Scalar Condensates in the Early Universe and Consequences of Their Non-perturbative Decay

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ACADEMIC DISSERTATION

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Abstract

The observations strongly support that the early Universe underwent an epoch of accelerated expansion called inflation. Inflation can explain many observed features of the Universe. For example it gives rise to the primordial seed required for the structure formation. Here we assume that inflation was caused by a slowly rolling scalar field called the inflaton. After the end of inflation the energy of the inflaton condensate converts into other degrees of freedom during the reheating process. A non-perturbative early stage of reheating is called preheating.

In addition to the inflaton field there exists at least one other scalar field - the Standard Model Higgs field. In this thesis we study the generation of the Higgs condensate during inflation and its decay after inflation during preheating. By using the stochastic approach one can find the probable value of the Higgs field during inflation in a random Hubble patch of the Universe. One then finds that after inflation the Higgs field decays into the weak gauge bosons in a few hundreds of Hubble times. We discuss the possible observational consequences of the Higgs condensate dynamics in the early Universe.

The decay of the inflaton field after the end of inflation during the preheating stage can produce a significant gravitational wave background. Here the focus is especially on the background produced by fermionic preheating. When the inflaton decays into fermions during preheating, the generated fermions develop a non-zero anisotropic stress, which serves as the source of gravitational waves. In addition the non-perturbative fermionic decay of a subdominant scalar field produces a gravitational wave background. The spectrum of the produced gravitational waves is typically peaked at very high frequencies and is unlikely to be detected by the planned gravitational wave observatories.
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Tuukka Meriniemi
List of Publications

The content of this thesis is based on the following research articles [1–3]:

1. K. Enqvist, D. G. Figueroa, and T. Meriniemi,
   *Stochastic Background of Gravitational Waves from Fermions*,
   Phys. Rev. D 86 (2012) 061301,

2. K. Enqvist, T. Meriniemi and S. Nurmi,
   *Generation of the Higgs Condensate and Its Decay after Inflation*,
   JCAP 1310 (2013) 057,

3. D. G. Figueroa and T. Meriniemi,
   *Stochastic Background of Gravitational Waves from Fermions – Theory and Applications*,
   JHEP 1310 (2013) 101,

In all of the papers the authors are listed alphabetically according to the particle physics convention.

Author’s Contribution

1: In the paper we have developed a formalism to compute stochastic background of gravitational waves generated by fermions and we have applied the formalism to the fermionic preheating scenario. In the paper, the author wrote a Mathematica code to compute numerical results and calculated analytical results jointly with the collaborators.

2: In the paper we have examined the generation of the Higgs condensate during inflation and its decay after inflation. Here, a Mathematica code to compute numerical results was developed by the author and the paper was written jointly by the collaborators.

3: In the paper we have computed the gravitational wave spectrum produced by fermions for several different scenarios with a wide range of parameters. The author improved the Mathematica code used in the first paper and wrote the first draft of the paper.
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Chapter 1

Introduction

According to the inflationary paradigm there was an epoch of accelerated expansion in the early Universe. It is strongly supported by the recent observations of the Planck satellite [4, 5] and the BICEP2 experiment [6]. A slowly rolling scalar field, the inflaton, gives a consistent and simple description for inflation although the exact model of inflation is strictly speaking unknown. In this thesis we assume that there was an epoch of inflation in the early Universe and that it was caused by the inflaton.

After inflation the energy of the Universe is stored in the coherent inflaton condensate. This energy must convert into particles in the Universe. This stage is called the reheating process. Reheating includes a non-perturbative early stage called preheating. The preheating process takes place right after the end of inflation, when the inflaton becomes massive and starts to oscillate in its potential. The inflaton field gives rise to an effective mass for the fields into which it decays. The effective mass is proportional to the amplitude of the inflaton field. The energy transfers from the inflaton condensate into these other fields when the inflaton crosses zero where the effective masses vanish momentarily. After reheating the particles thus created eventually thermalize and the evolution of the Universe follows the hot Big Bang (HBB) theory.

The $\Lambda$CDM model is a simple six parameter parametrization of the hot Big Bang theory. It is in good agreement with observations. In the $\Lambda$CDM model the Universe contains a cosmological constant ($\Lambda$) and cold dark matter (CDM) in addition to ordinary matter and radiation. The $\Lambda$CDM model with inflation has been extremely predictive. Thereby, it is deservedly the prevailing model of modern cosmology. The model provides a good description of the abundances of elements in the Universe formed during the Big Bang nucleosynthesis (BBN) [7–9] and is in very good agreement with the Planck observations of the cosmic microwave background (CMB) [4].

In 2012 the LHC experiment discovered a Higgs-like particle with a mass of 125-126 GeV [10, 11] which later was confirmed to be consistent with the Standard Model (SM) Higgs boson [12, 13]. In principle, the Higgs field could also act as the inflaton [14–16] but here we assume that inflation was caused by some other scalar field. In this thesis we study the generation of the Higgs condensate during inflation and its decay after inflation in preheating. By using the stochastic approach [17–19] we find the likely value of the Higgs amplitude in a random Hubble patches of the Universe. The
Higgs turns out to be a subdominant light spectator field during inflation [2].

After inflation the Higgs field decays. Perturbative decay turns out to be negligible, because the decay into the weak gauge bosons and the top quark is kinematically blocked and decay rates into the other Standard Model particles are very small.

Non-perturbative decay begins when the Higgs field becomes massive after inflation and starts to oscillate. Analogously to the inflaton decay, during preheating the Higgs field gives rise to effective masses for the SM fields into which it is coupled, and it decays into these SM degrees of freedom. The Higgs field is most strongly coupled into the weak gauge bosons and the top quark. The Higgs field also gives rise to effective mass for itself, due to its quartic self-coupling. The generated effective masses of the SM fields are proportional to the Higgs amplitude, and the energy transfers effectively from the Higgs condensate into these field each time the Higgs field crosses around zero. The non-perturbative decay into the Higgs particles is anomalously ineffective and also decay into top quarks is minor, because the Pauli exclusion principle restricts the energy transfer into fermions. In paper [2] we find that the Higgs field decays into the weak gauge bosons in few hundreds of Hubble times after the onset of Higgs oscillations. Although the Higgs field is subdominant during inflation it could contribute later to the metric perturbations via modulated reheating [20,21] or curvaton scenarios [22–26].

The second main topic studied in this thesis is the gravitational wave (GW) background produced during preheating. In particular, we introduce the formalism to calculate GW background produced by fermions during preheating that was first developed in [1,3]. The fermion distribution produced from non-perturbative decay of a coherently oscillating scalar field has an anisotropic stress, which is the source of gravitational waves. The calculations are much more complicated than in bosonic preheating because of technical problems related e.g. to regularization issues; these will be discussed in more detail in section 6.3. The produced GW spectrums are typically peaked at very high frequencies. Therefore, the planned GW observatories are not likely be able to detect these backgrounds of gravitational waves.

In the thesis we use natural units, and thereby we set the reduced Planck constant and speed of light to unity, $\hbar = c = 1$. Moreover, we define the reduced Planck mass as $M_p = 1/\sqrt{8\pi G} \simeq 2.435 \cdot 10^{18}$GeV, where $G$ is the Newton’s gravitational constant.

### 1.1 Friedmann-Robertson-Walker Metric

According to the Cosmological principle the Universe is homogeneous and isotropic at large scales. The metric of such a Universe is given by the Friedmann-Robertson-Walker (FRW) metric [27–31]

$$ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - Kr^2} + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right]. \tag{1.1}$$

Here $K$ is a constant which is related to the curvature of the space. For closed models $K$ is positive whereas for open models $K$ is negative. For spatially flat models $K$ is zero. The Universe is observed
1.1 Friedmann-Robertson-Walker Metric

To be to a very high accuracy spatially flat [4], and therefore we set \( K = 0 \). The spatially flat FRW metric can be written as

\[
ds^2 = -dt^2 + a^2(t) \left[ dx^2 + dy^2 + dz^2 \right],
\]

where \( a(t) \) is the scale factor and \( t \) is the cosmic time. Often it is convenient to write the metric in the conformal form

\[
ds^2 = a^2(\eta) \left[ -d\eta^2 + dx^2 + dy^2 + dz^2 \right],
\]

where \( \eta \) is the conformal time related to cosmic time \( t \) as

\[
d\eta = \frac{dt}{a(t)}.
\]

According to the theory of general relativity, the dynamics of the space-time is determined by the Einstein equations [32]

\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} = M_p^{-2} T_{\mu\nu},
\]

where on the left-hand side the Ricci tensor \( R_{\mu\nu} \) and the Ricci scalar \( R \) include the information of the curvature of the space-time, determined by the metric \( g_{\mu\nu} \).

In order to solve the Ricci tensor and the Ricci scalar for the given metric \( g_{\mu\nu} \) we need to compute the connection coefficients,

\[
\Gamma^\sigma_{\mu\nu} = \frac{1}{2} g^{\sigma\rho} \left( \partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} \right).
\]

From the connection coefficients we can then calculate the Riemann tensor, which is defined as

\[
R^\rho_{\sigma\mu\nu} = \partial_\rho \Gamma^\sigma_{\nu\mu} - \partial_\sigma \Gamma^\rho_{\nu\mu} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma}.
\]

The Ricci tensor and scalar are then given by

\[
R_{\mu\nu} = R^\lambda_{\mu\lambda\nu}, \quad R = R^\mu_{\mu}.
\]

On the right-hand side of the Einstein equations (1.5) \( T_{\mu\nu} \) is the energy-momentum tensor and it contains the information of the energy content of the space-time. The energy momentum tensor of the background has the same form as a perfect fluid, and it is given by

\[
T^\mu_{\nu} = \text{diag} \left( -\rho, p, p, p \right),
\]

where \( \rho \) and \( p \) are the energy density and pressure of the energy in the Universe respectively.

For the spatially flat FRW metric given in eq. (1.3) the solutions of the Einstein equations (1.5) are given by the Friedmann equations

\[
\mathcal{H}^2 = \left( \frac{a'}{a} \right)^2 = \frac{1}{3M_p^2} \rho a^2,
\]
\[ H' = -\frac{1}{6M_p^2} (\rho + 3p) a^2, \]  

(1.11)

where the over primes denote derivatives with respect to the conformal time \( \eta \) while \( H = \frac{a'}{a} = aH \) is the conformal Hubble rate and \( H = \frac{1}{a}\frac{da}{dt} \) is the Hubble rate in cosmic time. From the Friedmann equations one can derive the energy continuity equation

\[ \rho' = -3(\rho + p)H. \]  

(1.12)

**Backgrounds**

The energy content of the Universe can be characterized by the equation-of-state parameter

\[ w = \frac{p}{\rho}. \]  

(1.13)

For the radiation-like background we have \( w = 1/3 \), whereas the matter-like background is pressure-less \( (p = 0) \) and thus \( w = 0 \). During inflation the equation-of-state parameter is \( w < -1/3 \).

In the radiation dominated background the energy density decreases as \( \rho = \rho_0 a^{-4} \). According to the first Friedmann equation (1.10) we then have \( H \propto a^{-1} \) leading to \( a \propto t^{1/2} \propto \eta \). Matter-like energy density decreases as \( \rho = \rho_0 a^{-3} \) and thus we have \( H \propto a^{-1/2} \) and moreover \( a \propto t^{2/3} \propto \eta^2 \). During inflation the background energy density is almost independent of the scale factor, thus also the Hubble rate in cosmic time is constant \( (H = H_\ast) \) corresponding \( H \propto a \). Thus, we obtain that the scale factor grows exponentially in cosmic time, \( a \propto e^{H_\ast t} \). In conformal time we have \( a = -1/(H_\ast \eta) \), where \( \eta \) is in the interval \( -\infty < \eta < 0 \) and it increases from a negative initial value towards 0.

