Three Essays on Money, Wealth and the Exchange Rate
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Three Essays on Money,
Wealth and the Exchange Rate


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OVERVIEW

The three essays in this study report work that was mainly undertaken at the Research Department of the Bank of Finland in 1983-85. Some of the origins date back to the early 1970s. The essays are based chiefly on material which has previously appeared in the Bank of Finland Research Papers Series and presented in various seminars. While the study as a whole focuses on a narrow area of open economy macroeconomics, each essay is a self-contained inquiry into a topic of its own and can be read independently of the others. The cost of this organization is some overlap in coverage.

With the exception of some later revisions and extensions, the research is reported here in the same order as it evolved. In this way the exposition preserves the process through which the central themes of the study developed. The second essay grows out of the open ends of the first one while the third one seeks to fill a gap in the results derived in the second essay. Reflecting the increasing complexity of the questions posed, the mathematical tools employed in the analysis vary from one essay to the next. The last essay resorts to stochastic calculus and continuous time stochastic optimization methods.

The study deals with issues that arise within the asset market theory of the exchange rate. This rapidly growing area of contemporary macroeconomic literature comprises a rich variety of models, which have in common that they treat the exchange rate as a relative price of assets denominated in different currencies. Consequently, the functioning of the foreign exchange market is governed by wealth owners' willingness at the margin to hold existing asset stocks in their portfolios, rather than by needs to finance international trade flows and other current transactions. The asset market approach in exchange rate theory is a logical, although surprisingly late-born, extension of Keynes' liquidity preference theory of interest. When the asset market approach is set in a general equilibrium context, the
implication is not that the ultimate forces governing international trade play a less important role for exchange rate movements than in earlier theories but that their influence is channelled either through gradual changes in asset stocks or through expectations affecting the demand for assets.

Up until only some years ago, it was fashionable to distinguish between different schools of thought within the asset market theories of the exchange rate, such as the Yale-influenced "portfolio balance approach" and the "monetary approach" stemming from the Chicago tradition. In the present vintage of models with rational expectations and optimizing agents it has become increasingly difficult to keep the earlier schools apart and to find some substance in the old borderlines. The present study is an example of this.

The first essay considers a stability problem that has arisen in the Boyer-Kouri-Branson type, directly postulated portfolio balance models. Dynamics in these models is governed by the interplay between the current balance of payments and the exchange rate. In the second essay we attempt to find the simplest possible way of setting the analysis of this linkage in a framework with optimizing agents. The outcome turns out to be a Sidrauski-type monetary growth model. When regarded as an exchange rate model, it belongs to the "monetary approach", since under the assumption of deterministic foresight all assets but money are perfect substitutes. In the stochastic model of the last essay assets appear as imperfect substitutes. Nevertheless, the essential structural properties of the analogous deterministic model are fully preserved.

All models considered in this study are general equilibrium models by construction but very partial in substance. They are designed to provide insight into a limited range of questions of analytical interest. Considerations that are not pertinent to the discussion are not allowed to enter. While some of the findings may be useful as building blocks for empirically-oriented models, they are hardly testable in isolation. Therefore, no reference is made to the empirical exchange rate literature.
The common analytical environment for all three essays consists of a single commodity world with national currencies and floating exchange rates. Equilibrium in such a world implies purchasing power parity and exchange rates that are merely ratios of national price levels. Variations in the terms of trade and the real exchange rate or issues of "competitiveness" stemming from sticky wages have no place in this setup. The sole source of real economy intrinsic dynamics is wealth accumulation. Keeping within such strict limits means that much of relevance for short-term and cyclical exchange rate dynamics escapes the analysis. The reward is that certain fundamental issues concerning consistent modelling of monetary equilibria in open economy models can be addressed with rigour while maintaining tractability.

The subject matter of the first essay is the so called "negative net foreign asset problem" encountered by several authors who have employed directly postulated portfolio balance models of the exchange rate. The puzzle is that models that at first sight appear quite plausible and well-behaving turn out to produce instability and perverse comparative statics when allowance is made for the occurrence of net debtor positions in foreign currencies. The origin of these difficulties is a destabilizing wealth effect, and the situation is analogous to reversing the real balance effect in a Patinkinesque model by assuming the government to be a net creditor vis-à-vis the private sector.

It has been argued that this problem can be solved by assuming rational expectations and stabilizing speculation. We first demonstrate, using a simple two-asset model as a point of reference, that the local saddle-point stability thereby achieved is not sufficient to remedy the shortcomings of the model. It is then argued that the negative net foreign asset problem is merely an illusion, which can be traced back to hidden inconsistencies in the model specification. In the first place, there is a lack of correspondence between the concepts of income and wealth employed in the model, and second, the assumed saving behaviour violates a national solvency constraint on feasible market equilibria. A revised, equally simple model is constructed that avoids these pitfalls without changing the basic idea of the model. Stability in the revised model is dependent
on the characteristics of the assumed consumption function, but does not depend on the sign of the net foreign currency position, nor on whether expectations are static or rational. In this sense the puzzle appears to be solved.

The stimulus to the research reported in the second essay came from the observation that the existence of a stable steady state in the revised model of the first essay critically depends on a Metzlerian or target saving function, where the propensity to save is assumed to decline with increasing wealth. It is evident that this assumption is rather specific and not in line with the simplest standard formulation of intertemporal consumer choice in the neoclassical growth literature. If consumers are all alike and maximize over an infinite horizon applying a constant rate of time discount to time-additive utilities, convergence must be brought about by falling marginal productivity of capital rather than by increasing impatience of consumers.

When this theory is applied in the context of open economy models, the result is rather uninteresting, unless we allow for the possibility that consumers, while alike nationally, have a different degree of impatience across nations. In this case, the world distribution of wealth will not converge to an interior state but rather towards the boundary of its feasible range. Such a Marxian view of the wealth of nations being in a perpetual process of accumulation or decumulation may not be very plausible when pushed to its logical end, in which the thriftiest nation owns all world wealth. However, the rival view that the world distribution of wealth would, after any disturbance, always return to the same interior steady state set by constant structural parameters is equally implausible.

A casual observation of history might suggest that over long periods of time there have been changes, reversals and perhaps even cycles in the relative thriftiness of countries as cultural, social and technological conditions have evolved. Against this background, it is then rather a pragmatic choice, if, for the purposes of an exchange rate model, we regard wealth redistribution as a trend force maintaining constant exchange rate changes, rather than as a temporary disequilibrium phenomenon. At least this is the second simplest way of
dealing with wealth accumulation in exchange rate models. The simplest is to freeze changes in wealth entirely and focus on other sources of exchange rate dynamics.

If there is a steady state in an asset market model of the exchange rate, it must be assumed stable or saddle-point stable, depending on the type of expectations. Otherwise the model is of little use. But if there is none, it is not immediately obvious what the status of the model should be. To find out, we employ the infinite horizon consumer choice theory without wealth saturation in an optimizing version of the model of the first essay. There is no uncertainty, so that rational expectations degenerate into perfect foresight. In an optimizing model the notion of earning assets being imperfect substitutes is not compatible with deterministic foresight. But a rudimentary dimension of asset choice can be rescued by incorporating money in the utility function. The model is first constructed, solved and studied in the small country framework and then extended to a two-country setting.

Despite the lack of a steady state, the model has a well-defined state-dependent equilibrium path, analogous to the saddle-path in the steady-state models. There are some differences, though, in the way the model functions. When the equilibrium is disturbed, the exchange rate jumps onto a new permanent equilibrium path, rather than onto a dampening adjustment path towards some steady state. When viewed in terms of this model, the exchange rate is thus like a particle that can jump from one orbit to another. Since there is no reference point for long-term equilibrium, the distinction between short-term and long-term effects vanishes, and there is little scope for such phenomena as over- and undershooting.

The model is then used to analyze monetary and fiscal policy in some detail. The findings are much what one could expect on the basis of earlier literature, although they are flavoured by the characteristic dynamics of this model.

The impetus for the research reported in the third essay came from a problem that was encountered in the two-country model of the second
essay. When bonds denominated in different currencies are traded internationally, policy changes implemented in one country can affect equilibrium in other countries by changing the real value of foreign holdings of domestic bonds. Since assets are perfect substitutes in the deterministic foresight framework, these potential "real bond effects" cannot be analyzed. The purpose of the third essay was to fill this gap in the analysis.

Much of the content of the third essay is about a consistent treatment of the fiat money system in a stochastic environment, where uncertainty stems both from the money supply and real output. For that purpose the model is first specified and solved in the closed economy framework. When the model is extended to the two-country setting, it turns out that, given the assumption of homogenous investment opportunities ("world investors"), equilibrium bond holdings and thus real bond effects are zero. This contradicts earlier partial equilibrium results in the international finance literature, according to which the international bond markets serve as a pool for national money supply risks. The reason why is that for a fiat monetary system in which seignorage is paid back to the residents the component of the purchasing power risk of national money supplies which is due to money supply randomness cancels out in general equilibrium. These results stress the importance in any coherent treatment of monetary economies of paying attention to the full implications of money for wealth, prices and risks.

This concludes the description of the main stream of the analysis reported in the present study. The research project also produced some findings on side issues that may be of some interest. The uniqueness of the perfect foresight monetary equilibrium in the small country model of the second essay is investigated in an appendix. The equilibrium is unique for non-negative money supply expansion rates, which, given the "necessity of money" implication of the logarithmic utility function, agrees with earlier findings in the literature. On the other hand, price paths that are implosive relative to the state dependent (constant nominal interest rate) path, turn out to be equilibria as well when the money supply rate is negative. This result reveals that a transversality argument employed in recent literature
to rule out hyperdeflations is in general not valid. The contribution of Seppo Salo for this result as well as for the proof of optimality presented in appendix C of the third essay has been decisive.

A basic result from the closed economy model investigated in the second essay is that the mere substitution of stochastic for deterministic foresight does not affect such a structural property as the superneutrality of money. Some of the earlier work in this field is found to be biased by an improper treatment of government monetary transfers. The survival of the monetary dichotomy in a stochastic closed economy environment is essential for the outcome of the analogous two-country model, viz. that imperfect asset substitutability as such does not invalidate the monetary approach to exchange rate determination.

The closed economy model of the last essay also yields some insight into how random shocks originating in the real and monetary spheres affect the monetary equilibrium of the economy. It is found that the randomness of money supply reduces the nominal interest rate, raises the price level and increases the variance of inflation, but does not affect the average inflation rate. The randomness of output increases both the mean and the variance of inflation. A negative correlation between money and output, whether due to automatic financing of fiscal drag by the central bank or to discretionary countercyclical monetary policy, increases both the mean and the variance of inflation. Conversely, positive correlation, due to accommodation of random changes in money demand, improves price stability by both lowering mean inflation and by increasing the predictability of inflation. The model thus lends some support to views that advocate the independence of central banking from government financing. These results remain valid when the model is extended to the two-country setup. It also turns out that in this model exchange rate changes depend only on monetary randomness. The reason is that, in the absence of wage income and with investors holding identical portfolios of investment goods, the real side randomness affects the demand for money equally in both countries.
STABILITY IN PORTFOLIO BALANCE MODELS OF THE EXCHANGE RATE

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1 INTRODUCTION AND SUMMARY

Following the pioneering work of Boyer (1976, 1978), Branson (1976, 1979) and Kouri (1976, 1980), portfolio balance models of the exchange rate have become a popular strand of current research. The appeal of these models has, however, been marred by problems concerning the existence and stability of asset market equilibrium which may arise when wealth owners have, or wish to have, net-debtor positions in assets denominated in foreign currencies. This "negative net foreign asset problem" represents a challenge to the usefulness of the model since excluding by assumption negative net holdings of foreign currency assets would make as little sense as allowing only surpluses in the theory of international trade.

Although several authors have dealt with the issue, the emphasis has been on local stability rather than the related, more basic problems. The instability derives from the fact that, for some combination of asset yields and risks, net foreign currency holdings can be inferior in the sense that the additional foreign currency borrowing induced by an increase in asset wealth is greater than the additional demand for foreign currency assets. Such a destabilizing wealth effect is most easily identified in models like that of Branson, where asset demands are assumed to be homogeneous in asset wealth. The destabilizing

1FRENKEL and RODRIGUEZ (1975) and DORNBUSCH (1980), Ch. 13, analyse the small country case in a two asset framework of domestic money and "world equity". In these models a net debtor position arises when national holdings of equities fall short of the domestically located capital stock. The problems associated with net foreign currency debt do not emerge, however, since equity holdings are in all cases positive.

wealth effect then occurs whenever the foreign currency net position is negative.

One obvious way to overcome the destabilizing wealth effect is to assume a stabilizing substitution effect stemming from endogenous exchange rate expectations. Kouri (1981), Masson (1981) and Henderson and Rogoff (1982) have shown that under perfect foresight (rational expectations) a steady state in this type of model is locally saddle-point-stable irrespective of the sign of the net foreign currency position. Henderson and Rogoff conclude that negative net foreign asset positions do not constitute an independent source of instability, which can only arise under nonrational expectations or because of destabilizing speculation, the latter being defined as the occurrence of adjustment paths other than the unique stable path to the steady state. The same argument also appears in Branson and Henderson (1985). In contrast, Masson doubts whether the perfect foresight assumption really assumes away the stability problem and argues that full rationality is too strong an assumption, so that the instability associated with net debtor positions is relevant for observed exchange market fluctuations.

The research reported in this paper has been stimulated by difficulties in accepting either of these conclusions. As to the Masson view, there does not seem to be any evidence that private sector borrowing abroad would, within reasonable limits at least, give rise to asset market instability. On the other hand, while rational expectations and speculation - especially when softened by a search process as suggested by Henderson and Rogoff - may well be needed for realistic modeling of such sophisticated markets as foreign exchange markets, it is quite a different matter to maintain that these characteristics are necessary for the markets to work at all. Usually, only assumptions concerning current economic behaviour without much reference to the future are needed to establish market stability in simple macro models. Why then should foreign exchange markets be so special as to collapse in the absence of fully informed speculators? This leads one to suspect that the net-debtor problems are instead caused by neglected aspects of non-expectational economics.
To support the view that the net debtor problems do not stem from the lack of endogenous expectations, we shall first demonstrate that the perfect foresight case is not as free of complications as to warrant the favourable conclusion of Henderson and Rogoff. When the analysis is extended some way in the global direction, it turns out that, unless both initial and desired steady state net holdings of foreign currency assets are positive, the model may fail, either because there is no steady state or because the stable arm is not defined for the relevant initial asset holdings. Although the perfect foresight assumption improves the odds for the frictionless functioning of the model compared to the static expectations case, the difference is not as substantial as might appear on the basis of local stability analysis alone. These results are derived in sections 2 and 3, where section 2 first employs a simple small country framework and section 3 then extends the analysis to the two-country model of Henderson and Rogoff. For comparison, the static expectations versions of the models are also reviewed.

In the last section it is argued that the negative net foreign asset problem is merely an illusion that stems from hidden inconsistencies in model specification. The problem arises because, irrespective of other wealth (the present value of output), the demand for money is assumed to vanish at zero asset wealth, and because situations which represent a state of national insolvency are erroneously identified as potential market equilibria. To illustrate the relevance of these points, a revised version of the small country model is constructed which preserves the basic idea and the simplistic structure of the section 2 model but avoids its inconsistencies. As is fairly typical of postulated macromodels of this type, stability in the revised model is not fool-proof, but requires additional constraints on behavioural relationships. The key point is that the stability of the revised model does not depend on the sign of the net foreign currency position nor on whether expectations are static or rational.
Consider the following model of asset market equilibrium:

\[ m(\varepsilon)w = \frac{M}{e} \]  
\[ \frac{M}{e} + F = w \]  
\[ F = S(w) \]

where \( M \) is the home country money supply (net home currency assets), \( F \) the net supply of foreign currency assets, \( w \) real private asset wealth, \( e \) the exchange rate and \( \varepsilon \) the expected rate of depreciation of the home currency. The supply of money is exogeneous and constant over time. It is furthermore assumed that \( m > 0 \) and \( m', S' < 0 \). The function \( m \) may be bounded from above, but it is essential that it can assume values in excess of one. Otherwise the model is unable to deal with negative \( F \). In order for the model to have a steady state equilibrium it is necessary that there exists a \( \bar{w} > 0 \) such that \( S(\bar{w}) = 0 \).

The background scenario is as follows. The small country produces homogeneous tradeable output, the world demand for which is fully elastic at the prevailing world market price. The world market price is constant and equals one in terms of foreign currency, while the home currency output price equals the exchange rate \( e \). Domestic output is constant as determined by full employment of the domestic labour force. There is no capital or investment. The government has no expenditure, nor does it collect taxes. The current account balance equals private saving \( S \). Home money is not traded internationally. Consequently, all existing \( M \) has to be held in domestic portfolios.

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3Since short-term home currency bonds bear the same purchasing power risk as non-interest bearing currency, the variable \( M \) in the model can also be interpreted as the net supply of a consolidated (non-traded) home currency asset. If domestic currency bonds are modelled separately as in BRANSON (1976) and KOURI (1980), the domestic interest rate appears as an endogenous variable determined by the relative supplies of money and bonds. The negative net foreign asset problem arises in such a three-asset model as well.
F can change only over time through a surplus or deficit on the current account. If home money were internationally tradeable, wealth holders would be able to swap domestic for foreign assets and neither of the asset supplies in the domestic markets of the small country would be momentarily predetermined. Interest rates are considered as exogenous constants. The asset market clears continuously in time.

Assume first static expectations, insert \( \varepsilon = 0 \) into the asset market clearing condition (1) and solve for \( e \) as

\[
e = \frac{(1-\bar{m})M}{\bar{m} F}
\]

where \( \bar{m} \) stands for \( m(0) \). In order for a meaningful equilibrium \( e > 0 \) to exist, it is necessary and sufficient that

\[
(1-\bar{m})F > 0
\]

This means that the actual and desired net holdings of foreign currency assets must conform in sign. If \( \bar{m} < 1 \), asset market equilibria exist for positive but not for negative \( F \). If \( \bar{m} > 1 \), asset market equilibria exist for negative \( F \) only. The solution for \( w \) equals \( F/(1-\bar{m}) \) and is positive when condition (5) holds.

---

\(^4\)The non-tradeability of domestic currency assumption is the simplest but not the only way of specifying the small country case. Henderson and Rogoff as well as Boyer derive the small country model as a limiting case of the more general two-country model.
Figure 1 illustrates the model under static expectations. The two alternative MM-curves indicate instantaneous equilibrium values of the exchange rate for different current values of $F$. When $\bar{m} < 1$, the MM-curve is defined for $F > 0$, and when $\bar{m} > 1$, the MM-curve is defined for $F < 0$. The $F = 0$ curve in Figure 1 plots the values of $e$ and $F$ that are compatible with wealth saturation at $\bar{w}$:

$$e = \frac{M}{\bar{w} - F} \quad (6)$$

Two alternative steady state equilibria, A and B, are shown by the intersection of the $F = 0$ curve with the relevant MM-curve.

In the case of static expectations the adjustment path of $e$ and $F$ coincides with the relevant MM-curve. By eq. (3), the movement of $F$ along the MM-curve proceeds in the direction of the $F = 0$ curve. Therefore, steady states at $F > 0$ are stable, while steady states at
F < 0 are unstable, as shown by the arrows in Figure 1.\(^5\) Since the MM-curves associated with steady states at F > 0 are defined for F > 0 only, both the initial and steady state F must be positive in order for the model to have a useful life. If we neglect the rare cases where either F or 1-\(\bar{m}\) or both are exactly zero, and, in the absence of other information, regard the four combinations of signs of initial and steady state F as equally likely, the model has a one in four chance of functioning properly. Note, in addition, that the comparative statics of the model (across steady state equilibria) does not make much sense when F < 0, and that there is no way the market could change the sign of its net holdings of foreign currency assets without government intervention. The model is simply not designed to be a useful tool for analysing cases where either the desired or the actual net foreign currency holdings or both can be negative.

Consider next the case of perfect foresight:\(^6\)

\[ \varepsilon = \frac{\dot{e}}{e} \]

Equation (7) makes \(\varepsilon\) endogenous without adding an independent restriction to the model and therefore turns the dynamic system into an underdetermined differential equation in F and e (or w). The extra

\(^5\)More formally, a steady state at F < 0 is unstable since \(F_F = S''/(1-\bar{m}) > 0\) for \(\bar{m} > 1\). If, instead of the real wealth channel of eq. (3), one prefers to assume favourable relative price effects and writes \(F = S(e), S' > 0\), the result is the same: \(F_F = -S(\bar{m})S''/F^2(1-\bar{m})/\bar{m} > 0\) for \(\bar{m} > 1\). If we drop the assumption of continuous market clearance, keep F constant in the short run and assume a tântonnement-type price adjustment process for the foreign exchange market, negative F also results in instability. In that case we have \(Z_e = -\bar{m}F > 0\) for F < 0, where Z stands for the excess demand for net foreign currency assets.

\(^6\)Since uncertainty is not modelled explicitly, rational expectations must be treated technically as perfect foresight. While deterministic foresight is in general inconsistent with imperfect asset substitutability, in the simple two-asset model we are dealing with we may think of domestic money as being differentiated from other assets (F) because of its role as a means of payment.
degree of freedom is used by assuming that speculation is stabilizing in the sense that, whenever feasible, the foreign exchange market selects an exchange rate that is compatible with a path to the steady state.

Figure 2: THE SMALL COUNTRY MODEL UNDER PERFECT FORESIGHT

Figure 2 illustrates the model under perfect foresight. Let ε first be a free parameter which can be varied at will. Keeping M constant, eq. (1) then defines a family of asset market equilibrium curves in the (F; e > 0)-space. Each of these hyperbolas represents combinations of e and F that are compatible with asset market equilibrium at a given expected rate of depreciation. Since the expected rate equals the actual rate of depreciation, each of these curves also represents the locus of points where the relative speed of adjustment of e is constant. The steady state is indicated by the intersection of the F = 0 curve with the particular asset market equilibrium curve at which ε = ̇e/e = 0. The steady state is, of course, identical to that of the static expectations case. Two such MM- curves are plotted in Figure 2, and the two alternative steady states are indicated by points A and B.
Points situated to the left (right) of the relevant MM-curve lie on asset market equilibrium curves where the exchange rate is falling (rising). Therefore, the steady state is a saddle point irrespective of the sign of the steady state $F$. The unique trajectories leading to the steady states $A$ and $B$ are marked with arrows.

Perfect foresight in combination with stabilizing speculation thus results in local stability independently of the sign of the steady state $F$. The model now also provides for adjustment paths that lead from positive to negative net foreign currency positions or vice versa. It is, however, important to note that the usefulness of this property is limited by the non-linearity of the model. The essential point is that all adjustment paths, whether stable or unstable, are bounded from the left. This follows from the fact that the family of asset market equilibrium curves has a limit curve on the left

$$e = \theta(M/F)$$

where

$$\theta = \lim_{e \to -\infty} (1-m)/m$$

The parameter $\theta$ is negative but not less than minus one. It equals minus one unless $m$ is bounded from above. In Figure 2 the limit curve (8) is labelled $\dot{e}/e = -\infty$. When approaching the limit curve, all trajectories tend to become vertical. No asset market equilibria

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7 For a proof that the steady state is of the saddle point variety, differentiate the model and solve for $de$ and $dF$. The Jacobian $\dot{e}F - \dot{F}e$ reduces to $-mS'/m'$, which is negative. Therefore, if linearized in the steady state, the differential equation in $e$ and $F$ has real roots of opposite sign.

