer. However, for \(SO(3)\) only the integer values are allowed. This means that \(SO(3)\) has irreducible representations of odd dimensions: 1, 3, \ldots; \(SU(2)\) has irreducible representations also of even dimensions.

The first non-trivial irreducible representation of \(SU(2)\), for which \(j = 1/2\), describes spin-1/2 particles. It is two-dimensional and thus the fundamental representation of the group. The representation with \(j = 1\) is three-dimensional; for \(SU(2)\) that is the adjoint representation as it has three generators \(J_i\). It describes spin-
1 particles and is isomorphic to the standard three-dimensional representation of \( SO(3) \) which describes physical rotations. The description of particles with \( SU(2) \) is non-relativistic and in order to better describe them, one has to consider Lorentz transformations.

**B The restricted Lorentz group \( SO^+(3, 1) \)**

The elements of the restricted Lorentz group \( SO^+(3, 1) \) are orthogonal matrices with unit determinant. The group is said to be proper if the inversions are excluded. This is because one cannot have an infinitesimal expression of an inversion transformation. The different behavior of time and space dimensions is encoded in the metric tensor \( g = \text{diag}(1, -1, -1, -1) \), which defines the invariant \( c^2 \tau^2 \) as

\[
c^2 \tau^2 = x_0^2 - \mathbf{x}^2 = x^T g x,
\]

(139)

where \( x^\mu = g^{\mu\nu} x_\nu \).

Consider a Lorentz transformation acting on a four-vector \( x \):

\[
x' = \Lambda x,
\]

where \( \Lambda \) is a \( 4 \times 4 \) matrix. This combined with the invariant of Eq. (139) gives the defining condition for the Lorentz transformation:

\[
x'^T g x' = (\Lambda x)^T g (\Lambda x) = x^T (\Lambda^T g \Lambda) x,
\]

from which it follows, since \( x'^T g x' = x^T g x \), that \( \Lambda^T g \Lambda = g \). In index form this reads

\[
g_{\mu\nu} = (\Lambda^T g \Lambda)_{\mu\nu} = (\Lambda^T)^{\rho}_{\mu} g_{\rho\sigma} \Lambda^{\sigma}_{\nu} = g_{\rho\sigma} \Lambda^{\rho}_{\mu} \Lambda^{\sigma}_{\nu} = \Lambda_{\rho\mu} \Lambda_{\sigma\nu}.
\]

(140)

Taking the determinant of this gives \( (\det(g) = -1) \)

\[
(\det \Lambda)^2 = 1.
\]

Choosing \( \det (\Lambda) = 1 \) gives the proper Lorentz group. The negative value, \( \det (\Lambda) = -1 \), corresponds to the improper Lorentz group, where inversions are included. Together they form the group \( O(3, 1) \). The unit determinant still does not determine the group \( SO^+(3, 1) \) in which we are interested. One can write Eq. (140) as (setting
\( \mu = \nu = 0 \)

\[ g_{00} = g_{\rho\sigma} \Lambda_{\rho 0}^\sigma = g_{00} (\Lambda_{0 0}^0)^2 + \sum_i g_{ii} (\Lambda_{0 i}^i)^2, \]

from which it follows that

\[ (\Lambda_{0 0}^0)^2 = 1 + \sum_i (\Lambda_{0 i}^i)^2 \geq 1, \]

which gives the condition for the restricted Lorentz group \( SO^+(3, 1) \); it is the subgroup with \( \Lambda_{0 0}^0 \geq 1 \), i.e., the transformations conserve the direction of time. Such transformations are called orthochronous. The solution \( \Lambda_{0 0}^0 \leq -1 \) of Eq. \( (141) \) is called antichronous and is denoted as \( SO^-(3, 1) \), but its elements do not form a group \[84\].

One notices that a parity transformation combined with a time reversal is a proper antichronous Lorentz transformation, whereas acting with only either of them is an improper transformation, parity being orthochronous and time reversal, by definition, an antichronous transformation. Proper orthochronous Lorentz transformations form the restricted Lorentz group \( SO^+(3, 1) \). It turns out that any Lorentz transformation can be expressed as a product of an element of \( SO^+(3, 1) \) with either the unit element \( 1 \), parity \( P \), time reversal \( T \), or a combination of them, \( PT \) \[48\].