**1.2 Inflation**

According to the inflationary paradigm originally proposed by Starobinsky and Guth [33–35] the very early Universe underwent an epoch of accelerated expansion. The inflationary scenario explains successfully some of the properties of the Universe, such as the flatness and homogeneity of the Universe at large scales. Moreover, the inflationary paradigm provides an understanding of the primordial origin of the structure in the Universe. After inflation the Universe reheats and thermalizes during the processes denoted as preheating and reheating. The main focus of this thesis is on preheating. After the Universe has reheated and thermalized the hot Big Bang theory gives a good description of the evolution of the Universe.

**Cosmic Microwave Background**

The main idea of the HBB theory is that the Universe is expanding. Therefore, the Universe becomes less dense and is cooling down. The early Universe was then very hot and dense, and matter was fully ionized. The mean free path of photons was very short because they scattered constantly from ions
1.2 Inflation

and electrons. When the Universe was about 380,000 years old the temperature of the Universe had decreased to around 3000 Kelvins. At that time the ions and electrons started to form electrically neutral atoms. After this epoch of recombination the photons have propagated almost freely. They are the photons that are observed as the cosmic microwave background (CMB).

The cosmic microwave background radiation was discovered by Penzias and Wilson in 1965 [36]. Dicke, Peebles, Roll and Wilkinson interpreted this radiation as an evidence of the hot Big Bang theory [37]. Already in the late 1940s Gamow, Alpher and Herman predicted the existence of the CMB radiation [38, 39]. The temperature of the CMB has decreased from about 3000 Kelvins down to 2.725 Kelvins [40] due to the expansion of the Universe. When the CMB was formed, the observed Universe was in thermal equilibrium. Therefore, the spectrum of the CMB photons is an almost perfect black-body spectrum. The CMB spectrum is very isotropic, i.e. the CMB has almost the same temperature in every direction on the sky. However, there are extremely small deviations from the isotropy of size one part in 100,000. The source of these deviations is quantum effects during inflation and in preheating and reheating. Therefore, extremely accurate measurements of CMB give valuable information from these epochs of the early Universe. CMB is the most important source of information of the early Universe. The features of the CMB spectrum have been measured over the full sky by several satellites [4, 40, 41]. The observations of the CMB spectrum have substantially constrained the possible inflationary models and have set serious requirements for preheating and other early Universe phenomena.

Successes of Inflation

Isotropy and Homogeneity

Inflation can explain naturally the isotropy of the CMB temperature. On the basis of HBB alone, at the time of recombination, the presently observable Universe would have consisted of several thousand causally disconnected patches. However, the observed CMB spectrum has an unique temperature in all the directions on the sky. Thus, this region must have been causally connected at some point before the recombination. During inflation the comoving size of the causally connected regions decrease. Therefore, it is possible that during inflation the observable Universe was indeed causally connected. Hence, the initial state after inflation was the same everywhere in the observable Universe and after thermalization the temperature was also equal. Hence, the isotropy of the temperature distribution of the CMB can be understood. The isotropy and homogeneity of the large scale structure of the Universe can be explained analogously.

Flatness

According to the observations the Universe is spatially almost flat i.e. in the metric (1.1) the parameter $K \approx 0$ [4]. In other words, the energy density of the Universe is very close to the critical density $\rho_c \equiv 3H^2M_p^2$ corresponding $\Omega \equiv \rho/\rho_c \approx 1$. The energy density is driven away from its critical value during radiation dominated epoch as $\Omega - 1 \propto t$. Similarly, during matter dominated
epoch we have $\Omega - 1 \propto t^{2/3}$. Hence, the curvature of the Universe increases during radiation and matter domination. However, during inflation the energy density is driven towards its critical value as $\Omega - 1 \propto e^{-2Ht}$. Thus, curvature of the Universe is driven towards zero exponentially fast, yielding a flat Universe.

**Negligible Level of Magnetic Monopole Density**

Production of very massive magnetic monopoles is a typical prediction of grand unified theories [42, 43]. If inflation takes place during or after the magnetic monopole production the expansion of the Universe dilutes the magnetic monopole density to negligible level. Similarly the density of other potential unwanted relics such as gravitons is diluted to an insignificant level during inflation.

**Origin of Structure**

Inflation gives the primordial origin for the small scale inhomogeneities of the Universe. This is the most important achievement of the inflationary scenario. During inflation the quantum fluctuations of the inflaton and other fields are stretched with the expanding Universe by many orders of magnitude. Eventually, the quantum fluctuations become classical (metric) perturbations. These perturbations are the primordial seed for the temperature perturbations we see in the CMB. Temperature perturbations on the CMB correspond to small density differences. Due to the gravitational attraction, overdense regions then tend to collapse. Finally the galaxies and the clusters of galaxies will form in these regions. Thus, the quantum fluctuations are the primordial seed of the structure in the Universe.

**1.3 Single Field Slow-roll Inflation**

During inflation the expansion of the Universe is accelerating, defined as $\ddot{a} > 0$. According to the second Friedmann equation (1.11) this implies negative pressure $p < -\frac{1}{3}\rho$. The single field slow-roll inflation is a simple model of inflation, but yet consistent with all the observations [5]. In the slow-roll model the energy density of the Universe is dominated by the potential energy of a homogeneous scalar field called the inflaton, $\phi$. During inflation the inflaton slow-rolls in its potential $V(\phi)$. In the simple chaotic inflation models the inflaton potential is quadratic $V(\phi) = \frac{1}{2}m^2\phi^2$ or quartic $V(\phi) = \frac{1}{4}\lambda\phi^4$. The shape of the potential affects the spectrum of the generated metric perturbations. Strictly speaking, the quartic slow-roll inflation model have been ruled out by the Planck observations [5] and also the quadratic slow-roll inflation model is disfavored. However, radiative corrections may modify these considerations [44]. During preheating the effective inflaton potential can differ from the slow-roll type valid during inflation because the CMB observations gauge only a very limited range of the potential. Hence, during preheating the effective inflaton potential can be quartic or quadratic without being disfavored by observations.

The energy density and pressure of the homogeneous inflaton field, $\phi$, are given by
\[ \rho = \frac{1}{2a^2} \varphi'^2 + V(\varphi), \]  
(1.14) 

\[ p = \frac{1}{2a^2} \varphi'^2 - V(\varphi). \]

By substituting these into the energy-momentum continuity equation (1.12), we obtain the Klein-Gordon field equation in a flat FRW background for the inflaton field as

\[ \varphi'' + 2H \varphi' + a^2 V(\varphi) = 0, \]  
(1.15)

where \( V_\varphi = \frac{dV}{d\varphi}. \)

During inflation, the energy density of the Universe is dominated by the inflaton field. We thus obtain from the first Friedmann equation (1.10) the Hubble rate as

\[ H^2 = \frac{1}{3M_p^2} \left[ \frac{1}{2} \varphi'^2 + a^2 V(\varphi) \right]. \]  
(1.16)

During slow-roll inflation the inflaton varies very slowly. The slow-roll conditions are given by

\[ \varphi'^2 \ll a^2 V(\varphi), \]
\[ |\varphi''| \ll |2H \varphi'|. \]  
(1.17)

These slow-roll requirements lead to the flatness conditions for the potential which can be expressed with the slow-roll parameters \( \epsilon, \eta \) as follows

\[ \epsilon \equiv \frac{1}{2} M_p^2 \left( \frac{\varphi'}{V} \right)^2, \quad \eta \equiv M_p^2 \frac{\varphi''}{V}, \]  
(1.18)

\[ \epsilon \ll 1, \quad |\eta| \ll 1. \]

One can estimate that inflation ends when one of the slow-roll parameters reaches unity:

\[ \epsilon(\varphi_{end}) = 1 \quad \text{or} \quad |\eta(\varphi_{end})| = 1. \]  
(1.19)

During slow-roll inflation in the exponentially expanding Universe, the energy density of everything else but the inflatons decreases by many orders of magnitude. After inflation the Universe is left with the energy density of the inflaton condensate. Hence, the Universe is almost at zero temperature and must be reheated to reach the initial stage of the HBB theory. The energy density of the coherently oscillating inflaton condensate is released into radiation during the reheating process.
Chapter 2

Preheating

In this chapter we discuss the decay of homogeneous scalar field condensates after the end of inflation during the reheating stage. Perturbative decay is described shortly and the main focus is on the non-perturbative decay via bosonic and fermionic preheating. We consider the decay of the inflaton field as well as the decay of subdominant spectator fields, such as the Higgs field.

2.1 Reheating

After the end of inflation the energy density of the Universe is dominated by the oscillating inflaton condensate and the Universe is practically at zero temperature. The initial stage of the Universe in the observationally successful HBB theory is very hot and in thermal equilibrium. This is very different from the extremely cold final stage of inflation. Reheating describes how the hot initial stage of the HBB theory is achieved after the end of inflation. During reheating the energy of the inflaton field transfers into relativistic Standard Model particles through decay processes. After production these particles thermalize and the Universe begins to evolve according to the HBB theory.

Perturbative Decay

The first proposal for the reheating model was a simple perturbative decay of the single scalar particles into other degrees of freedom. For a scalar field \( \phi \) with potential \( V = \frac{1}{2} m^2_\phi \phi^2 \) coupled to another scalar field \( \chi \) through interaction \( M \phi \chi^2 \), the decay rate is

\[
\Gamma_{\phi \rightarrow \chi \chi} = \frac{M^2}{8\pi m_\phi}.
\]  

where \( M \) is a coupling constant with mass dimension one. Yukawa coupling \( h \phi \bar{\psi} \psi \) of the decaying scalar field \( \phi \) with fermions \( \psi \) leads to decay rate

\[
\Gamma_{\phi \rightarrow \bar{\psi} \psi} = \frac{h^2 m_\phi}{8\pi},
\]

where \( h \) is a dimensionless coupling constant.
The perturbative decay scenario does not take into account that the effective masses of the
decay products can be time-dependent and even vanish, which makes the evolution of the system
non-perturbative \cite{45}. In preheating these non-perturbative effects are noticed and the particle
production turns out to differ considerably from the perturbative treatment. Preheating is a non-
perturbative process which effectively transfers the energy from an oscillating scalar field condensate
to other degrees of freedom.

2.2 Non-perturbative Decay - Preheating

A homogeneous scalar field e.g. the inflaton oscillates after inflation and this gives rise to time-
dependent effective masses for the fields that the decaying scalar field is coupled to. When the
effective masses vanish the energy transfers very efficiently and non-adiabatically from the decaying
scalar condensate to relativistic degrees of freedom. After the inflaton decay in preheating the
produced particles can decay further into other particles perturbatively. Finally the produced particles
thermalize and the Universe reaches the initial state required by the successful HBB theory.

To gain better understanding of the preheating process we need to find ways to observe conse-
quences of the process in cosmological experiments. In this thesis we will focus on the dynamics of
the SM Higgs field during inflation and its decay during preheating and we will discuss its possible
consequences in the cosmological observables. Another main focus is the spectrum of gravitational
waves produced by fermions generated during preheating. Gravitational waves propagate almost
freely after production and therefore carry valuable information from the time they were produced.

In the case of inflaton decay during preheating the background evolution is determined by the
oscillating inflaton field itself, because it dominates the energy density of the Universe. If the inflaton
potential is quadratic ($V \propto \varphi^2$) during preheating the background evolves as matter-dominated
Universe, i.e. $a \propto t^{2/3} \propto \eta^2$. Whereas if the inflaton potential is quartic ($V \propto \varphi^4$) the background
is radiation-dominated-like and the scale factor evolves as $a \propto t^{1/2} \propto \eta$. The effective inflaton
potential during inflation can differ from the potential during preheating. Hence, although the chaotic
inflation models with quadratic or quartic potential are strongly disfavored by observations \cite{5}, the
effective inflaton potential may be quadratic or quartic during preheating without being in conflict
with observations. In \cite{1, 3} we have studied gravitational wave production from non-perturbative
fermionic decay of inflaton with quadratic and quartic potential during preheating. We have also
considered gravitational wave production from non-perturbative fermionic decay of a subdominant
field oscillating in a quadratic potential in a radiation-dominated background.

