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Contracting with Long-Term Consequences*

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Abstract

I examine optimal managerial compensation and turnover policy in a principal-agent model in which the firm output is serially correlated over time. The model captures a learning-by-doing feature: higher effort by the manager increases the quality of the match between the firm and the manager in the future. The optimal incentive scheme entails an inefficiently high turnover rate in the early stages of the employment relationship. The optimal turnover probability depends on the past performance and the likelihood of turnover decreases gradually with superior performance. Following weak performance, the contract implements a permanently inefficient turnover rate. With correlated outcome, a permanent inefficiency is needed to save on information rents to the agent, even when the agent does not have persistent private information.

Keywords: Dynamic moral hazard, managerial turnover, pay for performance

JEL Classification: C73, D82, D86.

1 Introduction

An extensive literature discusses the impact of the CEO ability on the firm value. Following weak performance, the CEO is replaced in a hope of finding a new manager who is better able to choose the right management strategy to enhance

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the firm value. However, empirical literature documents that the CEO turnover decisions do not only depend on the firm value, but the probability of the CEO turnover decreases with tenure. In particular, Dikolli, Mayew, and Nanda (2014) report that the turnover probability decreases with superior performance. Furthermore, Cremers and Palia (2011) find that a CEO's pay to performance increases with tenure. Thus, CEO compensation increases with both the firm value and the length of the manager's employment in the firm.

The main contribution of this paper is to show that the optimal contract entails a permanently inefficient turnover rate, even when the manager does not have persistent private information. In the beginning of the employment relationship, the manager is compensated for high performance by promising him a higher job security. Over time, the turnover policy becomes fixed, but the turnover probability depends on the performance at early stages. In particular, early weak performance cannot be undone by performing well in the future. Moreover, the manager's compensation depends on his entire history of past performance.

To be more concrete, I study the managerial compensation and turnover in a model with a firm and a manager in which the manager's actions have long-term consequences. The value of the firm under the current management depends both on the manager's past efforts and on exogenous circumstances. By investing in the firm, the manager has the opportunity to increase the quality of the match between himself and the firm. The manager's investment is productive and increases the firm value under the current management. This captures the additional value that the right choice of manager adds to the firm profit. The model includes a learning-by-doing component: higher managerial effort today increases the firm profit in all future periods.

My starting point is a dynamic model with a principal (firm owner) and an agent (manager). Both players are risk-neutral, but the agent is protected by limited liability. Limited liability implies that the contract cannot impose negative payments to the agent, and that eventual losses have to be covered by the principal. The firm produces a stochastic output that is serially correlated over time.[1] The agent can exert effort to increase the quality of the match and thereby the firm profit. Effort is unobserved by the principal and related with opportunity cost for the agent. If the agent shirks, he receives a private benefit. To prevent the agent from enjoying private benefits, the principal has to reward him for good performance and punish him following low outcomes. Since the manager's quality within the current firm is correlated over time, the agent's actions have long-term consequences. Shirking today decreases the firm profit under the current manager for all future periods.

[1]I assume that there is a one-to-one mapping between the quality of the match between the players and the output under the current manager.

At any point in time, the firm has the opportunity to fire the manager and hire a new one. If the firm replaces the manager, a new one is drawn from a time invariant distribution. Thus, I assume that both the expected firm value and the manager's expected utility following a turnover are constant while the value of the firm under the current manager depends on both the past performance and on the manager's effort. The firm potentially has two motivations to replace the manager. Firstly, the new manager eventually provides a better match to the firm, such that turnover improves efficiency. Secondly, the turnover threat provides the agent incentives to exert effort. Vice versa, the firm can reward the manager for good performance by promising him a higher level of job security in the future.

Turnover is efficient if the quality of the match falls too low, and the optimal contract never retains the manager unless it is efficient to do so. The optimal dynamic contract between the firm owner and the manager implies that the manager's compensation increases and the probability of turnover decreases with superior performance. In particular, I find that the optimal long-term contract entails an inefficiently high turnover rate that is needed to provide the manager incentives in the beginning of the employment. Following superior performance, the turnover threat is relaxed and the manager is rewarded by promising him a higher job security in future periods. However, since output is correlated over time, inefficient turnover is a permanent phenomenon. Moreover, the manager's compensation depends both on the firm value and on the tenure level. Finally, I find strong evidence indicating that tenure decisions correlate with past performance. Tenure does not only reflect manager's ability but also rewards him for his past efforts on the firm development.

