EXAMINING AND MODELING THE DYNAMICS
OF THE VOLATILITY SURFACE
- AN EMPIRICAL STUDY OF THE DAX AND ESX OPTIONS MARKET

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Key words: Implied Volatility, Volatility Smiles and Surfaces, Implied Volatility Function Models, the Term-Structure of Implied Volatility

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Examining and Modeling the Dynamics of the Volatility Surface
- An empirical study of the DAX and ESX options market

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Abstract

The objective of this paper is to investigate and model the characteristics of the prevailing volatility smiles and surfaces on the DAX- and ESX-index options markets. Continuing on the trend of Implied Volatility Functions, the Standardized Log-Moneyness model is introduced and fitted to historical data. The model replaces the constant volatility parameter of the Black & Scholes pricing model with a matrix of volatilities with respect to moneyness and maturity and is tested out-of-sample. Considering the dynamics, the results show support for the hypotheses put forward in this study, implying that the smile increases in magnitude when maturity and ATM volatility decreases and that there is a negative/positive correlation between a change in the underlying asset/time to maturity and implied ATM volatility. Further, the Standardized Log-Moneyness model indicates an improvement to pricing accuracy compared to previous Implied Volatility Function models, however indicating that the parameters of the models are to be re-estimated continuously for the models to fully capture the changing dynamics of the volatility smiles.

Keywords: Implied Volatility, Volatility Smiles and Surfaces, Implied Volatility Function Models, the Term-Structure of Implied Volatility

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1 Introduction

The failure of the original Black & Scholes (1973) model (BSM) to correctly price options in- and out-of-the-money is one of the most covered topics in modern finance literature. This bias is often referred to as the volatility smile, due to its graphical form over strike prices, $X$, meaning that out-of-the-money puts and calls are traded at higher volatilities than at-the-money (ATM) options. In practice the volatility is different with regard to its temporal component, $t$, as well, implying that the volatility $\sigma$ is a function of the level of the underlying asset, $S$, strikes prices, $X$ and maturity $t$, that is $\sigma(S, X, t)$. The combined phenomenon is denoted as the volatility surface.

The reason for the existence of the smile is subject to several opinions and there is no simple answer. However, the smile seemed to appear after the stock market crash in 1987 and is therefore believed to depend at least on the theory of supply and demand, as the market is willing to pay a premium for the downside protection generated by out-of-the-money puts. This crash-o-phobia, as referred to by Rubinstein (1994), implies that the market has a different view on the distribution of the underlying asset than assumed by the BSM. More specifically, returns $\varphi(h)$ over any period of length $h$ are not normally distributed with mean and variance that are linear in $h, \varphi(h) \sim N(\mu, \sigma^2 h)$, as presumed by the Geometric Brownian Motion which the BSM is based on.

It is a fact that the assumption of normality in returns for the underlying asset, which is a prerequisite for the assumption of constant volatilities across different strike prices, is continuously violated on most modern markets. As Bates (1996) and others have observed, the existence of the smile is strongly suggestive of the presence of leptokurtosis (positive excess kurtosis) in the conditional return distribution. As noted by Das & Sundaram (1997), this kurtosis is believed to be the cause of the smile, while the presence of skewness is argued to cause its typical asymmetric attributes (out-of-the-money calls and puts do often not imply the same patterns).

Over the past two decades, finance literature has been flooded by models and attempts to cover the characteristics of these smiles. In 1982 Jarrow & Rudd presented a model that builds on the deviations from normality in the conditional returns distributions which was later tested against market prices by Corrado & Su (1997). The usage of lattice models is explored in Derman & Kani (1994), Dupire (1994) and
Rubinstein (1994), while the impact of liquidity and transaction costs on the volatility smile is documented for example in Dumas et al. (1996) and in Longstaff (1995). Further Hull & White (1987), Heston (1993) and Bates (1996) propose stochastic volatility models as Gemmill (1996), Peña et al. (1999), Duque & Lopes (1994) all contribute to this topic employing a few different approaches. Additionally, for example Jackson et al. (1998), Avellaneda et al. (1997) and Lagnado & Osher (1997) attempt to cover the volatility surface using optimization techniques.

