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The heat kernel on curvilinear polygonal domains in surfaces

Medet Nursultanov^{1,2} · Julie Rowlett³ · David Sher⁴

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Abstract

We construct the heat kernel on curvilinear polygonal domains in arbitrary surfaces for Dirichlet, Neumann, and Robin boundary conditions as well as mixed problems, including those of Zaremba type. We compute the short time asymptotic expansion of the heat trace and apply this expansion to demonstrate a collection of results showing that corners are spectral invariants.

Résumé

Nous construisons le noyau de la chaleur pour des domaines polygonaux curvilignes dans des surfaces arbitraires avec des conditions aux bords de Dirichlet, Neumann et Robin ainsi que des conditions mixtes, y compris celles de type Zaremba. Nous calculons l'expansion asymptotique de la trace quand le temps approche zéro et nous utilisons cette expansion pour démontrer un ensemble de résultats montrant que les coins sont des invariants spectraux.

Keywords Curvilinear polygon · Surface with corners · Corner · Edge · Conic singularity · Heat kernel · Heat trace · Spectrum · Isospectral · Spectral invariant · Inverse spectral problem · Robin boundary condition · Vertex · Dirichlet boundary condition · Neumann boundary condition · Zaremba boundary condition · Mixed boundary conditions

Mathematics Subject Classification Primary 58J35 · 35K08 · 58J53 · 58J50; Secondary 44A10 · 35A22 · 47A60

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1 Introduction

If two compact Riemannian manifolds (M, g) and (M', g') are isospectral, meaning they have the same Laplace spectrum, then they need not be isometric. However, isospectrality does imply that M and M' both have the same dimension, n . Moreover, they also have the same n -dimensional volume. Thus, both dimension and volume are spectral invariants, in the sense that they are determined by the spectrum. This fact follows from Weyl's law [52], proven over one hundred years ago. It is natural to ask: what other geometric features are spectral invariants?

The next geometric spectral invariant was discovered by Pleijel [44] some forty years after Weyl's law. For an n -dimensional Riemannian manifold with smooth boundary, the $n - 1$ dimensional volume of the boundary is a spectral invariant. About ten years later, McKean and Singer [37] proved that certain curvature integrals are also spectral invariants. For smooth surfaces and smoothly bounded planar domains, McKean and Singer [37] and independently M. Kac [22] proved that the Euler characteristic is a spectral invariant. By the Gauss–Bonnet theorem, this shows that the number of holes in a planar domain is a geometric spectral invariant for a planar domain.

The tactic of both McKean and Singer and Kac was to use the existence of a short time asymptotic expansion for the heat trace, together with the calculation of the coefficients in this expansion. Recall that the *heat kernel* $H^M(t, z, z')$ on a Riemannian manifold (M, g) is the fundamental solution of the heat equation on M :

$$\begin{cases} (\partial_t + \Delta_z)H^M(t, z, z') = 0; \\ \lim_{t \rightarrow 0} H^M(t, z, z') = \delta_z(z'). \end{cases} \quad (1.1)$$

As long as the eigenvalues are discrete and approach ∞ sufficiently quickly, which is the case in all geometric settings considered here, the *heat trace* is the trace of this kernel, and satisfies

$$\text{Tr } H^M(t) = \int_M H^M(t, z, z) dz = \sum_{j=1}^{\infty} e^{-\lambda_j t}. \quad (1.2)$$

Above, λ_j are the eigenvalues of the Laplacian Δ on M , arranged in increasing order. As a consequence, the heat trace is a spectral invariant. Therefore the coefficients in its asymptotic expansion as $t \rightarrow 0$ are also spectral invariants. The existence and calculation of an asymptotic expansion for the heat trace is a powerful method for producing spectral invariants. This program has been carried out extensively both for smooth manifolds and for manifolds with boundary [37].

Here, we are interested in the heat kernel on curvilinear polygonal domains which are subsets of smooth surfaces. This includes curvilinear polygonal domains in the plane, as well as more exotic non-planar examples. We are interested in the heat kernel for such domains in part because it may allow us to determine new geometric spectral invariants. Indeed, we show in Sect. 6 that in general, the presence or lack of vertices is a spectral invariant for Dirichlet, Neumann, Robin, and mixed boundary conditions. Moreover, we shall see there that a jump in boundary condition is also a spectral invariant.

Let us now introduce our geometric setting.

Definition 1.1 We say that Ω is a *curvilinear polygonal domain* if it is a compact subset of a smooth Riemannian surface (M, g) with piecewise smooth boundary and a vertex at each

non-smooth point of $\partial\Omega$. A *vertex* is a point p on the boundary of Ω at which the following are satisfied.

1. The boundary in a neighborhood of p is defined by a continuous curve $\gamma(t) : (-a, a) \rightarrow M$ for $a > 0$ with $\gamma(0) = p$. We require that γ is smooth on $(-a, 0]$ and $[0, a)$, with $\|\dot{\gamma}(t)\| = 1$ for all $t \in (-a, a)$, and such that

$$\lim_{t \uparrow 0} \dot{\gamma}(t) = v_1, \quad \lim_{t \downarrow 0} \dot{\gamma}(t) = v_2,$$

for some vectors $v_1, v_2 \in T_p M$, with $-v_1 \neq v_2$.

2. The *interior angle* at the point p is the *interior angle* at that corner, which is the angle between the vectors $-v_1$ and v_2 .

Note that requiring $-v_1$ and v_2 to be distinct means that the interior angle will be an element of $(0, 2\pi)$, which rules out inward and outward pointing cusps. An angle of π is allowed.

A vertex in a curvilinear polygonal domain is an example of a conical singularity where the link is a one-dimensional manifold with boundary. Moreover, it is a “non-exact” conical singularity in the sense that the curve γ defining the boundary near a vertex may have non-zero geodesic curvature on the entire interval $(-a, a)$. For curvilinear domains in the plane, this means that there need not be a neighborhood of the vertex in which the edges are straight. This geometric setting is therefore not contained within the literature for either (1) conical singularities whose link is a compact manifold without boundary nor for (2) planar polygons for which the edges are straight near the vertices.

There is substantial work in the literature on heat trace expansions in the settings (1) and (2). For conical singularities with no boundary on the link, a non-exhaustive list of works concerning the heat kernel and its trace is: [12, 13, 24, 26, 29, 30, 41, 46]. In the case (2) of vertices which locally have straight edges, a non-exhaustive list includes [23, 27, 42, 48].

For polygonal domains in the plane with the Dirichlet boundary condition, Fedosov showed in the 1960s that the vertices produce an extra term in the short time asymptotic expansion of the heat trace [9, 10]. This term appears in the coefficient of t^0 . Its most simplified form and calculation can be found in a paper of van den Berg and Srisatkunarajah [51], although the expression there is originally due to unpublished work of Ray, and is mentioned in both [22] and [37].

Although it has been widely assumed that analogous results for the heat trace expansion hold for curvilinear polygons, a rigorous proof even in the planar Dirichlet case was not given until [31]. Similar results hold for Neumann boundary conditions, see [34]. Although Robin conditions have been studied on manifolds with boundary [14, 54], to our knowledge there is no work in the literature about heat trace expansions with Robin conditions in the presence of corners of arbitrary angles, even in the plane. For certain corner angles, however, we refer to the physical approach of [3]. Outside the planar case, or even in the planar case with mixed boundary conditions, less is known. For the mixed boundary condition, also known as Zarembo boundary condition, references include [4, 21, 28, 47].

Our geometric microlocal methods allow us to handle the general case of compact curvilinear polygonal domains in surfaces, with any combination of Dirichlet, Neumann, and/or Robin boundary conditions on the various smooth boundary components. The sign convention for our Laplacian, in local coordinates, with respect to the Riemannian metric, g , on a surface is

$$\Delta = -\frac{1}{\sqrt{\det(g)}} \sum_{i,j=1}^2 \partial_i \sqrt{\det(g)} g^{ij} \partial_j.$$

Our convention for the Robin boundary condition on any portion of the boundary is:

$$\frac{\partial u}{\partial \nu} \Big|_{\partial \Omega} = \kappa u \Big|_{\partial \Omega}.$$

Here, the derivative on the left is the *inward* pointing normal derivative, and therefore, on the right, κ is a *non-negative* function. Under this condition the spectrum is non-negative. We assume throughout, for simplicity, that κ is smooth.

Our main result is:

Theorem 1.2 *Let Ω be a curvilinear polygonal domain in a smooth surface with finitely many vertices V_1, \dots, V_n of angles $\alpha_1, \dots, \alpha_n$. Define its edges E_1, \dots, E_n by letting E_j be the segment of the boundary between V_{j-1} and V_j , with subscripts taken mod n . Let \mathcal{E}_D , \mathcal{E}_N , and \mathcal{E}_R be three disjoint sets whose union is $\{1, \dots, n\}$. For each $j \in \mathcal{E}_D$, \mathcal{E}_N , and \mathcal{E}_R , we impose Dirichlet, Neumann, and Robin conditions with parameter $\kappa_j(x)$, respectively, along E_j . Assume that all functions $\kappa_j(x)$ are non-negative and smooth.*

Let $\mathcal{V}_=$ be the set of j for which vertex V_j has either zero or two Dirichlet edges adjacent to it, i.e. either both j and $j+1 \in \mathcal{E}_D$ or neither are. Conversely, let \mathcal{V}_\neq be the set of j for which V_j has exactly one adjacent Dirichlet edge. Also let $K(z)$ and $k_g(x)$ be the Gauss curvature and geodesic/mean curvature of Ω and $\partial \Omega$ respectively.

Then the heat trace $\text{Tr } H^\Omega(t)$ for the Laplacian with those boundary conditions, and with the Friedrichs extension at each vertex, has a complete polyhomogeneous conormal expansion in t as $t \rightarrow 0$. Moreover, the first few terms of this expansion have the form

$$\text{Tr } H^\Omega(t) = a_{-1}t^{-1} + a_{-1/2}t^{-1/2} + a_0 + O(t^{1/2} \log t),$$

where:

$$a_{-1} = \frac{A(\Omega)}{4\pi}; \tag{1.3}$$

$$a_{-1/2} = \frac{1}{8\sqrt{\pi}} \left(\sum_{j \notin \mathcal{E}_D} \ell(E_j) - \sum_{j \in \mathcal{E}_D} \ell(E_j) \right); \tag{1.4}$$

$$a_0 = \frac{1}{12\pi} \int_{\Omega} K(z) dz + \frac{1}{12\pi} \int_{\partial \Omega} k_g(x) dx - \frac{1}{2\pi} \sum_{j \in \mathcal{E}_R} \int_{E_j} \kappa_j(x) dx \tag{1.5}$$

$$+ \sum_{j \in \mathcal{V}_=} \frac{\pi^2 - \alpha_j^2}{24\pi\alpha_j} + \sum_{j \in \mathcal{V}_\neq} \frac{-\pi^2 - 2\alpha_j^2}{48\pi\alpha_j}. \tag{1.6}$$

Remark 1.3 It is well-known that if the boundary of Ω is smooth then there are no logarithmic terms in the heat expansion. We do not characterize the nature of logarithmic terms in the expansion in our more general setting.

The proof of this result contains several ingredients which may be of independent interest. The main strategy is to use geometric microlocal analysis to construct the heat kernel on a heat space, denoted by Ω_h^2 , which is created by blowing up $\Omega \times \Omega \times [0, 1)$ along various p -submanifolds. On this heat space we show, in Theorem 5.8, that the heat kernel has a polyhomogeneous conormal expansion at every boundary hypersurface. Indeed we construct the heat kernel by solving suitable model problems at the various boundary hypersurfaces. This gives a full description of the heat kernel on a curvilinear polygonal domain in a surface, in all asymptotic regimes. As such this construction is useful for any application in which fine structure information about the heat kernel near $t = 0$ is needed.

A major advantage of this method is that a complete asymptotic description of the heat *kernel*, rather than just its trace, is obtained. This allows precise asymptotic analysis for expressions such as the gradient of the heat kernel and is likely of interest for future work.

The paper is organized as follows. In Sect. 2, we develop an integral representation of the heat kernel for infinite circular sectors with Dirichlet, Neumann, and mixed boundary conditions. We do this by first obtaining an integral representation of the Green’s function for the corresponding boundary condition. Using functional calculus, we prove that the heat kernel is obtained by taking the inverse Laplace transform of the Green’s function. By the uniqueness of the heat kernel, we thereby obtain the equivalence of this integral representation of the heat kernel and the more common series representation of the heat kernel [6]. In Sect. 3, we construct the heat spaces and demonstrate the composition rule for operators with polyhomogeneous conormal Schwartz kernels. To construct the heat kernel, we proceed in Sect. 4 to solve the model problem for the smooth parts of the boundary for the Dirichlet, Neumann, and Robin boundary conditions. In Sect. 5 we solve the model problem for the vertices with the various boundary conditions and combinations thereof. In this way, we construct the heat kernel on a curvilinear polygonal domain in a surface. In Sect. 6, we use this construction together with our integral representation of the heat kernels obtained in Sect. 2 to compute the heat trace and prove Theorem 1.2. We conclude in Sect. 6 with applications of Theorem 1.2 showing contexts in which corners (vertices) are spectral invariants.

1.1 Notation

For the benefit of readers, we collect here all notations occurring in at least two non-consecutive pages.