8 On the limit curve, $w$ vanishes, but $F$ must remain finite, if only because net exports cannot exceed output.
exist to the left of this curve. Since the trajectories to the steady states are downward sloping, they are always defined for a part of the negative F-region only.

Assume that the markets are initially in the steady state at point A. Then let the steady state shift to point B, say, because of a rise in the foreign interest rate. Since the initial F is fixed at A, speculators, having noticed the change in the steady state, should cause the exchange rate to jump upwards to the trajectory leading to the new steady state B. But, this trajectory does not exist above point A, and there is nothing the speculators can do about it. In the case shown in the figure, the new steady state is situated in the positive F-region, but the problem can also arise for a steady state B in the $F < 0$ region. If asset demands are sensitive to yield differentials, little disturbance may be needed to cause a situation like this.

From Figure 2 it is apparent that the admissible range of negative initial F depends on the position of the steady state. Other things being equal, this range becomes shorter as the steady state shifts further towards the northeast. Obviously, the maximum initial net foreign currency debt that the model can absorb can be made arbitrarily small by shifting the steady state towards the northeast. In this sense the case of non-negative F is the only one where we can be sure that the model does not fail.⁹

The conclusion from the above analysis is that, while perfect foresight remedies the local instability problem of the static expectations model, global saddle-point stability is secured for $F > 0$ only. Thus the negative net foreign asset problem is transformed into another form rather than being removed. Moreover, the steady state comparative statics of the model remains dependent on the sign of F and is perverse for $F < 0$.

⁹Note, however, that the stable arm connected with a given steady state is defined for $F$ greater than or equal to the steady state $F$. The model thus secures one-way traffic from steady states involving positive $F$ to steady states involving negative $F$ and likewise from positions with little foreign currency debt to positions with more foreign currency debt. It is the return ticket that is uncertain.
3 THE TWO-COUNTRY MODEL

The world consists of two countries, which trade in a homogeneous world output and in two national currencies. The world supplies of the two national "monies", one denoted by M and the other by F, are exogenous and regarded as positive constants in the following analysis. National holdings of the two assets add up to the existing world supplies. Since both assets are tradeable, wealth owners may at any time swap one currency for the other. Therefore, asset market equilibrium and portfolio balance enter as two independent restrictions in the model. This, in turn, necessitates a distinction between actual and "initial" asset holdings. Actual national holdings are denoted by \( M_1, M_2, F_1 \) and \( F_2 \) respectively and they are currently endogenous variables. Initial holdings are denoted by \( \bar{M}_1, \bar{M}_2, \bar{F}_1 \) and \( \bar{F}_2 \) respectively and they are (in the absence of open market operations and exogenous wealth transfers) defined as

\[
\bar{M}_1 = \lim_{t \to t_0} M_1, \quad t < t_0
\]

and similarly for \( \bar{M}_2, \bar{F}_1 \) and \( \bar{F}_2 \). These time limits from the past of the actual holdings are predetermined for each instant of time. They equal the actual holdings, except for those particular moments when portfolio balance undergoes a discrete jump.10

The world asset market clearance condition is

\[
m_1(e)W_1 + m_2(e)W_2 = M
\]

where

10Here we have followed Boyer rather than Henderson and Rogoff, who do not make an explicit distinction between initial and actual asset holdings. The role of the predetermined initial holdings is equivalent to that of the exclusion of disequilibrium trading in general equilibrium theory.
The condition for portfolio balance in the first country is

\[ W_1 = \frac{M_1}{M_1 + eF_1} \]  \hspace{1cm} (12)

\[ W_2 = \frac{M_2}{M_2 + eF_2} \]

The condition for portfolio balance in the first country is

\[ M_1 = m_1(e)W_1 \]  \hspace{1cm} (13)

World asset market equilibrium and portfolio balance in the first country imply portfolio balance in the second country. The role of the initial asset holdings entering the wealth constraints (12) is to prevent currency swaps from affecting the equilibrium exchange rate. Otherwise the asset market equilibrium would not be well-defined.\(^{11}\)

The asset market variables are subject to the following sign constraints:

\[ M, F, M_1, F_2, m_1, 1 - m_2 > 0 \]  \hspace{1cm} (14)

\[ m_1^*, m_2^* < 0 \]

In addition, it is assumed that for any given \( e \)

\[ m_1 > m_2 \]  \hspace{1cm} (15)

and that

\[ \frac{M_1}{M} > \frac{F_1}{F} \]  \hspace{1cm} (16)

---

\(^{11}\)If the distinction between initial and actual asset holdings is not made, the asset market equations can be merged by elimination of identities into a single equation: \( m_1(M_1 + eF_1) + m_2(M - M_1 + e(F - F_1)) = M \). This would not be sufficient to determine the three endogenous variables \( M_1, F_1 \) and \( e \).
These latter restrictions are not necessary for our general conclusions, but merely serve to exclude less likely cases from the analysis.

The world goods market equilibrium condition is

\[ S_1(W_1/P) + S_2(W_2/P) = 0 \]  

(17)

where \( P \) is the world output price in terms of the first country's currency, and \( S_1 \) and \( S_2 \) are the saving functions of the two countries. In both countries saving is a decreasing function of real asset wealth, and it is assumed that real asset wealth \( \bar{w}_1 > 0 \) and \( \bar{w}_2 > 0 \) exists such that \( S_1(\bar{w}_1) = 0 \) and \( S_2(\bar{w}_2) = 0 \). The goods market is dynamically linked to the asset market through the balance of payments identity

\[ \dot{M}_1 + e\dot{F}_1 = P S_1 \]  

(18)

It follows from eq. (17) that the rate of asset purchases of the first country equals the rate of asset sales of the second country.

Consider first the case of static expectations, where \( m_1 \) and \( m_2 \) are fixed at \( m_1(0) = \bar{m}_1 \) and \( m_2(0) = \bar{m}_2 \) respectively. Under static expectations the structure of the model is recursive. The asset market equations determine \( e, W_1, W_2 \) and the actual asset holdings as functions of the initial asset holdings. The goods market then solves for the world price level \( P \), real asset wealth \( w_1 \) and \( w_2 \), national savings \( S_1 = -S_2 \), and, through the balance of payments, sets the rate of change in the asset market equilibrium. International trade in assets, together with the moving exchange rate equilibrium determines the rate of wealth redistribution among the two countries.

For the purposes of this paper it is helpful to consider the asset
market equilibrium in terms of wealth distribution. Let \( \rho \) stand for the first country's share of world wealth:

\[
\rho = \frac{M_1 + eF_1}{M + eF}
\]  

Eq. (19) defines the locus of \( e \) and \( \rho \) that are compatible with a given set of initial asset holdings. In the following we shall refer to this locus as the "jump curve", since in the case of a shift in asset demands the subsequent jump in asset market equilibrium has to take place along this curve. Portfolio balance requires that asset wealth in both countries be positive. Otherwise the sign constraints (14) would be violated. Therefore the jump curve is defined for those \( e > 0 \) only, which map into \( 0 < \rho < 1 \).

Substituting from eq. (19) into the asset market clearance condition (11) and solving for \( e \) gives

\[
e = \frac{(1-m)M}{eF}
\]  

where

\[
m = \rho \tilde{m}_1 + (1-\rho)\tilde{m}_2
\]  

Eq. (20) defines the locus of \( e \) and \( \rho \) that are compatible with asset market equilibrium. Wealth redistribution through the current balance of payments takes place along this "adjustment curve". Apart from possible points of discontinuity, the adjustment curve is defined for those \( \rho \) in \( 0 < \rho < 1 \) that map into \( e > 0 \). This implies that eq. (21) is defined for those \( \rho \) in \( 0 < \rho < 1 \) that map into \( 0 < m < 1 \).

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12 Henderson and Rogoff analyse the dynamics of the model in terms of the first country's wealth valued at the steady state exchange rate. This helps to improve the transparency of the analysis in some respects, but requires the existence of the steady state. A similar transformation is also employed by Boyer.
The existence of an instantaneous asset market equilibrium requires that the jump curve (19) intersects the adjustment curve (20) at some $e > 0$, $0 < \rho < 1$. For this to happen, it is sufficient that in both countries the initial and desired holdings of net foreign assets conform in sign, i.e. that $(1-\tilde{m}_1)E_1, \tilde{m}_2M_2 > 0$, and necessary that $(1-\tilde{m}_1)E_1, \tilde{m}_2M_2, (1-\tilde{m}_1)\tilde{m}_2$ are not all negative.\(^{13}\)

Consider next the existence of a steady state. In a steady state the distribution of wealth is fixed at

$$\tilde{\rho} = \frac{\bar{w}_1}{\bar{w}_1 + \bar{w}_2} \quad (22)$$

For $\tilde{\rho}$ to be compatible with a positive steady state exchange rate, it is necessary and sufficient that $0 < \bar{m} < 1$, or

$$0 < \bar{m}_1 + (1-\tilde{\rho})\bar{m}_2 < 1 \quad (23)$$

Since $m$ in eq. (20) can be interpreted as the aggregate desired share of the first country's money in the world portfolio, and $1-m$ as the aggregate desired share of the second country's money, condition (23) states that the steady state desired world holdings of both assets must be positive. Otherwise no positive exchange rate is consistent with asset market equilibrium and wealth saturation in both countries.

Condition (23) is fulfilled for all $0 < \tilde{\rho} < 1$, if and only if the desired net foreign currency holdings in both countries are positive. If $1-\tilde{m}_1 < 0$ ($\tilde{m}_2 < 0$), condition (23) is violated for high (low) values of $\tilde{\rho}$. For a steady state to exist, the relative target wealth of a

\(^{13}\)These conditions are easily derived from a graphical inspection of equations (19) and (20). In general, the existence of a fixed point for the mapping defined by equations (19) and (20) does not depend on the respective signs alone. A set of necessary and sufficient conditions can be derived by solving the linear equations (11) and (12) for $e$, $W_1$ and $W_2$ and imposing the condition for this solution vector to be positive. Note that, while in the small country model condition (5) for the solution $e$ to be positive implied a solution $w > 0$, in the present model $e > 0$ only implies $W_1 + W_2 > 0$. 
debtor country must not be too high, or put in another way, the desired net foreign currency debt in a relatively large country must not be too high. If both \(1-m_1\) and \(m_2\) are negative - a less likely case - condition (23) holds only for intermediate values of \(\delta\).

In the small country model no condition like (23) appeared and the assumption of wealth saturation was sufficient for the existence of the steady state. It is now clear that this was only because of the restrictive assumptions employed. The non-tradeability of the small country's home currency precluded the possibility that world demand at the level of wealth saturation for the small country's currency would be negative.

As in the small country case, the stability of the steady state depends on the wealth effect associated with a change in the exchange rate. Saving behaviour alone would move the wealth distribution towards \(\delta\), but the model can be destabilized if the co-movement of \(\delta\) works in the opposite direction. Differentiating equations (19) and (20) with respect to time, substituting from equations (16) and (18) and solving for \(\dot{\delta}\) yields

\[
\dot{\delta} = \frac{m(1-m)(S_1/w)}{\psi}
\]

(24)

where \(w\) equals \((W_1+W_2)/\delta\) and

\[
\psi = \rho\tilde{m}_1(1-\tilde{m}_1) + (1-\rho)\tilde{m}_2(1-\tilde{m}_2)
\]

(25)

Stability requires that \(\dot{\delta}\) has the same sign as \(S_1\). This is the case when

\[
\psi > 0
\]

(26)

Therefore, a steady state is locally stable if and only if condition

\[
\psi > 0 \quad (26)
\]

implies \(0 < m < 1\). Letting \(\sigma^2\) denote the common variance of the \(\rho\)-weighted distributions of the desired portfolio shares \(\tilde{m}_i\) and \(1-\tilde{m}_i\), \(i = 1,2\), condition (26) can also be expressed as \(m(1-m) > \sigma^2\).
(26) holds for $\bar{\rho}$.\textsuperscript{15} Equation (24) has a discontinuity, at $\psi = 0$, which corresponds to the discontinuity at $F = 0$ in the small country model. For such a point equations (19) and (20) lack a well-defined solution.

If $1-m_1$ and $m_2$ are both positive, the stability condition (26) holds for all $\bar{\rho}$. Even in this case the model is not globally stable, since the (instantaneous) asset market equilibrium does not exist for all initial asset holdings. If $1-m_1$ and $m_2$ are both negative, the model is definitely unstable. When $1-m_1 (m_2)$ is negative and $m_2 (1-m_1)$ positive, the steady state is locally stable only for sufficiently low (high) values of $\bar{\rho}$. Since condition (26) is more stringent than condition (23) for the existence of a steady state, the set of possible steady states is divided in these cases into a stable and an unstable region according to the size of $\bar{\rho}$. Moreover, the adjustment path leading to a locally stable $\bar{\rho}$ is bounded by the discontinuity at $\psi = 0$.

Figure 3 illustrates the functioning of the model in the case where it is well-behaved with certainty: Both initial and steady state net holdings of foreign currency assets are positive in both countries. Markets are initially in a steady state at point A. The steady state then shifts upwards to point C, say, because of a rise in the rate of interest payable on the second country's assets. The change in asset demands makes the market jump to point B on the adjustment curve associated with the new steady state. The upward jump in the exchange rate causes a windfall transfer of real wealth in favour of the second country. This stimulates saving in the first country and dissaving in the second. The implied transfer of assets, enforced by the gradual fall in $e$, pushes the markets towards point C and the old $\bar{\rho}$. Since in this case the jump curve is defined for all $e > 0$ (the jump curve has a vertical asymptote at $\rho = F_1/F$) and the adjustment curve for all $\rho$ in $0 < \rho < 1$, there necessarily exists a point of intersection like B in Figure 3. Had the initial holdings of foreign currency been negative

\textsuperscript{15}Henderson and Rogoff derived the stability condition in terms of the determinant of equations (11) and (12) as $m_1 F_1 + m_2 F_2 > 0$. Multiplying this by $e/(W_1+W_2)$ and substituting from equations (13) and (19) yields condition (26).
in the first country \( \frac{F_1}{F} < 0 \), this would not necessarily have been the case.

Figure 3: THE TWO-COUNTRY MODEL UNDER STATIC EXPECTATIONS:
A STABLE CASE

If the shift in asset preferences in favour of the second country's currency is strong enough to make \( \bar{m}_2 \) negative, the model can fail for any of the following reasons: (a) a steady state does not exist, (b) the steady state is unstable, (c) a locally stable steady state exists, but the instantaneous equilibrium lies on an unstable adjustment path, or (d) an instantaneous equilibrium does not exist. Figure 4 illustrates case (c): The jump curve from the initial state A intersects the new adjustment curve at point B. A new, locally stable steady state exists at C, but the adjustment path passing through the instantaneous equilibrium at B is separated from the stable path to C at \( \rho^* \), where \( \psi = 0 \).
Let us now turn to the perfect foresight version of the model and start with a summing-up. It was shown above that, unless the desired steady state net foreign currency holdings are positive in both countries, the model may not have a steady state. When a steady state exists, i.e. when condition (23) holds, it is unique and can be shown to be a saddle point independently of the signs of the desired net foreign currency holdings. This is the central result of Henderson and Rogoff. But again it turns out that, unless the initial foreign currency holdings in both countries are positive, the stable arm leading to the steady state may not be defined for the relevant initial holdings. Therefore, the only safe case occurs when both initial and desired steady state net foreign currency holdings in both countries are positive. Although the odds of the model not failing outside this safe case are clearly better in the perfect foresight version, it is worth noting that exactly the same general conclusion emerged from the analysis of the static expectations version of the model.
In the present framework, which focuses on the variables $e$ and $\rho$, the stability analysis can be briefly summarized as follows. Since $e = \hat{e}/e$ is an endogenous variable, the asset market clearance condition defines a differential equation:

$$\dot{e} = \phi^1(e, \rho)$$  \hspace{1cm} (27)

Differentiating eq. (19) and substituting from eqs. (12), (13) and (18) yields

$$\dot{\rho} = (S_1/w) - \rho(1-\rho)(m_1-m_2)(\hat{e}/e)$$  \hspace{1cm} (28)

With further substitution from eq. (27) and the goods market equilibrium condition (17), we arrive at

$$\dot{\rho} = \phi^2(e, \rho)$$  \hspace{1cm} (29)

Following the usual procedure of linear approximation, the saddle point property of a steady state of eqs. (27) and (29) can be established by showing that the Jacobian of the system is negative in the steady state. When evaluated in an assumed steady state $(\bar{e}, \bar{\rho})$, the Jacobian reduces to

$$\phi^1 \phi^2 - \phi^2 \phi^1 = \left( \frac{\bar{m}(1-\bar{m})S_1'S_2'}{\bar{m}'S'} \right)$$  \hspace{1cm} (30)

where the bars refer to the steady state values of the respective variables, $\bar{m}' = \bar{\rho}m_1' + (1-\bar{\rho})m_2' < 0$ and $\bar{S}' = \bar{\rho}S_1' + (1-\bar{\rho})S_2' < 0$. The Jacobian (30) is negative if $0 < \bar{m} < 1$, i.e, whenever the steady state exists.

Figure 5 illustrates the model under perfect foresight. In the case shown, the desired net foreign currency holdings are negative in the first country and positive in the second. The $\hat{e}/e = 0$ curve is thus not defined for $\rho \geq \rho^* = (1-\bar{m}_2)/(\bar{m}_1-\bar{m}_2) < 1$. However, the stable arm
leading to the steady state $C$ is defined for all $\rho$ in $0 < \rho < 1$. Therefore, perfect foresight not only removes the problems caused by instability in the traditional sense but also reduces the number of cases where instantaneous asset market equilibrium does not exist.

The remaining two reasons why the model may fail are also indicated in Figure 5. Should the relative target wealth of the first country be as high as $\rho'$, the steady state and the associated stable arm would not exist. To eliminate the possibility of this case, the desired steady state

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16To see why, assume that the stable arm is defined for $\rho < \rho^{**} < 1$ only. Then $e \to 0$ and $\dot{e}/e \to +\infty$ along the stable arm as $\rho \to \rho^{**}$. But $(e = 0, \rho = \rho^{**})$ is the limiting point on an asset market equilibrium curve at some finite $\epsilon > 0$. Approaching $(e = 0, \rho = \rho^{**})$ would thus imply $\dot{e}/e \neq \epsilon$, a contradiction of the perfect foresight assumption.
state net foreign currency holdings should be positive in both
countries. The second reason is the same as was found in the small
country case, viz. the stable arm is not defined for the relevant
initial asset holdings. In Figure 5 this is shown by the jump curve,
starting from the initial state A, not having an intersection with the
stable arm to the steady state C. When \( F_1/F < 0 \), the domain of the
jump curve is bounded by \( e < -(M_1/F_1) \). On the other hand, along the
stable arm \( e + e^* \), \( e < e^* < \left([1-m_2]/m_2\right)(M/F) \), as \( \rho \to 0 \). Again, by
shifting the steady state upwards and to the right, the amount of
negative initial \( F_1 \) that the model can cope with can be made
arbitrarily small. By symmetry, the same problem arises when the
second country initially has a negative net foreign currency position.
To eliminate the possibility of this second type of failure, the
initial net foreign currency holdings must be positive in both
countries.
4 A REVISED MODEL

The findings of the previous sections show that the difficulty caused by negative net foreign assets is not really solved by the substitution of rational for static expectations, but rather transformed from a local stability problem into a global one. We should therefore look more carefully at the non-expectational economics of the model. The negative net foreign asset problem appears to be unknown in that strand of related literature in which macro relationships are explicitly derived from the optimizing behaviour of agents rather than postulated directly. While no general superiority can be claimed for such micro-macro models, they seem to offer some protection against inconsistencies that otherwise easily slip into macroeconomic reasoning.

A reasonable requirement in any model containing both flow and stock relationships is that there is a correspondence between the concepts of income and wealth employed in the model. In the models of the previous sections a constant real output was assumed. Therefore, the relevant total wealth variable for these models includes the present value of output, say \( Y/r \), where \( Y \) stands for output and \( r \) for the real interest rate. Having noted this, it clearly becomes implausible to assume that the demand for money would vanish at zero asset wealth. We might instead assume that the demand for money is proportional to total wealth rather than asset wealth. Let us thus redefine \( w \) in the small country model as

\[
\frac{M}{e} + F + \frac{Y}{r} = w \tag{2a}
\]

and consider the model consisting of eqs. (1), (2a) and (3).

The effect of the wider concept of wealth (2a) is to shift the critical point of the model from \( F=0 \) to \( F+(Y/r)=0 \). The region of well-behaved asset market equilibria in Figures 1 and 2, respectively, now extends to the negative \( F \) zone down to \( F=-(Y/r) \). This alone does not solve our puzzle, however. It would still appear as if, beyond this critical point, there looms another potential set of misbehaving asset market equilibria.
Another obvious requirement for any macro model is that behavioural equations are not inconsistent with accounting identities. In the small country model the accounting identity for the current account balance is

\[
F = Y + rF - C \quad (3a)
\]

where \(C\) stands for consumption. Since \(C\) must be non-negative, the net foreign debt of the country will grow without limit, if initially we have \(F < -(Y/r)\). Such a state of national bankruptcy is not a market equilibrium. This observation has two consequences. First, no asset market equilibria exist for \(F < -(Y/r)\). The mysterious mirror hyperbolas beyond the critical point are thus meaningless, a mere illusion stemming from a misspecified model.\(^{17}\) Second, eq. (3) is not an admissible hypothesis of saving behaviour, since it contradicts the implications of (3a). As a close substitute for (3) we might consider (3a) transformed into a behavioural equation by adding the assumption

\[
C = C(w), \quad C' > 0 \quad (3b)
\]

By (3a) and (3b), \(\dot{F}\) is not only a decreasing function of \(w\) but also an increasing function of \(F\).

Basically, the fault of equation (3) is that it fails to make a distinction between two concepts of saving. Given the assumptions of the small country model, \(\dot{F}\) is identically equal to saving according to the national accounts definition of unconsumed current income. If consumption (as a first approximation) is a function of total wealth, saving by this definition must depend both on total wealth and its composition. From a behavioural point of view, a more relevant concept of saving is \(\dot{w}\), which includes the capital gains or losses due to

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\(^{17}\)If one keeps to the assumption that wealth consists of \(M/e + F\) and nothing else, then \(Y = 0\) and equilibria do not exist for \(F < 0\). In part of the literature on the net foreign asset problem (e.g. KOURI (1980), MASSON (1981)) the current account dynamics is specified in a way that is consistent with (3a). The implied constraint on feasible asset market equilibria appears, however, to have escaped the analysis.
currency depreciation. Saving in this sense may well be assumed independent of the composition of total wealth, without contradicting the accounting identity (3a).

