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Paying Attention to Payoffs in Analogy-based Learning*

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Abstract

The *analogy-based expectation equilibrium*, or simply *analogy equilibrium* (AE), analyzes equilibrium stereotypes by imposing consistency of infinitely large action samples with the expectation that broad classes of opponent types behave identically. This paper introduces the *payoff-confirming analogy equilibrium* (PAE) to refine the set of analogy equilibria. The concept imposes additionally that sample marginal of own payoffs be consistent with one's expectations. Robust incorrect equilibrium stereotypes, i.e. non-Bayesian Nash PAE are shown to exist. General conditions are given for the prevalence of such stereotypes under correct expectations on exogenous uncertainty. In monotone selection games susceptible to winner's curse, *naive behavioral equilibrium* leading to aggravation of adverse selection has been shown to match plausible informational assumptions of experienced, but behaviorally biased, equilibrium play. Here, behavioral equilibrium is matched with a corresponding PAE with an incorrect prior and correct prior is shown to imply correct overall expectations.

JEL: C72, D82 Analogy expectations Bounded rationality Learning Stereotypes Winner's curse C72 D82

1 Introduction

Analogy-based expectations equilibrium, or simply *analogy equilibrium*, (AE; Jehiel 2005; Jehiel and Koessler 2008) provides a powerful tool to understand how steady-state-like equilibrium behavior may differ from Nash equilibrium behavior when players use stereotypical classifications, analogy classes, in learning about others. Stereotypes must be confirmed by experience. Not only the observed behavior of others, but also own successes and failures, provide data to test one's stereotypical beliefs.

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Along these lines, this paper introduces the *payoff-confirming analogy equilibrium (PAE)* to refine the set of analogy equilibria and the associated admissible analogy partitions. The analogy equilibrium is related to and can be considered as a self-confirming equilibrium (Fudenberg and Levine 1993; Fudenberg and Levine 1995; Dekel et al. 2004) and conjectural equilibrium (Battigalli 1987). Thus, it is based on the implicit assumption that over time each player gains experience about the characteristics and behavior in the opponent's population(s). Each player records which actions are chosen and which characteristics prevail in that population(s). In the PAE when observing the opponents, each player classifies opponents according to some characteristics, whether strategically and informationally relevant or not. She then organizes samples of observed actions into these classes (called *analogy classes*), one sample for each class of opponent characteristics, and expects that the opponents, with the characteristics of the class, play the sample average strategy of the class thus leading to an underestimation of correlation between oppponent types and strategies. This has been claimed to give rise to winner's curse which has been evidenced in many studies both in the field (Capen et al 1971, for instance) and in the lab (Kagel and Levin 2002).

Stereotyping gender, clothing or ethnicity provides examples of analogy classifications which typically miss the strategically relevant aspects of the characteristics; however, they may capture some coarse correlations of characteristics and strategies. In a bilateral labor market, for instance, an employer might conjecture that a particular ethnic group is in expected terms less qualified for a job than others or that people in the group do not use highly specialized or selective search strategies when looking for a job. The employer thus expects lower productivity among this ethnic group and might therefore make lower wage offers to that particular group. Such statistical discrimination[1] may be a well-founded rational strategy if the expectations are correct. But if expectations are incorrect, there is little justification for such discriminatory employment strategies. The question is under which conditions we can expect such incorrect conjectures to be held in equilibrium?

When equilibrium consistency merely requires that players' expectations about payoff-relevant uncertainty and marginal strategies must be correct, few restrictions are imposed on potential stereotypical classifications: all applicants might be conjectured to be playing the marginal strategy independent of their qualifications, or various compositions of qualifications might be bundled together and expected to behave identically independently of their specific qualifications, at least in some etnhnic groups, thus giving a reason for very selective statistical discrimination. This is why Fudenberg and Levine (2009, pp. 409), for instance, have called for gaining sense of "which sorts of false analogies are relevant... and to ideally endogenize the analogy classes" of the AE. This paper contributes to exactly this by making the analogy grouping part of the equilibrium description and by suggesting an additional, payoff-consistency criterion which allows the reckoning of which incorrect equilibrium stereotypes might be more viable in the long

[1]See Arrow (1998) for instance.

run.

Arguably, people often observe and remember how successful they have been over time when learning about interacting with others. When modeling learning with stereotypes, it is thus often plausible to assume that players have at least a coarse track-record of their own payoffs. The successes and the failures are what players care about - if they do not, why should they best-reply in the first place. Thus, the failures and successes tend to be remembered. If the sampled payoffs differ from the distribution that the player expects (given her beliefs about opponent types and actions) she should eventually realize the inconsistency and abandon her incorrect conjectures. The payoff samples thus provide a natural additional consistency criterion to study the robustness of the analogy classification and the corresponding equilibrium.

Two considerations should be kept in mind here. First, a player needs a large amount of data to test his conjectures, i.e. whatever data used when testing must be restored in memory over time. Thus data on payoffs, as relevant success experiences, may be used when testing conjectures although own types are not used because the latter are not rememebered over time.[2] Second, although success experiences are used to test conjectures, they may be only coarsely recalled (as are opponent types). Payoff realizations in broader intervals may be coded into a single payoff class when testing conjectures. Both these considerations suggest that recalling own payoffs does not necessarily imply a very demanding test on one's stereotypical conjectures.

A weak robustness check requires that the conjectured marginal payoff distribution must be consistent with the marginal sample payoff distribution. Assuming that players carry out a more sophisticated consistency check would require careful consideration: when one is interested in implications of underappreciation of correlation between types and actions, one would beg the question by requiring beliefs to be consistent with the sample joint marginals of own payoffs and opponent actions, for instance[3].

This paper studies such a weak refinement, the *payoff-confirming AE* (PAE) with one primary question in mind: when is an AE payoff-confirming also? We first show by means of an example that there are PAE which differ from Bayesian-Nash equilibria even when the prior is correct. We then provide general sufficient conditions for an AE with a correct prior to be payoff-confirming. These conditions are also necessary if one requires robustness to perturbations of the prior or if one confines attention to two-player games with two actions and two states. Since a Bayesian-Nash equilibrium is a PAE, the latter always exists when the former does. In particular, a PAE always exists in finite environments.

AE with finer than private information analogy partitions have been shown to provide a lack-

[2]This may happen when own type varies from one interaction to another and thus is not part of inherent personality characteristics. See Mullainathan (2002), for instance, for a model with memory limitations.

[3]Dekel, Fudenberg and Levine (2004) consider such refinements in contexts with more sophisticated players who keep track of their own types and actions in addition to other signals and perfectly understand correlations.

ing learning justification[4] for the cursed equilibria (Eyster and Rabin 2005; Jehiel and Koessler 2008; Miettinen 2009). Esponda (2008) also studies the learning foundations of the cursed equilibrium assuming that players learn others' actions and their own payoffs and that they occasionally learn something about the opponents' characteristics. He illustrates that in the steady states of such learning (i) there is even less trade than in any Nash equilibrium in bilateral common value trade, (ii) players bid less aggressively in common value auctions and (iii) less effort is provided in team work. This is to be contrasted with Eyster and Rabin's (2005) finding that there is more trade in the cursed equilibrium than in a Nash equilibrium of the bilateral trade, for instance.

We show that Esponda's conclusions hold also in the present context by pointing out that his equilibria correspond to PAE with private information partitions and with incorrect priors. As Esponda informally conjectured and as is formally shown here, the fact that observing payoffs leads to stronger selection problems hinges upon this incorrect-prior assumption, at least in well-behaved settings with differentiable payoffs and unique best-replies. This is rather intuitive: in the setups with monotone payoffs studied by Esponda, players can make sharp inferences based on payoffs.

One may argue, as Fryer and Jackson (2008), Mohlin (2009), and Schwartzstein (2010) do, that memory limitations may imply categorical or stereotypical classifications. Fryer and Jackson (2008) and Mohlin (2009) illustrate how such coarse stereotyping and discrimination of minorities may come about as one tries to minimize prediction errors whereas in the setting of Schwartzstein (2010) this may be implied by selective attention based on estimates of the predictiveness of additional information. The point of this paper is to take categorical simplifications on opponent types as given and study when they can survive in equilibrium when category marginals on action choices, on the one hand, and marginals on own payoffs, on the other hand, are kept track of and used to test one's stereotypical beliefs in a strategic interaction setting.

The paper is organized as follows. In Section 2 the model and the concepts are presented. In Section 3, it is assumed that the prior is known and a quite general class of games is analyzed. It is shown that generally there are PAE that differ from BNE and conditions are provided when this is the case. In Section 4 the games with monotone selection studied by Esponda (2008) are considered. The results are briefly discussed in Section 5.

2 Model

2.1 The underlying game, strategies and expectations

Consider a static game of incomplete information, $(A_i;\ u_i;\ \Theta;\ p(\theta);\ i = 1, ..., N)$. There are N players indexed by $i = 1, ..., N$. An action of player i is a_i and the finite set of actions[5]

[4] See Fudenberg (2006).

[5] The set of actions is the same in every type profile.

available to her is A_i. The actions of players other than i are denoted by $a_{-i} \in \times_{j \neq i} A_j$. An action profile is $a \in A = \times_{i=1}^{N} A_i$.

Exogenous uncertainty in the stage game is modeled by letting nature draw a type profile, θ, with probability $p(\theta)$ prior to the play of the stage game. The type profile is a vector of types, one for nature and one for each player, $\theta \in \Theta = \times_{i=0}^{N} \Theta_i$ where $\theta_i \in \Theta_i$ is the type of player i. Nature may have its own type, $\theta_0 \in \Theta_0$, to allow for cases where the player types alone do not determine payoffs. For simplicity, we suppose that the set of type profiles is finite. The vector of types of players other than i is $\theta_{-i} \in \Theta_{-i} = \times_{j \neq i} \Theta_j$. The outcomes are type and action profile combinations, (a, θ). The payoff depends on the actions and on the type profile: $u_i : A \times \Theta \rightarrow R$ for $i = 1, ..., N$.

A strategy of player i is a function of her type, $\sigma_i : \Theta_i \rightarrow \Delta(A_i)$ and the probability that type θ_i chooses action $a_i \in A_i$ is denoted by $\sigma_i(a_i|\theta_i)$. The strategies of players other than i are denoted by $\sigma_{-i} : \Theta_{-i} \rightarrow \times_{j \neq i} \Delta(A_j)$ and a strategy profile is $\sigma : \Theta \rightarrow \times_{j=1}^{N} \Delta(A_j)$. The conjecture of i about the state and about the strategy of the opponents[6] is denoted by $\widehat{\mu}^i : \Theta \rightarrow \Delta(\Theta)$ and $\widehat{\sigma}_{-i}^i : \Theta \rightarrow \Delta(A)$, respectively.

The equilibria in our context are understood as steady states of a learning process.[7] Essentially the described learning process is a ficticious-play like process (Fudenberg and Levine 1998) where each player presumes that analogy classes capture others' strategically relevant types. Thus players best-respond to beliefs which are sample averages of past actions in each analogy-class.[8] Stable steady states of such a process are a subset of analogy equilibria as described here.

The equilibrium concept of main interest is the payoff-confirming analogy equilibrium which assumes that players observe opponent actions, characteristics and own payoffs during the learning process while failing to understand correlations in their samples due to simplified observations or organization of these. This concept is defined in the next subsection.