Notation	Meaning	Sections
$\mathbb{R}, \mathbb{C}, \mathbb{N}$	Real, complex, and natural numbers	
\Re, \Im	Real and imaginary parts	
\mathbb{Z}, \mathbb{N}_0	Integers and non-negative integers	
$H^\Omega, \text{Tr } H^\Omega$	The Heat kernel and Heat trace on a curvilinear polygon Ω	1
E_j, V_j	Edges and vertices	1
K, k_g	The Gauss and geodesic curvature	1
G, G_D, G_N, G_{DN}	Green functions	2.1
K_ν, I_ν	The modified Bessel functions	2.1
$\mathfrak{L}, \mathfrak{L}^{-1}$	The Laplace transform and its inverse	2.2
F, \mathcal{F}	The index set and index family	3.1
pc	Polyhomogeneous conormal distributions	3.1
β	Blow-down map	3.1
E, V	The set of edges and vertices	3.2
$tf, e_j, sv_j, pv_j, pe_j$	Boundary hypersurfaces of single heat space	3.2
$M_h^2, M_{rh}^2, M_{rh}^3, M_{rh,c}^3$	Heat spaces	3.3, 3.5
$E_{j0}, E_{0j}, hvff_{jk}, hvr_{fj}, hvlf_j$	Boundary hypersurfaces of double heat space and their components	3.3
ff, sf, ff_j, sf_j		
π_C, π_L, π_R	Lifted projections from triple heat space to double heat space	3.4
ρ, ρ_X	Boundary defining functions	3.4

Notation	Meaning	Sections
$e_f(\cdot, \cdot)$	Boundary exponents	3.4
Π_C, Π_L, Π_R	b -maps	3.5
$F_A, F_{AB}, F_{ABC}, F_{ABCD}$	Boundary hypersurfaces of triple heat space	3.5
$\mathcal{A}_h^{\mathcal{F}}$	Spaces of pc functions	3.8, 4.2
$\Psi_h^{\mathcal{F}}, \Psi^{a,b,c,d}$	Set of pseudodifferential operators	3.8, 4.2
\mathcal{L}^h	Lifted heat operator	4.1.1
χ	Euler characteristic	6.1.2

2 Analytic preliminaries

The Laplace operator on a curvilinear polygonal domain, even with specified boundary conditions on each side, is emphatically *not* guaranteed to be self-adjoint. The angles at the vertices as well as the possibility of different boundary conditions on either side of a vertex can give rise to interesting phenomena [7, 8, 17]. We shall consider a Friedrichs type extension of the Laplace operator here. The Laplacian is a priori a symmetric operator on smooth, compactly supported functions on our domain, Ω .

Our sign convention for the Robin boundary condition is

$$\frac{\partial v}{\partial n} = \kappa v, \quad \text{for the inward pointing normal derivative.}$$

The Robin parameter is smooth on each boundary component and is non-negative. We define the Laplace operator corresponding to the mixed boundary conditions in the following way, as in [11, 45]. Consider the form

$$a(u, v) = \int_{\Omega} \nabla u(z) \overline{\nabla v(z)} dz + \int_{\partial\Omega_R} \kappa(z) u(z) \overline{v(z)} d\sigma(z)$$

with domain

$$D(a) = \{u \in H^1(\Omega) : u|_{\partial\Omega_D} = 0\}.$$

Above, $\partial\Omega_D$ and $\partial\Omega_R$ are the unions of the boundary components on which we impose the Dirichlet and Robin boundary conditions, respectively, and $\kappa(z)$ is the Robin parameter. Then a is a closed, densely defined, symmetric form. Therefore, by [25, Theorem 2.23 Chap. 6], it generates a self-adjoint operator, which we call the Laplace operator corresponding to the boundary conditions we mentioned above.

2.1 Green's functions

The general approach to study the heat kernel via the associated Green's function and Kantorovich–Lebedev transform is well documented in the literature, dating at least back to Fedosov in the 1960s [10]. This approach has continued to produce interesting results in modern work as well; see for example the doctoral thesis of Uçar [49] who considers polygonal domains in hyperbolic surfaces. Although the general technique is well known, the details of the calculations are often omitted. To maintain the flow and focus of this work, we present here the results of our calculations, and for the sake of completeness include the details in Appendix A.

We obtain integral expressions for Green’s functions for the Laplacian on an infinite circular sector with Dirichlet, Neumann, and mixed boundary conditions. Let γ be the interior angle of the sector; we need only assume $\gamma \in (0, 2\pi)$. The Green’s function solves the following equation:

$$\begin{cases} sG - \frac{\partial^2 G}{\partial r^2} - \frac{1}{r} \frac{\partial G}{\partial r} - \frac{1}{r^2} \frac{\partial^2 G}{\partial \phi^2} = \frac{1}{r} \delta(r - r_0) \delta(\phi - \phi_0), \\ \left(\alpha G + \beta \frac{\partial G}{\partial \phi} \right) \Big|_{\phi=0, \gamma} = 0, \end{cases} \tag{2.1}$$

with $\alpha = 1$ and $\beta = 0$ for the Dirichlet boundary condition or $\alpha = 0$ and $\beta = 1$ for the Neumann boundary condition, and in all cases with spectral parameter $s > 0$.

For the Dirichlet boundary condition we compute in Appendix A that the Green’s function is

$$\begin{aligned} G_D(s, r, \phi, r_0, \phi_0) &= \frac{1}{\pi^2} \int_0^\infty K_{i\mu}(r\sqrt{s}) K_{i\mu}(r_0\sqrt{s}) \\ &\times \left\{ \cosh(\pi - |\phi_0 - \phi|)\mu - \frac{\sinh \pi \mu}{\sinh \gamma \mu} \cosh(\phi + \phi_0 - \gamma)\mu \right. \\ &\left. + \frac{\sinh(\pi - \gamma)\mu}{\sinh \gamma \mu} \cosh(\phi - \phi_0)\mu \right\} d\mu, \end{aligned} \tag{2.2}$$

where K_ν is the modified Bessel function of second kind. For the Neumann boundary condition, we obtain

$$\begin{aligned} G_N(s, r, \phi, r_0, \phi_0) &= \frac{1}{\pi^2} \int_0^\infty K_{i\mu}(r\sqrt{s}) K_{i\mu}(r_0\sqrt{s}) \\ &\times \left\{ \cosh(\pi - |\phi_0 - \phi|)\mu + \frac{\sinh \pi \mu}{\sinh \gamma \mu} \cosh(\phi + \phi_0 - \gamma)\mu \right. \\ &\left. + \frac{\sinh(\pi - \gamma)\mu}{\sinh \gamma \mu} \cosh(\phi - \phi_0)\mu \right\} d\mu. \end{aligned} \tag{2.3}$$

For the mixed Dirichlet–Neumann boundary condition, taking the Dirichlet condition at $\phi = 0$ and the Neumann condition at $\phi = \gamma$ we obtain the Green’s function

$$\begin{aligned} G_{DN}(s, r, \phi, r_0, \phi_0) &= \frac{1}{\pi^2} \int_0^\infty K_{i\mu}(r\sqrt{s}) K_{i\mu}(r_0\sqrt{s}) \\ &\times \left\{ \cosh(\pi - |\phi_0 - \phi|)\mu + \frac{\sinh(\pi \mu)}{\cosh \gamma \mu} \sinh((\phi + \phi_0 - \gamma)\mu) \right. \\ &\left. - \frac{\cosh(\pi - \gamma)\mu}{\cosh \gamma \mu} \cosh((\phi - \phi_0)\mu) \right\} d\mu. \end{aligned} \tag{2.4}$$

2.2 The Heat kernel and the Green’s function

Let Δ be a self-adjoint, non-negative Laplace operator whose domain is contained in $\mathcal{L}^2(\Omega)$ associated with certain boundary conditions

$$B(u) = 0 \quad \text{on} \quad \partial\Omega,$$

where Ω is a domain with a piecewise smooth boundary $\partial\Omega$ which is contained in a larger smooth ambient manifold. Assume that $G(x, y, s)$ is the Green’s function of the operator $s + \Delta$, that is the solution of the system

$$\begin{cases} (s + \Delta)G(x, y, s) = \delta(x - y), \\ B(G) = 0. \end{cases} \quad (2.5)$$

Before stating the next result, we recall the definition of the Laplace transform and its inverse.

Definition 2.1 Let f be a continuous function such that there exists a constant $c > 0$ with

$$\int_0^\infty |f(t)|e^{-ct} dt < \infty.$$

The Laplace transform of f is defined to be

$$g(s) := \mathfrak{L}(f(t))(s) := \int_0^\infty f(t)e^{-st} dt, \quad \Re(s) \geq c.$$

The inverse Laplace transform is then

$$f(x) := \mathfrak{L}^{-1}(g(s))(t) := \frac{1}{2\pi i} \lim_{k \rightarrow \infty} \int_{a-ik}^{a+ik} g(s)e^{st} ds,$$

for $t > 0$ and $a > c$.

Proposition 2.2 With the notations above, let $H(x, y, t)$ be the heat kernel corresponding to Δ . Then

$$\mathfrak{L}[H](x, y, s) = G(x, y, s),$$

where \mathfrak{L} is the Laplace transform.

Proof Let $\{e^{-t\Delta}\}_{t \geq 0}$ be the semigroup generated by $-\Delta$. We note that $-\Delta$ is a non-positive, self-adjoint operator, so that this semigroup is well defined on $\mathcal{L}^2(\Omega)$. Moreover, the self-adjointness gives

$$\|(\lambda + \Delta)^{-1}\| \leq \frac{1}{\text{dist}(\lambda, \sigma(-\Delta))} \quad \lambda \in \rho(-\Delta), \quad (2.6)$$

where $\sigma(-\Delta)$ and $\rho(-\Delta)$ are, respectively, the spectrum and resolvent set of $-\Delta$. Therefore, by the Hille–Yosida theorem [53], $\{e^{-t\Delta}\}_{t \geq 0}$ is a contracting semigroup. Hence, by Theorem 8.2.2 in [53], it follows

$$\mathfrak{L} \circ e^{-t\Delta}(s) = (s + \Delta)^{-1}, \quad s > 0, \quad (2.7)$$

where \mathfrak{L} is the Laplace transform acting in t variable.

On the one hand, since the heat kernel $H(x, y, t)$ is \mathfrak{L} transformable, we may express

$$\mathfrak{L}[e^{-t\Delta}\phi] = \mathfrak{L} \int_\Omega H(t, x, y)\phi(y)dy = \int_\Omega \mathfrak{L}[H]\phi(y)dy$$

for $t, s > 0$ and

$$(s + \Delta)^{-1}\phi = \int_\Omega G(x, y, s)\phi(s)dy.$$

for $s > 0$. Therefore, the uniqueness of Schwartz kernels and (2.7) imply the statement. \square

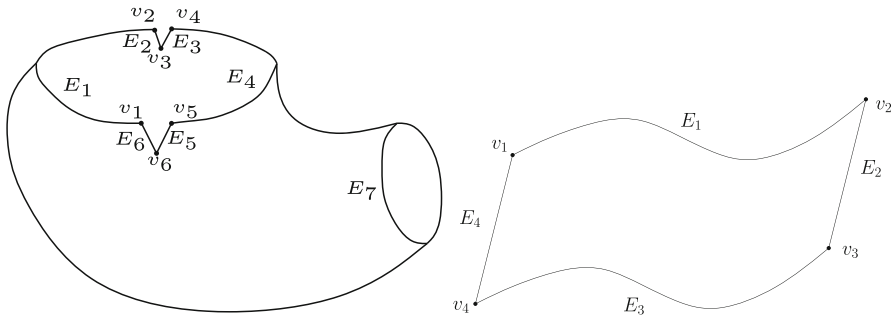


Fig. 1 Two examples of curvilinear polygonal domains, with edges and vertices

Remark 2.3 Due to Proposition 2.2, by applying the inverse Laplace transform to (2.2), (2.3), and (2.4), we obtain expressions for the heat kernels for the Laplacian on an infinite circular sector with Dirichlet, Neumann, and mixed boundary conditions, respectively. The heat kernels for an infinite circular sector were computed by Cheeger using separation of variables in polar coordinates [6, p. 592 (3.42)]. Cheeger’s formula simplifies in our setting to:

$$H(t, r, \theta, r', \theta') = \frac{1}{2t} \exp\left[-\frac{r^2 + (r')^2}{4t}\right] \sum_{j=1}^{\infty} I_{\mu_j}\left(\frac{rr'}{2t}\right) \phi_j(\theta)\phi_j(\theta'). \tag{2.8}$$

Here I_{μ_j} are the modified Bessel functions, and (ϕ_j, μ_j) are the eigenfunctions, and corresponding eigenvalues, of the appropriate eigenvalue problem (D–D, N–N, or D–N) on the interval $[0, \gamma]$. By the uniqueness of the heat kernel we therefore obtain the equality of these expressions with the inverse Laplace transform of the expression for the Greens functions.

3 Heat spaces

We consider curvilinear polygonal domains as in Definition 1.1; see examples illustrated in Fig. 1.

3.1 Manifolds with corners and polyhomogeneity

Near a vertex, Ω has the differentiable structure of a manifold with corners after blowing up the vertex. Specifically, an open neighborhood of a vertex is diffeomorphic to a sector, $(0, \varepsilon] \times C$, where C is a circular arc. If we include the point $\{0\} \times C$, then we obtain a smooth manifold with corners. This is what is meant by “blowing up the vertex,” in the sense that we replace the vertex with a copy of the link of the sector, namely C . This process may be thought of as pretending that polar coordinates are actually valid down to the origin. Doing this construction at each vertex yields a smooth surface with corners which we call Ω_0 . In this way, we may identify the differentiable structure of all the surfaces we consider here as the differentiable structure of *manifolds with corners*, defined below. The definition is first due to Melrose [39], here we use the version which is introduced in [38].

In order to define a manifold with corners, we must first define t-manifolds.

Definition 3.1 [38, Definition 1.6.1] An n -dimensional t-manifold X is a paracompact Hausdorff space such that at each point $x \in X$ there is a non-negative integer k such that

a neighborhood of x is homeomorphic to a neighborhood of the origin in the product $[0, \infty)^k \times \mathbb{R}^{n-k}$, with all transition maps being smooth with respect to the subspace topology on $[0, \infty)^k \times \mathbb{R}^{n-k} \subseteq \mathbb{R}^n$.

Now we define a manifold with corners.

Definition 3.2 [38, Definition 1.8.5] A manifold with corners is a t-manifold such that each boundary hypersurface is embedded.

Since with this definition we may have $k = 0$, we see that smooth manifolds without boundary also fit into the general class of “manifolds with corners.”