The revised small country model consisting of equations (1), (2a), (3a) and (3b) lacks the inconsistencies of the section 2 model and, subject only to the national solvency constraint, it is able to cope with any amount of external debt without giving rise to complications that are specific to a net debtor position. However, as is usually the case with postulated macro models of this type, additional restrictions are needed to guarantee local and global stability. A set of sufficient conditions for the existence of a unique stable (or saddle point stable) steady state is

\[ 0 < \bar{m} < 1 \]  
\[ C'' > 0, \quad C(0) = 0, \quad C'(0) = 0, \quad C'(\infty) < r \]  

Figure 6 illustrates the functioning of the model in the ideal case, where all the conditions (1a) and (3c) are fulfilled. The upper panel of Fig. 6 is drawn in the \((M/e,F)\)-space, which is more illuminating in this case. To facilitate comparison with Figures 1 and 2, the lower panel of the figure depicts the same relationships in the \((e,F)\)-space. The following comments refer to the upper panel of Fig. 6.

The MM curve depicts the points representing asset market equilibrium for a zero expected rate of depreciation. Along the \(F=0\) curve the current account is in balance. The curves meet at \(M/e = 0, \quad F + (Y/r) = 0\), at which point total wealth is zero. The MM-curve is a straight line, which by (1a) has the slope \(0 < \bar{m}/(1-\bar{m}) < \infty\). By (3c) the slope of the \(F=0\) curve, which equals \((r-C')/C'\), initially exceeds the slope of the MM-curve, but tends to become non-positive as \(F\) grows. The curves thus have a unique intersection for \(F > -(Y/r)\).
Under static expectations adjustment takes place along the MM-curve. Since the current account is in surplus below the $F=0$ curve and in deficit above it, the movement proceeds towards the steady state at $F$. Under perfect foresight the steady state is saddle-point stable, since the domestic currency is expected to depreciate below the MM-curve and appreciate above it. In the absence of speculative behaviour, the markets move along a saddle path which lies between the MM- and the $F=0$ curves and is thus defined for all feasible $F$.

No matter whether expectations are static or rational, the market equilibrium paths are monotonic and currency appreciation is
associated with a surplus and depreciation with a deficit on the current account. This result depends on the simplifying assumptions of the model. Since there is no investment, a current surplus implies growing domestic wealth, which raises the demand for real balances. As the nominal supply of money is constant, the growing demand for real balances deflates domestic prices and the price of foreign exchange. When allowance is made for an endogeneous capital stock and investment, w and F need not move in the same direction. If, for example, a capital inflow and a corresponding current deficit is induced by an exogeneous increase in productivity, one would expect the exchange rate to appreciate rather than depreciate. If the money supply is allowed to change, the exchange rate need not move in the opposite direction to wealth. All this shows that the simple model here deals with only one channel of intrinsic exchange rate dynamics.

Let us now look more closely at the restrictions (1a) and (3c) that were needed to secure the global stability of the model. Condition (1a) states that the desired domestic currency holdings must not exceed total wealth when the exchange rate is expected to stay constant. This sounds a reasonable constraint, given the wider concept of wealth employed in the revised model. Recall that in the section 2 model we needed $m>1$ to make the model cope with a negative F.

The constraints (3c) lie on much weaker ground. Clearly, if we only assume $c'>0$, there could be multiple steady states or no steady state at all. However, if we only seek a model that would serve as a useful tool of analysis over some relevant range of F, weaker conditions than (3c) will do. The necessary and sufficient condition for local stability (saddle-point stability) of a steady state is that (in the upper panel of Fig. 6) the slope of the MM-curve exceeds the slope of the $F=0$ curve in the vicinity of the steady state.\(^\text{18}\) The

\(^{18}\)Henderson and Rogoff draw attention to the unusual property of the section 3 model that instability (when $F<0$) under static expectations is transformed into saddle-point stability under perfect foresight. This curiosity is due to the inconsistencies of the model. In the revised model saddle-point stability goes hand in hand with ordinary stability.
basic behavioural assumption that is needed for the existence of such a locally stable steady state is that, for relatively low levels of wealth, saving is sufficiently lucrative to keep the $F=0$ curve above the MM-curve, but that the marginal propensity to consume out of wealth falls with growing wealth. While such a tendency for wealth accumulation to saturate is far from obvious, this is about as far as we can go without entering the microeconomics of consumer behaviour.

To sum up, by correcting the rather obvious structural inconsistencies of the small country model of section 2, it is possible to construct an equally simple macromodel which has at least a fair chance of working properly both under static and rational expectations. Although stability in the revised model ultimately has to be assumed rather than based on any very plausible behavioural invariances, this is often the case in directly postulated macromodels. The key result is that the complications which may arise in this model are in no way dependent on the sign of $F$. In this sense the puzzle of "the negative net foreign asset problem" appears to be solved. An analogous revision of the two-country model of section 3 would probably yield a similar conclusion. It appears to be more fruitful, however, to carry out further analysis of this and related issues in terms of models which are based on optimizing behaviour of agents. The other two essays in this study will demonstrate that well-behaved models of this type can in fact be based on more solid foundations, and in particular that useful models in the Kouri-Branson tradition can be specified in which there is no interior steady state.
WEALTH ACCUMULATION AND EXCHANGE RATE DYNAMICS IN THE ABSENCE OF A STEADY STATE

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1 INTRODUCTION

Much of the recent work on exchange rate dynamics assumes rational expectations or perfect foresight and employs models in which wealth accumulation provides the driving force. A key assumption of these models is that there exists some "desired" level of wealth at which accumulation ceases. In the absence of exogenous growth of population and productivity, this saturation point of savings fixes a steady state for real asset demands, which, in turn, determines, together with money supplies, the long-run behaviour of exchange rates. The intrinsic dynamics in these models thus consists of a movement of wealth and the exchange rate along a saddle path towards the long-run equilibrium.

In descriptive macromodels, wealth saturation is typically secured by postulating a Metzlerian or target saving function, where the propensity to save is assumed to decline with increasing asset wealth.¹ From the point of view of optimizing behaviour, it is by no means clear why, as a rule, the accumulation of wealth should by itself lead to a saturation point. Some analytical frameworks for optimizing consumer behaviour are, by their very construction, capable of producing feasible steady states consistent with the "desired" wealth concept. This is not the case for the standard model of consumer's intertemporal choice, which employs an infinite horizon, constant-rate-of-time-preference utility integral, and where asset wealth attracts utility only indirectly by facilitating future consumption. Consumers maximizing such a utility integral continue to accumulate or decumulate wealth as long as the yield on wealth deviates from their subjective rate of time discount.

Under diminishing returns to capital, a world economy consisting of such consumers will nevertheless converge, not because growing prosperity reduces the appetite for saving but because the declining return on savings makes it less lucrative to provide for the future. In order to make this a fruitful approach in open economy applications one has to assume that the representative consumers of different countries have a different rate of time preference. If consumers were alike worldwide, then the exchange rate in a one good model would degenerate to a purely monetary phenomenon and the international distribution of wealth would merely remain an arbitrary constant. Assuming that time preferences differ across countries allows for the plausible proposition that the wealth of nations has something to do with their thrift. Above all, it preserves the central idea of the asset market approach that the redistribution of wealth through current account deficits and surpluses is a significant determinant of exchange rate developments.

But, unlike in models incorporating a wealth saturation mechanism, the current account imbalances and the accompanying exchange rate movements are persistent. What happens in the long run depends on what is assumed to be the ultimate constraint on consumer behaviour. Assuming that capital markets are perfect, thus allowing deficit spending down to zero wealth, the world economy ultimately approaches a state where aggregate private wealth is owned by the least impatient nation. Since such a boundary state can hardly be considered to be an equilibrium, we may characterize the model as one that lacks a steady state. More stringent constraints on borrowing would not change the key property that wealth converges towards a boundary of its feasible range rather than to a regular interior steady state.²

It is probably this ultimate Marxian prediction that has discouraged the use of the infinite-horizon-constant-time-preference utility function in open economy models. With the notable exceptions of Obstfeld and Stockman (1985) and Helpman and Razin (1982), those

²Uncertainty may justify the imposition of a more stringent limit for borrowing than the standard solvency constraint. For stochastic convergence of wealth decumulation, see CLARIDA (1982).
writers that have relied on this framework have tended to focus on the special case where time preferences are equal across countries (Lipton and Sachs (1983)), or, if working in the small country setting, the case where the rate of time preference just happens to equal the world real interest rate (Hodrick (1982)). In our opinion, the lack of an interior steady state is not a sufficient reason for not examining the implications of this type of model from the point of view of exchange rate theory. More justification is offered in the concluding remarks of the paper, where alternative optimizing approaches currently used in open economy models are compared.

What difference does it make for the exchange rate dynamics if wealth converges towards a boundary of its feasible range rather than to an interior steady state? The aim of the present paper is to examine this question in some detail. The essential structural difference is that we are forced to look at exchange rate movements without any useful concept of long-run equilibrium. Hence, there is no meaningful distinction between impact and final policy effects and little scope for such phenomena as over- or undershooting. Unexpected policy changes cause the exchange rate to jump onto a new equilibrium path, but, unlike the steady state models, this need not imply a change in the limiting state of the economy. A further contrast with the steady state models is that we are not tempted to assume that economies are currently functioning anywhere close to their limiting states. The focus of the analysis is thereby shifted from comparing alternative long runs, which in wealth accumulation models could be very long indeed, to examining some currently relevant horizons of exchange rate dynamics.

The scope of the paper is limited in several respects. We shall only focus on the single-good, full employment framework of simple neoclassical growth models. Terms of trade and real exchange rate dynamics are thus not covered by the analysis. We also wish to avoid dead weight by keeping population and technology constant, and abstract from genuine investment dynamics by assuming instantaneous costless adjustment of the capital stock. In order to highlight the role of international redistribution of wealth, the domestic distribution of wealth within each country is kept constant by assuming nationally identical consumers.
Second, we shall employ the simple deterministic framework where all assets except money are perfect substitutes. National monies are distinguished from other assets by including home currency real balances in the utility functions. Moreover, only logarithmic utility is considered. While this set-up is sufficient to yield an asset market model of the exchange rate that is capable of throwing light on the role of wealth accumulation, it is not fully adequate to deal with policy analysis in a world where bonds of different currency denominations are traded internationally. The difficulty stems from an indeterminate currency composition of bond portfolios and could be overcome by the use of a stochastic optimization framework.

In the following sections the model is built up step by step using the small country framework, where the outside world is exogenous and domestic assets are not traded internationally. The derivation closely parallels Obstfeld and Stockman (1983), but we attempt to be more explicit in analysing economic policy effects. In the small country framework countries appear either as savers or dissavers according to whether their national rate of time preference falls short of or exceeds the exogenous world real interest rate. One should note that the small country specification of a saver economy is not fully consistent. It overlooks the fact that, by the logic of perfect foresight, the agents in a country that is in the process of becoming big should be able to foresee the ultimate relevance of world market feedbacks. The final outcome is dealt with in section 8, where the small country model is extended to a world model of two countries. In section 9 alternative approaches for generating wealth saturation in open economy macro models are surveyed and compared with the framework examined in this paper. Some observations on the uniqueness of the perfect foresight monetary equilibrium when there is no steady state are presented in the appendix.
We start by considering a small open economy in which the government issues domestic money to pay for its consumption outlays. While such a money issuing mechanism blurs the distinction between the monetary and fiscal policy roles, it provides a simple starting point for the more complex models of the following sections. 3

The economy consists of identical firms and consumers, which produce and consume a homogeneous world commodity, and of a government, which issues money and consumes goods. Technology is neo-classical, with diminishing returns to labour and capital and constant returns to scale. The labour force and productivity stay constant over time. The foreign currency price of output and the real interest rate are determined in the world markets and exogenous to the small country. The foreign currency price of output equals one. The domestic output price equals the exchange rate \( P \), as determined in the asset markets. The domestic capital stock has settled down at its equilibrium level, at which the marginal productivity of capital equals the real interest rate \( r \). The real interest rate is expected to remain at its prevailing level. Output is fixed at its full employment level.

Private real wealth per capita equals

\[
W = m + F + \frac{Y}{r}
\]

where \( m \) stands for real money balances \( M/P \), \( F \) for net holdings of foreign currency bonds and \( Y/r \) for the present value of domestic output. There is no market for physical capital or equities. Thus labour income and income from the domestic capital stock combine in a non-tradeable component of wealth. Money is traded only domestically and the predetermined nominal money supply \( M \) has to be held in domestic portfolios. The government issues new money at a fixed non-negative rate

3Tying money supply to government expenditure may provide some insight into the chronic inflation problems in some developing countries with institutional constraints on taxes and government borrowing.
and the new money comes into existence through government expenditure on goods. The real value of government expenditure equals \( \mu m \), and is thus an endogenous variable.

While wealth holders are free to create foreign currency assets by issuing foreign currency debt, the net supply of foreign currency bonds in the domestic asset market is fixed at each moment of time and changes over time as determined by the current account of the balance of payments:

\[
\dot{F} = Y + rF - C - \mu m
\]

Equation (3) states that the change in net foreign currency bonds equals the excess of national income over domestic absorption, where \( C \) stands for private and \( \mu m \) for government consumption. Since feasible paths of private and government consumption are non-negative and \( m \) is not traded internationally, market equilibrium cannot exist unless

\[ Y + rF > 0 \]  

(4)

A violation of (4) implies national insolvency.

Consumer choice at each instant of time obeys the identity

\[ C + (r+\Pi)m + \dot{W} = rW \]  

(5)

where \( \Pi = \frac{P}{P} \) is the domestic rate of inflation (depreciation). The domestic asset market clears continuously. Therefore, at each instant of time \( P \) and \( W \) are given to an individual consumer, but the consumer is free to choose the composition of his marketable asset portfolio \( m+F \). Consumers maximize

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4 Expressed in nominal terms, eq. (5) reads: \( PC + \dot{M} + \dot{PF} = PY + rPF \).
\begin{equation}
\int_{t}^{\infty} e^{-\rho(s-t)} \left[ \alpha \log C + (1-\alpha) \log m \right] ds = 0 \tag{6}
\end{equation}

subject to the wealth constraint

\begin{equation}
\int_{t}^{\infty} e^{-r(s-t)} \left[ C + (r+\Pi) m \right] ds = W(t) \tag{7}
\end{equation}

where \( \rho, r, W(t) > 0 \) and \( 0 < \alpha < 1 \). In order not to deprive the model of its only source of intrinsic dynamics, it is assumed that \( \rho \neq r \).

From the first order conditions we get

\begin{equation}
(r+\Pi)m = \frac{1-\alpha}{\alpha} C \tag{8}
\end{equation}

\begin{equation}
\frac{\dot{C}}{C} = r - \rho \tag{9}
\end{equation}

Eq. (9) solves for

\begin{equation}
C(s) = C(t)e^{(r-\rho)(s-t)} \tag{10}
\end{equation}

Inserting (8) and (10) into (7) and using (5) yields the familiar demand functions

\begin{equation}
C(t) = \alpha \phi W(t) \tag{11}
\end{equation}

\begin{equation}
m(t) = \phi(t) W(t) \tag{12}
\end{equation}

\begin{equation}
\dot{W}(t) = (r-\rho) W(t) \tag{13}
\end{equation}

where

\begin{equation}
\phi(t) = \frac{(1-\alpha)\rho}{r + \Pi(t)} \tag{14}
\end{equation}

In order to determine the perfect foresight equilibrium of the economy we have to find a price path \( \{P(s|t)\} \) such that eq. (12) holds for all
s > t. The price path must be sufficiently smooth for \( \pi \) to exist for 

\[ 0 < \{ \phi(s \geq t) \} < 1 \]  

(15)

Moreover, feasible equilibria satisfy

Since wealth stays positive along equilibrium paths, \( \phi < 0 \) would imply 

\[ m < 0. \]  

Since \( \mu m > 0, \phi > 1 \) violates (4).

Assuming that \( m(t) > 0 \), i.e. that a monetary economy exists at least 

initially, logarithmic differentiation of (12) yields\(^6\)

\[
\frac{\dot{m}}{m} = \frac{\dot{\phi}}{\phi} + \frac{\dot{W}}{W}  
\]  

(16)

Eq. (16) describes the evolution of the demand for real balances. On 

the supply side we have

\[
\frac{\dot{m}}{m} = \mu - \Pi  
\]  

(17)

Using (13), (16) and (17) to eliminate \( \dot{W}/W \) and \( \dot{m}/m \) and substituting 

from (14) for \( \pi \) yields

\[
\dot{\phi} = (\mu + \rho) \phi - (1-\alpha) \rho  
\]  

(18)

This linear differential equation has its stationary value at

\[
\phi = \frac{(1-\alpha) \rho}{\mu + \rho}  
\]  

(19)

---

5If \( t \) is the starting time for a new perfect foresight equilibrium 

path created by an unexpected disturbance taking place at \( t \), then 

\( \Pi(t) \) exists only as a limit from the future (forward-looking inflation 

rate).

6Under the assumed logarithmic utility function equilibrium paths of 

\( m \) are bounded away from zero.
Since $\mu + \rho > 0$, $\tilde{\phi}$ is unstable. Unless $\phi \neq \tilde{\phi}$ initially, $\phi$ will not remain bounded as required by (15). Hence only those $\{P(s;t)\}$ that keep $\phi$ constant at $\tilde{\phi}$ are potential equilibrium paths. But substituting $\tilde{\phi}$ in (12) determines a unique initial $P(t)$. Thus the perfect foresight equilibrium path of the economy (in the particular model of this section) is unique, and the rate of inflation is constant along the equilibrium path.

Since $\phi$ is constant along the perfect foresight path, $C$, $m$ and $F+Y/r$ all grow or decline at the same constant rate $r-\rho$. Substituting $\tilde{\phi}$ from (19) into (11) - (14) yields

$$C = \alpha \rho \left[ \frac{\rho + \mu}{\alpha \rho + \mu} \right] (F + \frac{Y}{r})$$

$$m = \left( \frac{1-\alpha}{\alpha} \right) \rho \left[ \frac{\rho + \mu}{\alpha \rho + \mu} \right] (F + \frac{Y}{r})$$

$$F = (r-\rho)(F + \frac{Y}{r})$$

$$P = \frac{\alpha \rho + \mu}{(1-\alpha) \rho} \left[ \frac{M}{F + \frac{Y}{r}} \right]$$

$$\Pi = \mu - (r-\rho)$$

The domestic price level (exchange rate) is proportional to the ratio of money to non-monetary real wealth and depends positively on the money supply rate. The rate of inflation (depreciation) equals the excess of $\mu$ over the wealth accumulation rate $r-\rho$. Consumption and real balances depend negatively on the money issuing rate, but the current account $F$ is independent of $\rho$. This means that there is complete crowding out of private by government consumption and that the authorities are unable to control the external balance of the economy.
3 TRANSFER MONEY

Let us next find out what difference it makes if, as is usually assumed in monetary growth models, money is injected in the economy through government transfers rather than through government consumption. We may now also consider a deflation of the money supply by letting \( \mu \) be negative. Such a money-issuing mechanism allows the analysis of monetary policy which does not involve any demands on real resources. In order to avoid biased results, one has to assume that government transfers, although proportional to aggregate real balances, are regarded as exogeneous income by the individual consumer and that \( \mu \) is not identified as part of the yield on real balances.

The following changes in the specification of the model are needed. First, in eq. (3) for the current balance the last term \( \mu m \), standing for government consumption, drops out:

\[
\dot{F} = Y + rF - C \tag{3a}
\]

Instead, the present value of government transfers, say \( V \), enters the definition of wealth:

\[
W = m + F + \frac{Y}{r} + V \tag{1a}
\]

where

\[
V = \int_{t}^{\infty} e^{-r(s-t)} \mu m(s) ds \tag{21}
\]

Present wealth now depends on the equilibrium path of \( m \). Assume that the integral in (21) converges and that \( W(t) > 0 \). Since the flow of transfers and thus \( V(t) \) are exogenous to an individual agent, the solution to the consumer's problem is given by (11) - (14), with wealth redefined as (1a). As before, market equilibria are characterized by the solutions to the differential equation (18), and the state dependent equilibrium is determined by \( \phi(t) = \delta \). For \( \mu > 0 \), this constant inflation path is again the unique equilibrium path.
The situation is more complicated if $\mu$ is negative. A feasible state dependent path only exists for

$$\mu > -\rho$$

(22)

Condition (22) states that a contraction of the money supply must not be faster than what is consistent with a positive nominal interest rate. Moreover, for $-\rho < \mu < 0$ the equilibrium is not unique. The reason is that when $\mu$ and thus $V$ are negative, $m > W$ need not violate the national solvency constraint (4). Therefore, the feasible range of $\phi$ is no longer bounded from above by (15). This non-uniqueness issue is considered more thoroughly in the Appendix. In what follows we shall assume that the economy stays on the state dependent path, along which the rate of inflation and the nominal interest rate $r + \pi$ keep constant. 7

Given the equilibrium inflation rate (20), the growth rate of real balances is

$$\frac{\dot{m}}{m} = \mu - \Pi = r - \rho$$

(23)

which solves for

$$m(s) = m(t)e^{(r-\rho)(s-t)}$$

(24)

Substituting into (21) yields

$$V(t) = \frac{\mu}{\rho}m(t)$$

(25)

so that from (12) and (19) we have for "monetary wealth"

7A simple and perhaps the most natural way to rule out speculative price paths is to assume that agents do not expect the authorities to commit themselves to a constant $\mu$, unless the economy stays on its state dependent path.
m + V = (1-\alpha)\omega \quad (26)

and thus from (1a)

W = \frac{1}{\alpha}(F + \frac{Y}{r}) \quad (27)

Using (11) - (13), (19) and (27) we may then solve for the market equilibrium

C = \rho(F + \frac{Y}{r})

m = \frac{(1-\alpha)}{\alpha}(\frac{\rho}{\mu+\rho})(F + \frac{Y}{r})

\dot{F} = (r-\rho)(F + \frac{Y}{r}) \quad (20a)

P = \frac{(\alpha)}{1-\alpha}(\frac{\mu+\rho}{\rho})(\frac{M}{F + \frac{Y}{r}})

\Pi = \mu - (r-\rho)

As a comparison with (20) reveals, the main difference between the two models is that, while in the case of "expenditure money" the level of private consumption reacts to changes in \mu so as to make room for the implied change in government consumption, in the case of transfer money private consumption is independent of \mu. Therefore, independently of whether money is issued through government consumption or as transfer payments, total domestic absorption and the external balance of the economy do not depend on monetary policy.\textsuperscript{8}

\textsuperscript{8}Given a constant \mu and r, this "superneutrality of money" property is not limited to the separable utility case. For the isoelastic utility function \(C = \alpha^{l-\alpha}\gamma/(1-\gamma)\), the solution for C in (20a) is only modified to the extent that \(r+\rho)/\gamma replaces \rho (see Obstfeld and Stockman (1983)). Hence, C is also independent of \mu when \gamma \neq 1. This result does not, however, extend to the two-country model of section 8, where r changes over time. For more on this, see appendix B of the third Essay.
4 MONETARY POLICY

Figure 1 illustrates how the economy reacts to unanticipated changes in \( \mu \). The curve labelled \( \mu = 0 \) is the path of \( P \) and \( F \) under what, in the context of the present model, could be considered a neutral stance of policy. It depicts the equation for \( P \) in (20a) when \( \mu = 0 \) and \( M \) consequently is constant.