For European options, replacing the constant volatility parameter, the scalar, of the BSM with a matrix of volatilities regarding strikes and maturities, efficiently captures the smile effect. Hence a model is needed to establish the relationship between the volatility and the respective strike and maturity. These models are often referred to as Implied Volatility Function Models, where the relationship between option and exercise prices is estimated by fitting a curve to market prices as documented in for example Rosenberg (2000). Other attempts to model this relationship can be found in Derman (1999) and Sundkvist (2000) that both focus on the relationship between the delta and smile components. The model presented in Sundkvist (2000) is also utilized in Essay 3.

This study aims not only at examining the existing volatility smiles and surfaces on the DAX-and ESX-index options markets, but also introduces the Standardized Log-Moneyness model as a contribution to the genre of Implied Volatility Functions. The model replicates the smile by fitting a curve to market prices, where each option is provided with an individual volatility as a function of the ATM volatility. The study is structured as follows. The methodology and notations are introduced in Chapter 2, the data in Chapter 3 and the empirical findings regarding the dynamics of the smiles in Chapter 4. The models for the surface are introduced in Chapter 5 and tested in Chapter 6, while the results and conclusions presented herein are summarized in Chapter 7.

2 Methodology, Notations and the Hypotheses

The purpose of this study is to examine the dynamics of the existing volatility smile/surface on the German DAX-index and the pan-European ESX-index options market and to construct a model that as accurately as possible replicates these.
Moreover, the possible similarities and differences between these two markets are discussed briefly. To accomplish this, a few notations are to be introduced.

Let $\xi(\delta)$ denote the observed implied volatility for an option with moneyness $\delta$, where $\delta = \ln(X / S)$, $S$ is the level of the underlying asset and $X$ is the strike price. Further, let $\xi^0$ denote the implied ATM volatility and let $VI_{\delta} = \xi(\delta) / \xi^0$ represent the volatility index for an option with moneyness $\delta$. This implies that the index is equal to 1 for the ATM option and larger than 1 for in-the-money calls, and vice versa.\(^3\)

For notational simplicity let $\Sigma$ represent the volatility smile, that is $\Sigma$ represents the complete set of observed implied volatilities so that $\Sigma = \{\xi(max), \ldots, \xi(min)\}$. Further, let $M(\Sigma^-)$ denote the magnitude of the ‘left-hand side’ of the smile ($X <$ ATM) and, accordingly, $M(\Sigma^+)$ the magnitude of the ‘right-hand side’ of the smile ($X >$ ATM) whilst $M(\Sigma)$ is to be perceived as the magnitude of the complete smile. Finally, let $\tau = T - t$, that is the time to maturity (days) for a specific option.

The volatility smile is documented to flatten, that is $M(\Sigma)$ decreases, as ATM-volatility and/or maturity increases and vice versa. This phenomenon is discussed for example in Duque & Lopes (1999) and Das & Sundaram (1999). Further, it is also argued by for example Cox & Ross (1976), Black (1976), Christie (1982), Koutmos (1996), Derman (1999) and Alexander (2000) that volatility increases when the market dives and, accordingly, that it decreases when the market moves upwards. The correlation between spot and volatility is negative. Hence, to establish the dynamics of the DAX/ESX-index smile, the following hypotheses are tested.

1. The markets are subject to a negative correlation between a change in the underlying asset and the observed implied ATM volatility, that is $\rho_{\delta \xi^0} < 0$.

2. The markets are subject to a positive correlation between time to maturity and the observed implied ATM volatility, that is $\rho_{\tau \xi^0} > 0$.

3. The magnitude of the smile decreases as implied ATM volatility increases, that is $\rho_{M(\Sigma) \xi^0} < 0$. 
The magnitude of the smile increases as time-to-maturity decreases, that is
\[ \rho_{M|\Sigma, \tau} < 0. \]

To test these hypotheses, the Ordinary Least Squares methodology is employed, meaning that for example hypothesis 1 is tested using

\[ \xi_{\tau}^0 = \phi + \beta S + \epsilon, \quad (1) \]

where the observations are daily returns. In this case, a significant negative beta-value is of course considered support for the hypothesis. The rest of the models are built in a similar fashion and presented in further detail below. This sort of model construction is also utilized in Duque & Lopes (1999) and Alexander (2000).

