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Robust expectations and uncertain models – A robust control approach with application to the New Keynesian economy

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Abstract

This paper extends Svensson and Woodford's (2003) partial information framework by allowing the private agents to achieve robustness against incomplete information about the structure of the economy by distorting their expectations in a particular direction. It shows how a linear rational expectations equilibrium under concern for robustness can be solved by exploiting the recursive structure of the problem and appropriately modifying the Bellman equations in their framework. The standard Kalman filter is then used for information updating under imperfect measurement of the state variables. The standard New Keynesian model is used for illustrating how concern for modelling errors interacts with imperfect information. Agents achieve robustness by simultaneously over-estimating the persistence of exogenous shocks, but under-estimating the policy response to the output gap. This under-estimation, combined with imperfect measurement, leads to larger and more persistent responses of private consumption to government expenditure shocks under robust expectations.

Key words: expectations, robust control, model uncertainty, monetary policy, imperfect information

JEL classification numbers: D81, C61, E52

Odotusten muodostus ja malliepävarmuus robustin säätöteorian valossa – sovellus uuskeynesiläiseen makromalliin

Suomen Pankin keskustelualoitteita 5/2004

Juha Kilponen
Tutkimusosasto

Tiivistelmä

Tässä tutkimuksessa laajennetaan Svenssonin ja Woodfordin (2003) kehittämää optimaalisten päätössääntöjen ratkaisumenetelmää tilanteeseen, jossa eteenpäin katsovat taloudenpitäjät ovat epävarmoja talouden dynaamisista ominaisuuksista. Ratkaisumenetelmässä oletetaan, että taloudenpitäjät varautuvat malliepävarmuuteen ratkaisemalla ns. robustin ennusteongelman. Tutkimuksessa osoitetaan, kuinka robustit odotukset ja eteenpäin katsovan mallin tasapaino voidaan ratkaista käyttämällä hyväksi mallin rekursiivisia ominaisuuksia ja Bellmanin yhtälöä. Muuttujat, joiden havaintoihin liittyy epävarmuutta, estimoidaan käyttämällä hyväksi Kalmanin suodinta.

Kehitettyä ratkaisumenetelmää sovelletaan uuskeynesiläiseen makromalliin. Käy ilmi, että ollessaan epävarmoja eksogeenisten sokkien dynaamisista ominaisuuksista sekä politiikkasäännön parametreista taloudenpitäjät aliarvioivat keskuspankin reagoinnin tuotantokuiluun. Samalla taloudenpitäjät kuitenkin yliarvioivat eksogeenisten sokkien keston. Tämä yhdistettynä epätäydelliseen informaatioon saa aikaan sen, että yksityisen sektorin kulutus tyypillisesti ylireagoi julkisen kulutuksen sokin kokoon sekä kestoon.

Avainsanat: odotusten muodostus, robusti säätöteoria, malliepävarmuus, rahapolitiikka, epätäydellinen informaatio

JEL-luokittelu: D81, C61, E52

Contents

1 Introduction

Concern for modelling errors in monetary economics dates back, at least, to Friedman's concern for unknown long and variable lags in the effects of policies, leading him to prefer policy rules that are invariant concerning detailed knowledge about the dynamics driving the economy. In this same spirit, Hansen and Sargent (2004) have been pioneering the application of robust control and filtering theory to economics and in particular to forward-looking models. The principal aim in this new approach is to ensure that resulting decisions and policies are relatively insensitive to modelling errors.

These papers mainly apply the elements of $\mathcal{H}_\infty$ -theory[1] and the risk-sensitive decision theory of Whittle (1981, 1990) to control and filtering problems. They develop techniques for numerically solving models in which the policymaker seeks an optimal policy but acknowledges that the underlying model is only an approximation of a true model. In the forward-looking context they typically solve the Stackelberg, or Ramsey problems, in which the government can commit itself to a policy at time 0 onwards. In the solution strategy, the government's policy instrument enters into the private sector's Euler equations as forcing variables, while the private sector's Euler equations become implementality constraints.[2]

What is new in the robust control approach, is the modification of the government's Lagrangian to incorporate a preference for robustness to model uncertainty. A typical result presented in these recent papers is that the robust decisions tend to be more aggressive when compared to those made without concern for robustness[3]. In particular, Kasa (2001) extends the Hansen-Sargent (1981) prediction formula for rational expectations in the frequency domain and shows that a robust forecaster, following new information, tends to revise the forecasts by more than a standard forecaster. This is due to the fact that a robust forecaster over-estimates the persistence of the shocks disturbing the economy. This suggests that forward-looking variables are likely to *over-react,* implying, for instance, that asset prices are more volatile than is justified by the fundamentals. Estimation errors will therefore have a forecastable component that would be absent in the RE solution (Giordani and Söderlind 2002; Tornell 2000).

This paper concentrates on developing and illustrating a method with which to solve the robust decision problems of the private sector when the policy itself is uncertain and some of the states are imperfectly observed. The first important difference compared to many of the current papers is that

[1] Seminal contribution of $\mathcal{H}_\infty$ -theory is that of Zames (1981). He showed that in order to achieve a minimum performance in the presence of model uncertainty, one may analyze traditional control problems by using the $\mathcal{H}_\infty$ -norm, rather than the standard sum-of-squares $\mathcal{H}^2$ -norm. This sparked a little revolution in control theory and led to the development of what is now called robust control theory.

[2] These solution methods have been developed in Miller and Salmon (1985), Currie and Levine (1987), Pearlman, Currie and Levine (1986) and Pearlman (1992) initially.

[3] See, for instance, Tetlow and von zur Muehlen (2001), Onatski and Stock (2002), Giannoni (2002), Hansen and Sargent (2004) and Kilponen (2003).

we abstract from the robust control problem of the policymakers.[4] This is motivated by the fact that most of the realistic policymaking decisions do not involve explicit optimisation, yet it has become increasingly common to use forward-looking models for policy analysis and forecasting purposes. When forecasting and policy analysis is conducted with forward-looking models, some version of Taylor-type (1993) rules is used to close the model, without explicitly optimising the parameters of the rule. Moreover, abstracting from the optimisation problem of the policymaker implies that any change in the dynamic properties of the model between the RE solution and the robust solution is due to the way in which forward-looking variables propagate into a model's distorted transition laws. This enables us to construct quite flexibly alternative simulation scenarios and to learn how different types of misspecifications influence the model's dynamics and the expected paths of different variables.

Second, we incorporate imperfect information into the analysis[5]. For instance Orphanides (2001) has stressed that monetary policy is often made on the basis of preliminary estimates that are revised substantially in subsequent months. Thus it is typical that the agents' real-time information set does not allow them to infer the values of all the variables accurately. As is well known and originally emphasised by Pearlman *et al* (1986), such partial information poses rather complex inference and signal extraction problems in the models with forward-looking expectations. In fact, many of the recent analyses abstract from partial information due to the technical difficulties of solving these models.

In order to incorporate partial information into our analysis, we closely follow the general framework of Svensson and Woodford (2003). They extended the method of Pearlman *et al* (1986) and show that even with forward-looking variables, optimal feedback coefficients of current state estimates under partial information are *identical* to those of the current states under full information, and that they are also invariant to unexpected shocks. This is the well-known separation principle, which allows us to separate the decision-making and estimation of hidden state variables. We exploit the same separation principle when solving the robust decision and estimation problem. Moreover, we use the Kalman filter for the estimation of the states under imperfect information. In our setting, the Kalman filter is, however, applied to the worst-case model. It thus acknowledges uncertainty in the transition laws of the model and how it propagates to forward-looking variables through distorted expectations.

In general, we solve the problem in two steps. In the first step, time-t distorted expectations of forward-looking variables are formed using all the information available, yet acknowledging that the underlying model might be misspecified. This is achieved by modifying the standard Bellman equation by adding a concern for robustness. In the second step, the model is recast

[4] Sims (2001) argues that min-max decisions are a more appropriate modeling device for the private sector than for the policymaker.

[5] Pearlman *et al* (1986) and Pearlman (1992) provided the first general solution to the inference problem of partial information in a linear rational expectations model. Hansen and Sargent (2002) have developed techniques to solve robust control and filtering problems with forward looking variables.

in terms of the pre-determined variables only and unobserved states are then estimated by applying the standard Kalman filter to the worst-case model.

The rest of the paper is organised as follows. Sections 2 sets up the problem and introduces the general framework. Section 3 discusses how robust decision-making and estimation can be solved within the framework and Section 4 applies the method to a New Keynesian type of monetary policy model. Section 5 concludes the discussion. The Appendix contains more detailed derivations of the method used for computing robust decisions.

2 Preliminaries

2.1 Nominal model

Throughout the paper, we operate in a linear framework in which a system of structural equations is determined by a vector of predetermined and forward-looking jump variables. The system can be thought of as representing the equilibrium conditions derived – and possibly linearized – from the optimising decisions of the private sector. Furthermore, it is assumed that the system is stationary; thus, the variables should be interpreted as deviations from some well defined long-run equilibrium. The optimisation problem of the policymaker is absent and thus the policy rule is already either explicitly or implicitly embedded into the systems dynamics. These structural equations are given by a system of the form

$$\begin{pmatrix} z_{t+1} \\ x_{t+1|t} \end{pmatrix} = A^1 \begin{pmatrix} z_t \\ x_t \end{pmatrix} + A^2 \begin{pmatrix} z_{t|t} \\ x_{t|t} \end{pmatrix} + B_1 \begin{pmatrix} v_{t+1} \\ 0 \end{pmatrix} \tag{2.1}$$

where z_t is the n_z vector of *predetermined* variables, x_t is the n_x vector of *jump* variables and v_{t+1} is the $n - n_x$ vector of normally distributed independent random variables with a mean of zero and the variance-covariance matrix Σ_v. For any variable a, $a_{t|t}$ denotes the best estimate of a_t, given the information available in period t. By $A^2 \neq 0$, it is possible to reduce the information set available to the decision-makers. This may include variables that are observed after a delay, or variables that are not observed at all, but need to be estimated consistently using the system's internal dynamics and observable variables. The system written in this form is relatively flexible in that it allows us to also consider the full information case where $z_t = z_{t|t}$, and $x_t = x_{t|t}$. In this case, only the aggregated matrix $A \equiv A^1 + A^2$ matters for the decision problem of the private sector[6]. Once the model is recast in terms of predetermined variables, and the model is appropriately extended with the Kalman filter, the system can be used to derive simulations and forecasts for policy analysis. Following the notation of Svensson and Woodford (2003, 2001), matrices A_1 and A_2 are defined as

[6]The system written in this form is somewhat more restrictive than the one studied, for instance, in Pearlman *et al.* (1986). Also the information assumptions are different.

$$
\begin{aligned}
A_{11} &\equiv A_{11}^1 + A_{11}^2 \\
A_{12} &\equiv A_{12}^1 + A_{12}^2 \\
A_{21} &\equiv A_{21}^1 + A_{21}^2 \\
A_{22} &\equiv A_{22}^1 + A_{22}^2 \\
A &\equiv A^1 + A^2
\end{aligned}
$$

Similarly, we partition a vector of observables as

$$O_t = D^1 \begin{pmatrix} z_t \\ x_t \end{pmatrix} + D^2 \begin{pmatrix} z_{t|t} \\ x_{t|t} \end{pmatrix} + \eta_t \tag{2.2}$$

and where

$$D^j = \begin{pmatrix} D_{11}^j & D_{12}^j \\ D_{21}^j & D_{22}^j \end{pmatrix}, \; j = 1, 2 \tag{2.3}$$

The information available for the decision-makers to make inferences about variables is given by vector O_t. Partitioning a vector of observables in this way allows us to analyse situations in which some of the states variables are observable only in the subsequent periods or only an estimated state is available at time t. Once more, in the full information case, only the aggregated matrix $D \equiv D^1 + D^2$ matters for the decision problems. Throughout the text, we assume that measurement errors η_t are Gaussian and that they are uncorrelated with v_{t+1}.