The movement along the curve proceeds to the right or to the left depending on whether the economy is a saver \( (r > \rho) \) or a dissaver \( (r < \rho) \). Note that under neutral monetary policy the model shares the property of many portfolio balance models of the exchange rate that currency depreciation is associated with a current account deficit and appreciation with a surplus. However, there is no steady state for \( P \) or \( F \).

**Figure 1: THE NATURAL RATE OF INFLATION AND THE EFFECT OF STABILIZING MONETARY POLICY**
Inserting \( \pi = 0 \) in (20a) gives the appropriate policy rule for price and exchange rate stability as \( \mu = r - \rho \); that is, the money issuing rate must be set so as to offset the "natural inflation rate" of the economy, which is positive for a dissaver and negative for a saver.\(^9\) Let us assume that there is an unexpected but permanent change in monetary policy from a neutral stance to one aiming at price and exchange rate stability. Although the policy objective is reached immediately, there will be a jump in the equilibrium \( P \) as shown in Figure 1. For dissavers there will be a once-and-for-all reduction in the price level (revaluation of the home currency). Disinflationary policy is therefore very efficient in this model. Not only does the inflation rate react immediately to the change in the money supply rule, but some of the past inflation (depreciation) will be undone by the change. Unlike the usual overshooting phenomenon in steady state models, this reduction in the price level remains permanent. In the case of a saver economy, the jump in the price level is in an upwards direction, so that part of past deflation (appreciation) is offset.

Since \( F \) is not affected by changes in \( \mu \), the movement of \( F + Y/r \) along the horizontal price stability paths in Figure 1 continues at the same relative speed \( r - \rho \) as before. In fact, the net foreign asset position of the economy at any time is independent of its monetary history.

Figure 2 illustrates the effect of anticipated price and exchange rate stabilization in a dissaver economy. Until time \( t \) the economy follows its neutral path with \( \mu = 0 \) and \( \pi = \rho - r \). At time \( t \), the authorities announce a new policy rule \( \mu = r - \rho \) to take effect from a future time \( T \) onwards. The news of this coming change in policy stance causes the market equilibrium at time \( t \) to jump onto a temporary adjustment path, which brings the economy to a constant price and exchange rate path at time \( T \).

\(^9\)Note that the constraint (22) does not prevent a dissaver economy from attaining price and exchange rate stability.
To find the temporary adjustment path, we have to solve the differential equation (18) under the old policy rule and the terminal condition that $\phi(T)$ equals the stationary value of $\phi$ under the new policy rule. In a general form, valid for any anticipated permanent change in $\mu$, such a solution is

$$
\phi(s) = (\bar{\phi}(T) - \bar{\phi})e^{-(\mu + \rho)(T-s)} + \bar{\phi}, \quad t < s < T
$$

(28)

where $\mu$ is the old, still applied, policy rule, while $\bar{\phi}$ and $\bar{\phi}(T)$ are the stationary values of $\phi$ corresponding to the old and anticipated new values of $\mu$, respectively. The initial jump $\phi(t) - \bar{\phi}$ is of the same sign, but smaller than the jump $\bar{\phi}(T) - \bar{\phi}$ that would take place, if the policy change at $T$ came as a surprise. The longer is the period of anticipation $T-t$, the smaller is the initial jump in $\phi$. The same holds for the relative size of the jump in $P$. Therefore, if the authorities wish to avoid discrete disturbances in the foreign exchange market, it pays to announce changes in policy well in advance. The final outcome of the policy change is, however, the same.
Return now to the case illustrated in Figure 2. Since anticipation decreases the size of the jump in $P$, the following question arises: Does the exchange rate after the initial jump under- or overshoot the stabilized exchange rate $P(T)$? One way to find out is to ask whether the home currency depreciates or appreciates during the adjustment period. Insert $\mu = 0$, $\phi = 1-\alpha$ and $\phi(T) = (1-\alpha)\rho/r$ into (28), substitute the resulting expression for $\phi$ in (14) and solve

$$\pi(s) = \left[ \frac{\rho}{r + (\rho-r)e^{-\rho(T-s)}} - 1 \right] r$$

This expression has the same sign as $\rho-r$. Hence the post-jump $P(t)$ undershoots $P(T)$, whereafter the home currency continues to depreciate, as it did before the news. In a saver economy, where the currency appreciates along the neutral path, there is a discrete devaluation on the news of the stabilizing policy and then a resumption in appreciation until the implementation of the new policy rule.
5 FISCAL POLICY

As a third exercise we shall specify the government accounts as

\[ G - T - \mu m = 0 \]  

(30)

where \( G \) stands for government consumption and \( T \) for a lump sum real tax. The government now has two degrees of freedom at its disposal. We shall assume that the government fixes \( G \) and \( \mu \), which leaves \( T \) as a residual. Our interest in this case lies primarily in showing that a separation of fiscal and monetary policy according to (30) brings about partial control of the current account in addition to the control of domestic inflation.

Assuming that current values of \( G \) and \( \mu \) are expected to be maintained for ever, the consumer's wealth is given by

\[
W(t) = m(t) + F(t) + \frac{Y}{r} - \int_{t}^{\infty} e^{-r(s-t)} T(s) ds
\]

\[ = m(t) + F(t) + \frac{Y-G}{r} + V(t) \]  

(1b)

where \( V \) is defined by (21). The current balance equals

\[ F = Y + rF - C - G \]  

(3b)

Going through the same steps as above, it can be shown that the model has the same reduced form as our previous version, with the exception that \( (Y-G)/r \) replaces the \( Y/r \) of (20a). The equations for the current account and the price level (exchange rate) are thus

\[
F = (r-\rho)(F + \frac{Y-G}{r})
\]

\[
P = \left( \frac{\alpha}{1-\alpha} \right) \left( \frac{1-\rho}{\rho} \right) \frac{M}{F + \frac{Y-G}{r}}
\]

(20b)
The novelty as compared to the earlier cases is that the government can affect \( F \) by changing \( G \). The reason for this is that the crowding-out of private by government consumption is no longer a one-to-one relationship. Instead, we have \( \frac{dC}{dG} = \frac{-\rho}{r} \), so that crowding-out is excessive in a dissaver economy, but only partial in a saver economy. The control of the external balance is limited in the sense that the authorities can only affect the size of \( F \) but not reverse its sign, which continues to be determined by the sign of \( r - \rho \). Raising \( G \) sufficiently to make \( F + (Y - G) / r \) negative would bring the economy to a state of insolvency rather than change the sign of \( F \) along an equilibrium path.

The control of domestic inflation by an appropriate choice of \( \mu \) is not affected by \( G \). Thus there is a clear-cut solution to the assignment problem: Only monetary policy is efficient in attaining price and exchange rate stability while only fiscal policy has some effect on external balance. There is no continuous trade-off between the two policy objectives. Unexpected changes in \( G \) as well as in \( \mu \) will, however, cause the price level (exchange rate) to jump.

Figure 3 illustrates the effect of an unexpected rise in government consumption in a dissaver economy under neutral monetary policy. When the shift in \( G \) takes place at time \( t \), the exchange rate jumps upwards to its new equilibrium path and thereafter the home currency continues to depreciate at its natural rate \( \rho - r \). But the current account deficit will be diminished for all future dates, so that at time \( t + \Delta t \) the net foreign position of the economy is less weak than it would have been without the increase in government consumption. Note that in a dissaver economy it is expansionary fiscal policy that helps to curb the external deficit. In a saver economy, however, an increase in \( G \) causes a fall in the current account surplus. In both cases expansionary fiscal policy works in the stabilizing direction by reducing disposable private wealth and hence the rate of saving or dissaving, whichever the case may be.
One implication of equations (20b) is that the jump in $P$ required by an increase in $G$ is in the opposite direction to the jump caused by a tightening of monetary policy. This means that the government of a dissaver economy can by the simultaneous use of its two policy instruments not only improve both price stability and external balance but also reduce the magnitude of the overall jump of the domestic price level and the exchange rate. The same goes for combined policy measures in a saver economy. In the real world, with its less-than-perfect foresight and price and wage rigidities, the jumps of the model economy are at best converted into periods of rapid inflation or deflation, which are not easy to separate from a more permanent type of price instability. A policy mix that avoids the jumps could therefore be of some value.

In Figure 4 the government of a dissaver economy stabilizes domestic prices and the exchange rate at their prevailing level by implementing a money supply deflation rate $\mu = r - \rho$ and at the same time increasing government consumption by an amount that exactly offsets the
deflationary jump (appreciation) which would have otherwise been caused by the fall in $\mu$ from zero to $r_p$. As well as attaining price stability without the initial jump, the government also achieves a smaller external deficit and thus a slower rate of accumulation of foreign debt. The required increase in $G$ can be calculated from (20b) as

$$\Delta G = \frac{\rho - r}{\rho} (rF + Y - G_0)$$

where $G_0$ is the previous level of government consumption.

Figure 4: MIXING MONETARY AND FISCAL POLICY TO ACHIEVE PRICE STABILITY WITHOUT A JUMP
6 GOVERNMENT BORROWING

Next we shall investigate what, if anything, can be achieved by government borrowing. For that purpose we introduce a government debt instrument other than money. For simplicity, it is assumed that the government bond is short-term, variable-interest-rate paper of fixed capital value, denominated in domestic currency and not tradeable internationally. Since our framework is non-stochastic and asset markets clear continuously, the government bond has to promise a real yield that equals the world market real interest rate. The nominal, domestic currency interest rate on government bonds is therefore $r + \pi$. At this interest rate, wealth holders are indifferent as to the composition of their portfolio of government and foreign currency bonds.

The government budget is now given by

$$G + rb = T + \mu m + b$$  \hspace{1cm} (30c)

where $b$ is the real value of outstanding government bonds. The variables $G$, $T$ and $\mu$ are government instruments, while $b$ is treated as a residual. As before, it is assumed that the current levels of $G$, $\mu$ and $r$ are expected to be maintained for ever and, in addition, that feasible tax policies are subject to the solvency constraint

$$\lim_{s \to \infty} e^{-r(s-t)} b(s) = 0.$$  

The capitalized budget is therefore

$$\frac{G}{r} = \hat{T}(t) + V(t) - b(t)$$  \hspace{1cm} (32)

where $\hat{T}(t)$ stands for the capitalized future taxes. Using (32), the consumer's wealth can be written as

$$W(t) = m(t) + b(t) + F(t) + \frac{Y}{r} - \hat{T}(t)$$

$$= m(t) + F(t) + \frac{Y-G}{r} + V(t)$$  \hspace{1cm} (1c)

which equals (1b). The model thus reduces to the balanced budget model
of the previous exercise. That government bonds do not appear as net wealth in (1c) is because more government debt now implies higher taxes later. The real resource burden of a given level of government consumption has to be reflected in the equilibrium price path, no matter how the government collects existing money from the private economy for financing its consumption expenditure.

The mere existence of government bonds will, however, increase the controllability of the economy by providing for the possibility to use open market operations to attain discrete changes in M and thus in P. Instead of mixing monetary and fiscal policies the government may use open market operations to avoid jumps in P. Assume, for example, that a dissaver economy is proceeding leftwards along its price stability path in Figure 1, and that the government, in order to slow down the deterioration of the net foreign position or for other reasons, wants to increase government consumption without disturbing price stability. From (20b) we may calculate that this can be done by an open market sale of government bonds in an amount corresponding to a reduction of \( \Delta G/(rF+Y-G_0) \) per cent in the nominal money supply.

Let us finally consider government transactions in foreign currency assets. Let \( F^g \) stand for the government sector's net holdings of foreign currency assets, including the foreign exchange reserves of the central bank, and write the government flow constraint as

\[
G + r(b-F^g) = T + \mu m + b - F^g
\]  

(30d)

where \( F^g \) is regarded as an additional instrument. A positive \( F^g \) means accumulation of foreign exchange reserves or redemptions of government foreign debt or of foreign currency domestic debt, while a negative \( F^g \) implies the running down of reserves or new foreign currency borrowing abroad or at home.

The capitalized government accounts equal

\[
\frac{G}{r} = \tilde{T}(t) + V(t) + F^g(t) - b(t)
\]  

(32a)
Letting $F^P$ stand for the net private foreign asset position and $F = F^P + F^G$ for the net national foreign position, we can net out the items in consumers' wealth as follows:

$$W(t) = m(t) + b(t) + F^P(t) + \frac{Y}{P} - \tilde{T}(t)$$

$$= m(t) + F(t) + \frac{Y-G}{P} + V(t) \quad (1d)$$

Thus government net foreign assets are part of consumers' net wealth, and the model of the previous section is still relevant on the understanding that $F$ includes government net holdings of foreign assets.

Given $\nu$ and $G$, government financing by foreign borrowing or domestic foreign currency borrowing or by running down foreign exchange reserves will only affect the composition of the national net foreign currency position among private and government holders, which is irrelevant for the perfect foresight equilibrium path of the economy. Open market operations in foreign currency assets (foreign exchange market intervention) will not change $F$ and their impact on $P$ is wholly determined by the implied change in $M$. An open market purchase of foreign currency bonds has the same effect on $P$ as an open market purchase of domestic currency bonds of the same nominal value. Swapping bonds of different denominations for each other has no effect on $P$.

10These results depend on domestic and foreign currency bonds being perfect substitutes.
7 WORLD INFLATION AND THE REAL RATE OF INTEREST

In the previous sections we examined to what extent monetary and fiscal policy can be used to control the dynamics of the economy. In this section we shall investigate the behaviour and controllability of the economy when confronted with unexpected external disturbances, such as a change in the world inflation rate, world price level or the world real interest rate.

So far we have assumed a world output price of unity and a zero world inflation rate, so that domestic currency output price coincides with the exchange rate and the domestic inflation rate with the rate of depreciation. Let $P$ and $\pi$ continue to stand for the domestic currency output price and domestic inflation rate, respectively, and let $P^*$ and $\pi^*$ denote the world output price and world inflation, and $e$ and $\varepsilon$ the exchange rate and the rate of depreciation, respectively. We then have

$$P = eP^* \quad (33)$$

$$\pi = \varepsilon + \pi^* \quad (34)$$

The real value of foreign currency bonds is no longer measured by their nominal foreign currency value $F$, but instead by a deflated variable $f = eF/P = F/P^*$. An unexpected discrete change in $P^*$ affects $f$ and thus wealth. But since non-monetary assets are perfect substitutes, any difference in inflation rates at home and abroad is reflected in the nominal interest rates and does not affect the movement of $f$ along its equilibrium path.

By inspection, it is easy to verify that the models of the previous sections remain valid if we replace $f$ for $F$ and $\dot{f}$ for $\dot{F}$ where ever $F$ or $\dot{F}$ appear. Given this change in the specification and the additional equations (33) and (34), let there be a change in $\pi^*$. Since $\pi^*$ does not appear anywhere else in the model than in eq. (34), the domestic inflation is independent of $\pi^*$ and a change in $\pi^*$ is fully absorbed by a compensating change in $\varepsilon$. It does not in general follow from this that the economy would be insulated against changes in
foreign policy. Changes in world monetary or fiscal policy may cause jumps in world output price $P^*$. For a debtor country with $f < 0$, a discrete rise in $P^*$ means a windfall gain in real wealth, and a consequent increase in consumption and in saving or dissaving, whichever is the case. Moreover, since the demand for real balances rises, domestic prices fall and the exchange rate therefore jumps downwards more than in proportion to the rise in $P^*$. A creditor country, on the other hand, experiences a windfall loss of wealth, a rise in the domestic price level and a smaller jumpwise appreciation of its currency.

A change in the world real interest rate will affect both consumer behaviour and the domestic capital stock as determined by the optimizing behaviour of domestic firms. We shall abstract from investment dynamics by assuming that the capital stock adjusts instantaneously to changes in the real interest rate. This implies discrete jumps in physical capital $K$ financed by discrete changes of the same size, but of opposite sign, in foreign currency assets $F$. Since assets are perfect substitutes, swaps of $K$ for $F$ do not affect asset wealth. But a change in $K$ implies a change in the present value of labour income and thus in consumers' total wealth.

Define capitalized labour income or human capital as

$$H = \frac{Y(K)}{r} - K \quad (35)$$

On the basis of the above results on the neutrality of money and government debt, wealth can be decomposed as

$$W = \frac{1}{\alpha}(F + K + H^d) \quad (36)$$

where

$$H^d = H - \frac{G}{r} \quad (37)$$

and the reduced form equations for $F$ and $P^*$ be written as
The variables $F$, $K$ and $H^d$ all depend on $r$, but the sum $F+K$ is predetermined at each instant of time. From the capital stock optimality condition

$$Y'(K) = r$$

and equations (35) and (37) we get

$$\frac{dH^d}{dr} = - \frac{Y-G}{r^2} < 0$$

Figure 5 illustrates the effect of an unexpected rise in the world real interest rate in a dissaver economy where $\mu = 0$. The horizontal axis now measures $F+K$ instead of $F$ alone. When $r$ rises, the economy jumps onto a new perfect foresight equilibrium path, which is situated above and to the right of the previous path. The upward jump in the exchange rate and the domestic price level reflects the increased scarcity caused by the decrease in human capital. By increasing $r-p$, the rise in the world real interest rate will, however, reduce the natural rate of inflation (depreciation) and improve the current account of the economy. If the change in $r$ is sufficient to make $r-p$ positive, the economy will turn from a dissaver into a saver, so that the direction of movement along the perfect foresight path is reversed.11

11It is obvious that the use of the small country framework to trace the effects of a change in the world market interest rate, or in other real variables that are common to all smaller countries, is likely to produce misleading results. The analysis in the text should be regarded as a technical excercise intended to illustrate the structural properties of the model.
The private sector of a debtor country may become bankrupt because of an unexpected rise in the world real interest rate. In the case shown in Figure 5, this does not happen since non-monetary wealth remains positive after the rise in r. But, should F+K initially equal, say, $-H_2^d$, the fall of $H^d$ from $H_0^d$ to $H_1^d$ would make the private sector insolvent. In that case no new perfect foresight equilibrium would be defined for the current value of F+K. Somewhat paradoxically, the occurrence of bankruptcy is more probable the larger is the reduction in foreign debt and K. The government may help to avoid a bankruptcy by reducing G so as to compensate for the fall that would otherwise take place in $H^d$. If the rise in r is sufficient to turn the economy into a saver, the previous level of government consumption can, in due course, be safely restored when the economy has reduced its foreign indebtedness sufficiently to make F+K > $H_1^d$. 
8 A TWO-COUNTRY MODEL

In this section we consider the simplest two-country extension of the model. Let the world consist of two countries that are otherwise similar but differ in the rate of time preference of their representative consumer. A homogeneous world commodity is produced with a common technology and an identical, constant labour force in both countries, and is either consumed or transformed into productive capital. The world real interest rate is determined by the marginal product of capital. There are no capital stock adjustment costs and, if necessary, physical capital "flies" across countries to equalize the marginal product of capital in firms residing in the two countries. The growth in the world capital stock and thereby the time path of the real interest rate is governed by world saving. Consumers hold asset wealth in the form of equities, bonds and money. While equities and bonds denominated in either of the two currencies are perfect substitutes, only bonds are traded internationally. The national capital stocks and money supplies of each country are wholly owned by their respective residents.

The intertemporal allocation problem faced by consumers in both countries is the same as in the small country models, with the exception that the real interest rate is expected to move along its equilibrium path rather than to stay constant. Since utility separates in goods and money, this does not affect the consumer's choice other than through the present values of the expected income flows. Consumer choice in both countries is thus given by (11) - (14), with \( r \) replaced by \( r(t) \), \( p \) and \( W(t) \) by \( p^* (\neq p) \) and \( W^*(t) \) for the second country, and \( W(t) \) and \( W^*(t) \) appropriately redefined to allow for future movements in \( r \). Since monies are not traded internationally, the money market equilibrium in both countries is characterized by the respective stationary values of the differential equation (18). Further, the ratio of monetary wealth \( m+V \) to total wealth is constant and equals \( 1-\alpha \) in both countries (eq. 26).

Using these results from the small country model, the world market equilibrium can be characterized as follows. Real private per capita wealth in the two countries is given by
\[ W = \frac{1}{\alpha} (K+H-G+a) \]  
(40)

\[ W^* = \frac{1}{\alpha'} (K+H-G^*-a) \]  
(40*)

where

\[ H = \int_t^\infty \exp\left[ -\int_t^s r d\tau \right] (Y(K)-rK) ds \]  
(41)

\[ \tilde{G} = \int_t^\infty \exp\left[ -\int_t^s r d\tau \right] G ds \]  
(42)

\[ \tilde{G}^* = \int_t^\infty \exp\left[ -\int_t^s r d\tau \right] G^* ds \]  
(42*)

\[ a = \frac{B}{P} + \frac{F}{P^x} \]  
(43)

and where \( B \) and \( F \) denote the first country's national (private and government) net holdings of bonds denominated in the two currencies. Equation (40) is derived from

\[ W = m + \frac{B_P}{P} + \frac{F_P}{P^x} + K + H - \tilde{T} \]

by substituting from

\[ \tilde{T} = \tilde{G} - V - \frac{B^G}{P} - \frac{F^G}{P^x} \]

and further from

\[ m + V = (1-\alpha)W \]

where \( B_P \) and \( F_P \) stand for private and \( B^G \) and \( F^G \) for government net holdings of bonds. The same procedure applied to the second country gives
\[ W^* = \frac{1}{a}(K^* + H^* - \tilde{G}^* + a^*) \]

Common technology and the "flying capital" assumption imply \( K^* = K \) and \( H^* = H \). Since world net supplies of both bonds are zero and countries have populations of the same size, \( B^* = -B \), \( F^* = -F \) and \( a^* = -a \). These equalities yield eq. (40*).

The other equations needed to describe the equilibrium are:

\[ C = \rho aW \]  
\[ C^* = \rho^* aW^* \]  
\[ \dot{W} = (r-\rho)W \]  
\[ \dot{W}^* = (r-\rho^*)W^* \]  
\[ \dot{K} = Y(K) - \frac{1}{2}|(C+G) + (C^*+G^*)| \]  
\[ a = \frac{1}{2}[(C^*+G^*) - (C+G)] + ra \]  
\[ r = Y'(K) \]  
\[ \pi = \mu - (r-\rho) \]  
\[ \pi^* = \mu^* - (r-\rho^*) \]  
\[ \varepsilon = (\mu-\mu^*) + (\rho-\rho^*) \]  
\[ m = \frac{(\rho-\mu)}{\mu+\rho}(1-\alpha)W \]  
\[ m^* = \frac{(\rho^*-\mu^*)}{\mu^*+\rho^*}(1-\alpha)W^* \]  
\[ p = \frac{(\mu+p)}{\rho}(\frac{M}{(1-\alpha)W}) \]  
\[ p^* = \frac{(\mu^*+p^*)}{\rho^*}(\frac{M^*}{(1-\alpha)W^*}) \]
Eq. (46) states that the common rate of capital growth equals world output less the world average of private and government consumption, all in per capita terms. By eq. (47), the current account balance depends on the difference in the national consumption rates, the real rate of interest and the real foreign position \( a \). Eqs. (49) - (53) refer to equilibrium paths of the monetary variables under constant money expansion rates. The variables \( m^* \) and \( M^* \) denote real and nominal holdings of the second country's currency by the residents of the second country.

