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Abstract

This paper considers equilibrium behavior in a descending price auction with two players that are asymmetrically informed. The "informed" player knows his valuation while the other does not. The uninformed player can acquire information about his valuation with a positive cost during the auction. We assume that the information acquisition activity is covert and we characterize the equilibrium behavior in the setting where players' valuations are independently and identically distributed. We derive the explicit "inverse bid" functions in the case of the uniformly distributed valuations and provide a revenue comparison between the ascending and descending price auctions in this case.

Keywords: Asymmetric auctions, information acquisition

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1 Introduction

It is commonly assumed in auction theory that the players that participate in the auction know their valuation of the object for sale. However, there are many instances where some of the participants may not know their valuation for the object. For instance, a license auction where an incumbent knows the value of the production license while the entrant may not have this knowledge, is an example that fits the setting considered in this paper. It may be the case that the incentives to acquire information are not satisfied in a sealed bid version of the auction. However, lowering the price may induce more interest from the uninformed player if information acquisition is possible during the auction. A firm that is not initially interested in the licence may want to assess how much the license is worth as the price is lowered enough.

In this paper we study the descending price auction in the case where the players are asymmetrically informed about their valuation for the object. We assume that one player is initially informed about his valuation while the other is not. We assume that the uninformed player can acquire information covertly and we characterize the asymmetric equilibrium. We provide closed form inverse bidding functions in the case of uniformly distributed valuations and compare revenues from the descending and ascending auctions.

1.1 Related literature

Papers that study information acquisition during an auction are by Compte and Jehiel [3], Rezende [15], Gretschko and Wambach [5] and Miettinen [12]. Compte and Jehiel [3] were the first to consider information acquisition during an English auction. They partially characterize the equilibrium strategies in an independent private values model and show that the revenue from an English auction is greater than from a sealed bid second price auction. Rezende [15] studies the English auction in a model with affiliated values. Like Compte and Jehiel he also finds that the English auction is superior to the second price auction. Wambach and Gretschko [5] and Miettinen [12] study information acquisition during a descending price auction in a symmetric setting with private values. Wambach and Gretschko assume that the players observe a noisy signal about their valuation and can learn their accurate valuation by incurring a cost that is private information for the players. They find that when the cost of information acquisition is sufficiently low, the first price auction generates more revenue than the descending price auction and that the descending price auction is more efficient than the first price auction. Miettinen [12] studies a setting where the information acquisition cost is the same for all players. The players may be informed or uninformed at the outset and they do not know whether other players are initially informed. He derives explicit bidding strategies and shows that the descending price auction produces more revenue than the first price auction if the number of players is sufficiently large.

These results contrast with the revenue equivalence theorem in the case of independent private values and symmetric players. In addition, the equilibrium strategies derived in Gretschko and Wambach [5] and Miettinen [12] show how the assumption about information acquisition can drive a wedge between the first price and descending price auctions that are strategically equivalent when information acquisition is not allowed during the auction. Other related work on information acquisition in static auctions and mechanism design are by Milgrom [13], Persico [14] and Bergemann and Välimäki [1].

This paper is organized as follows. We first provide some intuition for the reader about the equilibrium behavior in section 2 and then delve into the construction of the equilibrium in section 3. In section 3.3 we construct an example of the equilibrium strategies with uniformly distributed valuations and in section 3.4 we use the example to discuss the incentives to acquire information. We compare revenues from the ascending and descending price auctions in section 3.5.

2 A sketch of the equilibrium

It is useful to study the equilibrium behavior in two phases. In the first phase - when the price is relatively high - the uninformed decides about information acquisition. If he discovers that his valuation is above a threshold w_2, he buys the object at the current price. Otherwise he waits for the price to descend further. In the second phase - with lower prices - it is common knowledge in equilibrium that information has been acquired and bidding is equivalent to an asymmetric first price auction.[1] The informed player uses a pure strategy in both phases.

In the first phase the uninformed player must randomize his decision to acquire information over an interval of prices[2] while the informed player bids according to a pure strategy. This complicates the analysis of the first phase since we need to find a bidding strategy for the informed players that makes the uninformed player indifferent about acquiring at any of the prices in the interval. At the same, time the uninformed player must randomize the information acquisition decision such that the informed player's best response is the one that makes the uninformed indifferent. Furthermore, the information

[1] Unlike in Miettinen [12] there is no initial phase where only the informed player with high valuations bid. This is due to the fact that there is only one informed player.

[2] A pure strategy (or a single point) is ruled out by the argument that the informed player would pre-empt the uninformed player's actions. This produces a mass point in the distribution of informed player's bids and provides a profitable deviation for the uninformed player.

acquisition interval must have a lower bound that is strictly above zero.[3] The phase 1 ”inverse bidding functions” are illustrated in the case of a uniform distribution in figure 1 below. Here prices to the right of the vertical line represent the behavior in phase 1. Note that since the uninformed player is using a mixed strategy when the price is in this range, there is no clear inverse bidding function for the uninformed player.[4]

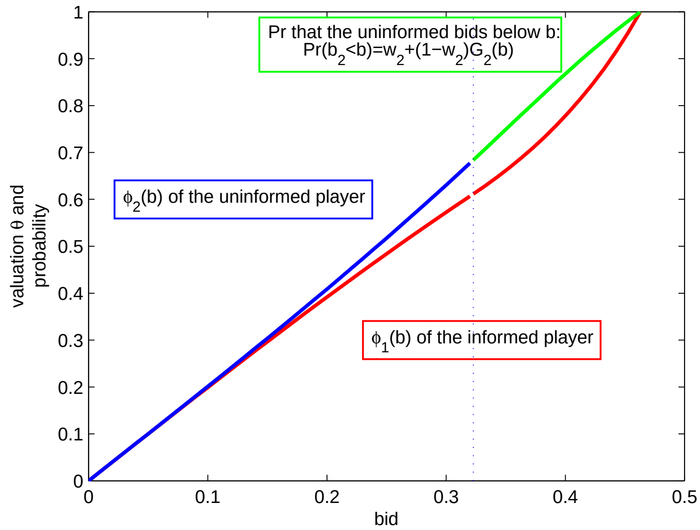

Figure 1: Inverse bidding functions and the distribution of bids for the informed and uninformed players

In equilibrium, both players are informed if the second phase is reached. The bidding in the second phase resembles an asymmetric first price auction where the valuations for the uninformed player are stretched. This case is studied by Maskin and Riley [8] and the contribution of this paper is characterizing the bidding behavior in the first phase, where also the valuation

[3]Otherwise the uninformed equilibrium payoff would need to be zero or we would need to have a mass point at the zero.