An infinitesimal Lorentz transformation is

\[ \Lambda_\mu^\nu = \delta_\mu^\nu - \epsilon \omega_\mu^\nu. \]

Inserting this into Eq. \( (140) \) gives the condition for \( \omega \),

\[ \omega_{\mu\nu} = -\omega_{\nu\mu}, \]

i.e., \( \omega \) is an antisymmetric \( 4 \times 4 \) matrix in the defining representation; it then has six independent parameters. This result determines the number of generators of \( SO^+(3, 1) \): consider a representation of the infinitesimal transformation:

\[ D (1 + \epsilon \omega) = 1 - \frac{i}{2} \epsilon \omega_{\mu\nu} J^{\mu\nu}, \]

where \( J^{\mu\nu} \) must also be antisymmetric, i.e., \( J_{\mu\nu} = -J_{\nu\mu} \), giving six generators, which satisfy the commutation relation

\[ [J_{\mu\nu}, J_{\rho\sigma}] = -i (g_{\nu\rho} J_{\mu\sigma} - g_{\mu\sigma} J_{\nu\rho} + g_{\nu\sigma} J_{\rho\mu} - g_{\rho\mu} J_{\nu\sigma}). \]

\( J_{\mu\nu} \) can be split into two parts: the angular \( (L_{\mu\nu}) \) and spin \( (S_{\mu\nu}) \) momentum operators, which both satisfy the above commutation relation separately. For the general-
ized angular momentum we have

\[ L_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu), \]

which rises from the expression of angular momentum, \( L_i = \epsilon_{ijk} x_j p_k = -i \epsilon_{ijk} x_j \partial_k. \)

With this \( J_{\mu\nu} \) becomes

\[ J_{\mu\nu} = i(x_\mu \partial_\nu - x_\nu \partial_\mu) + S_{\mu\nu}, \]

and it can be split into two sets

\[ J_i = \frac{1}{2} \epsilon_{ijk} J_{jk}, \quad K_i = J_{0i}, \]

where the \( J_i \) correspond to three dimensional rotations and \( K_i \) refer to boosts. The commutation relations work out to be

\[
\begin{align*}
[J_i, J_j] &= i \epsilon_{ijk} J_k \\
[K_i, K_j] &= -i \epsilon_{ijk} J_k \\
[J_i, K_j] &= i \epsilon_{ijk} K_k,
\end{align*}
\]

from which one notices that the generators of rotations form an \( SO(3) \) subalgebra. One can define two linear combinations of \( J_i \) and \( K_i \), namely

\[ T^L_i = \frac{1}{2} (J_i + i K_i), \quad T^R_i = \frac{1}{2} (J_i - i K_i), \]

where the superscripts \( L \) and \( R \) are chosen with a little fore-sight and for which the commutation relations become

\[
\begin{align*}
[T^L_i, T^L_j] &= i \epsilon_{ijk} T^L_k \\
[T^R_i, T^R_j] &= i \epsilon_{ijk} T^R_k \\
[T^L_i, T^R_j] &= 0.
\end{align*}
\]

This means that the Lorentz symmetry can be broken into two disjoint \( SU(2) \) algebras. As the two disjoint \( SU(2) \) algebras commute, the group \( SO(3,1) \) can be represented as a direct product of them. In fact, the representations of the restricted Lorentz group can be labeled using the quantum numbers of the two \( SU(2) \)s. Each of the \( SU(2) \) has the dimension \( 2j + 1 \). Let’s denote the values of \( j \) corresponding to \( T^L(T^R) \) as \( j_L \) (\( j_R \)). The dimension of a representation of \( SO(3,1) \) is then \((2j_L + 1)(2j_R + 1)\). Let’s study the properties of representations of \( SO(3,1), (j_L, j_R) \), with different values of \( j_L, j_R \).
Left-handed \((\frac{1}{2}, 0)\) spinors \(\phi\)

Let’s first study the situation where \(j_R = 0\) and \(j_L = 1/2\). The \(j_L = 1/2\) representation corresponds to the defining representation of \(SU(2)\). The generators must be as in Eq. (132):

\[
T^L_i = \frac{1}{2} (J_i + i K_i) = \frac{1}{2} \sigma_i,
\]

which is satisfied if

\[
J_i = \frac{1}{2} \sigma_i; \quad K_i = -\frac{i}{2} \sigma_i.
\]

This implies that

\[
T^R_i = \frac{1}{2} (J_i - i K_i) = \frac{1}{2} \left( \frac{1}{2} \sigma_i - i \left( \frac{1}{2} \sigma_i \right) \right) = 0.
\]