2.1.1 Payoff-confirming analogy equilibrium

In the *analogy equilibrium* each player i partitions the support of type profiles, Θ, into analogy classes. Player i's partition, $\mathcal{A}_i$, is called the *analogy partition of player i*. An element of $\mathcal{A}_i$ is a set of type profiles denoted by α_i, the element of $\mathcal{A}_i$ containing θ is $\alpha_i(\theta)$. An analogy system $(\mathcal{A}_1, ..., \mathcal{A}_N)$ describes the partitions of each player $i = 1, ..., N$. Whereas a player's information partition describes how precisely the player observes the type profile at the interim stage, the

[6]The player may be unaware of the opponent's private information partition. Therefore, the conjecture is conditioned on the whole type profile rather than on the profile of opponents' types.

[7]Implicit in the model, there is a learning process where, in each round, each player plays against randomly chosen opponents, one player drawn from each opponent population; and all randomly matched players then receive a random draw of types, their stage game specific private information.

[8]See Huck et al. (2009) for a simple explicit model of analogy-based learning.

analogy partition describes the player's stereotypes. This translates into how precisely he keeps track of the type profile realizations ex-post when the game is played.[9]

The analogy equilibrium is specific about how the conjectures are formed. Each player i conjectures that, in a given type profile, each opponent plays her average strategy of the analogy class of i where that type profile belongs to. This is the simplest theory consistent with observing only the analogy class rather than the precise opponent type. Moreover, this is the only consistent theory where each opponent plays a pooling strategy in each analogy class. To formalize this idea, we define the *opponents' average strategy* in a set of type profiles $B \subset \Theta$ as follows:

$$\overline{\sigma}_{-i}(B) = \frac{\sum_{\theta \in B} p(\theta)\sigma_{-i}(\theta_{-i})}{\sum_{\theta \in B} p(\theta)}. \tag{1}$$

We are now ready to define the analogy equilibrium:

The triple $(\sigma_i, \widehat{\sigma}_{-i}, \mathcal{A}_i)_{i=1}^N$ is an analogy equilibrium with a correct prior if, first, for all $\theta \in \Theta$, for all i and $a_i^* \in \text{supp}[\sigma_i(\theta_i)]$

$$a^* \in \arg\max \sum_{\theta'_{-i} \in \Theta_{-i}} p(\theta'_{-i}|\theta_i) \sum_{a_{-i} \in A_{-i}} \widehat{\sigma}^i_{-i}(\theta_i, \theta'_{-i}) u_i(a; \theta_i, \theta'_{-i}) \tag{2}$$

and, second, for all $\theta \in \Theta$, for all i $\quad \widehat{\sigma}^i_{-i}(\theta) = \overline{\sigma}_{-i}(\alpha_i(\theta))$.

Notice that in the definition the coarseness of the partitions is part of the equilibrium description rather than exogenous. This is where the concept differs from the definition in Jehiel and Koessler (2008) who assume that the analogy partitions are exogenous. Notice also that player i's analogy classification may depend on her type as in Jehiel and Koessler. That is, given θ'_{-i}, there may be θ'_i and θ''_i such that $(\theta'_i, \theta'_{-i}) \in \alpha'_i$ and $(\theta''_i, \theta'_{-i}) \in \alpha''_i$.

Jehiel and Koessler coin the *private information analogy partition AE* the analogy equilibrium where each player's analogy partition coincides with her information partition, $\mathcal{A}_i = \{\{\theta_i\}_{\theta_i \in \Theta_i} \times \Theta_{-i}\}$. This AE coincides with the *cursed equilibrium* of Eyster and Rabin (2005). This analogy partition is rather natural since each type fully understands her own information but fails to understand any correlations between opponent types and strategies conditional on this information. This special case of the concept will take a central role in Section 4 on incorrect prior where issues related to winner's curse are studied. In many trading mechanisms, such as auctions and bilateral trade, there is a substantial body of experimental evidence on the existence of winner's curse at least in enviroments where opportunities for learning are limited[10] and where prior distribution of a privately known quality is given. Yet, often in practice the experienced buyers have learned the quality conditional on trade, and prior information on the

[9] In the incorrect prior case, the ex-post observation structure is captured by the intersection of the visible states and the analogy partition.

[10] See Kagel and Levin (2002) for a review.

quality distribution is limited. To better isolate the influence of these features, a model with an incorrect prior AE is needed.

Clearly, if quality is observable only conditional on trade, a player may attach a zero probability to a quality of positive prior probability. This is formalized by partitioning type profiles into visible ones $\widehat{\Theta}^i_V$ and invisible ones $\widehat{\Theta}^i_I$ the latter of which are incorrectly attached a zero probability. The partitioning into visible and invisible type profiles is also part of the equilibrium description as is the analogy partition; but the two are distinct, as opposed to the case of correct prior where analogy classes may be understood to characterize the ex-post observation about the type profile. All types, including the invisible types, belong to some analogy group - the behavior of types is always observed even if the type profiles responsible for this behavior are not. The behavior in the invisible states of an analogy class influence the average strategy of the class thus biasing further the strategy conjectures held in that analogy group - the player deems opponent actions that actually stem from invisible type profiles to be chosen by visible type profiles of the group.

The quadruple $(\sigma_i, \widehat{\sigma}_{-i}, \mathcal{A}_i,\ \widehat{\Theta}^i_V)_{i=1}^N$ is an analogy equilibrium *with an incorrect prior* if,

- first, for all $\theta \in \Theta$, for all i and $a_i^* \in$supp$[\sigma_i(\theta_i)]$,

$$a^* \in \arg\max \sum_{\theta'_{-i} \in \Theta_{-i}} \widehat{\mu}^i(\theta'_{-i}|\theta_i) \sum_{a_{-i} \in A_{-i}} \widehat{\sigma}^i_{-i}(\theta_i, \theta'_{-i}) u_i(a; \theta_i, \theta'_{-i}),$$

- second, for every $\theta \in \widehat{\Theta}^i_V$, $\widehat{\mu}^i(\theta) = \frac{p(\theta)}{\sum_{\theta \in \widehat{\Theta}^i_V} p(\theta)}$ and for every $\theta \in \widehat{\Theta}^i_I$, $\widehat{\mu}^i(\theta) = 0$,
- third, for all i, for each α_i and each $\theta \in \alpha_i \in \mathcal{A}_i, \widehat{\sigma}^i_{-i}(\theta) = \overline{\sigma}_{-i}(\alpha_i(\theta))$.

The first equilibrium condition merely requires best-replying to conjectures $\widehat{\mu}^i, \widehat{\sigma}^i_{-i}$. With an incorrect prior, the conjecture about the distribution of types, $\widehat{\mu}^i$, and even the conjecture about its support may be incorrect while still consistent with what one observes. The conjectured probability of a visible type profile coincides with its actual prior probability conditional on visibility. Each type may have a different set of visible type profiles. That is, given θ'_{-i}, there may be θ'_i and θ''_i such that $(\theta'_i, \theta'_{-i}) \in \widehat{\Theta}^i_V$ and $(\theta''_i, \theta'_{-i}) \in \widehat{\Theta}^i_I$. This means that various types of a player may have different priors since their equilibrium strategies may select different visible type profiles. The third condition imposes consistency of the marginal sample distribution of actions by requiring that average strategies be played in each analogy class.

In this section we formally define the refinement of the AE, the payoff-confirming AE (PAE). The PAE studies a mild robustness check of the AE, where the player fails to keep track over time of how own payoffs are correlated with other signals. Thus, the marginal expected and sample payoff distribution must coincide. It would be unnatural to assume that players perceive correlations between own payoffs and opponents' actions or types although they do not perceive correlations between actions and types in each analogy class.

An extension of the concept allows for imprecise payoff recollection - the mapping $\sqcap : \mathbb{R} \to \mathcal{U}$ codes the payoffs into payoff recollection. Notice that if $\sqcap$ is not an isomorphism, then evaluation of past successes is coarser than the player's cardinal decision preferences. Although all results of Section 3 are written for the general case, in Section 4 and in the main text we will confine attention to the case of $\sqcap$ being an isomorphism or in fact an identity function if not mentioned otherwise.

Payoff-confirming analogy equilibrium is an AE where for all i and $\overline{u}_i$

$$\sum_{\{a \in A, \theta \in \Theta | \sqcap(u_i(a_i, a_{-i}, \theta_i, \theta_{-i})) = \overline{u}_i\}} \widehat{\mu}^i(\theta)\sigma_i(a_i|\theta_i)\widehat{\sigma}^i_{-i}(a_{-i}|\theta) \tag{3}$$
$$= \sum_{\{a \in A, \theta \in \Theta | \sqcap(u_i(a_i, a_{-i}, \theta_i, \theta_{-i})) = \overline{u}_i\}} p(\theta)\sigma_i(a_i|\theta_i)\sigma_{-i}(a_{-i}|\theta_{-i}).$$

If the prior is correct, then $\widehat{\mu}^i(\theta) = p(\theta)$.

To illustrate the concepts, let us consider the following simple two-player game whose payoff matrices are given in Figure 1. The row player is an employer and the column player can be either a male or a female employee, $g \in M, W$ respectively. The gender is entirely payoff irrelevant and illustrates here how payoff-irrelevant uncertainy may function as a way to construct steareotypical classifications, analogy partitions.

Most of the time due to well-planned incentive schemes perhaps, the interests of the employer and the employee are aligned. Let's assume that in the state of aligned interests, θ_g^A, occurring with probability $p_g^A = p^A$ for $g = \{M, W\}$, the parameter k is smaller than one in absolute value. In the state of aligned interests, players have dominant strategies T and R, respectively, and the Nash equilibrium outcome is the preferred one for both. In state θ^C where $p_g^C = p^C = 1 - p^A$ for $g = M, F$, the interests of the employer and the employee are entirely conflicting and they play a constant sum game with the only (Nash) equilibrium in mixed strategies, $\sigma_1(T) = 1/2, \sigma_2(L) = 3/4$. It is known to both whether interests are aligned or not and thus the above state-conditioning strategies constitute the unique Nash-equilibrium strategies of the game.

θ_g^A	L	R
T	2+k, 2	4, 4
B	1, 1	3, 3

θ_g^C	L	R
T	3, 2	1, 4
B	2, 3	4, 1

Figure 1: Stereotyping in workplace relationship, $g = M, W$.

Let us consider the following AE. Suppose that employees have fine analogy partitions $\{\{\theta^A\}, \{\theta^C\}\}$. Suppose that the employer organizes his experiences along the following analogy partition $\{\{\theta_M^A\}, \{\theta_M^C\}, \{\theta_W^A, \theta_W^C\}\}$. Thus the employer keeps track of male employee behavior conditional on the alignment of interests whereas he bundles states of conflicting and aligned interests together when facing a female emploee. This may be due to diverted attentiveness when playing with women or due to a belief that female employees only are unable to detect the strategic aspects of the situation. Let the male employees play the Nash equilibrium strategies and let the employer also play Nash equilibrium strategies when facing a male employee. Let the female employee play R in θ_W^A and L in θ_W^C and let the employer play T in θ_W^A and B in θ_W^C . Let us verify that this constitutes an AE and moreover a PAE.

By the second condition of Definition 1, the employer expects the average strategy being played by men in each class, and thus expectations must be correct conditional on the state being either θ_M^A or θ_M^C. By Condition 1, the employer rationally best-replies and thus his strategy must be a (Nash-)best-response. By the same token, the male employee must have correct expectations and (Nash-)best-response. This illustrates that with fine analogy-partitions, the only analogy-equilibrium, let alone payoff-confirming one, is the a Nash equilibrium where expectations are correct.