The purpose of the heat space construction is to create spaces on which the heat kernel and its trace are polyhomogeneous conormal distributions, abbreviated pc. This is a natural class of functions within which to study partial differential equations on manifolds with corners; see [33] and references therein. We briefly recap the definition here. To begin, we say that a subset $F \subseteq \mathbb{C} \times \mathbb{N}_0$ is an *index set* if F is a discrete set satisfying the following properties:

- For all N , $F \cap \{\Re(z) < N\}$ is finite;
- If $(s, p) \in F$, then $(s + 1, p) \in F$;
- If $(s, p) \in F$ and $p > 0$, then $(s, p - 1) \in F$.

The latter two conditions are sometimes omitted from this definition, but they give pc functions nice invariance properties; see [16].

Let X be an n -dimensional manifold with corners, and let $\{M_i\}_{i=1}^J$ be the set of its boundary hypersurfaces, that is, the set of all boundary faces of codimension one. We say that $\mathcal{F} = (F_1, \dots, F_J)$ is an *index family* for X if each $F_i = \{(s_{ij}, p_{ij})_{j=1}^\infty\}$ is an index set, ordered so that $s_{ij} \in \mathbb{R}$ are non-decreasing and p_{ij} are non-increasing whenever s_{ij} is unchanged. For each i , let x^i be a *boundary defining function* for M_i ; that is, a smooth, non-negative function $x^i : X \rightarrow \mathbb{R}$ such that x^i vanishes precisely at M_i but the differential dx^i is non-zero on M_i . We may use boundary defining functions as coordinates on X . Finally, let \mathcal{V}_b denote the space of smooth vector fields on X which are tangent to all boundaries. With this terminology, we define $\mathcal{A}^{\mathcal{F}}(X)$, the space of polyhomogeneous conormal, or pc functions, to be the space of functions f smooth on the interior of X which have:

- generalized Taylor-like expansions at each boundary hypersurface M_i of the form

$$f \sim \sum_{j=1}^\infty (x^i)^{s_{ij}} (\log(x^i))^{p_{ij}} a_{ij}(x^1, \dots, x^{i-1}, x^{i+1}, \dots, x^n),$$

where for each i , the set $\{(s_{ij}, p_{ij})\}$ is the index set F_i , enumerated so that s_{ij} is non-decreasing;

- product type expansions of the same form at each corner (polyhomogeneous),
- and for which Vf has expansions of the same type whenever V is a product of elements of \mathcal{V}_b (conormal).

The union of these spaces over all possible index sets is denoted $\mathcal{A}^*(X)$. Note that by definition these spaces are invariant under \mathcal{V}_b , in the sense that for any $V \in \mathcal{V}_b$ and any $u \in \mathcal{A}^{\mathcal{F}}(X)$, $Vu \in \mathcal{A}^{\mathcal{F}}(X)$ as well. Observe also that smooth functions on X are pc with each index set consisting of $\mathbb{N}_0 \times \{0\}$.

3.1.1 Blowups

Consider the finite cone, $(0, 1]_r \times \mathbf{S}^1_\theta$ with the Riemannian metric, $dr^2 + r^2d\theta^2$, where $d\theta^2$ is the standard metric on \mathbf{S}^1 , and the conical point is at $r = 0$. The simplest example of blowing up is replacing the conical point at $r = 0$ with a copy of \mathbf{S}^1 , so that the finite cone is now topologically identified with the cylinder $[0, 1] \times \mathbf{S}^1$. In this example, the point at $r = 0$ is replaced with the set of all directions, that is all values of θ , with which one can approach the point, $r = 0$. This type of blowup is known as a *radial blowup*, or a *normal blowup*.

More generally, we shall consider blowups along *p-submanifolds*. An embedded submanifold Y contained in a manifold with corners, X , is a *p-submanifold* if near each point $q \in Y$, there exist local coordinates so that Y is defined by the vanishing of a subset of these local coordinates. For example, the boundary faces of X are *p-submanifolds*. The intersection of two or more boundary faces of X is also a *p-submanifold*. The normal blowup of X around Y is denoted by

$$[X; Y] = \text{ff} \sqcup (X \setminus Y).$$

Above, ff is the inward pointing spherical normal bundle of Y which has replaced Y in $[X; Y]$. There is a unique minimal differentiable structure with respect to which $[X; Y]$ is a manifold with corners such that the following two conditions hold.

1. There is a smooth “blow-down” map

$$\beta : [X; Y] \rightarrow X$$

which is the identity on $(X \setminus Y)$.

2. Cylindrical coordinates around Y are smooth coordinates on $[X; Y]$.

In case we wish to blow up two or more *p-submanifolds*, we write

$$[X; Y_1; Y_2]$$

to indicate that we first blow up Y_1 and next blow up the lift of Y_2 to $[X; Y_1]$. This lift is the usual lift if $Y_2 \subseteq Y_1$, and otherwise is the closure of $Y_2 \setminus Y_1$ in $[X; Y_1]$.

3.2 The single heat space

The first of the heat spaces we construct is the single heat space. Let Ω be a curvilinear polygonal domain. Let E be the set of edges of Ω (maximal smooth boundary components) and V the set of vertices. Throughout, we let Ω_0 be Ω with the vertices blown up, so that Ω_0 is a surface with corners. We also let \tilde{V} be the lift of V to Ω_0 , that is, the union of the faces $\{r = 0\}$ at each vertex.

Throughout, we will use the time coordinate $T = \sqrt{t}$. This changes the smooth structure at $t = 0$ somewhat, but allows us to avoid the use of parabolic blow-ups.

The heat kernel restricted to the diagonal is defined on $\Omega \times [0, \infty)_T$ and is dubbed *the diagonal heat kernel*. The single heat space is a natural habitat of the diagonal heat kernel in the sense that the diagonal heat kernel lifted to the single heat space is pc. Note that the single heat space is the same for all the possible boundary conditions we consider. To create the single heat space, we begin with the manifold with corners $\Omega_0 \times [0, 1)_T$. We denote its $T = 0$ boundary hypersurface¹ by tf . The remainder of the boundary hypersurfaces

¹ A boundary hypersurface will often be referred to as a boundary face or face. A “side face” is a boundary hypersurface arising from the boundary in Ω . This is in contrast to the *tf* face as well as to the boundary hypersurfaces created by blowing up along *p-submanifolds*.

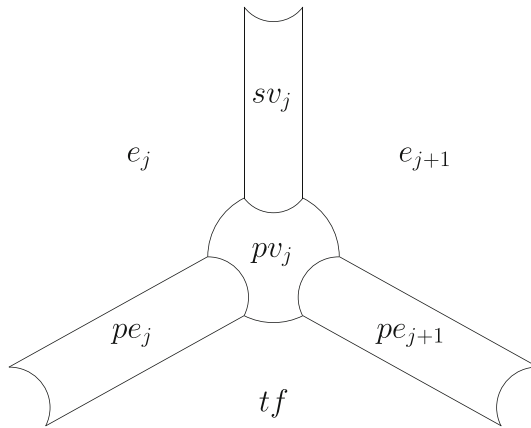


Fig. 2 Single heat space

correspond either to an edge or to a vertex (which has been blown up, so there is a boundary hypersurface for each vertex). Denote the edge/side faces for positive T by $\{e_j\}_{j=1}^{|E|}$, and the vertex faces $\tilde{V}_j \times [0, 1)_T$ by

$$\{sv_j\}_{j=1}^{|V|}. \tag{3.1}$$

Next we perform blow-ups, first of the vertices at $T = 0$ and then of the edges at $T = 0$, to create the *single space*²

$$M_h = [\Omega_0 \times [0, 1)_T; \{sv_j\}_{j=1}^{|V|} \cap \{T = 0\}; E \times \{T = 0\}].$$

We call the new faces obtained $\{pv_j\}_{j=1}^{|V|}$ and $\{pe_j\}_{j=1}^{|E|}$.

The sequence of blowups is therefore:

1. the normal blowup about each vertex for all time (implicit in the starting point of $\Omega_0 \times [0, 1)_T$);
2. the blowup of each vertex at $T = 0$;
3. the blowup of each edge at $T = 0$.

The space M_h has $2|V| + 2|E| + 1$ boundary hypersurfaces in total.

To motivate this construction, consider the diagonal heat kernel on an infinite sector, which from (2.8) is

$$H(t, r, \theta, r, \theta) = \frac{1}{2T^2} \exp\left[-\frac{r^2}{2T^2}\right] \sum_{j=1}^{\infty} I_{v_j}\left(\frac{r^2}{2T^2}\right) |\phi_j(\theta)|^2.$$

The pre-factor is not a pc function of (r, T) , but it is a pc function of $(r, T/r)$. And indeed, after creating pv_j , T/r may be taken as a boundary defining function for tf near pv_j .

In relation to the literature, our heat space construction can be seen as a hybrid combining elements of Mooers’s heat space for manifolds with isolated conical singularities [41] and Mazzeo and Vertman’s heat space for manifolds with edges [35]. The first step in Mooers’s

² In the notation M_h for the single heat space, M acts as a place-holder for the various model geometries we shall use to construct the heat kernel on our surface Ω (the analogous notation is used for the double and triple heat spaces) (Fig. 2).

construction is to blow up the conical singularity by replacing the point with the cross-section (link) of the cone. Next she takes the product with time. It is completely equivalent to first take the product with time, and then perform a normal blowup of the cone point for all time. This is the procedure we follow at the vertices (and cone points). Next, we perform blowups at $T = 0$ of the edges (and smooth boundary), analogous to [35].

In fact our construction can be viewed as an iterated version of [35], where we perform each of their blow-ups at the vertices (of codimension 2) and then at the edges (of codimension 1). We expect that a similar construction, with additional iteration, could work for polyhedral domains in manifolds of arbitrary dimension.

3.3 The double heat space

The double heat space is a natural habitat of the heat kernel in the sense that the heat kernel, initially defined on $\Omega \times \Omega \times [0, \infty)$, lifts to be pc on it. As with the single space, our models guide the construction of the double heat space by indicating which p-submanifolds should be blown up to ensure that the heat kernel will be pc. The general philosophy is to mimic [35], performing each of their blow-ups first at the vertices and then at the edges.

Begin with $M^2 := \Omega_0 \times \Omega_0 \times [0, 1)_T$. As we are using Ω_0 rather than Ω , this is now a manifold with corners, the analogue of the space with which Mazzeo and Vertman begin [35, Sect. 3.1]. Denote its $T = 0$ boundary hypersurface by tf . All other boundary hypersurfaces are of one of the following forms:

- $E_j \times \Omega_0 \times [0, 1)$, which we call E_{j0} ,
- $\Omega_0 \times E_j \times [0, 1)$, which we call E_{0j} ,
- $\check{V}_j \times \Omega_0 \times [0, 1)$, which is now a boundary hypersurface which we call $hvrff_j$;
- $\Omega_0 \times \check{V}_j \times [0, 1)$, which we call $hvlff_j$.

As a guide to the nomenclature here, “h” indicates “height” because these blowups persist for all time, and time is usually the vertical axis in figures of this type. As usual, “v” indicates vertices, and “ff”, “rf”, and “lf” indicate left, right, and front faces respectively.³ Now, for each j and k , blow up the intersection $hvlff_j \cap hvrff_k$ to create a new boundary hypersurface⁴ $hvvff_{jk}$. At this point we call the new space M_0^2 :

$$M_0^2 = [\Omega_0 \times \Omega_0 \times [0, 1); \cup_{j,k} hvlff_j \cap hvrff_k].$$

The next step is to blow up the union over all j of $hvvff_{jj} \cap \{T = 0\}$. This blowup creates N boundary faces, one at each vertex, denoted by ff_j . We shall collectively refer to these as ff for “front face(s).” The resulting space is

$$[M_0^2; \cup_j hvvff_{jj} \cap \{T = 0\}].$$

This construction at the vertices needs to be imitated at the edges, so now lift the triple intersection of the diagonal, boundary, and $T = 0$, namely $\{(z, z', 0) : z = z' \in E\}$, to $[M_0^2; hvvff_{jj} \cap \{T = 0\}]$. As the lift of a p-submanifold is a p-submanifold, this lift is a boundary p-submanifold which meets ff . We blow it up, creating a new boundary hypersurface at each side face. We denote their union by sf , for “side face(s)”. When we need to distinguish components, we shall refer to the j^{th} component as sf_j . Observe that the side faces are

³ Some authors reverse the roles of “right” and “left” here—our terminology is chosen to match [35].

⁴ This is different from [35]. It may be that only the blow-ups with $j = k$ are necessary, but doing all of them makes the proof of the composition theorem easier to read.

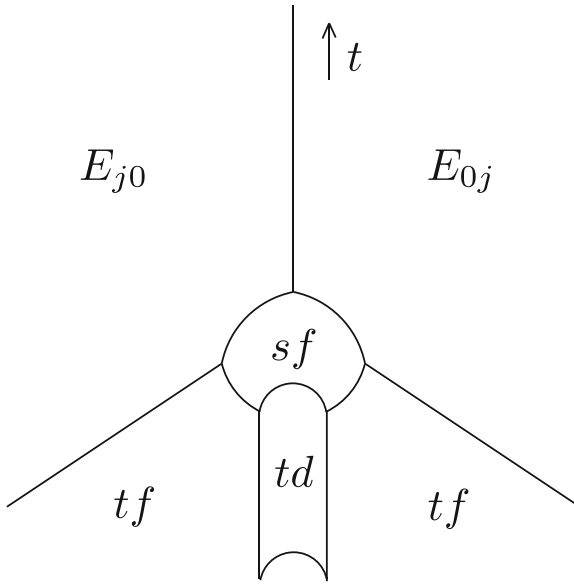


Fig. 3 This is a schematic depiction of the double heat space

pairwise disjoint, as their intersection in $\Omega_0 \times \Omega_0 \times [0, 1)$ has already been blown up to create *ff*. Let us call the space at this point the “reduced double heat space” M_{rh}^2 . It is the analogue in our setting of the “intermediate heat space” in [35, Sect. 3.1]. Specifically, we have

$$M_{rh}^2 = [M_0^2; \cup_j hvff_{jj} \cap \{T = 0\}; \{(z, z', 0) : z = z' \in E\}].$$

The final blowup is at the lift of the diagonal at $T = 0$. This blowup produces the double heat space:

$$M_h^2 = [M_{rh}^2; \{(z, z', 0) : z = z' \in \Omega\}].$$

Call the new front face *td*, for “time diagonal.” It intersects both *sf* and *ff*, as well as *tf*, though no other boundary hypersurfaces (in particular none of the *hvff*, *hvlf*, or *hvr* components, as the intersection of the diagonal with the boundary has already been blown up). The double heat space is depicted in Fig. 3.