This gives the transformation properties of a \((\frac{1}{2}, 0)\) spinor under Lorentz transformations. From the expression of a general element of \(SU(2)\), Eqs. (137)-(138), one gets for spinors \(\phi\) under rotations

\[
\phi \to e^{-i J \cdot n \theta} \phi = \left[ 1 \cos \left( \frac{1}{2} \theta \right) - i (\sigma \cdot n) \sin \left( \frac{1}{2} \theta \right) \right] \phi,
\]

where \(n \theta\) describes a rotation of angle \(\theta\) in direction \(n\). For transformations under boosts \(\eta = \eta n\) we have

\[
\phi \to e^{-i K \cdot n \eta} \phi = e^{i \left( \frac{1}{2} \sigma \cdot n \eta \right)} \phi = \left[ 1 \cosh \left( \frac{1}{2} \eta \right) - i (\sigma \cdot n) \sinh \left( \frac{1}{2} \eta \right) \right] \phi.
\]

Right-handed \((0, \frac{1}{2})\) spinors \(\chi\)

Let’s then look at the case with \(j_L = 0\) and \(j_R = 1/2\). Similarly to the case of \((\frac{1}{2}, 0)\) spinors, we have

\[
T^R_i = \frac{1}{2} (J_i - i K_i) = \frac{1}{2} \sigma_i,
\]

which is satisfied if

\[
J_i = \frac{1}{2} \sigma_i; \quad K_i = \frac{i}{2} \sigma_i,
\]

giving \(T^L_i = 0\). Transformations under rotations are the same for \((0, \frac{1}{2})\) spinors \(\chi\) as for \((\frac{1}{2}, 0)\) spinors \(\phi\). However, due to the sign difference of \(K_i\), the transformation under boosts differs with a sign:

\[
\chi \to e^{-i K \cdot n \eta} \chi = \left[ 1 \cosh \left( \frac{1}{2} \eta \right) + i (\sigma \cdot n) \sinh \left( \frac{1}{2} \eta \right) \right] \chi.
\]
Relation between \((\frac{1}{2}, 0)\) and \((0, \frac{1}{2})\) spinors

The representations \((\frac{1}{2}, 0)\) and \((0, \frac{1}{2})\) are related by a complex conjugation of the spinor fields. As \(\sigma_2\) is purely imaginary, and other Pauli matrices are real, one notices that (using the property that \(\sigma_i^2 = 1\))

\[
\sigma_2 \sigma^* \sigma_2 = -\sigma. \tag{145}
\]

Using this one sees that the complex conjugation of the field \(\phi\) multiplied by factor \(i\sigma_2\) transforms as \(\chi\). Under rotations it transforms as

\[
\begin{align*}
i\sigma_2 \phi^* &\rightarrow i\sigma_2 \left( e^{-iJ \cdot n \theta} \phi \right)^* = i\sigma_2 e^{\frac{i}{2} \sigma_i \cdot n \theta} \phi^* = i \left( \sigma_2 e^{\frac{i}{2} \sigma_i \cdot n \theta} \sigma_2 \right) \sigma_2 \phi^*, \\
&= \left[ 1 \cos \left( \frac{1}{2} \theta \right) - i (\sigma \cdot n) \sin \left( \frac{1}{2} \theta \right) \right] i\sigma_2 \phi^*,
\end{align*}
\]

and under boosts as

\[
\begin{align*}
i\sigma_2 \phi^* &\rightarrow i\sigma_2 \left( e^{-iK \cdot \eta \phi} \right)^* = i\sigma_2 e^{-\frac{i}{2} \sigma_i \cdot \eta} \sigma_2 \sigma_2 \phi^* \\
&= \left[ 1 \cosh \left( \frac{1}{2} \eta \right) + i (\sigma \cdot n) \sinh \left( \frac{1}{2} \eta \right) \right] i\sigma_2 \phi^*,
\end{align*}
\]

which is the transformation of a spinor \(\chi\) under a boost. This means that the representations \((\frac{1}{2}, 0)\) and \((0, \frac{1}{2})\) of \(SO(3,1)\) are complex conjugates of each others (up to a unitary transformation \(i\sigma_2\)), and thus with such a transformation one can identify

\[
i\sigma_2 \phi^* = \chi, \quad i\sigma_2 \chi^* = \phi. \tag{146}
\]