Consider then the states θ_W^A and θ_W^C which are bundled into the same analogy class by the employer. As required by the second condition of Definition 1, the employer adopts the simplest belief that all types in this class behave indentically. Can such an incorrect (analogy-based) stereotype survive in equilibrium? Can it be confirmed by experiences? Female employees play R in θ_W^A and L in θ_W^C, and thus by the second condition in Definition 1, the employer believes that female employees, in each state, play the average strategy: R with probability p^A and L with probability p^C. Best-responding to the believed strategy of a female employee amounts to playing T in θ_W^A and B in θ_W^C for any prior probability of state A greater than 1/4 as stated in the equilibrium description.[11] It is easy to see that playing L is indeed a best-response strategy of a female employee in state θ_W^C.[12] Dominance implies that remaining best-responses are optimal. Thus, we have an AE, an incorrect equilibrium stereotype confirmed by experience. But how robust is it?

As argued above, if anything, economic agents averting failures and striving for successes should recall how well they fared. Economic agents capable of deeming how much better one outcome is than another (as required by the implicitly assumed cardinality of preferences when best-responsing to beliefs), should also have the capacity of at least approximately recalling of how much better they did at various instances of a recurring interactive situation. In Definition 3, the mapping coding a payoff realization into a payoff class in one's memory is denoted by $\sqcap : \mathbb{R} \to \mathcal{U}$. In the example, suppose that the employer recalls past payoffs at precision κ so that payoff realizations u' and u'' more apart than κ are never coded into the same class of recollection, i.e. $|u' - u''| > \kappa \Rightarrow \sqcap(u') \neq \sqcap(u'')$, then $\kappa < k$ implies that the employer will eventually notice that he never observes $\sqcap(2 + k)$ in his recollection of payoffs and yet his

[11] Given the analogy-based expectation, the employer's payoff to playing T in state θ_W^C is $3p^C + p^A$ and the payoff to playing B in that state is $2p^C + 4p^A$ where $p^C = 1 - p^A$and thus B is a best-reponse for any $p^A \geq 1/4$.

[12] The female employee has correct expectations and thus playing L in state θ_W^C yields 3 for the female employee while playing R yields 1.

analogy-based expectations deem that event having a probability $(p^A)^2$. Thus the only PAE is the Bayesian-Nash equilibrium of the game in that case.

Yet when $k \leq \kappa$, incorrect conjecture stereotype on females may be robust to payoff-recollection. Stereotypes that are in such a manner robust to payoff information are probably particularly deeply grounded. As a special case, suppose that $\kappa = 0$. Then if $k = 0$, the employer will not be able to infer that his stereotype is misleading him since he cannot associate each payoff realization with the state or the actions.[13] In particular when facing a female, the employer's payoff is 2 with probability p^C and 4 with probability p^A which is exactly what the employer expects to get according to his incorrect beliefs.

Crucial for the payoff-consistency of the non-Bayesian-Nash AE is that once a choice of the female employee is fixed, the payoff is the same in both states given the employer's own equilibrium actions. It will be shown in Section 3, that this type of condition guarantees that a non-Nash analogy equilibrium is payoff confirming. Alternatively, in each state of the analogy class, the payoff must be the same whatever positive probability actions in the class the opponents choose.

3 Correct prior

In this section, we study the payoff-confirming equilibria with a correct prior and thus, $\widehat{\mu}^i(\theta) = p(\theta)$. Notice that in the workplace relationship example in the previous section, the employer has the correct expectations about the nature of the situation. The employer bundles together both states when learning about the female employee choices. This leads her to believe that the female employee plays a mixed strategy in each state. This in turn, justifies non-best-responding to the actual choice of a female employee when interests are conflicting. The example illustrates that there can be PAE (with a correct prior) that do not correspond to Bayesian-Nash equilibria.

The main purpose of this section is to identify cases when this can happen under a correct prior. Clearly, when strategies and conjectures in an AE coincide with those in a Bayesian Nash equilibrium, then conjectures about others' strategies must be correct by the definition of the Nash equilibrium. Then surely, if conjectures are correct, payoff-information cannot reveal anything which was not known already. Thus, an AE with Bayesian-Nash conjectures and strategies must be payoff-confirming also. There are two simple cases when this happens[14]: first, using the terminology of Battigalli et al. (1992), when the *ex-post observation* of types is more precise than the interim information at the time when the strategy is chosen (as in the

[13]Notice that the expected payoff of the employer is lower in θ^C_W when facing a female employee than when facing a male, θ^C_M. In θ^C_M, the expected payoff in the mixed strategy equilibrium is $4p^A + 5/2p^C$ whereas the payoff in state θ^C_W gives only $4p^A + 2p^C$, giving ultimately a motive for statistical discrimination, yet, this discrimination is based on an incorrect belief.

[14]See also Jehiel and Koessler (2008).

case of male employees in the example of the previous section); second, pooling strategies.

Proposition 1 *AE is BNE if, for each player, her opponents' types play a pooling strategy in each of the player's analogy classes.*

Formally, AE is BNE if $\theta' \in \alpha_i(\theta)$ implies that $\sigma_{-i}(\theta) = \sigma_{-i}(\theta')$.

The proposition is stated without a proof but the idea is simple. In at least one analogy partition, two types must play differently for AE to differ from BNE. When there are two such types then necessarily the average strategy differs from the actual strategies of these two types and thus the AE differs from BNE. Thus only if the analogy partition is coarser than some opponents' information partition and the opponent plays a separation strategy, may a PAE differ from a Bayesian-Nash equilibrium. In this case, some non-generic payoffs[15] are needed in order for a PAE to differ from a Bayesian-Nash equilibrium. Otherwise, the payoff information would reveal any mistaken conjectures. In two-player two-action two-state games of incomplete information, it is fairly easy to characterize the set of pure strategy AE that are PAE.

Proposition 2 *Let $N = 2$, $\Theta = \{\theta^1, \theta^2\}$ and $A_i = \{a_i^1, a_i^2\}$. Suppose that σ is a pure strategy AE.*

The AE is payoff-confirming if and only if

- $(\sigma_i, \widehat{\sigma}_{-i})_{i=1}^N$ is a Bayesian-Nash equilibrium

or

- for each i such that $\sigma_i(\theta^m) \neq \sigma_i(\theta^n)$ and $\mathcal{A}_j = \{\{\theta^1, \theta^2\}\}$

$$\text{for all } m,\ \sqcap(u_j(\sigma_j(\theta^m), r_i(\theta^m); \theta^m)) = \sqcap(u_j(\sigma_j(\theta^m), s_i(\theta^m); \theta^m)) \tag{4}$$

or

$$\text{for all } m,\ \sqcap(u_j(\sigma_j(\theta^m), r_i(\theta^m); \theta^m)) = \sqcap(u_j(\sigma_j(\theta^n), \sigma_i(\theta^n); \theta^n)) \tag{5}$$

where $r_i(\theta^m)$ is the action not chosen by i at θ^m.

In the appendix.

To gain some intuition with regard to this result, notice that each player is trying to detect a correlation between the opponents' actions and the opponents' type profile using payoff realizations as evidence. If there is evidence for correlation between actions and types, then clearly, the presumption that the opponents play the average strategy in each state of the analogy class must be incorrect. The fact that either Condition (4) or Condition (5) holds prevents inferring anything about the joint distribution: If Condition (4) holds, the player's payoff is the same

[15]Notice that this may be due to coarse partioning of payoffs.

given opponent type whatever the opponent chooses. In other words, there is no strategic uncertainty about own payoffs in each state. Alternatively, if (5) holds, the payoff is the same whatever the state given the action of the opponent. That is, there is no exogenous uncertainty about own payoffs given the action of the opponent. In either case, the payoffs do not provide any additional information about the joint distribution of actions and types of others.

In games with more states, more players and more actions, conditions parallel to (4) or (5) are sufficient but not necessary for an AE to be PAE.

Proposition 3 *Let in an AE σ differ from a Bayesian-Nash equilibrium. If for each j and α_j such that there are $\theta^m, \theta^n \in \alpha_j$ with $\sigma_{-j}(\theta^m) \neq \sigma_{-j}(\theta^n)$,*

- *either for all $\theta \in \alpha_j$, and for all action profiles of players other than j, $a^*_{-j} = (a^*_1, ..., a^*_{j-1}, a^*_{j+1}, ..., a^*_N)$ such that for each $i \neq j$ there is $\theta' \in \alpha_j$ such that $a^*_i \in supp[\sigma_i(\theta')]$, there exists u_θ such that for all $a_j \in supp[\sigma_j(\theta)]$*

$$\sqcap(u_j(a_j, a^*_{-j}; \theta)) = u_\theta \tag{6}$$

- *or for all action profiles of players other than j, $a^*_{-j} = (a^*_1, ..., a^*_{j-1}, a^*_{j+1}, ..., a^*_N)$ such that for each $i \neq j$ there is $\theta' \in \alpha_j$ such that $a^*_i \in supp[\sigma_i(\theta')]$, there exists $u_{a^*_{-j}}$ such that for all $\theta \in \alpha_j$ and $a_j \in supp[\sigma_j(\theta)]$*

$$\sqcap(u_j(a_j, a^*_{-j}; \theta)) = u_{a^*_{-j}}, \tag{7}$$

then the AE is payoff-confirming.

In the appendix.

Now for any coarse analogy partition of the opponent such that an opponent plays different strategies in two nodes of an analogy class, either (6) or (7) holds. Again, the former condition aggregates over strategic uncertainty and the latter condition aggregates over exogenous uncertainty. Thus, a player cannot infer anything about the joint distribution of type profiles and action profiles of other players in each analogy class. Notice moreover that Proposition 3 does not impose restrictions on the distribution of types. Therefore, a PAE satisfying the conditions of the proposition is robust to changes in the distribution of exogenous payoff uncertainty.

As will be illustrated in the next example, a PAE which is not BNE may fail (6) and (7) but then it cannot be similarly robust to perturbations in the prior.

There are two players and three states of nature, $\{\theta^1, \theta^2, \theta^3\}$, each drawn with probability 1/3. This prior is known to both players and the realization of the state is revealed to both. Each state of nature is associated with a simultaneous move two-player game. In each of the games, each player has three actions, $A_i = \{a^1, a^2, a^3\}$. Player 2 gets payoff +1 if he matches the state ($u_2 = 1$ if $a_2 = a^k$and θ^k is drawn by nature) and −1 if his action does not match the state. Payoffs of Player 1 are indicated in matrices below.

θ^1	a^1	a^2	a^3
a^1	1	0	-1
a^2	2	-2	-2
a^3	2	-2	-2

θ^2	a^1	a^2	a^3
a^1	-1	0	-1
a^2	0	-1	1
a^3	-1	0	-1

θ^3	a^1	a^2	a^3
a^1	-1	-1	1
a^2	-1	-1	1
a^3	-1	1	0

Consider the following equilibrium: Player 1 has the coarsest analogy partition and Player 2 has the finest. Each player plays a pure separation strategy, each player's choice at state θ^j is a^j. The conjecture of Player 1 is $\widehat{\sigma}_2^1(a^j|\theta^k) = \frac{1}{3}$ for all $j, k = 1, 2, 3$. Player 2 matches his action with the state and thus he is best-replying. Also, Player 1 is best-replying since in state θ^k choosing a_k gives expected payoff zero whereas other actions give negative expected payoffs given that Player 1 expects two to choose each action with probability $\frac{1}{3}$. Thus, this is an AE but certainly not a Nash equilibrium, since Player 1 is not choosing her best-response in a single state.