3.4 The triple heat space

The triple heat space, unlike the single and double heat spaces, is *not* a natural habitat. Instead, it is an artificial environment to which we shall lift the Schwartz kernels of operators from their natural habitats on the double space in order to compose them. With the correct construction of the triple space, the process of composition returns an element which is *pc* on the double space. Consequently, the construction of the triple space is guided by the desire to be able to lift and compose Schwartz kernels which live on the double heat space.

Our construction is based on that of Mazzeo and Vertman [35], and indeed is identical when $V = \emptyset$, that is when there are no vertices. Our guiding principle is that whenever Mazzeo and Vertman blow up a boundary, we first blow up V and then E . However, our

setting is further complicated by the additional blow-ups at the vertices for positive time. To begin, we consider the original triple space, with the vertices blown up in each factor so that we have the structure of a manifold with corners:

$$M^3 := \Omega_0 \times \Omega_0 \times \Omega_0 \times [0, 1) \times [0, 1) \tag{3.2}$$

along with the three projections $\pi_C, \pi_L,$ and π_R defined by

$$\pi_C : M^3 \rightarrow \Omega_0^2 \times \mathbb{R} \sqrt{(T')^2 + (T'')^2}, (z, z', z'', T', T'') \rightarrow (z, z'', \sqrt{(T')^2 + (T'')^2}); \tag{3.3}$$

$$\pi_L : M^3 \rightarrow \Omega_0^2 \times \mathbb{R}_{T'}, (z, z', z'', T', T'') \rightarrow (z, z', T'); \tag{3.4}$$

$$\pi_R : M^3 \rightarrow \Omega_0^2 \times \mathbb{R}_{T''}, (z, z', z'', T', T'') \rightarrow (z', z'', T''). \tag{3.5}$$

These projections will be used to re-interpret operator composition in terms of pullbacks and push-forwards. Modulo all the technical details, if we have two operators, one A with Schwartz kernel K_A and the other B with Schwartz kernel K_B , the Schwartz kernel K_C of the composition $C = A \circ B$ is given by

$$K_C = (\pi_C)_*(\pi_L^* K_A \pi_R^* K_B).$$

Fortunately, we do not need the full composition formula, just the version in which one of the operators vanishes to infinite order at td. This is because we use the composition formula to run a Neumann series argument to construct our heat kernel, and with a good enough initial parametrix for the heat kernel, the error will vanish to infinite order at td. For this reason, we will construct a reduced triple heat space M_{rh}^3 in order to prove a special case of the composition formula, the case in which K_B has order ∞ at the face td. In this case, K_B is pc on M_{rh}^2 .⁵

The triple space construction is guided by the following conditions, which are necessary and sufficient to obtain the composition formula we require:

1. we need the projection π_C to lift to a b-fibration $\Pi_C : M_{rh}^3 \rightarrow M_{rh}^2$;
2. we need the projection π_L to lift to a b-map $\Pi_L : M_{rh}^3 \rightarrow M_h^2$;
3. since K_B is pc on M_{rh}^2 , we need only that the projection π_R lift to a b-map $\Pi_R : M_{rh}^3 \rightarrow M_{rh}^2$.

We therefore recall the definitions of these important ‘‘b-notions.’’

Definition 3.3 [39, p. 51] Let $f : X \rightarrow Y$ be a smooth map between manifolds with corners, X with boundary hypersurfaces $M_1(X)$, and Y with boundary hypersurfaces $M_1(Y)$. Then f is a *b-map* if for each $H \in M_1(Y)$, and boundary defining function ρ_H

$$f^*(\rho_H) = 0 \quad \text{or} \quad f^*(\rho_H) = a \cdot \prod_{G \in M_1(X)} \rho_G^{e_f(G,H)}, \quad 0 < a \in C^\infty(X).$$

In the latter case, the numbers $e_f(G, H)$ are called the boundary exponents of f . In this case, writing $M_1(X) = \{G_j\}_{j=1}^n$ and $M_1(Y) = \{H_k\}_{k=1}^m$, the *exponent matrix* is the matrix whose entries are $\{e_f(G_j, H_k)\}_{j,k=1}^{n,m}$.

Definition 3.4 [39, p. 53] A b-map is a *b-submersion* if the b-differential is surjective for all $x \in X$. For the definition of the b-differential, we refer to [39, pp. 53–54].

⁵ In general, the operator kernel would be pc on M_{rh}^2 but not on M_{rh}^2 .

Definition 3.5 [39, p. 53] A b-map is *b-normal* if ${}^b f_*$, defined as in [39, (7), p. 53] is surjective.

Definition 3.6 [19, p. 124] A b-map is a *b-fibration* if f_* , acting on the b-tangent bundle, is surjective on each fibre, and the image of each boundary hypersurface in X is either Y or one boundary hypersurface $H \subset Y$. We note that this holds if and only if the b-map is both b-normal and a b-submersion [39, p. 53].

Definition 3.7 [19, p. 124] A total boundary defining function for X , which we denote ρ_X , is the product of boundary defining functions for all the boundary hypersurfaces. We say that a b-fibration is *simple* if

$$f^* \rho_Y = a \cdot \rho_X, \quad 0 < a \in C^\infty(X).$$

In terms of the exponent matrix, this is equivalent to requiring that the elements are either 0 or 1, and moreover for each $G \in M_1(X)$ there exists precisely one $H \in M_1(Y)$ with $e_f(G, H) = 1$.

To start constructing M_{rh}^3 , we begin with (3.2). This space has a number of boundary hypersurfaces: the $T'' = 0$ face, which we denote F_{TL} , the $T' = 0$ face, which we denote F_{TR} , and a number of faces of the form $E_j \times \Omega_0 \times \Omega_0 \times [0, 1)_{T''} \times [0, 1)_{T'}$, which we call F_{j00} , with similar notation for products where E_j is in the second or third factor. We also have a number of faces of the form $\tilde{V}_j \times \Omega_0 \times \Omega_0 \times [0, 1) \times [0, 1)$, which we call F_{Vj00} and collectively F_{V00} , et cetera. The notation used in the triple space is a bit different from the single and double spaces, in particular the use of capitals. This is in part to distinguish the triple space as an artefact to be used for the purpose of composition and in part to draw a parallel to related constructions in the literature [1, 50] which use an analogous notation.

A few blow-up facts will be useful throughout.

Proposition 3.8 *The following are true (each is well-known in the geometric microlocal literature):*

1. *Blow-ups which are nested, disjoint, or transverse commute* [19, Lemma 2.1].
2. *Blow-down maps are b-maps* [16, Sect. 2.3.3], [19, Proof of Lemma 2.7]. *Moreover, if $Y \subset X$ is an intersection of boundary hypersurfaces of X , then the blow-down map from $[X; Y] \rightarrow X$ is a b-submersion. (This reduces to the case $X = \mathbb{R}_+^n, Y = 0$, and then follows from a computation in local coordinates).*
3. *The composition of b-maps is a b-map* [16, Sect. 2.3.3]. *Further, it follows immediately from the definition that the composition of b-submersions is a b-submersion.*
4. *Once a b-map is known to be a b-submersion, checking b-normality is a matter of ensuring, by checking the exponent matrix, that no boundary hypersurface is mapped into a face of codimension > 1 in the image* [16, Definition 3.9], [1, Remark B.4].

We will also use the following.

Lemma 3.9 *Suppose A, B, C , and D are p -submanifolds of a manifold with corners X , and suppose that*

$$A \subseteq B \subseteq D, \quad A \subseteq C \subseteq D, \quad \text{and} \quad B \cap C \subseteq A.$$

Then

$$[X; C; D; A; B] \cong [X; A; B; C; D].$$

Proof Nested blow-ups commute, so we may do A before C and D and thus

$$[X; C; D; A; B] \cong [X; A; C; D; B].$$

Similarly, we may do B before D and thus

$$[X; C; D; A; B] \cong [X; A; C; B; D].$$

Now since $B \cap C \subseteq A$, the lifts of B and C are disjoint in the space $[X; A]$, and since disjoint blow-ups commute they may be done in either order. This completes the proof. \square

Our strategy will be to repeatedly take advantage of the following lemma of Hassell, Mazzeo, and Melrose [19]:

Lemma 3.10 [19, Lemma 2.5] *Suppose $f : X \rightarrow Y$ is a simple b-fibration of compact manifolds with corners. Suppose $U \subset Y$ is a closed p -submanifold. Then, with S the minimal collection of p -submanifolds of X into which the lift of U under f decomposes, f extends from the complement of $f^{-1}(U)$ to a b-fibration $f_U : [X, S] \rightarrow [Y, U]$, for any order of blow-up of the elements of S .*

This lemma guides our construction of the triple space using the construction of the double space.

3.5 Lifting the projection maps

To construct the triple space from M^3 we first blow up $\mathcal{O} = \{T' = T'' = 0\}$. The spatial variables are unaffected by this blow-up and so the space

$$[M^3; \mathcal{O}] = \Omega_0^3 \times [[0, 1) \times [0, 1); \{0, 0\}]. \tag{3.6}$$

Denote the new front face by $F_{\mathcal{O}}$, and as usual continue to denote the $T'' = 0$ face by F_{TL} and the $T' = 0$ face by F_{TR} . We claim:

Lemma 3.11 *The projections $\pi_C, \pi_L,$ and π_R lift by continuity to projections $\Pi_C, \Pi_L,$ and Π_R with domain $[M^3; \mathcal{O}]$ and ranges as in (3.3), (3.4), (3.5). Moreover, $\Pi_C, \Pi_L,$ and Π_R are all b-fibrations. Under Π_C , the image of $F_{\mathcal{O}}$ is the face $\{T = 0\}$, and the faces at $T' = 0$ and $T'' = 0$ are mapped into the interior.*

Proof This follows immediately from (3.6) and the corresponding statement considering only the time variables. Specifically, the map

$$\Pi : [[0, 1)_{T'} \times [0, 1)_{T''}; \{(0, 0)\}] \rightarrow [0, 1)_T, \quad T = \sqrt{(T')^2 + (T'')^2}$$

is a b-fibration, where the image of the front face is $\{T = 0\}$, and the image of the other two faces is $[0, 1)$. Now the lifted projection Π_C is simply Π in the time variables and the usual projection from M^3 to M^2 in the spatial variables, and thus is itself a b-fibration. Similar arguments take care of Π_L and Π_R , as left and right projection lift to b-fibrations from $[\mathbb{R}_+^2; \{0\}]$ to \mathbb{R} . \square

We will now make further blow-ups to $[M^3; \mathcal{O}]$ which allow all three of these maps to be b-fibrations onto M_0^2 . In each we use 3.10. Recalling (3.6), we define submanifolds $\mathcal{P}_{VVV}, \mathcal{P}_{VV0}, \mathcal{P}_{V0V}$, et cetera of $[M^3; \mathcal{O}]$ by restricting to \tilde{V} (the lift of V) each of the spatial variables which have index V rather than 0. For example,

$$\mathcal{P}_{V0V} = \tilde{V} \times \Omega_0 \times \tilde{V} \times [[0, 1) \times [0, 1); \{0, 0\}].$$

Using this notation, the lift of hvff under Π_C is \mathcal{P}_{V_0V} , under Π_L is \mathcal{P}_{VV_0} , and under Π_R is \mathcal{P}_{0VV} . Then the lifts of hvlf and hvrf may be computed under each map. Application of Lemma 3.10 shows that

$$\begin{aligned} \Pi_C &: [M^3; \mathcal{O}; \mathcal{P}_{V_0V}] \rightarrow M_0^2, \\ \Pi_L &: [M^3; \mathcal{O}; \mathcal{P}_{VV_0}] \rightarrow M_0^2, \\ \Pi_R &: [M^3; \mathcal{O}; \mathcal{P}_{0VV}] \rightarrow M_0^2, \end{aligned}$$

are each b-fibrations. In fact we can define a common domain: let

$$M_0^3 := [M^3; \mathcal{O}; \mathcal{P}_{VVV}; \mathcal{P}_{V_0V}; \mathcal{P}_{0VV}; \mathcal{P}_{VV_0}].$$

Denote the new faces created by F_{VVV} , F_{V_0V} , et cetera.

Proposition 3.12 Π_C , Π_L , and Π_R all lift to b-fibrations from M_0^3 to M_0^2 .

Proof First observe that M_0^3 is a blow-up of each of the three domain spaces for Π_C , Π_L , and Π_R . Indeed, begin with the domain space for Π_C and blow up the lift of \mathcal{P}_{VVV} . This blow-up is nested with the blow-up of \mathcal{P}_{V_0V} , so it may be done before that. So a blow-up of the domain space for Π_C is

$$[M^3; \mathcal{O}; \mathcal{P}_{VVV}; \mathcal{P}_{V_0V}].$$

Now we can blow up the lifts of \mathcal{P}_{0VV} and \mathcal{P}_{VV_0} , which are disjoint from the lift of \mathcal{P}_{V_0V} and thus can be done in any order. Analogous arguments show that M_0^3 is a blow-up of each of the three domain spaces. Since all our p-submanifolds are intersections of boundary hypersurfaces, by Proposition 3.8b), Π_C , Π_R , and Π_L are all b-submersions from M_0^3 to M_0^2 .