The decay of a subdominant spectator field in preheating is similar with the inflaton decay. The
main difference is that the background evolution is not determined by the decaying field. In \cite{2} we
have studied the decay of the Higgs condensate non-perturbatively via preheating after the end of
inflation. The energy density of the Universe is then dominated by the inflaton oscillating coherently
in a quadratic potential. Hence, the evolution of the scale factor resembles the one in matter
dominated background, $a \propto t^{2/3} \propto \eta^2$. 
After the slow-roll phase a homogeneous scalar field, for example the inflaton or a spectator field, becomes massive \((m^2 = V_{\phi,\phi} \sim H^2)\) and begins to oscillates around its minimum with decreasing amplitude. The equation of motion for a homogeneous scalar field in FRW background is given by

\[ \varphi'' + 2H\varphi' + a^2V_{\phi} = 0, \]  
(2.3)

The solution of eq. (2.3) can be written in form

\[ \varphi(\eta) \approx \Phi(\eta)F(\eta), \]  
(2.4)

where \(\Phi(\eta)\) is a decreasing amplitude and \(F(\eta)\) is an oscillatory periodic function with amplitude equal to unity.

The oscillating field gives rise to time-dependent effective masses for the fields it is coupled to. In the bosonic preheating scenario the interaction term is \(L_{\text{int}} = -\frac{1}{2}g^2 \varphi^2 \chi^2\) and this coupling gives rise to effective mass \(m_{\chi}^2 = g^2 \varphi^2\), whereas in the fermionic preheating scenario we have Yukawa coupling \(L_{\text{int}} = h\varphi \bar{\Psi}\Psi\) and the effective mass is then \(m_{\Psi} = h\varphi\). The quanta of \(\chi\) or \(\Psi\) field can be effectively produced when the effective mass crosses around zero, because then the evolution of the fields is non-adiabatic and the fields are excited.

### 2.3 Bosonic Preheating

In the bosonic preheating scenario a coherently oscillating scalar field, \(\varphi\), decays non-perturbatively into other bosons \([46–48]\). Here we assume that the decay is into another scalar field, \(\chi\). In a FRW background \(\chi\) field evolves according to the Klein-Gordon equation

\[ \chi'' + 2H\chi' - \nabla^2 \chi + a^2m_{\chi}^2\chi = 0, \]  
(2.5)

where \(m_{\chi}^2 \equiv \frac{d^2 V(\chi)}{d\chi^2}\) is the effective mass with \(V(\chi)\) the potential of \(\chi\). Here the scalar field \(\chi\) is coupled to the oscillating homogeneous scalar field \(\varphi\) via interaction \(L_{\text{int}} = -\frac{1}{2}g^2 \varphi^2 \chi^2\) and this coupling gives rise to effective mass, \(m_{\chi}^2 = g^2 \varphi^2\). For rescaled scalar field \(\chi = a\chi\) the equation of motion (2.5) at sub-horizon scales (\(k \gg H, \frac{a'}{a}\)) is

\[ \chi'' - \nabla^2 \chi + a^2m_{\chi}^2\chi = 0. \]  
(2.6)

The quantized scalar field is given as

\[ \chi(x, \eta) = \int \frac{dk}{(2\pi)^3} e^{-i\mathbf{k} \cdot \mathbf{x}} \left[ \hat{a}_k \chi_k(\eta) + \hat{a}^+_k \chi_k^*(\eta) \right], \]  
(2.7)

where the creation and annihilation operators satisfy the canonical commutation relation

\[ [\hat{a}_k, \hat{a}^+_k] = (2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k}'), \]  
(2.8)

and other commutators vanish. The definition of the vacuum state is given by \(\hat{a}_k |0\rangle = 0\). By substituting eq. (2.7) into eq. (2.6) we obtain the equation of motion for the mode functions, \(\chi_k\), as

\[ \chi_k'' + \left(k^2 + a^2m_{\chi}^2\right) \chi_k = 0, \]  
(2.9)
which is a harmonic oscillator with a time-dependent angular frequency, $\omega_k^2 = k^2 + a^2 m_\chi^2$. Thus, the energy transfer from the oscillating condensate $\varphi$ to the modes of $\chi$ field take place through parametric resonance. The number density of the $\chi$ particles with momentum $k$ is given by [2,47,48].

$$n_k = \frac{\omega_k}{2} \left( \frac{|\chi_k|^2}{\omega_k^2} + |\chi_k|^2 \right) - \frac{1}{2}. \quad (2.10)$$

The total number density of created scalar particles at time $\eta$ is then computed by integrating the number density over the momentum space as [47]

$$n(\eta) = \frac{1}{2\pi^2 a^3} \int_0^\infty dk k^2 n_k. \quad (2.11)$$

The positive-frequency initial condition is written as [47, 48]

$$\chi_k = e^{-i\omega_k \eta} \frac{\sqrt{2}}{\sqrt{2\omega_k}}. \quad (2.12)$$

It gives initially vanishing number density $n_k = 0$. By setting the initial time as $\eta_I = 0$ we find that the corresponding initial amplitudes are

$$\chi_k^{(I)} = \frac{1}{\sqrt{2\omega_k^{(I)}}}, \quad \chi_k'^{(I)} = -i\frac{\sqrt{\omega_k^{(I)}}}{\sqrt{2}}, \quad (2.13)$$

where index $I$ refers to the initial values.

The effectiveness of preheating is determined by the resonant parameter $q$. For a decaying scalar field oscillating coherently in quadratic or quartic potential and coupled to a scalar field $\lambda$ via $L_{\text{int}} = -\frac{1}{2}g^2 \varphi^2 \lambda^2$ the resonant parameters are given as

$$q = \frac{g^2 \Phi_I^2}{m^2}, \quad V(\varphi) = \frac{1}{2} m^2 \varphi^2 \quad \text{and} \quad (2.14)$$

$$q = \frac{g^2}{\lambda}, \quad V(\varphi) = \frac{1}{4} \lambda \varphi^4,$$

where $\Phi_I = \Phi(\eta_I)$ is the initial value of the scalar field defined in eq. (2.4).

The effectiveness of particle production in preheating in average grows with the resonant parameter, but the growth is not monotonic. For the conformal model with quartic potential there are instability and stability bands in $(k^2, q)$ parameter space [48]. Inside the stability bands the occupation number does not grow much but instead it oscillates, whereas inside the instability bands the occupation number of the bosons produced grows exponentially, and the energy transfer is very effective. For the zero-momentum quanta ($k = 0$) the instability bands are located at the resonant parameter ranges $n(n + 1)/2 < q < (n + 1)(n + 2)/2$, where $n$ is an odd integer [48]. Thus, the first three instability bands are $1 < q < 3, 6 < q < 10$ and $15 < q < 21$.

Due to the expansion of the Universe, in the case of the quadratic potential, a mode $\chi_k$ with momentum $k$ does not stay within a single instability or stability band but scans over several bands.
so that the resonance becomes stochastic process [47]. The quartic model is conformal and although the Universe expands a given mode, $\chi_k$ stays within a single instability or stability band. In the quartic model the scalar field also decays through self-resonance. The resonant parameter of the self-resonance is $q = 3$ and this turns out to be anomalously ineffective decay channel, because it is located on the boundary of a stability and instability band [2,47].

In the case of quartic potential the number density of created bosons inside the instability bands grows exponentially as

$$n_k \propto \exp \left(2\mu_k \sqrt{\lambda} \Phi_I (\eta - \eta_I)\right), \quad (2.15)$$

where $\mu_k$ is a characteristic exponent, which can be evaluated numerically and $\Phi_I$ is the initial value of the scalar field.

Figure 2.1: The solid line shows the evolution of the number density of bosons ($n_k$) in preheating for mode $k = 0.2q^{1/4}\sqrt{\lambda}\Phi_I$ with resonant parameter $q = g^2/\lambda = 100$. The scalar field potential is quartic ($V = \frac{1}{4}\lambda\phi^4$) and the background energy density is dominated by the scalar field. The initial value of the scalar field is $\phi = \Phi_I = \sqrt{12}M_p$ which corresponds to the slow-roll parameter values $\eta_I = 1$ and $\epsilon_I = 2/3 < 1$. The dotted line is $100|\phi|/M_p$. The number density increases in a step-like manner when the scalar field crosses around zero, because then the effective mass of the bosonic modes vanishes and they are non-adiabatically excited. The mode is chosen to be inside an instability band, and thus the number density grows exponentially (straight line in average on logarithmic scale).

2.4 Fermionic Preheating

In the fermionic preheating scenario [49–53] the decaying oscillating scalar field $\varphi$ is coupled to a fermionic field $\Psi$. The Dirac equation for a rescaled spin-$\frac{1}{2}$ fermion field $\psi(x,\eta) = a^{3/2}\Psi(x,\eta)$ in a
Preheating

flat FRW background is

\[
[i\gamma^\mu \partial_\mu - am_\psi] \psi(x, \eta) = 0,
\]

(2.16)

where \( m_\psi \) is the effective mass of the fermions. The coupling is assumed to be of the Yukawa type with \( \mathcal{L}_{\text{int}} = h \varphi \bar{\Psi} \Psi \), with \( h \) a dimensionless coupling constant. The scalar field \( \varphi \) generates an effective mass for the fermionic field as \( m_\psi = h \varphi \). The \( \gamma^\mu \) in (2.16) are the flat-space gamma matrices verifying the anti-commutation relation

\[
\{ \gamma^\mu, \gamma^\nu \} = -2\eta^{\mu\nu}1_{4\times4},
\]

(2.17)

with \( \eta^{\mu\nu} = \text{diag}(−1, 1, 1, 1) \) the Minkowski metric, and \( \gamma^\mu = \eta^{\mu\nu}\gamma^\nu \). ¹

The quantized Dirac field is given by

\[
\psi(x, \eta) = \int \frac{dk^3}{(2\pi)^3} e^{-i k \cdot x} \left[ \hat{a}_{k,r} u_{k,r}(\eta) + \hat{b}^\dagger_{-k',r} v_{k',r}(\eta) \right],
\]

\[
\bar{\psi}(x, \eta) = \psi^\dagger(x, \eta) \gamma_0 = \int \frac{dk^3}{(2\pi)^3} e^{i k \cdot x} \left[ \hat{a}^\dagger_{k,r} \bar{u}_{k,r}(\eta) + \hat{b}_{-k,r} \bar{v}_{k,r}(\eta) \right]
\]

(2.18)

with the creation and annihilation operators satisfying the canonical anti-commutation relations

\[
\{ \hat{a}_{k,r}, \hat{a}^\dagger_{k',r'} \} = \{ \hat{b}_{k,r}, \hat{b}^\dagger_{k',r'} \} = (2\pi)^3 \delta_{r,r'} \delta^{(3)}(k-k'),
\]

(2.19)

and other anti-commutators vanish. The vacuum state \( |0\rangle \) is defined as

\[
\hat{a}_{k,r}|0\rangle = \hat{b}_{k,r}|0\rangle = 0.
\]

(2.20)

The four-component spinors in eq. (2.18) can be written as

\[
\begin{align*}
\psi_{k,r} &\equiv \begin{pmatrix} u_{k,+} S_r \\ u_{k,-} S_r \end{pmatrix}, \\
\bar{\psi}_{k,r} &\equiv \begin{pmatrix} v_{k,+} S_{-r} \\ v_{k,-} S_{-r} \end{pmatrix},
\end{align*}
\]

(2.21)

with \( S_r \) 2-component spinors normalized as \( S_r^\dagger S_r = 2 \), and \( S_{-r} \equiv -i \sigma_3 S_r^* \). ² The spinors \( v_{k,r} \) and \( u_{k,r} \) are related by the charge conjugation [54] as \( v_{k,r} = i \gamma^0 \gamma^2 \bar{u}_{k,r}^T \). Hence, the mode functions are related as well

\[
v_{k,\pm} = \pm u_{k,\mp}^*.
\]

(2.22)

¹In the Dirac basis we have \( \gamma^0 = \begin{pmatrix} 1_{2\times2} & 0 \\ 0 & -1_{2\times2} \end{pmatrix} \) and \( \gamma^i = \begin{pmatrix} 0 & 0 \\ 0 & -\sigma^i \end{pmatrix} \), where \( \sigma^i \) (\( i = 1, 2, 3 \)) are the Pauli matrices satisfying \( \{ \sigma^i, \sigma^j \} = 2\delta^{ij}1_{2\times2} \).