Solving the model, given an arbitrary initial set of values for the asset stocks \( K, M, M^*, B \) and \( F \), the exogenous constants \( \mu \) and \( \mu^* \) and the exogenous time paths of \( G \) and \( G^* \), requires simultaneous determination of future dynamics and present state variables and is beyond the scope of this study. The increased complexity of the model is due to two features that were absent in the small country model. First, current wealth determines saving and thus the future time path of \( K \) and \( r \), but these feed back to current wealth via \( H, G \) and \( G^* \). Second, given \( K, H, G, G^* \) and the money supply expansion rates \( \tilde{\mu} \) and \( \tilde{\mu}^* \), we can solve for the national price levels and thus for the real foreign position \( a \) from the nominal asset stocks \( M, M^*, B \) and \( F \). But \( a \) determines the distribution of wealth among the countries, which again affects saving and \( H, G \) and \( G^* \). It follows from this link through \( a \) that \( M \) and \( M^* \), as well as \( \mu \) and \( \mu^* \), do in general affect the present and future values of the real variables, so that we do not have the clear-cut dichotomy of the small country model.

Another difficulty is that, since bonds denominated in the two currencies are perfect substitutes, the model only determines the time path of \( a \), and not the paths of \( B \) and \( F \) separately. The currency composition of the net foreign position at any point of time is therefore arbitrary, and need not be related in any way to the sign and size of \( a \), nor to its own history.

Things are less complicated when we observe the evolution of the world economy starting from an interior point on a perfect foresight
equilibrium path. Wealth and prices can be considered as predetermined variables, and money is supernormal in the sense that growth is independent of $\mu$ and $\mu^*$, except to the extent that these are reflected in the inherited distribution of wealth. Moreover, the currency composition of the net foreign position is irrelevant, as long as there are no unexpected disturbances.

Along an equilibrium growth path capital and domestic output per head grow at equal pace in both countries, but the relative rate of wealth accumulation is faster in the less impatient country by the difference in their rate of time preference. In the early phase of growth, when capital intensity is low and the real rate of interest high, we would expect both countries to be savers. Later, the falling real rate of interest turns the impatient country into a dissaver and reduces the rate of wealth accumulation in the patient country. But capital stock and output continue to grow in both countries, albeit at a diminishing rate. In the limiting state, which the world economy approaches but never reaches, the real interest rate equals the rate of time preference in the patient country, and the patient country, possibly together with the government of the impatient country, consumes all of world output.

As can be seen from eq. (47), any systematic relationship between private savings and the current account can be masked by short-term variation in government consumption rates. To avoid this, assume for the moment that, analogously to private consumption, $G$ and $G^*$ are chosen to be proportional to wealth, say

\[ G = \rho \beta W \]  
\[ G^* = \rho^* \beta^* W^* \]

with $\beta$, $\beta^*$ expected to remain constants. At this point we depart from the small country model, where $G$ rather than $\beta$ was held constant in time. Since wealth grows at rates $r-\rho$ and $r-\rho^*$, respectively, we then have
\[ \tilde{G} = \beta W \]  
(55)

\[ \tilde{G}^* = \beta^* W^* \]  
(55*)

\[ W = \frac{1}{\alpha + \beta}(K+H+a) \]  
(56)

\[ W^* = \frac{1}{\alpha + \beta^*}(K-H-a) \]  
(56*)

\[ K = Y(K) - \frac{1}{2}[\rho(K+H+a) + \rho^*(K+H-a)] \]  
(57)

\[ \dot{a} = \frac{1}{2}[(r-\rho)(K+H+a) - (r-\rho^*)(K+H-a)] \]  
(58)

In this variant of the model, government consumption rates have no effect on the time paths of \( K \) and \( a \) other than what has been the contribution of \( \beta \) and \( \beta^* \) in fixing the initial values of \( H \) and \( a \). In the early phase of growth when \( r > \rho, \rho^* \), the sign of \( \dot{a} \) depends on the distribution of wealth as well as on the relative magnitudes of \( \rho \) and \( \rho^* \), and it is possible that the patient country has to borrow from its less thrifty but wealthier neighbour in order to finance the expansion of its capital stock at the required common rate set by world average saving. But as soon as \( r < \max(\rho, \rho^*) \), \( \dot{a} \) has the same sign as \( \rho^* - \rho \), given the assumed stability of government spending.

The equilibrium inflation rates \((49, 49^*)\) corresponding to constant money supply expansion rates are no longer constants over time, but instead keep changing with the falling real interest rate so as to keep the nominal yields on both bonds constant. On the other hand, by (50) \( \varepsilon \) is independent of \( r \). For equal money expansion rates, \( \varepsilon = \rho - \rho^* \). If in addition \( G \) and \( G^* \) are kept in a constant proportion to wealth and \( r < \max(\rho, \rho^*) \), then a current deficit implies depreciation and a current surplus appreciation of the respective home currency.

Consider now policy effects. An unexpected change in either fiscal or monetary policy shifts the demand for money in one or both of the countries and calls for a discrete change either in goods prices or, alternatively, in money supplies for the asset market equilibrium to
be maintained. If $P$ or $P^*$ is allowed to jump, there is a change in $a$ and thus in the distribution of wealth, which greatly complicates the analysis. These real bond effects will be considered at the end of the section. In the meantime, it is assumed that the monetary authorities of the two countries, jointly if necessary, prevent discrete changes in $P$ or $P^*$ by open market operations in the bond market, or, what comes to the same thing, peg the exchange rate against jumps. Note that open market purchases or sales of bonds in either denomination - it does not matter which - only affect the nominal money supply of the respective country without altering the national holdings of bonds. When used to accommodate shifts in the demand for money, they are thus devoid of wealth effects other than preventing unexpected wealth transfers through $a$.

In general, unexpected changes in government consumption affect world saving even when the real foreign position is kept unchanged. The interdependence of present wealth and future growth then complicates the analysis, but some results are still readily available. Recall that in the small country model, with $K = 0$ and $G$ kept constant in time, we had $dC/dG = -(p/r)$, implying a partial crowding-out of private consumption in a saver economy and excessive crowding-out in a dissaver economy. In the present model we may reason as follows: Let there be an unexpected permanent increase in $G$ by one unit relative to whatever was its previously expected path. Since in a growing world economy the average future interest rate is lower than the present rate, the direct effect of the increase in $G$ - keeping $r$ unchanged - is to reduce $C$ by more than $p/r$ units. Thus for a dissaver economy we know for sure that the direct crowding-out of $C$ is excessive and $K$ is therefore increased. Accelerated capital growth means a faster increase in the real wage and a faster decline in the real interest rate, and hence an increase in $H$, $\widehat{G}$ and $\widehat{G^*}$ and feedback to $C$, $C^*$ and $K$. Although the feedback effect on $K$ is likely to counteract the direct effect, it cannot cancel it, for if it did, it would not exist. Therefore, a permanent increase in government consumption in a dissaver economy, with real bond effects neutralized, not only improves the current account of the dissaver economy but also adds to world growth by accelerating investment. The effects of a changed $G$ in a saver economy are uncertain.
There is one special case where fiscal policy is growth-neutral and lacks international repercussions. This is the fiscal policy rule that keeps government consumption proportional to wealth. As seen from (57) - (58), a change in $\beta$, with a kept unchanged, affects neither $K$ nor $a$, but is fully absorbed by a change in $C$.

Consider next monetary policy. Assume, for reasons not explained by the model, that the authorities are interested in keeping both the domestic output price and the exchange rate constant over time. Stabilizing the exchange rate alone is simple. Given a constant money supply expansion rate in one country, the monetary authority in the other country can determine its own money expansion rate such that $\epsilon$ in (50) is zero. On the other hand, since $r$ is moving, no constant money expansion rates are consistent with maintained price stability. To derive the variable money supply rules consistent with sustained zero inflation, consider the money market equilibrium condition

$$\frac{M}{P} = \left(1 - \alpha \right) \frac{\rho}{r + \Pi} \frac{\dot{W}}{W} \quad (59)$$

and its counterpart in the second country. In order for the money market to stay in equilibrium with $P$ constant and $\Pi = 0$, $M$ must grow at the relative rate

$$\mu = r - \rho - \frac{\dot{r}}{r} \quad (60)$$

where $\dot{r}/r = (Y''/Y')K$. Similarly, for the second country

$$\mu^* = r - \rho^* - \frac{\dot{r}}{r} \quad (60^*)$$

Since $\dot{r}/r$ is negative, these variable money expansion rates consistent with sustained zero inflation are higher than the constant money expansion rates (49) and (49*) that would be consistent with momentarily zero rates of inflation. Neither of the countries alone can stabilize both its domestic price level and the exchange rate. But, if both countries stabilize their price levels by applying the
money expansion rates (60) and (60*), the exchange rate stays constant as well.

If monetary policy does not prevent price level jumps by accommodating policy-induced discrete changes in the demand for money, there will, in general, be a change in a that modifies the fiscal policy effects and breaks the neutrality of monetary policy. The trouble is that, with an arbitrary currency composition of a, the sign and size of these real bond effects is arbitrary as well. Moreover, there is no guarantee that a new asset market equilibrium exists after a policy disturbance. Equilibrium fails to exist if the capital loss required to re-establish asset market equilibrium exceeds the wealth of the losing country. Since the potential magnitude of the capital loss is unbounded for any discrete policy change, the existence of equilibrium is not warranted except in a strictly local sense, i.e. for infinitesimal policy changes only.

Consider the case where government consumption is determined as a fraction of wealth. The conditions for asset market equilibrium are

\[ m = \psi(K+H+a) \]  \hspace{1cm} (61)

\[ m^* = \psi^*(K+H-a) \]  \hspace{1cm} (61*)

where

\[ \psi = \left( \frac{\rho}{\mu+\rho} \right) \left( \frac{1-\alpha}{\alpha+\beta} \right) \]

\[ \psi^* = \left( \frac{\rho^*}{\mu^*+\rho^*} \right) \left( \frac{1-\alpha}{\alpha+\beta^*} \right) \]

It is assumed that \( \mu+\rho > 0 \) and \( \mu^*+\rho^* > 0 \), so that \( \psi, \psi^* > 0 \). For an asset market equilibrium with \( m, m^* > 0 \) to exist we must have

\[ -(K+H) < a < (K+H) \]  \hspace{1cm} (62)

Although a change in a as such means only a shift in the distribution of wealth, the feedback from a change in the growth path of K affects
H and thus world average wealth as well. If the distribution of wealth changes in favour of the less impatient country, world saving, investment and output growth accelerate and H increases. Bearing that in mind, let us proceed as if H were a predetermined variable. Rewrite eq. (43) as

$$a = \frac{B}{M}m - \frac{F^*}{M^*}m^*$$

which expresses a in terms of the endogenous variables m and m* and the arbitrary but predetermined constants (B/M) and (F*/M*). Eqs. (61), (61*) and (63) solve for

$$m = \frac{2\psi_0\delta^*}{\Delta}(K+H)$$

(64)

$$m^* = \frac{2\psi_0\delta}{\Delta}(K+H)$$

(64*)

$$a = \frac{\delta^* - \delta}{\Delta}(K+H)$$

(65)

where

$$\delta = \frac{1}{2} - \psi_0\frac{B}{M}$$

$$\delta^* = \frac{1}{2} - \psi_0\frac{F^*}{M^*}$$

$$\Delta = \delta + \delta^*$$

In terms of the new parameters the condition for m and m* to be positive and $|a| < K+H$ is

$$\delta \delta^* > 0$$

(66)

While any discrete change in $\mu$, $\mu^*$, $\beta$ or $\beta^*$ may invalidate condition (66), we know that the condition is (almost everywhere) valid for
84 points lying on a perfect foresight equilibrium path. We may thus assume that the system is differentiable and investigate the comparative statics in terms of \( dm, dm^* \) and \( da \). It turns out that if \( (B/P)+(F^*/P^*) < K+H \), that is, if the holdings of home currency bonds in the two countries are not too large on the average, the policy reactions are as could be expected, viz. \( dm/d\psi (dm^*/d\psi^*) \) is positive and \( da/d\psi (da^*/d\psi^*) \) and \( -dm^*/d\psi (-dm/d\psi^*) \) have the same sign as \( B \) (\( F^* \)). Then an unexpected rise in \( \mu \) or \( \beta \) reduces demand for real balances in the first country causing \( P \) to jump upwards and, in addition, moves \( P^* \) up or down according to whether the first country is a net debtor or creditor in its home currency bonds. An opposite, perverse pattern of signs will materialize, if at the time of the disturbance \( (B/P)+(F^*/P^*) > K+H \). By (62), such a currency composition implies that the countries are net creditors in their home currency and net debtors in their respective foreign currency. When allowance is made for the endogeneity of \( H \), the above results become modified of course, but the possibility of perverse policy effects will not be removed.

No doubt, the difficulties stemming from the real bond effects point to a weakness of the model rather than to any real fragility in the international financial markets that the model attempts to describe. In an uncertain world, investors would have to be extreme risk lovers to hold portfolios that would not provide a hedge for them against insolvency in case of anything other than infinitesimal policy changes, or such skew currency positions that produce the perverse policy effects.

It is of course possible to neglect the real bond effects by assuming that all bonds are indexed (e.g. Dornbusch (1980)) or that only equities are traded internationally (e.g. Lipton and Sachs (1983)). Alternatively, as we did above, one can assume that the authorities prevent discontinuities in the exchange rate path, or then simply assume that the currency compositions are always such that the model

\[ 12 \text{The exception is that for a point on the perfect foresight equilibrium path it is also possible that } \delta, \delta^* = 0. \text{ We assume that this is not the case at the time of the disturbance.} \]
remains well-behaved. The point is that any of these ways of avoiding rather than facing the complications is likely to mean a loss of relevant information. The bulk of internationally traded liabilities are denominated in monetary units, so that systematic real bond effects are potentially significant. There is thus a price to be paid for using the simple perfect foresight, money vs. other assets framework, even when the focus of analysis is not on the effects of uncertainty as such.\textsuperscript{13}

\textsuperscript{13}The determination of optimal bond portfolios in an analogous stochastic foresight framework will be considered in section 4 of the last essay. It turns out that under homogenous investment sets ("world investors") bond holdings are zero in both countries and real bond effects thus do not appear.
9 REMARKS ON ALTERNATIVE APPROACHES

It was mentioned in the introduction that there exist optimizing models consistent with the wealth saturation hypothesis. Let us briefly consider some of these in order to see what it is that makes this possible. The theory of consumer choice in its general form is of little help. Maximizing $u(C_t, C_{t+1}, \ldots)$ subject to a wealth constraint only yields homogeneity and the Slutsky-equation, but no information concerning time preference and wealth saturation. Wealth saturation or lack of it is therefore due to more or less ad hoc restrictions built into the model. In continuous time or with several periods it is difficult to avoid assuming time-additive preferences and some systematic rule for discounting future utilities. It appears, however, that these assumptions alone are not decisive for wealth saturation.

Infinite horizon models, where asset wealth or all of its components enter the utility function, are capable of producing wealth saturation if assets are not too lucrative in comparison to current consumption. This approach is perhaps closest to the Metzlerian saving hypothesis in that it yields a consumption function in which assets appear as a separate argument. The objection is that including asset wealth in the utility function implies that the consumer is assumed to value permanent income streams deriving from assets higher than those originating from human wealth, which is difficult to justify in a frictionless neoclassical environment. Liviatan (1981) places both domestic and foreign money in the utility function in order to show that the currency substitution model of Calvo and Rodriguez (1977) may not be superneutral over steady states. If one "pure" asset were added, i.e. an asset that only indirectly produces utility by facilitating future consumption of goods, the Liviatan model would lose its steady state.

An alternative way (e.g. Findlay (1978), Kouri (1980), Obstfeld (1981)) to yield steady states from infinite horizon utility maximization is to assume in keeping with Uzawa (1968) that the rate of time preference increases with the level of utility. Under this hypothesis, the international economy converges to an interior steady state.
state distribution of wealth, determined by the fixed point of the falling marginal product of capital and the national time preference schedules. It is, of course, possible that the impatience of some consumers increases with the level of utility, but, despite the axiomatic treatment of such a consumer choice provided by Koopmans et al. (1964), it is not very obvious why this should hold as a general principle, comparable to, say, the falling marginal product of capital. Some might even argue that, if anything, the opposite is intuitively more plausible, or that it is the poor man rather than the rich who has to worry about the present and leave the future in the hands of the Almighty. One expression of such reasoning is the Cambridge view of saving as a luxury good available for capitalists only. In any case, given the vagueness of the concept of time preference, the assumption that it rises with utility is a rather specific one.

Yet another way of generating wealth saturation, while maintaining the convenience of the infinite horizon, is the "uncertain lifetime" approach by Yaari (1965), which has recently been used in a closed economy context by Blanchard (1983) and in an open economy model by Buiter (1984). In these models, the prospect of death as an ever-present probability results in a higher discount rate for human than for non-human wealth. A related wealth saturation mechanism is provided by the overlapping generations model, due to Samuelson (1958) and Diamond (1965), and employed in a two-country setting by Buiter (1981). In the overlapping generations model, aggregate wealth becomes saturated because the growing dissaving of the old eventually catches up with the saving of the young.

It is a merit of these two approaches that they bring some of the hard facts of individual life cycles to the macro level without blundering into a mess of aggregation problems. It is not for this reason, however, that these models generate wealth saturation. The key difference is the assumption of no bequests. In the infinite horizon model we may well let the individuals be subject to finite life expectancies and allow for overlapping generations. In that framework age simply does not matter, given the way bequests are planned and valued.
Although the infinite horizon model has a long standing in the optimal growth literature, its modern popularity in the positive theory of saving seems at least to be partly due to the difficulties in dealing with bequests in a rational way. Arrow and Kurz (1969) argued that the finite horizon model with a bequest motive is inadequate, since the bequest function itself is part of the problem, the individual's bequest motive being founded on his concern over the welfare of future members of the family. In addition, the finite period model implied artificial discontinuities in intergenerational transfers. Although the infinite horizon model avoids these weaknesses of the finite horizon - bequest motive model, one may question whether the implied degree of concern with future generations' welfare is empirically an overstatement of the representative man's family affections. On the other hand, the existence of bequests and other significant intergenerational transfers of wealth is a hard fact of reality, too. If an exogenous bequest motive is added to the uncertain lifetime or the overlapping generations model, we are back where we started. Loosely speaking, wealth becomes saturated if the aggregate Engel curve is such that actual consumption catches up with permanent consumption, and this may or may not be the case.

The above considerations suggest a rather pragmatic choice of model according to the desired implications and analytical usefulness. It is of interest to note that wealth saturation tends to be related to various non-neutralities of both money and government debt, although different results may arise depending on the mechanism that generates the steady state. In the Uzawa-type models an increase in \( \mu \) raises steady state consumption of goods. This is because the exogenous parameters of the model fix the steady state level of utility, so that lower steady state real balances have to be compensated with increased steady state consumption of goods. Whether this kind of long-term non-neutrality of money is a desirable property is another question. If so desired, the two-country model of section 8 can be tuned to produce non-neutrality of money by changing the utility function. The uncertain lifetime and the overlapping generations models imply non-neutrality of government debt. To the extent there is reason to believe that full discounting of future tax liabilities is an
unrealistic assumption, these models may have a comparative advantage when the focus of analysis is on fiscal policy. In the infinite horizon model some kind of liquidity constraint is needed for non-neutrality of debt.

In the context of exchange rate models the key question would seem to be whether it serves some useful purpose to let wealth accumulation fix a stationary value for the exchange rate. Perhaps such a reason exists, although we have not found one. One reason for the hesitation in using the infinite horizon model with constant but internationally different rates of time preference has possibly been that during the modern era of perfect foresight and rational expectations models we have become accustomed to analyzing macroeconomic phenomena in terms of saddle path dynamics. To have a saddle path by a narrow definition of the concept, one obviously needs a saddle, and without steady states there are no saddles. We have endeavoured to demonstrate that, at least from a technical and practical point of view, one may well do without interior steady states. In fact, the equilibrium trajectories towards a boundary state are like one-sided saddle paths, and if far enough from the final state the implied difference in the movement of variables need not be much more than of a second order of magnitude, i.e. concerning rates of acceleration rather than velocity.

A more basic structural difference is that, unlike an interior saddle, a boundary state does not define its unique convergent path. As has been amply demonstrated in this paper, we may have shifts in the adjustment path without any change in the relevant boundary state. In steady state models, a shift in the adjustment path normally implies a shift in the steady state and vice versa. Related to this is the absence in the non-steady-state model of a distinction between impact and permanent policy effects and the inapplicability of the conventional technique of linearizing around the steady state. These features need not be disadvantages, however.

In any case, the infinite horizon - constant time preference framework is the simplest of the current alternatives and the easiest to handle if new elements are added to the model. For example, the Mertonian theory of simultaneous determination of optimal saving and portfolio
is fully developed for this framework only. After all, wealth accumulation cannot be more than one of the relevant factors affecting exchange rate movements. If the various currently held views stressing wealth, money, risk, terms of trade, sticky prices or capital market imperfections are ever to be merged into a tangible whole, it is better that the way wealth accumulation enters the picture is not too complex.

A more complicated and potentially richer role for wealth accumulation emerges in models that are capable of producing endogenous movements in relative prices. If wealth accumulation implies changing terms of trade (the price of importables in terms of exportables) or a changing real exchange rate (the price of tradeables in terms of home goods), the relevant concept of the real rate of interest for saving decisions may move differently for different countries, depending on national consumption preferences and technologies for producing home goods. Likewise, movements in the relative price of investment goods in terms of consumption goods (Tobin's q) may interfere with investment and wealth accumulation differently across countries. Whether and under which conditions endogenous relative price movements can contribute to wealth saturation and the existence of interior steady states in the case of constant time preferences would seem to be a useful issue for future research.

14 In the context of a portfolio balance model of the exchange rate in which wealth saturation is postulated, BRANSON and HENDERSON (1985) use the mean variance model to derive asset demands. The justification offered is the MERTON (1969) result that under certain conditions the portfolio allocation decision is separable from the saving decision. They do not, however, prove that separability holds in an optimizing framework that is consistent with wealth saturation.

15 For analyses of the effect of the real exchange rate and terms of trade variations on intertemporal consumer choice, see HELPMAN and RAZIN (1982), DORNBUSCH (1983) and OBSTFELD (1983). Optimal foreign borrowing in a non-monetary small economy model with an investment good has been studied by BAZDARICH (1978). SACHS (1983) emphasizes the importance of national investment dynamics for the determination of the current balance.
APPENDIX

In this appendix, we investigate whether in the transfer money case of section 3 there are other monetary equilibria than the constant inflation path given by (20a). Our findings apply equally to the extended model with a more complete government sector.

The recent literature (Brock (1974) and (1975), Calvo (1979), Obstfeld and Rogoff (1983) and Gray (1984)) analyses these issues using models that possess a steady state for wealth accumulation. The lack of steady state in the present framework means that the state dependent path of real balances does not differ in kind from other divergent perfect foresight paths, and explosive price paths do not necessarily imply implosive real money holdings or vice versa. Another difference is that both Gray as well as Obstfeld and Rogoff keep the supply of money constant, while we allow the money supply to change over time. For these reasons it is of some interest to look at the uniqueness issue in the context of the present model.