Following the work of Rosenberg (2000), the stability of the ATM-volatility \((\xi^0)\) is tested, suggesting that it depends on a constant \((\phi)\), its first lag to incorporate mean reversion, and the most recent asset return \((r)\) to incorporate asymmetric return effects, giving

\[ \xi_{\tau}^0 = \phi + \beta_{\xi_{\tau-1}}^0 + \gamma[\max(0,-r)] + \epsilon, \quad (2) \]

The formulation of the asymmetry term is related to that of Glosten et al. (1993) and Engle & Ng (1993), that find this specification to exhibit superior performance in objective volatility modeling. The empirical findings are documented below.

3 The Data

The data consist of daily observations on implied volatilities forming the volatility smile (surface) and the closing prices of the indices, collected from the two markets over a period ranging from 1.3 to 30.6.1999, unless not clearly stated otherwise. This relatively short time-period is motivated by the fact that the dynamics of the smile change over time, thus calling for continuous re-estimation of the parameters. This implies that a shorter time-period should enable more accurate model building. Restricting the moneyness \((\delta)\) of the observed option volatilities to ±15% and the covered maturities \((\tau)\) to 10 to 340 days, narrows the observed vectors of implied
volatilities down to a size of \((5,448 \times 1)\) for the DAX and \((1,819 \times 1)\) in the case of the ESX. These vectors should be well sufficient for the aims of this study.

The implied volatilities are iterated from call option settlement prices provided by the exchange (EUREX™) using the familiar BSM. The close price on the respective performance index, where the dividends are constantly re-invested, is used as the underlying asset and the three-month EURIBOR interest rate as a proxy for the risk-free rate. The compilation of data from different databases is expected to cause some synchronization problems especially in the case of the ESX where the performance index is not available on a continuous basis during trading. On the whole these problems should be minor, though, and are not expected to affect the aims of the study, since the methodology is in focus and the models are tested against each other on the same set of data.

4 Examining the Dynamics of the Smile

To test the hypotheses put forward in this study regarding the dynamics of the volatility smile, simple regression models are estimated in line with the methodology presented above. Equation (1) is for example used to determine whether a negative correlation between a change in the level of the underlying asset and the level of ATM implied volatility (hypothesis 1) exists. Further, to test hypotheses 3 and 4, regression models with the volatility index \((V_{i6})\) as the dependent variable and the implied ATM volatility \((\xi^0)\) and time to maturity \((\tau)\), respectively, as the independent variable, are estimated. The results and models are summarized in Table 1 and Table 2.

Support for hypothesis 1 is found in the correlation between the level of the underlying asset and the implied ATM volatility, which in fact turns out to be significantly negative. The results also indicate support for hypothesis 3 as the correlation between \((V_{1.5\%})\) and the implied ATM volatility \((\xi^0)\) is negative, implying that \(M(\Sigma^-)\) increases when volatility decreases whereas the positive correlation between \(V_{1.5\%}\) and \(\xi^0\) indicates that \(M(\Sigma^+)\) decreases as volatility decreases. Thus, the magnitude of the smile increases as the level of implied ATM volatility decreases.
The negative correlation exists. The results for the ESX are very much in line with those of the DAX.

### Table 1
**Modeling the interrelation between the smile and ATM volatility**
The monitored time period is 3.3 to 31.5.1999 and the covered maturities 10 to 340 days where $n$ represents the number of observations. The time-period for equation (1) is 3.3 to 31.12.1999 and the observed maturities 14 to 40 days, covering the options closest to maturity. The results are presented as DAX / ESX.

