ON GEOMETRY OF MANDELBROT CASCADES AND MULTIPLICATIVE CHAOS AT CRITICALITY

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Academic dissertation

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This thesis consists of an introduction and the following four articles:

[**I**] J. Barral, A. Kupiainen, M. Nikula, E. Saksman, C. Webb: Critical Mandelbrot Cascades. Communications in Mathematical Physics 325, No. 2, p. 685–711. (2014)

[**II**] J. Barral, A. Kupiainen, M. Nikula, E. Saksman, C. Webb: Basic properties of critical lognormal multiplicative chaos. arXiv:1303.4548 (to appear in The Annals of Probability)

[**III**] M. Nikula: Small deviations in lognormal Mandelbrot cascades. arXiv:1306.3448

[**IV**] M. Nikula: Inhomogeneous cascades and fine multifractal properties of critical cascades. *manuscript*

Contents

1. Overview

1.1. **Introduction.** The main mathematical objects of this thesis are two related constructions of random multifractal measures: Mandelbrot cascades and lognormal multiplicative chaos. In its simplest form the Mandelbrot cascade is a very natural construction of a random measure on the unit interval $[0, 1]$. Let W be a positive random variable and $\beta > 0$ a parameter. Starting from the Lebesgue measure $\mu_{\beta,0}(\,\mathrm{d}x) = \mathrm{d}x$, one bisects the interval to the left and right halves and multiplies the density of the measure on each interval by an independent realization of the variable $W^\beta / \mathbb{E}W^\beta$ to obtain the measure $\mu_{\beta,1}(\,\mathrm{d}x)$. The halves are then bisected into quarters and the densities again multiplied by independent and identically distributed random factors, and iterating the process of bisections and multiplications is continued indefinitely to obtain a sequence of measures $(\mu_{\beta,n}(\,\mathrm{d}x))_{n=0}^{\infty}$. The density of the measures at any given point $x \in [0, 1]$ is a martingale with respect to the filtration generated by the successive bisections, so by the martingale convergence theorem one obtains a limit measure $\mu_\beta(\,\mathrm{d}x)$ with strong stochastic self-similarity properties.

The self-similarity properties of Mandelbrot cascade measures reflect the dyadic structure of the bisections rather than the natural geometry of $[0, 1]$. Multiplicative chaos is a generalization of the Mandelbrot cascade which allows for random measures that respect the geometric structure of the underlying space better. To fix ideas, lognormal multiplicative chaos on the unit interval can loosely be described as the measure

$$\nu_\beta(\,\mathrm{d}x) = \frac{e^{\beta X(x)}}{\mathbb{E}e^{\beta X(x)}}\,\mathrm{d}x, \tag{1}$$

where $\beta > 0$ is a parameter and $(X(x))_{x\in[0,1]}$ is a centered Gaussian process whose covariance has a logarithmic singularity, i.e.

$$\mathbb{E}X(x)X(y) \sim \log \frac{1}{|x-y|} \quad \text{as } |x-y| \to 0.$$

To make this construction rigorous, one constructs a sequence of approximating fields $(X_n)_{n=0}^{\infty}$ so that the sequence $\left(\frac{e^{\beta X_n(x)}}{\mathbb{E}e^{\beta X_n(x)}}\,\mathrm{d}x\right)_{n=0}^{\infty}$ of approximating measures is a (measure-valued) martingale.

The first mathematical question concerning both Mandelbrot cascades and multiplicative chaos is the question of possible degeneracy of the limit. In both constructions the total mass of the limit measure is obtained as a limit of a positive martingale, but the limit might be almost surely 0. It was shown by Kahane and Peyrière [24] for Mandelbrot cascades and Kahane [22] for lognormal multiplicative chaos that the models exhibit a phase transition: there exists a $\beta_c \in [0, \infty]$ depending on details of the model such that for any $\beta < \beta_c$ almost surely the limit measure gives nonzero mass to any interval, and that for any $\beta > \beta_c$ almost surely the limit measure is null. The value of β_c can typically be exactly calculated and in most interesting cases one has $\beta_c \in (0, \infty)$. Kahane and Peyrière [24] also give a characterization for the nondegeneracy of the limit measure in the case $\beta = \beta_c$; in the lognormal case the limit is almost surely null.

The limit measures in the critical ($\beta = \beta_c$) and supercritical ($\beta > \beta_c$) cases are degenerate, but these cases are still of great interest. The Mandelbrot cascade has been studied as a toy model of phase transition in the statistical physics of disordered systems, where the primary objects of study are the normalized measures $\frac{\mu_{\beta,n}(\,dx)}{||\mu_{\beta,n}||}$, interpreted as the Gibbs measure of a single particle in a random potential, and the asymptotics of the partition function $||\mu_{\beta,n}||$ as $n \to \infty$. The $\beta \nearrow \infty$ limit is clearly relevant for the study of the maxima of the field X (in lognormal multiplicative chaos) or branching random walks (in Mandelbrot cascades). Building on recent progress in the theory of branching random walks [2, 36, 29], it has been shown that for any $\beta \geq \beta_c$ there exist deterministic renormalizations $(c_n(\beta))_n$ to the martingales in Mandelbrot cascades and lognormal multiplicative chaos [7, 30] such that the sequences $\left(c_n(\beta)\mu_{\beta,n}(\,dx)\right)_n$ and $\left(c_n(\beta)\nu_{\beta_c,n}(\,dx)\right)_n$ converge to nontrivial limit measures. These limit measures are the critical cascade and chaos measures that are studied in this thesis.

The random measures discussed above have very different geometric properties in the phases $\beta < \beta_c$, $\beta = \beta_c$ and $\beta > \beta_c$. The subcritical (or high-temperature) case $\beta < \beta_c$ is the oldest and best understood. The subcritical measures are natural examples of multifractal measures. To very briefly explain what this means, let $E_\alpha \subset [0,1]$ be the (random) set in which μ_β has the local scaling exponent $\alpha \geq 0$, i.e. denote

$$E_\alpha = \left\{ x \in [0,1] \,\middle|\, \lim_{r \searrow 0} \frac{\log \mu_\beta([x-r, x+r])}{\log r} = \alpha \right\}.$$

It was shown already by Kahane and Peyrière in [24] that for any given $\beta \in (0, \beta_c)$ there exists a unique deterministic $\alpha = \alpha_\beta > 0$ for which E_{α_β} carries all the mass of μ_β. Nevertheless, the sets E_α are nonempty for a range of $\alpha \neq \alpha_\beta$, i.e. the measure μ_β exhibits multifractality. Molchan [32] and Barral [5] showed that their Hausdorff dimensions are deterministic quantities and may be computed through the so-called multifractal formalism. As $\beta \nearrow \beta_c$, the scaling exponent α_β for which E_{α_β} carries all the mass of μ_β satisfies $\alpha_\beta \searrow 0$ and $\dim(E_{\alpha_\beta}) \searrow 0$. In the supercritical (or low-temperature) case $\beta > \beta_c$, physicists had predicted (e.g. [13, 12]) that the limit measures are atomic. On the mathematics side, these predictions were recently rigorously proven [7, 30]. It also follows from the recent mathematical work that the laws of the supercritical measures can be expressed as a point process (which depends on $\beta > \beta_c$) with the intensity given by the critical measure.

1.2. **On the results in this thesis.** The article [**I**] is devoted to the study of the critical Mandelbrot cascade measure. It is shown that while the measure μ_{β_c} is almost surely concentrated on E_0, a set of Hausdorff dimension 0, the measure does not have atoms. An estimate for the modulus of continuity of (the cumulative distribution function of) μ_{β_c} is provided and the multifractal properties of the measure studied. In [**II**] the result on the modulus of continuity is extended to the case of lognormal multiplicative chaos. This requires careful study of the tail behavior of the limit variable $||\nu_{\beta_c}||$ and considerable technical work in addition to the method used in [**I**]. The article [**III**] studies the probabilities of the variables $||\mu_\beta||$ and $||\nu_\beta||$ having small values and represents progress towards understanding their full laws.

The right tails of these variables have been intensely studied and precise asymptotics had been known, but in the lognormal case only the finiteness of absolute moments of negative order had been written down in the literature. In [**IV**] the local scaling properties of Mandelbrot cascade measures are studied on a finer scale than in classical multifractal analysis.

More precise descriptions of the articles included in this thesis are given at the end of this introduction in Section 5.

1.3. **Remarks on nomenclature.** Mandelbrot cascades are intimately related to branching processes, have strong connections with stochastic fixed point equations and have been studied in the statistical physics of disordered systems. In the past research in these different fields was often carried out by different communities and independently of each other. This has had the consequence that the same (or at least essentially same) mathematical objects have different names for different communities. For example, Mandelbrot [31] defined the cascades later named after him as a comment on a theory of turbulence proposed by Kolmogorov and Yaglom. By analogy with statistical physics, he called them canonical cascades to contrast them with another model he called microcanonical. A more general form of what we have here called the total mass martingale $(||\mu_{\beta,n}||)_n$ of Mandelbrot cascades has been studied since the 70's, starting with Kingman [25] and Biggins [9], in connection with branching random walks and in that context it is nowadays called the additive martingale of the branching random walk. Durrett and Liggett [18] studied fixed points of the smoothing transform, which is an operator on the space of probability distributions of positive random variables. The fixed points have turned out to correspond precisely to the limit variables $||\mu_\beta||$. In the 80's Derrida and Spohn [13] studied directed polymers in random environments on Cayley trees – their model is precisely the Mandelbrot cascade.

The focus for researchers in different fields has been different. The name (Mandelbrot) cascade measure has been most used by researchers working on the limit measures, often from the standpoint of geometric measure theory. For many probabilists the fundamental objects have been either the branching random walk or the stochastic fixed point equation satisfied by the total mass, and the limit measure has been at most a tool for studying them. For the statistical physics community the most interesting features of the model have been the phase transition and the asymptotics of the partition function (total mass) for large n. Since the main theorems of this thesis feature the properties of the limit measures, the tradition with the most emphasis on the measures themselves is followed and the model is called the Mandelbrot cascade.

1.4. **Context and motivation.** In the last decade a new motivation for studying lognormal multiplicative chaos has emerged. The exponential of the Gaussian free field on a two-dimensional domain can naturally be interpreted as a lognormal multiplicative chaos measure. The Gaussian free field is a fundamental object in two-dimensional random geometry, often said to be a two-dimensional analog of one-dimensional Brownian motion in that it is a scaling limit of many natural discrete processes. Formally, the Gaussian free field on a domain $D \subset \mathbb{R}^2$ is the Gaussian field $(X(x))_{x\in D}$ with the covariance

$$\mathbb{E}X(x)X(y) = G_D(x,y),$$

where G_D is the harmonic Green's function of the domain D. Since

$$G_D(x,y) \sim \log \frac{1}{|x-y|} \quad \text{as } |x-y| \to 0,$$

the field X is not a random function on D. Instead, X may be interpreted as a distribution (generalized function; see e.g. [34]). Yet what motivates the work in this thesis is the exponential of the Gaussian free field, formally

$$e^{\beta X(x)}\,\mathrm{d}x \quad \text{for a parameter } \beta > 0,$$

and this object may be interpreted as a lognormal multiplicative chaos measure. The recent survey of Rhodes and Vargas [33] describes many applications of the theory of lognormal multiplicative chaos. The two applications to be detailed next have been the most central for the work in this thesis.