²Choosing eigenstates of the Pauli matrix \( \sigma_3 \) to be the basis of the spinors \( S_r \), these read \( S_1 = -S_{-2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( S_2 = S_{-1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \).
By substituting the decomposition of eq. (2.18) into the Dirac equation (2.16) we obtain the decoupled equations of motion for the mode functions $u_{k,+}$ and $u_{k,-}$ as

$$u''_{k,\pm} + \left(k^2 + a^2m^2_\psi \pm i(am_\psi)^r\right) u_{k,\pm} = 0,$$

which correspond to oscillators with complex time-dependent frequencies.

The number density i.e. the occupation number of fermions with momentum $k$ is given by [3,52]

$$n_k = \frac{\omega_k - E_k}{2\omega_k},$$

where

$$\omega^2_k = k^2 + a^2m^2_\psi,$$

$$E_k = \frac{i}{2} \left[ u_{k,+} u'^*_{k,+} + u_{k,-} u'^*_{k,-} \right].$$

The total number density of produced fermions at time $\eta$ (equal to the number of produced anti-fermions) [52] is computed analogously with the bosonic case as

$$n(\eta) = \frac{1}{2\pi^2a^3} \int_0^{\infty} dk k^2 n_k.$$

The initial amplitudes corresponding vanishing occupation number are given as [3,52]

$$u^{(I)}_{k,\pm} = \sqrt{1 \pm a_I m^{(I)}_\psi \omega^{(I)}_k},$$

$$u'^{(I)}_{k,\pm} = -iku^{(I)}_{k,\mp} \mp ia_I m^{(I)}_\psi u^{(I)}_{k,\pm}.$$

For these initial values we obtain $\omega^{(I)}_k = E^{(I)}_k$, whereupon the initial occupation number indeed vanishes, $n^{(I)}_k = 0$.

The occupation number of fermions cannot exceed unity according to the Pauli exclusion principle. Therefore, the occupation number, $n_k$, cannot grow exponentially as in the bosonic preheating scenario, and the energy transfer to fermions is limited.

The effectiveness of the fermionic preheating is also determined by a resonant parameter analogously with bosonic preheating. The resonant parameters in the case of quadratic and quartic scalar field potentials are given as

$$q = \frac{h^2 \Phi^2_1}{m^2}, \quad V(\varphi) = \frac{1}{2}m^2 \varphi^2$$

and

$$q = \frac{h^2}{\lambda}, \quad V(\varphi) = \frac{1}{4}\lambda \varphi^4.$$
In the fermionic preheating process the fermions fill up an Fermi-sphere with radius $k_F$, which is given as \[50,51\]

$$k_F \sim \left(\frac{a}{a_I}\right)^{1/4} q^{1/4} m, \quad V = \frac{1}{2} m^2 \varphi^2,$$

and

$$k_F \sim q^{1/4} \sqrt{\lambda} \Phi_I, \quad V = \frac{1}{4} \lambda \varphi^4.$$ \hfill (2.29)

Outside the Fermi-sphere the adiabaticity condition $\omega_k' < \omega_k^2$ is satisfied and the fermionic modes evolve adiabatically without being excited. Inside the Fermi-sphere the adiabaticity condition is most strongly violated when the effective mass ($m_\psi$) vanishes, then the fermionic modes are excited non-adiabatically. The effective mass vanishes each time the inflaton crosses zero. Then the occupation number of fermions changes drastically. Therefore, it evolves in a step-like manner inside the Fermi-sphere as can be seen from the figure (2.2). Outside the Fermi-sphere the occupation number vanishes.

![Figure 2.2: The solid line depicts the evolution of the occupation number of fermions ($n_k$) in preheating for mode $k = 0.2 q^{1/4} \sqrt{\lambda} \Phi_I$ with resonant parameter $q = h^2 / \lambda = 100$. The scalar field potential is quartic ($V = 1/4 \lambda \varphi^4$) and the background energy density is dominated by the scalar field. The initial value of the scalar field is $\varphi = \Phi_I = \sqrt{12} M_p$ which corresponds to the slow-roll parameter values $\eta_I = 1$ and $\epsilon_I = 2/3 < 1$. The dotted line shows the evolution of the scalar field $\varphi / M_p$. The number density changes in a step-like manner each time the scalar field crosses around zero, because then the effective mass of the fermionic modes vanishes and the adiabaticity condition is violated. Due to the Pauli exclusion principle the occupation number of fermions cannot exceed unity.](image-url)
Chapter 3

The Standard Model Higgs

3.1 The Higgs mechanism

The Higgs mechanism explains how the masses of the weak gauge bosons $W^\pm$, $Z$ are generated through the electroweak symmetry breaking (EWSB). The generation of the fermion masses can also be explained by the same mechanism. Introduction of the Higgs mechanism predicted existence of a new scalar particle, which was named as the Standard Model (SM) Higgs particle. The existence of the Higgs-like boson was confirmed at LHC in July 2012 with mass of $125 - 126$ GeV [10, 11]. In 2013 it was confirmed that the properties of the discovered boson are consistent with the Standard Model Higgs particle [12,13], and thereby we justly assume here that the found particle indeed is the SM Higgs boson. The Higgs mechanism was originally proposed by Higgs, Brout, Englert, Guralnik, Hagen and Kibble [55–59]. Higgs and Englert were awarded by the Noble Prize in Physics for their contribution in 2013.

The Higgs mechanism in the Standard Model is accomplished by adding a complex $SU(2)$ doublet of scalar fields in the theory. In the unitarity gauge this scalar field doublet can be parametrized as [54,60]

$$\Phi(x,t) = \frac{U(x,t)}{\sqrt{2}} \begin{pmatrix} 0 \\ h(x,t) + v \end{pmatrix},$$

where $U(x,t)$ is a general $SU(2)$ gauge transformation, $v$ is the vacuum expectation value of the Higgs field and $h(x,t)$ is the Higgs field with vanishing vacuum expectation value, $\langle h(x,t) \rangle = 0$. $v$ and $h(x,t)$ are real-valued. The Lagrangian involving the scalar field doublet, $\Phi$, is given by

$$\mathcal{L}_\Phi = |D_\mu \Phi|^2 - \mu^2 \Phi^\dagger \Phi - \lambda \left( \Phi^\dagger \Phi \right)^2,$$

where $D_\mu$ is the covariant derivative, $\mu^2$ is the mass term and $\lambda$ is the dimensionless Higgs self-coupling. By solving the minimum of the potential of the Lagrangian (3.2) for $\mu^2 < 0$, we find the vacuum expectation value to be

$$v = \sqrt{-\frac{\mu^2}{\lambda}}.$$
The Lagrangian involving the SM Higgs field, $h$, in the unitarity gauge can be solved by substituting the parametrization (3.1) into the Lagrangian (3.2) and by using eq. (3.3). We obtain

$$L_V = \frac{1}{2} \left( \partial \mu h \right)^2 - V(h),$$

(3.4)

where $V(h)$ is the effective potential of the Higgs field written as

$$V(h) = \lambda v^2 h^2 + \lambda vh^3 + \frac{1}{4} \lambda h^4.$$

(3.5)

From eq. (3.5) we can read the effective mass of the Higgs field,

$$m_h^2 \equiv \frac{d^2 V(h)}{dh^2} = 2\lambda v^2 + 6\lambda vh + 3\lambda h^2.$$

(3.6)

The Higgs mass evaluated at the Higgs field vacuum expectation value is given by

$$m_h = \sqrt{2\lambda v^2} = \sqrt{-2\mu^2}.$$

(3.7)

This is the Higgs boson mass which has a value of $125 - 126$ GeV according to the measurements at LHC published in July 2012. The vacuum expectation value ($v \approx 246$ GeV) was previously known from the measurements of muon decay [60]. Thereby, after the determination of the Higgs mass we can find the value of the Higgs self-coupling at low energy scale, $\lambda = m_h^2/(2v^2) \approx 0.13$.

3.2 The Standard Model Higgs Couplings

The interaction term between the Higgs field and the weak gauge bosons in the unitarity gauge is written as [60]

$$L_{\text{int}} = m_W^2 W^+ W^- + \frac{1}{2} m_Z^2 Z^+ Z^-,$$

(3.8)

where $m_W$ and $m_Z$ are the masses of the weak gauge bosons generated by the Higgs field, given as

$$m_W = \frac{g}{2} (h + v),$$

(3.9)

$$m_Z = \sqrt{g^2 + g'^2} (h + v).$$

The Higgs boson couples to quarks and leptons via the Yukawa interaction,

$$L_{\text{int}} = -m_i \bar{f} \not \! f,$$

(3.10)

where the effective mass of the fermions is given by

$$m_i = \frac{y_i}{\sqrt{2}} (h + v).$$

(3.11)

Before the electro-weak symmetry breaking in the early Universe, at high energy scale or equivalently at high temperature ($T \gtrsim v$), the expectation value of the scalar field doublet (3.1) vanishes.
3.3 Running of the Self-coupling $\lambda$

(or equivalently $\langle h + v \rangle = 0$) due to the thermal mass which dominates over the other interactions. Thus, the effective masses of the weak gauge bosons and fermions vanish, and the electro-weak symmetry is restored. Then the expanding Universe cools down, and at the electroweak energy scale ($T \sim v$) the electroweak symmetry breaking takes place, and after that the potential in the Lagrangian (3.2) has a minimum at $v$ corresponding

$$
\langle \Phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}, \quad \langle h \rangle = 0.
$$

Therefore, after EWSB the weak gauge bosons and SM fermions at low energy scales ($T \lesssim v$) have the masses

$$
m_W = \frac{g}{2} v, \quad m_Z = \frac{\sqrt{g^2 + g'^2}}{2} v, \quad m_i = \frac{y_i}{\sqrt{2}} v.
$$

### 3.3 Running of the Self-coupling $\lambda$

The Standard Model couplings run with the renormalization scale $\mu$ due to loop corrections. The running of the couplings is computed up to three-loop precision in [61,62]. The Higgs self-coupling $\lambda(\mu)$ decreases towards larger renormalization scales $\mu$. For the best fit Standard Model parameter values the self-coupling eventually vanish at $\mu_{\text{inst}} \sim 10^{11}$ GeV, $\lambda(\mu_{\text{inst}}) = 0$. For larger renormalization scales the self-coupling is negative. Use of the Standard Model at this high energy scales is speculative, because SM has been experimentally tested only for much lower energy scales. Thus, SM is not necessarily valid up to these extremely high energy scales. Here we assume that the Standard Model is valid up to inflationary energy scales. We require that the energy scale of inflation is below the instability scale, so that the Higgs self-coupling is always positive.

Also the other SM parameters run with the renormalization scale. The Higgs field is most strongly coupled to the weak gauge bosons and the top quark. Therefore, these couplings, in addition to the self-coupling itself, mainly determine the running of the Higgs self-coupling, $\lambda(\mu)$, while other Yukawa couplings are negligible for the running.

At one-loop order the renormalization group equations (RGE) for $\lambda$, $y_t$, $g'$, $g$ and $g_s$ are given by

$$
\beta_\lambda = \frac{1}{(4\pi)^2} \left( \frac{24}{2} \lambda^2 - 6 y_t^4 + \frac{3}{8} (2g^4 + (g^2 + g'^2)^2) + (9g^2 - 3g'^2 + 12y_t^2) \lambda \right),
$$

$$
\beta_{y_t} = \frac{y_t}{(4\pi)^2} \left( \frac{9}{2} y_t^2 - \frac{9}{4} g^2 - \frac{17}{12} g'^2 - 8g_s^2 \right),
$$

$$
\beta_{g'} = \frac{41}{6(4\pi)^2} g'^3,
$$

$$
\beta_g = -\frac{19}{6(4\pi)^2} g^3,
$$

$$
\beta_{g_s} = -\frac{7}{(4\pi)^2} g_s^3.
$$

(3.14)
where the Beta functions are defined as

$$\beta_X = \frac{\partial X}{\partial \log \mu}, \quad X = \lambda, y_t, g', g, g_s. \quad (3.15)$$

In the literature these RGEs have been solved and computed up to three-loop precision [61, 62]. The figures 3.1 and 3.2 have been computed by using publicly available computer code at http://www.inr.ac.ru/~fedor/SM/ [61, 63].