Dirac representation as a direct sum of the Weyl representations

Using the \(\gamma^\mu_W\) of Eq. (9) in Eq. (4) the generators of the Lorentz group become

\[
\begin{align*}
J_i &= \frac{1}{2} \epsilon_{ijk} L_{jk} = \frac{1}{2} \epsilon_{ijk} \frac{i}{4} (\gamma_j \gamma_k - \gamma_k \gamma_j) = \frac{i}{4} \epsilon_{ijk} \gamma_j \gamma_k = \left( \frac{i}{2} \sigma_i \quad 0 \quad 0 \right), \\
K_i &= L_{0i} = \frac{i}{4} (\gamma_0 \gamma_i - \gamma_i \gamma_0) = \left( -\frac{i}{2} \sigma_i \quad 0 \quad 0 \right),
\end{align*}
\]

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where the generators in the Weyl representations are in the diagonals. In the Weyl basis of $\gamma^\mu$ the role of $\gamma^0$ is particularly interesting, since

$$\gamma^0 J_i \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{i}{2} \sigma_i & 0 \\ 0 & \frac{i}{2} \sigma_i \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{i}{2} \sigma_i & 0 \\ 0 & \frac{i}{2} \sigma_i \end{pmatrix} = J_i,$$

$$\gamma^0 K_i \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -\frac{i}{2} \sigma_i & 0 \\ 0 & \frac{i}{2} \sigma_i \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{i}{2} \sigma_i & 0 \\ 0 & -\frac{i}{2} \sigma_i \end{pmatrix} = -K_i,$$

which implies that $\gamma^0$ acts as a parity operator. In this form it becomes clear why this representation of $\gamma^\mu$ is called 'chiral': the generators of the Weyl representations are in the diagonals of the generators; the direct sum form constructed from two Weyl representations of $SO(3,1)$ is visible.

Earlier we noticed that a spinor $\phi$ acted after a transformation $i\sigma_2 \phi^*$ as a spinor $\chi$, and vice versa, under Lorentz transformations. This means that the Weyl representations are conjugate representations. We can then write for massless Weyl fields,

$$\psi'_L = i\sigma_2 \phi^*_R, \quad \psi'^\dagger_L = i\phi^T_R \sigma_2.$$

We noticed that when writing a Dirac field as a direct sum of two Weyl spinors, one gets a pair of equations that decouple in the case of massless fermions; left and right-handed fields become disjoint (Eq. (14)). Using the transformation above, we can write the right-handed part of the massless fermion Lagrangian $\mathcal{L} = i\psi'_{R,i} \sigma^\mu D_\mu \psi_R + i\psi'_{L,i} \bar{\sigma}^\mu D_\mu \psi_L$, where the covariant derivative is of the form $D_\mu = \partial_\mu - igW^a_\mu t^a_r$, as

$$\int d^4x \psi'_{R,i} \sigma^\mu D_\mu \psi_R = i \int d^4x \psi'_{R,i} \sigma^\mu \left( \partial_\mu - igW^a_\mu t^a_r \right) \psi_R = i \int d^4x \psi'^\dagger_L (1, -\sigma^*) \left( \partial_\mu - igW^a_\mu t^a_r \right) \psi'^*_L$$

where we used integration by parts, the nature of the action and fermion fields, the identity of Eq. (145), and we marked $\sigma \equiv (1, \sigma)$ and $\bar{\sigma} \equiv (1, -\sigma)$, as before. In the last row we defined $- (t^a_\mu)^T \equiv t^a_\mu$, which is a general result for representation matrices of a conjugate representation $\bar{r}$ of a representation $r$ [131]. In weak interactions, the gauge fields interact only with the left-handed Weyl spinors $(\frac{1}{2}, 0)$. By looking at the transformation of the covariant derivative one notices that the transformed fields $\psi'_L$ belong to the conjugate representation $\bar{r}$. This states once more that the Dirac spinor is a representation $R = r \oplus \bar{r}$, where $r$ and $\bar{r}$ are the Weyl spinor representations.
C Properties of the Poincaré group