Furthermore, only outcomes (a^k, a^k, θ^k), $k = 1, 2, 3$, have a positive actual probability and each results with probability $\frac{1}{3}$. Thus, the sample distribution of Player 1's payoff assigns probability $\frac{1}{3}$ to payoffs -1, 0 and 1 respectively. Since this is Player 1's expectation of payoffs given her equilibrium strategy, we have a PAE. Yet, neither is there for each a^k, a payoff u^k such that for all l, $u_1(a^l, a^k, \theta^l) = u^k$, nor is there for each θ^k, a payoff u^k such that for all l, $u_1(a^k, a^l, \theta^k) = u^k$. Thus, we have a PAE even if neither Condition (6) nor Condition (7) in Proposition 3 are satisfied.

The PAE in this example imposes restrictions on the prior distribution unlike Proposition 3: each state must have probability 1/3. Therefore it is not similarly robust to changes in exogenous payoff uncertainty. More generally, only equilibria satisfying the condition in Proposition 3 are robust to all perturbations of the prior of a given small extent.

The following proposition illustrates. For the sake of simplicity, we consider only pure strategy strict equilibria. The equilibrium must be strict to avoid small perturbations affecting the *optimality of the strategies* themselves.

Proposition 4 *Let $\widehat{\sigma}$ be a profile of PAE conjectures and $\times_{i=1}^N \mathcal{A}_i$ the corresponding analogy partitions given prior p. For any $\varepsilon > 0$, there exists a perturbation of the prior, $p' = (1-\varepsilon)p + \varepsilon p^o$ where $p^o \neq p$, and a player i with an analogy class α in her analogy partition $\mathcal{A}_i$ such that*

$$\widehat{\sigma}(\theta) = \frac{\sum_{\widetilde{\theta} \in \alpha(\theta)} p'(\widetilde{\theta}) \sigma_{-i}(\widetilde{\theta}_{-i})}{\sum_{\widetilde{\theta} \in \alpha(\theta)} p'(\widetilde{\theta})}.$$

is not a part of a profile of PAE conjectures given p' if and only if the condition in Proposition 3 is not satisfied.

In the appendix.

The assumption that payoffs are perfectly recalled, i.e. they are identity-mapped into payoff recollection, imposes a very stringent robustness criterion. Had we identified a PAE that differed from a Bayesian-Nash equilibrium, even such that the condition in Proposition 3 is satisfied, perturbing one of the equilibrium payoffs just slightly would lead to a violation of the payoff consistency criterion.

Proposition 5 *Let $\widehat{\sigma}$ be a profile of PAE conjectures. Consider a payoff perturbation of Player i, $\widetilde{u}_i(a_i, a_{-i}, \theta; \varepsilon)$, such that $\lim_{\varepsilon\to 0} \widetilde{u}_i(a_i, a_{-i}, \theta; \varepsilon) = u_i(a_i, a_{-i}, \theta)$ for all (a, θ) but there is (a, θ) such that $\widetilde{u}_i(a_i, a_{-i}, \theta; \varepsilon) \neq u_i(a_i, a_{-i}, \theta)$ for all $\varepsilon \neq 0$. If $\sqcap$ is an identity map, there exists Player i and a payoff perturbation of his payoff such that $\widehat{\sigma}$ is a conjecture profile of a PAE iff $\widehat{\sigma}$ is a Bayesian-Nash equilibrium profile.*

Suppose that a PAE is not a BNE. Then there is i and there are $\theta^m, \theta^n \in \alpha'_i$ with $\sigma_{-i}(\theta^m) \neq \sigma_{-i}(\theta^n)$. Then picking up the type profile θ' in that analogy class, and the payoff $u_j(a'_j, a'_{-j}, \theta')$ such that $a' \in \text{supp}[\sigma(\theta\prime)]$ and perturbing that payoff by k, i.e. $\widetilde{u}_i(a'_i, a'_{-i}, \theta') = u_i(a'_i, a'_{-i}, \theta') + k$ implies that

$$\begin{aligned}
& \sum_{\substack{\{a_{-i}\in A_{-i}, \theta\in\alpha'_i | \\ \widetilde{u}_i(a_i, a_{-i}, \theta_i, \theta_{-i}) = u_i(a'_i, a'_{-i}, \theta')\}}} \widehat{\mu}^i(\theta)\sigma_i(a_i|\theta_i)\widehat{\sigma}^i_{-i}(a_{-i}|\theta) \\
= & \sum_{\substack{\{a_{-i}\in A_{-i}, \theta\in\alpha'_i | \\ \widetilde{u}_i(a_i, a_{-i}, \theta_i, \theta_{-i}) = u_i(a'_i, a'_{-i}, \theta')\}}} p(\theta)\sigma_i(a_i|\theta_i)\frac{\sum_{\widehat{\theta}\in\alpha'_i} p(\widehat{\theta})\sigma^i_{-i}(a_{-i}|\widehat{\theta})}{\sum_{\widehat{\theta}\in\alpha'_i} p(\widehat{\theta})} \\
= & \sum_{\{a_{-i}\in A_{-i}, \theta\in\alpha'_i\}} p(\theta)\sigma_i(a_i|\theta_i)\frac{\sum_{\widehat{\theta}\in\alpha'_i, \widehat{\theta}\neq\theta'} p(\widehat{\theta})\sigma^i_{-i}(a_{-i}|\widehat{\theta}) - p(\theta')\sigma^i_{-i}(a_{-i}|\theta')}{\sum_{\widehat{\theta}\in\alpha'_i} p(\widehat{\theta})} \\
\neq & \sum_{\{a_{-i}\in A_{-i}, \theta\in\alpha'_i\}} p(\theta)\sigma_i(a_i|\theta_i)\sigma_{-i}(a_{-i}|\theta_{-i}) \\
= & \sum_{\substack{\{a_{-i}\in A_{-i}, \theta\in\alpha'_i | \\ \widetilde{u}_i(a_i, a_{-i}, \theta_i, \theta_{-i}) = u_i(a'_i, a'_{-i}, \theta^m)\}}} p(\theta)\sigma_i(a_i|\theta_i)\sigma_{-i}(a_{-i}|\theta_{-i})
\end{aligned}$$

where the inequality holds if $\frac{p(\theta')\sigma^i_{-i}(a_{-i}|\theta')}{\sum_{\widehat{\theta}\in\alpha'_i} p(\widehat{\theta})} \neq \sigma_{-i}(a_{-i}|\theta'_{-i})$ - a sufficient condition is that there are $\theta^m, \theta^n \in \alpha'_j$ with $\sigma_{-j}(\theta^m) \neq \sigma_{-j}(\theta^n)$ which holds since our PAE is not BNE.

As an example, consider again the workplace example of Section 2. If k differed at all from 0 and $\sqcap$ is an identity map, then the employer would realize that he is holding an incorrect conjecture. In this sense, the correct prior PAE with perfect recollection is a very stringent, non-generic equilibrium notion. It is plausible that payoffs are not perfectly recalled and thus, as illustrated in the example of Section 2, non Bayesian-Nash PAE may generically exist. Instead of imperfect payoff recollection, generic existence of a non Bayesian-Nash PAE may also be due to imperfect payoff observation. This will be illustrated in the following section where incorrect priors are allowed for.

4 Incorrect prior

With incorrect priors, there can be many more steady states than with correct priors. Despite this potential multiplicity, results in Esponda (2008) suggest that payoff-consistency reduces the number of equilibria sufficiently to allow clear cut implications on the set of equilibria even in rather general settings. Esponda (2008) considers monotone selection setups in which lower actions select a lower distribution of quality (given strategies); and lower beliefs about quality induce players to choose lower actions. These setups comprise bilateral trade (Akerlof 1970), for instance, and many applications where Eyster and Rabin's (2005) cursed equilibrium, or the equivalent correct prior AE with private information analogy partitions, has been shown to alleviate selection problems and induce thicker markets. Esponda, quite surprisingly, shows that in games with monotone selection, markets are even thinner than in Nash equilibria when players learn the others' actions, their own payoffs, and occasionally the types of others, but fail to pay attention to correlations as in the cursed equilibrium.

We will begin with illustrating how private information analogy partition PAE with an incorrect prior yields similar insights as Esponda's *behavioral equilibrium* in a bilateral trading game with one-sided asymmetric information (Akerlof 1970; Samuelson and Bazerman 1985). We will then show that the only correct prior PAE of the game are Bayesian-Nash equilibria. We will then generalize these results by showing that Esponda's *naive behavioral equilibrium* has an equivalent payoff-confirming analogy equilibrium with an incorrect prior in the games of monotone selection he considers. Finally we will show how an additional assumption of differentiable payoffs implies that in this class of games the only correct prior PAE of the game are Bayesian-Nash equilibria. Throughout this section, it is assumed that payoff recollection is perfect and thus $\sqcap$ is an identity map.

4.1 Example

Let's consider the bilateral trading game with one-sided asymmetric information of Samuelson and Bazerman (1985). The seller values the object at s, while the buyer values the object at $b = s + x$, where s is the realization of a random variable S that is uniformly distributed on the interval $[0, 1]$ and $x \in (0, 1]$ is a parameter that captures gains from trade. The seller knows her valuation, but the buyer has no private information about either s or v. The buyer and seller simultaneously make offers to buy at price π and to sell at price π_{ask}, respectively. If $\pi_{ask} > \pi$, there is no trade; the seller keeps the object and the buyer obtains her reservation utility of zero. If $\pi_{ask} \leq \pi$, the object is traded, the buyer pays π, and obtains utility $u_S(\pi; b) = b - \pi$. We will restrict attention to equilibria where the seller plays his weakly dominant strategy, $\pi_{ask} = s$.

Now suppose that the buyer's expectations are analogy-based. The simplest such expectations with a correct prior are built on a unique analogy class implying, by the second condition in Definition 1, that the buyer expects all sellers to behave in an identical manner and asking randomly any price between zero and one with equal probability. A best response to her incorrect expectation would be to offer the price $\max_\pi \{(E_S(s) + x - \pi) \Pr[s \leq \pi]\}$ or $\pi^* = \frac{1/2+x}{2}$. This constitutes an analogy equilibrium of the game.[16] Is this AE payoff-confirming also? No, since the buyer expects to end up occasionally buying a product that the seller values higher than the offered price $s > \pi^*$ which never happens given the sellers weakly dominant strategy of asking exactly s. Therefore a buyer knowing the ex-ante distribution of quality and having access to payoff-information conditional on trade cannot hold her payoff-inconsistent analogy-based expectations.