Figure 3.1: The running of the Higgs self-coupling $\lambda(\mu)$ in the Standard Model computed at three-loop order for the best fit parameters: $m_t = 173.1$ GeV, $\alpha_s(m_Z) = 0.1184$ and $m_h = 125.5$ GeV. The coupling decreases towards larger renormalization scales and it eventually vanish at $\mu_{\text{inst.}} \sim 10^{11}$ GeV, $\lambda(\mu_{\text{inst.}}) = 0$ and after that the coupling is negative.
Figure 3.2: The running of the Standard Model couplings $y_t$, $g'$, $g$ and $g_s$ computed at three-loop order for the best fit parameters: $m_t = 173.1$ GeV, $\alpha_s(m_Z) = 0.1184$ and $m_h = 125.5$ GeV.
Chapter 4

The Standard Model Higgs in the Cosmological Framework

4.1 Generation of the Higgs Condensate during Inflation

After the discovery of the SM Higgs boson at LHC, all the SM parameters are known within the error limits. Then it is possible to study the cosmological consequences of the SM Higgs boson without free parameters. The Higgs field could in principle act as the inflaton [14–16], but here we assume that inflation is caused by another scalar field in slow-roll. We study the generation of the Higgs condensate during inflation and its decay into other SM degrees of freedom via preheating. We also discuss about the possible observational consequences of the Higgs field dynamics in the early Universe. The results presented in this chapter are based on the publication [2].

The quantum fluctuations of the Higgs field stretched to super-horizon scales during inflation effectively make the Higgs field amplitude to perform a random walk during inflation. According to the stochastic approach, where the UV modes have been integrated out, the probability distribution $P(h,t)$ of the Higgs field during inflation is given by the Fokker-Planck equation [17–19]

$$\frac{\partial P}{\partial t} = \frac{1}{3H} \frac{\partial}{\partial h} \left( \frac{\partial V}{\partial h} P \right) + \frac{H^3}{8\pi^2} \frac{\partial^2 P}{\partial h^2},$$

(4.1)

where $t$ is the cosmic time and $H = \frac{1}{a} \frac{da}{dt}$.

The equilibrium is achieved when $\frac{\partial P}{\partial t} = 0$. After a simple calculation we find that the equilibrium probability distribution for the mean field value during inflation is given by

$$P(h) = C \exp \left( \frac{-8\pi^2 V(h)}{3H^4} \right),$$

(4.2)

where $C$ is a normalization constant.

For large field values ($h \gg v$) the Higgs potential (3.5) is well approximated by

$$V(h) \approx \frac{1}{4} \lambda(\mu) h^4.$$

(4.3)
Thus, the probability distribution (4.2) is

\[ P(h) \approx C \exp \left( -\frac{2\pi^2 \lambda h^4}{3H^4} \right). \]  

(4.4)

The equilibrium probability distribution (4.4) is an asymptotical state, and strictly speaking, it is reached only after an infinity of efolds. If the probability distribution is initially peaked around some field value, the distribution will both spread out and its central value shift so that it approaches the equilibrium probability distribution. The decoherence time \( N_{\text{dec}} \) gives the timescale during which the distribution spreads out, in such a way that the variance of the distribution reaches its equilibrium value. In addition, the relaxation time \( N_{\text{rel}} \) measures the timescale during which the central value of the distribution achieves that of the equilibrium probability distribution. For a quartic coupling the decoherence time is given by [64]

\[ N_{\text{dec}} \approx 6\lambda^{-1/2}. \]  

(4.5)

The relaxation time is roughly twice the decoherence time. For inflationary scale \( H_* = 10^{10} \) GeV corresponding to \( \lambda \simeq 0.005 \) we obtain \( N_{\text{dec}} \simeq 85 \) and for inflationary scale \( H_* = 10^4 \) GeV we find \( \lambda \simeq 0.09 \) and thus \( N_{\text{dec}} \simeq 20 \). It is reasonable to assume that the horizon crossing of the largest observable scale was preceded by at least few tens of efolds. Therefore, it is justified to assume that on the observable scales the Higgs fluctuations follow the equilibrium probability distribution.

The expectation value of the Higgs field amplitude during inflation, \( h_* \), vanishes \( \langle h_* \rangle = 0 \). However, the typical value of the Higgs field is non-zero in a random patch of the size of the observable Universe, and it is given by the root mean square value

\[ h_* \sim \sqrt{\langle h_*^2 \rangle} \simeq 0.36\lambda_*^{-1/4} H_*, \]  

(4.6)

where \( \langle h_*^2 \rangle \) have been computed by using the equilibrium probability distribution given in eq. (4.4). At large field values \( (h \gg v) \) the effective mass of the Higgs field is given by

\[ m^2_h \approx 3\lambda h^2. \]  

(4.7)

Thus, we find that the Higgs mass in average during inflation is given as

\[ m^2_{h_*} = 3\lambda_* \langle h_*^2 \rangle \simeq 0.40\lambda_*^{1/2} H_*^2, \]  

(4.8)

which indicates that the Higgs field is light during inflation, \( \eta_{h_*} = \frac{m^2_{h_*}}{3H_*^2} \lesssim O(0.01) \). Therefore, the Higgs field slow-rolls during inflation, and the energy density of the Higgs field is simply given by the potential energy

\[ \rho_{h_*} \simeq \frac{1}{4} \lambda_* h_*^4 \sim 4 \cdot 10^{-3} H_*^4. \]  

(4.9)

From the total energy density \( \rho_{\text{tot.}} = 3M_p^2 H_*^2 \) the Higgs energy density is only a small fraction
\begin{equation}
\frac{\rho_{h^*}}{\rho_{\text{tot.}}} \sim 10^{-3} \left( \frac{H_*}{M_p} \right)^2.
\end{equation}

Hence, the Higgs is a light spectator field during inflation.

\section*{4.2 Higgs Decay after the end of Inflation}

\subsection*{Perturbative Decay}

In principle, the perturbative decay of the Higgs field into the weak gauge bosons and the top quark could be effective. However, the coupling with the Higgs condensate generates large effective masses for these fields. Unfortunately, these masses turn out to be so large that the perturbative decay channels are kinematically blocked at the onset of Higgs oscillations. For large Higgs field values ($h \gg v$) the effective masses of the weak gauge bosons are written as

\begin{equation}
m_W \approx \frac{g}{2} h,
\end{equation}

\begin{equation}
m_Z \approx \frac{\sqrt{g^2 + g'^2}}{2} h.
\end{equation}

Similarly for the fermions we have

\begin{equation}
m_t \approx \frac{y_t}{\sqrt{2}} h.
\end{equation}

Moreover, the Higgs field generates an effective mass for itself, given as

\begin{equation}
m_h \approx \sqrt{3} \lambda h.
\end{equation}

The perturbative decays $h \to W^+ W^-$, $h \to ZZ$ and $h \to t\bar{t}$ are all indeed kinematically blocked, because for all these channels we find that $m_h < 2m_{W,Z,t}$ at all renormalization scales $\mu < \mu_{\text{inst.}}$. The decay rate of the Higgs condensate into other SM particles turns out to be negligible compared to Hubble rate. Hence, those decay channels can be neglected.

\subsection*{Non-perturbative Decay via Preheating and Timescale of the Decay}

After the end of inflation the Higgs field eventually becomes heavy at $m_h \sim H$ and then starts to oscillate. Each time the Higgs field crosses zero the effective masses of the weak gauge bosons, the top quark and the Higgs boson itself vanish. When this happens, some of the energy of the Higgs condensate is effectively transferred into these fields, and quanta of these fields can be copiously produced. The Higgs decay into the top quarks is limited, because the Pauli blocking of fermions limits energy transfer into quarks. Also the non-perturbative decay of the Higgs condensate into the Higgs particles turns out to be anomalously inefficient [2, 48]. Therefore, non-perturbative decay into the weak gauge bosons turns out to be the dominant decay channel of the Higgs condensate.
The Standard Model Higgs in the Cosmological Framework

The Higgs coupling to the weak gauge bosons in the unitarity gauge at large field values \((h \gg v)\) is written as

\[
\mathcal{L}_{\text{int}} = q_W \lambda h^2 W^\mu W^-\mu + \frac{1}{2} q_Z \lambda h^2 Z^\mu Z_\mu, \tag{4.14}
\]

where the resonant parameters are defined as

\[
q_W = \frac{g^2}{4\lambda}, \quad q_Z = \frac{g^2 + g'^2}{4\lambda}. \tag{4.15}
\]

The effective Higgs mass terms generated by the \(W^\pm\) and \(Z\) weak gauge bosons are then given by

\[
m_h^2(W) = 2q_W \lambda \langle W^\mu W^-\mu \rangle = 2q_W \lambda \langle W^2 \rangle, \tag{4.16}
\]

and

\[
m_h^2(Z) = q_Z \lambda \langle Z^\mu Z_\mu \rangle = q_Z \lambda \langle Z^2 \rangle.
\]

The resonantly produced Higgs particles also contribute to the effective mass of the Higgs condensate, but this is a subdominant effect as the Higgs quanta are produced at a much lower rate than the weak gauge bosons.

Table 4.1: Numerical values of the characteristic exponent \(\mu_k\) of \(k = 0\) modes for a set of different inflationary energy scales, \(H_\star\), and corresponding estimates for the decay time of the Higgs condensate after the onset of Higgs oscillations.

<table>
<thead>
<tr>
<th>(H_\star/\text{GeV})</th>
<th>(\lambda)</th>
<th>((q_W, \mu_k))</th>
<th>((q_Z, \mu_k))</th>
<th>(H_{\text{osc}}/H_{\text{dec}})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(10^4)</td>
<td>0.09</td>
<td>(1.1, 0.14)</td>
<td>(1.5, 0.26)</td>
<td>370</td>
</tr>
<tr>
<td>(10^6)</td>
<td>0.04</td>
<td>(2.3, 0.25)</td>
<td>(3.2, 0.00)</td>
<td>360</td>
</tr>
<tr>
<td>(10^8)</td>
<td>0.02</td>
<td>(4.4, 0.00)</td>
<td>(6.2, 0.14)</td>
<td>630</td>
</tr>
<tr>
<td>(10^{10})</td>
<td>0.005</td>
<td>(16, 0.22)</td>
<td>(24, 0.00)</td>
<td>340</td>
</tr>
</tbody>
</table>

Non-perturbative decay becomes relevant for the dynamics of the Higgs field when the generated mass term given in eq. (4.16) is of the same order with the mass term \(m_h = 3\lambda h^2\). We approximate the timescale of the Higgs condensate decay after the onset of Higgs oscillations by the time when \(m_h(W)\) or \(m_h(Z)\) reaches \(m_h\).

As discussed in section 2.3, if the resonance parameter is inside an instability band, the number density of (weak gauge) bosons grows exponentially as \(n_k \propto \exp \left(2\mu_k \sqrt{\lambda} h_0 (\eta - \eta_0) \right)\). The Higgs field can decay effectively via preheating into the weak gauge bosons if the corresponding resonance parameter is inside an instability band. In the conformal model the first three instability bands for mode \(k = 0\) were \(q = 1\)\ldots3, \(q = 6\)\ldots10 and \(q = 15\)\ldots21, which seem to describe the resonance very well also in this case. This can be seen from the table (4.1) where we have listed resonance parameters and numerically evaluated characteristic exponents corresponding several inflationary

\[1\] The difference between the conformal model and this situation is that in the conformal case the background is radiation-dominated-like, whereas here the background is matter-dominated-like.
energy scale, $H_*$. In the table the characteristic exponent is non-zero inside an instability band and vanishes outside.