Pearlman *et al* (1986) derive the R.E. solution by extending the method of Blanchard and Kahn (1980) to the partial information case. Here, we follow the route taken by Svensson and Woodford (2003) in that their method is somewhat easier and more intuitive[7]. We next explain the necessary steps needed for finding the R.E. solution of the model.

In the first step, it is assumed that the estimates of the forward-looking variables relate linearly with the predetermined variables. In particular, it can be shown that[8]

$$x_{t|t} = G z_{t|t} \tag{2.4}$$

where G is a transfer matrix that maps the estimates of the pre-determined variables ($z_{t|t}$) to those of the forward-looking variables. In solving G we can exploit the recursive structure of the problem (see, for instance, Ljungqvist and Sargent 2000), or use the method of Blanchard and Kahn (1980) extended to the partial information case by Pearlman *et al* (1986). In both cases, the solution to the partial information case is based on an important separation principle. Namely, it can be shown that the solution to G is independent of the solution to $z_{t|t}$, meaning that $z_{t|t}$ can be solved independently in the first stage of the problem.[9] Appendix A shows that transfer matrix G satisfies the following non-linear matrix equation at R.E. equilibria

$$G = (GA_{12} - A_{22})^{-1} (A_{21} - GA_{11}) \tag{2.5}$$

[7]See Svensson and Woodford (2000) for a discussion on the asymmetric information case.

[8]Equation (2.4) results from the fact that $z_t - z_{t|t} = v_t - v_{t|t}$, so that $x_{t|t} = G(z_t - (v_t - v_{t|t}))$. Naturally, the same non-linear matrix equation (2.5) applies in the full information case too.

[9]For proof, see Svensson and Woodford (2003).

where $(GA_{12} - A_{22})$ is assumed to be invertible.

In order for stable, non-explosive rational expectations solution to exists, (2.1) must have a saddle-point property. This requires that there should be $(n_z - n_x)$ eigenvalues of $A \equiv A^1 + A^2$. within the unit circle, and exactly n_x eigenvalues of A outside the unit circle. These conditions can of course be easily be checked for the specific problem.

IIn the case of partial information, the next step in the solution strategy is to re-cast the structural equations in terms of the predetermined variables only. This is extremely useful in that it allows us to use an (almost) standard Kalman filter for the estimation. Afterwards, the complete system can be used to make simulations and forecasts for policy analysis. Leaving explicit derivations for Appendix B, it can be shown that the whole system's dynamics can be expressed compactly as

$$\begin{aligned} z_{t+1} &= Hz_t + Jz_{t|t} + B_1 v_{t+1} \\ O_t &= Lz_t + Mz_{t|t} + B_2 \eta_t \end{aligned} \tag{2.6}$$

where

$$H = A^1_{11} - A^1_{12} T \tag{2.7}$$

$$J = A^1_{12} (T + G_*) + A^2_{11} + A^2_{12} G_* \tag{2.8}$$

$$L \equiv D^1_1 - D^1_2 T \tag{2.9}$$

$$M \equiv D^1_2 (T + G_*) + D^2_1 + D^2_2 G_* \tag{2.10}$$

and where $z_{t|t}$ is given by the Kalman filter

$$z_{t|t} = z_{t|t-1} + K(O_t - Lz_{t|t-1} - Mz_{t|t}). \tag{2.11}$$

In (2.11), matrix K provides optimal weights of the observable variables. Svensson and Woodford (2003) show that K is and determined by

$$K = PL' \left(LPL' + \Sigma_\eta\right)^{-1} \tag{2.12}$$

where P satisfies the matrix Riccati equation

$$P = H[P - PL'(LPL + \Sigma_\eta)^{-1} LP]H' + \Sigma_w \tag{2.13}$$

Upper block of the equation (2.6) provides system's dynamics in terms of the predetermined variables, while the lower block of (2.6) summarizes the observations of the predetermined and forward looking variables[10].

We next demonstrate how the above procedure needs to be modified when the model itself is uncertain and expectations are made conditional on the misspecified model. This requires us to consider an *auxiliary robust expectation* problem, where the agents seek to limit the degree at which their loss degenerates as the model misspecifications grow larger. This requires assigning an appropriate quadratic loss function to the private agents.

[10] Gerali and Lippi (2003) provide algorithms and MatLab based toolkit that solves this class of models and analyses their properties using simulations, impulse response functions and other techniques with both commitment and discretion.

2.2 Uncertain model

The rapidly developing literature on robustness considers decision problems in circumstances where the true model is not exactly known. In this robustness literature, concern for model uncertainty is imputed into the actual decision-making problem of the agents. The agents have a reference model or the nominal model that reflects their best knowledge of the believed laws of motion of the economy, yet they acknowledge that their information about the complete dynamics of the economy is limited. The agents hedge against the uncertainty by making mental constructs of the model sets. In particular, they seek for alternative robust decision rules, which work relatively well, even if the true dynamics may be different from their reference model. In order to construct a robust decision rules, the decision-maker computes a Markov Perfect Equilibrium of a particular zero-sum game, where each player chooses sequentially and simultaneously in each period, taking the other player's decision rule as given (Hansen and Sargent 2004). In our case, the decision-makers are the private sector, which forms the expectations, and nature, which chooses the distortions to the nominal model's dynamics. The private sector solves a pure expectation problem, where the expectations are conditioned on the model chosen by nature. Agents thus achieve robustness with the help of evil nature's distorted laws of motion.

Following the standard setup in the robust control literature, we assume that the private sector considers the model uncertain in the sense that

$$\begin{pmatrix} z_{t+1} \\ x_{t+1|t} \end{pmatrix} = A^1 \begin{pmatrix} z_t \\ x_t \end{pmatrix} + A^2 \begin{pmatrix} z_{t|t} \\ x_{t|t} \end{pmatrix} + B_1 \begin{pmatrix} w_{t+1} + v_{t+1} \\ 0 \end{pmatrix} \quad (2.14)$$

$$O_t = D^1 \begin{pmatrix} z_t \\ x_t \end{pmatrix} + D^2 \begin{pmatrix} z_{t|t} \\ x_{t|t} \end{pmatrix} + B_2 \eta_t \quad (2.15)$$

In (2.14) w_{t+1} is a process of which $t+1$ time component is a measurable vector process that can feedback in general but restrained way on the history of the predetermined variables. In other words,

$$w_{t+1} = g_t(z_t, z_{t|t}, z_{t-1}, z_{t-2}...) \quad (2.16)$$

wherer $\{g_t\}$ is a sequence of measurable functions. w_{t+1} is a vector process, where the size of the model approximation errors is constrained such that

$$E_0 \sum_{t=0}^{\infty} \beta^t w'_{t+1} w_{t+1} \leq \mathcal{W} \quad (2.17)$$

and where E_0 denotes a mathematical expectation conditioned on the initial values of z and where β is a respective discount factor. $\mathcal{W}$ defines a set of models and provides an intertemporal budget for the maximising agent to distort the model's dynamics. The set of models, that are 'possible' around the approximating model are therefore constraint by $\mathcal{W}$. Given that the uncertainty surrounding the model is presented here in an unstructured way, it can be thought of as capturing a wide range of misspecified dynamics. As will be seen in due course, w_{t+1} appears as a control sequence which maximises the assigned loss function of the private agents. This then determines the optimal value of current expectations, given information available at t. w_{t+1} enters into the actual laws of motion of the worst worst-case model, but is of course absent from the actual transition laws of the economy.

Finally, note that since the laws of motion of z_{t+1} is only an approximation of the true laws of motion, and there is a possible circularity between predetermined and forward looking variables, w_{t+1} will affect *the actual path of forward and pre-determined variables.* We will demonstrate this in the next section[11].

Another way to interpret the rules that are obtained under a preference for robustness is to interpret them as a version of Epstein and Zin's (1989) specification of recursive preferences. As discussed in Hansen and Sargent (2004), despite their very different motivations, a risk-sensitive control problem yields precisely the same decision rules in the linear quadratic world as corresponding to the robust control problem. (See also Jacobson 1973; Whittle 1990; Hansen and Sargent 1995). More formally, risk-sensitive control is based on a recursive utility function

$$U_t = L_t + \beta \mathcal{R}(U_{t+1}) \tag{2.18}$$

where

$$\mathcal{R}(U_{t+1}) = \left(\frac{2}{\sigma}\right) \log E \left[\exp \left(\frac{\sigma U_{t+1}}{2} | \mathcal{I}_t \right) \right] \tag{2.19}$$

$\mathcal{I}_t$ denotes information set available at time t and $\sigma \leq 0$ is the risk sensitivity parameter. This risk-sensitivity parameter relates to the robust control problem by $\theta = -\sigma^{-1}$ and where θ is a constant Lagrange multiplier associated with the natures, or fictious player's, intertemporal constraint (2.17). As emphasised by Kasa (2002), this relationship is important, since robust decision control problems can be related to problems that have plausible decision-theoretic foundations. Moreover, Kasa (1999) shows how concern for robustness in the presence of model uncertainty can substitute for the ad hoc incorporation of adjustment costs into the dynamic policy models.

[11] When the model is backward looking, Giordani and Söderlind (2002) illustrate that only the forecasts are affected by model misspecification, not the actual path of the the variables.

3 Solving robust problem

3.1 Decision

In general, robust decision rules are derived from the iteration of a two-player zero-sum game and an appropriate version of a certainty equivalence principle. Yet, due to the feedback nature of w_{t+1}(and possibly η_t) , the volatility matrix B_1 will influence the robust decision rules and the forecasts. Given that we are only concerned with a pure robust forecasting problem, there is no control on the part of the policymaker. The absence of control, then, eliminates the minimisation part of the problem. However, due to the forward-looking part of the problem, expectations must be consistent with nature's control sequence w_{t+1} which maximises the assigned loss function of the private agents.

The solution algorithm, which solves the standard rational expectation models, can be adapted to the above robust program. This is because they typically solve the first order conditions, which are the same for the maximum and for the minimum. In every period, the evil agent's choice is disciplined by an intertemporal budget constraint (2.17) in formulating the robust policies. The evil agent optimises in every period, taking the expectations of the private sector as given. In equilibrium, the expectations of the private sector converge to a stable non-explosive solution, where the expectations are consistent with evil nature's decisions and the consequent distorted transition laws of the variables.

This type of robust program is implemented, for instance, in Giordani and Söderlind (2002). Yet, their solution algorithm is based on a generalised Schur decomposition, where the policymaker is present and information is perfect.[12]In this paper we provide a somewhat simpler algorithm, which exploits a recursive structure of the problem and allows us to consider the case of imperfect information. That is, we express the decision problem of the private sector and the evil agent as an optimal value problem in period t, by means of the Bellman equation

$$z_{t|t}V_t z_{t|t} + d_t \equiv \max_{\{w_{t+1}\}} E_t \left\{ L_t + \beta E[z'_{t+1}V_{t+1}z_{t+1}] + d_{t+1}|I_t \right\} \tag{3.1}$$

V_t is positive semidefinite matrix of the value function[13] of the problem and L_t is the appropriately modified quadratic loss criteria. This problem reflects the pure forecasting problem of the agents, where L_t is the periodic loss function of a representative agent in the economy and the agent's concern for modelling errors is directly introduced into this loss function.

Writing the problem in this way provides us with a convenient way of solving the robust decision problem also under partial information, since V_t can be computed from the ordinary linear regulator problem with the usual methods. Application of the ordinary optimal linear regulator is justified

[12]See a detailed explanation in Söderlind (1999).