In a Bayesian-Nash equilibrium of this game the buyer, correctly conjecturing the seller's strategy, realizes that lower offers select a lower quality. She optimally trades off the lower price with the lower quality and probability of getting the product and offers $\max_\pi \{(E(s|s \leq \pi) + x - \pi) \Pr[s \leq \pi]\}$ or $\pi^* = x$. This Bayesian-Nash equilibrium of the game corresponds to the AE where there are infinitely many singleton analogy classes, one for each realization of S, and the payoffs only confirm the correct analogy-expectations in this case.[17]

There are in fact no other PAE when the prior is correct in this example (as will be shown below). Yet, there is a PAE with an incorrect prior. In this equilibrium the seller believes that the quality range coincides with the range of qualities she ends up buying given her price offer π^{PAE} thus violating the correct prior assumption, i.e. $\widehat{\Theta}_V^B = [0, \pi^{PAE}]$ and the buyer thus expects that the density of quality is $\frac{1}{\pi^{PAE}}$ (see the second condition in Definition 2). The buyer bundles all types of the seller into a single analogy class and mistakenly expects all sellers to use the same average strategy asking each price in the zero-one interval with equal probability (as required by the third condition in Definition 2). Thus, given that the seller has only one type, this is an AE with private information analogy partition. This PAE satisfies the following fixed-point equation $\pi^{PAE} = \arg\max_\pi \left\{(E_S(s|s \leq \pi^{PAE}) + x - \pi) \Pr[s \leq \pi]\right\}$ yielding $\pi^{PAE} = \frac{2}{3}x$. The buyer now expects to observe all qualities in the range $[0, \pi^{PAE}]$ with equal probability and never to observe any other quality. This is indeed what she ends up observing. We have thus established a PAE with an incorrect prior.

We can also remark that all PAE with a correct prior coincide with the unique BNE of the game. To see this, let us first consider analogy partitions where each analogy class is a subinterval of the support of the prior $[0, 1]$. Thus an arbitrary analogy partition is of the form:

$$\{[0, s_1], (s_1, s_2], ..., (s_k, s_{k+1}], ..., (s_n, 1]\}.$$

If $s_k \to s_{k+1}$ and $n \to \infty$, then all analogy classes would tend to singletons and AE would approach a BNE. Since the prior is correct, $\widehat{\Theta}_V^B = [0, 1]$.

[16] The one corresponding to the fully cursed equilibrium of Eyster and Rabin (2005).

[17] Although the definitions in Section 2 are only defined for finite-action, finite-state setups, it is easy to extend those notions to the infinite game setup considered here.

The expected payoff when playing π reads[18],

$$\pi(\frac{s_{K-1}+s_K}{2}+x-\pi)-\frac{s_K s_{K-1}}{2}.$$

To see this, notice that given the seller's strategy, the probability of trade in this case equals π. Conditional on trade, the buyer expects the sellers in each class to choose any price in the interval $(s_{k-1}, s_k]$ for $k = 2, ..., K$ with equal probability (class $[0, s_1]$ is expected to select each price in $[0, s_1]$ with equal probability) yielding

$$(\pi - s_{K-1})(\frac{s_K + s_{K-1}}{2} + x - \pi) + \sum_{k=1}^{K-1}(s_k - s_{k-1})(\frac{s_k + s_{k-1}}{2} + x - \pi)$$

$$= \pi(\frac{s_K + s_{K-1}}{2} + x - \pi) - \frac{s_K s_{K-1}}{2},$$

where $s_K \equiv \inf_k\{s_k | s_k \geq \pi\}$ and $s_0 = 0$.

Taking the derivative w.r.t π yields $\frac{s_K+s_{K-1}}{2} + x - 2\pi$ and thus the optimal π within the analogy-group is characterized by $\pi^* = \frac{s_K+s_{K-1}}{4} + \frac{x}{2}$ if there is such π^* in $(s_{K-1}, s_K]$. If $\pi > \frac{s_K+s_{K-1}}{4} + \frac{x}{2}$ for all $\pi \in (s_{K-1}, s_K]$, then within the analogy class, it is optimal to choose π as close as possible to s_{K-1}. If $\pi < \frac{s_K+s_{K-1}}{4} + \frac{x}{2}$, then within the group, it is optimal to choose $\pi = s_K$.

In a payoff-confirming AE, the conjectured equilibrium payoff distribution must coincide with the sample payoff distribution and therefore π^* must be an upper bound of an analogy grouping implying in fact that $\pi \leq \frac{s_K+s_{K-1}}{4} + \frac{x}{2}$ must hold for all $\pi \in (s_{K-1}, s_K]$. Otherwise, the buyer would expect to perceive valuations in the interval $(\pi, s_K]$ but never actually perceives these.

[18] Playing $\pi \in (s_l, s_{l+1}]$ where $l \neq K$ gives

$$\begin{aligned}
\sum_{i=0}^{l}\Pr(s &\in [s_i, s_{i+1}])[E(\tilde{v}|s_i \leq \tilde{s} \leq s_{i+1}) - \pi] \\
+\Pr(s &\leq \pi, s \in [s_l, s_{l+1}])[E(\tilde{v}|s_l \leq \tilde{s} \leq s_{l+1}) - \pi] \\
&= \sum_{i=1}^{l}(s_i - s_{i-1})[E(\tilde{v}|s_i \leq \tilde{s} \leq s_{i+1}) - \pi] + (\pi - s_l)[E(\tilde{v}|s_i \leq \tilde{s} \leq s_{i+1}) - \pi] \\
&= \sum_{i=1}^{l}(s_i - s_{i-1})[\frac{s_i + s_{i+1}}{2} + x - \pi] + (\pi - s_l)[\frac{s_l + s_{l+1}}{2} + x - \pi] \\
&= s_1(\frac{s_1}{2} + x - \pi) \\
&\quad +(s_2 - s_1)(\frac{s_1 + s_2}{2} + x - \pi) \\
&\quad ... \\
&\quad +(s_l - s_{l-1})(\frac{s_{l-1} + s_l}{2} + x - \pi) \\
&\quad +(\pi - s_l)(\frac{s_l + s_{l+1}}{2} + x - \pi) \\
&= \pi(\frac{s_l + s_{l+1}}{2} + x - \pi) - \frac{s_l s_{l+1}}{2}.
\end{aligned}$$

The expected payoff to the strategy $\pi = s_K$ yields

$$\begin{aligned} & s_K(\frac{s_{K-1}+s_K}{2} + x - s_K) - \frac{s_K s_{K-1}}{2} \\ = \quad & s_K(x - \frac{s_K}{2}). \end{aligned}$$

Let us now consider a small upward deviation $\pi' > s_K$. The payoff to this strategy yields

$$s_K(\frac{s_K}{2} + x - \pi') + (\pi' - s_K)(\frac{s_l + s_{l+1}}{2} + x - \pi').$$

Taking the derivative of this expression with respect to π' yields

$$-s_K + (\frac{s_K + s_{K+1}}{2} + x - \pi') - (\pi' - s_K) = (\frac{s_K + s_{K+1}}{2} + x - 2\pi'),$$

which, given that $\pi = s_K$ is an equilibrium, must be non-positive arbitrarily close to $\pi' = s_K$ implying $(\frac{s_{K+1}}{2} + x - \frac{3}{2}s_K) \leq 0$, or

$$\frac{s_{K+1}}{3} + \frac{2}{3}x \leq s_K. \tag{8}$$

On the other hand, since $\pi = s_K$ is optimal within $(s_{K-1}, s_K]$ we must have

$$s_K \leq \frac{s_{K-1}}{3} + \frac{2x}{3}. \tag{9}$$

The Conditions (8) and (9) yield a contradiction since $s_{K+1} > s_{K-1}$.

More generally, considering arbitrary analogy partitions, the analogy groups close to π^* must be singletons since otherwise the two marginal optimality conditions $\frac{s_{K+1}}{3} + \frac{2}{3}x \leq s_K$ and $s_K \leq \frac{s_{K-1}}{3} + \frac{2}{3}x$ could not be simultaneously satisfied. But if this holds, then the first inequality reads $\frac{s_K}{3} + \frac{2}{3}x \leq s_K$ and the second reads $s_K \leq \frac{s_K}{3} + \frac{2}{3}x$ yielding $s_K = x$. Thus the equilibrium strategy of the buyer must coincide with the Bayesian-Nash equilibrium strategy.

4.2 General environment

The main points of the previous example can be stated generally. To achieve this, let us define the primary class of games of monotone selection that Esponda studies.[19] In the case of "trade", each player's payoff only depends on the player's own action and a common value component, v_0, the exact value of which is typically unknown at the interim (time of decision) to at least one of the players. If the player does not trade, then payoff is zero. That the player trades is captured by the event $(a_{-i}, t_0) \in \Phi_i(\widehat{a}_i)$ to be determined shortly. Player i's payoff can now be written as

$$u_i(a, t_0, v_0) = \begin{cases} \widetilde{u}_i(a_i, v_0) & \text{for } (a_{-i}, t_0) \in \Phi_i(\widehat{a}_i) \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

where $\theta_0 = (t_0, v_0) \in T_0 \times V_0$ represents payoff uncertainty and the vector $\theta_1, ..., \theta_N \in \Theta_1 \times ... \times \Theta_N$ is the vector of types, one type for each of the players. Therefore, the type profile $\widetilde{\theta}$ is a $N + 2$-dimensional random variable. The following assumptions are made:

[19]The setup we use here is that of the on-line appendix of Esponda (2008) which is used to provide an example of a setup where the assumptions (some of which are on endogenous elements) of his main theorems hold.

1. $\widetilde{t_0}$ is independent of $\widetilde{\theta}_{-0}$,

2. $\widetilde{v}_0$ and $\widetilde{\theta}_{-0}$ are affiliated,[20]

3. $\widetilde{u}_i$ is increasing in v_0,

4. $\Phi_i(\widehat{a}_i)$ is non-decreasing in $\widehat{a}_i$ in the strong set order[21] and in the inclusion set order,

5. $\widetilde{u}_i(a_i, v_0)$ is supermodular.

Esponda illustrates his selection result by means of the so called *naive behavioral equilibrium* concept. Proposition 6 establishes that for each naive behavioral equilibrium one can find a corresponding PAE with an incorrect prior in the class of games satisfying (10) and Assumptions 1 to 5. The proof and the definition of the behavioral equilibrium is relegated to the appendix. Denote by $\varphi_i(\sigma_i, \sigma_{-i}, \theta_i)$ the probability that $(a_{-i}, t_0) \in \Phi_i(\widehat{a}_i)$ given σ_i, σ_{-i}, and θ_i.

Proposition 6 *Let for each θ_i, $\varphi_i(a_i, a_{-i}, \theta_i) > 0$ for some $a_i \in$ supp$[\sigma_i(\theta_i)]$ and $\sigma_{-i}(\theta_{-i}|\theta_i) > 0$ such that $a_{-i} \in$ supp$[\sigma_{-i}(\theta_{-i}|\theta_i)]$. Then every pure strategy naive behavioral equilibrium is equivalent to a PAE with $\Theta_i^V = \{(t_0, v_0, \theta)|$ there is $(a_{-i}, t_0) \in \Phi_i(\widehat{a})$ where $\widehat{a} \in$ supp$[\sigma_i(\theta_i)]$ and $a_{-i} \in$ supp$[\sigma^i_{-i}(.|\theta_i)]\ \}$ and the analogy partitions are the private information analogy partitions.*

To build a correspondence between the naive behavioral equilibrium and the PAE, we need one analogy group for each i. Moreover, the set of visible type profiles of i should coincide with the types of others from which this type "buys the product".

As illutrated in the bilateral trade example, the incorrect prior assumption plays a key role in cutting the markets even thinner than in the Bayesian-Nash equilibrium. When prior is known, equlibrium strategies in every PAE with a correct prior coincide with those in the unique BNE. The following proposition generalizes this point.