[4]The green section represents the distribution of bids induced by the mixed strategy to become informed and buy immediately if the value discovered exceeds w_2.

ranges for the second phase are endogenously determined.

The uninformed player's threshold valuation w_2 is pinned down by the information acquisition cost, while the informed player's threshold w_1 is determined with a technical condition for the distribution of the uninformed player's mixed strategy distribution.[5] We can show that $w_1 < w_2$ holds whenever the information acquisition cost is low. The intuition for this is that the informed player is using his information advantage to postpone his bidding and trades off lower price with a lower probability of winning the item. However, in order to capitalize on his information advantage he bids more aggressively once the bidding starts. More aggressive bidding results in the lowest type for the informed player that bids in the information acquisition range to be smaller than the threshold valuation w_2 for the uninformed player. Once the price reaches the lower bound of the information acquisition range $\underline{b}$, the uninformed player has acquired information. In equilibrium, it is then common knowledge that the informed player has a lower valuation in expectation which makes him bid more aggressively than the (initially) uninformed player also in the second phase. I.e. for any given valuation $t \in [0, w_1)$ player 1's bid is higher than that of player 2. The phase 2 inverse bidding functions are illustrated in figure 1 to the left of the vertical line.

3 The model

We consider a descending price auction with two players. Each player's valuation θ is independently distributed on $[0, 1]$ according to an absolutely continuous distribution function $F(\cdot)$ with a density $f(x) > 0$ for all $x \in [0, 1]$. We denote the valuation of player 1 by θ and the valuation for player 2 by

[5] We find a value w_1 such that the mixed strategy distribution for the uninformed player equals 1 at the high end of the bidding range $\bar{b}$. The threshold w_1 is only formally defined in equation 9 .

ω. Player 1 is informed about his valuation θ whereas player 2 knows only the distribution from which his valuation is drawn and can thus calculate his expected valuation. Player 2 can choose to become informed about his valuation at any point during the auction by incurring a cost of $c > 0$. If player 2 decides to acquire information he learns the true value of ω. We assume that player 1 cannot observe if player 2 has acquired information or not.

We analyze the descending price auction where the auctioneer begins with a high asking price which is lowered until some participant announces his willingness to buy the object at the current price. This participant wins the auction and pays the current price.

Let us start with defining a function $u^a(p)$ that gives the expected value of becoming informed (for free) and having the opportunity to buy the object at price p.

$$u^a(p) = \int_p^{\bar{\theta}} (\omega - p) f(\omega) d\omega.$$

We need to assume that the information acquisition cost c is not too large:

Assumption 1. *The information acquisition cost $c > 0$ satisfies*

$$c < u^a(v), \tag{1}$$

where $v = \mathbb{E}[\omega]$ denotes the expected valuation of the uninformed bidder. Let us define w_* as a threshold valuation that satisfies $c = u^a(w_*)$. Assumption 1 guarantees that w_* exist and that $v < w_*$.

We characterize an asymmetric equilibrium where the uninformed player randomizes his decision to acquire information over an interval $[\underline{b}, \bar{b}]$ and the informed player uses a pure strategy. Once the price reaches $\underline{b}$, in equilibrium, it is common knowledge that both players are informed about their valuations and that the valuations of players' 1 and 2 respectively are distributed on $[0, w_1]$

and $[0, w_2]$, with $1 > w_2 > w_1$. If at price $\underline{b}$ the auction has not ended, the bidding continues as in an asymmetric first price auction. We denote the bid function of player i by $\beta_i(\cdot)$ and the inverse bid function with $\phi_i(\cdot) \equiv \beta_i^{-1}(\cdot)$, for $i \in \{1, 2\}$.[6]

Let us first define

$$u_2(\omega, \hat{\omega}) = \begin{cases} (\omega - \beta_2(\hat{\omega}))F(\phi_1(\beta_2(\hat{\omega}))), \text{ for } \omega < w_2 \\ (\omega - \tilde{\beta}_2(\hat{\omega}))H_1(\hat{\omega}), \text{ for } \omega \geq w_2 \end{cases}$$

as the utility of player 2 with type ω that behaves like a type $\hat{\omega}$ player and where $H_1(\hat{\omega}) = \Big(F(w_1) + (1 - F(w_1))G_1(\tilde{\beta}_2(\hat{\omega}))\Big)$. The informed player's bid is surely below a price $\tilde{\beta}_2(\hat{\omega}) \in [\underline{b}, \bar{b}]$ if his valuation is below w_1. If his valuation is above w_1 then he nevertheless bids below $\beta_2(\hat{\omega})$ with probability $(1 - F(w_1))G_1(\tilde{\beta}_2(\hat{\omega}))$, where $G_1(\cdot)$ is the distribution of the informed player's bids conditional on $\theta \geq w_1$.

In analogous fashion we define

$$u_1(\theta, \hat{\theta}) = \begin{cases} (\theta - \beta_1(\hat{\theta}))F(\phi_2(\beta_1(\hat{\theta}))), \text{ for } \theta < w_1 \\ (\theta - \tilde{\beta}_1(\hat{\theta}))H_2(\hat{\theta}), \text{ for } \theta \geq w_1 \end{cases}$$

where $H_2(\hat{\theta}) = \Big(F(w_2) + (1 - F(w_2))G_2(\tilde{\beta}_1(\hat{\theta}))\Big)$. The uninformed player's bid is below a price $\tilde{\beta}_1(\hat{\theta}) \in [\underline{b}, \bar{b}]$ if his valuation is below w_2. The uninformed chooses the point at which he acquires information according to $G_2(\cdot)$. Conditional on his valuation being above w_2 he bids below $\beta_1(\hat{\theta})$ with probability $(1 - F(w_2))G_2(\tilde{\beta}_1(\hat{\theta}))$.