The values of the resonance parameters vary with the renormalization scale due to the running of the couplings. Therefore, we find different timescales for the decay of the Higgs condensate for different inflationary energy scales as given in table (4.1). The resonance parameters $q_W$ and $q_Z$ are slightly different but typically at least one of the parameters will be inside the instability band. Therefore, the Higgs field decays very effectively into the weak gauge bosons after the onset of Higgs oscillations via non-perturbative preheating. In [2] we found that the Higgs condensate decays into the gauge bosons typically in few hundreds of Hubble times after the onset of Higgs oscillations as listed in the table (4.1).

### 4.3 Generation of Metric Perturbations

The perturbations of the Higgs field can be converted into observable metric perturbations after the end of inflation, if the expansion history is sensitive to the exact value of the Higgs condensate, and its slight variation on super-horizon scales, as for example in the modulated reheating scenario [20, 21] and the curvaton scenario [22–26].

**Modulated Reheating**

In the modulated reheating scenario [20, 21] the inflaton couplings to the fields the inflaton decays into depend on another light field called the moduli field, $\chi$. Hence, the inflaton decay width also depends on the moduli field. The perturbations of the light moduli field generated during inflation can convert into metric fluctuations when the inflaton decays during reheating [20, 21]. The decay width of the inflaton can be formally expressed as a sum of two parts

$$\Gamma_\phi = \Gamma_0 + \Gamma(\chi), \tag{4.17}$$

where $\Gamma(\chi)$ depends on the amplitude of the moduli field. The perturbations of the light moduli field cause the decay rate $\Gamma_\phi$ to vary slightly on super-horizon scales. Therefore, the inflaton decay occurs at slightly different times in different horizon patches. This causes variation in the expansion history of the Universe, because the energy density of the Universe changes from matter-like to radiation at reheating. This generates metric perturbations.

The Higgs field is a natural moduli field candidate. It is light during inflation, as we showed in section 4.1, and therefore quantum fluctuations give rise to perturbations in the Higgs field. The fluctuations of the Higgs field can convert into metric perturbations during reheating if the decay width of the inflaton depends on the amplitude of the Higgs field. Due to the quartic self-coupling, the Higgs field can generate sizable non-Gaussianity. The observation of the Planck satellite have set severe constraints on the amount of non-Gaussianity [65]. Therefore, the contribution of the Higgs field into the total metric perturbations is limited by the observations [66].
As discussed in section 4.2, the Higgs condensate decays in few hundreds of Hubble times after inflation. The modulated reheating scenario with the Higgs field can produce significant amount of metric perturbations only if the inflaton decay occurs before the decay of the Higgs condensate. To satisfy this requirement the inflaton decay rate has to satisfy $\Gamma_\phi \gg \Gamma_h$, where $\Gamma_h$ is the decay rate of the Higgs field. Therefore, the inflaton couplings should be large enough during reheating or the perturbations of the Higgs field do not generate a significant contribution to the primordial metric perturbations.

**Curvaton Scenario**

In the curvaton scenario [22–26] there exists a subdominant scalar field during inflation called the curvaton. The curvaton decays into radiation long after the end of inflation whereby its relative energy density has been enhanced considerably with respect to the radiation background produced by the inflaton decay. At this time the perturbations of the curvaton field can be converted into metric fluctuations.

Here we discuss briefly if the Higgs field could act as a curvaton. During inflation the Higgs field is light and subdominant as we have shown in section 4.1. Here we have assumed that inflation is caused by the inflaton field, which oscillates in quadratic potential after the end of inflation. Therefore, in this scenario the background is matter-dominated-like after inflation, and the total energy density decreases as $\rho_{\text{tot.}} \propto a^{-3}$. Higgs oscillations start relatively soon after the end of inflation [2]. Since the Higgs field oscillates in quartic potential its energy density decreases as $\rho_h \propto a^{-4}$. Hence, the ratio of the Higgs energy density to the total one decreases as $\rho_h/\rho_{\text{tot.}} = a^{-1}$ until the inflaton decays into radiation. After the inflaton decay the ratio of the energy densities remains. Anyhow, the energy density of the Higgs field cannot become significant at any time in this scenario. Therefore, the Higgs field perturbations cannot be converted into large enough metric perturbations to be observable, so the Higgs field cannot act as a curvaton [66].

However, if there is a so called kination epoch after the end of inflation, the perturbations of the Higgs field can convert into observable metric fluctuations [67]. The kination phase occurs in inflationary models in which the Universe transfers suddenly from inflation to a kinetic energy dominated phase. Then right after the end of inflation the energy density of the inflaton field decreases as $\rho_\phi \propto a^{-6}$. Thus, the Higgs field energy density can become significant and the perturbations of the Higgs field can convert into metric perturbations at observable level.
Chapter 5

Tensor Perturbations

Here we study the perturbations of the flat FRW metric. This is called the cosmological perturbation theory \[68\]. At the linear order the metric perturbations decouple into scalar, vector and tensor perturbations. The scalar perturbations are responsible of the structure formation of the Universe, and therefore the most important class of the metric perturbations. The vector perturbations decay, and their importance is minor. Here we are especially interested on the tensor perturbations which are gravitational waves (GW). Although GW do not have a significant contribution to the structure formation in the Universe, they can carry valuable information from the epoch when they were produced. Gravitational waves are generated during several early Universe epochs such as inflation, preheating and phase transitions.

The evolution of the metric perturbations is determined by the solutions of their Einstein equations (1.5). For the perturbed metric the solutions of the Einstein equations can be separated into equation of motion (EOM) of the background metric and EOM of the perturbations. For tensor perturbations we find that after production they propagate freely at the speed of light.

5.1 Metric Perturbations

The Universe is well described by the spatially flat Friedman-Robertson-Walker background, as discussed in section 1.1. Here we are interested in small perturbations around the homogeneous background metric. The most general perturbed flat FRW line element can be written in the form \[69–71\]

\[
\begin{align*}
\text{ds}^2 &= a^2(\eta) \left\{ - (1 + 2A)d\eta^2 - 2B_i d\eta dx^i + \left[ (1 + 2D)\delta_{ij} + 2E_{ij} \right] dx^i dx^j \right\}. 
\end{align*}
\]

(5.1)

In the metric \(E_{ij}\) is traceless, \(E_{ii} = 0\), and \(D\) contains the diagonal part of the curvature perturbations. Since the scalar, vector and tensor perturbations are decoupled at the linear order, they can be studied separately.

For the scalar perturbations the metric at the linear order is given by

\[
\begin{align*}
\text{ds}^2 &= a^2(\eta) \left\{ - (1 + 2A)d\eta^2 - 2B_i^S d\eta dx^i + \left[ (1 + 2D)\delta_{ij} + 2E_{ij}^S \right] dx^i dx^j \right\}, 
\end{align*}
\]

(5.2)
where $B_i^S$ is the curl-free part of the vector $B_i$ and $E_{ij}^S = \left( \partial_i \partial_j - \frac{1}{3} g_{ij} \nabla^2 \right) E$ is the scalar perturbation part of $E_{ij}$.

The metric of vector perturbations at the linear order can be written as
\[ ds^2 = a^2(\eta) \left[ -d\eta^2 - 2B_i^V dx^i dx^j + \left( \delta_{ij} + 2E_{ij}^V \right) dx^i dx^j \right], \tag{5.3} \]
where $B_i^V$ is the divergenceless, or in other words, the transverse part of the vector $B_i$ and $E_{ij}^V = \left( \partial_i E_j + \partial_j E_i \right)$ is the vector perturbation part of $E_{ij}$. The vector perturbations have only decaying solutions.

The tensor perturbations corresponding to gravitational waves are the transverse-traceless (TT) degrees of freedom of the metric perturbations. Here the main focus is on the tensor perturbations. After their production the tensor perturbations propagate freely at the speed of light. The metric of tensor perturbations at the linear order is given by
\[ ds^2 = a^2(\eta) \left[ -d\eta^2 + \left( \delta_{ij} + 2E_{ij}^T \right) dx^i dx^j \right], \tag{5.4} \]
where $E_{ij}^T$ is the transverse part of the traceless tensor $E_{ij}$ ($\partial_i E_{ij}^T = 0$ and $E_{ii}^T = 0$). For convenience from now on we will use notation $h_{ij} = 2E_{ij}^T$ which coincide with notation used in [1,3].

The left hand side of the Einstein equations (1.5) for the scalar, vector and tensor perturbations can be solved by computing the Ricci tensor and scalar for the corresponding perturbed metric (5.2), (5.3) or (5.4) by using eqs. (1.6), (1.7) and (1.8).

5.2 Energy Momentum Tensor Perturbations

In the previous section we discussed perturbations of the space-time metric. Here we focus on the perturbations of the energy momentum tensor which depict the fluctuations of the matter content of the Universe. The perturbed energy momentum tensor gives us the right hand side of the Einstein equations (1.5) for the metric perturbations.

The energy momentum tensor can be written in separated form as
\[ T^\mu_\nu = \tilde{T}^\mu_\nu + \delta T^\mu_\nu, \tag{5.5} \]
where $\tilde{T}^\mu_\nu$ is the energy momentum tensor of the background given by eq. (1.9), and $\delta T^\mu_\nu$ contains the perturbations of the energy momentum tensor. $\delta T^\mu_\nu$ is given by
\[ \delta T^\mu_\nu = \begin{pmatrix} -\delta \rho & (\rho + p) (v_i - B_i) \\ - (\rho + p) v_i & \delta \rho \delta^i_j + \Pi_{ij} \end{pmatrix}, \tag{5.6} \]
where $\delta \rho$ and $\delta p$ are the perturbations of the energy density and pressure and $v_i$ is the velocity perturbation. Moreover, $\Pi_{ij}$ is the traceless anisotropic stress tensor, which contains the perturbations that are not included in the perfect fluid description.

The density and pressure perturbations are purely scalar perturbations. The velocity perturbation can be divided into scalar and vector perturbation parts (analogously to $B_i$) as
\[ v_i = v_i^S + v_i^V, \tag{5.7} \]
where \( v^S_i \) is curl-free and \( v^V_i \) is divergenceless (transverse). The anisotropic stress tensor can be separated similarly with \( E_{ij} \) into scalar, vector and tensor perturbation parts as

\[
\Pi_{ij} = \Pi^S_{ij} + \Pi^V_{ij} + \Pi^T_{ij},
\]

(5.8)

where \( \Pi^S_{ij} = \left( \partial_i \partial_j - \frac{1}{3} \delta_{ij} \nabla^2 \right) \Pi \), \( \Pi^V_{ij} = -\frac{1}{2} (\partial_j \Pi_i + \partial_i \Pi_j) \) and \( \Pi^T_{ij} \) is the transverse part of the traceless tensor \( \Pi_{ij} \) i.e. \( \partial_i \Pi^T_{ij} = 0 \) and \( \Pi^T_{ii} = 0 \). The transverse-traceless (TT) tensor perturbation part of the anisotropic stress tensor (\( \Pi^T_{ij} \)) is the source of gravitational waves. In the literature it is often written as \( \Pi^\text{TT}_{ij} \), and from now on we use this convention.

Generally the scalar and vector perturbations are gauge dependent. In the cosmological perturbation theory the gauge refers to the chosen coordinate system of the perturbed metric. There are several such coordinate systems, and coordinate transformations between these are called gauge transformations. However, here the focus is on the tensor perturbations which are gauge-invariant [71]. Therefore, we do not discuss gauge transformations here.

## 5.3 Gravitational Waves

The existence of gravitational waves is a prediction of general relativity. So far GW have not been detected directly. However, there are indirect evidences, such as B-mode polarization of the CMB radiation [6] and decreasing of the rotational frequency of compact binaries [72, 73].

Gravitational waves produced during inflation leave an imprint on the B-mode polarization of the CMB [74] and also GW generated by cosmic defects may leave an imprint [75]. The BICEP2 experiment recently reported about the observation of the CMB B-mode polarization [6]. The Planck Surveyor is expected to publish its measurements of the CMB polarization during 2014, and there are also other polarization optimized ground-based CMB experiments such as PolarBear [76] and QUIET [77].