One line of recent mathematical work on the exponential of the Gaussian free field was initiated by the rigorous formulation of the Knizhnik–Polyakov–Zamolodzhikov relation by Duplantier and Sheffield [17]. The KPZ relation was originally formulated in physics literature in the context of conformal field theory and it was used, among other things, to predict various scaling exponents appearing in models of statistical physics. Roughly speaking, the relation connects the scaling exponents of a lattice model in the Euclidean geometry to the scaling exponents of the same model in a random ("quantum gravity") geometry. For example, Duplantier [14] predicted the so-called nonintersection exponents of the simple random walk on $\mathbb{Z}^2$ by using the KPZ relation and an exact solution of the problem on the random geometry side. This work and the method used is (as of yet) mathematically nonrigorous. The rigorous formulation of the KPZ relation established in [17] is a relation between the Hausdorff dimensions of a set $A \subset \mathbb{R}^2$ with respect to the Euclidean geometry and with respect to the geometry of what is called Liouville quantum gravity. While there is no rigorous construction for what should be the Riemannian metric or even the distance function in the geometry of Liouville quantum gravity, the volume form associated to the metric is the exponential of the Gaussian free field $e^{\beta X(x)}\,\mathrm{d}x$. If x denotes the dimension of A w.r.t. the Euclidean geometry and Δ the dimension of A w.r.t. Liouville quantum gravity, the KPZ relation states that almost surely

$$x = c_\beta \Delta^2 + (1-c_\beta)\,\Delta,$$

where the constant $c_\beta \in (0,1)$ is a function of the parameter β. The work of Duplantier and Sheffield, which did not use the theory of multiplicative chaos, was soon followed by Benjamini and Schramm [8] who proved a relation of the same type between dimensions of sets w.r.t. the length measure on $[0,1]$ and w.r.t. Mandelbrot cascade measures.

Another line of recent work in which the exponential of the Gaussian free field features prominently is the construction of random planar curves by conformal welding. In conformal welding one associates a homeomorphism

ϕ of the unit circle onto itself to a planar curve, but the process is easiest to explain by describing the inverse situation where one starts from the planar curve. Suppose one is given a Jordan curve Γ on $\mathbb{R}^2 \cong \mathbb{C}$. By the Jordan curve theorem one may write $\mathbb{C} = G_+ \cup \Gamma \cup G_-$, where G_+ and G_- are disjoint domains such that G_+ is bounded and G_- unbounded. By the Riemann mapping theorem there exist conformal maps $\varphi_+ : \mathbb{D} \to G_+$ and $\varphi_- : \mathbb{C} \setminus \overline{\mathbb{D}} \to G_-$, and by Carathéodory's extension theorem the maps φ_+ and φ_- may be uniquely extended to continuous maps $\overline{\mathbb{D}} \to G_+ \cup \Gamma$ and $\mathbb{C} \setminus \mathbb{D} \to G_- \cup \Gamma$, respectively. Restricting the extensions to the unit circle $\partial\mathbb{D}$, one obtains continuous maps $\varphi_+|_{\partial\mathbb{D}} : \partial\mathbb{D} \to \Gamma$ and $\varphi_-|_{\partial\mathbb{D}} : \partial\mathbb{D} \to \Gamma$. Finally, the welding homeomorphism $\phi : \partial\mathbb{D} \to \partial\mathbb{D}$ associated to the curve Γ is the composition $\phi = \varphi_+^{-1} \circ \varphi_-$. Note that for any conformal automorphism τ of the plane (i.e. a Möbius map), the curves Γ and $\tau(\Gamma)$ have the same welding homeomorphism. The fundamental theorem of conformal welding states that if one starts with a quasisymmetric homeomorphism $\phi : \partial\mathbb{D} \to \partial\mathbb{D}$ this process may be inverted, so that one obtains a planar curve Γ associated to ϕ which is moreover uniquely defined modulo the equivalence relation on planar curves induced by Möbius maps. Thus random planar curves may be constructed out of random self-homeomorphisms of the unit circle. A particularly interesting class of random planar curves is obtained by taking $\phi(e^{2\pi i t}) = \exp(2\pi i \frac{\mu([0,t])}{\mu([0,1])})$, $t \in [0,1)$, for a lognormal multiplicative chaos measure μ. Random homeomorphisms defined this way are far from being quasisymmetric and the existence of the corresponding welding curves is due to Astala, Jones, Kupiainen and Saksman [3]. The random curves obtained this way inherit stochastic self-similarity properties from the multiplicative chaos measures used in the construction. A related result due to Sheffield [35] shows that the welding homeomorphisms associated to the well-known Schramm–Loewner evolution curves can be expressed in terms of the exponential of the Gaussian free field.

2. Mandelbrot Cascades

The aim of the rest of this introduction is to give precise definitions and fundamental theorems concerning Mandelbrot cascades and lognormal multiplicative chaos. The theory of Mandelbrot cascades can by now be called classical. Lognormal multiplicative chaos has seen a surge of interest in recent years and the theory is younger. However, many properties of the chaos measures can be proved almost exactly in the same way as the properties of the cascade measures once the correct viewpoint has been found. In this section we treat Mandelbrot cascades and once the theory has been introduced, understanding lognormal multiplicative chaos measures in the next section has become an easier task.

We will construct Mandelbrot cascades on binary trees and the geometric realization of the measures will be on the system of dyadic subintervals of $[0,1]$. While this is not the most general viewpoint, it illustrates all the features of the theory that are necessary for understanding multiplicative chaos.

Notation. Let $\Sigma_n = \{0,1\}^n$ and $\Sigma = \bigcup_{n=1}^{\infty} \Sigma_n$. We consider Σ an infinite binary tree (with the root node missing; more than anything else this is for reasons of convention). The generic $\sigma \in \Sigma_n$ is denoted by $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) = \sigma_1\sigma_2 \ldots \sigma_n$. The tree structure is given in the obvious way: the node $\sigma = \sigma_1\sigma_2 \ldots \sigma_n$ is connected to its parent node $\sigma_1\sigma_2 \ldots \sigma_{n-1}$ and the child nodes $\sigma_1\sigma_2 \ldots \sigma_n 0$ and $\sigma_1\sigma_2 \ldots \sigma_n 1$. The length of σ is denoted by $|\sigma|$, i.e. $|\sigma| = n$ if $\sigma \in \Sigma_n$. If $\sigma, \sigma' \in \Sigma$ the concatenation of the strings is denoted by $\sigma\sigma' = \sigma_1\sigma_2 \ldots \sigma_{|\sigma|}\sigma'_1\sigma'_2 \ldots \sigma_{|\sigma'|}$. We will often need to index over a certain branch of the tree. For this purpose, if $\sigma = \sigma_1\sigma_2 \ldots \sigma_n \in \Sigma_n$ we denote $\sigma|k = \sigma_1\sigma_2 \ldots \sigma_k \in \Sigma_k$ for integers $1 \le k \le n$.

For the geometric realization of the cascade measures on $[0,1]$ we need notation to connect the tree to the system of dyadic intervals. Every $\sigma \in \Sigma_n$ naturally corresponds to a unique half-open dyadic subinterval I_σ of $[0,1]$ of length 2^{-n}.[1] If $x \in [0,1]$, by $I_n(x)$ we denote the unique half-open dyadic interval of length 2^{-n} containing x and by $\sigma(x)$ we denote the unique infinite binary string such that $I_{\sigma(x)|n} = I_n(x)$ for all n.

2.1. Mandelbrot cascades and branching random walks.

Definition 1. *Let W be a positive random variable such that $\mathbb{E}W = \frac{1}{2}$. Let $\{W_\sigma : \sigma \in \Sigma\}$ be an i.i.d. collection of copies of W. For every $n \in \mathbb{N}$, we define μ_n as the measure on $[0,1]$ that has constant density with respect to the Lebesgue measure on each dyadic interval I_σ, $\sigma \in \Sigma_n$, of length 2^{-n} and that has the values $\mu_n(I_\sigma) = \prod_{k=1}^{n} W_{\sigma|k}$. Explicitly in terms of density with respect to the Lebesgue measure, this says that*

$$\text{for each } \sigma \in \Sigma_n, \quad \frac{d\mu_n}{dx}(x) = \prod_{k=1}^{n} (2W_{\sigma|k}) \quad \text{for all } x \in I_\sigma.$$

The limit $\mu = \lim_{n\to\infty} \mu_n$, with the limit interpreted in the sense of weak convergence of measures, is the Mandelbrot cascade measure*. The total masses of the measures are denoted by*

$$Y = \mu([0,1]) \quad \text{and} \quad Y_n = \mu_n([0,1]) = \sum_{\sigma\in\Sigma_n} \prod_{k=1}^{n} W_{\sigma|k}.$$

Remark 2. Since $\{W_\sigma : \sigma \in \Sigma\}$ is an independent family and $\mathbb{E}W = \frac{1}{2}$, for any $x \in [0,1]$ the densities of the μ_n at x form a martingale with respect to the filtration $(\mathcal{F}_n)_n$ defined by $\mathcal{F}_n = \sigma(\{W_\sigma : |\sigma| \le n\})$. This implies that for any $I \subset [0,1]$ the sequence $(\mu_n(I))_n$ is a positive martingale and as such almost surely convergent, by the martingale convergence theorem. The almost sure existence of the weak limit $\mu = \lim_n \mu_n$ then follows.

As indicated in the introduction, Mandelbrot cascades naturally occur in one-parameter families. For any $\beta \in \mathbb{R}$ one may replace W by the variable $W^\beta/(2\mathbb{E}W^\beta)$ in the definition above to obtain the sequence of measures $(\mu_{\beta,n})_n$ and the limit measure μ_β. Embedding the cascade in a one-parameter family this way might seem ad-hoc, but there are reasons for

[1]For definiteness, $I_\sigma = [\sum_{k=1}^{n} \sigma_k 2^{-k}, \sum_{k=1}^{n} \sigma_k 2^{-k} + 2^{-n}[$. All the random measures to be considered here are such that they will, by easy scaling arguments, have no deterministic atoms and further they are defined by giving their densities with respect to the Lebesgue measure. Thus the choice of where to include the endpoints is immaterial.

it. One is multifractal analysis, in which the one-parameter family $(\mu_\beta)_\beta$ is indispensable. Another, more in line with the scope of this introduction, is that it becomes natural when the total mass martingale $(\mu_n([0,1]))_n$ is interpreted in terms of the branching random walk.

Let V be a real random variable. The binary branching random walk on the real line is defined as follows. At time 0 there is a single particle at 0. The particle splits into two particles, each of which is displaced from 0 by an independent realization of the variable V. Thus at time 1 there are two particles at positions V_0 and V_1, with $V_0 \perp V_1$ and $V_0 \overset{d}{=} V_1 \overset{d}{=} V$. Each of these particles splits into two, with the displacements of the offspring again being independent of each other and distributed like V, giving rise to four particles at time 2 with the positions V_0+V_{00}, V_0+V_{01}, V_1+V_{10} and V_1+V_{11} where all the summands are independent and distributed like V. Continuing this way, we arrive at the next definition.