I build on the leading continuous-time model by DeMarzo and Sannikov (2006) and extend the model to allow for the output to be serially correlated over time.[2] I show that allowing for correlation of output, adds an additional distortion: a permanently inefficient turnover rate. Sannikov (2014) examines a related model in which the agent's action affects the future output.[3] DeMarzo and Sannikov (2016) examine a related model in which the players learn about the unknown mean of the cash-flow process; Vasama (2017) considers real options. While permanent distortions are prevalent in models with persistent private information, I show that they arise when output is serially correlated over time, even if the agent's private information is instantaneous.

The empirical fact that turnover decisions become more lenient over time is often interpreted as a consequence of managerial entrenchment in the literature. In their seminal paper, Shleifer and Vishny (1989) predict that managers attempt

[2]See also DeMarzo and Fishman (2007) and Biais, Mariotti, Plantin, and Rochet (2007).

[3]The model builds on Sannikov (2008) in which the agent's action only has an instantaneous effect on the output. Sannikov (2012) provides an excellent survey of the earlier literature.

to gain power by choosing specific investments that enhance their value inside the firm when compared to outsiders. As a consequence, the model predicts an inefficiently low turnover rate for managers with longer tenure. Garrett and Pavan (2012) show that such a phenomenon is a natural consequence of the principal's profit maximization when there is asymmetric information about the manager's quality: the optimal contract entails excessive retention of the manager over time.[4] In contrast, my model predicts a permanently inefficient turnover rate.

This paper is not the first one to examine moral hazard with long-term consequences of the agent's action. In He (2009), both the firm cash-flow and the outside options depend on the firm scale that is affected by the agent's action. Kwon (2011) proposes an alternative approach to model correlation of output in a model with two states. Interestingly, the long run effects vanish over time while my model entails a permanent distortion.

To model serial correlation of output, I adopt an approach that defines the cash-flow as the level of a diffusion process, rather than as the increment. In the principal-agent framework, the approach was first introduced by Williams (2011) who examines a model in which the output is persistent, and the agent is risk-averse. Strulovici (2011) extends the model to allow for the players to renegotiate the contract. Again, our model differs from these important papers by the information structure: the agent has no persistent information in our framework. Besides, Strulovici (2011); Williams (2011) allow for a more general utility function, but do not consider turnover.

2 The Model

I examine a game in which a principal (she) hires an agent (he) who is necessary to operate a company. Time is continuous and the time horizon is infinite. The principal has access to unlimited funds, but the agent is protected by limited liability. In our framework this implies that the agent cannot make negative payments. Both players discount the future by a common rate $r > 0$.

At any time t during the employment of the current agent, the firm produces an output X_t. The output is stochastic and changes following a standard Brownian motion. Formally, the output at period t is

$$X_t = X_0 + \int_0^t a_s ds + \int_0^t \sigma dZ_s, \tag{1}$$

where Z is a standard Brownian motion, $a_t \leq \mu$ denotes the agent's effort and $\sigma > 0$ is the variance. At time 0, the output process starts from the initial value

[4] Empirical evidence provides stronger support on the turnover decisions reflecting differences in managerial abilities than managerial entrenchment, see for example Rose and Shepard (1997).

X_0 that is common knowledge. (1) implies that $E[X_t|X_0] = X_0$, i.e., the output follows a Markov process.[5]

The output process $\{X_t : t \geq 0\}$ is observed by both players, but the effort $\{a_t : t \geq 0\}$ is the agent's private information. (1) implies that actions have long-term consequences. If the agent shirks today, the firm value is lower in all future periods. Thus, the model entails a learning-by-doing component: a higher effort today increases the expected match value between the principal and the agent in the future. This increases the expected output in the future and therefore the firm value.

Effort is costly for the agent and shirking generates an instantaneous private benefit of

$$\lambda(\mu - a_t)dt.$$

I assume that $\lambda \in (0, 1)$ such that private benefits are inefficient. Hence shirking is related with a social loss of $1 - \lambda$ for each unit of labor.