[13]Notice that in the above problem,V_t is essentially the same matrix as the one obtained from the Lagrangian approach, where the stable solution is characterized by the stabilizing condition $\mu_{t|t} = Vz_{t|t}$, and where $\mu_{t|t}$ are the appropriate Lagrange multipliers of the problem. See, for instance, Söderlind (1999).

here by recognizing that the Riccati equation for the optimal linear regulator emerges from the first order conditions alone. By suitably modifying the standard regulator problem, the first order conditions yield the maximum with respect to nature's control variable w_{t+1} instead of the minimum.

Leaving more detailed derivations for Appendix B, the solution to the robust decision problem is characterised by a pair of decision rules

$$w_{t+1|t} = F z_{t|t} \quad (3.2)$$
$$x^r_{t|t} = G^r z_{t|t} \quad (3.3)$$

for evil nature and private agents respectively. (3.2) is the worst-case shock, which induces a distorted transition law for the model's dynamic equations described in (2.14), while (3.3) gives the expectations of the forward looking variables consistent with the distorted transition laws of the model. In particular, it is shown in the appendix that G^r is a solution of the recursion

$$G^r_t \equiv \left(G^r_{t+1} A_{12} - A_{22}\right)^{-1} \left(A_{21} - G^r_{t+1}(A_{11} + B_1 F_t)\right) \quad (3.4)$$

As usual in robust decision problems, the volatility matrix B_1 enters into the robust decision problem and G^r now depends upon F. Furthermore, the appendix shows how F depends upon the parameters of the loss function through (3.1) and in particular, on the penalty parameter assigned to the intertemporal budget constraint of evil nature (θ). The decision-maker thus achieves robustness by distorting his expectations. Given that there is isomorphism between robust control and risk-sensitive control in the LQ case, we can interpret G^r_t as being a result of a decision rule where the agents put additional correction for risk into the evaluation of the continuation utility.

After G^r_t and thus $x^r_{t|t} = G^r_t z_{t|t}$ has been found, we can proceed with re-casting the system's dynamics in terms of the predetermined variables only. Yet, the difference compared to the earlier case is that there are essentially two models that can be analysed. The first is the so-called worst-case model, and the other one is the nominal model with distorted expectations (D.E). Once more, leaving exact derivations for the appendix, it can be shown that the worst-case model, written in terms of the predetermined variables only, reduces into two equations on transition laws and observables

$$z^w_{t+1} = H^r z_t + J^r z_{t|t} + B_1 v_{t+1} \quad (3.5)$$
$$O^w_t = L^r z_t + M^r z_{t|t} + B_2 \eta_t \quad (3.6)$$

where

$$H^r = H \equiv \left(A^1_{11} - A^1_{12} T\right)$$
$$J^r \equiv A^2_{11} + A^2_{12} G^r + A^1_{12} (T + G^r) + B_1 F$$
$$L^r = L \equiv D^1_1 - D^1_2 T$$
$$M^r \equiv D^1_2 (T + G^r) + D^2_1 + D^2_2 G^r$$

Distorted expectations and distorted transition laws are now embedded in the system's matrices J^r and M^r. This model reflects the transition laws under the worst possible scenario of the shocks. The nominal model with distorted expectations can be obtained from above by simply changing

$$J^n \equiv A_{11}^2 + A_{12}^2 G^r + A_{12}^1 (T + G^r) \tag{3.7}$$

and thus ignoring the additional control sequence w_{t+1}. This is in fact the relevant model against which the model should be evaluated. The reason is that nature's decisions are used only as an instrument to obtain more robust expectations, and they are of course not present in the actual true laws of motion of the economy.

3.2 Estimation

The Kalman filter[14] is the optimal linear least-mean-squares estimator for the systems that are described by linear state-space Markov models. Since its development in the 1960s it has played an important role in numerous fields ranging from engineering applications to finance and economics. The Kalman filter is essentially a recursive algorithm for computing the mathematical expectation $E(z_t|O_t...O_0)$ of an imperfectly observed state vector z_t, conditional on observing the current state and the history $O_t...O_0$ of a vector of noisy signals on imperfectly observed states. At the same time when Kalman filter was developed (Kalman (1960)), Muth (1960) wanted to understand under which conditions Cagan's (1956) and Friedman's (1956) adaptive forecasting scheme would be optimal. Cagan and Friedman suggested that when people wanted to form expectations of future values of some scalar z_t, they would use the following adaptive expectations scheme

$$z_{t+1|t} = K \sum_{j=0}^{\infty} (1-K)^j z_{t-j} = z_{t|t} + K(z_t - z_{t|t}) \tag{3.8}$$

where $z_{t+1|t}$ denotes people's expectation and K is the degree at which people update their expectation upon time-t estimation errors (Kalman gain). Friedman used this scheme to describe people's forecasts of future income in particular. Muth realised that the stochastic model given by

$$\begin{aligned} z_{t+1} &= z_t + v_{t+1} \\ O_t &= z_t + \eta_t \end{aligned} \tag{3.9}$$

[14] A central premise of Kalman filter theory is that the underlying state-space model is accurate. When this assumption is violated, the performance of the filter can degenerate very fast. This filter sensitivity to modeling errors has led to several works in the literature on the development of robust state-space filters; robust in the sense that they attempt to limit, in some particular way, the effect of model and measurement uncertainties on the overall performance of the filter. Within this literature, perhaps one of the most promising approaches has been the development of $\mathcal{H}_\infty$ filtering. This $\mathcal{H}_\infty$ approach attempts to construct filters that bound the 2-induced norm of the transfer function which maps the disturbances to the estimation errors to some pre-specified level. This $\mathcal{H}_\infty$ approach has been shown to correspond to minimum entropy and risk sensitive control in the linear quadratic case. (See, for instance, Whittle 1990, 1996).

would be the one in which the updating rule (3.8), and therefore Kalman filter is optimal.

Svensson and Woodford (2003) show that by expressing the filtering problem associated to (2.6) in terms of the prediction errors $z_t - z_{t|t-1}$ and $O_t - O_{t|t-1}$, optimal steady state Kalman gain is given by

$$K = PL' \left(LPL' + \Sigma_\eta\right)^{-1} \tag{3.10}$$

where P satisfies the matrix Riccati equation

$$P = H[P - PL'(LPL + \Sigma_\eta)^{-1}LP]H' + \Sigma_w \tag{3.11}$$

Σ_w and Σ_η denote variance-covariance matrices associated with the system and measurement noise respectively. $P \equiv E(z_t - z_{t|t-1})(z_t - z_{t|t-1})'$ is the variance covariance matrix of the prediction errors. The Kalman filter provides a natural computational tool for the real-time determination of state. Recursion for the estimate $z_{t|t}$ takes a convenient form

$$z_{t|t} = z_{t|t-1} + K(O_t - Lz_{t|t-1} - Mz_{t|t}) \tag{3.12}$$

Under the assumption that measurement errors are Gaussian instead of energy-bounded as in the robust filtering problem, we can use this very same Kalman filter to obtain estimates for the state of the economy, even in the case where the private sector is hedging against the model misspecification errors. We only need to make the following substitutions in the above filtering equations

$$\begin{aligned} G &\rightarrow G^r \\ M &\rightarrow M^r \\ J &\rightarrow J^r \end{aligned}$$

In this case, the standard Kalman filter delivers the worst-case predictions, by exploiting the transition laws and expectations of the worst-case model. Forecasting errors are white noise with respect to the worst-case model. However, when these estimates are evaluated with respect to the nominal model, forecasting errors will contain a predictable component.

3.3 A little detour to the robust filter

The standard Kalman filter described above requires an accurate model of the process under consideration. It assumes only additive uncertainty concerning the process and measurement equations in the form of Gaussian noise, assumptions that are now implicit in the model once recast in terms of the predetermined variables only. The standard Kalman filter discussed above and applied to either the nominal or the worst-case model, aims at producing white noise forecast errors. By construction, such a standard Kalman filter is robust in the sense that it minimises the maximum of a one-step ahead forecasting error[15]. However, if the agents would care about past forecast

[15] Hansen and Sargent (2003) explain this in detail.

errors, committing in advance to a given filter, then an alternative robust filter would be more appropriate.

The idea of such a 'backward-looking' robust filter is similar to robust control, but there an evil agent chooses a disturbance process that feeds back on the estimation errors. Typically, this leads to an increase in the persistence and variance of the forecast errors. In order to safeguard against this possibility, the forecaster tends to increase the gain parameter, leading to larger initial reactions to the shocks. Kasa (2003) suggests that this makes agents less susceptible to low-frequency misspecifications, which are especially damaging.

In this paper, we abstract from the consideration of robust, or risk-sensitive filtering (for details see Whittle 1999). This means that the procedure adopted here is valid only in the special case that the measurement errors are assumed to be Gaussian and uncorrelated with v_{t+1} and w_{t+1}. Alternatively, we may interpret the situation here as one in which the state estimation errors do not cause additional "stress" for the decision-maker[16].

4 Application to the New Keynesian economy

In order to illustrate how robust expectations and imperfect information influence the properties of the typical monetary policy model of the day, we use the New Keynesian model[17] where price stickiness creates a channel by which monetary policy can influence the real economy. This model is based on two log-linear approximations of Euler equations that describe decisions of representative households and firms

$$c_t \equiv y_t - g_t = -\varphi(i_t - E_t\pi_{t+1}) + E_t(y_{t+1} - g_{t+1}) \tag{4.1}$$

$$\pi_t = \lambda(y_t - z_t) + \beta E_t\pi_{t+1} \tag{4.2}$$

where

$$g_t \equiv -\log(1 - \frac{G_t}{Y_t}) \tag{4.3}$$

G_t is exogenously given government consumption and y_t denotes the stochastic component of output. Typically, the output gap is defined as $x_t \equiv (y_t - \omega_t)$, where ω_t denotes the stochastic component of the natural level of output (the one that prevails in flexible-price equilibria). The first equation is obtained by log-linearizing the consumption Euler equation, arising from the household's optimal saving decision, and using the equilibrium condition that consumption equals output less government spending ($Y_t - G_t = C_t$). Expectations $E_{t+1}(.)$ are conditional upon the full information set of the private sector to be specified

[16] Whittle (1999) shows how estimation and decisiondecision-making in the context of risk-sensitive control can be separated into two recursions of matrix Riccati equations, then coupling them in the final extremization of stress. However, he only considers backward-looking models and it is not clear whether such separation also holds in the forward-looking models.