The solution for the differential equation (18) is

\[ \phi(s) = (\phi(t) - \phi) e^{(\mu + \rho)(s-t)} + \phi \]  

(A1)

where \( \phi \) is given by (19). We are interested in the feasibility and optimality of paths for which \( \phi(t) \neq \phi \). For \( \mu > 0 \), such paths are ruled out by the feasibility condition (15), which requires \( \phi(s) \) to remain in the interval \([0,1]\). When \( \mu \) is negative, the feasible range of \( \phi(s) \) is not bounded from above as in (15). The reason for this is that when \( V < 0 \) we may well have \( m > W > 0 \) and \( F+(Y/r) > 0 \), as required by (4). However, no equilibria exist for \( \mu < -\rho \), since all paths violate \( \phi(s) > 0 \) in a finite time period. If \( -\rho < \mu < 0 \), paths emanating from \( \phi(t) < \phi \) are infeasible for the same reason. The only remaining case is \( \phi(t) > \phi \), \( -\rho < \mu < 0 \). We now proceed to show that these
"hyperdeflationary" paths are equilibria.16

Let us first check whether all paths \( \phi(t) > \bar{\phi} \), \(-\rho < \mu < 0\) are feasible in the sense that the implied wealth is positive and finite and that the constraint (4) is not violated. For feasible Euler paths the solution for \( W \) equals

\[
W(s) = W(t)e^{(r-\rho)(s-t)} \quad \text{(A2)}
\]

Assume that \( V(t) \) and thus \( W(t) \) are in fact well defined and try the solution

\[
m(s) = \phi(s)W(s) = (\phi(t)-\bar{\phi})W(t)e^{(r+\mu)(s-t)} + \bar{\phi}W(t)e^{(r-\rho)(s-t)} \quad \text{(A3)}
\]

From (21) this gives

\[
V(t) = \mu \int_t^\infty e^{-r(s-t)m(s)}ds
\]

\[
= \mu \int_t^\infty (\phi(t)-\bar{\phi})W(t)e^{u(s-t)}ds \quad \text{(A4)}
\]

\[
+ \mu \int_t^\infty \bar{\phi}W(t)e^{-\rho(s-t)}ds
\]

Since \( \mu < 0 \) by assumption, (A4) converges and equals

\[
V(t) = ||\frac{\mu+\rho}{\rho} \bar{\phi} - \phi(t)||W(t) \quad \text{(A5)}
\]

\[
= ||(1-\alpha)\phi(t)||W(t)
\]

16 Along these paths prices fall at a relative rate that tends to \(-r\) and the nominal interest rate \( r+\Pi \) hence approaches zero. Real balances move at a relative rate \( \mu-\Pi \), which tends to \( \mu+r \). In a saver economy, \( r > \rho > 0 \) and \(-\rho < \mu < 0\) imply \( \mu+r > 0 \). But in a dissaver economy with \( \rho > r \) we may have the case where \( \mu+r < 0 \), so that real balances fall along with falling prices.
Therefore

\[ m(t) + V(t) = (1-\alpha)W(t) \tag{A6} \]

and from (1a)

\[ W(t) = \frac{1}{\alpha}(F(t)+\frac{V}{r}) \tag{A7} \]

which equals (27) and yields the same reduced form (20a) for \( C \) and \( \pi \) as the state dependent path \( \phi(t) = \bar{\phi} \).

We shall next demonstrate that all feasible paths (including the paths \( \phi(t) > \bar{\phi}, \ -\rho < \mu < 0 \)) that are generated by the general solution (11) - (14) of the model are optimal for an individual agent. Since these paths fulfil the Euler conditions and the Hamiltonian

\[ H = e^{-\rho(s-t)}u_C(C,m) + \lambda(rW-C-Rm) \tag{A8} \]

is concave in \( C, m \) and \( W \), they are optimal (see Seierstad and Sydsæter (1977)) if

\[ \lim_{s \to \infty} \lambda(s)|W(s) - W^*(s)| > 0 \tag{A9} \]

where \( W^*(s) \) is the suggested path of the state variable, as given by (A2), and \( W(s) \) is any other admissible path. For the costate variable we have

\[ \lambda(s) = e^{-\rho(s-t)}u_C = \left[ \frac{1}{\rho W(t)} \right] e^{-r(s-t)} \tag{A10} \]

Since

\[ \lim_{s \to \infty} \lambda(s)W^*(s) = 0 \tag{A11} \]

and \( W(s) > 0 \), condition (A9) is verified.

To sum up, for \( \mu > 0 \) the equilibrium is unique and given by the constant inflation path \( \phi(s) = \bar{\phi} \), while for \( -\rho < \mu < 0 \) paths that are explosive in \( m \) and implosive in \( P \) relative to the constant
inflation path are also equilibria. The result that hyperinflationary paths are not equilibria conforms to earlier findings in steady state models, and is likely to be dependent on the properties of the utility function.\footnote{Obstfeld and Rogoff (1983) show that for separable (\(u_{cm} = 0\)) utility functions "the necessity of money" property \(\lim m \to 0\) is necessary for hyperinflations to be infeasible.} A more interesting result is that hyperdeflationary equilibria cannot be ruled out, if \(\mu\) is negative. While this does not contradict earlier findings that under a constant \(M (\mu = 0)\) hyperdeflations are not equilibria, it reveals that a transversality argument used by Gray (1984) and by Obstfeld and Rogoff (1983) to rule out hyperdeflations is in general not correct.\footnote{This has been pointed out to the author by Seppo Salo.} These authors claim that for optimality over an infinite horizon it is necessary that

\[
\lim e^{-\rho(s-t)} \left( \frac{u_C(s)}{P(s)} \right) = 0
\]  

(A12)

In so far as wealth is the only state variable, the relevant transversality condition is instead (A11), which also appears as part of the sufficient conditions considered above.\footnote{The intertemporal budget constraint (7) contains the implicit assumption that consumption paths whose present value is smaller than the present wealth are not optimal. If the strict equality in (7) is replaced by (\(<\)), or alternatively, if the state space restriction \(W(s) > 0\) is used, the solution (11)-(12) is difficult to derive without resorting to the transversality condition (A11).} If \(\mu = 0\), (A11) implies (A12), but this is not the case if \(\mu < 0\). Indeed, the case \(\phi(t) > 0\), \(-\rho < \mu < 0\) contradicts (A12), but is nevertheless an equilibrium.
INFLATION AND GROWTH WITH STOCHASTIC MONEY AND OUTPUT

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1 INTRODUCTION

This paper examines monetary policy in a stochastic general equilibrium model with rational expectations and optimizing agents. The model is developed in the closed economy framework and then extended to a two-country model.

With the exception of the nominal bond rate of interest, the variables of the model are specified as Ito-processes. Uncertainty derives from two ultimate sources, random variations in output, on the one hand, and in money supply, on the other. We allow for the possibility that money supply is correlated with output. The probability distributions of other variables are endogeneously determined by market interaction. Money is modelled as a consumer good and utility is logarithmic. The agents maximize expected discounted utility over an infinite horizon so as to decide on current consumption and portfolio composition. The production function is linear in the sense that the mean and variance of output are proportional to the capital stock, and there is no labour income. Money is issued as lump sum transfer income and the government lays no claims to real resources. Output that is not consumed is added to the capital stock. The agents are identical and constant in number. The model is solved for its state dependent path and the implications of the stochastic structure for monetary policy are then examined.

As expected on the basis of results derived in deterministic models (Sidrauski (1967), Brock (1974), Fischer (1979), Cohen (1985)), money turns out to be superneutral in the sense that capital accumulation and thus growth are independent of the parameters of the money supply process. Although in the present paper we explicitly consider only the simplest case, where utility is logarithmic and output and money supply processes have constant parameters, monetary dichotomy or the lack of it should not in general be affected by the mere replacement of deterministic by stochastic foresight. We would thus expect
superneutrality to generalize to stochastic models in which either utility separates in goods and money or the distribution of the return on capital is stationary.

Given the neutrality result, the interesting features of the model concern the effects of the stochastic environment on the monetary equilibrium. We find that the randomness of the money supply, as measured by its instantaneous variance, reduces the nominal interest rate and therefore increases the demand for real balances and lowers the current level of output price. It also increases the variance of inflation, but does not affect the average inflation rate. The randomness of output increases both the mean and the variance of inflation.

An interesting result concerns the potential covariation of money supply and output. A negative correlation of money and output, whether due to automatic financing of fiscal drag by the central bank or to discretionary countercyclical monetary policy, increases both the mean and the variance of inflation. Conversely, a positive correlation, due to accommodation of random changes in money demand, improves price stability by both lowering mean inflation and increasing the predictability of inflation. Such accommodative practices might arise in institutional circumstances where the central bank acts as the bank of the banks rather than of the government. The model thus lends some support to views that advocate the independence of central banking from government financing.

In the final section the model is extended to a two-country world economy with national monies. Monies produce utility for the respective residents only, so that there is no "currency substitution". While agents are free to borrow or lend internationally in either currency, it turns out that equilibrium holdings of both bonds are zero in both countries. Thus the monetary sectors of the two countries are fully separated and the closed economy model as such remains relevant for each country. As a consequence of zero bond holdings, domestic investment absorbs all domestic saving and the current account is in balance. Purchasing power parity holds in this single commodity world and the exchange rate thus follows the ratio of the national price
levels. The mean rate of depreciation depends on money supply randomness but not on output volatility.

It follows from zero bond holdings that, contrary to earlier views in finance literature, international bond markets do not serve as a vehicle for pooling the purchasing power risks of national money supplies. The reason is that the component of purchasing power risk which is due to money supply randomness nets out against the risk in the present value of the monetary transfer. The total monetary wealth in a country only bears the component of inflation risk which stems from output randomness. Since this risk is common for agents in both countries, it cannot be diversified away by swapping bonds. The result highlights the importance of general equilibrium analysis in finding out the full consequences of a fiat monetary system for prices, wealth and risk.

A similar closed economy model, but with the difference that money supply and output are assumed independent, has been studied by Gertler and Grinols (1982). In their analysis investment and growth depend positively on the mean money expansion rate and negatively on its variance. In appendix A it is demonstrated that the non-neutrality of money in the Gertler-Grinols model is not due to uncertainty but to a structural bias which arises from treating the transfer income from the government as part of the return on asset wealth. ¹

Appendix B contains a review of the super-neutrality-of-money issue in Sidrauski-type deterministic models, and serves the discussion in section 3 about the likely degree of generality of the analysis in this paper.

Appendix C contains a proof that the market equilibrium solution of the model, as derived in section 2, is optimal.

¹It is clear that results derived in the deterministic perfect foresight framework would be rather meaningless, unless they hold at least qualitatively in the stochastic framework as well.
2 THE MODEL

The economy consists of a constant number of identical, immortal agents and a government. The agents employ capital to produce homogeneous output, which they consume or invest in the capital stock. The government issues fiat money through lump sum transfers but does not have any claims on real resources. Output is random but endowed in an equal amount to each agent by an Itô-process

\[ dY = rKdt + \alpha K dZ_K \]  

(1)

where \( K \) is capital per agent, \( r \) and \( \alpha \) positive constants and \( dZ_K \) a normally and time-independently distributed random variable with zero mean and variance \( dt \). Output not consumed is added to the capital stock:

\[ dK = dY - C dt \]  

(2)

Besides capital, agents hold money balances and may lend or borrow at the nominal interest rate \( R \). Bonds mature instantaneously and bear no nominal value risk. \( R \) may thus be considered as a deterministic function of time. Since the economy is closed and the agents are identical, bond holdings are zero in market equilibrium. The bond market is nevertheless needed to provide an asset with the same risk as money and thereby to fix the opportunity cost of holding money. The real (marketable) asset wealth \( A \) of each agent thus consists of

\[ A = m + b + K \]  

(3)

where \( m = M/P \) stands for real money balances and \( b = B/P \) for real net bond holdings.

The government issues (or deissues) money at the rate
\[
\frac{dM}{M} = \mu \, dt + \sigma_M dZ_M
\]  
(4)

where \(\mu\) and \(\sigma_M\) are constant parameters. The instantaneous covariance of \(dZ_M\) and \(dZ_K\) is denoted by \(\sigma_{MK}\) and is likewise constant over time.

The agents perceive the output price to obey

\[
\frac{dP}{P} = \pi \, dt + \sigma_P dZ_P
\]  
(5)

where \(P\), \(\pi\), \(\sigma_P\) and \(dZ_P\) are endogenous variables to be determined by the model. It is assumed that the agents expect \(\pi\), \(\sigma_P\) and \(\sigma_{KP}\) to stay constant over time. It will turn out that this assumption is valid along the state dependent path of the economy, which is the only one that we shall be considering.

Given (4) and (5), Ito's lemma implies that the real money supply evolves according to

\[
\frac{dm}{m} = (\mu-\pi+\sigma_P^2-\sigma_{MP})dt + \sigma_M dZ_M - \sigma_P dZ_P. 
\]  
(6)

The change in real money supply given by (6) can be decomposed as follows

\[
dm = [r_{bm} dt - m\sigma_P dZ_P] + [(u-\sigma_{MP})m dt + m\sigma_M dZ_M] - [Rm \, dt] 
\]  
(6a)

where

\[
r_{b} = R - \pi + \sigma_P^2
\]  
(7)

is the (mean) real bond rate of interest. In (6a), the terms in the first brackets stand for real gross asset income from real balances, the terms in the second brackets for real transfer income from the government and the last term for the consumption of money services valued at their opportunity cost \(R\). Real asset income from bond holdings equals

\[
r_{bb} dt - b\sigma_P dZ_P
\]  
(8)
Since money and bonds bear the same risk, they can be treated as a composite asset \((m+b)\). Using the portfolio constraint (3) to eliminate this composite asset, the change in asset wealth can be written

\[
dA = [(r-r_b)K + r_bA + (\mu - \sigma_{MP}) \bar{m} - C - Rm] \, dt
\]

\[
+ K \sigma_K dZ_K - (A-K) \sigma_p dZ_p + \bar{m} \sigma_M dZ_M
\]

(9)

where a bar has been added over \(m\) in those terms that are to be considered as exogeneous market data from the viewpoint of an individual agent. Let \(t\) stand for the initial time and \(s\) for future time. Agents know \(M(t)\) and \(P(t)\) and thus \(\bar{m}(t)\), and expect \(\bar{m}(s)\) to obey (6). Initial asset wealth \(A(t)\) is predetermined and savings are determined by (9). Markets are expected to clear for all \(s \geq t\).

Subject to this information, the agents maximize

\[
U = \mathbb{E}_t \int_t^\infty e^{-\rho (s-t)} u(C, m) \, ds
\]

\[u = \alpha \log C + (1-\alpha) \log m\]

\(\rho > 0, \, 0 < \alpha < 1\)

A technical complication, which prevents us from relying on standard methods, arises from the presence of \(\bar{m}\) as a second state variable. To solve the model we shall impose a trial solution for \(C(t)\) and then derive the state dependent path of the economy by using market equilibrium conditions and two necessary conditions for optimum which are independent of the presence of non-asset income. At the end we are able to check the optimality of the suggested solution.

The analogous model in the deterministic perfect foresight setting (see appendix A) solves for

\[2\]

While \(K\) is predetermined for the economy as a whole, the state variables for an individual agent are \(A\) and \(\bar{m}\), rather than \(K\). From the point of view of an agent, eq. (11) thus expresses an equilibrium relationship between two controls.
\( C = \rho K \) \hspace{1cm} (11)

Since there is nothing in the stochastic structure that would cause us to expect money to lose its neutrality and since in the case of logarithmic utility (unitary relative risk aversion) consumption is immune to real side uncertainty, we take (11) as the trial solution.\(^3\)

Substituting (11) and (1) into (2) yields

\[
\frac{dC}{C} = \frac{dK}{K} = (r-\rho)dt + \alpha_K dZ_K
\]

(12)

There are two necessary conditions for interior optimum that do not depend on the presence of non-asset income. First, along an optimum consumption path we have

\[ u_m - Ru_C = 0 \]

that is

\[ m = \frac{1 - \alpha}{\alpha R} C \]

(13)

Second, if consumption is an Itô-process, it holds according to the consumption-beta model of Breeden (1979) and Grossman and Shiller (1982)\(^4\) that the mean excess return on any tradeable asset equals the covariance of the excess return with consumption multiplied by the coefficient of relative risk aversion (one in our case). This result is directly applicable to a model with a single consumption good only. In the present model there are two consumption goods, \( C \) and \( m \).

Assuming, as will be validated below, that \( R \) is constant along the state dependent path, we may for this purpose aggregate the two goods into "total consumption" \( \tilde{C} = C + Rm \). From (13) it follows that \( \frac{d\tilde{C}}{\tilde{C}} = \)

---

\(^3\)For a constant relative risk aversion utility function (see Appendix B) we would, given the assumptions of the model, expect \( C \) to depend on \( r \) and \( \alpha_K \), but not on \( \mu \) or \( \sigma_M \).

\(^4\)Grossman and Shiller proved that the Breeden model is also valid in the presence of non-tradeable risky assets. This result is needed here.
dC/C, which by (12) is an Ito-process. Therefore, for the mean excess return on bonds over capital we have from (1), (8) and (12)

\[ r_b - r = -\sigma_{KP} - \sigma_K^2 \]  

(14)

It has been assumed that \( \pi, \sigma_P^2 \) and \( \sigma_{KP} \) stay constant over time. It then follows from (14) and (7) that \( R \) is constant as well. Hence, from (13) and (12)

\[ \frac{dm}{m} = (r - \rho)dt + \sigma_K dZ_K \]  

(15)

Equation (15) describes the movement of the demand for real balances over time. Equation (6) gives the movement of the real supply of money. Markets clear for all \( s > t \). This implies that equality holds separately for the deterministic and stochastic parts of (6) and (15):

\[ \pi = \mu + \sigma_P^2 - \sigma_{MP} - r + \rho \]  

(16)

\[ \sigma_p dZ_p = \sigma_M dZ_M - \sigma_K dZ_K \]  

(17)

From (17)

\[ \sigma_P^2 = \sigma_M^2 + \sigma_K^2 - 2\sigma_{MK} \]  

(18)

\[ \sigma_{MP} = \sigma_M^2 - \sigma_{MK} \]

\[ \sigma_{KP} = \sigma_{MK} - \sigma_K^2 \]

Thus, from (16)

\[ \pi = \mu + \sigma_K^2 - \sigma_{MK} - r + \rho \]  

(19)

and further from (14) and (7)

\[ r_b = r - \sigma_{MK} \]  

(20)
Substituting (21) into (13) and (11) gives

\[ m = \left( \frac{1-\alpha}{\alpha} \right) \left( \frac{-\rho}{\mu^2 - \sigma^2} \right) K \]  

(22)

after which we may solve for the output price from \( P = M/m \). The solution for \( P \) is uniquely determined by the current values of \( K \) and \( M \), and is thus the state dependent solution. A feasible solution for \( m \) must be positive. Therefore the state dependent solution exists only for

\[ \mu > \frac{\sigma^2}{\alpha^2} - \rho \]  

(23)

We next want to prove that the suggested path of the economy is consistent with optimal behaviour of the agents. In general, consumption under logarithmic utility is proportional to wealth. The problem is that from the outset we do not know what wealth is in the present context. Besides having command of a given amount of asset wealth, the agent faces a probability distribution, defined for \( s > t \), of receiving transfer income. If there is to be a control rule for \( C \) in terms of a single state variable, called wealth, it must be possible for the relevant information concerning the uncertain prospect of transfer income to be condensed into a real number, call it the wealth equivalent of the transfer income. Let us assume that this is the case and define

\[ W = A + V \]  

(24)

where \( W \) stands for wealth and \( V \) for the wealth equivalent of the transfer income, as defined in (6a). Now the problem is in a familiar setting, where the closed form solution for \( C \) can be expected to be\(^5\)

\(^5\)Merton (1969).
\[ C = \alpha \rho W \]  

In order for (25) to be consistent with the trial solution (11), we must have

\[ W = \frac{K}{\alpha} \]  

(26)

Since \( b = 0 \), it follows from (3), (24) and (26) that

\[ m + V = \left( \frac{1 - \alpha}{\alpha} \right) K \]  

(27)

and further from (22) that

\[ V = \left( \frac{1 - \alpha}{\alpha} \right) \left( \frac{\mu - \sigma_M}{\mu + \rho - \sigma_M} \right) K \]  

(28)

Since \( m = \bar{m} \) in equilibrium, (22) and (28) imply

\[ V = \left( \frac{\mu - \sigma_M}{\rho} \right) \bar{m} \]  

(29)

We are now ready to check the optimality of the suggested solution. Let \( J(A, \bar{m}) = e^{\rho t \max} U \) be the current value transformation of the indirect value function. The foregoing reasoning amounts to the proposition

\[ J(A, \bar{m}) = \frac{1}{\rho} \log W + h = \frac{1}{\rho} \log [A + (\frac{\mu - \sigma_M}{\rho}) \bar{m}] + h \]  

(30)

where \( h \) is a constant independent of \( A \) and \( \bar{m} \). For an interior optimum it is necessary that
\[0 = u(C,m) - \rho J + J_A E(dA) + \frac{1}{2} J_{AA} E(dA)^2 + J_{m} E(d\bar{m}),\]

\[+ \frac{1}{2} J_{mm} E(d\bar{m})^2 + J_{A\bar{m}} E(dA\bar{m})\]

\[0 = u_C - J_A\]

\[0 = u_m - RJ_A\]

\[0 = J_A (r - r_b) + J_{AA} [\left( \sigma_K^2 + \sigma_P^2 + 2\sigma_{KP} \right) K - (\sigma_P^2 + \sigma_{KP}) A + (\sigma_{MK} + \sigma_{MP}) \bar{m}]\]

\[+ J_{Am} (\sigma_{MK} - \sigma_{KP} - \sigma_P^2 + \sigma_{MP}) \bar{m}\]

where \(E(dA)\), \(E(dA)^2\), etc. are the means, variances and the covariance, respectively, of the processes (9) and (6). Substituting for the endogeneous variables from (3), (11), (18)-(22), and from \(m = \bar{m}\) and \(b = 0\), the means become proportional to \(K\) and the second order terms proportional to \(K^2\). From (30) and (26)

\[J_A = \frac{\alpha}{\rho K}\]

\[J_m = \frac{\alpha (\mu - \sigma_M^2)}{\rho^2 K}\]

\[J_{AA} = -\frac{\alpha^2}{\rho^2 K^2}\]

(32)

\[J_{mm} = -\frac{\alpha^2 (\mu - \sigma_M^2)}{\rho^3 K^2}\]

\[J_{Am} = -\frac{\alpha^2 (\mu - \sigma_M^2)}{\rho^2 K^2}\]

With these values (31.2) - (31.4) hold as identities. In (31.1.), \(\log K\) nets out from the first two terms and \(K\) cancels in the other terms, so that the equation becomes an identity with a suitable choice of the integration constant \(h\). We have thus shown that the suggested constant inflation equilibrium path does not contradict the necessary conditions for optimum. A direct proof of optimality is given in appendix C.
Before turning to the interpretation of the results, we wish to make a few general comments on this type of model. In the analogous deterministic model, which is summarized in appendix A, one is able to solve the model in the natural order of first finding a general solution for consumption and asset demands in terms of $W$ which is valid for any (sufficiently bounded) expectations and then imposing the additional restrictions implied by perfect foresight and market clearing. This is possible because - assets being perfect substitutes - there is no ambiguity about the appropriate discount factor, and because the non-tradeability of the government transfer is of no relevance as long as borrowing or short sales of assets are allowed up to total wealth.