At this point, proving these maps are b-fibrations is simply a matter of checking their exponent matrices to make sure no boundary hypersurface is mapped into a corner. This calculation is combinatorial, very similar to the proof of Lemma 3.14 in [33], and as a result we omit it. □

Having proven that Π_C is a b-fibration from $M_0^3 \rightarrow M_0^2$, we shall now lift Π_C to be a b-fibration onto M_{rh}^2 . Let $\mathcal{O}_{V_0V} = F_{\mathcal{O}} \cap \{z = z' \in \tilde{V}\}$, with a similar definition for \mathcal{O}_{E_0E} , \mathcal{O}_{VVV} , \mathcal{O}_{EEE} , etc. Using Lemma 3.10, we obtain that Π_C lifts to a b-fibration

$$\Pi_C : [M_0^3; \mathcal{O}_{V_0V}; \mathcal{O}_{E_0E}] \rightarrow M_{rh}^2.$$

Call the new faces $F_{\mathcal{O}_{V_0V}}$ and $F_{\mathcal{O}_{E_0E}}$. We are now in good shape with Π_C but we need to do more work in order for Π_L and Π_R to be b-maps onto M_{rh}^2 and M_{rh}^2 respectively.

To begin the extra work, we do some more blow-ups which preserve the b-fibration property of Π_C . Begin with \mathcal{O}_{VVV} , which in M_0^3 is $F_{\mathcal{O}} \cap \{z = z' = z'' \in \tilde{V}\}$. Using notation as before, we claim that

$$\Pi_C : M_{cen}^3 := [M_0^3; \mathcal{O}_{V_0V}; \mathcal{O}_{E_0E}; \mathcal{O}_{VVV}; \mathcal{O}_{EEE}] \rightarrow M_{rh}^2$$

is a b-fibration. Indeed, this follows from two applications of [19, Lemma 2.7].⁶ Moreover, by an application of Lemma 3.9, we have

$$M_{cen}^3 = [M_0^3; \mathcal{O}_{VVV}; \mathcal{O}_{EEE}; \mathcal{O}_{V_0V}; \mathcal{O}_{E_0E}].$$

⁶ See in particular the comment after the proof in [19]; in this case Π_C restricts to a b-fibration from \mathcal{O}_{VVV} onto ff and also from \mathcal{O}_{EEE} onto sf, which is sufficient.

Continuing in this vein, let $L_{0VV} = F_{TL} \cap \{z' = z'' \in \tilde{V}\}$, with similar notations for L_{0EE} , R_{VV0} , R_{EE0} , and let $R_{diag} = F_{TL} \cap \{z = z'\}$. Let

$$M_{rh,c}^3 := [M_{cen}^3; R_{VV0}; R_{EE0}; R_{diag}; L_{0VV}; L_{0EE}].$$

Denote the five new boundary hypersurfaces by F_{RVV0} , F_{REE0} , F_{TRD} , F_{L0VV} , and F_{L0EE} . None of the five new boundary hypersurfaces are mapped into a corner by Π_C ; their images are hvf , $\cup_j E_{j0}$, the interior, hvf , and $\cup_j E_{0j}$ respectively. Thus by the same argument with [19, Lemma 2.17], $\Pi_C : M_{rh,c}^3 \rightarrow M_{rh}^2$ is a b-fibration.

Finally, using similar notation, we define p-submanifolds \mathcal{O}_{VV0} , \mathcal{O}_{EE0} , \mathcal{O}_{0VV} , \mathcal{O}_{0EE} , and \mathcal{O}_D , which is the interior lift of $\{t' = t'' = 0, z = z'\}$ in our new space. Define the *reduced triple heat space*

$$M_{rh}^3 := [M_{rh,c}^3; \mathcal{O}_{VV0}; \mathcal{O}_{EE0}; \mathcal{O}_{0VV}; \mathcal{O}_{0EE}; \mathcal{O}_D].$$

Call the new boundary hypersurfaces $F_{\mathcal{O}VV0}$, $F_{\mathcal{O}EE0}$, $F_{\mathcal{O}0VV}$, $F_{\mathcal{O}0EE}$, and $F_{\mathcal{O}D}$. It is no longer true that Π_C is a b-fibration from M_{rh}^3 onto M_{rh}^2 , but it is a b-map.

We must also know something about Π_L and Π_R :

Proposition 3.13 Π_L and Π_R lift by continuity to well-defined b-maps from M_{rh}^3 to M_h^2 and M_{rh}^2 respectively.

Proof It is immediate by composing with the blow-down map that Π_L and Π_R lift to well-defined b-maps from M_{rh}^3 to M_0^2 . The question is whether they still lift to b-maps when M_0^2 is blown up to create ff , then sf , then (for Π_L) td . But this may be checked directly: computing the pullbacks of the boundary defining functions for the boundary hypersurfaces of M_h^2 and M_{rh}^2 shows that each is a product of boundary defining functions on M_{rh}^3 . The specific products are given below, in Lemma 3.14 (Fig. 4). \square

3.6 Combinatorics of b-maps

Now we come to the key combinatorial lemma for composition. Recall that $\Pi_L : M_{rh}^3 \rightarrow M_h^2$ and $\Pi_R : M_{rh}^3 \rightarrow M_{rh}^2$ are b-maps and $\Pi_C : M_{rh,c}^3 \rightarrow M_{rh}^2$ is a b-fibration.

Lemma 3.14 *The exponent matrix entries for the b-maps $\Pi_L : M_{rh}^3 \rightarrow M_h^2$, $\Pi_R : M_{rh}^3 \rightarrow M_{rh}^2$, and $\Pi_C : M_{rh,c}^3 \rightarrow M_{rh}^2$ are all zero, except for the following, which are 1:*

$$\begin{aligned} & \text{For } \Pi_L : (F_{TR}, tf), (F_{\mathcal{O}}, tf), (F_{\mathcal{O}V_j0V_j}, tf), (F_{\mathcal{O}E_j0E_j}, tf), (F_{\mathcal{O}0V_jV_j}, tf), \\ & (F_{\mathcal{O}0E_jE_j}, tf), (F_{TRD}, td), (F_{\mathcal{O}D}, td), (F_{\mathcal{O}V_jV_jV_j}, ff_j), (F_{RV_jV_j0}, ff_j), (F_{\mathcal{O}V_jV_j0}, ff_j), \\ & (F_{\mathcal{O}E_jE_jE_j}, sf_j), (F_{RE_jE_j0}, sf_j), (F_{\mathcal{O}E_jE_j0}, sf_j), (F_{V_jV_kV_l}, hvff_{jk}), (F_{V_jV_k0}, hvff_{jk}), \\ & (F_{V_j0V_l}, hvrf_j), (F_{V_j00}, hvrf_j), (F_{\mathcal{O}V_j0V_j}, hvrf_j), \\ & (F_{\mathcal{O}V_kV_l}, hvlf_k), (F_{\mathcal{O}V_k0}, hvlf_k), (F_{\mathcal{O}0V_jV_j}, hvlf_j), (F_{L0V_jV_j}, hvlf_j), \\ & (F_{\mathcal{O}E_j0E_j}, E_{j0}), (F_{j00}, E_{j0}), (F_{\mathcal{O}0E_jE_j}, E_{0j}), (F_{0j0}, E_{0j}), (F_{L0E_jE_j}, E_{0j}), \end{aligned} \tag{3.7}$$

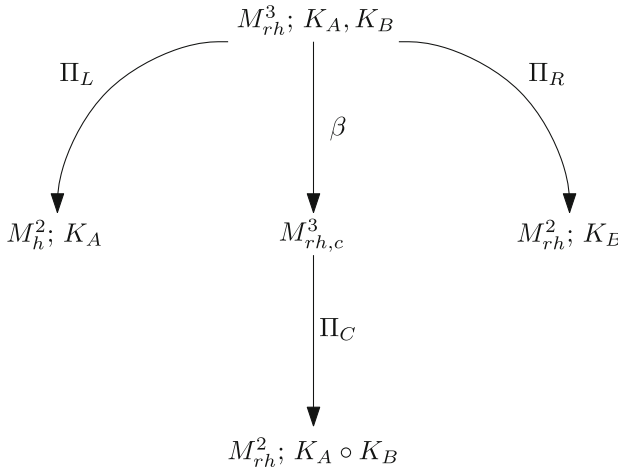


Fig. 4 The schematic diagram for the construction of the reduced triple space as required here for our composition rule. The kernel K_A lifts from the double heat space, M_{rh}^2 , to the triple heat space via pullback by the projection map Π_L . The kernel K_B vanishes to infinite order at td , so it is pc on the reduced double heat space. It lifts to the triple heat space via pullback by the projection map Π_R . On M_{rh}^3 the two kernels are composed, and the result is then pushed forward by the blow-down map to $M_{rh,c}^3$, followed by the projection map Π_C to M_{rh}^2

with F_{TL} , F_{00l} , and F_{00V_l} mapping to the interior:

$$\begin{aligned}
 \text{For } \Pi_R : & (F_{TL}, tf), (F_{\mathcal{O}}, tf), (F_{\mathcal{O}D}, tf), (F_{\mathcal{O}V_j0V_j}, tf), (F_{\mathcal{O}E_j0E_j}, tf), \\
 & (F_{\mathcal{O}V_jV_j0}, tf), (F_{\mathcal{O}E_jE_j0}, tf), (F_{\mathcal{O}V_jV_jV_j}, \text{ff}_j), (F_{L0V_jV_j}, \text{ff}_j), (F_{\mathcal{O}0V_jV_j}, \text{ff}_j), \\
 & (F_{\mathcal{O}E_jE_jE_j}, \text{sf}_j), (F_{L0E_jE_j}, \text{sf}_j), (F_{\mathcal{O}0E_jE_j}, \text{sf}_j), (F_{V_jV_kV_l}, hv\text{ff}_{kl}), (F_{0V_kV_l}, hv\text{ff}_{kl}), \\
 & (F_{V_j0V_l}, hv\text{lf}_l), (F_{00V_l}, hv\text{lf}_l), (F_{\mathcal{O}V_j0V_j}, hv\text{lf}_j), \\
 & (F_{V_jV_k0}, hv\text{rf}_k), (F_{0V_k0}, hv\text{rf}_k), (F_{\mathcal{O}V_jV_j0}, hv\text{rf}_j), (F_{RV_jV_j0}, hv\text{rf}_j), \\
 & (F_{\mathcal{O}E_j0E_j}, E_{0j}), (F_{00j}, E_{0j}), (F_{\mathcal{O}E_jE_j0}, E_{j0}), (F_{0j0}, E_{j0}), (F_{RE_jE_j0}, E_{j0}), \quad (3.8)
 \end{aligned}$$

with F_{TR} , F_{j00} , F_{V_j00} , and F_{TRD} mapping to the interior:

$$\begin{aligned}
 \text{For } \Pi_C : & (F_{\mathcal{O}}, tf), (F_{j00}, E_{j0}), (F_{RE_jE_j0}, E_{j0}), (F_{00j}, E_{0j}), \\
 & (F_{L0E_jE_j}, E_{0j}), (F_{\mathcal{O}V_jV_jV_j}, \text{ff}_j), (F_{\mathcal{O}V_j0V_j}, \text{ff}_j), (F_{\mathcal{O}E_jE_jE_j}, \text{sf}_j), (F_{\mathcal{O}E_j0E_j}, \text{sf}_j), \\
 & (F_{V_jV_kV_l}, hv\text{ff}_{jl}), (F_{V_j0V_l}, hv\text{ff}_{jl}), (F_{V_jV_k0}, hv\text{rf}_j), (F_{V_j00}, hv\text{rf}_j), (F_{RV_jV_j0}, hv\text{rf}_j), \\
 & (F_{0V_kV_l}, hv\text{lf}_l), (F_{00V_l}, hv\text{lf}_l), (F_{L0V_lV_l}, hv\text{lf}_l), \quad (3.9)
 \end{aligned}$$

with F_{TR} , F_{TL} , F_{0j0} , F_{00V_j0} , and F_{TRD} mapping to the interior:

Proof All of the exponent matrices are computed the same way: by computing pullbacks of boundary defining functions. Consider, for example, the face ff_j of M_{rh}^2 . The faces of $M_{rh,c}^3$ which are in the preimage of ff_j under Π_C are all of the faces where $z = z'' = V_j$ and $\sqrt{(T')^2 + (T'')^2} = 0$, that is $T' = T'' = 0$. These are precisely $F_{\mathcal{O}V_jV_jV_j}$ and $F_{\mathcal{O}V_j0V_j}$, so those two faces map to ff_j , and the corresponding exponent matrix entries are 1. Computing these pullbacks for each boundary hypersurface of M_{rh}^2 yields the desired exponent matrix for Π_C . By a similar process we obtain the exponent matrices for Π_L and Π_R .

Note also that the fact that Π_C is a b-fibration can be observed directly: for each boundary hypersurface G of $M_{rh,c}^3$ there is at most one boundary hypersurface H of M_{rh}^2 such that the (G, H) exponent matrix entry for Π_C equals 1. \square

3.7 Densities

Our kernels on the double space are most naturally considered as “full right densities” with respect to the usual metric on Ω . For example, the kernels of our operators A and B will be

$$K_A(t', z, z') dt' dz'; K_B(t'', z', z'') dt'' dz'', \quad \text{respectively.}$$

Multiplying the two, then integrating over $t = t' + t''$ and z' , yields

$$K_{A \circ B}(t, z, z') dt dz'.$$

If we multiply everything in the expressions by dz , we have the full-density form we need for the pushforward theorem; see [31, Theorems 4 and 5] and [19, Theorem 2.3].

To apply the aforementioned pushforward theorem, we need to transform our natural metric densities into canonical full densities and b-densities on $M_{rh,c}^3$ and M_{rh}^2 . Here are the formulas for those transformations. Throughout, let $v(X)$ and $v_b(X)$ be canonical densities and b-densities on a manifold with corners X , and let $\rho_{tot}(X)$ be a product of boundary defining functions for all boundary hypersurfaces of X . It is immediate that $v_b(X) = \rho_{tot}^{-1}(X)v(X)$.