[17] Woodford (1998), Goodfriend and King (1997), Rotemberg and Woodford (1997), Clarida, Gali and Gertler (1999).

below. (4.1) is typically written in terms of the stochastic version of an output gap

$$x_t = -\varphi(i_t - E_t\pi_{t+1}) + E_t x_{t+1} + \epsilon_t$$

and where

$$\epsilon_t \equiv E_t\{\Delta\omega_{t+1} - \Delta g_{t+1}\} \tag{4.4}$$

In the simulation and impulse response analysis, it is typical to assume that ϵ_t follows some stochastic law of motion, typically an $AR(1)$ process with a mean of zero. Moreover, since ϵ_t shifts the IS curve, it can be thought of as representing the demand shock to the economy. Notice however, that ϵ_t captures expectations of two different types of shocks: expenditure shocks due to changes in government consumption, and technology shocks. In order to distinguish between these two shocks, we write first that

$$y_t - g_t = -\varphi(i_t - E_t\pi_{t+1}) + E_t(y_{t+1} - g_{t+1}) \tag{4.5}$$

$$\pi_t - \lambda(y_t - \omega_t) = \beta E_t\pi_{t+1} \tag{4.6}$$

and derive that

$$y_t = -\varphi(i_t - E_t\pi_{t+1}) + E_t(y_{t+1}) - E_t(g_{t+1} - g_t)$$

Let us denote $\frac{G_t}{Y_t} \equiv \gamma_t$ and let $\frac{G_t}{Y_t}$ be the exogenously given government consumption share. Now, since $g_t \equiv -\log(1-\gamma_t) \approx \gamma_t$, we find that the stochastic component of the output gap behaves as

$$y_t = -\varphi(i_t - E_t\pi_{t+1}) + E_t(y_{t+1}) - E_t(\gamma_{t+1} - \gamma_t) \tag{4.7}$$

$E_t(\gamma_{t+1} - \gamma_t)$ denotes the one-step ahead expectation error of the government consumption share, given the information available at time t. This expectation error affects negatively the current stochastic component of the output. Notice also that there is an implicit crowding out effect in the model; the expected increase in government consumption will crowd out private consumption today. In order to make the analysis simple, we now make the assumption that both the stochastic component of the natural level of output and the government consumption follow exogenous $AR(1)$ processes given by

$$\omega_{t+1} = \rho_\omega\omega_t + (w_{\omega,t+1} + v_{\omega,t+1}) \tag{4.8}$$

$$\gamma_{t+1} = (1-\rho_\gamma)\gamma^* + \rho_\gamma\gamma_t + (w_{\gamma,t+1} + v_{\gamma,t+1}) \tag{4.9}$$

where $\gamma^* > 0$ denotes a long run target level of the government consumption share. This long run target of government consumption now enters into the output equation through $E_t(\gamma_{t+1} - \gamma_t)$[18].

$$E_t(\gamma_{t+1} - \gamma_t) = (1-\rho_\gamma)\gamma^* + \left(\rho_\gamma - 1\right)\gamma_t + E_t(w_{\gamma,t+1} + v_{\gamma,t+1})$$

[18] In a more realistic situation γ_t should be treated as an endogenous variable due to active fiscal policy, and/or automatic stabilisers.

Finally $w_{j,t+1}$, $j = \omega, \gamma$ introduces specification errors into the dynamics of the exogenous shocks. v_j are the stochastic *iid* components of the errors with zero mean and some variance σ_j^2, representing productivity and government expenditure shocks to the economy. Expectations operator E_{t+1} must then be interpreted as distorted expectations operator, applied to distorted transition laws.

Substituting inflation expectations into the output equation, we can finally write down the fundamental equations of the model:

$$\begin{aligned}
\omega_{t+1} &= \rho_\omega \omega_t + w_{\omega,t+1} \\
\gamma_{t+1} &= (1-\rho_\gamma)\gamma^* + \rho_\gamma \gamma_t + w_{\gamma,t+1} \\
E_t(y_{t+1}) &= (1-\rho_\gamma)\gamma^* + \varphi(i_t - \beta^{-1}\pi_t) + (1+\beta^{-1}\lambda\varphi)y_t - \beta^{-1}\lambda\varphi\omega_t \\
&\quad + (\rho_\gamma - 1)\gamma_t \\
\beta E_t \pi_{t+1} &= \pi_t - \lambda(y_t - \omega_t)
\end{aligned}$$

4.1 Policy rule

Finally, in order to close the model, we need to specify the monetary policy rule. We specify it in the format of a Taylor type of rule, based on observables y_t and π_t.

$$i_t = \delta_y y_t + \delta_\pi \pi_t + \sigma_i w_{i,t}$$

We introduce $w_{i,t}$ in the policy rule in order to allow uncertainty in the policy rule as well. One may, of course, speculate on a number of reasons as to why the private sector may know the form of the policy rule, but not the exact coefficients of the rule. This type of policy rule is nowadays a common workhorse assumption in monetary economics, but without an additional disturbance term $w_{i,t}$. The optimal weights δ_y and δ_π depend upon a complicated way on the structure of the model, the preferences of the policymakers and expectations, but they nevertheless measure the sensitivity of the interest rate response to movements in observed output and inflation. The policy parameters are an implicit result of a complex policymaking process, which may consist of optimisation and subjective assessment of the state of the economy and the calibration of different models.

Another type of rule is the so-called forecasting, or forecast-based rule, written as

$$i_t = \delta'_y y_{t+1|t} + \delta'_\pi \pi_{t+1|t} + w_{i,t} \tag{4.10}$$

Here $y_{t+1|t}$ and $\pi_{t+1|t}$ denote one-step ahead estimates (expectations) of the output gap and inflation. However, if $w_{i,t} = 0$, the linear rational expectations structure of the model would actually make the forecasting rule equivalent to the outcome-based rule. This is simply due to the fact that in the rational expectation model, $y_{t+1|t}$ and $\pi_{t+1|t}$ become linear functions of the observables. This useful linear property can be easily illustrated in this setup as well.

Namely, since $x_{t|t} \equiv \begin{pmatrix} y_{t|t} \\ \pi_{t|t} \end{pmatrix} = G z_{t|t}$, we find that

$$x_{t+1|t} = A_{21} z_{t|t} + A_{22} x_{t|t} = (A_{21} G^{-1} + A_{22}) x_{t|t} \tag{4.11}$$

and where $A_{21} \equiv A_{21}^1 + A_{21}^2$, $A_{22} \equiv A_{22}^1 + A_{22}^2$. Therefore, we could express the forecasting rule as a function of observable forward looking variables at rational expectations equilibria by appropriately combining (4.10) and (4.11). If we would allow policy rule uncertainty by allowing $w_{i,t}$ feedback in an arbitrary but constrained way on observable state variables, the linear property would still be preserved. However, the problem would be more complicated due to the presence of G in the equation for forward-looking variables.

5 Nominal model with full information

In order to benchmark the model, we assign typical parameter values for the model as shown in Table (1).

Parameters									
θ	ρ_z	ρ_γ	λ	β	φ	δ_y	δ_π	λ_y	γ^*
∞	.7	.5	0.05	.99	2	.15	1.5	1	.05

Innovations			Measurement errors			
σ_ω	σ_γ	σ_i	$\sigma_{z,\eta}$	$\sigma_{\gamma,\eta}$	$\sigma_{y,\eta}$	$\sigma_{\pi,\eta}$
.005	.015	0	0	0	0	0

Table 1: Benchmark parameter values

When benchmarking the model, we assume that the agents perceive the policy rule and the model as accurate. We assume that the agents dislike fluctuations in private consumption $(y_t - \gamma_t)$ and the deviation of inflation from the target level of zero, with equal weights of 1. In other words, we postulate that infinitely lived representative household (and firm) welfare criteria can be approximated by

$$-\sum_{t=0}^{\infty} \beta^t L_t \tag{5.1}$$

where L_t represents a periodic loss criterion

$$L_t \equiv \frac{1}{2}(\pi_t - \pi^*)^2 + \lambda_y (y_t - \gamma_t - y^*)^2 - \theta w_{t+1}' w_{t+1} \tag{5.2}$$

π^* is inflation target and y^* is the output target, which are from now on set to zero. Thus private agents dislike fluctuations in private consumption as well as inflation. This differs from the original log-linear approximation of the welfare of the household of Woodford (1997) in that there is additional correction for

	Output	Inflation	Nominal rate	Consumption	Real rate		SS(y)	SS(π)
Benchmark model (I)	0.0115	0.0022	0.0033	0.0116	0.0037	0	0	0
Model uncertainty (I)	0.0115	0.0023	0.0033	0.0117	0.0038	0.001	–.0026	0.0013
Model uncertainty (II)	0.0118	0.0026	0.0038	0.0123	0.0043	0.1309	–.117	.057
Benchmark model (II)*	0.0263	0.0024	0.0059	0.0264	0.0210	0	0	0
Policy uncertainty (I)	0.0265	0.0024	0.0059	0.0268	0.0268	0.038	–.001	–.001
Policy uncertainty (II)	0.0283	0.0021	0.0058	0.0301	0.0301	.355	–0.012	–0.014

* This is different from benchmark model I only to the extent that it includes policy uncertainty.

Table 2. Standard errors of the main variables of interest and steady state

risk introduced by $-\theta w'_{t+1} w_{t+1}$ and where w_{t+1} is a vector of nature's decision variables.

In the benchmark model we assume that all the variables are observed without a measurement error and the agents perceive the model as accurate. This can be achieved in the framework by setting $\theta \to \infty$, which assigns an infinite penalty to nature in making w_{t+1} different from zero.

Using the methodology described above, we can then derive rational expectations equilibria computationally and then use various methods to describe the dynamic aspects of the model. For instance, unconditional standard errors of output, inflation, the nominal interest rate, consumption and real interest rates are described in Table 2 under various assumptions of model uncertainty.

The Benchmark Model (1) is analysed under the assumption that the decision-makers perceive the model as certain. The following 2 rows (Model Uncertainty, 1–2) describe the model under 2 different degrees of uncertainty associated with the dynamics of exogenous shocks, while keeping the interest rate rule as certain. We measure this uncertainty by calculating the induced sum of the standard errors of the worst-case shocks ($\sum \sigma_w$).

The first observation here is that the model uncertainty, which in this setting is due to the uncertainty regarding the dynamic properties of the expenditure shock and productivity shocks, induces only slightly larger unconditional variations in the model's variables. The initial response of inflation to expenditure shock is larger in the nominal model with distorted expectations, when compared to the nominal model with R.E. It also takes somewhat longer for inflation to return back to the long-run equilibrium. Regarding consumption, additional concern for robustness leads consumers to engage in a form of precautionary saving[19], which shows up as a slightly stronger dynamic response of consumption to expenditure shocks. The key difference, however, appears in the steady-state results of the model. Agents are prepared for the deviation of exogenous government expenditure from its long-run level, and this in turn induces a permanent deviation of output and inflation from their corresponding steady-state values in the nominal model, as shown in Table 2 in columns $SS(y)$ and $SS(\pi)$. In the case without policy uncertainty, output will remain belows its target, while inflation will remain above its target value of zero.

5.1 Policy uncertainty

Next, we also introduce into the model policy uncertainty by allowing shocks to the interest rate equation. Effectively this is done by setting σ_i as different from zero. We set $\sigma_i = 0.015$, thus keeping it within the range of the other shocks. This uncertainty about the policy may reflect many things, including a lack of transparency and/or the credibility of the announced policy rule.

[19] Hansen, Sargent and Wang (2002) show that in the basic permanent income model, a consumer with a preference for robustness prefers future over current consumption, thus delivering a kind of precautionary savings motive.

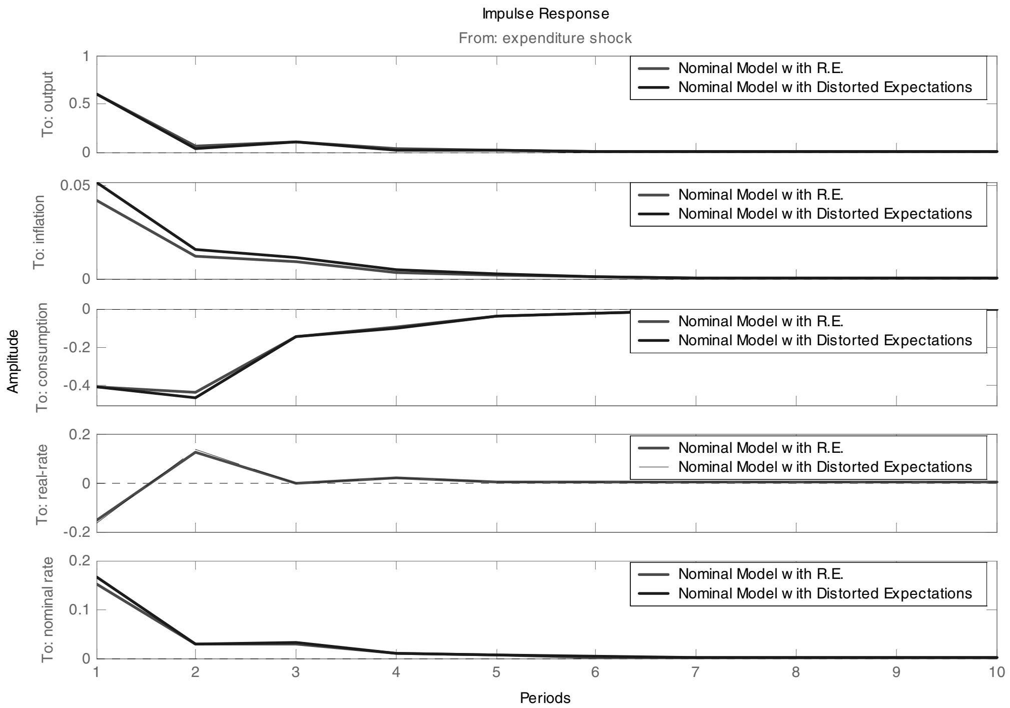

Figure 1: Expenditure shock in the nominal model

Agents know the form of the policy rule, but at the same time they know that there are other elements that can make the policymaker deviate from such a mechanical rule. In order to hedge against this uncertainty, they construct worst-case missperceptions of the policy rule and form expectations on the basis of this scenario. Since the interest rate rule is written as a feedback rule of observable forward-looking variables of output and inflation, and forward-looking variables depend upon the interest rate, missperceptions of the interest rate rule will also affect the paths of the forward-looking variables.