Proposition 7 *Assume that the game satisfies (10) and MSP (1-4) and that $\widetilde{u}_i(a_i, v_0)$ is twice differentiable with $\frac{\partial \widetilde{u}_i^2(a_i,v_0)}{\partial a_i \partial v_0} > 0$ and $\frac{\partial^2 \widetilde{u}_i(a_i,v_0)}{\partial a_i^2} < 0$ and that for all σ_{-i} ,θ_i, $\varphi_i(a_i, \sigma_{-i}, \theta_i)$ is twice differentiable with $\frac{\partial \varphi_i(a_i,\sigma_{-i},\theta_i)}{\partial a_i} > 0$, and for all i, a_{-i},and θ the μ^i-best-response of Player i is interior and unique. Then every strategy profile of a PAE with a correct prior coincides with a Bayesian-Nash equilibrium strategy profile.*

Let σ^* be a PAE strategy profile under a correct prior and assume to the contrary that this profile does not coincide with a BNE. Then there is Player i and $a_i^* \in \text{supp}[\sigma_i(\theta_i)]$ such that

[20] θ are affiliated if for all θ' and $\theta'' \in \Theta$, $p(\theta' \vee \theta'')p(\theta' \wedge \theta'') \geq p(\theta')p(\theta'')$ where $\theta' \vee \theta'' = (\max(\theta'_0, \theta''_0), ..., \max(\theta'_N, \theta''_N))$ and $\theta' \wedge \theta'' = (\min(\theta'_0, \theta''_0), ..., \min(\theta'_N, \theta''_N))$.

[21] A set $A \subset R$ is greater than a set $B \subset R$ in the strong set order if for any $a \in A$ and any $b \in B$, $max(a, b) = a \vee b \in A$ and $min(a, b) = a \wedge b \in B$.

$\frac{\partial \mathcal{E}^{\sigma,\mu}_{a_{-i},\theta} u_i(a_i^*, a_{-i}, \theta_i)}{\partial a_i} \neq 0$ where $\mathcal{E}^{\widehat{\sigma},\widehat{\mu}}_{a_{-i},\theta} u_i(a_i, a_{-i}, \theta_i)$ refers to the expected payoff of Player i playing a_i and holding conjectures $\widehat{\sigma}, \widehat{\mu}$, and σ, μ are the correct conjectures.

Suppose first that $\frac{\partial \mathcal{E}^{\sigma,\mu}_{a_{-i},\theta} u_i(a_i^*, a_{-i}, \theta_i)}{\partial a_i} > 0$ and let us show that this implies

$$lim_{a_i' \to_+ a_i^*} \frac{\mathcal{E}^{\widehat{\sigma}^i,\widehat{\mu}^i}_{a_{-i},\theta} u_i(a_i', a_{-i}, \theta_i) - \mathcal{E}^{\widehat{\sigma}^i,\widehat{\mu}^i}_{a_{-i},\theta} u_i(a_i^*, a_{-i}, \theta_i)}{a_i' - a_i^*} > 0$$

for correct prior PAE conjectures $\widehat{\sigma}^i, \widehat{\mu}^i$ of Player i. First notice that by set inclusion and strong set order, $\Phi(a_i^*) \subset \Phi(a_i')$ and if there are $(a'_{-i}, t_0') \in \Phi(a_i')$ and $(a''_{-i}, t_0'') \in \Phi(a_i^*)$, then $(a'_{-i}, t_0') \vee (a''_{-i}, t_0'') \in \Phi(a_i')$ and $(a'_{-i}, t_0') \wedge (a''_{-i}, t_0'') \in \Phi(a_i^*)$. Since for each $j \neq i$, $\widetilde{u}_j(a_i, v_0)$ is supermodular and v_0 and θ_{-0} are affiliated, the set of qualities v_0 conditional on $(a_{-i}, t_0) \in \Phi(a_i^*)$ is lower than the set of qualities conditional on $(a_{-i}, t_0) \in \Phi(a_i')$ in the same set orders. Denote the set of qualities conditional on $(a^*_{-i}, t_0) \in \Phi(a_i^*)$ where $a^*_{-i} \in \text{supp}[\sigma^*_{-i}(\theta)]$ by V_0^* and the set of qualities conditional on $(a^*_{-i}, t_0) \in \Phi(a_i')$ by V_0'.

Consider an arbitrary analogy class of i, α_i'' such that there is v_0'' such that $(t_0'', v_0'', \theta) \in \alpha_i''$ and $(a^*_{-i}, t_0'') \in \Phi(a_i^*)$ for some $a^*_{-i} \in \text{supp}[\sigma^*_{-i}]$. Consider then a $v_0' \in V_0' \backslash V_0^*$ and denote an arbitrary analogy class α_i' such that there exists t_0' s.t. $(a^*_{-i}, t_0') \in \Phi(a_i')$ for some $a^*_{-i} \in \text{supp}[\sigma^*_{-i}(\theta_{-i})]$ and $(t_0', v_0', \theta) \in \alpha_i'$. Clearly α_i' and α_i'' can never coincide, otherwise i would expect (where the expectations is taken using $\widehat{\mu}_i$) to observe v_0' when playing $\sigma_i^*(\theta_i)$ but never actually does this. Second in a PAE, i observes payoffs and thus $\mathcal{E}^{\widehat{\sigma}^i,\widehat{\mu}^i}_{a_{-i},\theta} u_i(a_i^*, a_{-i}, \theta_i) = \mathcal{E}^{\sigma,\mu}_{a_{-i},\theta} u_i(a_i^*, a_{-i}, \theta_i)$. Finally $lim_{a_i' \to_+ a_i^*} \frac{\mathcal{E}^{\widehat{\sigma}^i,\widehat{\mu}^i}_{a_{-i},\theta} u_i(a_i', a_{-i}, \theta_i) - \mathcal{E}^{\widehat{\sigma}^i,\widehat{\mu}^i}_{a_{-i},\theta} u_i(a_i^*, a_{-i}, \theta_i)}{a_i' - a_i^*} \geq lim_{a_i' \to_+ a_i^*} \frac{\mathcal{E}^{\sigma,\mu}_{a_{-i},\theta} u_i(a_i', a_{-i}, \theta_i) - \mathcal{E}^{\sigma,\mu}_{a_{-i},\theta} u_i(a_i^*, a_{-i}, \theta_i)}{a_i' - a_i^*}$ > 0 since if there is another $(\widehat{t}_0, \widehat{v}_0, \widehat{\theta}) \in \alpha_i'$ such that $\widehat{v}_0 \notin V_0'$ this necessarily has to satisfy $\widehat{v}_0 > v_0'$ since α_i' and α_i'' cannot coincide for any α_i'' as defined above.

A similar argument implies that $lim_{a_i' \to_- a_i^*} \frac{\mathcal{E}^{\widehat{\sigma}^i,\widehat{\mu}^i}_{a_{-i},\theta} u_i(a_i', a_{-i}, \theta_i) - \mathcal{E}^{\widehat{\sigma}^i,\widehat{\mu}^i}_{a_{-i},\theta} u_i(a_i^*, a_{-i}, \theta_i)}{a_i' - a_i^*} < 0$ and thus $\frac{\partial \mathcal{E}^{\sigma,\mu}_{a_{-i},\theta} u_i(a_i^*, a_{-i}, \theta_i)}{\partial a_i} < 0$ implies that $\frac{\partial \mathcal{E}^{\widehat{\sigma}^i,\widehat{\mu}^i}_{a_{-i},\theta} u_i(a_i^*, a_{-i}, \theta_i)}{\partial a_i} < 0$ for analogy-based expectations $\widehat{\sigma}^i, \widehat{\mu}^i$.Thus we have a contradiction.

Notice that the result hinges upon further qualifications beyond (10) and Conditions 1 to 5 characterizing the monotone selection environment. What is needed in addition is differentiability and unique best-replies. These imply that for every strategy profile that is not a BNE, there is a player with a marginal deviation that pays off. One can then show that the payoff-confirming requirement implies that the qualities not purchased, on the one hand, and the purchased ones, on the other, cannot belong to the same analogy class since otherwise a player would have to expect to purchase qualities that he never purchases thus violating the payoff-consistency condition. Coarse inference then implies that marginal upward deviations are too optimistically expected to select a larger range of non-observable higher qualities than actually is the case. Thus, if an upward deviation pays off for a standard player, it pays off for a player building expectations on coarse analogy-partitions. A similar argument implies that selection effects due to downward deviations are underestimated and thus, if a downward deviation pays off for a standard player, it pays off for a coarse analogy-partition player as well.

5 Conclusion

Jehiel and Koessler (2008) illustrate that rational players' optimal strategies can differ from a Bayesian-Nash equilibrium strategies when they base their conjectures on experience about others' behavior using stereotypical classifications. In this paper we consider such stereotypical learning, assuming that players use their performance (payoffs) as a means to verify the consistency of the learned conjectures. We have identified conditions under which such Payoff-confirming analogy equilibria can differ from Bayesian-Nash equilibria. By considering the analogy-partition as a description of a player's stereotypes, the PAE provides an interesting avenue for a game-theoretic analysis of these latter.

In a related paper, Esponda (2008) points out that payoff-information may have surprising consequences on steady states of learning if individuals are unable to detect correlations, as in the analogy equilibria, and base their understanding of the uncertainty of the environment merely on their observations:[22] if players' conjecture on the prior distribution of types may be incorrect and correlation between types and strategies is not understood, learning leads to an aggravation of adverse selection problems in common value environments. This paper complements the findings of Esponda by pointing out that the solution concept he uses can be considered as a Payoff-confirming AE in the class of monotone selection games studied by Esponda. Moreover, the current paper identifies conditions when a non-Bayesian Nash PAE exists even when conjecture on the prior is correct.

PAE is considered as a steady state of an underlying of ficticious play learning dynamic where players best-respond to the beliefs infuenced by others' history of play. Is there evidence that (stochastic) ficticious play is a prevalent mode of learning? On the one hand knowing one's best-response mapping and others' history of play, does not seem to be necessary for convergence (Van Huyck et al., 2007) and even mere reinforcement learning models (which do not require such information) fit data and predict well (Erev and Roth 1998; Erev et al. 2007). Yet, adding information about one's best-response mapping and the history of others' actions (as required by ficticious play models) speeds up convergence (Van Huyck et al. 2007), and adding components of best-responding to reinforcement learning models improves both fit and predictive power (Erev and Roth 1998; Camer and Ho, 1999; Erev et al. 2007) in contexts where all necessary information for best-response learning is available. Given that theoretically the main difference in empirical patterns of reinforcement and stochastic ficticious play dynamics is the speed of convergence (Hopkins 2002), the evidence is certainly not dismissive of the importance of ficticious play learning.

[22] Esponda points out that the mere existence of naive players is sufficient for the phenomenon - not every player needs to fail to account for correlations, i.e. to be *naive*.

There are few experimental studies of cross-game learning. Cooper and Kagel (2008) point out that exposure to an analogous but different game prior playing a game speeds up learning and convergence. In their design, the two games do not appear in alternating or random order but the repeat interactions are organized by design into two very distinct regimes. Thus their design does not allow for testing the present theory. Huck et al. (2009) carry out a very direct test of the AE model and study learning in an environment where two complete information stage games are being played in random order and players receive feedback about opponents' play in the two games in varying degrees of coarseness. Players receive no feedback about own payoffs and thus any channel for reinforcement learning is shut down by design. They find that very explicit game-specific feedback ultimately leads to the play of Nash equilbrium in each game but coarser forms of feedback lead players to converge to the unique analogy equilibrium. By allowing subjects to learn own payoffs, their design could be easily extended to experimentally test the relevance of the payoff-confirming refinement. This is left for future research.

6 Appendix

6.1 Proof of Proposition 2

If AE is BNE, then by Proposition 1 the AE is PAE.