The Poincaré group extends the Lorentz group with space-time translations. It is a semidirect product of translations and the Lorentz group: \( \mathbb{R}^{1,3} \times O(1,3) \). We are interested in it because all the particles can be treated as irreducible representations of it. The Poincaré group is the full symmetry group of special relativity, its general transformation being

\[
x'_\mu = \Lambda^\nu_\mu x_\nu + a_\mu,
\]

where \( \Lambda^\nu_\mu \) is a \( 4 \times 4 \) Lorentz transformation matrix, \( x_\nu \) is a four-vector, and \( a_\mu \) is a constant four-vector describing the translation. The generators of the Poincaré group include the ones of the Lorentz group, but in addition there are four generators of translations: the four-momentum operators \( P_\mu = i \partial_\mu \), which commute:

\[
[P_\mu, P_\nu] = 0,
\]

which implies that the translation group is Abelian. Translations and Lorentz transformations do not commute:

\[
[P_\mu, L_{\rho\sigma}] = i (g_{\mu\rho} P_\sigma - g_{\mu\sigma} P_\rho),
\]

from which one gets commutation relations for \( P_\mu, K_i, \) and \( J_i \):

\[
\begin{align*}
[P_0, J_i] &= \frac{i}{2} \epsilon_{ijk} (g_{0j} P_k - g_{0k} P_j) = 0, \\
[P_i, J_j] &= \frac{i}{2} \epsilon_{jkl} (g_{ik} P_l - g_{il} P_k) = \frac{i}{2} (\epsilon_{jkl} P_k - \epsilon_{jil} P_l) = i \epsilon_{ijk} P_k \\
[P_0, K_i] &= (i \partial_0) i (x_0 \partial_i - x_i \partial_0) = i (x_0 \partial_i - x_i \partial_0) (i \partial_0) = i (i \partial_i) = i P_i \\
[P_j, K_i] &= (i \partial_j) i (x_0 \partial_i - x_i \partial_0) = i (x_0 \partial_i - x_i \partial_0) (i \partial_j) = i (-i \delta_{ij} \partial_0) = -i P_0 \delta_{ij}.
\end{align*}
\]

These describe the algebra of the Poincaré group. As it is seen from the relations, a boost in a direction \( i \) affects \( P_0 \) and \( P_i \), the translations in the time and \( i \) directions. One can also note that a translation in time does not affect rotations, but instead, a space translation does have an effect on rotations about other axes. The commutation relations between \( K_i \) and \( J_i \) remain the same as in \( SO(3,1) \).

C.1 Massive representations and helicity

In the case of the \( SU(2) \) algebra, we had an operator \( J^2 \), which commuted with all of the generators. One wants to find the corresponding Casimir operators for the
Poincaré group as well. It turns out that there are two such operators, each having their own eigenvalues. It is not a coincidence that a massive particle is characterized with two quantum numbers, mass and spin. The invariance of mass follows from the invariance of four-momentum: \( p^2 = m^2 \). The spin describes properties of a particle under an internal transformation, i.e., in the rest-frame of a particle. The operator corresponding to the invariance of mass is \( P^2 = P_\mu P^\mu \). As said, it commutes with \( P_\mu \) and \( L_{\mu\nu} \), which follow from Eq. (148), implying that \( P^2 \) indeed is an invariant under Lorentz transformations and translations.

The Casimir operator corresponding to spin is defined as \( W^2 = W_\mu W^\mu \) where \( W_\mu \) is the Pauli-Lubanski pseudovector

\[
W_\mu = -\frac{1}{2} \epsilon_{\mu\nu\rho\sigma} L^\nu P^\rho P^\sigma,
\]

where \( \epsilon_{\mu\nu\rho\sigma} \) is an anti-symmetric tensor for which \( \epsilon_{0123} = 1 \). Using the identity \([A, BC] = [A, B] C + B [A, C]\) and Eq. (149) one notices that it also commutes with \( P_\mu \) and \( L_{\mu\nu} \). Let’s study \( W^2 \) in the rest-frame which is denoted by a subscript \( RF \).