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>$VI_{-5%} = \phi + \beta \xi^0 + \varepsilon$</th>
<th>$\xi^0 = \phi + \beta S + \varepsilon$</th>
<th>$VI_{5%} = \phi + \beta \xi^0 + \varepsilon$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi$</td>
<td>1.353 / 1.246</td>
<td>0.006 / 0.005</td>
<td>0.768 / 0.821</td>
</tr>
<tr>
<td>$\beta$</td>
<td>-1.061 / -0.632</td>
<td>-3.131 / -2.905</td>
<td>0.632 / 0.447</td>
</tr>
<tr>
<td>$n$</td>
<td>210 / 61</td>
<td>209 / 209</td>
<td>188 / 81</td>
</tr>
<tr>
<td>$Adj. R^2$</td>
<td>0.294 / 0.048</td>
<td>0.341 / 0.380</td>
<td>0.459 / 0.104</td>
</tr>
</tbody>
</table>

All parameters are significant at the 1%-level.

Further, considering the results presented in Table 2, support for the respective hypotheses, 2 and 4 is also found. Although indicating small figures, the correlation between the time to maturity ($\tau$) and the level of implied ATM volatility is significantly positive, suggesting that hypothesis 2 holds. Further, the correlation between the time to maturity ($\tau$) and the volatility indices ($VI_{5\%}$) is negative implying that $M(\Sigma^-)$ increases when maturity decreases while the opposite correlation is again found regarding ($VI_{5\%}$) and $M(\Sigma^+)$. Thus, hypothesis 4 also seems to hold.

Similar results and support for the hypotheses are indicated for the ESX market, that again shows results very close to those of the DAX.

### Table 2
**Modeling the interrelation between the smile and maturity**
The monitored time period is 3.3 to 31.5.1999 and the observed maturities 10 to 340 days where $n$ represents the number of observations. The results are presented as DAX / ESX.

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>$VI_{-5%} = \phi + \beta \tau + \varepsilon$</th>
<th>$\xi^0 = \phi + \beta \tau + \varepsilon$</th>
<th>$VI_{5%} = \phi + \beta \tau + \varepsilon$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi$</td>
<td>1.109 / 1.099</td>
<td>0.250 / 0.239</td>
<td>0.915 / 0.933</td>
</tr>
<tr>
<td>$\beta$</td>
<td>-0.0003 / -0.0002</td>
<td>0.0001 / 0.0001</td>
<td>0.0001 / 0.0001</td>
</tr>
<tr>
<td>$n$</td>
<td>210 / 86</td>
<td>267 / 99</td>
<td>188 / 78</td>
</tr>
<tr>
<td>$Adj. R^2$</td>
<td>0.365 / 0.250</td>
<td>0.347 / 0.195</td>
<td>0.541 / 0.115</td>
</tr>
</tbody>
</table>

All parameters are significant at the 1%-level.
Finally, the stability of the ATM volatility is tested using the model in equation (2). As the results in Table 3 indicate, the level of the implied ATM volatility on the previous day is quite helpful when predicting the subsequent ATM volatility. The explanatory power is also quite strong for the model on both markets and all variables are statistically significant. It is also fairly obvious that negative returns significantly increase volatility, while adding a positive return component to the model failed to improve the accuracy of the model.

### Table 3

**Empirical results for the stability of the ATM volatility**

Testing the stability of the ATM volatility, estimated using equation (2) generated the following results, where $\phi$ represents the constant, $\beta$ the first lag and $\gamma$ the negative return. The data consists of observations on the implied ATM-volatility for a period ranging from 5.3.1999 to 31.1.2000 for maturities between 14 to 40 days, covering the options closest to maturity.

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>DAX</th>
<th>ESX</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi$</td>
<td>1.987</td>
<td>1.798</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.894</td>
<td>0.906</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>-113.860</td>
<td>-94.188</td>
</tr>
<tr>
<td>$n$</td>
<td>232</td>
<td>231</td>
</tr>
<tr>
<td>Adj. $R^2$</td>
<td>0.888</td>
<td>0.915</td>
</tr>
</tbody>
</table>

All parameters are significant at the 1%-level

### 5 Modeling the Surface –The Standardized Log-Moneyness Model

From an academic and mathematical point-of-view the problem statement and modeling of the volatility smile are naturally ill-posed, as the smile itself is not driven by any mathematical process. It is in fact impossible to determine whether the smile is an existing phenomenon or if it is a result of a model mis-specification, for example due to liquidity factors and the obvious deviations from normality in returns. However, the smile prevails and cannot be ignored.