3.1 Equilibrium: phase 2, $p < \underline{b}$

Consider the situation where the price has descended to $\underline{b}$. If the players abide by the proposed strategies they are both informed and their valuations are

[6] The bid functions $\beta_i(\cdot)$ refer to bidding in phase 2 while $\tilde{\beta}_i(\cdot)$ refers to bidding in phase 1.

distributed as

$$J_1(\theta) = \frac{F(\theta)}{F(w_1)}$$
$$J_2(\omega) = \frac{F(\omega)}{F(w_2)},$$

Notice that the distribution of valuations of the uninformed player 2 stochastically dominates the distribution of the informed player 1 valuations.

$$J_1(x) = \frac{F(x)}{F(w_1)} > \frac{F(x)}{F(w_2)} = J_2(x), \text{ for all } x \in [0, w_1].$$

Now we have that the "conditional stochastic dominance" as defined by Maskin and Riley [8] with $\gamma = w_1$ and $\lambda = \frac{F(w_1)}{F(w_2)} < 1$ is satisfied. [7] Then conditional on $p < \underline{b}$ the equilibrium bid functions are characterized with the following system of differential equations

$$\phi_j'(p) = \frac{J_j(\phi_j(p))}{J_j'(\phi_j(p))} \frac{1}{\phi_i(p) - p}, \text{ for } i, j \in \{1, 2\}, i \neq j,$$

with the boundary conditions $\phi_i(0) = 0$ and $\phi_i(\underline{b}) = w_i$ for $i = 1, 2$. Maskin and Riley [9] show that a solution to this system of differential equations exists. The bid functions $\beta_i(\cdot)$ that appear in the definition of $u_i(\cdot, \cdot)$ are determined with the inverse bid functions $\phi_i(\cdot)$ for $i \in \{1, 2\}$ respectively.

3.2 Equilibrium: phase 1, $p \geq \underline{b}$

We now analyze the stage where the uninformed acquires information. The uninformed mixes his decision to acquire information over an interval $[\underline{b}, \bar{b}]$. If it turns out that $\omega > w_2$, then the uninformed will buy immediately, otherwise he waits and bids according to $\omega = \phi_2(p)$. The informed player uses a pure strategy on the interval $[\underline{b}, \bar{b}]$, that makes the uninformed willing to mix and correspondingly the uninformed mixture is such that the pure strategy of the

[7] For $x \in (w_1, w_2]$ we have that $\frac{J_2'(x)}{J_2(x)} = \frac{f(x)}{F(x)} > 0 = \frac{J_1'(x)}{J_1(x)}$. Here $J_1(x)$ is taken to be equal to 1 when $x \in (w_1, w_2]$.

informed player is indeed optimal. We denote the expected equilibrium payoff to the uninformed player by W. We define $U_2(\omega) = u_2(\omega, \omega)$. For every $b = \tilde{\beta}_2(\omega) \in [\underline{b}, \bar{b}]$ we must have that the expected utility from information acquisition is equal to[8]

$$W = \int_0^{w_2} U_2(y) f(y) dy + \int_{w_2}^1 u_2(y, \omega) f(y) dy - cH_1(\omega). \tag{2}$$

Substituting $p = \tilde{\beta}_2(\omega)$ we solve for $G_1(p)$ from (2) and the boundary condition $G_1(\underline{b}) = 0$ gives us

$$G_1(p) = \frac{F(w_1)}{1 - F(w_1)} \left(\frac{\int_{w_2}^1 (y - \underline{b}) f(y) dy - c}{\int_{w_2}^1 (y - p) f(y) dy - c} - 1 \right). \tag{3}$$

The uninformed player that has acquired information at some price $p \in [\underline{b}, \bar{b}]$ needs to find the optimal price to bid. The player with valuation $\tilde{\omega}$ solves

$$\max_{p'} (\tilde{\omega} - p') \Big(F(w_1) + \big(1 - F(w_1)\big) G_1(p') \Big). \tag{4}$$

Calculating the first order condition and substituting for G_1 (and g_1) from (3) we get a condition

$$\int_{w_2}^1 (y - \tilde{\omega}) f(y) dy = c, \tag{5}$$

which can be satisfied by our assumption 1 with $w_2 = \tilde{\omega} = w^*$. This allows us to determine w_2 from (5) by imposing that $w_2 = w^*$.

The condition for the threshold w_2 in 5 allows us to rewrite $G_1(\cdot)$ as

$$G_1(p) = \frac{F(w_1)}{1 - F(w_1)} \left(\frac{w_2 - \underline{b}}{w_2 - p} - 1 \right),$$

[8]We are abusing notation here slightly as we are calculating the expected utility. The first term in (2) reads $\mathbb{E}[(t - \beta_2(t)) J_1(\phi_1(\beta_2(t))) \| \ t < w_2] F(w_1) F(w_2)$, but the denominator terms cancel out with $F(w_1) F(w_2)$.

and we can rewrite the problem for player with valuation $\tilde{\omega}$ (4) as

$$(\tilde{\omega} - p')\Big(F(w_1) + \big(1 - F(w_1)\big)G_1(p')\Big) = \frac{(\tilde{\omega} - p')(w_2 - \underline{b})}{w_2 - p'}F(w_1).$$

This allows us to observe that when $\tilde{\omega} > w_2$ then the expected utility is increasing in price. Therefore, the uninformed player that acquires information at p and observes a valuation $\omega > w_2$ buys the object at the price p. If $\omega < w_2$ then the expected utility is decreasing in price. So, the uninformed player that acquires information at p and observes a valuation $\omega < w_2$ waits and bids $\beta_2(\omega) < \underline{b}$.