Definition 3. *Let V be a real random variable and $\{V_\sigma : \sigma \in \Sigma\}$ and independent collection of copies of V. Denote*

$$X_\sigma = \sum_{k=1}^{|\sigma|} V_{\sigma|k}$$

for all $\sigma \in \Sigma$. Define the binary branching random walk *as the stochastic process indexed by integers $n \geq 0$ whose state at time n is the 2^n-tuple of points $(X_\sigma)_{\sigma\in\Sigma_n}$.*

Any given branch of the branching random walk is nothing but a simple random walk with the step distribution V. The collective behavior of all the particles is, however, not so simple to describe. For example, the maximum process $(M_n)_n$ defined by

$$M_n = \max\left\{X_\sigma \,\middle|\, \sigma \in \Sigma_n\right\}$$

obviously has very different behavior than a simple random walk. The process $(M_n)_n$ has received much attention in the literature (see e.g. [1] and the references therein) and the results and techniques of this study have been extremely significant also for the purpose of understanding the critical and supercritical cascade measures. A naive analytic approach towards finding the maximum of the branching random walk at any given moment n would be to consider the exponential sums

$$Z_{\beta,n} = \sum_{\sigma\in\Sigma_n} e^{\beta X_\sigma} \tag{2}$$

and their asymptotics as $\beta \to \infty$. To analyze $Z_{\beta,n}$ we denote

$$\tau(\beta) = 1 + \log_2 \mathbb{E}e^{\beta V} \quad \text{so that} \quad \mathbb{E}e^{\beta V} = 2^{\tau(\beta)-1}.$$

We then have

$$\begin{aligned}\mathbb{E}Z_{\beta,n} &= \mathbb{E}\sum_{\sigma\in\Sigma_n} e^{\beta X_\sigma} = \mathbb{E}\sum_{\sigma\in\Sigma_n} e^{\sum_{k=1}^n \beta V_{\sigma|k}} \\ &= 2^n\left(2^{\tau(\beta)-1}\right)^n = 2^{n\tau(\beta)}\end{aligned}$$

and

$$\frac{Z_{\beta,n}}{\mathbb{E}Z_{\beta,n}} = 2^{-n\tau(\beta)} \sum_{\sigma\in\Sigma_n} e^{\beta X_\sigma} = \sum_{\sigma\in\Sigma_n} e^{\sum_{k=1}^n (\beta V_{\sigma|k} - \tau(\beta)\log 2)}$$
$$= \sum_{\sigma\in\Sigma_n} \prod_{k=1}^n e^{\beta V_{\sigma|k} - \tau(\beta)\log 2}.$$

In the notation for Mandelbrot cascades introduced above, the last expression is precisely the total mass $\mu_{\beta,n}([0,1])$ of the cascade associated to the variable $W = e^{V-\tau(1)\log 2}$. In what follows we will assume that the variables W and V are related to each other through this relation and use these notational conventions interchangeably. In keeping with the assumption that $\mathbb{E}W = \frac{1}{2}$, we have $\tau(1) = 0$ and thus generally

$$\tau(\beta) = 1 + \log_2 \mathbb{E}W^\beta. \tag{3}$$

We remark that in the literature on branching random walks, the sequence $(Z_{\beta,n}/\mathbb{E}Z_{\beta,n})_n$ is called the *additive martingale* of the branching random walk.

Remark 4. We have not strived for generality in these definitions. Cascade measures have perhaps most often been considered on a b-ary tree with $b \geq 2$ an arbitrary integer, with the tree projected onto the system of b-adic intervals of the unit interval. It is also natural to project the cascade measure on a b^n-ary tree onto the system of b-adic hypercubes of $[0,1]^n$. The independence of the family $\{W_\sigma : \sigma \in \Sigma\}$ could also be relaxed. Instead of the binary branching random walk introduced here in which at every step each particle independently splits into two particles with the displacements of the offspring being independent, it is common to consider the general branching random walk in which the offspring distribution of each particle is simply some point process on the real line. Even in this context the additive martingale can be defined as above. Yet while one can (and often does) consider measures on boundaries of Galton–Watson trees, it is perhaps fair to say that cascade measures associated to general branching random walks have no obvious geometric realizations.

2.2. **Self-similarity in Mandelbrot cascades.** The construction implies stochastic self-similarity relations for the total mass and for the measure itself. For any dyadic interval $I_\sigma \subset [0,1]$, $\sigma \in \Sigma$, one has

$$\mu(I_\sigma) = \lim_{n\to\infty} \mu_n(I_\sigma) = \lim_{n\to\infty} \sum_{\sigma'\in\Sigma_{n-|\sigma|}} \prod_{k=1}^n W_{\sigma\sigma'|k}$$
$$= \left(\prod_{k=1}^{|\sigma|} W_{\sigma|k}\right) \lim_{n\to\infty} \sum_{\sigma'\in\Sigma_n} \prod_{k=|\sigma|+1}^{n} W_{\sigma\sigma'|k}$$
$$\stackrel{d}{=} \left(\prod_{k=1}^{|\sigma|} W_{\sigma|k}\right) Y',$$

where Y' is independent of $\{W_{\sigma|k} : k = 1, 2, \ldots, |\sigma|\}$ and distributed like the total mass Y. Thus the mass of any dyadic interval is distributed like the total mass multiplied by an independent factor, the expectation of which is the length of the interval. In fact the same calculation gives even more: the measure μ restricted onto the interval I_σ is distributed like the measure on $[0, 1]$, multiplied by an independent factor and properly rescaled. More explicitly,

$$(\mu(A))_{A\subset I_\sigma} \overset{d}{=} \left(\left(\prod_{k=1}^{|\sigma|} W_{\sigma|k} \right) \mu'((A - y_\sigma)/|I_\sigma|) \right)_{A\subset I_\sigma} \tag{4}$$

where y_σ denotes the left endpoint of I_σ and μ' is an independent realization of μ. This is known as the *exact scaling property* of μ for dyadic intervals.

A similar calculation also gives a recursion for the distributions of the total masses $(Y_n)_n$ and a fixed point equation for the distribution of the limit Y. Namely,

$$\begin{aligned} Y_n &= \mu_n([0, 1]) = \mu_n(I_0) + \mu_n(I_1) \\ &\overset{d}{=} W_0 Y_{n-1}^{(0)} + W_1 Y_{n-1}^{(1)}, \end{aligned} \tag{5}$$

where $Y_{n-1}^{(0)}$ and $Y_{n-1}^{(1)}$ are independent copies of Y_{n-1} that are also independent of the pair $\{W_0, W_1\}$. By independence and the martingale convergence theorem we may take the limit $n \to \infty$ to get the distributional equality

$$Y \overset{d}{=} W_0 Y^{(0)} + W_1 Y^{(1)}, \tag{6}$$

where again $Y^{(0)}$ and $Y^{(1)}$ are independent copies of Y that are also independent of the pair $\{W_0, W_1\}$.

2.3. **Nondegeneracy and moments.** The existence of the limits Y and μ was obtained by the martingale convergence theorem. However, the martingale convergence theorem only ensures that the limits almost surely exist and nothing more. In particular, it is possible and in many cases true that almost surely $Y = 0$. If this is the case, we say that the cascade is degenerate. It turns out that the characterization of degeneracy in terms of the generating variable W is rather simple.

By Fatou's lemma it is clear that $\mathbb{E}Y = \mathbb{E} \liminf_n Y_n \le \liminf_n \mathbb{E}Y_n = 1$ for any W. Proving the nondegeneracy of the cascade generated by W is achieved if one can show that in fact equality holds, and in martingale theory the basic condition for interchanging the order of limit and expectation is uniform integrability. If $\sup_n \mathbb{E}\phi(Y_n) < \infty$ for some nondecreasing function $\phi : [0, \infty) \to [0, \infty)$ that satisfies $\phi(x)/x \to \infty$ as $x \to \infty$, the order of limit and expectation may be interchanged and the cascade is nondegenerate. Taking $\phi(x) = x^p$ for some $p > 1$, this approach also connects to the existence of absolute moments of the total mass variable.

Estimates for the sequence of moments $(\mathbb{E}Y_n^p)_n$ are easy to obtain, especially if p is an integer. In the case $p = 2$ we have, by the distributional recursion (5),

$$\begin{aligned}\mathbb{E}Y_n^2 &= \mathbb{E}\left(W_0 Y_{n-1}^{(0)} + W_1 Y_{n-1}^{(1)}\right)^2 \\ &= 2\mathbb{E}W^2\,\mathbb{E}Y_{n-1}^2 + \frac{1}{2} \\ &= 2^{\tau(2)}\mathbb{E}Y_{n-1}^2 + \frac{1}{2}.\end{aligned}$$

Since $Y_0 \equiv 1$, it is clear that $\sup_n \mathbb{E}Y_n^2 < \infty$ if and only if $\tau(2) < 0$. From this it follows that the martingale $(Y_n)_n$ in fact converges in $\mathcal{L}^2$ and that $\mathbb{E}Y^2 < \infty$. The same direct computation may be performed for integer moments $p > 2$ with the same consequences. Even more, the same idea carries through to arbitrary $p > 1$. For $p \in (1,2)$ one may use the subadditivity of the function $x \mapsto x^{p/2}$ to estimate

$$\begin{aligned}\mathbb{E}Y_n^p &= \mathbb{E}\left(\left(W_0 Y_{n-1}^{(0)} + W_1 Y_{n-1}^{(1)}\right)^2\right)^{p/2} \\ &= \mathbb{E}\left(\left(W_0 Y_{n-1}^{(0)}\right)^2 + \left(W_1 Y_{n-1}^{(1)}\right)^2 + 2W_0W_1Y_{n-1}^{(0)}Y_{n-1}^{(0)}\right)^{p/2} \\ &\leq 2\mathbb{E}W^p\,\mathbb{E}Y_{n-1}^p + 2^{p/2}\left(\mathbb{E}W^{p/2}\right)^2\left(\mathbb{E}Y_{n-1}^{p/2}\right)^2 \\ &\leq 2^{\tau(p)}\mathbb{E}Y_{n-1}^p + 2^{2\tau(\frac{p}{2})-2+\frac{p}{2}},\end{aligned}$$

where Jensen's inequality and the fact that $\mathbb{E}Y_{n-1} = 1$ for all n were applied in the last step. We deduce that if $\tau(p) < 0$, $\sup_n \mathbb{E}Y_n^p < \infty$ which implies nondegeneracy of the cascade and $\mathcal{L}^p$-convergence of the martingale $(Y_n)_n$. With the same subadditivity estimate applied to the distributional equation (6) one obtains a converse conclusion: if $\mathbb{E}Y^p < \infty$ for some $p > 1$, then $\tau(p) < 0$.

The cumulant generating function τ characterizes nondegeneracy of the cascade and the existence of moments of the total mass. From the definition of τ we see that $\tau(0) = 1$ and the assumption $\mathbb{E}W = \frac{1}{2}$ is equivalent to $\tau(1) = 0$. It is a straightforward consequence of Hölder's inequality that τ is finite and convex on some interval containing $[0,1]$. Above we saw that if $\tau(p) < 0$ for some $p > 1$, the cascade generated by W is nondegenerate. By convexity and $\tau(1) = 0$, if $\tau(p) < 0$ for some $p > 1$, $\tau(h) > 0$ for all $h \in (0,1)$. Conversely if $\tau(h) < 0$ for some $h \in (0,1)$, $\tau(p) > 0$ for all $p > 1$. This suggests that in this case the cascade is degenerate. Indeed, one may argue directly from the recursion (5) and the subadditivity of $x \mapsto x^h$ for $h \in (0,1)$ as follows. For any n and $h \in (0,1)$,

$$\begin{aligned}\mathbb{E}Y_n^h &= \mathbb{E}\left(W_0 Y_{n-1}^{(0)} + W_1 Y_{n-1}^{(1)}\right)^h \\ &\leq 2\mathbb{E}W^h\,\mathbb{E}Y_{n-1}^h \;=\; 2^{\tau(h)}\mathbb{E}Y_{n-1}^h.\end{aligned}$$

Since $\mathbb{E}Y_0^h = 1$ we obtain $\mathbb{E}Y_n^h \leq 2^{n\tau(h)}$, which clearly implies $Y \equiv 0$ if $\tau(h) < 0$.

It takes only slightly more work to prove the following theorem of Kahane and Peyrière which completely characterizes the nondegeneracy and existence of moments of the total mass.

Theorem 5 (Kahane, Peyrière [24]). *Let W be a positive random variable with $\mathbb{E}W = \frac{1}{2}$. For any $p > 1$ the following are equivalent.*

(1) $\tau(p) < 0$.
(2) $\sup_n \mathbb{E}Y_n^p < \infty$. *In this case the martingale $(Y_n)_n$ converges in $\mathcal{L}^p$, and especially $\mathbb{E}Y^p < \infty$.*

Further, the limiting total mass is nondegenerate if and and only if

$$\tau'(1) = \mathbb{E}W \log_2 W < 0,$$

and if this condition holds one also has $\mathbb{E}Y = 1$.

2.4. **Peyrière probability.** A natural next step in presenting the fundamentals of the theory of Mandelbrot cascades is to study the properties of the total mass variable through the distributional equation (6) which is also of considerable independent interest. Before that we need to introduce a technical tool of great importance: the Peyrière probability, introduced already in [24].