As mentioned in the introduction, the smile effect is efficiently captured for European options by replacing the constant volatility parameter, the scalar, of the BSM with a matrix of volatilities regarding strikes and maturities. Hence a model is
needed to establish the relationship between the volatility and the respective strike and maturity. Attempts to model parts of these non-linear relationships can be found for example in Derman (1999) and Sundkvist (2000), that both focus on the relationship between the delta and smile components and in Rosenberg (2000) where different Implied Volatility Function models are tested. As these models successfully reconstruct the volatility smile with regard to the moneyness of options (the spatial component) in relation to the level of implied ATM volatility, they do not attempt to cover the temporal component, that is the term-structure of the implied volatility. Thus, the relationship between ATM volatilities over time is to be added.

First, combining the volatility-by-moneyness model,

\[ \sigma [\ln(X / S_t)] = \beta_0 + \beta_1 \ln(X / S_t) + \beta_2 \ln^2(X / S_t) \] (3)

as presented in Rosenberg (2000), with the standardization \( \sigma [\ln(X / S_t)/\sqrt{T-t}] \), suggested by Natenberg (1994), gives the Standardized Log-Moneyness (SLM) model as

\[ VI_{\delta} = \phi + \sum_{i=1}^{n} \beta_i \left( \frac{\delta}{\sqrt{\tau}} \right)^i + \varepsilon. \] (4)

Recall that \( \delta = \ln(X / S) \) and \( \tau = T - t \). Further, \( \phi \) is a constant expected to equal 1, that is the volatility of the ATM option. Higher orders are needed to account for the non-linearity of the volatility smile. The model is estimated using historical data and tested out-of-sample against market prices. This procedure is also applied to the Sticky-delta rule as introduced in Derman (1999) and refined into the Sticky Delta Skew Approximation (SDSA) model by Sundkvist (2000). The SDSA model is deduced as

\[ VI_{\delta} = \phi + \sum_{i=1}^{n} \beta_i (\Delta_{\delta} - \Delta_{ATM})^i + \varepsilon, \] (5)

where \( \Delta_{\delta} \) is the delta of the option with moneyness \( \delta \). The SDSA parameters are of course estimated using the same vector of data.
Adding the temporal dimension needs a model that determines the relationship between the level of the implied ATM volatility ($\xi^0$) and time to maturity ($\tau$). This study suggests that this relationship be modeled as

$$\ln\left(\frac{\xi^0_{long}}{\xi^0_{short}}\right) = \phi + \beta \ln\left(\frac{\tau_{long}}{\tau_{short}}\right) + \epsilon,$$

where long/short refers to the respective time to maturity. Provided that the volatility for the month closest to maturity is known, this model provides the volatility for longer maturities. Thus, solving for the parameter $\xi^0_{long}$ using standard algebra gives

$$\frac{\xi^0_{Long}}{\xi^0_{short}} = e^{\phi+\epsilon} \cdot \left(\frac{\tau_{long}}{\tau_{short}}\right)^\beta.$$

As $\phi$ is close to 0 (see Table 5) and if the noise term $\epsilon$ is dropped implying that $e^{\phi+\epsilon}$ equals 1, multiplying both sides with $\xi^0_{short}$ gives

$$\xi^0_{long} = \xi^0_{short} \cdot \left(\frac{\tau_{long}}{\tau_{short}}\right)^\beta.$$

Estimating the parameters for the Standardized Log-Moneyness model in equation (4) generated the results presented in Table 4, while the term structure model yielded the estimates as presented in Table 5. For the results regarding the SDSA-model, consult Appendix A. It is obvious when considering the adjusted $R^2$ statistics of Table 4 that both models perform best for longer maturities where the smile effect is small of magnitude. It also seems that the model is better suited for the DAX-index market as the $R^2$ statistics are well above those of the ESX-index market for all maturities. This is also the case when studying the results of the SDSA-model. Intuitively, it is difficult to explain this phenomenon, but one possible explanation is that the ESX-index surface might be of a bit more irregular shape due to less liquidity than the highly efficient DAX-index market.
Table 4

Estimating the parameters for the SLM

The beta estimates for the Standardized Log-Moneyness model in equation 4. The covered time period is 1.3 to 31.5.1999 and the maturities 10 to 340 days. The results are presented as DAX / ESX.