A compact binary is a gravitationally bound system of two very massive objects e.g. neutron stars. The objects rotate around common center of mass. According to general relativity this kind of system of accelerated masses in an asymmetric distribution generate gravitational waves as the compact binary loses energy. Thereby, the rotational frequency of the system decreases in a manner that has been observed in neutron star systems [72, 73].

In the early Universe there are several possible sources of GW. The quantum fluctuations during inflation generate metric perturbations, including GW [33]. After inflation but before the big bang nucleosynthesis there are several non-equilibrium phenomena which are possible GW sources, such as preheating [78–84], phase transitions [85–96], turbulent motions [97–101] and cosmic defects [102–107].

The lack of GW observations have set an upper bound for the amplitudes of potential GW backgrounds [108–112]. There are several plans for laser interferometric based observatories, such as LIGO, NGO/eLISA, BBO, DECIGO and ET [113–119]. The typical operative frequency for these observatories range from \( \sim 10^{-5} \) Hz to \( \sim 10^{3} \) Hz.

More detailed derivation of the results presented in this section can be found in [3].
The Solution of the Einstein Equations of Gravitational Waves

The perturbed flat FRW line element with the tensor perturbations as the only perturbation is written as

\[ ds^2 = a^2(\eta) \left[ -d\eta^2 + (\delta_{ij} + h_{ij}) \, dx^i \, dx^j \right], \]

with \( a(\eta) \) the scale factor and \( \eta \) the conformal time. The tensor perturbations \( h_{ij} = 2E_{ij}^T \) satisfy the transversality \( (\partial_i h_{ij} = 0) \) and tracelessness \( (h_i^i = 0) \) conditions. For convenience we rescale the tensor perturbations as \( \bar{h}_{ij}(k, \eta) = a(\eta) h_{ij}(k, \eta) \).

By splitting the Einstein equations (1.5) of the perturbed metric (5.9) into the background equations and the linearized tensor perturbation equations, we find that the equations of motion in the momentum space for GW in a flat FRW background at sub-horizon scales \( (k \gg H) \) are given by [109]

\[ \ddot{\bar{h}}_{ij}(k, \eta) + k^2 \bar{h}_{ij}(k, \eta) = 16\pi G a(\eta) \Pi^{TT}_{ij}(k, \eta). \]

The source \( \Pi^{TT}_{ij} \) is the transverse-traceless-part of the anisotropic stress tensor, \( \Pi_{ij} \). It satisfies the transverse \( (\partial_i \Pi^{TT}_{ij} = 0) \) and the tracelessness \( (\Pi^{TT}_{ii} = 0) \) conditions.

The source of GW can be written as

\[ \Pi^{TT}_{ij}(k, \eta) = \Lambda_{ij,lm}(\hat{k}) \Pi_{lm}(k, \eta). \]

where \( \Lambda_{ij,lm}(\hat{k}) \) is the TT-projection operator defined as

\[ \Lambda_{ij,lm}(\hat{k}) = P_{il}(\hat{k})P_{jm}(\hat{k}) - \frac{1}{2} P_{ij}(\hat{k}) P_{lm}(\hat{k}), \]

\[ P_{ij} = \delta_{ij} - \hat{k}_i \hat{k}_j, \quad \hat{k}_i = k_i / k. \]

The operator fulfills the transverse-traceless conditions in the momentum space, \( k_i \Pi^{TT}_{ij}(\hat{k}, \eta) = \Pi^{TT}_{ii}(\hat{k}, \eta) = 0, \) since \( P_{ij} \hat{k}_j = 0 \) and \( P_{ij} P_{jm} = P_{jm}. \)

The solution of eq. (5.10) is given by a Green’s function, \( G(k, \eta - \eta') = \frac{1}{4\pi} \sin(k(\eta - \eta')) \), assuming that at the initial time, \( \eta_I \), there are no gravitational waves, i.e. \( h_{ij}(k, \eta) = h_{ij}'(k, \eta) = 0 \). Thus, we obtain

\[ h_{ij}(k, \eta) = \bar{h}_{ij}(k, \eta) a(\eta) \int_{\eta_I}^{\eta} d\eta' a(\eta') \sin[k(\eta - \eta')] \Pi^{TT}_{ij}(k, \eta'). \]

5.4 The Spectrum of Gravitational Waves

By expanding the Einstein equations (1.5) up to the second order in the tensor perturbations, we recognize that the energy density of a GW background is given by [109]

\[ \rho_{GW}(\eta) = \frac{1}{32\pi G a^2(\eta)} \left\langle \bar{h}_{ij}(x, \eta) h_{ij}'(x, \eta) \right\rangle_V, \]

where \( \langle \ldots \rangle_V \) is the spatial average over the volume \( V \) \( (V^{1/3} \gg \lambda_*) \) containing all the relevant wavelengths \( \lambda_* \) of the tensor perturbations, \( h_{ij} \). Here the source of gravitational waves is stochastic.
and thereby also the spatial distribution of the produced GW is stochastic. In such circumstances, according to the Ergodic hypothesis, the spatial average \( \langle \ldots \rangle_V \) can be replaced by the ensemble average \( \langle \ldots \rangle \) [3].

By substituting the solution (5.13) into the eq. (5.14) we obtain the GW energy density spectrum per logarithmic interval as

\[
\frac{d\rho_{GW}}{d \log k}(k, \eta) = \frac{2}{\pi} \frac{G k^3}{a(\eta)} \int_{\eta_s}^{\eta} d\eta' a(\eta') \cos[k(\eta' - \eta'')] \Pi^2(k, \eta, \eta''),
\]

where \( \Pi^2(k, \eta, \eta') \) is the unequal-time-correlator (UTC) defined by

\[
\langle \Pi^T_{ij}(k, \eta)\Pi^T_{ij}(k', \eta') \rangle \equiv (2\pi)^3 \Pi^2(k, \eta, \eta'') \delta^{(3)}(k - k').
\]

After production the gravitational waves propagate freely. Thus, the GW energy density spectrum today can be computed from the spectrum right after the production by simply redshifting the frequency and amplitude. The frequency and amplitude of the GW spectrum today are given as [3]

\[
f \equiv \frac{a_* k}{a_0} \approx \left( \frac{a_*}{a_{RD}} \right)^{(1-3w)/4} \left( \frac{a_*}{a_*} \right) \left( \frac{k}{\rho_*} \right) \times 5 \cdot 10^{10} \text{Hz},
\]

where subscript \( I \) denote the initial time without GW, \( * \) indicate the moment when GW production ends, RD points out the time when the Universe became radiation dominated and finally 0 refers to today. In eq. (5.17) \( h^2 \Omega_{GW} \approx 4 \cdot 10^{-5} \), and the ratio of the number of relativistic degrees of freedom today to those active at end of GW production is \( (g_0/g_*)^{1/3} \sim \mathcal{O}(0.1) \), so the total prefactor is approximately \( h^2 \Omega_{rad} (\frac{g_0}{g_*})^{1/3} \sim \mathcal{O}(10^{-6}) \). This formalism applies to any stochastic GW source in a flat FRW background, characterized by its UTC, \( \Pi^2(k, \eta, \eta') \). In the next chapter we use this formalism to compute the GW spectrum produced in bosonic and fermionic preheating.
Chapter 6

Gravitational Waves from Preheating

As mentioned before, there are several mechanisms in the early Universe in which gravitational waves are possibly produced. Here we focus on the gravitational wave production from preheating after inflation. The bosonic and fermionic preheating scenarios were described in the chapter 2. The particle distribution produced during preheating have non-zero anisotropic stress tensor which is the source of gravitational waves.

First we discuss the GW production from the bosonic preheating. This scenario has been studied in great detail previously [78–84]. It is helpful for understanding to first consider the bosonic preheating scenario, because it is computationally simpler and better-known than GW production from fermionic preheating.

However, here the main focus is on the GW production from the fermionic preheating. The analysis of this part is based on the papers [1,3]. In [1] we computed for the first time the spectrum of GW produced by fermionic preheating. In [3] we have addressed the derivation of the results in great detail and explored systematically the range of the relevant parameters for several scenarios.

6.1 Gravitational Waves from Bosonic Preheating

The GW production from bosonic preheating has been studied previously by several groups [78–84]. In the basic model a coherently oscillating scalar field decays into other bosonic degrees of freedom - typically into other scalar field(s). Here we assume that the decay is into another scalar field, denoted as \( \mathcal{X} \). The non-perturbatively produced boson distribution has a non-zero TT-part of the anisotropic stress tensor, which is then the source of GW. The analytic calculations of GW production for the bosonic case are less complicated than for the fermionic preheating scenario. In addition, there is no need for regularization of the outcome in the bosonic preheating scenario, whereas in the fermionic preheating scenario regularization is required, as will be discussed in section 6.3.

The source of gravitational waves is the TT-part of the anisotropic stress tensor, \( \Pi^{TT}_{ij} \). For a scalar field, \( \mathcal{X} \), it is given by

\[
\Pi^{TT}_{ij} = \left( \partial_i\mathcal{X} \partial_j\mathcal{X} \right)^{TT}.
\] (6.1)
By substituting the quantized form of the scalar field $\chi'$ given in eq. (2.7) into eq. (6.1) we find the TT-part of the anisotropic stress tensor in the momentum space, $\Pi_{ij}^{TT}(k, \eta)$. Then we can find the spectrum of GW at the time of production by substituting $\Pi_{ij}^{TT}(k, \eta)$ in eq. (5.15) to obtain

$$\frac{d\rho_{GW}}{d\log k} (k, \eta) = \frac{Gk^3}{2\pi^3a^4(\eta)} \int dp d\theta p^6 \sin^5 \theta \left( |I_{(c)}(k, p, \theta, \eta)|^2 + |I_{(s)}(k, p, \theta, \eta)|^2 \right),$$

where we have defined

$$I_{(c)}(k, p, \theta, \eta) = \int_{\eta_I}^{\eta} \frac{dn}{a(\eta')} \cos(k\eta') \chi_{k-p}(\eta') \chi_p(\eta'),$$

$$I_{(s)}(k, p, \theta, \eta) = \int_{\eta_I}^{\eta} \frac{dn}{a(\eta')} \sin(k\eta') \chi_{k-p}(\eta') \chi_p(\eta'),$$

where $\eta_I$ is the initial time with no gravitational waves. By using formulas (6.2) and (6.3) we can compute the spectrum of GW right after production. The present frequency and amplitude of the GW spectrum produced by bosonic preheating can then be calculated by using eq. (5.17).

### 6.2 Gravitational Waves from Fermionic Preheating

In this section we describe the formalism to calculate the spectrum of GW created by fermions developed in [1, 3].

The spatial-spatial components of the energy-momentum tensor of a spin-$\frac{1}{2}$ fermion field are given by

$$T_{ij} = i \frac{a}{2} \left[ \bar{\Psi} \gamma_i(\hat{D}_j) \Psi - \bar{\Psi} \hat{D}_i(\gamma_j) \Psi \right],$$

where $D_\mu$ is the covariant derivative$^1$.