Proposition 3.15 *The density bundles transform under blow-ups as follows:*

$$\begin{aligned} \beta^*(v(\Omega \times \Omega \times [0, 1))) &= \rho_{ff}^4 \rho_{sf}^3 \rho_{hvff}^3 \rho_{hvlf} \rho_{hvr} v(M_{rh}^2); \\ \beta^*(v(\Omega \times \Omega \times \Omega \times [0, 1))) &= \rho_{\mathcal{O}} \rho_{\mathcal{O}VVV}^7 \rho_{\mathcal{O}EEE}^6 \rho_{\mathcal{O}V0V}^5 \rho_{\mathcal{O}E0E}^4 \rho_{VVV}^5 \rho_{V0V}^3 \\ &\cdot \rho_{V0V}^3 \rho_{VV0}^3 \rho_{V00} \rho_{0V0} \rho_{00V}^4 \rho_{RVV0}^3 \rho_{REE0}^4 \rho_{LOVV}^3 \rho_{LEE}^2 \rho_{TRD} v(M_{rh,c}^3). \end{aligned}$$

Proof When blowing up a submanifold F of a manifold with corners W , blow-up introduces a factor of $\rho^{\dim(W) - \dim(F) - 1}$, equivalently $\rho^{\text{codim}(F) - 1}$, with ρ the defining function for the new (blow-up) face (see Proposition C.5 of [1]). We repeatedly apply this.⁷

For M_{rh}^2 , $\dim(W) = 5$. The blow-up to produce ff blows up a finite collection of points, so the codimension is 5 and we acquire a ρ_{ff}^4 . The blow-up to produce sf , on the other hand, requires $t' = 0, z = z' \in E$, so the codimension is 4 and we acquire a ρ^3 . The blow-up to produce $hvff$ has codimension 4, and the blow-ups to produce $hvlf$ and hvr have codimension 2. Putting this all together yields the result.

A similar analysis works for $M_{rh,c}^3$, being careful about repeated blow-ups. For example, blowing up \mathcal{O} introduces a factor of $\rho_{\mathcal{O}}$ at first. However, when blowing up a submanifold of \mathcal{O} , $\rho_{\mathcal{O}}$ itself continues to lift. For example, when creating $F_{\mathcal{O}VVV}$, $\rho_{\mathcal{O}}$ lifts to $\rho_{\mathcal{O}} \rho_{\mathcal{O}VVV}$. \square

Another important observation is that rather than t' and t'' , we are treating their square roots as the boundary defining functions, so canonical densities have dT' and dT'' . For example,

$$v(\Omega \times \Omega \times [0, 1)) = dT dz dz'',$$

and recalling that $dt = 2T dT$,

$$K_{A \circ B} dt dz dz'' = 2T K_{A \circ B} v(\Omega \times \Omega \times [0, 1)).$$

⁷ Note that even though Ω with the usual metric is not technically a manifold with corners, the same analysis works to write $v(\Omega)$ in terms of $v(\Omega_0)$.

Similarly,

$$\begin{aligned}
 &K_A(t', z, z')K_B(t'', z', z'')dt'dt''dzdz'dz'' \\
 &= 4T'T''K_A(t', z, z')K_B(t'', z', z'')\nu(\Omega \times \Omega \times \Omega \times [0, 1)^2).
 \end{aligned}$$

3.8 Composition theorem

To define various classes of operator kernels we use notation which is similar but not identical to [35]. For an index family,

$$\mathcal{F} = (F_{ff_j}, F_{sf_j}, F_{hvff_{jk}}, F_{hvr_f_j}, F_{hvl_f_j}, F_{j0}, F_{0j}, F_{td}),$$

define $\mathcal{A}_h^{\mathcal{F}}(M_h^2)$ to be the space of functions on M_h^2 which are pc with index sets given at the respective faces by \mathcal{F} and which also vanish to infinite order at tf. We also define $\Psi_h^{\mathcal{F}}$ as the set of pseudodifferential operators whose Scheartz kernels, as functions on M_h^2 , are elements of $\mathcal{A}_h^{\mathcal{F}}(M_h^2)$.

We may also use the notation

$$\mathcal{A}_h^{\alpha_{ff_j}, \alpha_{sf_j}, \alpha_{hvff_{jk}}, \alpha_{hvr_f_j}, \alpha_{hvl_f_j}, \alpha_{j0}, \alpha_{0j}, \alpha_{td}}$$

to indicate functions on M_h^2 which are elements of $\mathcal{A}_h^{\mathcal{F}}$ for some index family \mathcal{F} whose index set at each face is bounded below by the corresponding α . In other words these are functions on M_h^2 which are pc and which have leading order at each face no worse than the corresponding α (and which furthermore do not have terms of the form $x^\alpha (\log x)^p$ for $p \geq 1$). We also use $\Psi^{\alpha_{ff_j}, \alpha_{sf_j}, \alpha_{hvff_{jk}}, \alpha_{hvr_f_j}, \alpha_{hvl_f_j}, \alpha_{j0}, \alpha_{0j}, \alpha_{td}}$ in the analogous way.

Theorem 3.16 *Suppose that A is an operator whose Schwartz kernel $K_A \in \mathcal{A}_h^{\mathcal{F}^A}$ with index family*

$$\mathcal{F}_A = (A_{ff_j}, A_{sf_j}, A_{hvff_{jk}}, A_{hvr_f_j}, A_{hvl_f_j}, A_{0j}, A_{j0}, A_{td}).$$

Suppose that B is an operator whose Schwartz kernel $K_B \in \mathcal{A}_h^{\mathcal{F}^B}$ with index family

$$\mathcal{F}_B = (B_{ff_j}, B_{sf_j}, B_{hvff_{jk}}, B_{hvr_f_j}, B_{hvl_f_j}, B_{0j}, B_{j0}, B_{td} = \infty).$$

Suppose, finally, that

$$A_{0j} + B_{j0} > -1, \quad A_{hvl_f_j} + B_{hvr_f_j} > -2, \quad \text{and} \quad A_{td} > -4. \tag{3.10}$$

Then the Schwartz kernel of the composition $A \circ B$ is an element of $\mathcal{A}_h^{\mathcal{F}}$, where \mathcal{F} has index sets

$$\begin{aligned}
 &A_{ff_j} + B_{ff_j} + 4 \text{ at } ff_j, \\
 &A_{sf_j} + B_{sf_j} + 4 \text{ at } sf_j, \\
 &(\bar{\cup}_k(A_{hvff_{jk}} + B_{hvff_{kl}} + 2))\bar{\cup}(A_{hvr_f_j} + B_{hvl_f_j}) \text{ at } hvff_{jl}, \\
 &(\bar{\cup}_k(A_{hvff_{jk}} + B_{hvr_f_k} + 2))\bar{\cup}(A_{hvr_f_j})\bar{\cup}(A_{ff_j} + B_{hvr_f_j} + 4) \text{ at } hvr_f_j, \\
 &(\bar{\cup}_k(A_{hvl_f_k} + B_{hvff_{kl}} + 2))\bar{\cup}(B_{hvl_f_l})\bar{\cup}(A_{hvl_f_l} + B_{ff_l} + 4) \text{ at } hvl_f_l, \\
 &A_{j0}\bar{\cup}(A_j + B_{j0} + 4) \text{ at } E_{j0}, \\
 &B_{0j}\bar{\cup}(A_{0j} + B_j + 4) \text{ at } E_{0j}, \\
 &\infty \text{ at } td.
 \end{aligned}$$

Proof By the pullback theorem (see [39, Theorem 3] or [19, Theorem 2.2]), $\Pi_L^* K_A$ is polyhomogeneous on M_{rh}^3 with index sets:

- A_{id} at F_{TRD} and $F_{\mathcal{O}D}$; A_{ff_j} at $F_{\mathcal{O}V_j V_j V_j}$, $F_{RV_j V_j 0}$, and $F_{\mathcal{O}V_j V_j 0}$;
- A_{sf_j} at $F_{\mathcal{O}E_j E_j E_j}$, $F_{RE_j E_j 0}$, and $F_{\mathcal{O}E_j E_j 0}$; $A_{\text{hvff}_{jk}}$ at $F_{V_j V_k V_l}$ and $F_{V_j V_k 0}$;
- A_{hvrf_j} at $F_{V_j 0V_l}$ and $F_{V_j 00}$; A_{hvlf_k} at $F_{0V_k V_l}$, $F_{0V_k 0}$, and $F_{L0V_k V_k}$;
- A_{j0} at F_{j00} ; A_{0j} at F_{0j0} and $F_{L0E_j E_j}$; 0 at F_{TL} , F_{00l} , and F_{00V_j} ; and finally
- ∞ at F_{TR} , $F_{\mathcal{O}}$, $F_{\mathcal{O}V_j 0V_j}$, $F_{\mathcal{O}E_j 0E_j}$, $F_{\mathcal{O}0V_j V_j}$, $F_{\mathcal{O}0E_j E_j}$.

Note in particular that at the four hypersurfaces in the domain mapped to intersections of two hypersurfaces in the range, the pullback index set is the sum of the index sets at the two range hypersurfaces. At each of these the sum ends up being infinity. Similarly, $\Pi_R^* K_B$ is polyhomogeneous on M_{rh}^3 with index sets

- B_{ff_j} at $F_{\mathcal{O}V_j V_j V_j}$, $F_{L0V_j V_j}$, and $F_{\mathcal{O}0V_j V_j}$;
- B_{sf_j} at $F_{\mathcal{O}E_j E_j E_j}$, $F_{L0E_j E_j}$, and $F_{\mathcal{O}0E_j E_j}$;
- $B_{\text{hvff}_{kl}}$ at $F_{V_j V_k V_l}$ and $F_{0V_k V_l}$; B_{hvlf_l} at $F_{V_j 0V_l}$ and F_{00V_l} ;
- B_{hvrf_k} at $F_{V_j V_k 0}$, $F_{0V_k 0}$, and $F_{RV_k V_k 0}$; B_{0j} at F_{00j} ; B_{j0} at F_{0j0} and $F_{RE_j E_j 0}$;
- 0 at F_{TR} , F_{j00} , $F_{V_j 00}$, and F_{TRD} ; and finally
- ∞ at F_{TL} , $F_{\mathcal{O}}$, $F_{\mathcal{O}D}$, $F_{\mathcal{O}V_j V_j 0}$, $F_{\mathcal{O}V_j 0V_j}$, $F_{\mathcal{O}E_j E_j 0}$ and $F_{\mathcal{O}E_j 0E_j}$.

It is also easy enough to compute the pullbacks of T' and T'' ; they each have order 1 at each face in the lift of $T' = 0$ and $T'' = 0$ respectively. Therefore the product $4T'T''(\Pi_L^* K_A)(\Pi_R^* K_B)$ is polyhomogeneous on M_{rh}^3 with index sets

$$\begin{aligned}
 &A_{j0} \text{ at } F_{j00}; A_{0j} + B_{j0} \text{ at } F_{0j0}; B_{0j} \text{ at } F_{00j}; \\
 &A_{\text{ff}_j} + B_{\text{ff}_j} + 2 \text{ at } F_{\mathcal{O}V_j V_j V_j}; A_{\text{sf}_j} + B_{\text{sf}_j} + 2 \text{ at } F_{\mathcal{O}E_j E_j E_j}; \\
 &A_{\text{hvff}_{jk}} + B_{\text{hvff}_{kl}} \text{ at } F_{V_j V_k V_l}; A_{\text{hvff}_{jk}} + B_{\text{hvrf}_k} \text{ at } F_{V_j V_k 0}; \\
 &A_{\text{hvrf}_j} + B_{\text{hvlf}_l} \text{ at } F_{V_j 0V_l}; A_{\text{hvlf}_k} + B_{\text{hvff}_{kl}} \text{ at } F_{0V_k V_l}; \\
 &A_{\text{hvrf}_j} \text{ at } F_{V_j 00}; A_{\text{hvlf}_k} + B_{\text{hvrf}_k} \text{ at } F_{0V_k 0}; B_{\text{hvlf}_l} \text{ at } F_{00V_l}; \\
 &A_{\text{ff}_j} + B_{\text{hvrf}_j} + 1 \text{ at } F_{RV_j V_j 0}; A_{\text{sf}_j} + B_{j0} + 1 \text{ at } F_{RE_j E_j 0}; \\
 &A_{\text{hvlf}_j} + B_{\text{ff}_j} + 1 \text{ at } F_{L0V_j V_j}; A_{0j} + B_{\text{sf}_j} + 1 \text{ at } F_{L0E_j E_j}; \\
 &A_{\text{id}} + 1 \text{ at } F_{TRD}; \\
 &\infty \text{ at } F_{TL}, F_{TR}, F_{\mathcal{O}}, F_{\mathcal{O}D}, F_{\mathcal{O}V_j V_j 0}, F_{\mathcal{O}E_j E_j 0}, F_{\mathcal{O}V_j 0V_j}, \\
 &F_{\mathcal{O}E_j 0E_j}, F_{\mathcal{O}0V_j V_j}, \text{ and } F_{\mathcal{O}0E_j E_j}.
 \end{aligned} \tag{3.11}$$

Now we make the observation that on the front faces of each of the five blow-ups needed to create M_{rh}^3 from $M_{rh,c}^3$, the product $(\Pi_L^* K_A)(\Pi_R^* K_B)$ vanishes to infinite order. Applying Proposition 3.8 five times, we see that $4T'T''(\Pi_L^* K_A)(\Pi_R^* K_B)$ is in fact polyhomogeneous conormal on $M_{rh,c}^3$ with index sets the same as in (3.11), with the exception of deleting the five extra faces.