I abstract away from private savings. Since both players are risk neutral and discount the future by the same rate, the assumption is without loss of generality. Any contract with private savings can be replicated by a contract that includes no private savings that yields the same payoff to the players.[6]

The principal has the possibility to terminate the agent's contract and hire a new agent. If the agent is fired, the principal hires a new agent under which the firm value is stochastic and depends on the quality of the match with the new agent. I assume that the expected firm value under the new agent is independently and identically distributed over time. Termination of the contract is irreversible such that the old agent returns to the pool of new agents and the match quality becomes unobserved. Without loss of generality, I normalize the expected payoff under the new agent to $L \geq 0$, net of the hiring cost of the new agent and possible firing cost. If the agent is fired, he receives an expected payoff of $R \geq 0$, which summarizes the expected cost of finding a new employment and the expected profit under the new employment contract.

In the beginning of each employment relationship, the players agree on a contract. The contract specifies a nonnegative payment process $\{dW_t \geq 0 : t \geq 0\}$ from the principal to the agent, and a stopping time τ at which the agent's contract is terminated. Both the payment process and the stopping time are adapted to the filtration generated by the public history of the past outcomes, $\mathcal{F}_t = \sigma\{X_s : 0 \leq s \leq t\}$. I assume that both players can fully commit to the

[5]The assumption is in sharp contrast with DeMarzo and Sannikov (2006) in which the output is $dX_t = \mu dt + \sigma dZ_t$. The expected flow of output is $E[dX_t] = \mu dt$ which is independent of the past performance.

[6]See DeMarzo and Fishman (2007); DeMarzo and Sannikov (2006) for a formal argument. The extension to our framework is straightforward.

contract.

For any incentive compatible contract, the principal's total expected profit is the discounted stream of output X_t minus the payments dW_t to the agent

$$v_0 = E\left[\int_0^\tau e^{-rt}(X_t dt - dW_t) + e^{-r\tau}L\right], \tag{2}$$

where the output X_t depends on the agent's action a_t as described in (1). Let U_0 denote the agent's promised utility from the contract at time 0. The promise-keeping constraint at time 0 guarantees that the agent receives his expected payoff consisting of the discounted stream of payments

$$U_0 = E\left[\int_0^\tau e^{-rt} dW_t + e^{-r\tau}R\right] \tag{3}$$

under the contract that implements full effort, $a_t = \mu$ for all $t \leq \tau$. The agent's incentive compatibility constraint guarantees that the agent receives at least the same utility for exerting full effort from t onwards than for any arbitrary effort strategy with $\{a_t \leq \mu : t \leq \tau\}$ with an additional stream of private benefits $\lambda(\mu - a_t)$. Formally,

$$U_0 \geq E^a\left[\int_0^\tau e^{-rt}(dW_t + \lambda(\mu - a_t)dt) + e^{-r\tau}R\right] \tag{4}$$

almost surely, for all t, and for all feasible strategies $a_t \leq \mu$.

The optimal contract determines an intertemporal payment rule and a turnover policy that satisfy the agent's promise-keeping condition, the incentive compatibility constraints, and the limited liability constraints, and maximize the principal's expected payoff. Formally, it chooses a nonnegative payment process W and a stopping time τ to maximize (2) subject to the constraints (3) and (4).

3 First-Best Solution

I first revisit the solution to the problem when there is no asymmetric information. Since both players are risk-neutral and discount the future by the same rate, the intertemporal allocation of payments is irrelevant for efficiency. The goal is to determine the efficient match value at which the old agent's contract is terminated and a new agent is hired.

The first-best value s solves a standard real option problem. The objective is to choose an optimal stopping time τ, measurable with respect to the output process X_t at time t, to maximize the expected discounted flow of output

$$s_0 = E\left[\int_0^\tau e^{-rt} X_t dt + e^{-r\tau}(L + R)\right]. \tag{5}$$

The optimal turnover decision is a threshold policy: the old agent is retained so long as the match value stays above a certain cutoff, $X_t \geq x^*$, and as soon as it reaches x^*, it is efficient to turn over the agent.