The reason why, in solving the stochastic model, we have proceeded in a more or less reverse order is as follows. When capitalizing the expected transfer income one has to allow for the non-diversifiable risk carried by that source of income, either by applying an appropriate discount factor or by subtracting an adequate risk premium from the mean income flow. Since none of the assets of the model bear the same risk as the transfer income, the discount factor or the risk premium is only determined as part of the market equilibrium solution of the model. Moreover, from the outset it is not clear whether a feedback policy rule for consumption and asset demands exists in terms of $W$, i.e. the sum of $A$ and $V$, unless $V$ is constrained to be consistent with the state dependent equilibrium path of the economy.

Now, since we know the reduced form of the model, $V$ as defined by (29) can be interpreted as the expected present value of a risk-corrected income flow. Suppose that we choose $r$ to be the discount rate and ask what will then be the risk premium that has to be subtracted from the mean transfer income flow $(\mu - \sigma_{MP})\bar{m}(s)$. Let us reason as follows. Bearing the risk (per unit wealth) of $\alpha_K dZ_K$ is remunerated by the markets with the mean yield $r$. Carrying the risk $(-\sigma_p dZ_p)$ is rewarded by the mean rate of return $r_b$. The unit risk of the transfer income is
But by (17), \( \sigma_d dM \) equals \( \sigma_K dZ_K - (\sigma_P dZ_P) \). Consequently, the implicit market valuation for this risk is \( (r - r_b) \). This suggests that the risk premium for the transfer income flow equals \( (r - r_b)\mbar \).

Therefore, in equilibrium

\[
V(t) = E_t^\infty e^{-r(s-t)} \left[ (\mu - \sigma_{MP})\mbar(s) - (r - r_b)\mbar(s) \right] ds
\]

\[
= \int_t^\infty e^{-r(s-t)} \left[ (\mu - \sigma_M^2 + \sigma_{MK})E_t\mbar(s) - \sigma_{MK}E_t\mbar(s) \right] ds
\]

\[
= \int_t^\infty e^{-r(s-t)} (\mu - \sigma_M^2)\mbar(t)e(r - \rho)(s-t) ds
\]

\[
= \frac{\mu - \sigma_M^2}{\rho} \mbar(t)
\]

(33)

which agrees with the indirectly derived value of \( V \) given by (29).

The main purpose of the exercise in section 2 was to demonstrate that a model which exhibits monetary dichotomy or "superneutrality of money" under deterministic perfect foresight continues to do so under stochastic foresight. This result reflects the basic structural properties of the underlying non-stochastic model, viz. that the money supply process does not absorb real resources nor distort the intertemporal substitution of consumption. We would therefore expect superneutrality to generalize to other stochastic models in which the analogous deterministic model has this property.

The neutrality issue in Sidrauski-type models is reviewed in appendix B. The conclusion is that for superneutrality to appear as a reasonably robust feature of the model, the marginal utility of consumption of goods has to be independent of money. It is primarily

6Alternatively, \( V(t) \) may be expressed as the present value of the uncorrected mean transfer flow \( (\mu - \sigma_{MP})E_t\mbar(s) \). The appropriate discount rate is \( r + \rho\sigma_{MK}/(\mu - \sigma^2) \), something not very easy to reason out directly.
for this reason that in this paper we have focused on the logarithmic rather than the apparently more general isoelastic (constant relative risk aversion) utility function.⁷ The point is that if superneutrality is desirable, isoelastic utility may be too permissive.

The fundamental problem - which also explains the coexistence of many rival approaches to modelling money in macroeconomics - is that monetary theory lacks a rigorous choice theoretic foundation. Treating money as a consumer durable merely means that the demand for money is supposed to obey the rather weak axioms of general demand theory. While this assumption may or may not be valid, the "true" microeconomics of money may also imply stronger restrictions, such as the superneutrality captured by the additive separability of the utility function.⁸ Answers to these questions obviously have to wait for further advances in monetary theory.

⁷Given the assumption of separable utility, the choice of the logarithmic form is a matter of convenience. For the class of utility functions that allow for explicit closed form solutions and for the underlying mathematical reasons, see HAKANSSON (1970) and MERTON (1971).

⁸Some institutional explanations for the demand for money do not favour superneutrality. If money is held to economize trips to the bank, the marginal utility of leisure is hardly independent of money (BROCK (1974)). On the other hand, cash-in-advance (constant velocity) models may imply monetary neutrality of an even stronger type than separable utility (see HELPMAN and RAZIN (1982)).
Given the result that monetary policy does not affect consumption or capital stock growth, it is of interest to observe how the monetary side of the model is influenced by the parameters of the random processes. The relevant results are given by equations (17)-(23).

Setting \( a_K^2 \), \( a_M^2 \) and \( a_{MK} \) equal to zero, yields the deterministic perfect foresight solution of the model, as it is reported in appendix A. The roles of the mean growth of money \( (\mu) \) and output \( (r) \) are therefore the same as in the non-stochastic model. The additional features that are due to a stochastic environment can be traced by focusing on the effects of the three parameters \( a_K^2 \), \( a_M^2 \) and \( a_{MK} \) only.

The way these distribution parameters enter the reduced form of the model is not easy to justify by commonsense arguments. As an example of how intuition might help to penetrate the jungle of stochastic calculus, consider the following short-cut to explaining the solution for the inflation rate.

The task of the movements in \( P \) is to fill all impending gaps between the movements of the nominal money supply and the real demand for money. Real balances follow the common real growth rate \( (r-\rho)dt + \sigma_K dZ_K \) of the economy. We thus obtain the inflation path directly by applying Ito's differentiation rule to the ratio \( M/K \):

\[
\frac{dP}{P} = \frac{d(M/K)}{M/K} = \frac{dM}{M} - \frac{dK}{K} + \left( \frac{dK}{K} \right)^2 - \left( \frac{dM}{M} \right) \left( \frac{dK}{K} \right) \\
= (\mu \ dt + \sigma_M dZ_M) - ((r-\rho)dt + \sigma_K dZ_K) + \sigma_K^2 dt - \sigma_{MK} dt \\
= [\mu - (r-\rho) - (\sigma_{MK} - \sigma_K^2)]dt + \sigma_M dZ_M - \sigma_K dZ_K
\]

The last component \( (\sigma_{MK} - \sigma_K^2) \) of the mean rate of inflation stems from the second order terms of Ito's formula and is equal to the covariance.
of the processes $dM/M - dK/K$ and $dK/K$.\textsuperscript{9}

Disregard $\sigma_{MK}$ for a moment and consider how $\sigma_K^2$ and $\sigma_M^2$ affect the monetary equilibrium. The effect of the randomness of output, as measured by $\sigma_K^2$, is to increase both the mean and the variance of inflation, but $\sigma_K^2$ does not affect the nominal interest rate, nor the equilibrium output price.\textsuperscript{10} The volatility of money supply, as measured by $\sigma_M^2$, also increases the variance of inflation, but not its mean. On the other hand, $\sigma_M^2$ reduces the nominal interest rate and thus increases the demand for real balances. Hence, an unexpected once-and-for-all increase in $\sigma_M^2$ will cause the price level to jump downwards, but has no effect on the mean inflation rate, while an increase in $\sigma_K^2$ raises $\pi$ without altering the current $P$.

Consider then the rather interesting role of $\sigma_{MK}$. What, in the first place, does a non-zero $\sigma_{MK}$ signify? A positive correlation between money and output means that the supply of money tends to accommodate random changes in money demand caused by non-anticipated variations in output. Such accommodative behaviour of the money supply may arise under institutional arrangements where the "lender of last resort" function dominates central banking. A negative $\sigma_{MK}$ again might be due to financial consequences of fiscal drag in an economy where the central bank is dominantly the bank of the government: An unexpected fall in output reduces tax revenue and causes additional social benefit outlays, and an extra run on the note printing press is then the easiest way of supplying what the treasury needs. An alternative interpretation for a negative $\sigma_{MK}$ might be that the authorities attempt to use money for countercyclical purposes, so that when output falls short of its trend growth money is pushed above its mean expansion rate.

\textsuperscript{9}Note that for the stochastic process of "the value of money" $(1/P)$ it does not hold that $d(1/P)/(1/P) = -dP/P$.

\textsuperscript{10}Since we have parameterized the model in terms of means, variances and covariances, an increase in uncertainty or randomness is taken to mean a rise in variance with unchanged covariances. Keeping the coefficients of correlation fixed, would modify the comparative statics somewhat.
A positive $\sigma_{MK}$ reduces both the mean and the variance of inflation and lowers the real bond rate of interest, whereas $R$ and $P$ are unaffected by correlation between money and output. Correspondingly, a negative $\sigma_{MK}$ is bad for price stability, since it both adds to the average rate of inflation and renders inflation less predictable. The model would thus appear to lend support to views, such as the bills-only doctrine, which advocate the independence of central banking from government finances.11

What is the message for discretionary conduct of monetary policy? First, for any target $\pi$, the mean money expansion rate $\mu$ should be set lower the greater is the output volatility (eq. 19). Second, randomness of the money supply should be minimized if one seeks to maximize the predictability of inflation (eq. 18). Third, a high $\sigma_{M}^{2}$ may also prevent the authorities from attaining their target $\pi$ by forcing them to maintain a too high $\mu$ (eq. 23). Fourth, to the extent that $\sigma_{MK}$ is not a purely institutional parameter but partly reflects the use of operational policy instruments, efforts to use it for countercyclical purposes will only increase $\pi$ and $\sigma_{P}^{2}$ and raise the real bond rate of interest.

11A full account of the effects of fiscal policy and government financing would require an explicit specification of the government sector. On the basis of analogous deterministic models it is here assumed that the impact of monetary policy is not materially affected by the inclusion of government consumption, taxes and borrowing.
5 TWO-COUNTRY MODEL

In this section we shall use the closed economy model of the previous sections to study the general equilibrium in a world economy consisting of two countries with their own national monies. The focus of the analysis is on showing that the superneutrality of money is not affected by the possibility of international trade in nationally denominated bonds. In particular, the analysis reveals that bonds do not act as hedges against the purchasing power risks of national money supplies. This is in contrast to earlier results from partial equilibrium models, where national inflation rates have been considered as exogeneous variables and the full implications of a fiat monetary system for portfolio choice have been neglected.

With these aims in mind, the framework of the analysis is constructed so as to minimize complexity. At the end of the section we shall briefly consider the generality of the main results. We assume homogeneous world consumer-investors with logarithmic preferences and a single world commodity, which is produced by an identical linear technology and is subject to the same stochastic process in both countries. National money supplies are stochastic and follow processes similar to that specified in section 2. Total real national per capita wealth in the two countries respectively equals

\[
W = m + b + f + K + V
\]

\[
W^* = m^* + b^* + f^* + K^* + V^*
\]

(35)

where \(m\) and \(m^*\) stand for real balances of the two national monies, \(b\) and \(b^*\) for real bond holdings denominated in the first country's currency, \(f\) and \(f^*\) for real bond holdings denominated in the second country's currency, \(K\) and \(K^*\) for holdings of physical capital and \(V\) and \(V^*\) for the present values of government transfers. While all assets are tradeable in principle, it is assumed that holdings of each national money produce utility for the respective nationals only. Since the residents of both countries have the option of holding
interest-bearing foreign currency bonds, there is no demand for foreign money. This equilibrium property of the model has been anticipated in the definitions (35), where holdings of foreign money have been left out.

It is also assumed that each government repays the seignorage earned by money creation to its own nationals. This stochastic transfer payment is a component of the total change in m (m*), as defined in eq. (6a) in section 2. V and V* stand for the expected present values of these transfer flows.

The world (net) supply of each bond is zero. To simplify the algebra, we assume that the size of the population in the two countries is the same. It then holds in equilibrium that

\[ b + b^* = 0 \]
\[ f + f^* = 0 \]  

(36)

Let w and w*, respectively, stand for non-monetary wealth, i.e. for the sum of capital and bonds. By (36) we may write

\[ w = K + b + f \]
\[ w^* = K^* - b - f \]  

(37)

Let us then assume, again as a trial hypothesis, that in equilibrium

\[ C = \rho w \]
\[ C^* = \rho w^* \]  

(38)

To prove (38), we shall once more resort to the Breeden consumption-beta model: Portfolio equilibrium in the first country implies that \( r_b - r \), the expected excess rate of return on b over K, equals the covariance of this excess return with \( dC/C \). But portfolio equilibrium
in the second country implies that \( r_{b-r} \) equals the covariance of this same excess return with \( dC*/C* \). It follows that in world equilibrium these two covariances are equal. Similarly, the covariances of the excess rate of return on \( f \) over \( K \) with \( dC/C \) and \( dC*/C* \) must be equal. From this it is rather obvious that \( dC/C \) equals \( dC*/C* \), which in turn implies that equilibrium bond holdings are zero.\(^{12}\) To show this formally, let us first note that the stochastic components of the excess rate of return on \( b \), the excess rate of return on \( f \), \( dC/C \) and \( dC*/C* \) are, in this order,

\[
-\sigma_p dz_p - \sigma_K dz_K
\]

\[
-\sigma_p^* dz_p^* - \sigma_K dz_K
\]

\[
\left( \frac{K}{w} \right) \sigma_K dz_K - \left( \frac{b}{w} \right) \sigma_p dz_p - \left( \frac{f}{w} \right) \sigma_p^* dz_p^*
\]

\[
\left( \frac{K^*}{w^*} \right) \sigma_K dz_K - \left( \frac{b^*}{w^*} \right) \sigma_p dz_p - \left( \frac{f^*}{w^*} \right) \sigma_p^* dz_p^*
\]

The conditions that the consumption betas are equal across the countries can be written in vector notation

\[
x'u = y'u \tag{40}
\]

\[
x'v = y'v
\]

where

\[
x' = \left( \frac{K}{w}, \frac{b}{w}, \frac{f}{w} \right)
\]

\[
y' = \left( \frac{K^*}{w^*}, \frac{b^*}{w^*}, \frac{f^*}{w^*} \right)
\]

\(^{12}\)A more direct method of justifying this would be to refer to the results in GROSSMAN and SHILLER (1982) and STULZ (1981) concerning the validity of the Breeden consumption-beta model in the case of heterogenous consumers.
Using (36) and (37), eqs. (40) can be reduced to a homogeneous linear system in $b$ and $f$:

\[
\begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix}
\begin{bmatrix}
b \\
f
\end{bmatrix}
=
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\]  

(41)

where

\[a_{11} = \sigma_k^2 + \sigma_p^2 + 2\sigma_{kp}\]
\[a_{12} = a_{21} = \sigma_k^2 + \sigma_{kp} + \sigma_{kp*} + \sigma_{pp*}\]
\[a_{22} = \sigma_k^2 + \sigma_{pp*} + 2\sigma_{kp*}\]

Assuming that the determinant of (41) does not vanish, the unique solution is $b = f = 0$.\(^{13}\) It follows that

\[C = \rho K\]
\[C* = \rho K*\]

(42)

This means that the closed economy model is, as such, relevant for both countries separately and that the earlier proof for eq. (11) in section 2 is equally valid for the trial hypothesis (38).

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\(^{13}\) The condition for a zero determinant is $\sigma_k^2 \sigma_{pp*} = \sigma_{M*}^2 \sigma_{MM*}$, i.e. that the money supplies are perfectly co-linear. This makes the bonds perfect substitutes (as in the deterministic perfect foresight case) so that the bond portfolio is indeterminate.
The implications of the model for the domestic economies of the two countries are the same as those of the closed economy model. Let us therefore consider the international implications. A consequence of zero bond holdings is that domestic saving is fully absorbed by domestic investment and therefore the current account is in balance. The capital stocks in both countries grow at the common rate \( r_p \). As to the exchange rate, let

\[ e = \frac{p}{p^*} \]  

(43)

stand for the domestic currency price of foreign currency as viewed by the agents in the first country. Since \( P = M/m \) and \( P^* = M^*/m^* \), the equilibrium \( e \) is determined by

\[ e = \frac{\left[ \mu + \rho - \frac{2}{M} \right] \left[ \frac{M}{K} \right]}{\left( \mu^* + \rho - \frac{2}{M^*} \right) \left[ \frac{M^*}{K^*} \right]} \]  

(44)

From (44) we obtain by Itô's lemma

\[ \frac{de}{e} = (\mu - \mu^* + \frac{2}{M^* - \sigma_{MM^*}})dt + \sigma_M dZ_M - \sigma_{M^*} dZ_{M^*} \]  

(45)

Thus the expected rate of depreciation, say \( \epsilon \), depends on monetary uncertainty but not on output volatility. Random changes in output affect money demand in both countries so that there is no net effect on the exchange rate. Comparing (45) with (19) reveals that, if only \( \mu \) and \( \mu^* \) can be used as policy instruments, it is not in general possible to achieve \( \pi = \pi^* = \epsilon = 0 \). For such stability of the world monetary system one would need some control of the variances and covariances of

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14In this model the current account would be in balance even if we let \( \rho \neq \rho^* \). Under diminishing returns on capital, however, common technology and zero bond holdings would require direct investments (and net holdings of foreign equities) to maintain an equal capital intensity across the countries, unless the rate of time preference is the same in the two countries.
the money supply processes. From (45) we also see that a positive correlation of the money supply rates increases exchange rate predictability by reducing the variance of de/e.

A notable result of the above analysis is that the purchasing power risk of each national money supply will be borne by the residents of the respective country instead of being spread among the residents of both countries. The opposite view, viz. that the international market for nationally denominated bonds serves as a pool in which the purchasing power risks of money supplies become redistributed and equalized among world investors, appears to have a strong standing in the international finance literature. Such a role for the international bond market emerges in Kouri (1977), is of central importance in the analysis of Fama and Farber (1979) and is still present in the recent study of Branson and Henderson (1985).

The reason why in the present model the bond market does not appear as an international equalizer of money supply risks is as follows: The nominal money supply risk \( \sigma_M \), which by (17) is a component of the inflation risk \( \sigma_p \), cancels out when equilibrium holdings of real balances are added to the present value \( V \) of the monetary transfer. Therefore monetary wealth \( m + V \) only bears the physical capital risk \( \sigma_K \). This other component of the purchasing power risk is the same in both countries and is non-diversifiable. Thus there is no motive for the residents of the two countries to pool national purchasing power risks by issuing domestic currency debt against foreign currency assets. The implication is that, although national monies and bonds of each country are equivalent in terms of risk, agents act as world investors vis-à-vis bonds rather than vis-à-vis the composite national currency assets. This result is due to the general equilibrium approach applied in this paper, which makes it possible to explore the full consequences of a fiat monetary system. In a partial equilibrium analysis, where national inflation processes are regarded as exogeneous, it is not easy to account for the interrelationships within the total monetary wealth and to avoid a bias in the analysis.

We may gain some further insight into this issue by considering briefly the case where \( M \) and \( M^* \) are assumed to be exogeneous
constants. Then $V$ and $V^*$ are zero and $W$ and $W^*$ as defined in (35) consist of asset wealth $A$ and $A^*$ only. In order to see how a partial equilibrium bias, or rather an inconsistency, may arise in this framework, assume first a "barter" economy where $M = M^* = 0$. Assuming that the nominal bond interest rates $R$ and $R^*$ are constants and that output, $P$ and $P^*$ follow exogenous processes of the type (1) and (5) with constant parameters, the desired portfolio balance in both countries will satisfy

$$
\begin{bmatrix}
  a_{11} & a_{12} \\
  a_{21} & a_{22}
\end{bmatrix}
\begin{bmatrix}
  \phi \\
  \psi
\end{bmatrix}
= \begin{bmatrix}
  r_b - r + \alpha_k^2 + \alpha_{kp} \\
  r_f - r + \alpha_k^2 + \alpha_{kp*}
\end{bmatrix}
$$

(46)

where the elements of the matrix $[a_{ij}]$ are those given in (41), and $\phi$ and $\psi$ stand for the desired portfolio shares of the two bonds. World supplies of both bonds are zero. Therefore the solution for equilibrium $r_b$ and $r_f$ is obtained by setting the RHS of (46) equal to zero. The corresponding nominal rates $R$ and $R^*$ are given by (7) and its counterpart for the second country.

Now add monies to the utility functions and assume that $M$ and $M^*$ are positive constants. Then condition (46) remains valid when $\phi$ and $\psi$ are re-defined to denote the portfolio shares of the composite assets $b+m$ and $f+m^*$, respectively. But it does not follow that the market equilibrium would continue to be characterized by agents in both countries holding the world portfolio

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15KOURI (1977) and BRANSON and HENDERSON (1985) consider this case. In FAMA and FARBER (1979), newly issued money is used to finance government consumption, so that in principle their model deals with "expenditure money" rather than with a pure fiat money system. The general equilibrium consequences of the real resources absorbed by money creation are not studied, however.

16Eq. (46) is the standard mean-variance portfolio rule. For an extensive treatment in the context of intertemporal optimization, see MERTON (1969), (1970) and (1973). Naturally, the solution agrees with eq. (14), which was derived using the consumption-beta model.
\[ \phi = \frac{m}{m + m^* + K + K^*} \]  

(47)

\[ \psi = \frac{m^*}{m + m^* + K + K^*} \]

The reason is that, in addition to (46), money markets must satisfy the equilibrium conditions

\[ m = \left[ \frac{(1-\alpha)\rho}{R} \right] A \]  

(48)

\[ m^* = \left[ \frac{(1-\alpha)\rho}{R^*} \right] A^* \]

This means that we can no longer adhere to the assumption that \( P \) and \( P^* \) are exogeneous. Otherwise there are four independent equations to determine the two endogeneous bond interest rates and the model is in general inconsistent.\(^\text{17}\)

When prices are endogenized, the nature and implications of conditions (46) undergo a fundamental change. Assuming, as before, constant \( R \) and \( R^* \), the equilibrium price paths must by (48) satisfy

\[ \frac{dm}{m} = \frac{dA}{A} \]

(49)

\[ \frac{dm^*}{m^*} = \frac{dA^*}{A^*} \]

Equating the stochastic components yields

\[ -\sigma_p dZ_p = -\phi\sigma_p dZ_p - \psi\sigma_{p^*} dZ_{p^*} + (1-\phi-\psi)\sigma_K dZ_K \]

\[ -\sigma_{p^*} dZ_{p^*} = -\phi^*\sigma_{p^*} dZ_p^* - \psi^*\sigma_{p^*} dZ_{p^*} + (1-\phi^*-\psi^*)\sigma_K dZ_K \]

\[ (50) \]

\(^\text{17}\)Note also that with exogeneous and stochastic \( P \) and \( P^* \), (48) implies stochastic \( R \) and \( R^* \), which contradicts the assumptions needed for (46). Modifying the underlying HJB-equation so as to allow for stochastic, state dependent \( R \) and \( R^* \) would not remove the overdeterminacy, however.
where \( \phi \) and \( \psi \) are the portfolio shares of the two composite assets in the first country and \( \phi^* \) and \( \psi^* \) the corresponding shares in the second country. Independently of the portfolio shares, eqs. (50) solve for

\[
\sigma_p dZ_p = \sigma_{p*} dZ_{p*} = -\sigma_K dZ_K
\]

(51)

Therefore \( \alpha_p^2 = \alpha_{p*}^2 = \alpha_{K*}^2 \) and \( \alpha_{KP^*} = -\alpha_K^2 \). This implies that \([a_{ij}]\) in (46) is a zero-matrix and that the vector on the RHS of (46) reduces to \((r_b - r, r_f - r)'\). The solution of (46) is thus \( r_b = r_f = r \).