<table>
<thead>
<tr>
<th>Maturity (days)</th>
<th>10-100</th>
<th>100-340</th>
<th>10-340</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \phi )</td>
<td>1.00 / 1.00</td>
<td>1.00 / 1.00</td>
<td>1.00 / 1.00</td>
</tr>
<tr>
<td>( \beta_1 )</td>
<td>-10.68 / -8.99</td>
<td>-13.22 / -13.11</td>
<td>-12.03 / -11.13</td>
</tr>
<tr>
<td>( \beta_2 )</td>
<td>18.27 / 84.20</td>
<td>15.94 / -107.36</td>
<td>37.09 / 67.10</td>
</tr>
<tr>
<td>( \beta_3 )</td>
<td>1.75<em>10^3 / -1.43</em>10^3</td>
<td>- / 16.66*10^3</td>
<td>5.98<em>10^3 / 7.25</em>10^3</td>
</tr>
<tr>
<td>( \beta_4 )</td>
<td>2.34<em>10^5 / 3.02</em>10^5</td>
<td>- / -</td>
<td>2.05<em>10^5 / 3.18</em>10^5</td>
</tr>
<tr>
<td>( \beta_5 )</td>
<td>- / -</td>
<td>- / -</td>
<td>-2.60<em>10^6 / -6.55</em>10^6</td>
</tr>
<tr>
<td>( n )</td>
<td>2.833 / 795</td>
<td>2.615 / 1.024</td>
<td>5.448 / 1.819</td>
</tr>
<tr>
<td>Adj. ( R^2 )</td>
<td>0.915 / 0.877</td>
<td>0.983 / 0.947</td>
<td>0.925 / 0.889</td>
</tr>
</tbody>
</table>

All parameters are significant at the 1%-level.

Overall, the models prove satisfactory considering their ability to model the historical surfaces. However, as the adjusted \( R^2 \) statistic is higher for the SLM over all maturities (except 100-340 where results are almost identical) compared with the SDSA (Appendix A), the SLM framework is considered to better capture the dynamics of the historical smiles.

Table 5

Estimating the parameters for the term structure model

The beta estimate for equation (8). The covered time period is 1.3 to 31.5.1999 and the maturities 10 to 340 days. For each day the observed ATM volatilities and maturities are grouped into five groups according to their maturity (10-39, 40-67, ..., 213-340). The observations hence consist of the differences in volatility and maturity between these groups, giving 4 to 5 observations per day (group 5 – group 1, group 4 – group 1 and so on).

<table>
<thead>
<tr>
<th>Coefficient</th>
<th>DAX</th>
<th>ESX</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \phi )</td>
<td>0.002</td>
<td>0.002</td>
</tr>
<tr>
<td>( \beta )</td>
<td>0.059</td>
<td>0.065</td>
</tr>
<tr>
<td>( n )</td>
<td>530</td>
<td>229</td>
</tr>
<tr>
<td>Adj. ( R^2 )</td>
<td>0.886</td>
<td>0.199</td>
</tr>
</tbody>
</table>

All parameters are significant at the 1%-level.

To visually demonstrate the characteristics of the SLM, a hypothetical volatility surface is created using the estimated parameters (Figure 1).
Testing the model
A graphical illustration of a hypothetical volatility surface for DAX-index (left) and ESX-index (right) options using the estimated parameters, where the ATM volatility for the options closes to maturity is 25%.

Overall, the volatility surfaces indicated by the two index markets seem to be quite similar considering their dynamics, but as visualized in Figure 2, the ESX market implies a surface that is of greater magnitude than the smile of the DAX.

The difference between the volatility indices on the two markets
Based on the hypothetical surfaces generated using the estimates from tables 4 and 5, the differences between the historical surfaces on the two markets are highlighted. The figure shows the differences between the estimated volatility indices between the ESX and the DAX markets (calculated as $VI_{ESX}^\delta - VI_{DAX}^\delta$).
Testing the Models on Market Data

As historical data fitting gives no guarantee of future smile accuracy, the two models are tested out of sample for the period ranging from 1 to 30.6.1999. The SLM and SDSA frameworks are tested against each other on maturities ranging from 10 to 100 days. Here, the implied ATM volatility is known for each maturity and the models are used to estimate the corresponding volatility smiles. Further, the predictability of the term structure model is tested on maturities ranging from 10 to 300 days. In this case, the ATM volatility for the options closest to maturity is the only known variable.