Like most of the concepts in the theory, the same essential idea with a slightly different emphasis is well known in the theory of branching random walks. In the setting of general branching random walks described in Remark 4, the following extension of probability space and the new measure introduced in the extension is known as a spine decomposition, following Lyons, Pemantle and Peres [28]. The idea is to extend the probability space on which the branching random walk is defined to include information on a special marked branch of the binary tree (or Galton–Watson tree in the general case), called the spine. Different regimes of the cloud of particles in the branching random walk can then be studied by weighting the spine differently and studying the behavior of the spine. This has the effect of reducing the study of the whole configuration of the branching random walk to studying the behavior of a single weighted branch, which can often be effected through the theory of simple random walks. In terms of the Mandelbrot cascade measure this, roughly speaking, corresponds to sampling a random point x from $[0, 1]$ according to a cascade measure μ_β with some parameter $\beta \in \mathbb{R}$. The spine of the branching random walk picture then corresponds to the nested sequence $(I_n(x))_n$ of dyadic intervals containing x and the corresponding sequence

$$(\mu_{\beta,n}(I_n(x)))_n = \left(\exp\left(\sum_{k=1}^{n}(\beta V_{\sigma(x)|k} - \tau(\beta)\log 2)\right)\right)_n.$$

For any given $\beta_0 \in \mathbb{R}$ such that the limit measure μ_{β_0} is nondegenerate, one may obtain precise information on the local behavior of μ_{β_0} by varying the parameter β according to which the random point x is chosen.

Definition 6. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote the original probability space, where $\mathcal{F}$ denotes the σ-algebra generated by the family $\{W_\sigma : \sigma \in \Sigma\}$. Suppose the cascade generated by $W^\beta/(2\mathbb{E}W^\beta)$ is nondegenerate. We define the probability measure $\mathbb{Q}^\beta$ and the extended probability space $(\Omega\times[0,1], \mathcal{F}\otimes\mathcal{B}([0,1]), \mathbb{Q}^\beta)$ by setting*

$$\mathbb{E}_{\mathbb{Q}^\beta} f(\omega, x) = \mathbb{E}\int_0^1 f(\omega, x)\mu_\beta(\omega,\, \mathrm{d}x)$$

for all bounded $\mathcal{F} \otimes \mathcal{B}([0,1])$*-measurable functions* $f : \Omega \times [0,1] \to \mathbb{R}$*, where* $\mathcal{B}([0,1])$ *denotes the Borel sets of* $[0,1]$.

In [24] the measure $\mathbb{Q}^\beta$ was used to prove the following theorem on the support of the measure μ_β. Since this is a good illustration of the properties of $\mathbb{Q}^\beta$, we present a sketch of this analysis. For this purpose it is sufficient to work with a fixed β, so we take $\beta = 1$ and omit it from the notation.

Theorem 7 (Kahane, Peyrière [24])**.** *For* $\alpha > 0$*, denote*

$$E_\alpha = \left\{ x \in [0,1] \, : \, \lim_{n\to\infty} \frac{\log \mu(I_n(x))}{\log |I_n(x)|} = \alpha \right\}.$$

Suppose the cascade generated by W *is nondegenerate, i.e. that* $\tau'(1) < 0$*, and that* $\mathbb{E}Y(\log Y)^2 < \infty$*. Then*

$$\mu(E_{-\tau'(1)}) = \mu([0,1]) \quad \textit{almost surely.}$$

Remark 8. Note that if by slight abuse of notation we extend the total mass variable Y to the probability space $(\Omega \times [0,1], \mathcal{F} \otimes \mathcal{B}([0,1]), \mathbb{Q})$ simply by $Y(\omega, x) = Y(\omega)$ for all $(\omega, x) \in \Omega \times [0,1]$, we have $\mathbb{E}Y(\log Y)^2 = \mathbb{E}_\mathbb{Q}(\log Y)^2$. Also, by Theorem 5 the extra condition $\mathbb{E}Y(\log Y)^2 < \infty$ holds at least if there exists some $p > 1$ such that $\tau(p) < 0$.

Sketch of proof. By the exact scaling property of μ, for any $x \in [0,1]$

$$\mu(I_n(x)) = e^{\sum_{k=1}^n V_{\sigma(x)|k}} Y(\sigma(x)|n)$$

for all n, where $Y(\sigma(x)|n) \stackrel{d}{=} Y$ and $Y(\sigma(x)|n) \perp \{V_{\sigma(x)|k} : k = 1, 2, \ldots, n\}$ for all n and x. It follows that

$$\log \mu(I_n(x)) = S_n + \log Y(\sigma(x)|n), \quad \text{where } S_n = \sum_{k=1}^n V_{\sigma(x)|k}.$$

The crucial property of Peyrière probability and the reason for its effectiveness is that under $\mathbb{Q}$, the process $(S_n)_n$ is a random walk with independent and identically distributed steps. To see this, simply let $f_1, f_2, \ldots, f_n$ be bounded and continuous functions $\mathbb{R} \to \mathbb{R}$ and calculate

$$\begin{aligned}
\mathbb{E}_\mathbb{Q} \prod_{k=1}^n f_k(V_{\sigma(x)|k}) &= \mathbb{E} \int_0^1 \left(\prod_{k=1}^n f_k(V_{\sigma(x)|k}) \right) \mu(\mathrm{d}x) \\
&= \mathbb{E} \sum_{\sigma \in \Sigma_n} \left(\prod_{k=1}^n f_k(V_{\sigma|k}) \right) \left(\prod_{k=1}^n e^{V_{\sigma|k}} \right) \\
&= \sum_{\sigma \in \Sigma_n} \prod_{k=1}^n \mathbb{E} \left(f_k(V_{\sigma|k}) e^{V_{\sigma|k}} \right) \\
&= \prod_{k=1}^n \left(2\mathbb{E} f_k(V) e^V \right).
\end{aligned}$$

Taking $f_k \equiv 1$ for all but one k we obtain

$$\mathbb{E}_\mathbb{Q} f(V_{\sigma(x)|k}) = 2\mathbb{E} f(V) e^V \quad \text{for all } k, \tag{7}$$

so we deduce that indeed the sequence $(V_{\sigma(x)|n})_n$ consists of i.i.d. variables, and moreover the common distribution is explicitly given by (7).

By the strong law of large numbers, $\mathbb{Q}$-almost surely

$$\frac{S_n}{n} \longrightarrow \mathbb{E}_{\mathbb{Q}} V_{\sigma(x)|1} = 2\mathbb{E}Ve^V \quad \text{as } n \to \infty.$$

It is not difficult, but also not very relevant for the purpose of this sketch, to prove that the assumption $\mathbb{E}_{\mathbb{Q}}(\log Y)^2 < \infty$ implies that $\frac{1}{n}\log Y(\sigma(x)|n) \to 0$ as $n \to \infty$, $\mathbb{Q}$-almost surely. We have thus deduced that for $\mathbb{Q}$-almost every $(\omega, x) \in \Omega \times [0,1]$,

$$\frac{\log \mu(I_n(x))}{\log |I_n(x)|} = \frac{S_n + \log Y(\sigma(x)|n)}{-n \log 2} \longrightarrow -\frac{2}{\log 2}\mathbb{E}Ve^V = -\tau'(1).$$

But from the definition of the probability $\mathbb{Q}$ it is clear that this is equivalent to the claim. □

Above we have only defined and discussed the Peyrière probability $\mathbb{Q}^\beta$ for $\beta \in \mathbb{R}$ such that the cascade generated by $W^\beta/(2\mathbb{E}W^\beta)$ is nondegenerate. However, the definition could be extended to all $\beta \in \mathbb{R}$. Though we will not need this extension, it is helpful to understand how this is done. Denote $\mathcal{B}_n = \sigma(\{I_\sigma : \sigma \in \Sigma_n\}) \subset \mathcal{B}([0,1])$ and recall that $\mathcal{F}_n = \sigma(\{W_\sigma : |\sigma| \le n\})$. Then $\mathcal{F}_n \otimes \mathcal{B}_n$-measurable functions $f : \Omega \times [0,1] \to \mathbb{R}$ are those that depend only on the cascade variables $\{W_\sigma\}$ up to level n and that are constant on dyadic intervals of length 2^{-n}. Now, since the density of any Mandelbrot cascade measure is a martingale with respect to $(\mathcal{F}_m)_m$, for any bounded $\mathcal{F}_n \otimes \mathcal{B}_n$-measurable $f : \Omega \times [0,1] \to \mathbb{R}$ we may consistently define

$$\mathbb{E}_{\mathbb{Q}^\beta} f(\omega, x) = \mathbb{E}\int_0^1 f(\omega, x)\mu_{\beta,m}(\omega, \mathrm{d}x) \quad \text{for some } m > n.$$

The sequence $(\mathcal{F}_n \otimes \mathcal{B}_n)_n$ of σ-algebras generates $\mathcal{F} \otimes \mathcal{B}([0,1])$, so the measure $\mathbb{Q}^\beta$ as defined above extends to the full σ-algebra $\mathcal{F} \otimes \mathcal{B}([0,1])$ as a probability measure. In the case where the limit measure μ_β is nondegenerate, this definition is readily checked to be equivalent with Definition 6. Especially, for random variables under the original probability measure $\mathbb{P}$, interpreted in the extended probability space as variables that do not depend on the random point x, the effect of the measure $\mathbb{Q}^\beta$ is equivalent to tilting the original probability measure $\mathbb{P}$ by the martingale $(Y_{\beta,n})_n$.

2.5. **Stochastic fixed point equations.** The distributional equation (6) is a key property of the total mass variable Y. However, the equation is a rich topic of study in itself. Let $\mathcal{P}$ denote the set of probability measures on $[0, \infty)$ and let $\{W_0, W_1\}$ be a fixed pair of positive random variables. Define the mapping $\mathbf{T} : \mathcal{P} \to \mathcal{P}$ as follows. Let $\eta \in \mathcal{P}$ and let Z be a (positive) random variable with the law η. Then $\mathbf{T}\eta$ is the law of the random variable $W_0 Z^{(0)} + W_1 Z^{(1)}$, where $Z^{(0)}$ and $Z^{(1)}$ are independent copies of Z that are also independent of the pair $\{W_0, W_1\}$. By a slight abuse of notation we also denote $\mathbf{T}Z \overset{d}{=} W_0 Z^{(0)} + W_1 Z^{(1)}$. The mapping $\mathbf{T}$ is known as the *smoothing transform* associated to the pair $\{W_0, W_1\}$. The distributional equation (6) can now be restated as follows: the total mass Y of a (nondegenerate) Mandelbrot cascade is a (nontrivial) fixed point of the smoothing transform associated to the pair $\{W_0, W_1\}$, where W_0 and W_1 are independent copies of the generating variable W.

Fixed points of the smoothing transform were studied and characterized by Durrett and Liggett [18]. Given a pair $\{W_0, W_1\}$, with weak assumptions a nonzero fixed point exists and it is unique up to the trivial issue of multiplication by a positive constant. Possible fixed points Y of smoothing transforms may be classified according to the asymptotics of the Laplace transform $\varphi_Y(t) = \mathbb{E}e^{-tY}$ as $t \searrow 0$. Total mass variables of Mandelbrot cascades, as defined above, have $\mathbb{E}Y = 1$ and therefore $\varphi_Y(t) \sim 1 - t$ as $t \searrow 0$. According to the following theorem, the only other possible asymptotics are $\varphi_Y(t) \sim 1 - t\log 1/t$ and $\varphi_Y(t) \sim 1 - t^\alpha$ for $\alpha \in (0,1)$.