Equating the means of (49) and employing the definitions (7) of \( r_b \) and \( r_f \) we then get \( \pi = \pi^* = \rho - r + \alpha_K^2 \) and \( R = R^* = \rho \).

The solution agrees with the more general model of this section, only in this case the portfolio shares remain fully arbitrary, since all three assets are perfect substitutes (cf. footnote 11). In the general case, with stochastic and less than perfectly correlated money supplies, the equilibrium portfolio is fully determined and characterized by zero holdings of bonds. It is now easier to understand why this must be so. The "barter" model was characterized both by identical portfolios across the countries and by zero bond holdings. When money is introduced, one of these two properties must be given up. When a fiat money system is modelled in a way that preserves monetary neutrality, it will not affect the equilibrium of the real variables. Therefore the portfolio shares of non-monetary wealth have to remain the same as in the barter model. On the other hand, with national monies in the national utility functions and in the presence of national non-traded transfer flows, there is no longer full homogeneity of the consumption-investment sets. This explains why the (composite) asset portfolios in the two countries no longer need be identical.

The significance of the zero bond position is that it implies a complete lack of monetary interdependence. Somewhat paradoxically, to modify this property of the model one has to introduce more national independence in the non-monetary relationships of the model. From this point of view the assumption of a single world commodity is not restrictive as such. As long as the consumption-investment set is
fully integrated, we may well have a world portfolio of capital goods
to replace the single K of the model and a common basket of
consumption goods instead of the single world consumption good.\textsuperscript{18} If,
however, there are national differences in consumer tastes, then in a
multi-goods framework there will also be a national segmentation of
real investment opportunities. This is because differences in national
consumption patterns imply differences in the composition of the
relevant domestic consumer price indices and therefore in the
probability distributions of appropriately deflated real asset
returns, although markets are perfectly integrated in terms of nominal
rates of return. In general, such a segmentation of the investment set
also implies different, non-zero bond portfolios in the two countries.

The device of nationally separated consumer price indices has been
used in the literature to provide theoretical justification for
empirically observed "habitat country motives" in portfolio
selection.\textsuperscript{19} But if it is true that home country consumption habits
produce an extra bonus in terms of a smaller perceived risk for assets
denominated in home country currency, the same argument should also
increase the attractiveness of home currency liabilities and thus
reduce the appetite for foreign currency debt. Since such tendencies
cannot materialize for the two countries in aggregate, it is not clear
in which direction the effect on bond portfolios should operate.\textsuperscript{20}

\textsuperscript{18}COX, INGERSOLL and ROSS (1985) study asset pricing in a non-monetary
model where a single commodity can be produced by several
stochastically linear technologies.

\textsuperscript{19}BRANSON and HENDERSON (1985) investigate (in the context of a
non-monetary small country model with exogeneous prices and fully
elastic asset supplies) the conditions under which there is a positive
association between the desired portfolio share of domestic currency
assets and the expenditure share of domestic goods. STULZ (1981)
employs the Breeden consumption-beta model to study international asset
pricing in a non-monetary model where (real) investment sets become
nationally segmented because of differences in national consumption
opportunity sets that are caused by non-traded goods rather than by
heterogeneous consumer tastes. LUCAS (1982) suggests that
international as well as regional habitat preferences might be
explained by locally limited information sets.

\textsuperscript{20}This raises the potentially interesting question as to what extent
the existence of multinational financial intermediaries could be
explained by a niche for institutions that are capable of pooling
conflicting habitat country preferences.
If the world asset market equilibrium is characterized by non-zero national bond holdings, the equilibrium values of the real variables, among them the bond holdings, become dependent on monetary policy in both countries. The non-neutrality effects stemming from segmented asset markets are in the nature of distribution effects and are likely to be both model-specific and hard to systematically associate with the structural characteristics of the two economies.

There is more scope for monetary interdependence and possibly also for systematic non-neutrality effects, if it is assumed that government taxes are not fully capitalized and, consequently, that government debt at least partly appears as private sector wealth. In such a framework the relevant world market net supplies of bonds are positive rather than zero. A third source of non-neutrality and interdependence effects appears if money is modelled in a way that implies non-neutrality even in the underlying closed economy model. The options range from combining non-separable utility with diminishing returns in production to cash-in-advance and overlapping generations models.\textsuperscript{21}

Obviously, the rather austere model developed in this section cannot compete against more sophisticated rivals in analytical richness or empirical accuracy. Our exercise was designed to stress one important point: When modelling monetary equilibria, money should not enter merely as an extra asset, commodity, factor or constraint. A fiat monetary system creates a divergence between private and social cost. It is the full implications of this for wealth, prices and risk that any coherent treatment of monetary economies has to account for.

\textsuperscript{21}For monetary general equilibrium models of the exchange rate that use the cash-in-advance approach, see STOCKMAN (1980) and LUCAS (1982).
APPENDIX A

The purpose of this appendix is to show that the non-neutrality of money in the Gertler-Grinols model is not due to uncertainty at all, but rather to a bias in the consumer choice problem that stems from the way they specify the transfer income from the government. Because the issue is much more transparent in the deterministic framework, we shall use the perfect foresight version of the model to demonstrate the existence and effect of this bias.

In order to get rid of the nuisance caused by a second state variable, Gertler and Grinols assume that the agents perceive the real transfer income, say dv, to remain proportional to asset wealth:

\[ \text{dv} = gA \, \text{dt} + A_\alpha \text{vdZ}_v \]  

Assumption (A1) turns the consumer choice problem into a standard exercise, with asset wealth as the single relevant state variable. This simplification may appear harmless, since it is consistent with the final market equilibrium solution of the model. The point is, however, that (A1) makes \( \dot{e} \) appear as if it were part of the return on asset wealth and in so doing seriously biases the intertemporal allocation of consumption.22

With (A1), our eq. (9) would read

---

22In their footnote (2), Gertler and Grinols justify assumption (A1) as follows: "We formulate the model so that the transfer rule does not inappropriately bias structural relationships. We assume (...) that the household's utility is logarithmic, which implies that the consumption/saving decision is independent of the return on wealth." This is not quite correct. The consumption/saving decision is never independent of the return on wealth since any change in income must affect either consumption or saving or both. Even under logarithmic utility the return on wealth affects consumption through the discounting of non-asset income flows.
\[ dA = [(r - r_b)K + (r_b + \sigma)A - C - Rm]dt \]

\[ + K(\sigma_d^2 dZ_k + \sigma_p dZ_p) + A(\sigma_d dZ_v - \sigma_p dZ_p) \] \hspace{1cm} (A2)

Maximizing expected utility (10) subject to (A2) yields consumption and asset demands as functions of \( A, r, r_b, \sigma, R \) and the variances and covariances of the Ito-processes of \( K, P \) and \( v \). Imposing market clearing under rational expectations, Gertler and Grinols then derive the final solution of their model, which is of the form

\[ C = \alpha A \]

\[ m = \left[ \frac{(1 - \alpha)\rho}{R} \right] A \]

\[ A = K + m \] \hspace{1cm} (A3)

\[ R = R(\mu, \sigma_M^2), R > 0, R \sigma_M^2 < 0 \]

The solution for \( R \) does not reduce to anything as simple as our eq. (21), but is nevertheless qualitatively similar. A rise in \( \mu \) increases \( \pi \) and \( R \), which in turn decreases real balances and thus asset wealth. A fall in asset wealth diminishes consumption and hence, given output, raises investment and speeds up growth. The effect of a rise in \( \sigma_M^2 \) proceeds through the same channel but with reversed signs, so that the result is a fall in capital formation and growth. Effects of \( \sigma_{MK}^2 \) do not appear in the Gertler-Grinols model, since money supply and output are assumed independent. Compared to the model of the main text, the key difference is that here consumption is proportional to asset wealth instead of total wealth. Therefore changes in \( m \) and \( A \) that are induced by shifts in \( \mu \) or \( \sigma_M^2 \) cannot net out against opposite changes in \( V \).

To clarify the matter further, consider now the analogous model in the perfect foresight setting. We first solve the model keeping the transfer fully exogeneous to the optimizing agent. The solution
technique is straightforward and requires no guesswork. We then repeat the procedure by imposing the non-stochastic equivalence of (A1), which enables us to follow step by step how a bias develops in the model.

Under perfect foresight, bonds and capital are perfect substitutes and the real bond rate of interest equals the real yield on capital. The equation of movement for real asset wealth reads

\[ \dot{A} = rA + \mu\tilde{m} - C - Rm \]  

where \( R = r + \pi \) and \( \mu\tilde{m} \) stands for the transfer income from the government. Define

\[ V = \int_{t}^{\infty} e^{-r(s-t)}\mu\tilde{m} \, ds \]  

and assume that \( V \) is finite and greater than \((-A)\). Then

\[ \dot{V} = rV - \mu\tilde{m} \]  

Add (A4) and (A6) together to get

\[ \dot{W} = rW - C - Rm \]  

where \( W = A + V \). Let the agent be constrained by the requirement\(^{23}\)

\[ e^{-r}\int_{0}^{\infty} W(s) \, ds = 0 \]  

The constraints (A7) and (A8) imply

\[ \int_{t}^{\infty} e^{-r(s-t)}(C+Rm) \, ds = W(t) \]  

Paths that are extremals under the utility functional (10) satisfy the Euler conditions

\(^{23}\)Without the borrowing limit (A8) the consumer choice is unconstrained and the optimum is \( C,m = \infty \), all \( s > t \).
\[ m = \frac{(1-\alpha)C}{\alpha R} \quad (A10) \]

\[ \frac{\dot{C}}{C} = r - \rho \quad (A11) \]

The solution for (A11) is

\[ C(s) = C(t)e^{(r-\rho)(s-t)} \quad (A12) \]

Substitute (A12) and (A10) into the wealth constraint (A9) to get the demand functions

\[ \dot{C} = \alpha \rho W \quad (A13) \]

\[ m = \frac{1-(1-\alpha)\rho}{R} \cdot W \quad (A14) \]

Substituting these into (A7) gives

\[ \frac{\dot{W}}{W} = r - \rho \quad (A15) \]

Let \( \phi \equiv (1-\alpha)\rho/R \) and differentiate (A14):

\[ \frac{\ddot{m}}{m} = \frac{\phi}{\phi} + \frac{\dot{W}}{W} \quad (A16) \]

On the supply side we have

\[ \frac{\ddot{m}}{m} = \mu - \pi \quad (A17) \]

Market clearing implies \( m(s) = \bar{m}(s) \) for all \( s > t \). Hence, from (A15)-(A17) and noting that \( r + \pi = R \equiv (1-\alpha)\rho/\phi \):

\[ \phi = (\mu + \rho)\phi - (1-\alpha)\rho \quad (A18) \]

A state dependent equilibrium path exists for \( \mu + \rho > 0 \), and it is determined by the unstable steady state value \( \bar{\phi} \) as
\[ \phi(s>t) = \frac{(1-\alpha)p}{\mu+p} \tag{A19} \]

Along the state dependent path \( R \) and \( \pi \) are constants:

\[ R = \mu + \rho \tag{A20} \]
\[ \pi = \mu - (r-\rho) \tag{A21} \]

Hence

\[ \bar{m}(s) = m(s) = m(t)e^{(r-\rho)(s-t)} \tag{A22} \]
\[ V = \frac{\mu}{p}m \tag{A23} \]
\[ m + V = \frac{R}{\rho}m = (1-\alpha)W \tag{A24} \]

and, since \( b = 0 \),

\[ W = \frac{K}{a} \tag{A25} \]

The state dependent perfect foresight solution for the real variables is thus

\[ C = \rho K \]
\[ m = \frac{1-\alpha}{\alpha} \left( \frac{p}{\mu+p} \right) K \tag{A26} \]
\[ \dot{K} = (r-\rho)K \]

Consider now what happens if, instead of (A4), we start from

\[ \dot{A} = (r+\theta)A - C - Rm \tag{A27} \]

where \( \theta A \) replaces \( \mu \bar{m} \) as the variable denoting transfer income from the government. A consumer faced with (A27) is led to believe that the amount of transfer income he will receive in the future depends on his future asset wealth and thus on his present saving. In other words, saving is remunerated at the rate of return \( r+\theta \) instead of \( r \) alone.
Thus \( r + \theta \) is the relevant discount rate for future consumption. Multiplying both sides of (A27) by \( e^{-(r+\theta)(s-t)} \), integrating by parts and imposing \( e^{-(r+\theta)\infty}A(\infty) = 0 \), yields

\[
\int_{t}^{\infty} e^{-(r+\theta)(s-t)}(C+Rm) \, ds = A(t)
\]

which is the counterpart of (A9) above. The condition (A11) now reads

\[
\frac{\dot{C}}{C} = (r+\theta) - \rho
\]

Instead of (A13) and (A14) we get the demand functions

\[
C = \alpha\rho A
\]

\[
m = \left(\frac{1-\alpha}{R}\right) A
\]

and (A15) is replaced by the saving function

\[
\frac{\dot{A}}{A} = (r+\theta) - \rho
\]

If \( \theta \) is positive, then for any given \( A \) the consumer spends less and saves more than in the previous model.

To derive the state dependent perfect foresight path, replace \( \theta A \) by its true equilibrium value \( \mu m \). Thus, by (A31)

\[
\theta = \mu \left( \frac{m}{A} \right) = \frac{(1-\alpha)\rho u}{R}
\]

Along the state dependent path \( R \) is again constant, as has already been implicitly assumed by regarding \( \theta \) as a constant. Hence, from (A17) and (A31)-(A33), money market equilibrium implies

\[
\mu - \pi = r - \rho + \frac{(1-\alpha)\rho u}{R}
\]

Recalling that \( r + \pi = R \), (A34) determines \( R \) as
where the other root can be ruled out by feasibility considerations.  

Eq. (A35), as well as the solution for $\pi$ implied by it, are identical to the Gertler-Grinols solutions when $\sigma_k^2$ and $\sigma_m^2$ are set equal to zero. It should be evident from this that the non-neutrality of money in the Gertler-Grinols model does not arise from uncertainty but from specifying the transfer income in a way which distorts the market price of waiting.  

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24 Gertler and Grinols, appendix.  
25 Gertler and Grinols suggest that the role of $\mu$ in their model is similar to that in Tobin (1965). As has been pointed out by Levhari and Patinkin (1968), the non-neutrality of money in the Tobin model is due to a concept of disposable income that neglects the liquidity services produced by money. In optimizing models like the present one such a source of non-neutrality cannot appear.
APPENDIX B

In order to determine the range of such extensions of the stochastic model that are likely to preserve the superneutrality of money, we shall consider a more general version of the non-stochastic optimization problem of Appendix A. For that purpose, replace logarithmic utility by \( u(C, m) \) and let \( r \) and \( \mu \) be exogenous functions of time rather than constants.

Since in market equilibrium \( \mu(s)\tilde{m}(s) \) equals \( \tilde{m}(s) + \pi(s)m(s) \), we may write for the perfect foresight present value of the monetary transfer

\[
V(t) = \int_t^s u(s)\tilde{m}(s)ds
\]

\[
= \int_t^s \tilde{m}(s)ds + \int_t^s \pi(s)m(s)
\]

\[
= \int_t^s m(s)ds + \int_t^s (r(s)+\pi(s))m(s)ds
\]

\[
= -m(t) + \int_t^s R(s)m(s)ds
\]  

(B1)

---

26The analysis in this appendix draws mainly on COHEN (1985) but attempts to be more general. An extensive treatment of the separable utility case is in MUSSA (1976), Appendix D.

27Note again that since we are studying competitive equilibrium, the time path of \( r \) has to be kept exogeneous for the optimizing agent, even if the market equilibrium \( r \) may well be endogeneous and depend on \( K \).
where it is assumed that \( \lim_{t \to \infty} m(s) = 0 \). If \( m(s) \) tends to grow at a faster rate than \( r(s) \), the optimization problem is not well-defined. Substituting (B1) into (A9) yields

\[
\int_t^s e^{-r(\tau) d\tau} \int_t^\tau C(s) ds = K(t)
\]

Therefore the wealth constraint (A9) dichotomizes under perfect foresight. Eq. (B2) reflects the basic assumption that fiat money is produced at zero social cost. A similar "wealth dichotomy" arises in models which are characterized by the independence of monetary policy and government expenditure on goods, full capitalization of the future tax burden (the Ricardo-Barro neutrality of government debt) and absence of distribution effects.

Given (B2), the superneutrality of money requires that the money supply process does not distort the intertemporal allocation in the consumption of goods and thereby the time path of \( K \) and of the equilibrium \( r \). For this to happen, the Euler conditions

\[
\dot{u}_C + (r(s) - \rho)u_C = 0 \quad (B3)
\]

\[
Ru_C - u_m = 0 \quad (B4)
\]

must "dichotomize" in \( \dot{C} \) and \( \dot{m} \). If \( u_{Cm} = 0 \), (B3) becomes

\[
\dot{C} = \frac{u_C(C)}{u_{CC}(C)} (\rho - r(s)) \quad (B4)
\]

so that the time path of \( C \) is independent of \( u(s) \). When utility does not separate in \( C \) and \( m \), additional constraints on \( u(C,m) \), \( r(s) \), \( u(s) \) and the nature of the equilibrium price path are needed for superneutrality. In the case of the isoelastic utility function
\[ u = \frac{(C^\alpha m^{1-\alpha})^{1-\gamma} - 1}{1-\gamma}, \quad 0 < \alpha < 1, \gamma > 0 \]  
(B5)

(B3) and (B4) imply

\[ \dot{C} = \frac{r(s) - \omega}{\gamma} - \frac{(1-\gamma)(1-\alpha)}{\gamma} \frac{\dot{R}}{R} \]  
(B6)

Thus for the non-separable cases \((\gamma \neq 1)\) of (B5) we have superneutrality only if \(\mu(s)\) is adjusted to \(r(s)\) in such a way as to peg a constant \(R\). A special case of this arises if both \(r\) and \(\mu\) are constants and markets move along the state-dependent path (see Obstfeld and Stockman (1983)). In the logarithmic case \((\gamma \neq 1)\) and for other separable utility functions superneutrality holds (subject to feasibility and the existence of equilibrium) for all \(r(s)\) and \(\mu(s)\) and is not limited to the state dependent path.

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28 FISHER (1979) found that under (B5) and with \(\mu\) kept constant over time, expansionary monetary policy (a high \(\mu\)) in the SIDRAUSKI (1967) growth model speeds up capital stock adjustment towards the steady state both if \(\gamma > 1\) and if \(\gamma < 1\), while money is superneutral along the adjustment path for \(\gamma = 1\). An explanation for this unintuitive result in terms of income and substitution effects was given by COHEN (1985). BROCK (1974) pointed out that in a model with endogenous labour supply, money is non-neutral even across steady states, unless utility separates in \(m\).
APPENDIX C

In section 2 it was shown that the suggested market equilibrium path verifies the necessary conditions for optimum. To complete our discussion on this topic, we shall in this appendix present a proof of optimality which has been suggested to the author by Seppo Salo. This method of proof allows a more direct approach than in Brock and Magill (1979), but requires a technical assumption concerning the controls $C$, $m$ and $K$ (C1).

Assume that the economy follows the market equilibrium path indicated in the main text. Then the choice $(C(s), m(s), K(s))$ of an agent is constrained by

\[
dA = [(r - \sigma_{MK})A + \sigma_{MK}K + (\mu - \sigma_M^2 + \sigma_{MK})\bar{m} - C - Rm]ds + A(\sigma_K dZ_K - \sigma_M dZ_M) + \sigma_M dZ_M + \bar{m}\sigma_M dZ_M
\]

\[
d\bar{m} = (r - \rho)\bar{m} ds + \bar{m}\sigma_K dZ_K
\]

\[
W = A + V = A + \left(\frac{\mu - \sigma_M^2}{\rho}\right)\bar{m} > 0
\]

\[
W(t) = W_0
\]

where $\bar{m}$ is exogeneous and $R = \mu + \rho - \frac{\sigma_M^2}{\rho}$.

Let $u^*$ be the path of utility received from choosing the market path $(C^*, m^*, K^*)$ and $u$ a comparison path of utility derived from following another path of $(C, m, K)$ consistent with (C1) - (C4). We want to prove that

\[
\lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} e^{-\rho(s-t)}(u^*-u)ds > 0
\]

It is helpful to formulate the problem in terms of $W^*$ and its dual process. The path of $V$ obeys
Adding (C1) and (C6) gives

\[ dW = [rW - \alpha_W(A - \bar{m}) - C - Rm]ds + \omega_KdZ_K - (A - \bar{m})\omega_MdZ_M \]  

(C7)

Let \( \xi \) be the dual process of \( W^* \):

\[ \xi = J_W = u^*_c = \frac{1}{R^*}u^*_m = \frac{\alpha}{C^*} \]  

(C8)

Then \( \xi \) is a solution of

\[ d\xi = (\rho - r + \alpha_k^2)\xi \, ds - \xi \omega_KdZ_K \]  

(C9)

Define

\[ D(s) = e^{-\rho(s-t)}\xi(s)(W^*(s) - W(s)) \]  

(C10)

In order to be sure that Ito's lemma holds for \( W^* - W \) (see Fleming and Rishel (1975), pp. 118-120), we next assume that there exists \( \eta > 0 \) such that

\[ 0 < C, m, K < \eta \]  

(C11)

Then, from (C10), (C7) and (C9)

\[ dD = e^{-\rho(s-t)}\{ -\rho \xi(W^*-W)ds + d\xi(W^*-W) + \xi d(W^*-W) + d\xi d(W^*-W) \} \]

\[ = -e^{-\rho(s-t)}\{ [\xi(C*-C) + R\xi(m*-m)]ds + [(A*-A) - (K*-K)]\xi dZ_M \} \]  

(C12)

By (C8) and the concavity of the utility function it holds that

\[ u^* - u > \xi(C^* - C) + R\xi(m^* - m) \]  

(C13)

Therefore, by (C12)

\[ e^{-\rho(s-t)}(u^* - u)ds > -dD - e^{-\rho(s-t)}[(A*-A) - (K*-K)]\xi dZ_M \]  

(C14)

and further
\begin{equation}
\begin{align*}
-E_t \int_t^T e^{-\rho(s-t)}(u^*-u)ds & > -e^{-\rho(T-t)}\xi(T)(W^*(T)-W(T)) \\
\text{Since } \xi(T)W^*(T) = \left(\frac{\alpha}{\alpha W^*}\right)W^* = 1, \xi(T) > 0 \text{ and } W(T) > 0;
\end{align*}
\end{equation}

\begin{equation}
\begin{align*}
\lim_{T \to \infty} [-e^{-\rho(T-t)}\xi(T)(W^*(T)-W(T))] &= \lim_{T \to \infty} e^{-\rho(T-t)}\xi(T)W(T) > 0
\end{align*}
\end{equation}
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