Once more the first and second moments of the variables of interest are shown in Table (2) on the rows labeled as Policy Uncertainty (I) and (II). In general, the introduction of policy rule uncertainty does not make the second moments of the variables differ much more from their corresponding values in the nominal model. However, impulse response analysis reveals that the initial response of consumption to expenditure shock is now markedly larger when expectations are distorted by uncertainty. On the contrary, the initial response of inflation and the nominal interest rate is now rather smaller in the uncertain case.

The larger consumption response can be understood intuitively by calculating the perceived worst-case rule of the private agents from the worst-case model. This delivers

$$i_t^w = 0.077 y_t + 2.01 \pi_t \tag{5.3}$$

This implies that agents expect the central bank to react to output fluctuations less and to inflation more than in the nominal model. In other words, agents under-estimate the dynamic responses of the interest rate to output, but

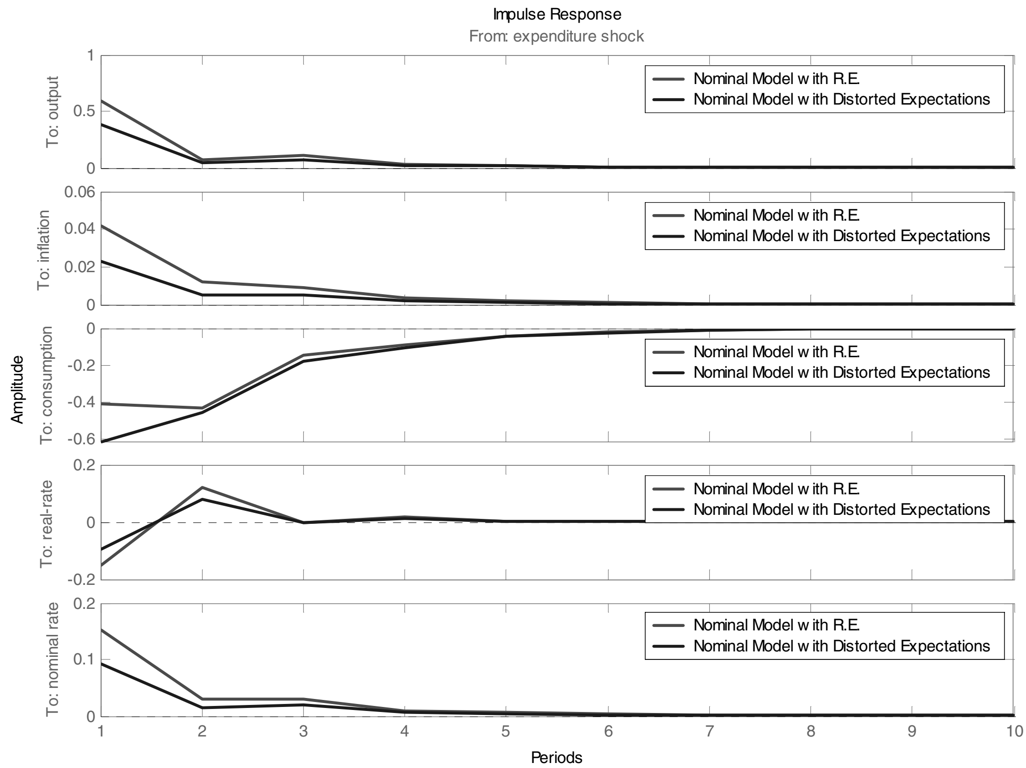

Figure 2: Nominal model with distorted expectations and policy rule uncertainty

over-estimate the response to inflation in achieving robustness. This causes private agents to cut consumption initially rather more than in the absence of uncertainty. Moreover, it can be shown that perceived paths for technology and expenditure shocks tend to be more persistent than those implied by the nominal values of ρ_γ and ρ_z in the worst-case equilibria: agents over-estimate the persistence of the exogenous shocks. Additional uncertainty regarding the policy rule and the agents' concern for robustness thus leads into a kind of precautionary saving, showing up as a larger initial response of consumption. This is also reflected in a relatively larger unconditional standard error of consumption around the steady state. Moreover, both inflation and output are now below their respective target values of zero.

5.2 Imperfect information

At the time the forecasts (and decisions) are made, it is typical that the agents' real-time information set does not allow them to infer the values of all the variables accurately. For instance, Orphanides (1998) has stressed that policy is made on the basis of preliminary estimates that are revised substantially in subsequent months. Kilponen and Salmon (2002) also show that the average standard error associated with measurement between preliminary and final estimates of U.K. GDP has been substantial during the last decade.

Measurement errors			
$\sigma_{z,\eta}$	$\sigma_{\gamma,\eta}$	$\sigma_{y,\eta}$	$\sigma_{\pi,\eta}$
0.01	.01	0.01	0.01

Table 2: Stds of the measurement errors

Under imperfect measurement, the decision-making problem is thus further complicated by simultaneous signal extraction and the inference problem. With imperfect information, the decision-maker forms an estimate about the true state of the economy using noisy observations of the relevant variables. The first obvious consequence of this is that the true pattern of the shock now differs from the one estimated by the decision-maker even if the model is perceived to be accurate. That is, $z_t \neq z_{t|t}$ and $x_t \neq x_{t|t}$ even if $\theta \to \infty$.

The signal extraction problem can, however, be solved by the Kalman filter. This leads the decision-maker to learn only gradually about the realizations of the actual shocks. The standard Kalman filter provides an optimal estimate of the true state of the economy and produces white noise forecast errors if the model is certain.

When the decision-maker perceives the model to be inaccurate, this leads to an additional error in estimating the true shocks. The eventual magnitudes of the forecast errors induced by the imperfect information on the one hand, and uncertainty about the model itself on the other, depend upon the dynamic properties of the nominal model, the signal to noise ratio, ie the measurement errors, as well various assumptions about where the misspecifications lie. What is important, however, is that forecast errors are no longer white noise, when evaluated with respect to the nominal model.

In order to illustrate these points, Figure (3) contrasts the actual technology shock to those estimated by the decision-maker under the perception that the model is [either] accurate or inaccurate. Measurement errors associated with productivity, government expenditure, output and inflation are assumed to be as given in Table (2).

The first observation here is that after a unitary transitory technology shock ($\omega_t = 1$), the contemporaneous estimate of the shock by the decision-maker is $\omega_{t|t} = .44$, inducing a rather large contemporaneous forecast error. This contemporaneous estimate of the shock turns out to be the same in both models, yet the dynamic path of the estimates differs in the two cases. In this setup, the difference between the nominal model and the worst-case model, however, is not very large, yet it is interesting to note that agents ‘learn’ more gradually under the worst-case model.

Through their effect on the expectations about the true state of the economy, imperfect information and model uncertainty affect the dynamics of the forward-looking variables, introducing different responses to the shocks. Figure (4) draws the impulse responses of inflation and the real rate on the unitary shock to government expenditures. As earlier, imperfect information causes contemporaneous estimates of inflation ($\pi_{t|t}$) and the real rate ($i_{t|t} - \pi_{t+1|t}$) to deviate from those paths under full information. Moreover,

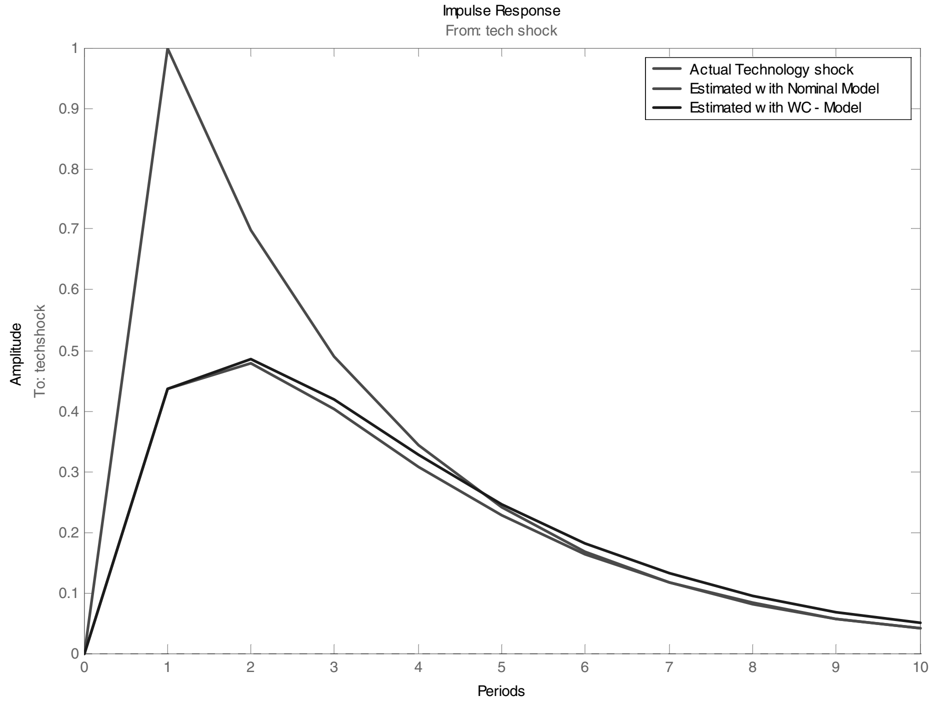

Figure 3: Actual and perceived technology shocks under imperfect information

contemporaneous estimates of inflation and the real rate differ between the nominal model and the worst-case model due to the fact that estimates $(\pi_{t|t})$ and $(i_{t|t})$ of the private sector are now based on laws of motion consistent with the worst-case model. In case of inflation, the contemporaneous forecast error $(\pi_t - \pi_{t|t})$ is larger and it takes much longer for the estimates to revert to being closer to the actual values when compared with the nominal model. This is due to the fact that the agents perceive that the shocks last longer in the worst-case model. This effect is actually magnified when the exogenous shocks do persist longer in the nominal model.

Combining imperfect information and robustness makes the agents over-react to innovations. This is illustrated in Figure (5), which contrasts the 'true' expenditure shock to the one estimated by the private agents under imperfect information and concern for robustness in the typical simulation.

The worst-case estimates of the expenditure shock can deviate from the actual ones for long periods of time and typically have a larger standard deviation than the true shock. The prediction errors $(z_{t|t} - z_t)$ – when evaluated with respect to the true shock – are no longer white noise, but have a predictable component due to the fact that estimates $z_{t|t}$ are distorted by concern for robustness. This is perfectly consistent with the fact that the agents over-estimate the persistence of exogenous shocks, as illustrated in Figure (5): Innovations persist longer in the estimates than in the actual series. However, forecast errors still have a mean of zero in long-simulations, implying that there is no systematic bias in the forecasts.