Now we can determine $\bar{b}$, from the requirement that $G_1(\bar{b}) = 1$ and get that

$$\bar{b} = \underline{b}F(w_1) + (1 - F(w_1))w_2.$$

Notice that

$$F(w_1) + (1 - F(w_1)G_1(p) = \Pr(b_1 \leq p) = \Pr(\theta_1 \leq \tilde{\phi}_1(p)) = F(\tilde{\phi}_1(p)),$$

where $\tilde{\phi}_1(p)$ is the inverse of player 1's bidding function when $p > \underline{b}$. Naturally from the requirement that $\tilde{\phi}_1(p) = \theta$, we can solve the bid function $\tilde{\beta}_1(\theta)$ for player 1 on the domain $\theta \in [w_1, 1]$. In order to do this we need to evaluate the quantile function $F^{-1}(\cdot)$ of player 1's valuations. The inverse bidding function conditional on $p > \underline{b}$ is

$$\begin{aligned}\tilde{\phi}_1(p) =& F^{-1}\big(F(w_1) + (1 - F(w_1)G_1(p)\big)\\ =& F^{-1}\Big(F(w_1)\Big(\frac{w_2 - \underline{b}}{w_2 - p}\Big)\Big).\end{aligned} \tag{6}$$

We have now pinned down the conditions for the informed player's inverse bidding functions $\phi_1(\cdot), \tilde{\phi}_1(\cdot)$ that also define $\beta_1(\cdot)$ and $\tilde{\beta}_1(\cdot)$. We need to determine a mixed strategy for the uninformed player 2 over the interval $[\underline{b}, \bar{b}]$, that he uses to determine the point at which he acquires information, such that the informed player 1 behaves according to $\tilde{\phi}_1(\cdot)$. The problem for the informed player 1 with a valuation $\theta \geq w_1$ is to solve

$$\max_p (\theta - p)\big(F(w_2) + (1 - F(w_2))G_2(p)\big),$$

where $G_2(\cdot)$, denotes the mixed strategy of player 2. The first order condition for this problem is

$$(\tilde{\phi}_1(p) - p)(1 - F(w_2))g_2(p) = F(w_2) + (1 - F(w_2))G_2(p), \tag{7}$$

where we have used the fact that $\tilde{\phi}_1(p) = \theta$. Solving this differential equation for the information acquisition strategy of the uninformed player 2 - the mixed strategy distribution $G_2(\cdot)$ - we obtain[9]

$$G_2(p) = \frac{F(w_2)}{1 - F(w_2)}\Big(\exp\Big(-\int_{\underline{b}}^{p} \gamma(x)dx\Big) - 1\Big), \tag{8}$$

where $\gamma(x) = -\frac{1}{\tilde{\phi}_1(x) - x}$. We need to find w_1 that satisfies[10]

$$G_2(\bar{b}) = 1. \tag{9}$$

Finally we need to make the following assumption about the cost of information acquisition.

Assumption 2. *The uninformed player always acquires information in equilibrium. Deviating to not acquiring information and behaving as if his valuation was v is not profitable for the uninformed player:*

$$W \geq (v - \beta_2(v))F(\phi_1(\beta_2(v))), \tag{10}$$

where W is the expected equilibrium payoff from information acquisition as defined in equation 2.

The constructed equilibrium of this paper builds on the assumption that the cost of information acquisition is small enough to allow for the uninformed

[9]We derive the solution in the appendix.

[10]We show that this can be done in Lemma 3 below.

player to acquire information in equilibrium in events where the price gets sufficiently low. We elaborate this assumption further in section 3.4.

We have thus constructed the equilibrium strategies.

Proposition 1. *The following strategies constitute an equilibrium of the descending price auction: The inverse bid functions* $\phi_1(\cdot), \phi_2(\cdot)$ *onto* $[0, w_i]$ *solve the pair of differential equations*

$$\phi_j'(p) = \frac{J_j(\phi_j(p))}{J_j'(\phi_j(p))} \frac{1}{\phi_i(p) - p}, \text{ for } i, j \in \{1, 2\},$$

with the boundary conditions $\phi_i(0) = 0$ *and* $\phi_i(\underline{b}) = w_i$ *for* $i \in \{1, 2\}$.

The informed player's bidding is determined by

$$\hat{\phi}_1(p) = \begin{cases} \tilde{\phi}_1(p), \text{ for } p \in [\underline{b}, \bar{b}] \\ \phi_1(p), \text{ for } p \in [0, \underline{b}) \end{cases},$$

where the inverse bid function $\tilde{\phi} : [\underline{b}, \bar{b}] \rightarrow [w_1, 1]$ *is determined by the distribution of bids*

$$\tilde{\phi}_1(p) = F^{-1}\Big(F(w_1)\Big(\frac{w_2 - \underline{b}}{w_2 - p}\Big)\Big),$$

the uninformed player acquires information over $[\underline{b}, \bar{b}]$ *according to*

$$G_2(p) = \frac{F(w_2)}{1 - F(w_2)}\Big(\exp\Big(\int_{\underline{b}}^{p} \frac{dx}{\tilde{\phi}(x) - x}\Big) - 1\Big),$$

and conditional on acquiring information at a price p *he bids according to*

$$\beta_2(\omega) = \begin{cases} p, \text{ for } \omega \in [w_2, 1] \\ \beta_2(\omega), \text{ for } \omega \in [0, w_2), \end{cases}$$

where $\beta_2(\cdot)$ *is characterized by* $\phi_2(\cdot)$. *The bids* $\underline{b}, \bar{b}$ *satisfy*

$$\underline{b} = \beta_1(w_1) = \beta_2(w_2),$$
$$\bar{b} = \underline{b}F(w_1) + (1 - F(w_1))w_2$$

and thresholds w_1 *and* w_2 *solve*

$$G_2(\bar{b}) = 1$$

$$\int_{w_2}^{1} (\omega - w_2) f(\omega) d\omega = c.$$

To prove that the strategies constitute an equilibrium it is sufficient to show that the uninformed player has no profitable deviations after the information acquisition and that the informed player has no profitable deviations from $\tilde{\phi}_1(\cdot)$ when $p \in [\underline{b}, \bar{b}]$. The proof that the inverse bid functions are an equilibrium is found from Maskin and Riley [8]. The uninformed is made indifferent regarding the information acquisition strategy and the incentive to acquire information is satisfied by assumption 2. The reasoning that followed the derivation of the critical type in equation (5) implies that the uninformed either bids immediately or waits until the price descends to $\beta_2(\theta)$. For the informed player we need to check that the uninformed player's mixing does not allow profitable deviations. Finally we check that we can find a $w_1 \in (0, 1)$ such that $G_2(\bar{b}) = 1$. We prove the proposition with the following three lemmas.