The efficient turnover policy is the unique solution of the Hamilton-Jacobi-Bellman equation

$$rs(x) = x + \mu s_x(x) + \frac{\sigma}{2}^2 s_{xx}(x) \tag{6}$$

with the boundary conditions $s(x^*) = L + R$ and $s_x(x^*) = 0$. The value function $s(x)$ and the first-best optimal turnover value x^* can be solved explicitly. The firm value at any point $x \in [x^*, \infty)$ is

$$s(x) = \frac{x}{r} + \frac{\mu}{r^2} - \left(\frac{x^*}{r} + \frac{\mu}{r^2}\right) e^{\alpha(x-x^*)} + (L+R)e^{\alpha(x-x^*)}. \tag{7}$$

Turnover is efficient if the match value reaches

$$x^* = r(L+R) + \frac{1}{\alpha} - \frac{\mu}{r} \tag{8}$$

with

$$\alpha = -\frac{\mu}{\sigma^2} - \sqrt{\frac{\mu^2}{\sigma^4} + \frac{2r}{\sigma}}. \tag{9}$$

4 Incentive Compatibility

In this section, I derive necessary and sufficient conditions that guarantee that the agent exerts full effort at the optimal contract. Since effort is unobserved, the principal does not know if fluctuations of output are consequences of the agent's actions or of exogenous events. This gives rise to a moral hazard problem in our framework. For the agent to be willing to exert effort, he needs to be compensated for his lost private benefit.

To provide the agent incentives to exert effort, the principal has to let his continuation value vary with the fluctuations of the output. Following the standards in the literature, I start by specifying how the agent's continuation value U_t depends on his actions. In continuous time, the agent's continuation value admits a convenient representation as a stochastic process.

At $t \leq \tau$, the agent's continuation value is

$$U_t = E_t\left[\int_t^\tau e^{-r(s-t)}(dW_s + \lambda(\mu - a_s)ds) + e^{-r\tau}R\right] \tag{10}$$

for any incentive compatible contract that implements the effort $\{a_s \leq \mu : t \leq s \leq \tau\}$ by the agent. (10) is sometimes called the agent's promise-keeping constraint.

It is a book-keeping constraint that guarantees that the agent receives his promised value at the optimal contract.

I first determine how the agent's continuation value evolves for an arbitrary effort strategy $\{\tilde{a}_s : 0 \leq s \leq \tau\}$ when the principal wants to implement the particular effort $\{a_s : 0 \leq s \leq \tau\}$. Then I derive conditions for the agent to choose full effort, $\tilde{a}_t = \mu$ for all $t \leq \tau$. Later I will show that it is indeed optimal for the principal to implement full effort.

The representation is summarized in the following lemma

Lemma 1. *Fix a contract $\{a, W, \tau\}$ with $U_t < \infty$ for all t. The process U_t is the agent's continuation value from the contract if and only if the following conditions are satisfied. (i) At $t \leq \tau$, U_t admits the representation*

$$dU_t = rU_t dt - dW_t + \beta_t(dX_t - a_t). \tag{11}$$

β is a progressively measurable process in L^* that describes the sensitivity of the agent's continuation value to his effort.[7] *(ii) U_t satisfies the transversality condition $\lim_{s\to\infty} E_t[1_{s\leq\tau}e^{-rs}U_{t+s}] = 0$ almost everywhere.*

Proof. See, for example, Sannikov (2008). □

To derive conditions, that guarantee that the agent exerts full effort, I compare his payoff from the full effort strategy $a_t = \mu$ to the payoff from an arbitrary effort strategy with $\tilde{a}_t \leq \mu$. If the agent shirks, he earns an additional flow of private benefits

$$\lambda(\mu - \tilde{a}_t)dt. \tag{12}$$

However, he loses a flow compensation of

$$\beta_t(\mu - \tilde{a}_t)dt \tag{13}$$

that he would have received from the principal if he had exerted higher effort. By comparing (12) and (13), I find that, for the agent to be willing to exert effort, his continuation value has to increase by $\beta_t \geq \lambda$ for each unit of effort required.

The result is summarized in the following proposition

Proposition 1. *A necessary and sufficient condition for $a_t = \mu$ to be incentive compatible is that $\beta_t \geq \lambda$ for all $t \leq \tau$.*

Proof. Using the standard argument I can write the agent's incentive compatibility constraint as

$$U_0 \geq U_0 + E^{\tilde{a}}\left[\int_0^\tau e^{-rt}(dW_t + \lambda(\mu - \tilde{a}_t) - dW_t - \beta_t(\mu - \tilde{a}_t)dt)\right],$$

[7] A process β is in L^* if $E\left[\int_0^t 1_{s\leq\tau}\beta_s^2 ds\right] < \infty$.

or,

$$-E^{\tilde{a}}\left[\int_0^\tau e^{-rt}(\beta_t-\lambda)(\mu-\tilde{a}_t)dt\right]\leq 0.$$

Since $\tilde{a}_t \leq \mu$, $\tilde{a}_t = \mu$ is incentive compatible if and only if $\beta_t \geq \lambda$. □

5 Contracting Problem

I next turn to the principal's problem of determining the optimal contract. This section presents a heuristic discussion. The optimal contract can be verified using a standard procedure;[8] the precise derivation is delegated to the Online Appendix. Since shirking is always inefficient, the optimal contract implements full effort by the agent. The incentive constraints bind and the payments are delayed until the agent's continuation value reaches a certain boundary.