All this eventually shows up in the way in which model's endogenous variables react to various shocks. In particular, Figure (6) shows the impulse

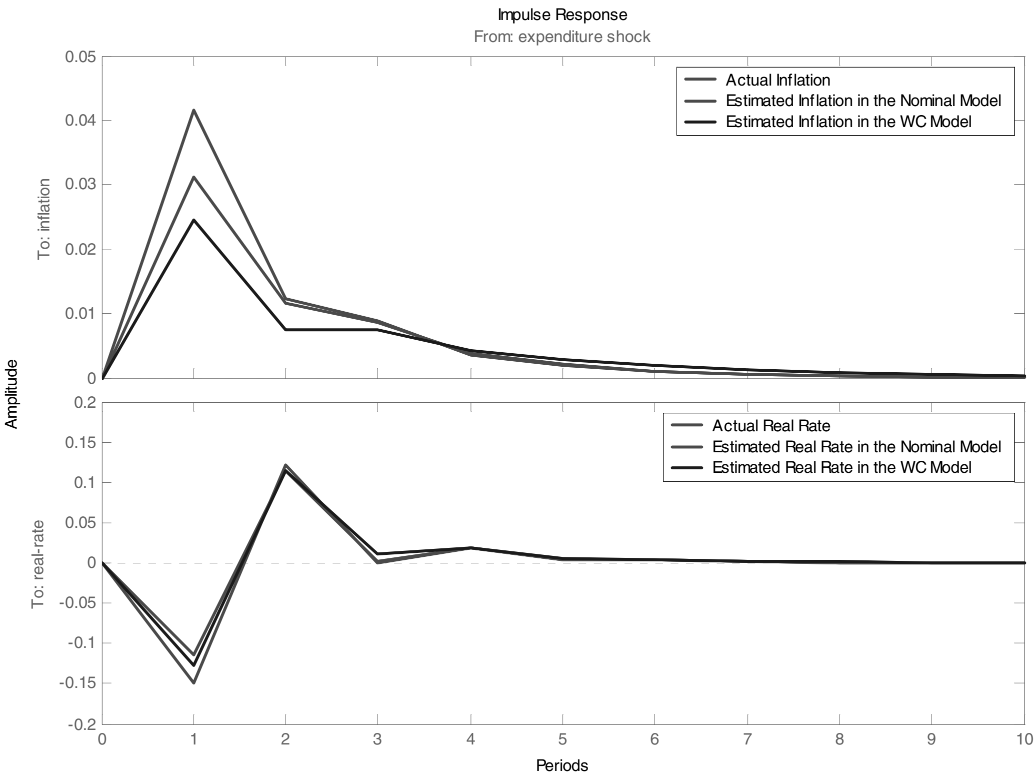

Figure 4: Response of inflation and real rate on unitary shock to government expenditures

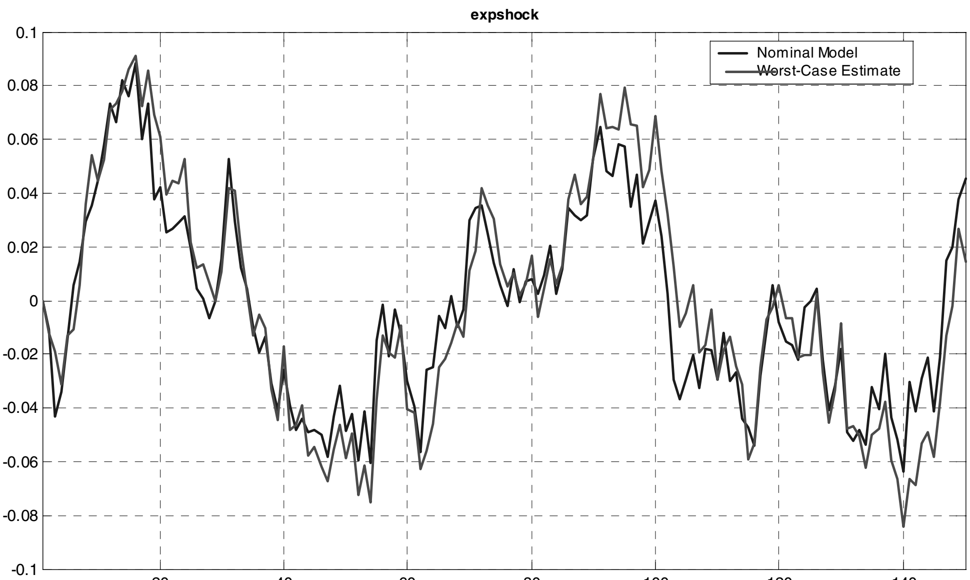

Figure 5: The 'true' expenditure shock and its worst-case estimate under imperfect information. *In these simulations, we have set* $\rho_\gamma = \rho_\omega = 0.9$ *and* $\theta = 0.019$. *Other parameters are as given in tables (1) and (2).*

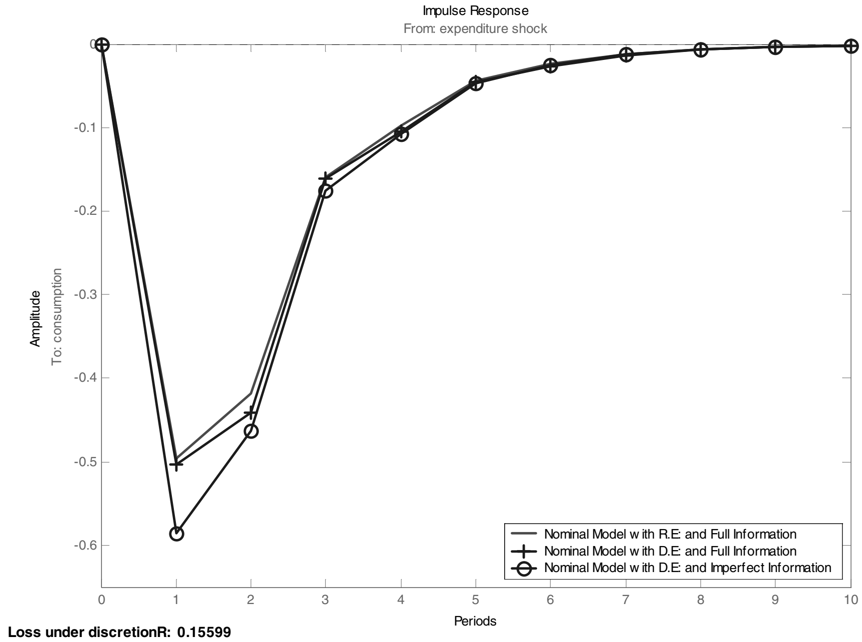

Figure 6: Consumption patterns after unitary shock to government expenditure under different informational assumptions.

responses of private consumption to temporary unitary shocks to government expenditure under various informational assumptions. Interestingly, when the agents are concerned with modelling errors, the 'precautionary' savings motive is magnified by the imperfect information associated with the measurements of the observables. This is due to the combination of under-estimation of dynamic responses of the interest rate to output and imperfect measurement of output and inflation. Imperfect information makes the estimated response of the interest rate to deviations of output and inflation from their targets even smaller than under full information. This triggers larger consumption reactions to expenditure shocks[20].

6 Conclusions

Uncertainty on the transition laws of the economy, as well as imperfect information are important factors affecting the decision problems of rational economic agents. This paper has illustrated how agents may achieve robustness by distorting their expectations in a particular direction and how these robust expectations then alter the observed dynamic paths of both exogenous and non-predetermined variables. It has been shown how the standard method of computing a linear, rational expectations equilibrium under discretion can be applied in order to solve robust expectation problems and how the standard

[20] It can be shown through simulations that if households are sure about the policy rule, the reaction of consumption to expenditure shocks is typically smaller than in the nominal model.

Kalman filter provides a way of solving a complicated signal extraction problem under imperfect information. The approach is novel in that we extend the partial information framework of Pearlman *et al* (1986) and Svensson and Woodford (2002) with model uncertainty and robustness.

For the purpose of illustrating how the model uncertainty and imperfect information interact in the dynamic forward-looking models, we have analysed the standard New-Keynesian model, closed with the Taylor type of interest rate rule. By concentrating solely on the robust expectation problem, we have been able to analyse the model under policy uncertainty as well as under uncertainty regarding dynamic properties of exogenous shocks.

In particular, we have shown that additional concern for robustness leads consumers to engage in a form of precautionary saving, which shows up in our setting as a stronger dynamic response of consumption to government expenditure shocks. This is perfectly consistent with the fact that agents achieve robustness by *over-estimating* the persistence of the shocks in a forward-looking context, as suggested by Kasa (2001). In addition, we have shown that when the policy rule is uncertain, agents achieve robustness against this by *under-estimating* the responses of the interest rate to the output gap. This under-estimation, combined with imperfect measurement of output and inflation, leads to larger initial responses of private consumption to government expenditure shocks under robust expectations.

Finally, imperfect information combined with concern for robustness brings a predictable component into the forecast errors of the exogenous variables, even if the standard Kalman filter is used for information updating. This is simply due to the fact that estimates under concern for robustness are based on the distorted transition laws of the model.

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A Appendix Derivation of G and a system with predetermined variables only

Following closely Svensson and Woodford (2002), but abstracting from the control problem, the decision problem of private agents can be expressed as

$$x_{t+1|t+1} = G_{t+1} z_{t+1|t+1} \tag{A.1}$$

where G_{t+1} is determined by the decision problem in period $t+1$. By using (A.1), taking expectations in period t of the upper block of (2.14) and using $A \equiv A^1 + A^2$, we first obtain that

$$x_{t+1|t} = G_{t+1} z_{t+1|t} = G_{t+1}(A_{11} z_{t|t} + A_{12} x_{t|t}) \tag{A.2}$$

Taking the expectation in period t of the lower block of (2.14) in the main text, yields also

$$x_{t+1|t} = A_{21} z_{t|t} + A_{22} x_{t|t} \tag{A.3}$$

Setting (A.2) and (A.3) equal to each other, then, yields

$$G^r_{t+1}(A_{11} z_{t|t} + A_{12} x_{t|t}) = A_{21} z_{t|t} + A_{22} x_{t|t} \tag{A.4}$$

$$\Rightarrow$$

$$x_{t|t} = \widetilde{A}_t z_{t|t} \tag{A.5}$$

and where

$$\widetilde{A}_t = \left(G^r_{t+1} A_{12} - A_{22}\right)^{-1} \left(A_{21} - G^r_{t+1} A_{11}\right) \tag{A.6}$$

Stable solution is found by comparing (A.6) and (A.1) and realising that G is a stable solution of a recursion

$$G_t = \left(G_{t+1} A_{12} - A_{22}\right)^{-1} \left(A_{21} - G_{t+1} A_{11}\right) \tag{A.7}$$

Naturally, this solution requires that $(G_{t+1}A_{12} - A_{22})$ is invertible. This method can be shown to be equivalent to the method of Blanchard and Kahn (1981). In particular, G_* which solves the recursion (A.7) is equivalent to Blanchard and Kahn (1981) solution $-M_{21}^{-1} M_{21}$, where M satisfies

$$MA = \lambda(A) M \tag{A.8}$$

and where $\lambda(A)$ is a diagonal matrix with eigenvalues of A on its diagonal and where M is a matrix with appropriate dimenssions.