Lemma 1. *The informed player has no profitable deviations when* $p \in [\underline{b}, \bar{b}]$.

Proof. We abuse notation by $u_1(\theta, \theta') = (\tilde{\phi}_1(p) - p')(F(w_2) + (1 - F(w_2))G_2(p'))$ where we have substituted for θ, θ' with $\tilde{\phi}_1(p) = \theta \neq \theta' = \tilde{\phi}_1(p')$, where $p, p' \in [\underline{b}, \bar{b}]$. The first order condition for the informed player in equation (7) with $\theta \geq w_1$ when is

$$\begin{aligned} \frac{\partial u_1(\theta, \theta')}{\partial p'} &= (\tilde{\phi}_1(p) - p')(1 - F(w_2))g_2(b') - \Big(F(w_2) + (1 - F(w_2))G_2(p')\Big) \\ &= F(w_2) \exp\Big(\int_{\underline{b}}^{p'} \frac{dx}{\tilde{\phi}(x) - x}\Big)\Big(\frac{\tilde{\phi}_1(p) - p'}{\tilde{\phi}_1(p') - p'} - 1\Big), \end{aligned}$$

where we have substituted in $G_2(\cdot)$ and $g_2(\cdot)$ from (8) and $\gamma(x) = -\frac{1}{\tilde{\phi}(x) - x}$. The sign of this expression is determined by the last term in the brackets.

Since $\tilde{\phi}_1(\cdot)$ is increasing[11] we have that for $p' < p$ $\frac{\partial u_1(\theta,\theta')}{\partial p'} > 0$ and for $p' > p$ we have that $\frac{\partial u_1(\theta,\theta')}{\partial p'} < 0$. □

Lemma 2. *The uninformed player has no profitable deviations from his information acquisition strategy.*

Proof. First notice that acquiring information at $p > \bar{b}$ is not profitable since it yields the same expected payoff as acquiring information at $\bar{b}$. Then consider deviating to $p = \beta_2(z) < \beta_2(w_2) = \beta(w_1) = \underline{b}$. The expected payoff of the uninformed player from acquiring information at a price $p = \beta_2(z) < \underline{b}$ is given by

$$W(z) = \int_0^z u_2(x,x)f(x)dx + \int_z^1 u_2(x,z)f(x)dx - cF(\phi_1(\beta_2(z))).$$

The expected payoff difference when comparing with information acquisition at $\underline{b}$ is

$$\begin{aligned} W(w_2) - W(z) = &\int_z^{w_2} \big(u_2(x,x) - u_2(x,z)\big) f(x)dx + \int_{w_2}^1 (u_2(x,w_2) - u_2(x,z))f(x)dx \\ &- c(F(\phi_1(\beta_2(w_2))) - F(\phi_1(\beta_2(z)))) \\ = &\int_z^{w_2} \big(u_2(x,x) - u_2(x,z)\big) f(x)dx \\ &+ \int_{w_2}^1 \big(u_2(w_2,w_2) - u_2(w_2,z)\big) f(x)dx \geq 0, \end{aligned}$$

where the second equality follows from the condition for c in equation (5). The inequality follows, as the bidding functions satisfy incentive compatibility and hence we have that $u_2(\omega,\omega) \geq u_2(\omega,z)$ for all $\omega, z \in [0, w_2]$. □

Next we show that we can find $w_1 \in (0,1)$ such that $G_2(\bar{b}) = 1$ whenever the information acquisition cost is small.[12]

[11] It is readily verified that $\tilde{\phi}_1'(p) = \frac{g'(p)}{f\left(F^{-1}\left(g(p)\right)\right)}$, where $g(p) = F(w_1)\frac{w_2 - b}{w_2 - p}$ and $g'(p) > 0$.

[12] Small information acquisition cost corresponds to a large w_2.

Lemma 3. *There exists a threshold* $w_1 \in (0,1)$ *such that* $G_2(\bar{b}) = 1$*, when* $F(w_2) \geq 1 - w_2$*.*

Proof. It is sufficient to show that we can find $w_1', w_1'' \in (0,1)$ such that

$$\Big(G_2(\bar{b}) - 1\Big)(1 - F(w_2)) = \exp\Big(-\int_{\underline{b}}^{\bar{b}} \gamma(x)dx\Big)F(w_2) - 1 \gtrless 0.$$

First notice that $\tilde{\phi}(\bar{b}) = 1 \geq \tilde{\phi}(p) \geq w_1 = \tilde{\phi}(\underline{b})$, for $p \in [\underline{b}, \bar{b}]$. In addition, it is readily verified from equation (6) that $\tilde{\phi}(\cdot)$ is strictly increasing.[13] Therefore, we have that

$$\begin{aligned}\exp\Big(-\int_{\underline{b}}^{\bar{b}} \gamma(x)dx\Big)F(w_2) - 1 &= \exp\Big(\int_{\underline{b}}^{\bar{b}} \frac{dx}{\tilde{\phi}(x) - x}\Big)F(w_2) - 1 \\ &> \exp\Big(\int_{\underline{b}}^{\bar{b}} \frac{dx}{1 - x}\Big)F(w_2) - 1 \\ &= \frac{1 - \underline{b}}{1 - \bar{b}}F(w_2) - 1.\end{aligned}$$

Now since $\phi_1(0) = 0$ and $\phi_1(\underline{b}) = w_1$ we have that as $w_1 \to 0$ then $\underline{b} \to 0$ and also $\bar{b} \to w_2$. Notice also that as $w_1 \to 0$ we have that $\tilde{\phi}_1(\cdot) \to 0$. Therefore, for small enough w_1 the approximation above holds and we have that,

$$\frac{1 - \underline{b}}{1 - \bar{b}}F(w_2) - 1 \to \frac{1}{1 - w_2}F(w_2) - 1 \geq 0,$$

which is satisfied for a large w_2.