The source of GW is the TT-part of the fermions anisotropic stress tensor corresponding the TT-part of the fermions energy-momentum tensor,

$$\Pi_{ij}^{TT}(k, \eta) = \Lambda_{ij,lm}(\hat{k})T_{lm}(k, \eta).$$

By substituting the quantized decomposition given in eq. (2.18) into eq. (6.4), we obtain the TT-part of the anisotropic stress tensor in the momentum space, eq. (6.5), as

$$\Pi_{ij}^{TT}(k, \eta) = \frac{\Lambda_{ij,lm}(\hat{k})}{a^2(\eta)} \int \frac{dp^3}{(2\pi)^3} \left( \hat{b}_{-p,s} \bar{v}_{p,s} + \hat{a}^\dagger_{p,s} \bar{u}_{p,s} \right) p(l(\gamma_m)) \left( \hat{a}_{k+p,r} \bar{u}_{k+p,r} + \hat{b}^\dagger_{-(k+p),s} \bar{v}_{k+p,r} \right).$$

Next we need to compute the UTC, $\Pi^2(k,t,t')$ defined in eq. (5.16). By substituting eq. (6.6) into eq. (5.16) we find 16 different quadrilinear combinations of the creation and annihilation operators. Of these, only the following one contribute to $\Pi^2(k,t,t')$:

$$\langle 0 | \hat{b}_{-p,s} \hat{a}_{k+p,r} \hat{a}^\dagger_{q,s} \hat{b}^\dagger_{k'-q,r'} |0 \rangle = (2\pi)^6 \delta^{(3)}(k + p - q)\delta^{(3)}(k - k')\delta_{s,r}\delta_{r',s'}.$$
6.3 UV Divergence - Regularization

Now we can find \( \Pi^2(k, \eta, \eta') \) as a function of the 4-spinors \( u_{p,\pm} \) and \( v_{p,\pm} \) by using eqs. (5.16), (6.6) and (6.7). We also use the property \( \Lambda_{ij,pq}(\hat{k})\Lambda_{pq,lm}(\hat{k}) = \Lambda_{ij,lm}(\hat{k}) \) together with eq. (2.22) to relate the mode functions \( v_{k,\pm} \) with \( u_{k,\pm} \). We then obtain

\[
\Pi^2(k, \eta, \eta') \equiv \frac{1}{2 \pi^2 a^2(\eta)a^2(\eta')} \int dp \, d\theta \, p^4 \sin^3 \theta \, W_{k,p}(\eta)W^*_{k,p}(\eta'),
\]

where we have defined

\[
W_{k,p} \equiv u_{k-p,+,u_{p,+}} - u_{k-p,-,u_{p,-}}.
\]

Finally, by substituting eq. (6.8) into eq. (5.15), we find that the spectrum of GW produced by fermions is given by

\[
\frac{d\rho_{GW}(k, \eta)}{d \log k} = \frac{Gk^3}{\pi^3 a^4} \int dp \, d\theta \, p^4 \sin^3 \theta \left( |I(c)(k, p, \theta, \eta)|^2 + |I(s)(k, p, \theta, \eta)|^2 \right),
\]

where

\[
I(c)(k, p, \theta, \eta) \equiv \int_{\eta_I}^{\eta} \frac{d\eta'}{a(\eta')} \cos(k\eta')W_{k,p}(\eta'),
\]

\[
I(s)(k, p, \theta, \eta) \equiv \int_{\eta_I}^{\eta} \frac{d\eta'}{a(\eta')} \sin(k\eta')W_{k,p}(\eta').
\]

The structure of the formulas in eqs. (6.10)-(6.11) and (6.9) resembles that of the scalar field sourcing GW, i.e. eqs. (6.2)-(6.3). Eqs. (6.9)-(6.11) are the master formulas to calculate the spectrum of GW produced by an asymmetric fermion distribution generated during preheating. They were first presented in [1]. However, this result cannot be used straightforwardly to calculate the spectrum of GW produced by fermions, as it must be properly regularized and the formulas reformulated, as explained in detail in [3].

6.3 UV Divergence - Regularization

The calculation presented above leads to a UV divergence in the momentum integral in eq. (6.10). In order to obtain a physical amplitude of the produced GW spectrum we have to regularize the result. This means that we have to remove the contribution of vacuum fluctuations which lead to the divergence. We have used the time-dependent normal-ordering scheme for the regularization. It resembles the ordinary normal-ordering, but here the background is time-dependent and the fermionic vacuum state varies in time. Therefore, we have to use the time-dependent equivalent. The regularized vacuum expectation value of an operator \( \mathcal{O}(\eta) \) in the time-dependent normal-ordering prescription is given as

\[
\langle \mathcal{O}(\eta) \rangle_{\text{reg}} \equiv \langle 0 | \mathcal{O}(\eta) | 0 \rangle - \langle 0_{\eta_I} | \mathcal{O}(\eta) | 0_{\eta} \rangle,
\]

where \( |0\rangle \) is the fermionic vacuum state initially \( (\eta = \eta_I) \) and \( |0_{\eta}\rangle \) is the vacuum state at time \( \eta \).

In section 5.3, where we calculated the spectrum of gravitational waves, we encountered in eq. (5.16) the vacuum expectation value of the operator \( \Pi^T_{ij}(k, \eta)\Pi^T_{ij}(k', \eta') \), which depends on two time parameters \( \eta \) and \( \eta' \). In eq. (6.12) the operator \( \mathcal{O}(\eta) \) depends only on one time parameter
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Therefore, eq. (6.12) cannot be used straightforwardly for regularization in the present situation. In [3] we have described in great detail how to make use of the time-dependent normal-ordering procedure to calculate the regularized vacuum expectation value of the unequal-time correlator,

\[
\left\langle \Pi_{ij}^{TT}(k, \eta) \Pi_{ij}^{TT}(k', \eta') \right\rangle_{\text{reg}}.
\] (6.13)

In order to remove the UV divergence of the GW spectrum we need to regularize all the formulas (6.10)-(6.11) and (6.9). In [3] we have argued that the regularized form can be obtained by the substitution

\[ u_{p,\pm} \rightarrow \tilde{u}_{p,\pm} \equiv \sqrt{2} |\beta_p| u_{p,\pm}, \] (6.14)

which leads to the replacement

\[ W_{k,p} \rightarrow \tilde{W}_{k,p} \equiv \tilde{u}_{k-p,\pm} \tilde{u}_{p,\pm} - \tilde{u}_{k-p,-} \tilde{u}_{p,-} = 2|\beta_p||\beta_{k-p}| W_{k,p}. \] (6.15)

In this framework the issue of regularization is very complicated. In the first publication [1], we used a slightly different approach than in [3]. The results were however qualitatively similar in both cases. The problem is also numerically very challenging, since the formulas contain integration of products of very rapidly oscillating functions, see eq. (6.11).

The spectrum of gravitational waves right after production can then be calculated by using eqs. (6.10) and (6.11), where \( W_{k,p} \) is replaced with its regularized equivalent according to eq. (6.15). The frequency and amplitude of the GW spectrum today produced by the fermion distribution generated during preheating can then be computed by using eq. (5.17).

### 6.4 Observational Prospects

In [3] we studied several early Universe scenarios in which a homogeneous oscillating scalar field decays non-perturbatively into fermions. In all the scenarios we performed a systematic survey of the parameters involved. We considered GW production from decay of the inflaton with quartic or quadratic potentials into fermions. We also studied the fermionic decay of a spectator field oscillating in a quadratic potential while the background is radiation dominated. In each of the scenarios we calculated numerically the GW spectrum by using eqs. (6.10), (6.11) and (6.15) and accounting for a wide range of parameters. The frequency and amplitude of the GW spectrum today were then evaluated by using eq. (5.17).

The fermions produced during preheating from a coherently oscillating scalar field will fill up Fermi-sphere with the radius \( k \propto q^{1/4} \), as mentioned in chapter (2.4). We discovered that the peak of the produced GW spectrum is located in the vicinity of the Fermi-sphere radius. The peak position can be understood, because the integrand in eq. (6.10) increases with the internal momentum \( p \) as \( p^4 \), whereas outside the Fermi-sphere the fermions are not excited substantially. Hence, the integrand grows up to the radius of the Fermi-sphere, and after that drops abruptly. As a consequence, the GW spectrum is peaked close to the radius of the Fermi-sphere.
Figure 6.1: The spectrum of GW from fermionic preheating right after the end of production. Here the inflaton oscillates coherently in quartic potential and decays non-perturbatively via preheating into fermions. The parameters adopted are $q = 10^6$, $h = 0.1$ and initial energy scale $(\rho_I)^{1/4} = E_I \approx 6.0 \cdot 10^{16}\text{GeV}$. The dashed line is a rough fitting, and it depicts the peaked shape of the GW spectrum. Due to the very rapidly oscillating functions in the integrand the computations suffered from numerical inaccuracy. The scattering of the computed points in the spectrum and relatively slow decrease of the spectrum in the UV tail are likely due to this inaccuracy. Essentially this figure shows that the GW spectrum is peaked at $\kappa = k \sqrt{\lambda \Phi_I} \sim q^{1/4}$, as expected.

Unfortunately, we discovered that in each of the scenarios considered the GW spectrum is naturally peaked at very high frequencies. For example, for the scenario in which inflaton oscillates in a quartic potential and decays into fermions we found that for parameters $q = 10^6$, $h = 0.1$ and initial energy scale $(\rho_I)^{1/4} = E_I \approx 6.0 \cdot 10^{16}\text{GeV}$, the GW spectrum is peaked at $k_{\text{peak}} \simeq 2 q^{1/4} \sqrt{\lambda \Phi_I}$ and has the amplitude $\Omega_{\text{GW}}^{(\ast)}(k_{\text{peak}}) \simeq 3 \cdot 10^{-10}$, as can be seen from figure (6.1). As expected, the peak is located in the vicinity of the radius of the Fermi-sphere. The corresponding frequency and amplitude of the GW spectrum peak today are $f_{\text{peak}} \sim 10^{10}\text{Hz}$ and $h^2 \Omega_{\text{GW}}(f_{\text{peak}}) \sim 10^{-15}$.

The observational range of the planned GW observatories is typically from $f \sim 10^{-5}\text{Hz}$ to $f \sim 10^3\text{Hz}$, hence the peak of the spectrum of GW in the example above has a significantly higher frequency than can be observed by the planned experiments. In principle, fermionic preheating can produce GW spectrum with a frequency in the observable range of the planned observatories, but then the parameters turn out to be such that the amplitude of GW is extremely small. In all the scenarios studied we have found qualitatively similar results as in the example above. That is, if the GW spectrum has a significant amplitude, then it is peaked at very high frequency, and therefore will not be detected by the planned GW observatories. In contrast, if the produced spectrum of GW has a frequency within the range of the planned observatories, then the amplitude of GW spectrum turns out to be extremely small and therefore unobservable. We can only hope that eventually GW observatories operating at high frequencies will be developed.
Chapter 7

Discussion

In this thesis we have studied the evolution and decay of scalar fields during inflation and preheating. We have assumed that inflation is caused by a slowly rolling scalar field called the inflaton. The thesis can be divided into two main topics. First we studied the generation of the Higgs condensate during inflation, and its non-perturbative decay via preheating. We also discussed the possible observational consequences of the Higgs field dynamics in the early Universe. Secondly we studied gravitational wave production from the non-perturbative decay of a scalar field into fermions via fermionic preheating.

In chapter (4) we computed the average amplitude and effective mass of the Higgs condensate generated during inflation by using the stochastic approach. We showed that the Higgs field is a light slowly rolling spectator field during inflation. After the end of inflation the Higgs field becomes massive and starts to oscillate. Then it decays non-perturbatively into the weak gauge bosons via bosonic preheating in a few hundreds of Hubble times. Perturbations of the Higgs condensate can contribute to the metric perturbations if for example the Higgs field acts as a curvaton, or if the Higgs field modulates the reheating.

In chapter (6) we computed the gravitational wave background produced by the non-perturbative decay of a coherently oscillating scalar field into fermions. We showed that this fermionic preheating mechanism can produce a significant gravitational wave background. We discussed about the gravitational wave production from the non-perturbative decay of the inflaton into fermions when the inflaton oscillates in quadratic or quartic potential. We also considered gravitational wave production in the case where a subdominant field in a radiation dominated background oscillates in a quadratic potential and decays into fermions non-perturbatively.

The fermion distribution produced during fermionic preheating has non-zero transverse-traceless part of the anisotropic stress tensor, which is the source of gravitational waves. The fermions produced during fermionic preheating fill up the Fermi-sphere with a radius of $k_F \propto q^{1/4}$, where $q$ is the resonant parameter. We have found that the produced gravitational wave spectrum is peaked close to the Fermi-sphere radius. In each of the scenarios considered we unfortunately found that the produced gravitational wave spectrum with a significant amplitude has a very high frequency, and therefore the planned gravitational wave observatories will not be able to detect this signal. If
the GW spectrum frequency is peaked within the range of the planned gravitational wave detectors, the amplitude turns out to be at a negligible level.
Bibliography