Now \( P_{RF,\mu} = (m, 0) \), which gives

\[
W_{RF,0} = 0, \quad W_{RF,i} = -\frac{1}{2} m \epsilon_{i0\rho0} L^\rho = \frac{1}{2} m \epsilon_{0i0\rho} L^\rho = \frac{1}{2} m \epsilon_{ijk} L^{jk} = m J_i,
\]

from which it follows that

\[
W_\mu W^\mu = W_{RF,i} W_{RF,i} = -m^2 J^2.
\]

One can choose another quantum number to label a state of a massive particle, but in order to do that we have to find an operator that commutes with \( P^2 \) and \( W^2 \). Both of them commute with \( P_\mu \). It turns out that there is another suitable operator; the inner product \( J \cdot P \) does commute with the Casimir operators. As \( J \cdot P = (r \times P) \cdot P + S \cdot P = S \cdot P \), it gives the value of the spin along the direction of the momentum. It is called the helicity operator. The eigenvalues of \( J^2 \) are known (see Appendix A). Operators \( P^2 \) and \( W^2 \) have common eigenstates for which the eigenvalues are

\[
W^2 |m, j, p, \lambda\rangle = -m^2 j(j+1) |m, j, p, \lambda\rangle \\
P^2 |m, j, p, \lambda\rangle = m^2 |m, j, p, \lambda\rangle \\
P_\mu |m, j, p, \lambda\rangle = p_\mu |m, j, p, \lambda\rangle \\
S \cdot P |m, j, p, \lambda\rangle = \lambda |p| |m, j, p, \lambda\rangle.
\]  

(150)
The structure of the internal symmetry of a massive particle can then be regarded as an $SU(3)$ symmetry \[144\]. Next we will see that for massless particles, internal symmetries are manifested in a profoundly different way.

## C.2 Massless representations and helicity

For massless particles, such as neutrinos of the Standard Model, there is no rest-frame, as $p^2 = 0$, and the derivation for the eigenvalues for Casimir operators is different. Let us consider a four-momentum (as in, e.g., \[102\]),

$$p_\mu = p (1, 0, 0, 1),$$

which has some energy and momentum along the $z$-direction. A rotation about the $z$-axis, $J_3$, leaves a state $|p_\mu \rangle$ invariant. One can choose $M_1 \equiv J_i + K_2$ and $M_2 \equiv J_2 - K_1$ as generators of the internal symmetry,

$$[J_3, M_1] = iM_2$$
$$[J_3, M_2] = -iM_1$$
$$[M_1, M_2] = 0,$$

and where $[M_1, p_\mu] |p_\mu \rangle = 0 \[102\]$. Now the Pauli-Lubanski pseudovector becomes

$$W_0 = -\frac{1}{2} \epsilon_{\nu\rho\beta} L^{\nu\rho} P^\beta = -\frac{1}{2} \epsilon_{ij3} L^{ij} P^3 = -\frac{1}{2} L^{12} P^3 + \frac{1}{2} L^{21} P^3 = pJ_3$$
$$W_1 = -\frac{1}{2} \epsilon_{\nu\rho0} L^{\nu\rho} P^0 - \frac{1}{2} \epsilon_{\nu\rho3} L^{\nu\rho} P^3 = pL^{23} + pL^{02} = p(J_1 + K_2) = pM_1$$
$$W_2 = pM_2$$
$$W_3 = pJ_3,$$

from which it follows that

$$W^2 = -p^2 \left( M_1^2 + M_2^2 \right),$$

which commutes with both $M_i$ and with $J_3$ and it is therefore a Casimir operator. One can then choose one of the generators $J_3$, $M_1$, or $M_2$, which is conventionally $J_3$. The other generators form the raising and lowering operators for $J_3$, as $M_\pm \equiv M_1 \pm iM_2$, which satisfy Eq. \[134\]. The situation is different from the case of $SU(2)$: First, the algebra is different; second, in $SU(2)$ $J_3$ is connected to $J^2$ which restricts their spectrums, but in massless representations there is no connection, and in general, a single particle could have infinite degrees of freedom. Therefore it is assumed that the
case of infinite eigenvalues of $J_3$ is prevented by choosing $M_1 \ket{\cdot} = M_2 \ket{\cdot} = 0$, which gives $(M_1^2 + M_2^2) \ket{\cdot} = 0$, i.e., it is zero for all possible states. Physically this means that in the case of massless particles, there is just one quantum number labeling the state, the eigenvalue of $J_3$, which now corresponds to the spin along the direction of motion. There is no spin value in the same sense as with massive particles, where the possible eigenvalues of $J_3$ vary between $+j$ and $-j$, and where spin manifests an $SU(2)$ symmetry. For massless particles the eigenvalue of $J_3$, the helicity, is the only defined value for the spin. That value is a direct property of an irreducible representation of the Poincaré group.