On the other hand, if (4) or (5) holds for each i such that $s_i(\theta^m) \neq s_i(\theta^n)$ and $\mathcal{A}_j = \{\{\theta^1, \theta^2\}\}$, then the AE is a PAE by lemma 1 below.

Consider now an AE which is a PAE and suppose to the contrary that the AE is not a BNE and there is i such that $s_i(\theta^m) \neq s_i(\theta^n)$ and $\mathcal{A}_j = \{\{\theta^1, \theta^2\}\}$ and neither (4) nor (5) holds. Thus, by lemma 2, AE is not a PAE - a contradiction.

Lemma 1 *Let $N = 2$, $\Theta = \{\theta^1, \theta^2\}$ and $A_i = \{a_i^1, a_i^2\}$. Suppose that s is a pure strategy profile of an AE. Let for each i such that $s_i(\theta^m) \neq s_i(\theta^n)$ and $\mathcal{A}_j = \{\{\theta^1, \theta^2\}\}$either (4) or (5) hold.*

Then the AE is PAE.

Define the probability of j getting payoff u given strategy profile σ as

$$g_\sigma^j(u) = \sum_{\{a,\theta | u = u_j(a,\theta)\}} p(\theta)\sigma_i(a_i|\theta)\sigma_j(a_j|\theta).$$

Let there be i such that $s_i(\theta^m) \neq s_i(\theta^n)$ and $\mathcal{A}_j = \{\{\theta^1, \theta^2\}\}$. There are three subcases to consider: first

$$\begin{aligned} u_j(s_j(\theta^m), r_i(\theta^m); \theta^m) &= u_j(s_j(\theta^m), s_i(\theta^m); \theta^m) \\ &= u_j(s_j(\theta^n), r_i(\theta^n); \theta^n) \\ &= u_j(s_j(\theta^n), s_i(\theta^n); \theta^n)) \end{aligned}$$

in which case both conditions hold. In this first subcase trivially $g^j_{(s_j,\widehat{\sigma}^j_i)}(u_j(s_j(\theta^m), s_i(\theta^m); \theta^m)) = 1 = g^j_s(u_j(s_j(\theta^m), s_i(\theta^m); \theta^m))$. Thus, s is a payoff confirming analogy equilibrium.

In the second case only (5) holds but not (4). For each θ^m, the perceived probability that $u_j(s_j(\theta^m), s_i(\theta^m); \theta^m)$ results is

$$\begin{aligned} g^j_{(s_j,\widehat{\sigma}^j_i)}(u_j(s_j(\theta^m), s_i(\theta^m); \theta^m)) &= [f(\theta^m)]\beta^j(s_i(\theta^m)) + [1 - f(\theta^m)]\beta^j(s_i(\theta^m)) \\ &= [f(\theta^m)]^2 + [1 - f(\theta^m)]f(\theta^m) \\ &= f(\theta^m) \\ &= g^j_s(u_j(s_j(\theta^m), s_i(\theta^m); \theta^m)). \end{aligned}$$

Thus, s is a payoff confirming analogy equilibrium.

Third, if only (4) holds and not (5), we have that

$$\begin{aligned} g^j_{(s_j,\widehat{\sigma}^j_i)}(u_j(s_j(\theta^n), s_i(\theta^n); \theta^n)) &= [f(\theta^n)][\beta^j(s_i(\theta^n)) + \beta^j(r_i(\theta^n))] \\ &= [f(\theta^n)] \\ &= g^j_s(u_j(s_j(\theta^n), s_i(\theta^n); \theta^n)). \end{aligned}$$

Thus, s is a payoff confirming analogy equilibrium.

Lemma 2 *Let $N = 2$, $\Theta = \{\theta^1, \theta^2\}$ and $A_i = \{a^1_i, a^2_i\}$. Suppose that s is a pure strategy profile of an AE.*

If there is i such that $s_i(\theta^m) \neq s_i(\theta^n)$ and $\mathcal{A}_j = \{\{\theta^1, \theta^2\}\}$ and neither

$$\text{for all } m,\ u_j(s_j(\theta^m), r_i(\theta^m); \theta^m) = u_j(s_j(\theta^m), s_i(\theta^m); \theta^m)$$

nor

$$\text{for all } m,\ u_j(s_j(\theta^m), r_i(\theta^m); \theta^m) = u_j(s_j(\theta^n), s_i(\theta^n); \theta^n))$$

where $r_i(\theta^m)$ is the action not chosen by i at θ^m,

then the AE is not a PAE.

We use proof by contradiction. There are two subcases to consider. Suppose first, that there is m such that

$$u_j(s_j(\theta^m), r_i(\theta^m); \theta^m) \notin \{u_j(s_j(\theta^m), s_i(\theta^m); \theta^m),\ u_j(s_j(\theta^n), s_i(\theta^n); \theta^n))\}. \tag{11}$$

Define the probability of j getting payoff u given strategy profile σ as

$$g^j_\sigma(u) = \sum_{\{a,\theta | u = u_j(a,\theta)\}} p(\theta)\sigma_i(a_i|\theta_i)\sigma_j(a_j|\theta_j).$$

Since $s_i(\theta^m) \neq s_i(\theta^n)$, j expects $u_j(s_j(\theta^m), r_i(\theta^m); \theta^m)$ to result with a positive probability,

$$g^j_{s_j,\widehat{\sigma}^j_i}(u_j(s_j(\theta^m), r_i(\theta^m); \theta^m)) > 0$$

But since (12) holds, $g^j_{s_j,\sigma_i}(u_j(s_j(\theta^m), r_i(\theta^m); \theta^m)) = 0$ which contradicts the consistency condition of PAE and thus the AE is not PAE.

In the second subcase, suppose in addition to $s_i(\theta^m) \neq s_i(\theta^n)$ that there is m and i such that

$$\begin{aligned} u_j(s_j(\theta^m), r_i(\theta^m); \theta^m) &= u_j(s_j(\theta^m), s_i(\theta^m); \theta^m) \\ &= u_j(s_j(\theta^n), r_i(\theta^n); \theta^n)) \\ &\neq u_j(s_j(\theta^n), s_i(\theta^n); \theta^n)) \end{aligned}$$

Then

$$\begin{aligned} g^j_{(s_j,\widehat{\sigma}^j_i)}(u_j(s_j(\theta^m), s_i(\theta^m); \theta^m)) &= f(\theta^m) + f(\theta^n)\beta^j(r_i(\theta^n)) \\ &\neq f(\theta^m) \\ &= g^j_s(u_j(s_j(\theta^m), s_i(\theta^m); \theta^m)) \end{aligned}$$

and thus AE is not PAE.

6.2 Proof of Proposition 3

Let for each j and α_j such that $\sigma_{-j}(\theta^m) \neq \sigma_{-j}(\theta^n)$ and $\theta^m, \theta^n \in \alpha_j$, for all $\theta \in \alpha_j$, for all action profiles of players other than j, $a^*_{-j} = (a^*_1, ..., a^*_{j-1}, a^*_{j+1}, ..., a^*_N)$ such that for each $i \neq j$ there is $\theta' \in \alpha_j$ such that $a^*_i \in \text{supp}[\sigma_i(\theta')]$ for all $a_j \in \text{supp}[\sigma_j(\theta)]$,

$$u_j(a_j, a^*_{-j}; \theta) = u_\theta.$$

Now,

$$\begin{aligned} &\sum_{\{a,\theta|u_j=u_j(a,\theta)\}} p(\theta)\widehat{\sigma}_{-j}(a_{-j}|\theta_{-j})\sigma_j(a_j|\theta_j) \\ =& \sum_{\alpha_j\in\mathcal{A}_j} \sum_{\{(a_j,\theta)|u_j=u_j(a,\theta),\ \theta\in\alpha_i\}} p(\theta)\widehat{\sigma}_{-j}(a_{-j}|\theta_{-j})\sigma_j(a_j|\theta_j) \\ =& \sum_{\alpha_j\in\mathcal{A}_j} \sum_{\theta^n\in\alpha_j} p(\theta^n)\sigma_j(a_j|\theta^n_j) \sum_{a^*_{-j}} \widehat{\sigma}_{-j}(a^*_{-j}|\theta^n_{-j}) \\ =& \sum_{\alpha_j\in\mathcal{A}_j} \sum_{\theta^n\in\alpha_j} p(\theta^n)\sigma_j(a_j|\theta^n_j) \\ =& \sum_{\alpha_j\in\mathcal{A}_j} \sum_{\theta^n\in\alpha_j} p(\theta^n)\sigma_j(a_j|\theta^n_j) \sum_{a_{-j}} \sigma_{-j}(a_{-j}|\theta_j) \\ =& \sum_{\alpha_j\in\mathcal{A}_j} \sum_{\{a,\theta|u_i=u_i(a,\theta),\theta\in\alpha_j\}} p(\theta)\sigma_{-j}(a_{-i}|\theta_{-j})\sigma_j(a_j|\theta_j) \\ =& \sum_{\{a,\theta|u_i=u_i(a,\theta)\}} p(\theta)\sigma_{-j}(a_{-i}|\theta_{-j})\sigma_j(a_j|\theta_j) \end{aligned}$$

where the second equality follows from the fact that, in an analogy class, for a state in the class and for an action that is chosen with a positive probability by j in that state, the payoff is the same for any action profile of players other than j to which $\widehat{\sigma}_{-j}$ assigns a positive probability. The third and the fourth equality follow because a conjecture and a strategy is a probability distribution and thus sums up to one. ($\sum_{a^*_{-j}} \widehat{\sigma}_{-j}(a_{-j}|\theta_{-j}) = 1 = \sum_{a_{-j}} \sigma_{-j}(a_{-j}|\theta_j)$) and only actions which are assigned a positive probability in the average strategy of the analogy class can be assigned a positive probability in the actual strategy. Let for each j and α_j such that $\sigma_{-j}(\theta^m) \neq \sigma_{-j}(\theta^n)$ and $\theta^m, \theta^n \in \alpha_j$, for all action profiles of players other than j, $a^*_{-j} = (a^*_1, ..., a^*_{j-1}, a^*_{j+1}, ..., a^*_N)$ such that for each $i \neq j$ there is $\theta' \in \alpha_j$ such that $a^*_i \in \text{supp}[\sigma_i(\theta')]$, for all $\theta \in \alpha_j$ and $a_j \in \text{supp}[\sigma_j(\theta)]$

$$u_j(a^*_j, a^*_{-j}; \theta) = u_{a^*_{-j}}.$$

Now,

$$\begin{aligned}
& \sum_{\{a,\theta|u_j=u_j(a,\theta)\}} p(\theta)\widehat{\sigma}_{-j}(a_{-j}|\theta_j)\sigma_j(a_j|\theta_j) \\
= & \sum_{\alpha\in\mathcal{A}_j} \sum_{\{(a,\theta)|u_j=u_j(a,\theta),\ \theta\in\alpha\}} p(\theta)\sigma_j(a_j|\theta_j)\widehat{\sigma}_{-j}(a_{-j}|\theta) \\
= & \sum_{\alpha\in\mathcal{A}_j} \sum_{a^*_{-j}} \widehat{\sigma}_{-j}(a^*_{-j}|\alpha) \sum_{\theta\in\alpha} p(\theta)\sigma_j(a_j|\theta_j) \\
= & \sum_{\alpha\in\mathcal{A}_j} \sum_{\theta\in\alpha} p(\theta)\sigma_j(a_j|\theta_j) \\
= & \sum_{\alpha\in\mathcal{A}_j} \sum_{\{(a_j,\theta)|u_j=u_j(a,\theta),\ \theta\in\alpha\}} p(\theta)\sigma_j(a_j|\theta_j) \sum_{a_{-j}} \sigma_{-j}(a_{-j}|\theta_{-j}) \\
= & \sum_{\alpha\in\mathcal{A}_j} \sum_{\{(a,\theta)|u_j=u_j(a,\theta),\ \theta\in\alpha\}} p(\theta)\sigma_j(a_j|\theta_j)\sigma_{-j}(a_{-j}|\theta_{-j}) \\
= & \sum_{\{a,\theta|u_i=u_i(a,\theta)\}} p(\theta)\sigma_{-j}(a_{-i}|\theta_{-j})\sigma_j(a_j|\theta_j)
\end{aligned}$$

where the second equality follows from the fact that for a given action profile $a^*_{-j} \in \text{supp}[\widehat{\sigma}_{-j}(\alpha)]$ the payoff $u_j(a_j, a^*_{-j}, \theta)$ is the same for each θ in α_j and (a_j, θ) such that $a_j \in \text{supp}[\sigma_j(\theta)]$. The third and the fourth equality follow because a strategy and a conjecture are probability distributions and only actions which are assigned a positive probability in the average strategy of the analogy class can be assigned a positive probability in the actual strategy.