Theorem 9 (Durrett, Liggett [18]). *Let $\{W_0, W_1\}$ be a pair of positive nonconstant random variables such that $\mathbb{E}(W_0 + W_1) = 1$ and denote*

$$\tilde{\tau}(\alpha) = \log_2 \mathbb{E}(W_0^\alpha + W_1^\alpha).$$

Then $\tilde{\tau}(1) = 0$. The fixed points of the smoothing transform are classified according to the following three exhaustive cases:

(1) *$\tilde{\tau}(\alpha) > 0$ for $\alpha \in (0,1)$ and $\tilde{\tau}'(\alpha) < 0$.*
(2) *$\tilde{\tau}(\alpha) > 0$ for $\alpha \in (0,1)$ and $\tilde{\tau}'(\alpha) = 0$.*
(3) *$\tilde{\tau}(\alpha) = 0$ for some unique $\alpha \in (0,1)$.*

In each case, there exists a unique (up to multiplication by a positive constant) fixed point of the smoothing transform associated to $\{W_0, W_1\}$. Supposing that the pair $\{\log W_0, \log W_1\}$ is nonlattice[2], the Laplace transform $\varphi_Y(t)$ of the fixed point Y can be chosen to satisfy the following:

Case (1): *$\varphi_Y(t) \sim 1 - t$ as $t \searrow 0$. In this case the limit Y is obtained as the total mass of the Mandelbrot cascade generated by the pair $\{W_0, W_1\}$.*

Case (2): *$\varphi_Y(t) \sim 1 - t\log 1/t$ as $t \searrow 0$.*

Case (3): *$\varphi_Y(t) \sim 1 - t^\alpha$ as $t \searrow 0$, where $\alpha \in (0,1)$ is the unique solution of $\tau(\alpha) = 0$.*

The separation to the three different cases is not an incidental property of the fixed points. The argument of Durrett and Liggett proceeds by studying the fixed point equation

$$Y \stackrel{d}{=} W_0 Y^{(0)} + W_1 Y^{(1)}$$

in terms of its Laplace transform

$$\varphi_Y(t) = \mathbb{E}\varphi_Y(W_0)\varphi_Y(W_1).$$

It is natural to try to find the fixed point above by defining a sequence of approximations by starting from some nonincreasing initial data $\varphi_0 : [0,\infty) \to [0,\infty)$ with $\varphi_0(0) = 1$ and defining iteratively

$$\varphi_{n+1}(t) = \mathbb{E}\varphi_n(W_0)\varphi_n(W_1).$$

[2]The assumption is that there is no number $s > 0$ such that with probability 1, both $\log W_0$ and $\log W_1$ are integer multiples of s. This requirement is related to issues in renewal theory, which Durrett and Liggett [18] also deal with. In the lattice case the fixed points are parametrized by a certain set of periodic functions which appears in the asymptotics for the Laplace transform, but again in the interest of simplicity we have preferred to not discuss this case here.

In case (1) this works from the constant initial data $\varphi_0(t) \equiv 1$ – this is precisely the convergence of the total mass in the Mandelbrot cascade. In cases (2) and (3) one must start from initial data with the same asymptotics as the fixed point itself.

The proof will not be studied here. Instead, we take a look at a related result concerning the fixed points in case (1) above. The following theorem was originally proved by Guivarc'h [20], while a more conceptual and general proof based explicitly on the idea of Peyrière probability was given by Liu [26].

Theorem 10 (Guivarc'h [20]). *Let Y be the total mass of the Mandelbrot cascade generated by W. Suppose $\tau(p) = 1 + \log_2 \mathbb{E}W^p = 0$ for some $p > 1$. (If such a $p > 1$ exists, it is unique.) Then there exists a constant $C > 0$ (depending on W) such that*

$$\mathbb{P}(Y \geq x) \sim C/x^p \quad \text{as } x \to \infty.$$

The following proof could be written utilizing the standard Peyrière probability as defined in Definition 6, but it is instructive to understand that the idea is flexible and may be used in different forms.

Sketch of proof. The total mass of a Mandelbrot cascade is a fixed point of the smoothing transform, i.e.

$$Y \stackrel{d}{=} W_0 Y^{(0)} + W_1 Y^{(1)}.$$

We define a new probability space which extends the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting the variables on the right hand side so that the variables are weighted like under the Peyrière probability. We define the probability space $(\Omega \times \{0,1\}, \mathcal{F} \otimes \sigma(\{0\},\{1\}), \mathbb{Q})$ by setting

$$\mathbb{E}_{\mathbb{Q}} f(\omega, j) = \mathbb{E}\left(W_0 Y^{(0)} f(\omega, 0) + W_1 Y^{(1)} f(\omega, 1)\right)$$

for all bounded measurable $f : \Omega \times \{0,1\} \to \mathbb{R}$. Define the variables $\widetilde{Y}$, $\widetilde{W}$ and $\widetilde{B}$ on this extended probability space by

$$\widetilde{Y}(\omega, j) = Y^{(j)}, \quad \widetilde{W}(\omega, j) = W_j \quad \text{and} \quad \widetilde{B}(\omega, j) = \begin{cases} W_1 Y^{(1)}, & j = 0 \\ W_0 Y^{(0)}, & j = 1 \end{cases}.$$

Let f, g and h be bounded continuous functions. From the definitions and the independence of $\{W_0, W_1, Y^{(0)}, Y^{(1)}\}$ we have

$$\begin{aligned} &\mathbb{E}_{\mathbb{Q}} f(\widetilde{Y}) g(\widetilde{W}) h(\widetilde{B}) \\ &\quad = \mathbb{E}\left(f(Y^{(0)}) g(W_0) h(W_1 Y^{(1)}) W_0 Y^{(0)} + f(Y^{(1)}) g(W_1) h(W_0 Y^{(0)}) W_1 Y^{(1)}\right) \\ &\quad = \big(\mathbb{E} f(Y) Y\big)\big(2\mathbb{E} g(W) W\big)\big(\mathbb{E} h(WY)\big). \end{aligned}$$

Thus the triple $\{\widetilde{Y}, \widetilde{W}, \widetilde{B}\}$ is independent and their distributions are given by

$$\mathbb{E}_{\mathbb{Q}} f(\widetilde{Y}) = \mathbb{E} f(Y) Y, \quad \mathbb{E}_{\mathbb{Q}} g(\widetilde{W}) = 2\mathbb{E} g(W) W, \quad \text{and} \quad \mathbb{E}_{\mathbb{Q}} h(\widetilde{B}) = \mathbb{E} h(WY).$$

Moreover by (6)

$$\begin{aligned}
&\mathbb{E}_{\mathbb{Q}} f(\widetilde{W}\widetilde{Y}+\widetilde{B}) \\
&\quad = \mathbb{E}\left(f(W_0 Y^{(0)} + W_1 Y^{(1)}) W_0 Y_0 + f(W_1 Y^{(1)} + W_0 Y^{(0)}) W_1 Y_1 \right) \\
&\quad = \mathbb{E} f(Y) Y.
\end{aligned}$$

Thus $\widetilde{Y}$ satisfies the distributional fixed point equation

$$\widetilde{Y} \stackrel{d}{=} \widetilde{W}\widetilde{Y} + \widetilde{B}.$$

The tails of the solutions to such equations have been studied intensively. A direct way to obtain the result, due to Grincevičjus [19], is to consider the function $F(t) = \mathbb{Q}(\widetilde{Y} \geq e^t)$ and write

$$\begin{aligned}
F(t) = \mathbb{Q}(\widetilde{Y} \geq e^t) &= \mathbb{Q}(\widetilde{W}\widetilde{Y} + \widetilde{B} \geq e^t) \\
&= \mathbb{Q}(\widetilde{W}\widetilde{Y} \geq e^t) + \mathbb{Q}(\widetilde{W}\widetilde{Y} + \widetilde{B} \geq e^t) - \mathbb{Q}(\widetilde{W}\widetilde{Y} \geq e^t) \\
&= \mathbb{Q}(\widetilde{Y} \geq e^{t-\log \widetilde{W}}) + \mathbb{Q}(e^t > \widetilde{W}\widetilde{Y} \geq e^t - \widetilde{B}) \\
&= \mathbb{E}_{\mathbb{Q}} F(t - \log \widetilde{W}) + \psi(t),
\end{aligned}$$

where the last two lines serve as the definition of ψ. Thus F satisfies the renewal equation. While the result is not yet immediately apparent from this, the utility of Peyrière probability in simplifying the question has been illustrated. □

3. Lognormal multiplicative chaos

Kahane [23] considered measures constructed out of multiplicative martingales in great generality. In the most generic situation one may consider a measure space (T, σ) and an independent sequence $(Q_n)_n$ of random functions $T \to [0, \infty)$ such that $\mathbb{E}Q_n(x) = 1$ for all $x \in T$, and form a measure-valued martingale $(\nu_n)_n$ by setting $\nu_n = \left(\prod_{k=1}^n Q_n\right)\sigma$. Some interesting general theory may be developed even if one only assumes that T is a compact metric space and the reference measures σ are Radon measures. Lognormal multiplicative chaos, also introduced by Kahane [22], combines Gaussian processes with the idea of multiplicative measure-valued martingales. In effect, one takes $Q_n(x) = e^{\beta V_n(x)}/\mathbb{E}e^{\beta V_n(x)}$ for some independent sequence $(V_n)_n$ of Gaussian processes on T.

3.1. Lognormal multiplicative chaos.

Lognormal multiplicative chaos. Lognormal multiplicative chaos is a rigorous way to define the measure

$$\nu(\,\mathrm{d}x) = \frac{e^{\beta X(x)}}{\mathbb{E}e^{\beta X(x)}}\,\mathrm{d}x,$$

where $(X(x))$ is a Gaussian process with a covariance kernel that has a logarithmic singularity, i.e.

$$K(x,y) = \mathbb{E}X(x)X(y) \sim \log \frac{1}{|x-y|} \quad \text{as } |x-y| \to 0.$$

While the parameter space of the process X could be taken as just a metric space, we restrict to $\mathbb{R}^d$ and in the next section we will consider special constructions in $\mathbb{R}$. It is well known that if K has such a logarithmic singularity, then a Gaussian process X with the covariance K can not be defined as a random function on $\mathbb{R}^d$. Instead K can be interpreted as the covariance of a Gaussian process X by interpreting X as a random distribution (i.e. generalized function). However, there is no canonical way to define the exponential of a distribution. While one could still work with the theory of distributions to define the measure, lognormal multiplicative chaos provides another and perhaps more direct way.

We consider the class of *σ-positive definite* kernels K on $\mathbb{R}^d$ that are of the form

$$K(x,y) = \log^+ \frac{1}{|x-y|} + g(x,y), \tag{8}$$

where $g : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a bounded continuous function and we have denoted $\log^+ x = \max(\log x, 0)$. That K is σ-positive definite means that there exists a sequence of continuous nonnegative definite functions (i.e. bounded covariances of Gaussian processes) $(k_n)_n$ such that

$$K(x,y) = \sum_{n=1}^{\infty} k_n(x,y). \tag{9}$$

Definition 11. *Let $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be σ-positive definite and satisfy* (8). *Let $(k_n)_n$ be a decomposition of K as in* (9) *and let $(V_n)_n$ be an independent sequence of centered Gaussian processes such that the covariance kernel of V_n is k_n. Denoting $X_n = \sum_{j=1}^n V_j$ and $K_n = \sum_{j=1}^n k_j$, for any $\beta \in \mathbb{R}$ we define the sequence $(\nu_{\beta,n})_n$ of measures by setting*

$$\frac{d\nu_{\beta,n}}{dx}(x) = e^{\beta X_n(x) - \frac{\beta^2}{2}K_n(x,x)},$$

where the left hand side denotes the Radon–Nikodym derivative of $\nu_{\beta,n}$ with respect to the Lebesgue measure on $\mathbb{R}^d$. The lognormal multiplicative chaos measure *associated to the kernel K is the measure*

$$\nu_\beta = \lim_{n\to\infty} \nu_{\beta,n},$$

where the limit is interpreted in the sense of weak convergence of measures.