Using the results obtained in Section 4, I can write the optimal contracting problem as one of maximizing the expected flow of outputs X_t minus the flow of payments W_t to the agent

$$v(u,x)=\max\ E\left[\int_0^\tau e^{-rt}(X_t dt-dW_t)\right]$$

subject to the state variables X_t and U_t evolving according to (1) and (11), the transversality condition

$$\lim_{s\to\infty}E_t[1_{s\leq\tau}e^{-rs}U_{t+s}]=0$$

almost everywhere, and the feasibility, incentive compatibility and limited liability constraints

$$a_t\leq\mu,\quad \beta_t\geq\lambda \ \text{ and } \ dW_t\geq 0 \ \text{ for all } \ t\leq\tau.$$

The optimal contract can be derived from the principal's Hamilton-Jacobi-Bellman equation[9]

$$rv(u,x)=f(u,x)+\max_{a\leq\mu,w\geq 0,\beta\geq\lambda}\Big\{-w(1+v_u(u,x)) + av_x(u,x)+\sigma^2\left(\frac{\beta^2}{2}v_{uu}(u,x)+\beta v_{ux}(u,x)\right)\Big\}. \quad (14)$$

with the terms that are independent of the controls collected in

$$f(u,x)=x+ruv_u(u,x)+\frac{\sigma^2}{2}v_{xx}(u,x).$$

[8]The proof follows along the same lines than in DeMarzo and Sannikov (2016); see also Vasama (2017).

[9]For simplicity, I assume the existence of an absolutely continuous payment process $dW_t = w_t dt$. See discussion in Section 6 below.

The boundary condition is $v(R, x) = L$. Because of limited liability, u cannot become negative. If the agent's continuation value decreases to R, the only way to provide him incentives to exert effort is to terminate his contract. Besides, since the agent is risk-neutral, it is (weakly) suboptimal to deliver him income after τ. Therefore, it is without loss of generality to concentrate on contracts that terminate the agent's contract if his continuation value hits R.

5.1 Full Effort Implemented

I show that the optimal contract implements full effort by the agent. Implementing full effort is optimal because cash-flows are valuable. Additional effort increases expected cash-flow, which is anticipated by the principal. For any incentive compatible contract, the agent's total value follows a martingale and any deviation from the expected value becomes as a surprise. As a consequence, all additional rents can be extracted from the agent.

Indeed, (14) implies that implementing the full effort $a = \mu$ is optimal since

$$v_x(u, x) \geq 0, \tag{15}$$

as confirmed by the next lemma

Lemma 2. *$v(u, x)$ is nondecreasing in x.*

Proof. Consider the processes $(U_s^i, X_s^i)_{s \geq 0}$, $i = 1, 2$, that follow (1) and (26) starting from the values that satisfy

$$X_0^1 - X_0^2 = \delta > 0 \text{ and } U_0^1 = U_0^2 \geq 0. \tag{16}$$

Let $\tau^1 \equiv \tau(U^1)$ and $\tau^2 \equiv \tau(U^2)$. (26) together with (16) implies that $U_t^1 = U_t^2$ for all t, and therefore, $\tau \equiv \tau^1 = \tau^2$. Then

$$v(U, x^1) - v(U, x^2) = E\left[\int_0^\tau e^{-rt}(X_t^1 - X_t^2)dt\right] = E\left[\int_0^\tau e^{-rt}\delta dt\right] \geq 0$$

almost surely. The result follows since δ was chosen arbitrarily. □

5.2 Payment Policy

I first discuss the optimal Since the agent is risk-neutral, the marginal cost of providing him immediate income is -1. The marginal cost of increasing the agent's continuation value is lower whenever the inefficient termination threat is reduced. Indeed, by optimizing (14) with respect to w, I find that if

$$v_u(u, x) \geq -1, \tag{17}$$

it is optimal to set $dW_t = 0$. The payments to the agent are delayed, and the principal rewards him by promising a higher continuation value for the future. When the condition (17) binds, the optimal contract relies implements turnover at a fixed cutoff.