We would like to use the pushforward theorem, but first we must view $4T'T''(\Pi_L^* K_A)(\Pi_R^* K_B)$, currently a section of $\nu(\Omega \times \Omega \times \Omega \times [0, 1]^2)$, as a section of $\nu(M_{rh,c}^3)$. By Proposition 3.15, as a section of $\nu(M_{rh,c}^3)$, its orders are:

$$\begin{aligned}
 & A_{j0} \text{ at } F_{j00}; A_{0j} + B_{j0} \text{ at } F_{0j0}; B_{0j} \text{ at } F_{00j}; \\
 & A_{\text{ff}_j} + B_{\text{ff}_j} + 9 \text{ at } F_{\mathcal{O}V_jV_jV_j}; A_{\text{sf}_j} + B_{\text{sf}_j} + 8 \text{ at } F_{\mathcal{O}E_jE_jE_j}; \\
 & A_{\text{hvff}_{jk}} + B_{\text{hvff}_{kl}} + 5 \text{ at } F_{V_jV_kV_l}; A_{\text{hvff}_{jk}} + B_{\text{hvff}_{kl}} + 3 \text{ at } F_{V_jV_k0}; \\
 & A_{\text{hvr}_fj} + B_{\text{hvl}_fj} + 3 \text{ at } F_{V_j0V_l}; A_{\text{hvl}_fk} + B_{\text{hvff}_{kl}} + 3 \text{ at } F_{0V_kV_l}; \\
 & A_{\text{hvr}_fj} + 1 \text{ at } F_{V_j00}; A_{\text{hvl}_fk} + B_{\text{hvr}_fk} + 1 \text{ at } F_{0V_k0}; \\
 & B_{\text{hvl}_fl} + 1 \text{ at } F_{00V_l}; A_{\text{ff}_j} + B_{\text{hvr}_fj} + 5 \text{ at } F_{RV_jV_j0}; \\
 & A_{\text{sf}_j} + B_{j0} + 4 \text{ at } F_{RE_jE_j0}; A_{\text{hvl}_fj} + B_{\text{ff}_j} + 5 \text{ at } F_{L0V_jV_j}; \\
 & A_{0j} + B_{\text{sf}_j} + 4 \text{ at } F_{L0E_jE_j}; A_{\text{id}} + 3 \text{ at } F_{TRD}; \\
 & \text{and } \infty \text{ at } F_{TL}, F_{TR}, F_{\mathcal{O}}, F_{\mathcal{O}V_j0V_j}, \text{ and } F_{\mathcal{O}0E_jE_j}.
 \end{aligned} \tag{3.12}$$

Now we apply the pushforward theorem and push forward by Π_C . There is a condition in the pushforward theorem, see [[19], Theorem 2.3] or [33]: any face which is mapped to the interior must have index set greater than -1 (or, equivalently, 0 as a b-density). This, however, is guaranteed by (3.10). By the pushforward theorem, $(\Pi_C)_*((\Pi_L^*K_A)(\Pi_R^*K_B))$ is polyhomogeneous on M_{rh}^2 with index sets given as a section of $\nu(M_{rh}^2)$ by

$$\begin{aligned}
 & A_{\text{ff}_j} + B_{\text{ff}_j} + 9 \text{ at } \text{ff}_j; A_{\text{sf}_j} + B_{\text{sf}_j} + 8 \text{ at } \text{sf}_j; \infty \text{ at } \text{tf}; \\
 & (\overline{U}_k(A_{\text{hvff}_{jk}} + B_{\text{hvff}_{kl}} + 5))\overline{U}(A_{\text{hvr}_fj} + B_{\text{hvl}_fj} + 3) \text{ at } \text{hvff}_{jl}; \\
 & (\overline{U}_k(A_{\text{hvff}_{jk}} + B_{\text{hvff}_{kl}} + 3))\overline{U}(A_{\text{hvr}_fj} + 1)\overline{U}(A_{\text{ff}_j} + B_{\text{hvr}_fj} + 5) \text{ at } \text{hvr}_fj; \\
 & (\overline{U}_k(A_{\text{hvl}_fk} + B_{\text{hvff}_{kl}} + 3))\overline{U}(B_{\text{hvl}_fl} + 1)\overline{U}(A_{\text{hvl}_fl} + B_{\text{ff}_l} + 5) \text{ at } \text{hvl}_fl; \\
 & A_{j0}\overline{U}(A_{\text{sf}_j} + B_{j0} + 4) \text{ at } E_{j0}; B_{0j}\overline{U}(A_{0j} + B_{\text{sf}_j} + 4) \text{ at } E_{0j}.
 \end{aligned}$$

Finally, we use Proposition 3.15 to go back from sections of $\nu(M_{rh}^2)$ to sections of $\nu(\Omega \times \Omega \times [0, 1))$, then divide by $2T$ to go back to $dt dz dz''$. The index sets of the composition are thus

$$\begin{aligned}
 & A_{\text{ff}_j} + B_{\text{ff}_j} + 4 \text{ at } \text{ff}_j; A_{\text{sf}_j} + B_{\text{sf}_j} + 4 \text{ at } \text{sf}_j; \infty \text{ at } \text{tf} \text{ and } \text{td}; \\
 & (\overline{U}_k(A_{\text{hvff}_{jk}} + B_{\text{hvff}_{kl}} + 2))\overline{U}(A_{\text{hvr}_fj} + B_{\text{hvl}_fj}) \text{ at } \text{hvff}_{jl}; \\
 & (\overline{U}_k(A_{\text{hvff}_{jk}} + B_{\text{hvff}_{kl}} + 2))\overline{U}(A_{\text{hvr}_fj})\overline{U}(A_{\text{ff}_j} + B_{\text{hvr}_fj} + 4) \text{ at } \text{hvr}_fj; \\
 & (\overline{U}_k(A_{\text{hvl}_fk} + B_{\text{hvff}_{kl}} + 2))\overline{U}(B_{\text{hvl}_fl})\overline{U}(A_{\text{hvl}_fl} + B_{\text{ff}_l} + 4) \text{ at } \text{hvl}_fl; \\
 & A_{j0}\overline{U}(A_{\text{sf}_j} + B_{j0} + 4) \text{ at } E_{j0}; B_{0j}\overline{U}(A_{0j} + B_{\text{sf}_j} + 4) \text{ at } E_{0j}.
 \end{aligned}$$

□

Remark 3.17 It is instructive to compare our composition formula to that of Mazzeo–Vertman [35, Theorem 5.3]. Indeed, the two settings coincide in the special case of a surface with boundary and no vertices. Our faces ff , hvff , hvr_f , and hvl_f do not exist in that case. Moreover, in the notation of Mazzeo–Vertman, $\ell = A_{\text{sf}_j} + 4$ and $\ell' = B_{\text{sf}_j} + 4$ (see Definition 3.1, and note that the dimension of the base $b = 1$), so our calculations are in agreement.

4 The heat kernel on a surface with boundary

In this section we will build the heat kernel for a surface with smooth boundary in the absence of conical singularities. This construction has been performed in the Neumann and Dirichlet settings by Grieser [16]; the Robin case we give here is new. The work we do here shall

bdry hypersurface	sf	td	tf
model hk	hk for half plane	hk for \mathbb{R}^2	∞ order vanishing
compatibility check	tf and td	sf and tf	sf and td

Fig. 5 The model heat kernels (abbreviated hk) and the boundary faces which intersect are given above. Note that the heat kernel for the half plane is taken with the corresponding boundary condition: Dirichlet, Neumann, or Robin

be used in the later construction of the heat kernel for a surface with corners. We follow the usual geometric microlocal strategy. Namely, we specify models at various boundary hypersurfaces to which $\{T = 0\}$ lifts in the heat space, then look for a pc function which has those models as its leading order behavior at each boundary hypersurface. In order for this method to work the models must be compatible with each other at the surfaces where they intersect, in the sense that their restrictions to the intersection must be identical, as otherwise no pc function with the specified leading order behavior can exist.

This setting is a special case of the setting of surfaces with corners, and as such the double heat space will be a special case of the space M_h^2 constructed in the previous section. Many of the blow-ups are now trivial; in this setting, we need just two blowups, the first of which is

$$[\Omega \times \Omega \times [0, \infty); \{(z, z, 0) : z \in \partial\Omega\}].$$

The new boundary face is sf, and in the absence of corners, the resulting space is the reduced heat space M_{rh}^2 . To complete the construction, we perform one more blowup,

$$[M_{rh}^2; \{(z, z, 0) : z \in \Omega\}] = M_h^2.$$

The resulting blown-up face is td. Thus, the heat space for a surface with smooth boundary has boundary faces sf, td, tf, as well as the two side faces E_{10} and E_{01} , which are the lifts of $\partial\Omega \times \Omega \times [0, \infty)$ and $\Omega \times \partial\Omega \times [0, \infty)$, respectively. The boundary hypersurfaces which comprise the lift of $\{T = 0\}$ to M_h^2 are therefore sf, td, and tf (Fig. 5).

4.1 Heat kernels on the half-plane

Let (x, y) be the usual Cartesian coordinates on \mathbb{R}^2 , and consider the half-space

$$\mathbb{R}_+^2 := \{(x, y) \in \mathbb{R}^2 : y \geq 0\}. \tag{4.1}$$

The heat kernel on all of \mathbb{R}^2 is

$$H_{\mathbb{R}^2}(t, x, y, x', y') := \frac{1}{4\pi t} \exp\left[-\frac{(x - x')^2 + (y - y')^2}{4t}\right]. \tag{4.2}$$

4.1.1 The Neumann and Dirichlet heat kernels

By the method of images, the Neumann heat kernel on \mathbb{R}_+^2 is

$$\begin{aligned} &H_{\mathbb{R}^2}(t, x, y, x', y') + H_{\mathbb{R}^2}(t, x, y, x', -y') \\ &= \frac{1}{4\pi t} \exp\left[-\frac{(x - x')^2}{4t}\right] \left(\exp\left[-\frac{(y - y')^2}{4t}\right] + \exp\left[-\frac{(y + y')^2}{4t}\right]\right). \end{aligned} \tag{4.3}$$

The first term above is known as the direct term, whereas the second term is known as the reflected term or image term. We examine the behavior of this Neumann heat kernel on the double heat space M_h^2 , albeit in the simpler setting where there are no corners. Although we will not prove that this heat kernel is pc on the double space in this section (it is a consequence of later work), we will use our examination to determine the appropriate pc models in the next section.

To examine the model heat kernels for the half-space, let us examine the blow-ups in local coordinates. To get M_h^2 , we first blow up

$$\{T = y = y' = 0; x = x'\},$$

then blow up the lift of the diagonal at $T = 0$:

$$\{T = 0; y = y'; x = x'\}.$$

After the first blow-up, coordinates near the interior of the new face sf, away from the intersection with $\{T = 0\}$ (where the second blow-up takes place), are given by

$$X := \frac{x - x'}{T}; \xi := \frac{y}{T}; \xi' = \frac{y'}{T}; x'; \text{ and } T. \tag{4.4}$$

In these coordinates, the expression (4.3) becomes

$$T^{-2} \frac{1}{4\pi} \exp\left[-\frac{1}{4}X^2\right] \left(\exp\left[-\frac{1}{4}(\xi - \xi')^2\right] + \exp\left[-\frac{1}{4}(\xi + \xi')^2\right] \right). \tag{4.5}$$

To encode this, we write

$$\mathcal{H}_{-2, sf, N} := \frac{1}{4\pi} \exp\left[-\frac{1}{4}X^2\right] \left(\exp\left[-\frac{1}{4}(\xi - \xi')^2\right] + \exp\left[-\frac{1}{4}(\xi + \xi')^2\right] \right) \tag{4.6}$$

and say that $\mathcal{H}_{-2, sf, N}$, which we view as a function on sf, is the leading order model of the Neumann heat kernel at the face sf, appearing at order T^{-2} . What this means is that the Neumann heat kernel, in a coordinate patch near the interior of sf, is given at least to leading order (in this case, exactly) by

$$T^{-2}\mathcal{H}_{-2, sf, N}.$$

After the second blow-up, coordinates near the interior of the new face td, away from $y = 0$, are given by

$$X = \frac{x - x'}{T}; Y := \frac{y - y'}{T}; x'; y'; \text{ and } T. \tag{4.7}$$

Consider (4.3) in these coordinates. Away from $y = 0$ the image term is $O(T^\infty)$, and so (4.3) becomes

$$T^{-2} \frac{1}{4\pi} \exp\left[-\frac{1}{4}X^2\right] \exp\left[-\frac{1}{4}Y^2\right] + O(T^\infty). \tag{4.8}$$

So we may similarly define

$$\mathcal{H}_{-2, td} := \frac{1}{4\pi} \exp\left[-\frac{1}{4}X^2\right] \exp\left[-\frac{1}{4}Y^2\right], \tag{4.9}$$

where we have omitted the N since the model will be the same for all boundary conditions. The leading order of the Neumann heat kernel at td is given by

$$T^{-2}\mathcal{H}_{-2, td}.$$

To check for compatibility, we show that our model for the Neumann heat kernel on \mathbb{R}_+^2 is compatible with the model we have defined at *td*. Specifically, we want

$$\mathcal{H}_{-2, sf, N} |_{sf \cap td} = \mathcal{H}_{-2, td} |_{sf \cap td}.$$

The coordinate patches we have described to this point are not necessarily valid systems of coordinates near the intersection *sf*∩*td*. However, it is easy enough to show that

$$\eta := \frac{T}{y'} = \frac{1}{\xi'}; \quad X; \quad Y; \quad x'; \quad y' \tag{4.10}$$

are valid coordinates in a neighborhood of this intersection, away from *tf*. In this new coordinate patch,

$$y = YT + y', \text{ so } \xi = Y + \frac{1}{\eta}, \text{ so } \xi + \xi' = Y + \frac{2}{\eta},$$

and we get

$$\begin{aligned} \mathcal{H}_{-2, sf, N} &= \frac{1}{4\pi} \exp\left[-\frac{1}{4}X^2\right] \left(\exp\left[-\frac{1}{4}Y^2\right] + \exp\left[-\frac{1}{4}\left(Y + \frac{2}{\eta}\right)^2\right] \right); \\ \mathcal{H}_{-2, td} &= \frac{1}{4\pi} \exp\left[-\frac{1}{4}X^2\right] \exp\left[-\frac{1}{4}Y^2\right]. \end{aligned}$$

Restricting to *sf*∩*td* means letting $\eta \rightarrow 0$ in the first term, which corresponds to approaching *sf*∩*td* from the interior of *sf*, and letting $y' \rightarrow 0$ in the second term, which corresponds to approaching *sf*∩*td* from the interior of *td*. We see immediately that the second exponential in $\mathcal{H}_{-2, sf, N}$ tends to zero when η tends to zero, and $\mathcal{H}_{-2, td}$ is independent of y' , so the restrictions are well-defined and they match. This proves compatibility of $\mathcal{H}_{-2, sf, N}$ and $\mathcal{H}_{-2, td}$.