There are remarks to be made about this definition. Writing

$$e^{\beta X_n(x) - \frac{\beta^2}{2}K_n(x,x)} = e^{\beta \sum_{j=1} V_j(x) - \frac{\beta^2}{2}\sum_{j=1}^n k_n(x,x)} = \prod_{j=1}^n e^{\beta V_j(x) - \frac{\beta^2}{2}k_n(x,x)}$$

and recalling that the moment generating function of a $V \sim \mathcal{N}(0,\sigma^2)$ is $\mathbb{E}e^{qV} = e^{\frac{q^2}{2}\sigma^2}$, it is clear that the densities of the sequence $(\nu_{\beta,n})_n$ with respect to the Lebesgue measure form a martingale at any given point $x \in \mathbb{R}^d$. By the martingale convergence theorem, for any compact set $C \subset \mathbb{R}^d$ the sequence $\nu_{\beta,n}(C)$ converges to a limit and one may then deduce the existence of the limit measure. Note also that the definition above is stated in terms of a σ-positive definite kernel K and its particular decomposition as $K = \sum_{n=1}^{\infty} k_n$, so the limit may depend on the chosen decomposition. It is a theorem of Kahane [22] that if the k_n are all nonnegative this is not so, i.e. in this case the limit measure ν_β depends only on K itself.

Before addressing the inevitable question of nondegeneracy of the limit we point out that the Mandelbrot cascade generated by a lognormal variable $W = e^{\beta V - \frac{\beta^2}{2}\mathbb{E}V^2}$ is a particular case of lognormal multiplicative chaos.[3] Simply observe that if for all n one defines the process $(V_n(x))_{x\in[0,1]}$ by setting $V_n(x) = V_{\sigma(x)|n}$, the respective definitions of Mandelbrot cascade and multiplicative chaos agree.

Theorem 12 (Kahane [22])**.** *A lognormal multiplicative chaos measure* ν_β *associated to the* σ*-positive definite kernel* (8) *on* $\mathbb{R}^d$ *is nondegenerate if and only if*

$$\beta^2 < 2d.$$

If this condition holds, then $\mathbb{E}\nu_\beta(C) = |C|$ *for all compact sets* $C \subset \mathbb{R}^d$*.*

Kahane's proof for this fact relied on more general theory of multiplicative martingales, and we are content to only describe the regime of $\mathcal{L}^2$-convergence. Fix a compact set C that contains some open set of $\mathbb{R}^d$. The fundamental calculation is to characterize when the martingale $(\nu_\beta(C))_n$ is bounded in $\mathcal{L}^2$. As in the case of Mandelbrot cascades, this is simple: we have

$$\begin{aligned}
\mathbb{E}\nu_{\beta,n}(C)^2 &= \mathbb{E}\int_C e^{\beta X_n(x) - \frac{\beta^2}{2}K_n(x,x)}\,\mathrm{d}x \int_C e^{\beta X_n(y) - \frac{\beta^2}{2}K_n(y,y)}\,\mathrm{d}y \\
&= \int_C \mathrm{d}x \int_C \mathrm{d}y\, \mathbb{E}e^{\beta X_n(x) + \beta X_n(y) - \frac{\beta^2}{2}K_n(x,x) - \frac{\beta^2}{2}K_n(y,y)} \\
&= \int_C \mathrm{d}x \int_C \mathrm{d}y\, e^{\beta^2 K_n(x,y)} \\
&\le \int_C \mathrm{d}x \int_C \mathrm{d}y\, e^{\beta^2 K(x,y)} \\
&\le C_{g,C} \int_C \mathrm{d}x \int_C \mathrm{d}y\, (|x-y| \wedge 1)^{-\beta^2},
\end{aligned}$$

where the constant $C_{g,C} > 0$ depends only on the bounded continuous function g on $C \times C$. The integral is finite if and only $\beta^2 < d$, which settles the issue of $\mathcal{L}^2$-boundedness and proves the nondegeneracy of ν_β for $\beta^2 \in (0, d)$.

3.2. **A white noise decomposition for multiplicative chaos on $\mathbb{R}$.** Mandelbrot cascade measures are very strongly bound to the system of dyadic subintervals of $[0, 1]$. For example, the exact scaling property is valid only for dyadic intervals. With the freedom available in lognormal multiplicative chaos, measures with more widely applicable scaling properties may be constructed. General constructions of this kind exist in any $\mathbb{R}^d$ but the ones on $\mathbb{R}$ are special, since they can be understood through geometric white noise decompositions. We are particularly interested in *exact scale invariance.*

[3] On $\mathbb{R}^1$ and restricted to $[0, 1]$, if the conventional geometric realization of the previous section is followed. Also, in Definition 11 the covariances K and $(k_n)_n$ were assumed to be continuous, but strictly speaking this is not necessary and is but a convenient way to get around possible regularity issues that could arise if, say, mere measurability was assumed.

The exact scaling property was stated for the Mandelbrot cascade measure in (4). More generally, we say that the random measure μ on $\mathbb{R}^d$ is exactly scale invariant if there exists an $R > 0$ so that for any $\lambda \in (0,1)$,

$$(\mu(\lambda A))_{A \subset \mathcal{B}(B(0,R))} \stackrel{d}{=} (W_\lambda \mu(A))_{A \subset \mathcal{B}(B(0,R))} \tag{10}$$

for some random variable W_λ which is independent of μ on the right hand side.[4]

We approach the scaling properties by interpreting the construction of the Mandelbrot cascade in a geometric way and generalizing it to a continuous geometry. These geometric ideas were first put forth by Barral and Mandelbrot [6] and in a sense completed by Bacry and Muzy [4]. For every $x \in [0,1]$, define the set $\mathcal{C}(x) \subset [0,1] \times \mathbb{R}^+$ by

$$\mathcal{C}(x) = \{(x', y') \in [0,1] \times \mathbb{R}^+ \,:\, |x' - x| < \max(2y, 1)\}$$

and the truncated sets $\mathcal{C}_\varepsilon(x)$ for $\varepsilon \in (0,1)$ by

$$\mathcal{C}_\varepsilon(x) = \mathcal{C}(x) \cap \{y' \geq \varepsilon\}.$$

Draw the binary tree on the upper half plane above the segment $[0,1]$ as indicated in Figure 1 and identify the node $\sigma \in \Sigma$ with the point corresponding to it on the upper half plane. Then the density of the Mandelbrot cascade measure $\mu_{\beta,n}$ at almost every $x \in [0,1]$ is given by

$$\exp\left(\beta \sum_{\sigma \in \mathcal{C}_{2^{-n}}(x)} V_\sigma\right) / \mathbb{E} \exp\left(\beta \sum_{\sigma \in \mathcal{C}_{2^{-n}}(x)} V_\sigma\right).$$

This suggests the following: instead of the deterministic set of points on the binary tree one could take a random point process on the upper half-plane, associate an independent positive random variable to each, and take as the density at x of the measure to be constructed the product of the random variables associated to the random points that lie inside the cone $\mathcal{C}(x)$. Even further, one may simply replace the atomic points and the random variables associated to them by a random measure or some more general set function W, and take as density at x the quantity $\exp(W(\mathcal{C}(x)))/\mathbb{E}\exp(W(\mathcal{C}(x)))$. In the constructions that follow, we will take W as the white noise on the upper half-plane.

Let λ denote the hyperbolic area measure on the upper half-plane, i.e.

$$\lambda(A) = \int_A \frac{\mathrm{d}x\,\mathrm{d}y}{y^2}.$$

The hyperbolic area measure is invariant with respect to scalings: for any $r > 0$ and any $A \subset \mathbb{R} \times \mathbb{R}^+$, $\lambda(A) = \lambda(rA)$. Let W be the white noise on the upper half-plane with control measure λ. Thus W is a random set function defined on Borel sets $A \subset \mathbb{R} \times \mathbb{R}^+$ of finite λ-measure that is characterized by the following properties:[5]

(1) $W(A) \sim \mathcal{N}(0, \lambda(A))$.

[4] To be precise this is exact scale invariance at the origin, but it is obvious how the scaling relation should read in disks $B(x,r)$ centered at arbitrary $x \in \mathbb{R}^d$.

[5] These properties are not enough to make W an actual random measure. However, for any fixed countable collection $\{A_j\}$ of disjoint Borel sets of finite λ-measure one almost

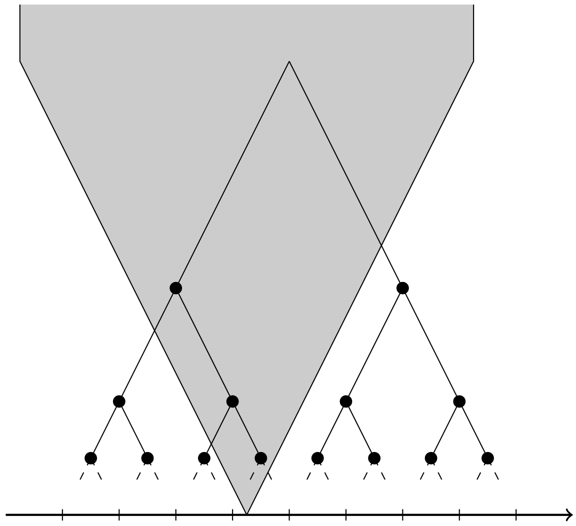

FIGURE 1. Mandelbrot cascades and a white noise decomposition.

(2) If $A \cap B = \emptyset$, then $W(A) \perp W(B)$.
(3) If $A \cap B = \emptyset$, then $W(A \cup B) = W(A) + W(B)$ almost surely.

Finally, define the process $(X_\varepsilon(x))_{x\in[0,1],\varepsilon\in(0,1]}$ by setting

$$X_\varepsilon(x) = W(\mathcal{C}_\varepsilon(x)).$$

The covariance structure of X may be computed as

$$\mathbb{E}X_\varepsilon(x)X_{\varepsilon'}(y) = \begin{cases} \log \frac{1}{\varepsilon\vee\varepsilon'} + 1 - \frac{1}{\varepsilon\vee\varepsilon'}|x-y|, & |x-y| < \varepsilon\vee\varepsilon' \\ \log\frac{1}{|x-y|}, & \varepsilon\vee\varepsilon' \le |x-y| \le 1 \end{cases}.$$

Indeed one could simply define X to be the Gaussian process with the covariance given above. However, the geometric white noise decomposition gives considerable additional insight to the process. We remark that it is easy to use Kolmogorov's criterion to show that X has a modification that is almost surely continuous. However, we will not discuss these (easy) regularity issues in detail.

By the process $(X_\varepsilon(x))_{x\in[0,1],\varepsilon\in(0,1]}$ constructed above we may give a very concrete and explicit definition for lognormal multiplicative chaos on $[0,1]$ corresponding to the kernel $K(x,y) = \log^+\frac{1}{|x-y|}$. In Definition 11 one may simply take $V_n(x) = X_{2^{-n}}(x)$ to obtain the processes $(V_n)_n$. Note also that the white noise decomposition allows us to define the limit measure as a limit of a continuous-parameter martingale. By direct computation one may confirm that the logarithmic singularity of the covariance is produced by the angle that the cones $\mathcal{C}(x)$ make with the x-axis, and so many other covariances K with a logarithmic singularity may be explicitly exhibited by

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surely has

$$W\left(\bigcup_j A_j\right) = \sum_j W(A_j).$$

a similar white noise construction by modifying the shape of the cones away from the angle.

The covariance $K(x,y) = \log^+ \frac{1}{|x-y|}$ is particularly interesting, as it produces exactly scale invariant measures. Consider $\lambda \in (0,1)$. Since

$$\log^+ \frac{1}{|\lambda x - \lambda y|} = \log \frac{1}{\lambda} + \log^+ \frac{1}{|x-y|},$$

if one could directly define it a centered Gaussian process $(X(x))_{x\in[0,1]}$ with the covariance K would satisfy

$$(X(\lambda x))_{x\in[0,1]} = (X_\lambda + X(x))_{x\in[0,1]},$$

where $X_\lambda \sim \mathcal{N}(0, \log \frac{1}{\lambda})$. The lognormal multiplicative chaos measure ν_β associated to K should thus satisfy

$$\begin{aligned}
(\nu_\beta(\lambda A))_{A\in\mathcal{B}([0,1])} &= \left(\int_{\lambda A} e^{\beta X(x) - \frac{\beta^2}{2}\mathbb{E}X(x)^2} \,\mathrm{d}x \right)_{A\in\mathcal{B}([0,1])} \\
&= \left(\lambda \int_{A} e^{\beta X(\lambda x) - \frac{\beta^2}{2}\mathbb{E}X(\lambda x)^2} \,\mathrm{d}x \right)_{A\in\mathcal{B}([0,1])} \\
&\overset{d}{=} \left(\lambda \int_{A} e^{\beta X_\lambda - \frac{\beta^2}{2}\mathbb{E}X_\lambda^2} e^{\beta X(x) - \frac{\beta^2}{2}\mathbb{E}X(x)^2} \,\mathrm{d}x \right)_{A\in\mathcal{B}([0,1])} \\
&\overset{d}{=} \left(\lambda e^{\beta X_\lambda - \frac{\beta^2}{2}\mathbb{E}X_\lambda^2} \nu_\beta(A) \right)_{A\in\mathcal{B}([0,1])},
\end{aligned}$$

i.e. exact scale invariance. This heuristic calculation can be made exact by using the process $(X_\varepsilon(x))_{x\in[0,1],\varepsilon\in(0,1]}$ constructed as above. The geometric reasoning that establishes exact scale invariance in the limit is illustrated in Figure 2.