D Discrete $A_4$ symmetry

The alternating group of cycles not longer than four, $A_4$, is a subgroup of the permutation group $S_4$. As it consists of only the even permutations of $S_4$, its order is half of $S_4$, i.e., $4!/2 = 12$. A permutation is even if it contains an even number of single transpositions of two indices, e.g. $(1\ 2\ 3) = (1\ 2)(2\ 3)$ has two transpositions and is thus counted as an even permutation.

The group $A_4$ can also be regarded as the symmetry group of rotations of a tetrahedron, to which it is isomorphic; instead, $S_4$ is the complete symmetry group of a tetrahedron, as it also includes reflections [102].

The rotation group of a tetrahedron is generated by generators satisfying $S^2 = T^3 = (ST)^3 = e$. The generators can be identified as permutations $S = (1\ 3)(2\ 4)$ and $T = (2\ 3\ 4)$, from which it follows $ST = (1\ 3\ 2)$. They span all the other elements of $A_4$, which divide into four equivalence classes

$$A_4 = \{ () ; (1\ 2\ 3), (1\ 4\ 2), (1\ 3\ 4), (2\ 4\ 3); 
(1\ 3\ 2), (1\ 2\ 4), (1\ 4\ 3), (2\ 3\ 4); 
(1\ 2)(3\ 4), (1\ 3)(2\ 4), (2\ 3)(1\ 4) \}.$$ 

The notation is such that e.g. $(1\ 2\ 3)$ describes a permutation which moves the first element into the second place, the second element into the third place, and the third element into the first place.

The first class consists of an identity operator, the second and third are disjoint classes of 3-cycle permutations, and the fourth permutations of two decoupled 2-cycles. Elements in the same equivalence class are conjugates to other elements of the class, i.e., they are related according to

$$a_1 = ga_2g^{-1},$$
where \( a_1, a_2 \) are elements belonging to the same equivalence classes, and \( g \) is an element of \( A_4 \). For example, if we choose \( a_2 = (1 2 3) \), \( g = (1 2)(3 4) \), we get \( a_1 = (1 2)(3 4)(1 2 3)(1 2)(3 4) = (1 4 2) \).

The number of irreducible representations of a finite group is restricted. In fact, there are as many inequivalent irreducible representations as there are conjugacy classes. We also have a condition \[ \sum_{\mu} n_{\mu}^2 = [g], \quad (151) \]

where \([g]\) is the number of elements of \( A_4 \) and \( n_{\mu} \) is the dimensionality of the matrix representing an irrep. Obviously, the dimensionality of each matrix has to be at least one. We have then only one possible solution for Eq. (151), namely

\[
1^2 + 1^2 + 1^2 + 3^2 = 12.
\]

**Character table**

The character table is achieved with following key steps. Firstly, \( \mathbf{1} \) is the trivial irreducible representation, which maps everything to unity. The corresponding equivalence class contains only one element, the identity \( e = () \).

For one dimensional representations the characters (or traces of the matrix elements) must satisfy the conditions of group representing matrices. This combined with the fact that elements of the same conjugacy class have the same character, gives us

\[
\chi(1 3)(2 4)\chi(2 3 4) = \chi(1 3 2) \implies \chi(1 3)(2 4) = 1,
\]
as \( \chi(1 3 2) \) and \( \chi(2 3 4) \) are of the same conjugacy class. We also know that

\[
(\chi(1 3 2))^3 = \chi ((1 3 2)^3) = \chi(e) = 1 \implies \chi(1 3 2) = e^{\frac{2\pi i}{3}}.
\]

This choice is made for ensuring the orthogonality of irreducible representations \[ \sum_k k_i \chi_i^{(\mu)} \chi_i^{(\nu)} = [g] \delta^{\mu\nu}. \quad (152) \]

Now for the trivial and other one-dimensional irreducible representation:

\[
\sum_i k_i \chi_i^{(1)} \chi_i^{(2)} = 1 + 4e^{\frac{2\pi i}{3}} + 4e^{\frac{4\pi i}{3}} + 3 = 0,
\]
which satisfies the orthogonality condition of Eq. [152]. The irreducible representations $1'$ and $1''$ turn out to be complex conjugates of each other.