Now assuming that $w_1 \geq \bar{b}$ and $w_1 > w_2$ we have that

$$\begin{aligned}\exp\Big(\int_{\underline{b}}^{\bar{b}} \frac{dx}{\tilde{\phi}(x) - x}\Big)F(w_2) - 1 &< \exp\Big(\int_{\underline{b}}^{\bar{b}} \frac{dx}{w_1 - x}\Big)F(w_2) - 1 \\ &= \frac{w_1 - \underline{b}}{w_1 - \bar{b}}F(w_2) - 1 \\ &= \frac{(w_1 - \underline{b})F(w_2) - (w_1 - w_2) - F(w_1)(w_2 - \underline{b})}{(w_1 - w_2) + F(w_2)(w_2 - \underline{b})},\end{aligned}$$

[13] We assumed the $f(x) > 0$ for all $x \in [0,1]$. See also footnote 11.

where we have substituted in for $\bar{b}$ and rearranged.[14] Then it is obvious that the denominator is positive and the nominator can be rearranged to $-(w_1 - w_2)(1 - F(w_2)) - (F(w_1) - F(w_2))(w_2 - \underline{b}) < 0$.

Since $F(\cdot)$ is strictly increasing and continuous, we have that $F^{-1}(\cdot)$ is continuous. Therefore $\gamma(x)$ is continuous and the result follows by the intermediate value theorem. □

Corollary 1. *From lemma 3 it follows that for a threshold w_1 satisfying $G_2(\bar{b}) = 1$ it must hold that $w_1 < w_2$ for small $c > 0$.*

3.3 Example with uniformly distributed valuations

We illustrate the equilibrium by deriving the inverse bidding functions explicitly in the case of uniformly distributed valuations on $[0, 1]$. For prices $p < \underline{b}$ the bidding is equivalent to an asymmetric first price auction[15] and the inverse bidding function is

$$\psi_i(b) = \frac{2b}{1 + k_i b^2}, \text{for } i \in \{1, 2\}$$

where $k_i = 1/w_i^2 - 1/w_j^2$. The bid $\underline{b}$ such that $\psi_i(\underline{b}) = w_i$ and upper bound $\bar{b}$ of the mixing interval are given by

$$\underline{b} = \frac{w_1 w_2}{w_1 + w_2}$$
$$\bar{b} = w_1 \underline{b} + (1 - w_1) w_2.$$

For $p \geq \underline{b}$ we have that the inverse bidding function for player 1 is

$$\tilde{\phi}_1(p) = w_1 \Big(\frac{w_2 - \underline{b}}{w_2 - p} \Big)$$

and the solution to the differential equation[16] for $G_2(\cdot)$ is given by

[14] One observation is that if $w_1 = w = w_2$, then this expression equals zero.

[15] An explicit solution to this problem, when valuations are uniform, can be found from Krishna [6].

[16] We derive the solution in the appendix together with the associated constants.

$$G_2(p) = \mathcal{C}\frac{1}{\sqrt{K - w_2 p + p^2}} \exp\left(\frac{w_2}{\sqrt{4K - w_2^2}} \arctan(\frac{2p - w_2}{\sqrt{4K - w_2^2}})\right) - \frac{w_2}{1 - w_2},$$

where $\mathcal{C} = \sqrt{K - w_2\underline{b} + \underline{b}^2} \exp\left(-\frac{w_2}{\sqrt{4K-w_2^2}} \arctan(\frac{2\underline{b}-w_2}{\sqrt{4K-w_2^2}})\right)\frac{w_2}{1-w_2}$ and $K = w_1(w_2 - \underline{b})$. The condition for the critical w_2 is $w_2 = 1 - \sqrt{2c}$. We do not have an analytical solution for w_1, but figure 2 illustrates the link between the information acquisition cost, the induced threshold w_2 and the threshold w_1 that satisfies $G_2(\bar{b}) = 1$.

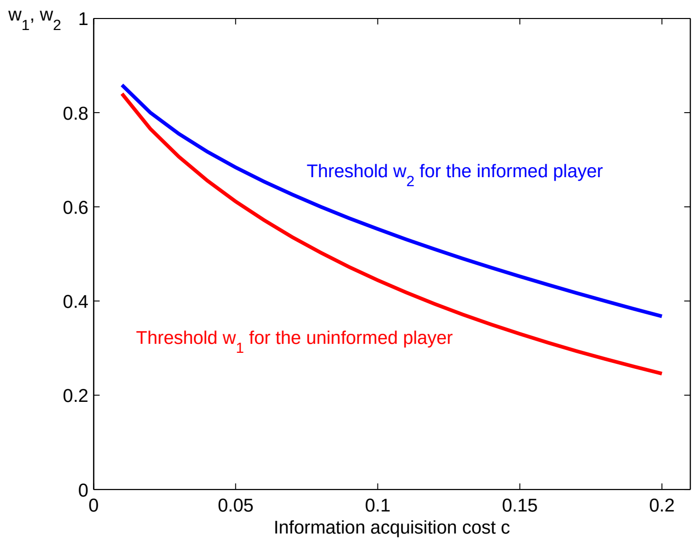

Figure 2: Threshold valuations for the informed and uninformed players

3.4 Incentives to acquire information

This paper considers equilibrium behavior when it is assumed that information acquisition is preferred to not acquiring information and behaving as if the uninformed player's type was v. In a symmetric model [11] it was shown that relaxing this assumption leads to an initial stage where uninformed players decide first whether to acquire information or not. Players that acquire information use a threshold strategy - similar to the one in this paper - to decide about their subsequent bids and the players who do not acquire information use a mixed strategy in bidding once the price descends enough. With only one uninformed player, the question boils down to a threshold for the information acquisition cost. Apart from the example with uniformly distributed valuations, we do not have explicit bidding functions and we do not pursue to solve this analytically. When valuations are uniform we can find a threshold for the information acquisition cost such that the derived equilibrium is obtained whenever the cost is below this threshold. For uniformly distributed values the threshold cost is approximately 0.05. The difference between the equilibrium expected payoff for the uninformed player and the payoff from deviating to bidding as if the uninformed player's valuation were v is depicted as Δ in figure 3.