When the firm output is correlated over time, relaxing the agent's continuation value guarantees him a higher expected information rent for all future periods. Relaxing the inefficient turnover threat is more costly to the principal than if the output process is uncorrelated over time. Formally, the left hand side of (17) depends both on x that follows a simple Markov process and on u that is a functional of the entire path of past outcomes. The optimal contract entails a permanent turnover threat following histories of weak early performance; the argument is made precise in Section 6 below.

5.3 Incentive Constraints Bind

To show that the incentive constraints bind at the optimum, I adopt the following approach, borrowed from DeMarzo and Sannikov (2016). I compare the principal's profit from the contract with $\beta = \lambda$ with her profit from any other contract with $\beta \geq \lambda$. I show that the contract with $\beta = \lambda$ attains the highest feasible profit for the principal.

For $\beta = \lambda$, the principal's Hamilton-Jacobi-Bellman equation (14) can be written as

$$\begin{aligned} rv(u,x) &= x + ruv_u(u,x) + \mu v_x(u,x) \\ &\quad + \sigma^2 \left(\frac{\lambda^2}{2} v_{uu}(u,x) + \lambda v_{ux}(u,x) + \frac{1}{2} v_{xx}(u,x) \right). \end{aligned} \tag{18}$$

By comparing (14) and (18), I find that $\beta = \lambda$ is optimal if and only if

$$\frac{1}{2}(\beta - \lambda)^2 v_{uu}(u,x) + (\beta - \lambda)(\lambda v_{uu}(u,x) + v_{ux}(u,x)) \leq 0. \tag{19}$$

The first term of (19) is negative since excess volatility of the agent's continuation value increases the risk of inefficient turnover;[10] i.e.

$$(\beta - \lambda)^2 v_{uu}(u,x) \leq 0, \tag{20}$$

is maximized at $\beta = \lambda$. The principal optimally exposes the firm to the minimal inefficient contract termination risk that is necessary to sustain incentives. The

[10]See Lemma 3 in the Online Appendix.

second term in (19) is negative since the cost of inefficient turnover is higher the higher the match value with the current agent[11]

$$(\beta - \lambda)(\lambda v_{uu}(u,x) + v_{ux}(u,x)) \leq 0. \tag{21}$$

Again, $\beta = \lambda$ is optimal. (19) together with (20) and (21) imply that $\beta = \lambda$ at the optimum. Principal always terminates the manager's contract at the lowest match value that is compatible with incentives.

6 Fixed Turnover Regime

In this section, I will characterize the optimal contract to the right of the payment boundary. In particular, the condition (17) is satisfied with equality. Let $x_L \geq x^*$ denote the fixed cutoff at which the agent's contract is terminated.

In the fixed turnover regime, the agent's expected payoff only depends on the current match value x and on the cutoff x_L. In particular, his value function $u(x, x_L)$ satisfies the following Hamilton-Jacobi-Bellman equation

$$ru(x, x_L) = c(x, x_L) + \mu u_x(x, x_L) + \frac{\sigma^2}{2} u_{xx}(x, x_L) \tag{22}$$

for all $x \geq x_L$ with the boundary condition $u(x_L, x_L) = R$.

Now I know that the agent's continuation value satisfies the stochastic differential equation (11) with $\beta_t = \lambda$. Also, I know that the process U_t has to hit R the same time that X_t hits x_L. Vice versa, for a fixed pair (x, x_L), the hitting probability has to be the same. By combining the facts, I can show that the agent's continuation value in the fixed turnover regime can be characterized by an ordinary differential equation.

The result is summarized in the following lemma; the details are delegated in the Appendix.

Lemma 3. *The agent's continuation value is the solution of*

$$r(u - R)^2 = \alpha^2 \frac{\lambda^2 \sigma^2}{2}(x - x_L)^2 + \alpha(x - x_L)(u - R)\left(\mu u_x + \frac{\sigma^2}{2} u_{xx}\right) \tag{23}$$

with the boundary conditions $u(x_L, x_L) = R$ and $u(x, x_L) \geq R$ for all $x \geq x_L$ and α as defined in (9). If $\mu = 0$, $u(x, x_L)$ admits the closed form solution

$$u(x, x_L) = \lambda(x - x_L) + R. \tag{24}$$

Proof. See Appendix. □

[11] See Lemma 4 in the Online Appendix.