Next, in order to recast the whole system in terms of only the backward-looking variables, notice first that the estimates of the future values of the predetermined variables can be expressed as a function of its current estimates. In other words,

$$z_{t+1|t} = A_t^* z_{t|t} \tag{A.9}$$

where

$$A_t^* \equiv A_{11} + A_{12} G_t \tag{A.10}$$

In the stable solution $G_t = G_{t+1} = G_*$, and therefore

$$z_{t+1|t} = \left(A_{11} + A_{12} G_*\right) z_{t|t} \tag{A.11}$$

where G_* is a solution to (2.5). Furthermore, taking expectations of the lower block of (2.1) at time t, subtracting it from itself and setting the difference to zero yields

$$A_{21}^1 \left(z_{t|t} - z_t\right) + A_{22}^1 (x_{t|t} - x_t) = 0$$

$$x_t - x_{t|t} = A_{22}^{-1} A_{21}^1 \left(z_{t|t} - z_t\right) \tag{A.12}$$

$$\Rightarrow$$

$$x_t = \left(T + G_*\right) z_{t|t} - T z_t \tag{A.13}$$

where $T \equiv \left(A_{22}^1\right)^{-1} A_{21}^1$ and where we have already utilized (2.4). The forward-looking variable x_t is now written in terms of only the observations of the predetermined variables z_t , while acknowledging the circularity of estimating the forward-looking variables from possibly unobserved current states. Finally, using again the upper block of (2.1) , we have a dynamic system with three linearly related equations

$$z_{t+1} = A_{11}^1 z_t + A_{12}^1 x_t + A_{11}^2 z_{t|t} + A_{12}^2 x_{t|t} + v_{t+1} \tag{A.14}$$

$$x_t = \left(T + G_*\right) z_{t|t} - T z_t \tag{A.15}$$

$$x_{t|t} = G_* z_{t|t} \tag{A.16}$$

Substituting (A.15) and (A.16) for (A.14) and re-organizing, we can write the dynamics of the pre-determined variables as

$$z_{t+1} = H z_t + J z_{t|t} + B_1 v_{t+1}$$

where

$$H = A_{11}^1 - A_{12}^1 T \tag{A.17}$$

$$J = A_{12}^1 \left(T + G_*\right) + A_{11}^2 + A_{12}^2 G_* \tag{A.18}$$

Similarly, recall that the observation equation can be partitioned as

$$O_t = D_1^1 z_t + D_2^1 x_t + D_1^2 z_{t|t} + D_2^2 x_{t|t} + B_2 \eta_t \tag{A.19}$$

Consequently, using (A.15) and (A.16) we obtain observation equation in terms of only the predetermined variables

$$O_t = L z_t + M z_{t|t} + B_2 \eta_t \tag{A.20}$$

where

$$L \equiv D_1^1 - D_2^1 T \tag{A.21}$$
$$M \equiv D_2^1 (T + G_*) + D_1^2 + D_2^2 G_* \tag{A.22}$$

and where $D^j = [D_1^j \; D_2^j]$ is decomposed according to predetermined variables z_t and forward looking variables x_t. Due to the separation principle, G, H, and J are unaffected by the vector of observables. In summary, we have recasted the system with forward looking variables into that of the predetermined variables only

$$\begin{aligned} z_{t+1} &= Hz_t + Jz_{t|t} + B_1 v_{t+1} \\ O_t &= Lz_t + Mz_{t|t} + \eta_t \end{aligned} \tag{A.23}$$

In order to complete the system, $z_{t|t}$ can be shown to be given by the Kalman filter (See the main text for brief discussion and Svensson and Woodford (2002) for detailed derivations)

$$z_{t|t} = z_{t|t-1} + K(O_t - Lz_{t|t-1} - Mz_{t|t}) \tag{A.24}$$

B Appendix
Solving robust decision problem

In this section, we derive a solution to the robust decision problem, using a recursive formulation of the problem and appropriate Bellman equations. The solution is derived under the assumption that an evil agent acts in a discretionary manner, taking expectations as given. Proceeding as earlier, we start by using (2.14), taking expectations in period t of the upper block of (2.14) and using $A \equiv A^1 + A^2$,

$$x_{t+1|t} = G^r_{t+1}(A_{11}z_{t|t} + A_{12}x_{t|t} + B_1 w_{t+1|t}) \tag{B.1}$$

Taking expectations in period t of the lower block of (2.14), yields also

$$x_{t+1|t} = A_{21}z_{t|t} + A_{22}x_{t|t} \tag{B.2}$$

Setting the above equations equal to each other, then, yields

$$G^r_{t+1}(A_{11}z_{t|t} + A_{12}x_{t|t} + B_1 w_{t+1|t}) = A_{21}z_{t|t} + A_{22}x_{t|t} \tag{B.3}$$
$$\Rightarrow$$

$$x_{t|t} = \widetilde{A}_t z_{t|t} + \widetilde{B}_t w_{t+1|t} \tag{B.4}$$

and where

$$\widetilde{A}_t = \left(G^r_{t+1}A_{12} - A_{22}\right)^{-1}\left(A_{21} - G^r_{t+1}A_{11}\right) \tag{B.5}$$
$$\widetilde{B}_t = \left(A_{22} - G^r_{t+1}A_{12}\right)^{-1} G^r_{t+1}B_1 \tag{B.6}$$

Using (B.4) in the expectation of the upper block of (2.14) gives, furthermore,

$$z_{t+1|t} = A^*_t z_{t|t} + B^*_t w_{t+1|t} \tag{B.7}$$

where

$$A^*_t \equiv A_{11} + A_{12}\widetilde{A}_t \tag{B.8}$$
$$B^*_t \equiv B_1 + A_{12}\widetilde{B}_t \tag{B.9}$$

An important insight one can draw from this is that the one-step ahead prediction of a predetermined variable z_{t+1} now depends upon the perceived laws of motion of the deterministic error term, ie $w_{t+1|t}$. In the absence of $w_{t+1|t}$, $B^*_t = 0$, and consequently, the R.E solution would collapse to (A.7)

Next we need to solve for $w_{t+1|t}$ and simultaneously force expectations to be consistent with these misspecifications. The logic is therefore exactly the same as when there is an additional decision-maker affecting the paths of the state variables! The expectation operator is then always applied to the distorted laws of motion of the economy. In the context of robust control, agents seek to achieve robustness against these distortions by letting evil nature pick $w_{t+1|t}$. Therefore, let

$$w_{t+1|t+1} = F_{t+1}z_{t+1|t+1} \tag{B.10}$$
$$x_{t+1|t+1} = G_{t+1}z_{t+1|t+1} \tag{B.11}$$

where F_{t+1} and G_{t+1} need to be determined simultaneously. The standard procedure in the solution strategy is to assign the private agents a quadratic loss function, and assume that the agents seek to achieve robustness by looking for a conservative way of evaluating the continuation value function. In other words, agents let a fictious player pick F_{t+1}, such that it maximises the assigned loss criteria of the private agents. This is essentially the procedure used by Hansen and Sargent (2003).

Assuming now that the loss function of the private agents (L_t) is quadratic and noticing that the constraints are linear, the optimal value of the problem will also be quadratic. In period $t+1$, the optimal value will depend on the estimate of the predetermined variable $z_{t+1|t+1}$ and therefore it can be written as $z'_{t+1|t+1}V_{t+1}z_{t+1|t+1}$, where V_{t+1} is a positive semidefinite matrix and d_{t+1} is scalar. The associated Bellman equation can therefore be written as

$$z'_{t|t}V_t z_{t|t} + d_t \equiv \max_{\{w_{t+1|t}\}} \left\{ L_{t|t} + \beta E[z'_{t+1|t+1}V_{t+1}z_{t+1|t+1}] + d_{t+1}|I_t \right\} \quad \text{(B.12)}$$

and where $L_{t|t}$ denotes the expected value of the quadratic loss criteria of the private agents and d_t is scalar. The loss function can be written conveniently by first defining a vector of target variables as

$$Y_t = C_1 \begin{pmatrix} z_t \\ x_t \end{pmatrix} + C_2 \begin{pmatrix} z_{t|t} \\ x_{t|t} \end{pmatrix} + C_w w_{t+1} \quad \text{(B.13)}$$

where w_{t+1} enters as nature's control and where C_1, C_2 and C_w are the selector matrices with appropriate dimensions. Let the loss function be a quadratic form of Y_t defined as $L_t = Y_t'QY_t$.Then, using the fact that, in general,

$$E(L_t) = E(Y_t'WY_t) = Y'_{t|t}WY_{t|t} + tr(E[(Y_t - Y_{t|t})'W(Y_t - Y_{t|t})]) \quad \text{(B.14)}$$

an expected value of the periodic loss function can be expressed as

$$L_{t|t} = \begin{pmatrix} z_{t|t} \\ x_{t|t} \end{pmatrix}' Q^* \begin{pmatrix} z_{t|t} \\ x_{t|t} \end{pmatrix} + 2 \begin{pmatrix} z_{t|t} \\ x_{t|t} \end{pmatrix}' Uw_{t+1|t} + w'_{t+1|t}Rw_{t+1|t} + l_{t|t} \quad \text{(B.15)}$$

where

$$C \equiv C_1 + C_2 \quad \text{(B.16)}$$
$$Q = C'WC \quad \text{(B.17)}$$
$$U = C'WC_w \quad \text{(B.18)}$$
$$R = -C'_w\beta\theta IC_w \quad \text{(B.19)}$$

and

$$l_{t|t} \equiv E\left\{ \begin{pmatrix} z_t - z_{t|t} \\ x_t - x_{t|t} \end{pmatrix}' C_1'WC_1 \begin{pmatrix} z_t - z_{t|t} \\ x_t - x_{t|t} \end{pmatrix}' \right\} \quad \text{(B.20)}$$

θ in (B.19) is the Lagrange multiplier of evil nature's budget constraint, as explained in more detail in Hansen and Sargent (2004). θ determines the set of models available for fictious player (nature). A very low θ allows nature to entertain with large misspecification. A very high θ in turn forces nature to choose very small misspecifications. The problem is now in fact in the form of a standard linear quadratic control problem with the control vector $w_{t+1|t}$. In order for decision-making and estimation to be separable, it must be that $l_{t|t}$ is independent of $w_{t+1|t}$. This is verified in Svensson and Woodford (2003) under a standard linear quadratic control problem with symmetric information.

Consequently, we can next proceed in the standard way by eliminating the forward-looking variables from the loss function by using the result

$$x_{t|t} = \widetilde{A}_t z_{t|t} + \widetilde{B}_t w_{t+1|t}. \tag{B.21}$$

as derived earlier. Without loss of generality[21], we restrict further derivations to the case where $U = 0$, which implies that there are no cross terms between control and state variables. The loss function then expands to

$$\begin{aligned}
L_{t|t} &= z'_{t|t}Q_{11}z + z'_{t|t}Q_{12}(\widetilde{A}_t z_{t|t} + \widetilde{B}_t w_{t+1|t}) + (\widetilde{A}_t z_{t|t} + \widetilde{B}_t w_{t+1|t})' Q_{21} z_{t|t} \\
&+ \left(\widetilde{A}_t z_{t|t} + \widetilde{B}_t w_{t+1|t}\right)' Q_{22} \left(\widetilde{A}_t z_{t|t} + \widetilde{B}_t w_{t+1|t}\right) \\
&= z'_{t|t}Q_{11}z_{t|t} + z'_{t|t}Q_{12}\widetilde{A}_t z_{t|t} + z'_{t|t}Q_{12}\widetilde{B}_t w_{t+1|t} + z_{t|t}\widetilde{A}'_t Q_{21} z_{t|t} \\
&+ w_{t+1|t}\widetilde{B}'_t Q_{21} z_{t|t} + z_{t|t}\widetilde{A}'_t Q_{22}\widetilde{A}_t z_{t|t} + z_{t|t}\widetilde{A}'_t Q_{22}\widetilde{B}_t w_{t+1|t} \\
&+ w_{t+1|t}\widetilde{B}'_t Q_{22}\widetilde{A}_t z_{t|t} + w_{t+1|t}\widetilde{B}'_t Q_{22}\widetilde{B}_t w_{t+1|t} - w_{t+1|t}R^* w_{t+1|t}
\end{aligned}$$

where Q_t^* has been partitioned accordingly. Combining the terms, we can derive somewhat easier expression[22]

$$L_{t|t} = z'_{t|t}Q_t^* z_{t|t} + z'_{t|t}U_{t1}^* w_{t+1|t} + w_{t+1|t}U_{t2}^* z_{t|t} + w'_{t+1|t}R_t^* w_{t+1|t} \tag{B.22}$$

where

$$\begin{aligned}
Q_t^* &\equiv Q_{11} + Q_{12}\widetilde{A}_t + \widetilde{A}'_t Q_{21} + \widetilde{A}'_t Q_{22}\widetilde{A}_t \\
U_{t1}^* &\equiv Q_{12}\widetilde{B}_t + \widetilde{A}'_t Q_{22}\widetilde{B}_t \\
U_{t2}^* &\equiv \widetilde{B}'_t Q_{21} + \widetilde{B}'_t Q_{22}\widetilde{A}_t \\
R_t^* &\equiv \widetilde{B}'_t Q_{22}\widetilde{B}_t - R^*
\end{aligned}$$

The problem is now transformed into a linear quadratic form without forward-looking variables, but with time-varying parameters. Time-varying parameters entered into the loss function upon substitution of forward-looking variables with the linear combination of evil nature's decision variable and the predetermined variables. In the absence of forward-looking variables, then, we

[21] See, for instance, Ljungqvist and Sargent (2000), Ch.21.