The volatility indices provided by the models are used to price options in- and out-of-the-money using the BSM, and these are compared to market settlement prices. The deviations ($\theta$) are established using

$$\theta = \frac{\hat{P}_{\text{model}} - \bar{P}_{\text{market}}}{\bar{P}_{\text{model}}},$$  \hspace{1cm} (9)

where $P$ stands for the respective prices. Again, the closing prices on the two indices are used as the level of the underlying asset, and the three-month EURIBOR as proxy for the risk-free interest rate.

Table 6
Testing the SLM and SDSA against market prices
The estimated errors, $\theta$, of the models compared with market prices from 1.6 to 30.6.1999. The results are presented as DAX / ESX.

<table>
<thead>
<tr>
<th>Moneyness</th>
<th>All</th>
<th>$-7.5% &lt; \delta &lt; 7.5%$</th>
<th>$\delta &gt; \pm 7.5%$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>SLM</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Average</td>
<td>5.2% / 5.8%</td>
<td>3.7% / 3.5%</td>
<td>8.3% / 12.6%</td>
</tr>
<tr>
<td>Max</td>
<td>81.7% / 94.0%</td>
<td>58.0% / 56.5%</td>
<td>81.7% / 94.0%</td>
</tr>
<tr>
<td>Stdev</td>
<td>9.7% / 11.5%</td>
<td>6.2% / 6.3%</td>
<td>12.9% / 19.1%</td>
</tr>
<tr>
<td><strong>SDSA</strong></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Average</td>
<td>6.8% / 6.3%</td>
<td>4.0% / 3.4%</td>
<td>13.4% / 15.6%</td>
</tr>
<tr>
<td>Max</td>
<td>94% / 190.0%</td>
<td>58.9% / 44.7%</td>
<td>94.1% / 190.0%</td>
</tr>
<tr>
<td>Stdev</td>
<td>14.5% / 14.4%</td>
<td>7.0% / 5.7%</td>
<td>22.1% / 27.6%</td>
</tr>
</tbody>
</table>
It seems from Table 6 that the SLM is a bit more accurate than the SDSA in all cases as expected when considering the $R^2$ statistics in the preceding chapter. Both models price fairly accurately within the moneyness range of $\pm 7.5\%$ and the large mispricings occur for call options close to maturity and deep-out-of-money that in practice are close to worthless. Again, the results for the ESX are very much in line with those of the DAX. The large pricing errors occur for deep-out-of-the-money calls, which can be seen from Figure B1 and Figure B2 (Appendix B). Overall, the pricing accuracy is satisfactory but these results also indicate that the model parameters are to be re-estimated on a continuous basis, in order to provide volatility smiles up-to-date with the markets. The dynamics of the smile change, which calls for re-estimation at least on a weekly basis, perhaps even daily.

Testing the term structure model generated the following results as outlined in Table 7.

**Table 7**

**Testing the term structure model**
The estimated errors of the model compared with market volatilities from 1.6 to 30.6.1999. The only known variable is the ATM volatility for the options closest to maturity whereas the volatility for the longer maturities is estimated using the model and compared to the respective market volatility.

<table>
<thead>
<tr>
<th>Error</th>
<th>DAX</th>
<th>ESX</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average</td>
<td>4.9%</td>
<td>4.2%</td>
</tr>
<tr>
<td>Max</td>
<td>14.7%</td>
<td>9.2%</td>
</tr>
<tr>
<td>Stddev</td>
<td>3.3%</td>
<td>2.7%</td>
</tr>
</tbody>
</table>