For now we denote the exactly scale invariant one-dimensional lognormal multiplicative chaos measure by $\widetilde{\nu}(\mathrm{d}x)$. The techniques for analyzing Mandelbrot cascades can often be used for exactly scale invariant multiplicative chaos measures as well and in the one-dimensional situation this is made particularly transparent through the white noise decomposition. For example, denoting $\widetilde{Y} = \widetilde{\nu}([0,1])$, we have

$$\widetilde{Y} = \widetilde{\nu}([0,\tfrac{1}{2}]) + \widetilde{\nu}([\tfrac{1}{2},1]) = W_0 \widetilde{Y}^{(0)} + W_1 \widetilde{Y}^{(1)}, \tag{11}$$

where $W_0 \overset{d}{=} W_1 \overset{d}{=} \frac{1}{2} e^{\beta V - \frac{\beta^2}{2}\mathbb{E}V^2}$ with $V \sim \mathcal{N}(0, \log 2)$, $\widetilde{Y}^{(0)} \overset{d}{=} \widetilde{Y}^{(1)} \overset{d}{=} \widetilde{Y}$ and further $W_0 \perp \widetilde{Y}^{(0)}$ and $W_1 \perp \widetilde{Y}^{(1)}$. In the second equality in (11) exact scale invariance was used in the intervals $[0,\frac{1}{2}]$ and $[\frac{1}{2},1]$, but more information was actually needed. The claim is that all the variables, on both sides of the equation, can be realized simultaneously on the same probability space. That this can be simply stated is a consequence of the white noise representation. The relation (11) (and its analogues in which the limit $\varepsilon \searrow 0$ is not taken) can be used very much in the same way to analyze the measure $\widetilde{\nu}$ and the total mass $\widetilde{Y}$ as the relation (6) was used to analyze Mandelbrot cascades. This is despite the fact that $\widetilde{Y}^{(0)}$ is not independent of either W_1 or $\widetilde{Y}^{(1)}$. For example, analogues to Theorems 7 and 10 can be established for $\widetilde{\nu}$ almost exactly in the same way as for the cascade.

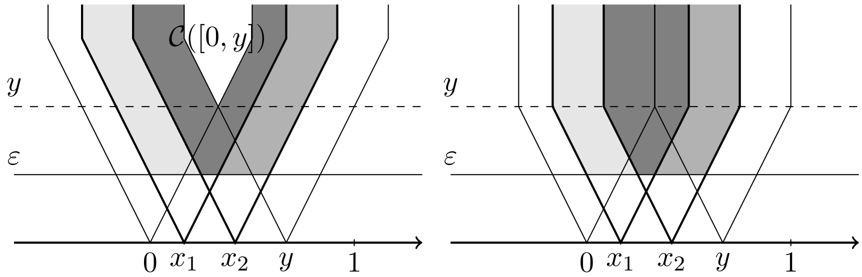

FIGURE 2. *Left.* Denote $\mathcal{C}([0,y]) = \mathcal{C}(0) \cap \mathcal{C}(y)$. The sets $\mathcal{C}_\varepsilon(x_1)\setminus\mathcal{C}_\varepsilon(x_2)$ and $\mathcal{C}_\varepsilon(x_2)\setminus\mathcal{C}_\varepsilon(x_1)$ are shaded light gray, while the intersection $\big(\mathcal{C}_\varepsilon(x_1)\cap\mathcal{C}_\varepsilon(x_2)\big)\setminus\mathcal{C}([0,y])$ is dark gray. The law of the Gaussian process $\big(X_\varepsilon(x) - W(\mathcal{C}([0,y]))\big)_{x\in[0,y]}$ is determined by the hyperbolic areas of these sets for all pairs $(x_1,x_2) \in [0,y]^2$. The set $\mathcal{C}([0,y])$, contained in every $\mathcal{C}_\varepsilon(x)$ for $x \in [0,y]$ and $\varepsilon \leq y$, has been left white. *Right.* Closing the gap left by the set $\mathcal{C}([0,y])$ does not affect the hyperbolic areas of any of the shaded regions. Scaling this picture by $1/y$ also leaves the hyperbolic areas invariant, giving the distributional equality $(X_\varepsilon(x))_{x\in[0,y]} = \big(X_y + X_{\varepsilon/y}(x/y)\big)_{x\in[0,y]}$, where X_y is independent of $(X_\varepsilon(x))$ on the right hand side and $X_y \overset{d}{=} W(\mathcal{C}([0,y]))$.

4. Critical measures

The primary objects of study of this thesis, the Mandelbrot cascade and lognormal multiplicative chaos measures *at criticality*, have still not been introduced. In Theorems 5 and 12 we stated that in both cases the martingale constructions that we have introduced are nondegenerate only for some range $\beta \in (0,\beta_c)$ and that outside this range[6] the martingale converges almost surely to 0. This raises the question of the speed of convergence to 0, and one can even ask whether it is possible to find sequences $(c_n(\beta_c))_n$ for which we would have

$$c_n(\beta_c)\mu_{\beta_c,n} \to \tilde{\mu}_{\beta_c} \quad \text{and} \quad c_n(\beta_c)\nu_{\beta_c,n} \to \tilde{\nu}_{\beta_c}$$

in some sense as $n \to \infty$ for some random measures $\tilde{\mu}_{\beta_c}$ and $\tilde{\nu}_{\beta_c}$. We restrict our discussion mostly to the cascade case, but even so the theory is more subtle than that of the subcritical measures and in the scope of this introduction we only introduce the objects.

4.1. **Derivative martingale.** For notational convenience, suppose $W = e^V$ satisfies

$$\mathbb{E}W = \mathbb{E}e^V = \frac{1}{2} \quad \text{and} \quad \mathbb{E}W\log W = \mathbb{E}Ve^V = 0.$$

[6]In lognormal multiplicative chaos and also in lognormal Mandelbrot cascades. However, a classical Mandelbrot cascade generated by an arbitrary W might be nondegenerate at the critical point as well.

The latter condition is equivalent to $\tau'(1) = 0$ for

$$\tau(q) = 1 + \log_2 \mathbb{E}W^q,$$

which by Theorem 5 is to say that the Mandelbrot cascade generated by W is critical (i.e. $\beta_c = 1$). Define the *derivative martingale* $(D_n)_n$ by

$$D_n = -\left.\frac{d}{d\beta}\right|_{\beta=1} Y_{\beta,n} = -\left.\frac{d}{d\beta}\right|_{\beta=1} \mu_{\beta,n}([0,1]).$$

As the name implies, (D_n) is a martingale. By differentiation,

$$\begin{aligned} D_n = -\left.\frac{d}{d\beta}\right|_{\beta=1} Y_{\beta,n} &= -\left.\frac{d}{d\beta}\right|_{\beta=1} \sum_{\sigma\in\Sigma_n} e^{\beta\sum_{k=1}^n V_{\sigma|k} - n\tau(\beta)\log 2} \\ &= \sum_{\sigma\in\Sigma_n}\left(\sum_{k=1}^n V_{\sigma|k}\right) e^{\sum_{k=1}^n V_{\sigma|k}} \end{aligned}$$

and therefore

$$\begin{aligned} \mathbb{E}\left(D_{n+1}\,|\,\mathcal{F}_n\right) &= \mathbb{E}\left(\sum_{\sigma\in\Sigma_{n+1}}\left(\sum_{k=1}^{n+1} V_{\sigma|k}\right) e^{\sum_{k=1}^{n+1} V_{\sigma|k}} \,\middle|\, \mathcal{F}_n\right) \\ &= \sum_{\sigma\in\Sigma_{n+1}}\sum_{k=1}^{n+1}\mathbb{E}\left(V_{\sigma|k} e^{\sum_{k=1}^{n+1} V_{\sigma|k}} \,\middle|\, \mathcal{F}_n\right) \\ &= \sum_{\sigma\in\Sigma_{n+1}}\sum_{k=1}^{n} V_{\sigma|k} e^{\sum_{k=1}^{n} V_{\sigma|k}}\,\mathbb{E}\left(e^{V_{\sigma|n+1}} \,\middle|\, \mathcal{F}_n\right) \\ &\quad + \sum_{\sigma\in\Sigma_{n+1}} e^{\sum_{k=1}^{n} V_{\sigma|k}}\mathbb{E}\left(V_{\sigma|n+1} e^{V_{\sigma|n+1}} \,\middle|\, \mathcal{F}_n\right) \\ &= D_n. \end{aligned}$$

In the same way one may define the measures $(\mu_n')_n$ corresponding to the derivative martingale by setting

$$\mu_n'(I) = -\left.\frac{d}{d\beta}\right|_{\beta=1} \mu_{\beta,n}(I) \quad \text{for } I \subset [0,1],$$

and a similar calculation shows that $(\mu_n'(I))_n$ is a martingale for any interval $I \subset [0,1]$.

The derivative martingale is not positive, so the martingale convergence theorem does not apply. It is nevertheless true that the derivative martingale converges almost surely *to a positive limit*. One way to show this is to consider the martingale $(D_n + \alpha Y_{1,n})_n$ and to show that by taking $\alpha > 0$ large enough, the probability that any of the branches of the tree contributes a negative term to the sum can be made arbitrarily small. Then, by applying the martingale convergence theorem to the modified positive martingale in which a branch is stopped as soon as it hits 0, one can show that $(D_n + \alpha Y_{1,n})_n$ converges to a positive limit with a probability close to 1. But since the cascade is critical $Y_{1,n} \to 0$ almost surely as $n \to \infty$, implying that in fact $(D_n)_n$ converges to a positive limit with probability close to 1. In the end one takes $\alpha \to \infty$ and obtains the almost sure convergence of

$(D_n)_n$. As a consequence, there also exists an almost sure limit measure $\mu' = \lim_{n\to\infty} \mu'_n$.

In the case of cascades there is another, more elegant, proof due to Liu [26] for the convergence of the derivative martingale. However, unlike Liu's proof the sketch of the argument above generalizes to lognormal multiplicative chaos [15].