[22] For the purpose of computations, it useful to notice that the loss function can be expressed in a convenient form $L_{t|t} = \begin{pmatrix} z_{t|t} \\ x_{t|t} \\ w_{t+1|t} \end{pmatrix}' \begin{pmatrix} Q & 0 \\ 0 & R \end{pmatrix} \begin{pmatrix} z_{t|t} \\ x_{t|t} \\ w_{t+1|t} \end{pmatrix}$.

can proceed in a standard way of solving the first-order conditions with respect to $w_{t+1|t}$ from (B.12), yielding

$$\begin{aligned} 0 &= z'_{t|t}U^*_{t1} + U^*_{t2}z_{t|t} + w'_{t+1|t}R^*_t + \beta E(z'_{t+1|t+1}V_{t+1}B^*_t|I_t) \\ &= z'_{t|t}U^*_{t1} + U^*_{t2}z_{t|t} + w'_{t+1|t}R^*_t + \beta E(z'_{t|t}A^{*\prime}_t + w'_{t+1|t}B'^*_t)V_{t+1}B^*_t) \end{aligned}$$

Next, using $w_{t+1|t+1} = F_{t+1}z_{t+1|t+1}$, we find that

$$\begin{aligned} 0 &= z'_{t|t}U^*_{t1} + U^*_{t2}z_{t|t} + z'_{t|t}F_tR^*_t + \beta E(z'_{t|t}A^{*\prime}_t + z'_{t|t}F_tB'^*_t)V_{t+1}B^*_t) \\ &= z'_{t|t}F_t(R^*_t + \beta B'^*_tV_{t+1}) + z'_{t|t}U^*_{t1} + U^*_{t2}z_{t|t} + \beta z'_{t|t}A^{*\prime}_tV_{t+1}B^*_t \\ &\Rightarrow \end{aligned}$$

$$F_t \equiv -(R^*_t + \beta B'^*_tV_{t+1}B^*_t)^{-1}(U^*_{t1} + U^{*\prime}_{t2} + \beta A^{*\prime}_tV_{t+1}B^*_t) \tag{B.23}$$

Recalling once more that $x_{t|t} = \widetilde{A}_tz_{t|t} + \widetilde{B}_tw_{t+1|t}$ we also have that

$$x_{t|t} = \left(\widetilde{A}_t + \widetilde{B}_tF_t\right)z_{t|t} \tag{B.24}$$

and so,

$$G_t \equiv \widetilde{A}_t + \widetilde{B}_tF_t \tag{B.25}$$

and where $\widetilde{A}_t$ and $\widetilde{B}_t$ are given by (B.5)–(B.6).

Furthermore, it is relatively easy to see that the value function has a recursive representation

$$V_t \equiv Q^*_t + U^*_{t1}F_t + F'_tU^*_{t2} + F'_tR^*_tF_t + \beta[(A^*_t + B^*_tF_t)'\,V_{t+1}\,(A^*_t + B^*_tF_t)] + d_{t+1}|I_t \tag{B.26}$$

Finally, equations (B.23)–(B.26) define a contraction mapping from

$$(F_{t+1}, G_{t+1}, V_{t+1}) \to (F_t, G_t, V_t)$$

and the solution to this problem is a fixed point (F, G, V) of the mapping. Consequently, the algorithm that solves an ordinary linear quadratic program can be straightforwardly applied to solve associated rational expectations equilibria in the linear quadratic economy where the agents hedge against unstructured model uncertainties. Specifically, it only requires us to specify an appropriate loss function for the private sector and the decision variable of nature.

After a solution to G^r has been found, for instance, by iterating (B.23)–(B.26) until convergence, the distorted model can be recast in terms of the predetermined variables as follows. First, using $x_{t|t} = G^rz_{t|t}$ we obtain

$$\begin{aligned} A^1_{21}\left(z_{t|t} - z_t\right) + A^1_{22}(G^rz_{t|t} - x_t) = 0 \\ \Rightarrow \\ x_t = \left(T + G^r\right)z_{t|t} - Tz_t \end{aligned} \tag{B.27}$$

where $T \equiv \left(A_{22}^1\right)^{-1} A_{21}^1$.Consequently, we have the following system of equations for the perceived worst-case law of motion of the private agents

$$z_{t+1} = A_{11}^1 z_t + A_{12}^1 x_t + A_{11}^2 z_{t|t} + A_{12}^2 x_{t|t} + B_1(v_{t+1} + w_{t+1}) \tag{B.28}$$
$$x_t = (T + G^r)\, z_{t|t} - T z_t \tag{B.29}$$
$$x_{t|t} = G^r z_{t|t} \tag{B.30}$$
$$w_{t+1|t} = F z_{t|t} \tag{B.31}$$

The matrices A_{ij} ,and T, B_1 reflect the values of the nominal model. G^r characterizes the private agent's expectations that are consistent with the perceived laws of motion of the error term $w_{t+1|t} = F z_{t|t}$. After some straightforward substitutions, we can write down the perceived worst-case laws of motion of the economy as

$$\begin{aligned} z_{t+1} &= A_{11}^1 z_t + A_{12}^1 x_t + A_{11}^2 z_{t|t} + A_{12}^2 x_{t|t} + B_1 w_{t+1} \\ &= \left(A_{11}^1 - A_{12}^1 \left(A_{22}^1\right)^{-1} A_{21}^1\right) z_t \\ &+ \left(A_{11}^2 + A_{12}^2 G^r + A_{12}^1 \left(\left(A_{22}^1\right)^{-1} A_{21}^1 + G^r\right)\right) z_{t|t} + B_1(w_{t+1} + v_{t+1}) \\ &= \left(A_{11}^1 - A_{12}^1 T\right) z_t + \left(A_{11}^2 + A_{12}^2 G^r + A_{12}^1 (T + G^r) + B_1 F_*\right) z_{t|t} + B_1 v_{t+1} \end{aligned}$$

The laws of motion for the predetermined variables that are consistent with distorted expectations and the misspecified model can be presented then in a compact form

$$z_{t+1} = H^r z_t + J^r z_{t|t} + B_1 v_{t+1} \tag{B.32}$$
$$O_t = L^r z_t + M^r z_{t|t} + B_2 \eta_t \tag{B.33}$$

where

$$\begin{aligned} H^r &= H \equiv \left(A_{11}^1 - A_{12}^1 T\right) \\ J^r &\equiv A_{11}^2 + A_{12}^2 G^r + A_{12}^1 (T + G^r) + B_1 F \\ L^r &= L \equiv D_1^1 - D_2^1 T \\ M^r &\equiv D_2^1 (T + G^r) + D_1^2 + D_2^2 G^r \end{aligned}$$

as in the main text.

C Appendix
State space representation of the model

Using the equations above, and notation elsewhere in this paper, the state space representation of the New Keyenesian mode can be written down as:

$$\begin{bmatrix} 1 \\ i_t \\ \omega_{t+1} \\ \gamma_{t+1} \\ y_{t+1|t} \\ \pi_{t+1|t} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \delta_1 & \delta_2 \\ 0 & 0 & \rho_\omega & 0 & 0 & 0 \\ \gamma^*(1-\rho_\gamma) & 0 & 0 & \rho_\gamma & 0 & 0 \\ \gamma^*(1-\rho_\gamma) & \varphi & -\frac{\lambda\varphi}{\beta} & (\rho_\gamma - 1) & \frac{(1+\lambda)\varphi}{\beta} & -\frac{\varphi}{\beta} \\ 0 & 0 & \frac{\lambda}{\beta} & 0 & -\frac{\lambda}{\beta} & \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} 1 \\ i_t \\ \omega_t \\ \gamma_t \\ y_t \\ \pi_t \end{bmatrix} +}_{A^1}$$

$$\begin{bmatrix} B_1 \\ \mathbf{0}_{(2\times3)} \end{bmatrix} \left(\begin{bmatrix} w_{i,t+1} \\ w_{\omega,t+1} \\ w_{\gamma,t+1} \end{bmatrix} + \begin{bmatrix} v_{i,t+1} \\ v_{\omega,t+1} \\ v_{\gamma,t+1} \end{bmatrix} \right)$$

$$\text{where } B_1 = \begin{bmatrix} 0 & 0 & 0 \\ \sigma_i & 0 & 0 \\ 0 & \sigma_\omega & 0 \\ 0 & 0 & \sigma_\gamma \end{bmatrix}.$$

$v_{.,t+1}$ represents fundamental shocks to the economy, while $w_{(.),t+1}$ represents distortions to the transition law. Matrix B_1 encodes precisely how these distortions influence the model's predetermined variables. The vector of the target variables is given as

$$Y_t \equiv \begin{bmatrix} \pi_t - \pi^* \\ y_t - \gamma_t - x^* \end{bmatrix} = \underbrace{\begin{bmatrix} -\pi^* & 0 & 0 & 0 & 0 & 1 \\ x^* & 0 & 0 & -1 & 1 & 0 \end{bmatrix}}_{C^1} \begin{bmatrix} 1 \\ i_t \\ \omega_t \\ \gamma_t \\ y_t \\ \pi_t \end{bmatrix} + \underbrace{\begin{bmatrix} \mathbf{0}_2 \\ I_{(3)} \end{bmatrix}}_{C_w} \mathbf{w}_{t+1}$$

where vector $\mathbf{w}_{t+1}$ enters as nature's decision variables.

Finally, the setup is closed by specifying an equation for observables

$$O_t \equiv \begin{bmatrix} i_{t,o} \\ \gamma_{t,o} \\ y_{t,o} \\ \pi_{t,o} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{D^1} \begin{bmatrix} 1 \\ i_t \\ \omega_t \\ \gamma_t \\ y_t \\ \pi_t \end{bmatrix} + B_2 \begin{bmatrix} \eta_{i,t} \\ \eta_{\gamma,t} \\ \eta_{y,t} \\ \eta_{\pi,t} \end{bmatrix}$$

$$\text{where } B_2 = \begin{bmatrix} \sigma_{\eta i} & 0 & 0 & 0 \\ 0 & \sigma_{\eta\gamma} & 0 & 0 \\ 0 & 0 & \sigma_{\eta y} & 0 \\ 0 & 0 & 0 & \sigma_{\eta\pi} \end{bmatrix}.$$

Elements of the B_2 matrix ($\sigma_{\eta.}$) correspond to the standard errors of the measurements. In the full information case, these are set numerically very close to zero ($1e-8$). In the imperfect information case, these entries are set as different from zero. Finally, $A^2 = C^2 = D^2 = 0$ with appropriate dimension.

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