We note that here it is assumed, that the uninformed player obtains his valuation for free in case he wins the auction. It could be argued, that even if he won the auction he would have to incur the same cost of information acquisition in order to reap the benefit from the purchase. In this case the cost of information acquisition needs to be deducted from the utility in case the uninformed wins.[17] In this case bidding without fist acquiring information is not profitable on a relevant range of information acquisition costs. We display

[17]This means that the relevant comparison is $W \geq (v - \beta_2(v) - c)F(\phi_1(\beta_2(v)))$.

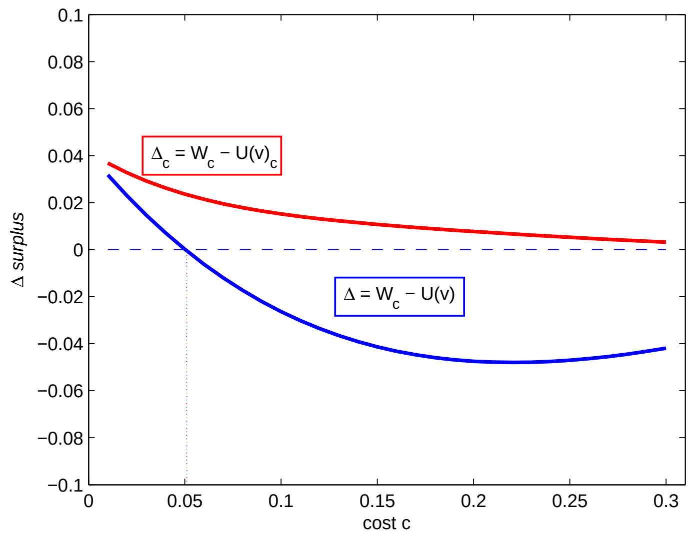

Figure 3: Difference between the expected payoff W and the payoff from deviating to bidding $\beta_2(v)$. Δ represents the difference when the uninformed does not incur the cost of information acquisition. Δ_c is the difference when the uninformed must pay the information acquisition cost after he wins the object.

this difference as Δ_c in figure 3.

3.5 Revenue comparison

Employed with the uniform distribution example we would like to understand how the revenue generated from the descending price auction compares with the revenue from the ascending auction. The equilibrium behavior in the ascending price auction with information acquisition was explored by Compte and Jehiel [4] in a setting that is comparable to ours. In equilibrium, the informed player stays in the auction until the price reaches his valuation and the uninformed player employs the following threshold strategy. The uninformed stays in and acquires information if price reaches p^{**} that is defined by $h(p^{**}) - c = v - p^{**}$, where $h(p) = \Pr(\theta > p)\mathbb{E}[\theta - p \mid \theta > p]$. After the information is acquired he immediately drops out, if his valuation is below the current price and otherwise stays in the auction until the price reaches his valuation. Thus, prior to information acquisition the uninformed may win, if the informed player's valuation is below p^{**}. Similar to Maskin and Riley [8] we study the equilibrium distribution of bids $\Pr(p_{win} \leq p)$ in these two settings. For the ascending auction this can be written as

$$\Pr(p)_a = \begin{cases} F(p) \text{ if } p < p^{**}, \\ 2F(p) - F(p)^2 \text{ if } p \geq p^{**} \end{cases}$$

and for the descending price auction as

$$\Pr(p)_d = \begin{cases} F(\phi_1(p))F(\phi_2(p)) \text{ if } p < \underline{b}, \\ \Big(F(w_1) + (1 - F(w_1))G_1(p)\Big)\Big(F(w_2) + (1 - F(w_2))G_2(p)\Big) \text{ if } p \geq \underline{b}. \end{cases}$$

In the descending auction the informed player benefits from his uninformed opponent as this allows him to postpone the start of bidding in comparison to the case without information acquisition cost. Although the ensuing bidding is more aggressive, the delay causes the revenue to decline for the seller in comparison to a case with no information acquisition. In the ascending auction the uninformed player's strategy to stay in the auction without acquiring information for low prices, pushes seller's revenue up in comparison to the case without information acquisition cost.

When the cost of information acquisition is equal to zero the revenue from the two auctions coincides by the revenue equivalence theorem.[18] We know from Compte and Jehiel [4] that p^{**} is increasing in c and this is readily verified in the case of uniformly distributed values where this threshold is equal to $p^{**} = \sqrt{2c}$. Then as $c > 0$ the probability of the winning bid being below p^{**}

[18]Then the parameters for the distributions $\Pr_a(\cdot)$ and $\Pr_d(\cdot)$ are $p^{**} = 0$ and $\underline{p} = \bar{p} = 0.5$.