The measure μ' is the correct replacement for the degenerate critical measure $\lim_{n\to\infty} \mu_{1,n}$. This was realized even before the question of deterministic renormalization of the martingale $(Y_{1,n})_n$ was settled. One reason for this is that μ' satisfies the correct scaling relations and that Y' is a fixed point of the smoothing transform associated to the generating variable W. The last claim follows from the calculation

$$\begin{aligned}
Y' &= \lim_{n\to\infty} \mu'_n([0,1]) = \lim_{n\to\infty} \left(\mu'_n([0,1/2]) + \mu'_n([1/2,1])\right) \\
&= \lim_{n\to\infty} \sum_{\sigma\in\Sigma_{n-1}} \left(V_0 + \sum_{k=1}^{n-1} V_{0\sigma|k}\right) e^{V_0 + \sum_{k=1}^{n-1} V_{0\sigma|k}} \\
&\quad + \lim_{n\to\infty} \sum_{\sigma\in\Sigma_{n-1}} \left(V_1 + \sum_{k=1}^{n-1} V_{1\sigma|k}\right) e^{V_1 + \sum_{k=1}^{n-1} V_{1\sigma|k}} \\
&= \lim_{n\to\infty} V_0 e^{V_0} \sum_{\sigma\in\Sigma_{n-1}} e^{\sum_{k=1}^{n-1} V_{0\sigma|k}} \\
&\quad + \lim_{n\to\infty} e^{V_0} \sum_{\sigma\in\Sigma_{n-1}} \left(\sum_{k=1}^{n-1} V_{0\sigma|k}\right) e^{\sum_{k=1}^{n-1} V_{0\sigma|k}} \\
&\quad + \lim_{n\to\infty} V_1 e^{V_1} \sum_{\sigma\in\Sigma_{n-1}} e^{\sum_{k=1}^{n-1} V_{1\sigma|k}} \\
&\quad + \lim_{n\to\infty} e^{V_1} \sum_{\sigma\in\Sigma_{n-1}} \left(\sum_{k=1}^{n-1} V_{1\sigma|k}\right) e^{\sum_{k=1}^{n-1} V_{1\sigma|k}} \\
&= e^{V_0} Y'(0) + e^{V_1} Y'(1),
\end{aligned}$$

where $Y'(0) = \lim_{n\to\infty} \sum_{\sigma\in\Sigma_{n-1}} \left(\sum_{k=1}^{n-1} V_{0\sigma|k}\right) e^{\sum_{k=1}^{n-1} V_{0\sigma|k}} \stackrel{d}{=} Y'$ and similarly $Y'(1) \stackrel{d}{=} Y'$, and furthermore the collection $\{W_0, Y'(0), W_1, Y'(1)\}$ is independent.

4.2. **Deterministic normalization.** The derivative martingale is not the only way to construct the critical measures. Namely, the question of finding a deterministic sequence $(c_n)_c$ for which $c_n\mu_{1,n} \to \widetilde{\mu}$ and $c_n\nu_{1,n} \to \widetilde{\nu}$ has a positive answer and the law of the limit measures agree with the laws obtained from the respective derivative martingales.

Theorem 13 (Aïdékon, Shi [2]). *Suppose V satisfies*

$$\mathbb{E}e^V = \frac{1}{2}, \quad \mathbb{E}Ve^V = 0 \quad \textit{and} \quad \mathbb{E}V^2e^V < \infty.$$

Then

$$n^{1/2} Y_{1,n} \to \widetilde{Y}_1 \quad \textit{in probability as } n \to \infty,$$

and the law of the limit variable $\widetilde{Y}_1$ is a deterministic multiple of the law of the limit of the derivative martingale.

Based on the techniques of Bramson [10], developed originally for the closely related model of branching Brownian motion, Webb [36] proved essentially the same theorem in the case of Gaussian V, though with convergence in probability replaced with convergence in distribution. The basic idea in this approach is to carefully analyze the action of the smoothing transform on a suitably reparametrized Laplace transform of the total mass variable. The proof of Aïdékon and Shi, on the other hand, utilizes a modification of the spinal decomposition (or Peyrière probability) based on the derivative martingale. What Aïdékon and Shi are able to prove is in fact even more general and applies, with appropriately modified assumptions, to general branching random walks with very weak assumptions on the general point process that describes the branching rule.

The last point of this introduction is the statement of the analogue of the previous theorem to the case of lognormal multiplicative chaos. The proof uses the same fundamental idea as the proof of Aïdékon and Shi, in that it is based on a modified Peyrière probability. We give the assumptions of the theorem in the same form as in [16], though as the authors themselves note, the proof actually gives the result for a somewhat wider class of kernels K and the one in the theorem has been chosen mostly for its scaling properties. Especially, the renormalization result holds for the exactly scale invariant one-dimensional measure constructed in the previous section.

Theorem 14 (Duplantier, Rhodes, Sheffield, Vargas [16]). *Let $k : [0, \infty) \to \mathbb{R}$ be a C^1-function with compact support and $k(0) = 1$ such that the map $\mathbb{R}^d \times \mathbb{R}^d \ni (x, y) \mapsto k(|x-y|)$ is nonnegative definite. Let $K_t : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, $t \geq 0$, be a family of kernels of the form*

$$K_t(x, y) = \int_1^{e^t} \frac{k(u|x-y|)}{u} \,\mathrm{d}u.$$

The limit $K = \lim_{t\to\infty} K_t$ is σ-positive definite and has the form

$$K(x, y) = \log^+ \frac{1}{|x-y|} + g(x, y)$$

for some bounded continuous function $g : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$. Let $(X_t(x))_{x\in\mathbb{R}^d, t\geq 0}$ be a Gaussian process with the covariance K_t and define the measures

$$\widetilde{\nu}_t(A) = \sqrt{t} \int_A e^{\sqrt{2d} X_t(x) - d\mathbb{E}X_t(x)^2} \,\mathrm{d}x, \quad A \in \mathcal{B}(\mathbb{R}^d).$$

Then, for any bounded open set $A \subset \mathbb{R}^d$,

$$\widetilde{\nu}_t(A) \to \widetilde{\nu}(A) \quad \text{in probability as } t \to \infty,$$

where $\widetilde{\nu}$ is a random measure with the same law as the limit of the derivative martingale measure.

5. The articles included in this thesis

[I] Critical Mandelbrot Cascades. Mandelbrot cascade measures in the critical case were studied already by Liu [26] and Barral [5] as the limit of the derivative martingale. However, the construction through the deterministic renormalization provided by Theorem 13 allows for a deeper study of the critical measure. In the joint work [**I**] with J. Barral, A. Kupiainen, E. Saksman and C. Webb it is proven that under weak assumptions on the generating variable V, the critical Mandelbrot cascade measure μ' almost surely has no atoms. Further, the following modulus of continuity estimate is proven.

Theorem 15 (Theorem 3 in [**I**])**.** *Suppose V satisfies the criticality conditions $\mathbb{E}e^V = \frac{1}{2}$ and $\mathbb{E}Ve^V = 0$ and the moment assumptions $\mathbb{E}e^{-hV} < \infty$ and $\mathbb{E}e^{(1+h)V} < \infty$ for some $h > 0$. Then for any $\gamma \in (0, 1/2)$*

$$\mu'(I) \le C(\omega)\left(\log\left(1+\frac{1}{|I|}\right)\right)^{-\gamma} \tag{12}$$

for all intervals $I \subset [0,1]$, where $C(\omega)$ is a random constant which is almost surely finite. Moreover, the statement is false for $\gamma > 1/2$.

The method of proof also gives new estimates for the modulus of continuity of the subcritical measures. Other geometric properties of the critical measures are also studied, including multifractal properties and a critical temperature version of the KPZ formula in the style of Benjamini and Schramm [8], and the renormalization theorems of Webb [36] are extended.

[II] Basic properties of critical lognormal multiplicative chaos. One of the key ingredients of the proof of the modulus of continuity estimate in [**I**] is the extension of Theorem 10 to the critical temperature, due to Buraczewski [11]:

$$\mathbb{P}\left(Y' \ge x\right) \sim C/x \quad \text{as } x \to \infty \tag{13}$$

for some constant $C > 0$. The first main result of the joint work [**II**] with J. Barral, A. Kupiainen, E. Saksman and C. Webb is the analogue of Buraczewski's result in the case of (one-dimensional) exactly scale invariant critical lognormal multiplicative chaos, to be denoted by ν'.

Theorem 16 (Theorem 1 in [**II**])**.** *The tail probability of the total mass of ν' has the asymptotic behavior*

$$\mathbb{P}\left(\nu'([0,1]) \ge x\right) \sim c/x \quad \text{as } x \to \infty,$$

where the constant $c > 0$ is explicitly given by

$$c = \frac{2}{\log 2}\mathbb{E}\,\nu'([0,1/2])\log\left(1+\frac{\nu'([1/2,1])}{\nu'([0,1/2])}\right) < \infty.$$

The second main result of [**II**] is the modulus of continuity estimate (12) for ν'. The article also discusses some further geometric properties of ν' and the extensions of the tail and modulus of continuity results to higher-dimensional cases. The main results extend directly to the important case of lognormal multiplicative chaos on $\mathbb{R}^2$, but higher-dimensional analogues need new methods.

[III] Small deviations in lognormal Mandelbrot cascades. The theorems of Guivarc'h [20] (Theorem 10) and Buraczewski [11] ((13) above) give precise information on the laws of the fixed points of smoothing transforms, as far as the probabilities of large values are concerned. It is natural to look for more complete descriptions of these laws. Molchan [32] gave a condition for the finiteness of negative moments and the probabilities of a fixed point having small values were further studied by Liu [27] and most recently by Hu [21]. These studies concentrate on cases where $\mathbb{E}W^{-s} = \infty$ for some $s > 0$ or where $W \leq c$ almost surely for some $c > 0$, but the important special case of lognormal W falls in neither case. The main result of [**III**] is the following.

Theorem 17 (Theorem 2 in [**III**]). *Suppose W is a positive random variable satisfying*

$$\lim_{x\to 0} \frac{\log\log 1/\mathbb{P}(W \leq x)}{\log\log 1/x} = \gamma > 1.$$

Let Y be a nontrivial fixed point of the smoothing transform associated to W, i.e. suppose

$$Y \overset{d}{=} W_0 Y^{(0)} + W_1 Y^{(1)}$$

where $W \overset{d}{=} W_0 \overset{d}{=} W_1$, $Y \overset{d}{=} Y^{(0)} \overset{d}{=} Y^{(1)}$ and the set $\{W_0, W_1, Y^{(0)}, Y^{(1)}\}$ is independent. Then

$$\lim_{t\to\infty} \frac{\log\log 1/\mathbb{E}e^{-tY}}{\log\log t} = \lim_{x\to 0} \frac{\log\log 1/\mathbb{P}(Y \leq x)}{\log\log 1/x} = \gamma.$$

This theorem applies to Mandelbrot cascades with a lognormal generator and, on a rough scale, says that the total mass variable behaves like the generating variable in the range of small values. The idea of the proof goes back to the original work of Molchan [32].

[IV] Inhomogeneous cascades and fine multifractal properties of critical cascades. In the context of Mandelbrot cascades, multifractal analysis has typically meant determining the (almost sure) Hausdorff dimensions of the sets

$$\begin{aligned} E_\alpha &= \left\{ x \in [0,1] \;\middle|\; \lim_{n\to\infty} \frac{\log \mu(I_n(x))}{\log |I_n(x)|} = \alpha \right\} \\ &= \left\{ x \in [0,1] \;\middle|\; -\log \mu(I_n(x)) = (\alpha \log 2) n + o(n) \text{ as } n \to \infty \right\}, \end{aligned}$$

in terms of quantities related to the law of W. Here the notation $f(n) = o(g(n))$ as $n \to \infty$ means that $f(n)/|g(n)| \to 0$ as $n \to \infty$. The general form of the question studied in [**IV**] is the following: what can be said about the sets

$$E_{\alpha,\psi} = \left\{ x \in [0,1] \;\middle|\; -\log \mu(I_n(x)) = (\alpha \log 2) n + \psi(n) + o(\psi(n)) \text{ as } n \to \infty \right\},$$

where ψ is some function satisfying $\psi(n) = o(n)$?

The main tool used in [**IV**] to study the question formulated above is the construction of inhomogeneous cascade measures. The fixed parameter $\beta \in \mathbb{R}$ in the construction of Mandelbrot cascade is replaced by a sequence $\bar{\beta} =$

$(\beta_n)_{n=1}^{\infty}$, and the inhomogeneous cascade measure $\mu_{\bar{\beta}}$ is the limit measure corresponding to the total mass given by

$$Y_{\bar{\beta}} = \lim_{n\to\infty} \sum_{\sigma\in\Sigma_n} e^{\sum_{k=1}^{n}\left(\beta_k V_{\sigma|k} - \tau(\beta_k)\log 2\right)}.$$

Among other results, a criterion for $\mathcal{L}^1$-convergence of the inhomogeneous cascade measures is obtained.

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