Remark. *I do not establish a formal verification for the existence of a absolutely continuous payment schedule that satisfies the limited liability condition* $c(x, x_L) \geq 0$ *apart from the case* $\mu = 0$*, which is trivial. Numerical experiments suggest that the conditions are not too restrictive. DeMarzo and Sannikov (2016); Vasama (2017) provide conditions in a framework with persistent private information.*

I next verify that the optimal contract entails a permanent distortion in the turnover policy. For instance, if the expected match value is very high at the contracting stage, it is optimal to threaten the agent with a permanently high turnover risk. Such an event has a very low probability if the agent exerts low effort and is very effective to deter shirking.

The result is summarized in the following proposition

Proposition 2. *The optimal contract entails a permanently inefficient turnover rate.*

Proof. See Appendix. □

The firm value resembles the standard real options problem, only with the difference that the turnover occurs at a fixed cutoff x_L. This can be solved in closed form similar to the first-best value. The principal's expected payoff is then the firm value minus the expected payoff of the agent, i.e.

$$v(x, x_L) = \frac{x}{r} + \frac{\mu}{r^2} - \left(\frac{x_L}{r} + \frac{\mu}{r^2}\right) e^{\alpha(x - x_L)} + (L + R)e^{\alpha(x - x_L)} - u(x, x_L) \qquad (25)$$

where $u(x, x_L)$ is the solution of (23).

7 Characterization of the Optimal Contract

I conclude by summarizing the results from the previous sections. The optimal contract relies on an inefficient turnover threat to provide the agent incentives. It delays payments in the beginning, and rewards him for high higher outputs by increasing his continuation value for the future. Over time, as the payments are delayed, the agent's continuation value u grows high enough compared to x, and the condition (17) is met with equality. The contract implements turnover at a fixed cutoff x_L and the payments to the agent are initiated. Turnover is the more lenient and the agent's salary the higher the stronger the performance in the early stages.

The optimal contract that implements full effort is summarized in the following theorem

Theorem 1. *Starting from the initial match quality* $X_0 > x^*$*, and the agent's initial value* $U_0 \in (0, u(X_0, x^*))$*, the optimal contract attains the profit* $v(U_0, X_0)$ *for the principal. The players' values evolve stochastically in response to the fluctuations of the output, and admit the following dynamics: When* $U_t \in (0, u(x, x_L))$*, where* $u(x, x_L)$ *is the solution of (23), it evolves according to*

$$dU_t = rU_t dt + \lambda\sigma dZ_t. \tag{26}$$

The payments to the agent are delayed, i.e. $dW_t = 0$*. The principal's expected profit at any point is* $v(u, x)$*, which is the unique solution of the following partial differential equation*

$$\begin{aligned} rv(u,x) &= x + ruv_u(u,x) + \mu v_x(u,x) \\ &\quad + \sigma^2 \left(\frac{\lambda^2}{2} v_{uu}(u,x) + \lambda v_{ux}(u,x) + \frac{1}{2} v_{xx}(u,x) \right) \end{aligned} \tag{27}$$

with the boundary conditions $v_u(u(x, x_L), x) = -1$*,* $v(x, x_L)$ *as defined in (25) and* $v(R, x) = L$*.*

Proof. Follows by repeating the argument in DeMarzo and Sannikov (2016); see the Online Appendix for details. □

8 Conclusions

This paper studies the optimal managerial compensation and turnover policy in a framework with serially correlated outcomes. My results deliver interesting insights to the optimal turnover policy that are in contrast with previous theoretical findings. The optimal contract implies an inefficiently high turnover rate in the beginning of the employment relationship and the likelihood of performance-related dismissal decreases with superior performance. Moreover, to save on incentive cost to the agent, the optimal contract relies on an inefficient turnover threat that is permanent, even if the manager's private information is not persistent. Also, the relative pay for performance increases with past performance. Managers who are able to show stronger performance in the beginning of their employment relationship, enjoy permanently higher compensation for their performance at later stages.