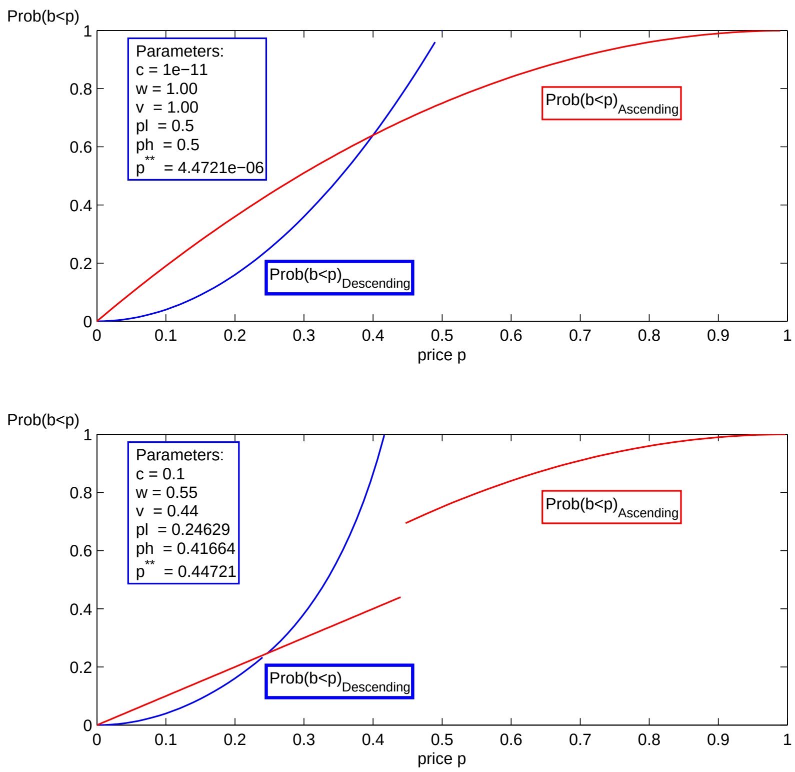

Figure 4: The distribution of bids in the ascending and descending auctions with uniformly distributed valuations. The picture above displays distributions with (virtually) no information acquisition cost. The picture below displays the distributions with costly information acquisition.

is smaller than when there is no information acquisition. This is shown in the lower panel in figure 4.

When the information acquisition cost increases from c to c' in the descending price auction, it leads to a decrease in w_2 to w_2' that follows from equation (5). This translates to a decrease in $\bar{b}$ to $\bar{b}'$ which has an impact on w_1 through the requirement that $G_2(\bar{b}') = 1$. Then corollary 1 suggests that we need that

$w_1' < w_1$ in order to satisfy $G_2(\bar{b}') = 1$.[19] By monotonicity of the bid functions the smaller values of w_1' and w_2' necessitate that $\underline{b}' < \underline{b}$. In the case of a uniform distribution the entire distribution of bids shifts to the right, making it more probable that a lower price wins when the information acquisition costs are higher.

Thus, the probability of lower prices increases in the descending price auction while the probability of lower prices decreases in the ascending price auction when the information acquisition costs are introduced. Thus, the ascending price auction produces more revenue than the descending price auction when information acquisition has a cost.

It is interesting to see if this comparison turns around when there are more than just two players. In the ascending auction information acquisition activity may be discouraged with more informed players. In contrast, more (informed and uninformed) players should lead to more aggressive bidding behavior in the descending price auction.

[19] To be precise, we have not computed comparative statics of $G_2(\bar{b})$ with respect to w_1 and w_2.

4 Omitted proofs

Lemma 4. *$G_2(\cdot)$ as defined in equation (8) constitutes a solution to the differential equation*

$$G_2'(b) + G_2(b)\gamma(b) = -\frac{F(w_2)}{1-F(w_2)}\gamma(b), \tag{A1}$$

with a boundary condition $G(\underline{b}) = 0$, where

$$\gamma(b) = -\frac{1}{\tilde{\phi}_1(b) - b}.$$

Proof. We solve the first order linear differential equation in (A1) with the method of an integrating factor. Denote the integrating factor as

$$\Gamma(b) = \int^b \gamma(x)dx.$$

Then we have that

$$\frac{d}{db}\Big(G(b)\exp\Big(\Gamma(b)\Big)\Big) = -\frac{F(w_2)}{1-F(w_2)}\gamma(b)\exp\Big(\Gamma(b)\Big).$$

Performing the integration we get that the general solution for the problem is

$$G(b)\exp\Big(\Gamma(b)\Big) = -\frac{F(w_2)}{1-F(w_2)}\int^b \gamma(y)\exp\Big(\Gamma(y)\Big)dy + \mathcal{C}. \tag{A2}$$

Now notice that

$$\frac{d}{db}\Big(\exp\Big(\int^b \gamma(x)dx\Big)\Big) = \exp\Big(\int^b \gamma(x)dx\Big)\gamma(b) = \exp\Big(\Gamma(b)\Big)\gamma(b).$$

Then we have that the general solution (A2) becomes[20]

$$G(b) = \mathcal{C}\exp\Big(-\Gamma(b)\Big) - \frac{F(w_2)}{1-F(w_2)}.$$

[20]Since the sign of the constant $\mathcal{C}$ is determined from the boundary conditions we drop the minus sign here.

Using the boundary condition $G_2(\underline{b}) = 0$ gives us the desired particular solution.

$$G(b) = \frac{F(w_2)}{1 - F(w_2)} \Big(\exp \Big(- \int_{\underline{b}}^{b} \gamma(x) dx \Big) - 1 \Big).$$

□

When the valuations are uniformly distributed we have that $\tilde{\phi}_1(b) - b = \frac{b^2 - w_2 b + K}{w_2 - b}$ where $K = w_1(w_2 - \underline{b})$.Then we have that

$$\begin{aligned} \Gamma(b) &= \int \gamma(b) db \\ &= \int \frac{b - w_2}{b^2 - w_2 b + K} db. \end{aligned}$$

The solution to this integral is[21]

$$\int \frac{b - w_2}{b^2 - w_2 b + K} db = \frac{1}{2} \log(b^2 - w_2 b + K) - \frac{w_2 \arctan\Big(\frac{2b - w_2}{\sqrt{4K - w_2^2}}\Big)}{\sqrt{4K - w_2^2}},$$

and hence the particular solution to the problem is

$$\begin{aligned} G_2(b) &= \mathcal{C} \frac{1}{\sqrt{K - w_2 b + b^2}} e^{\frac{w_2}{\sqrt{4K - w_2^2}} \arctan\Big(\frac{2b - w_2}{\sqrt{4K - w_2^2}}\Big)} - \frac{w_2}{1 - w_2}, \text{ where} \\ \mathcal{C} &= \sqrt{K - w_2 \underline{b} + \underline{b}^2} e^{-\frac{w_2}{\sqrt{4K - w_2^2}} \arctan\Big(\frac{2\underline{b} - w_2}{\sqrt{4K - w_2^2}}\Big)} \frac{w_2}{1 - w_2}. \end{aligned}$$

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[21] See for instance [16] pp. 160, formula 164.

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