Again, the results can be regarded as satisfactory as the errors mainly seem to occur at time-points where the term structure is of different shape than the historical one, that is when the longer maturities imply a lower volatility than does the shortest maturity. To predict whether the term-structure should be rising or falling might in fact be considered the real modeling challenges. However, this kind of model can still be used as such as long as the beta parameter is re-estimated continuously enabling the model to react according to the prevailing (changing) market conditions. During this monitored period, the term-structure is rising (see also Figure 1) as the beta coefficient is larger than zero, whereas a falling term-structure (often characterized by volatile markets) would imply a negative beta value for equation (8).
7 Summary and Concluding Remarks

The purpose of this study was to investigate and model the characteristics of the prevailing volatility smiles on the DAX- and ESX-index options markets. Continuing on the trend of Implied Volatility Functions, the Standardized Log-Moneyness model was introduced and fitted to historical data. The model replaces the constant volatility parameter of the Black & Scholes (1973) pricing model with a matrix of volatilities with respect to moneyness and maturity and was tested on out-of-sample data together with the Sticky Delta Skew Approximation model.

The results show support for the hypotheses put forward in this study, implying that the smile increases in magnitude when maturity and ATM volatility decreases and that there is a negative/positive correlation between a change in the underlying asset/time to maturity and implied ATM volatility. Further, the introduction of the SLM indicate an improvement to pricing accuracy compared to the models suggested by Derman (1999) and Sundkvist (2000). The SLM also provides an estimate for the term structure of the volatility, thus covering the whole volatility surface. Testing the models against market prices proved satisfactory results, however indicating that the parameters of the models are to be re-estimated continuously for the models to capture the changing dynamics of the volatility smiles. Regarding the similarities between the examined markets, it seems that the ESX market implies a surface of greater magnitude than that of the DAX. Overall, the results are quite similar in both cases.

Endnotes:

1 See also Essay 2.
2 See also Essay 5.
3 This implies that the smile is fixed ATM, and that the volatility of options with other moneynesses is explained using the ATM volatility.
4 This also supports the statements regarding the asymmetric attributes of the smile mentioned in the introduction, that is that out-of-the-money puts and calls do seldom exhibit the same volatility patterns.
5 Surprisingly, considering the results of the SDSA-model overall model accuracy (the adjusted $R^2$ statistic) is well below those found in Sundkvist (2000), but this is likely to be due to the different sets of data used. Sundkvist (2000) also limits the observed option volatilities to those with a delta between 0.1 and 0.9 and the examined maturities are also a bit different. The study also accounts for insignificant variables in the estimation of the adjusted $R^2$ statistic.
References


Appendix A

Estimating the parameters for the SDSA-model in equation (5) generated the following results.

Table A1
Estimating the parameters for the SDSAM
The beta estimates for the Sticky Delta Skew Approximation model. The covered time period is 4.3 to 31.5.1999 and the maturities 10 to 340 days. The results for the ESX are shown within brackets below the results for the DAX.

<table>
<thead>
<tr>
<th>10-100</th>
<th>100-340</th>
<th>10-340</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi$</td>
<td>1.00 / 1.00</td>
<td>1.00 / 1.00</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>0.349 / 0.338</td>
<td>0.503 / 0.480</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>0.623 / 0.715</td>
<td>0.383 / -0.058</td>
</tr>
<tr>
<td>$\beta_3$</td>
<td>1.127 / 0.819</td>
<td>0.348 / -0.947</td>
</tr>
<tr>
<td>$\beta_4$</td>
<td>- / -</td>
<td>-0.959 / -</td>
</tr>
<tr>
<td>$\beta_5$</td>
<td>- / -</td>
<td>- / -</td>
</tr>
<tr>
<td>$n$</td>
<td>2,833 / 795</td>
<td>2,615 / 1,024</td>
</tr>
<tr>
<td>$\text{Adj. } R^2$</td>
<td>0.791 / 0.699</td>
<td>0.985 / 0.939</td>
</tr>
</tbody>
</table>

All parameters are significant at the 1%-level.
Appendix B

A graphical summary of the mispricings of the SLM model for the DAX and ESX.

Figure B1
Testing the model for against the DAX market
A graphical illustration of the estimated mispricings when using the SLM model on the DAX index market.

Figure B2
Testing the model for against the ESX market
A graphical illustration of the estimated mispricings when using the SLM model on the ESX index market.