9 Appendix

Proof of Lemma 3. The agent's continuation value $u(x, x_L)$ satisfies the Hamilton-Jacobi-Bellman equation (22). Moreover, it has to satisfy the stochastic differential equation

$$dU_t = (ru(X_t, x_L) - c(X_t, x_L))dt + \lambda\sigma dZ_t.$$

The agent's continuation value $u(x, x_L)$ has to hit R the same time that x hits x_L. From the standard real options problem, I know that

$$f(u(x, x_L)) = f(x, x_L) \equiv E[e^{-r\tau}] = e^{\alpha(x - x_L)}. \tag{28}$$

Moreover, $f(u(x, x_L))$ has to satisfy the Kolmogorov equation

$$rf(u) = \underbrace{\left(\mu u_x + \frac{\sigma^2}{2}u_{xx}\right)}_{=ru-c} f'(u) + \frac{\lambda^2\sigma^2}{2} f''(u)$$

with the boundary conditions $f(R) = 1$ and $\lim_{u\to\infty} f(u) = 0$. A straightforward calculation shows that the solution is

$$f(u) = \exp\left\{-\left(\mu u_x + \frac{\sigma^2}{2}u_{xx} + \sqrt{\left(\mu u_x + \frac{\sigma^2}{2}u_{xx}\right)^2 + 2r\lambda^2\sigma^2}\right)\frac{u - R}{\lambda^2\sigma^2}\right\}.$$

Combining this with (28) I find that $u(x, x_L)$ solves

$$-\left(\mu u_x + \frac{\sigma^2}{2}u_{xx} + \sqrt{\left(\mu u_x + \frac{\sigma^2}{2}u_{xx}\right)^2 + 2r\lambda^2\sigma^2}\right)\frac{u - R}{\lambda^2\sigma^2} = \alpha(x - x_L)$$

or,

$$r(u - R)^2 = \alpha^2\frac{\lambda^2\sigma^2}{2}(x - x_L)^2 + \alpha(x - x_L)(u - R)\left(\mu u_x + \frac{\sigma^2}{2}u_{xx}\right)$$

with the boundary conditions $u(x_L, x_L) = R$ and $u(x, x_L) \geq R$ for all $x \geq x_L$. For $\mu = 0$, a straightforward calculation confirms (24). □

Proof of Proposition 2. By combining (22) and (23), I find that

$$\begin{aligned} ru(x, x_L) - c(x, x_L) &= \mu u_x(x, x_L) + \frac{\sigma^2}{2}u_{xx}(x.x_L) \\ &= \frac{r(u(x, x_L) - R)}{\alpha(x - x_L)} - \frac{\lambda^2\sigma^2}{2}\frac{\alpha(x - x_L)}{u(x, x_L) - R}. \end{aligned}$$

The principal's expected profit in the fixed turnover regime can be written as

$$
\begin{aligned}
v(u,x) &= E\left[\int_0^\tau e^{-rs} X_s ds\right] - U_0 \\
&= E\left[\int_0^\tau e^{-rs} X_s ds\right] - U_t + E\left[\int_0^t rU(X_s,x_L) - c(X_s,x_L) ds\right] \\
&= E\left[\int_0^\tau e^{-rs} X_s ds\right] - U_t + E\left[\int_0^t \frac{r(U(X_s,x_L)-R)}{\alpha(X_s-x_L)} - \frac{\lambda^2\sigma^2}{2}\frac{\alpha(X_s-x_L)}{U(X_s,x_L)-R} ds\right].
\end{aligned}
$$

The second line uses (11). The expectation is taken with respect to the information set at the contracting stage.

Suppose that the optimal contract always delays payments until the first-best solution is reached. Then if $x_L = x^*$, the directional derivative of $v(u,x)$ satisfies

$$
\frac{\partial}{\partial \varepsilon}(v(u+\varepsilon,x) - v(u,x))\bigg|_{\varepsilon\to 0} = -1
$$

for all x and u. That is, evaluating the derivative and letting $\varepsilon \to 0$, I find that

$$
-1 + E\left[\int_0^t \frac{r}{\alpha(X_s-x^*)} + \frac{\lambda^2\sigma^2}{2}\frac{\alpha(X_s-x^*)}{(U(X_s,x^*)-R)^2} ds\right] = -1.
$$

Since $X_t \geq x^*$ and $\alpha < 0$, the integrand is always negative, a contradiction.[12] $\square$

[12] Notice in particular that x_L has to depend on the path.

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