The three-dimensional representation, $3$, can be achieved considering the rotations of a tetrahedron in 3-dimensional coordinates. For example, the permutation $\chi(1\ 2\ 3)$ corresponds to the $2\pi/3$-rotation about z-axis, which can easily be represented as a rotation matrix, for which the trace is 0. We can re-orient the tetrahedron so that a permutation $(1\ 2)(3\ 4)$ corresponds to a rotation in which the orientation of the tetrahedron stays as it is, but the direction of axes $x$ and $z$ is changed. The corresponding rotation matrix is a diagonal matrix diag($-1, 1, -1$), and thus $\chi((1\ 2)(3\ 4)) = -1$.

The character table is then

<table>
<thead>
<tr>
<th>$A_4$</th>
<th>$(e)$</th>
<th>e.g. $(1\ 2\ 3)$</th>
<th>e.g. $(2\ 3\ 4)$</th>
<th>e.g. $(1\ 3)(2\ 4)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1$ (trivial rep)</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$1'$</td>
<td>1</td>
<td>$e^{2\pi i/3}$</td>
<td>$e^{2\pi i/3}$</td>
<td>1</td>
</tr>
<tr>
<td>$1''$</td>
<td>1</td>
<td>$e^{-2\pi i/3}$</td>
<td>$e^{-2\pi i/3}$</td>
<td>1</td>
</tr>
<tr>
<td>$3$ (standard rep)</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>-1</td>
</tr>
</tbody>
</table>

Table 2: The character table of $A_4$. Each row corresponds to an irreducible representation and each column to a conjugacy class.

The decomposition of the product group $3 \otimes 3$ is achieved from the Clebsch-Gordan series:

$$3 \otimes 3 = \sum_{\sigma} a_{\sigma} 1^{(\sigma)},$$

where the summation goes over all irreducible representations of $A_4$ and where the coefficients $a_{\sigma}$ are given by

$$a_{\sigma} = \frac{1}{[g]} \sum_{g} \chi(g)\chi^{\sigma}(g^{-1}),$$

where $\chi(g)$ is the compound character, whereas $\chi^{(\sigma)}(g)$ is a character of an irreducible representation, and $[g]$ is now 12, i.e., the number of elements of $A_4$. As the character of the product representation is the product of the characters, we have $\chi(g) = \chi^{(3)} \otimes \chi^{(3)} = (9, 0, 0, 1)$, using the character table.

From Eq. [153] the decomposition reads $a = (1, 1, 1, 2)$. Then we have

$$3 \otimes 3 = 1 \oplus 1' \oplus 1'' \oplus 3_{\text{asym}} \oplus 3_{\text{sym}}.$$
products decompositions can also be calculated from Eq. (153), they are

\[ 1 \otimes 1 = 1 \]
\[ 1' \otimes 1'' = 1 \]
\[ 1'' \otimes 1' = 1. \]

These Kronecker products above are basis-independent, but in order to get rules for behavior of multiplets in products, we have to choose a basis.

**Altarelli-Feruglio basis**

In the basis used in Ref. [22] the generator \( T \) is chosen to be diagonal, i.e.,

\[
\begin{align*}
1 : & \quad S = 1, \quad T = 1 \\
1' : & \quad S = 1, \quad T = e^{2\pi i/3} \equiv \omega \\
1'' : & \quad S = 1, \quad T = e^{4\pi i/3} \equiv \omega^2 \\
3 : & \quad S = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}
\end{align*}
\]

By denoting a 3 object as \((a, b, c)\) and using the properties of the basis vectors of the invariant subspaces of the kronecker product, one can compute, as in Ref. [116], the expressions for the \((a_1, a_2, a_3)\) and \((b_1, b_2, b_3)\) in the representations of the Clebsch-Gordan decomposition, \(3 \otimes 3 = 1 \oplus 1' \oplus 1'' \oplus 3_{\text{asym}} \oplus 3_{\text{sym}}\), namely [23]

\[
\begin{align*}
1 &= a_1 b_1 + a_2 b_3 + a_3 b_2 \\
1' &= a_1 b_2 + a_2 b_1 + a_3 b_3 \\
1'' &= a_1 b_3 + a_2 b_2 + a_3 b_1, \\
3_{\text{asym}} &= \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_1 b_2 - a_2 b_1 \\ a_3 b_1 - a_1 b_3 \end{pmatrix} \\
3_{\text{sym}} &= \begin{pmatrix} 2a_1 b_1 - a_2 b_3 - a_3 b_2 \\ 2a_3 b_3 - a_1 b_2 - a_2 b_1 \\ 2a_2 b_2 - a_3 b_1 - a_1 b_3 \end{pmatrix}
\end{align*}
\]
References


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