6.3 Proof of Proposition 4

Consider an analogy class of a player, denote her by i, where the condition in Proposition 3 is violated. Let K be the number of states in the analogy class. Let s^l_{-i} be the action profile of others chosen at state l. Let us construct a square matrix $U := [u(s^*_i(\theta^k_i), s^l_{-i}, \theta^k)]^K_{k,l=1}$. The payoff-confirming condition can be written as a system of $\#\{u_i(s^*_i(\theta^k_i), s^*_{-i}(\theta^k), \theta^k)|k = 1, ..., K\}$ equations, that is, the number of actual generic payoffs which cannot be higher than K by construction. Define $f^u(p)$ as the difference in the expected and the actual probability of payoff

u given p. An equation of the system for a given generic payoff u reads

$$\begin{aligned} f^u(p) &= \sum_{k=1}^{K} p_k \sum_{l=1}^{K} \widehat{\sigma}^i_{-i}(s^l_{-i}|\theta^k)\mathcal{I}(U_{lk}=u) - \sum_k p_k \mathcal{I}(U_{kk}=u) = 0 \\ &\Leftrightarrow \sum_{k=1}^{K} p_k \sum_{l=1}^{K} \frac{\sum_m p_m \mathcal{I}(s^m_{-i}=s^l_{-i})}{\sum_m p_m}\mathcal{I}(U_{lk}=u) - \sum_k p_k \mathcal{I}(U_{kk}=u) = 0 \end{aligned}$$

where $\mathcal{I}(U_{lk}=u)$ is the indicator that the element at the lth row and kth column of U satisfies $U_{lk}=u$. The first term of the difference is the probability by which the player expects payoff u and the second term is the actual probability of payoff u. The condition says merely that these probabilities must coincide.

The effect of a change of the prior probability p_κ of a given state θ^κ on $f^u(p)$ is given by

$$\begin{aligned} f^u_\kappa(p) &= \sum_l \widehat{\sigma}^i_{-i}(s^l_{-i}|\theta^\kappa)\mathcal{I}(U_{l\kappa}=u) - \mathcal{I}(U_{\kappa\kappa}=u) \\ &+ \sum_k p_k \frac{\partial \sum_l \widehat{\sigma}^i_{-i}(s^l_{-i}|\theta^k)}{\partial p_\kappa}\mathcal{I}(U_{lk}=u) \\ &= \sum_l \frac{\sum_m p_m \mathcal{I}(s^m_{-i}=s^l_{-i})}{\sum_k p_k}\mathcal{I}(U_{l\kappa}=u) - \mathcal{I}(U_{\kappa\kappa}=u) \\ &+ \sum_k p_k \sum_l \mathcal{I}(U_{lk}=u)\left(\frac{1}{\sum_m p_m} - \frac{\sum_m p_m \mathcal{I}(s^m_{-i}=s^l_{-i})}{(\sum_m p_m)^2}\right) \end{aligned}$$

Since the condition in Proposition 3 is violated, there must be two different states $k \neq \kappa$ and actions $l \neq \lambda$ chosen with a positive probability in the analogy class such that $U_{kl} = U_{\kappa\lambda}$.

Consider a perturbation of the prior probabilities of two states, k and κ only, so that $\Delta p_k = -\Delta p_\kappa$. The prior probabilities of the other states remain unaltered. Thus, the initial strategies do not constitute a PAE of the perturbed game if $-f^u_k(p)\Delta p_k \neq f^u_\kappa(p)\Delta p_\kappa$ or

$$-\frac{\Delta p_k}{\Delta p_\kappa} \neq \frac{f^u_\kappa(p)}{f^u_k(p)}$$

where by construction $f^u_k(p), f^u_\kappa(p) \neq 0$. Thus, almost all such perturbations destabilize the original PAE and such perturbations exist for any $\varepsilon > 0$.

The converse holds by Proposition 3.

6.4 Behavioral equilibrium and PAE

Assumptions on player's conjectures and how they handle information in a behavioral equilibrium are the following:

(Behavioral equilibrium[23])

[23] This corresponds to Esponda's naive behavioral equilibrium.

B1 Players believe that $\widetilde{t_0}$ is independent of $(\widetilde{\theta}, \widetilde{a}_{-i})$.

B2 Every player has correct conjectures about the probability of $(\widetilde{t_0}, \widetilde{a}_{-i}) \in \Phi_i(\widehat{a})$ (and therefore about $(\widetilde{t_0}, \widetilde{a}_{-i}) \notin \Phi_i(\widehat{a})$) given θ_i.

B3 Payoffs are observed.

B4 Players suppose that the opponents' actions are independent of the opponents' types.

[Proof of Proposition 6]

We will first show that the properties of the behavioral equilibrium, B1-B4, in the given context, (10) and properties 1 to 4, imply that players must hold certain equilibrium conjectures. We then show that a PAE with an incorrect prior, with the private information analogy partitions, and with $\widehat{\Theta}^i_V = \{(t_0, v_0, \theta)|$ there is $(a_{-i}, t_0) \in \Phi_i(\widehat{a})$ where $\widehat{a} \in$ supp $[\sigma_i(\theta_i)]$ and $a_{-i} \in \text{supp}[\sigma^i_{-i}(.|\theta_i)]\}$ satisfies these conjectures.

Given strategies σ, B4 implies the following conjectures on player strategies: type θ_i of Player i conjectures that every opponent type profile that she conjectures to have a positive probability plays the marginal distribution of actions conditional on θ_i

$$\widehat{\sigma}^i_{-i}(a_{-i}|\theta_i) = \sum_{\theta_{-i} \in \Theta_{-i}} \sigma_{-i,0}(a_{-i}|\theta_{-i,0}) p(\theta_{-i}|\theta_i). \tag{12}$$

B4 and the fact that actions are observed implies that conjectures about strategies satisfy (13). B1, B2, that a_{-i} are observed for each θ_i and the fact that $\varphi_i(\sigma_i, \sigma_{-i}, \theta_i)$ depends only on (a_{-i}, t_0) for each σ_i implies that the marginal conjecture about t_0 must be correct. If i had an incorrect conjecture about the distribution of t_0, she would necessarily have an incorrect probability estimate about $\varphi_i(\sigma_i, \sigma_{-i}, \theta_i)$. Thus,

$$\widehat{\mu}^i(t_0) = \sum_{v_0, \theta_{-i} \in V_0 \times \Theta_{-i,0}} p(v_0, \theta_{-i,0}, t_0|\theta_i). \tag{13}$$

Given θ_i, by assumption $\varphi_i(\sigma_i, \sigma_{-i}, \theta_i) > 0$ and therefore, by B2,
$\varphi_i(\sigma_i, \sigma_{-i}, \theta_i) = \varphi_i(\sigma_i(\theta_i), \widehat{\sigma}^i_{-i}(.|\theta_i), \theta_i) > 0$. B3 and the fact that u_i is increasing in v_0 implies that Player i observes the realization of v_0 conditional on $(a_{-i}, t_0) \in \Phi_i(\widehat{a})$ where $\widehat{a} \in$ supp $[\{\sigma_i(\theta_i)]$, and $a_{-i} \in \text{supp}[\widehat{\sigma}^i_{-i}(.|\theta_i)]$, call such a v_0 an observed v_0 and a v_0 which is not observed a non-observed v_0.

By B3, the equilibrium conjectures μ^i must deem all observed v_0 and none of the non-observed v_0 to have a positive probability (i.e. $\theta \in \widehat{\Theta}^i_V$ if and only if it is observed). Since by B4 actions are non-correlated with types, every expected type must play the same strategy: the strategy described in (13). Therefore

$$\sum_{\theta_{-i,0}\in\Theta_{-i,0}} \widehat{\mu}^i(v_0,\theta_{-i,0})$$
$$= \frac{1}{\varphi_i(\sigma_i(\theta_i),\sigma_{-i}(\theta_{-i}|\theta_i),\theta_i)} \times \sum_{\substack{\{(a_i,a_{-i},t_0,\theta_{-i,0})| \\ (a_{-i},t)\in\Phi_i(\widehat{a}_i)\}}} \{\sigma_i(a_i|\theta_i)\sigma_{-i}(a_{-i}|\theta_{-i}) \times p(v_0,\theta_1,...,\theta_n,t_0|\theta_i)\}.$$

Let's verify that a PAE with an incorrect prior, with the private information analogy partitions, and with $\widehat{\Theta}^i_V = \{(t_0,v_0,\theta)|$ there is $(a_{-i},t_0)\in\Phi_i(\widehat{a})$ where $\widehat{a}\in \text{supp}[\sigma_i(\theta_i)]$and $a_{-i}\in \text{supp}[\sigma^i_{-i}(.|\theta_i)]\}$ satisfies these restrictions. It is easy to see that given private information analogy classes, for each θ_i, equation (13) imposes the third condition in Definition 2. Distribution of t_0 is independent and thus $\frac{p(t_0,v_0,\theta_{-0}|\theta_i)}{\sum_{\theta\in\widehat{\Theta}^i_V}} = \frac{p(t_0)p(v_0,\theta_{-0}|\theta_i)}{\sum_{\theta\in\widehat{\Theta}^i_V}} = \frac{\widehat{\mu}^i(t_0)p(v_0,\theta_{-0}|\theta_i)}{\sum_{\theta\in\widehat{\Theta}^i_V}}$ by (14). The behavioral equilibrium puts no particular restriction on what should be expected about $\theta_{-i,0}$. We can just as well adopt the assumption that $\widehat{\mu}^i(v_0,\theta_{-i,0}) = \frac{p(v_0,\theta_{-i,0}|\theta_i)}{\sum_{\theta\in\widehat{\Theta}^i_V} p(\theta|\theta_i)}$ which implies that $\frac{p(t_0,v_0,\theta_{-0}|\theta_i)}{\sum_{\theta\in\widehat{\Theta}^i_V}} = \widehat{\mu}^i(t_0)\widehat{\mu}^i(v_0,\theta_{-i,0}) = \widehat{\mu}^i(t_0,v_0,\theta_{-i,0})$ as required by the second condition of Definition 2.[24] Therefore, for each BE we have established a corresponding PAE with an incorrect prior.

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