An identical analysis works for the Dirichlet heat kernel; the only difference is the sign of the image term. So we write

$$\begin{aligned} \mathcal{H}_{-2, sf, D} &:= \frac{1}{4\pi} \exp\left[-\frac{1}{4}X^2\right] \left(\exp\left[-\frac{1}{4}(\xi - \xi')^2\right] - \exp\left[-\frac{1}{4}(\xi + \xi')^2\right] \right), \end{aligned} \tag{4.11}$$

and just as before, $\mathcal{H}_{-2, sf, D}$ is compatible with $\mathcal{H}_{-2, td}$.

Note that the models $\mathcal{H}_{-2, sf, N}$ and $\mathcal{H}_{-2, sf, D}$ themselves satisfy Neumann or Dirichlet boundary conditions, respectively. In particular, looking at the expansion of $\mathcal{H}_{-2, sf, N}$ in ξ at $\{\xi = 0\}$ (i.e. $y = 0$), we observe that there is a complete Taylor expansion with no first-order term. If we take $\frac{\partial}{\partial \xi} \mathcal{H}_{-2, sf, N}$ and restrict to $\{\xi = 0\}$, we get zero. Similarly, $\mathcal{H}_{-2, sf, D}$ has a Taylor expansion with no zeroth-order term; that is, its restriction to $\{\xi = 0\}$ is zero.

It is also useful to consider the heat operator, lifted from the left (that is, acting in the unprimed coordinates):

$$\mathcal{L} := \partial_t - \partial_{xx} - \partial_{yy} = \frac{1}{2T} \partial_T - \partial_{xx} - \partial_{yy}.$$

This operator lifts under the blow-down maps to an operator on the double heat space, which we also call \mathcal{L} , abusing notation. The lift of the operator $t\mathcal{L} = T^2\mathcal{L}$ is more useful, because $T^2\mathcal{L}$ lifts to the double heat space to be a b-operator, except in a neighborhood of E_{10} , where it is merely smooth. In particular $\rho_{E_{10}}^2 T^2\mathcal{L}$ is a b-operator. Therefore $T^2\mathcal{L}$ (1) preserves polyhomogeneity and (2) preserves infinite order vanishing at *tf*. These two facts shall be useful.

The operator $T^2\mathcal{L}$ can be analyzed in the coordinate systems (4.4), (4.7), and (4.10). In the coordinates (4.4) we have by a chain rule calculation:

$$T^2\mathcal{L} = \frac{1}{2}T\partial_T - \partial_{XX} - \frac{1}{2}X\partial_X - \partial_{\xi\xi} - \frac{1}{2}\xi\partial_\xi - \frac{1}{2}\xi'\partial_{\xi'}. \tag{4.12}$$

In the coordinates (4.7), we have

$$T^2\mathcal{L} = \frac{1}{2}T\partial_T - \partial_{XX} - \frac{1}{2}X\partial_X - \partial_{YY} - \frac{1}{2}Y\partial_Y. \tag{4.13}$$

Finally, in the coordinates (4.10), we get

$$T^2\mathcal{L} = \frac{1}{2}\eta\partial_\eta - \partial_{XX} - \frac{1}{2}X\partial_X - \partial_{YY} - \frac{1}{2}Y\partial_Y. \tag{4.14}$$

The point of all of this is to show that our leading order models solve model problems at their designated boundary hypersurfaces. Specifically, since the heat kernel solves the heat equation, we must have

$$\begin{aligned} \beta^*(T^2\mathcal{L})\beta^*(T^{-2}\mathcal{H}_{-2,\text{td}})|_{\text{td}} &= \beta^*(T^2\mathcal{L})\beta^*(T^{-2}\mathcal{H}_{-2,\text{sf},N})|_{\text{sf}} \\ &= \beta^*(T^2\mathcal{L})\beta^*(T^{-2}\mathcal{H}_{-2,\text{sf},D})|_{\text{sf}} = 0. \end{aligned} \tag{4.15}$$

Observe additionally that when lifting to the double heat space, in the coordinate systems near these boundary faces, the three factors of \mathcal{H} are independent of the T coordinate. Consequently,

$$\frac{1}{2}T\partial_T(T^{-2}\mathcal{H}) = -T^{-2}\mathcal{H},$$

and so

$$\begin{aligned} (T^2\mathcal{L} - \text{Id})|_{\text{sf}}\mathcal{H}_{-2,\text{sf},N} &= (T^2\mathcal{L} - \text{Id})|_{\text{sf}}\mathcal{H}_{-2,\text{sf},D} \\ &= (T^2\mathcal{L} - \text{Id})|_{\text{td}}\mathcal{H}_{-2,\text{td}} = 0. \end{aligned} \tag{4.16}$$

Remark 4.1 The Neumann and Dirichlet heat kernels for a half-space are indeed pc on M_h^2 . This can be seen directly in local coordinates, and alternately follows from a reflection argument similar to the proof of Lemma 5.1.

4.1.2 The Robin heat kernel

Now we consider the heat kernel on \mathbb{R}_+^2 with a Robin boundary condition, namely

$$\left. \frac{\partial u(x, y)}{\partial y} \right|_{y=0} = \kappa u(x, 0), \text{ for some constant } \kappa > 0. \tag{4.17}$$

We recall that this condition is for the *inward* pointing normal derivative. The explicit expression for this heat kernel is known [5]. It is

$$H_{\mathbb{R}_+^2, \text{Robin}} := H_{\mathbb{R}_+^2, \text{Neumann}} + H_{\text{Corr}},$$

where

$$H_{\text{Corr}}(t, x, y, x', y') := -\frac{\kappa e^{\kappa(y+y')} e^{\kappa^2 t}}{\sqrt{4\pi t}} \exp\left[-\frac{1}{4t}(x-x')^2\right] \operatorname{erfc}\left(\frac{y+y'}{\sqrt{4t}} + \kappa\sqrt{t}\right).$$

Recall that the complementary error function is smooth in z , bounded by 1 for $z \geq 0$, and decaying to infinite order as $z \rightarrow \infty$:

$$\operatorname{erfc}(z) = 1 - \operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-s^2} ds.$$

Let us examine the behavior of H_{corr} in the coordinate systems (4.4) and (4.10). In (4.4):

$$H_{corr} = -T^{-1} \frac{\kappa e^{\kappa T(\xi + \xi')}}{2\sqrt{\pi}} e^{\kappa^2 T^2} \exp\left[-\frac{1}{4}X^2\right] \operatorname{erfc}\left(\frac{1}{2}(\xi + \xi') + \kappa T\right).$$

The restriction of TH_{corr} to sf , that is to $T = 0$, is well-defined. Based on our previous notation, we give it a name:

$$\mathcal{H}_{-1,\text{sf},R} := -\frac{\kappa}{2\sqrt{\pi}} \exp\left[-\frac{1}{4}X^2\right] \operatorname{erfc}\left(\frac{1}{2}(\xi + \xi')\right). \tag{4.18}$$

On the other hand, in the coordinate system (4.10) that is valid near the intersection of sf and td , we have

$$H_{corr} = -T^{-1} \frac{\kappa e^{\kappa y'(\eta Y + 2)} e^{\kappa^2 \eta^2 (y')^2}}{2\sqrt{\pi}} \exp\left[-\frac{1}{4}X^2\right] \operatorname{erfc}\left(\frac{1}{2}Y + \frac{1}{\eta} + \kappa \eta y'\right).$$

We approach $\text{td} \cap \text{sf}$ from the interior of sf by letting $\eta \rightarrow 0$. As this happens, the erfc function decays to infinite order due to the $1/\eta$ term in its argument, and we see that H_{corr} vanishes to infinite order at $\text{td} \cap \text{sf}$. Thus $\mathcal{H}_{-1,\text{sf},R}$ vanishes to infinite order at td , and so adding H_{corr} to the Neumann heat kernel does not disrupt compatibility at $\text{td} \cap \text{sf}$.

The Robin correction term also solves a model problem at sf . The model problem in this case is slightly different, because \mathcal{H} has a factor of T^{-1} rather than T^{-2} . Consequently,

$$\frac{1}{2}T \partial_T (T^{-1}\mathcal{H}) = -\frac{1}{2}T^{-1}\mathcal{H},$$

so here our model problem is

$$\left(t\mathcal{L} - \frac{1}{2}\text{Id}\right)\Big|_{\text{sf}} \mathcal{H}_{-1,\text{sf},R} = 0. \tag{4.19}$$

For the sake of completeness we include this calculation. First, we compute (dropping the subscripts for notational simplicity)

$$\partial_X \mathcal{H} = \frac{\kappa X}{4\sqrt{\pi}} e^{-X^2/4} \operatorname{erfc}\left(\frac{\xi + \xi'}{2}\right) \implies -\frac{X}{2} \partial_X \mathcal{H} = \frac{X^2}{4} \mathcal{H}.$$

Similarly we compute

$$\partial_{XX} \mathcal{H} = \frac{\kappa}{4\sqrt{\pi}} e^{-X^2/4} \operatorname{erfc}\left(\frac{\xi + \xi'}{2}\right) - \frac{\kappa X^2}{8\sqrt{\pi}} e^{-X^2/4} \operatorname{erfc}\left(\frac{\xi + \xi'}{2}\right),$$

thus

$$-\partial_{XX} \mathcal{H} = \frac{1}{2}\mathcal{H} - \frac{X^2}{4}\mathcal{H}, \quad -\frac{X}{2}\partial_X \mathcal{H} - \partial_{XX} \mathcal{H} = \frac{1}{2}\mathcal{H}.$$

Noting that $\operatorname{erfc}'(z) = -e^{-z^2}$, we also compute

$$\partial_\xi \mathcal{H} = \frac{\kappa}{4\sqrt{\pi}} e^{-X^2/4} e^{-(\xi + \xi')^2/4}, \quad -\frac{\xi}{2} \partial_\xi \mathcal{H} = \frac{-\xi \kappa}{8\sqrt{\pi}} e^{-X^2/4} e^{-(\xi + \xi')^2/4}.$$

By the symmetry in ξ and ξ' , we also have

$$-\frac{\xi'}{2} \partial_{\xi'} \mathcal{H} = \frac{-\xi' \kappa}{8\sqrt{\pi}} e^{-X^2/4} e^{-(\xi+\xi')^2/4}.$$

Finally, we compute

$$\partial_{\xi\xi} \mathcal{H} = -\frac{\kappa(\xi + \xi')}{8\sqrt{\pi}} e^{-X^2/4} e^{-(\xi+\xi')^2/4}, \quad -\partial_{\xi\xi} \mathcal{H} = \frac{\kappa(\xi + \xi')}{8\sqrt{\pi}} e^{-X^2/4} e^{-(\xi+\xi')^2/4}.$$

In total, we therefore have

$$\begin{aligned} T^2 \mathcal{L}\mathcal{H} &= \frac{1}{2} \mathcal{H} + \frac{-\xi\kappa}{8\sqrt{\pi}} e^{-X^2/4} e^{-(\xi+\xi')^2/4} + \frac{-\xi'\kappa}{8\sqrt{\pi}} e^{-X^2/4} e^{-(\xi+\xi')^2/4} \\ &\quad + \frac{\kappa(\xi + \xi')}{8\sqrt{\pi}} e^{-X^2/4} e^{-(\xi+\xi')^2/4} = \frac{1}{2} \mathcal{H}, \end{aligned}$$

verifying (4.19).

The upshot of all of this is that the Robin heat kernel on a half-space may be seen as a correction of the Neumann heat kernel, where the correction is lower order in the sense that it appears in the asymptotic behavior half an order below the leading order in the asymptotic regime corresponding to sf, namely at T^{-1} rather than T^{-2} . The correction also vanishes to infinite order at the $T = 0$ diagonal in the interior, indicating that it has no effect on the interior heat asymptotics. So the Robin heat kernel on a half space has:

- Leading order behavior at td of order T^{-2} given by $\mathcal{H}_{-2,td}$, in particular the same as the Neumann and Dirichlet heat kernels;
- Leading order behavior at sf of order T^{-2} given by $\mathcal{H}_{-2,sf,N}$, in particular the same as the Neumann heat kernel; and
- Additional sub-leading order behavior at sf of order T^{-1} given by $\mathcal{H}_{-1,sf,R}$ (and some subsequent terms at higher powers of T). However, there is no additional sub-leading order behavior at td.

4.2 Construction of the heat kernel on a surface with boundary

We are now poised to construct the Dirichlet, Neumann, and Robin heat kernels on a surface with boundary, prove they are pc on the double heat space, and identify their leading order terms in their pc expansion. Our construction is inspired by [35], [16] and indeed has already been done in [35] for Dirichlet boundary conditions (which are the Friedrichs extension for a cone-edge structure with one-dimensional edge and zero-dimensional link). These references were both largely inspired and guided by [40].

We use the same double space M_h^2 that we have been using, which in the boundary-only case for surfaces is the same as the double space of [35], with a one-dimensional edge and a zero-dimensional cone link. Note that the faces have the same names, with the exception of our E_{10} which corresponds to rf in [35] and our E_{01} which corresponds to lf. Our composition formula, Theorem 3.16, agrees with that of [35] as well.

Throughout, we will use boundary normal coordinates (x, y) on our surface with boundary Ω . In these coordinates, the boundary is defined by $y = 0$. The Riemannian metric in these boundary normal coordinates near the boundary takes the form

$$g(x, y)dx^2 + dy^2,$$

with $g(0, y) = 1$, and $g(x, y)$ smooth in x and y . The Laplacian has the following expression:

$$\Delta := -\partial_{xx} - \partial_{yy} + a_1(x, y)\partial_{xx} + a_2(x, y)\partial_x + a_3(x, y)\partial_y,$$

where a_1, a_2 , and a_3 are smooth, with $a_1(x, 0) = 0$ and $a_2(x, 0) = 0$. In the interior, we let z be a local coordinate patch on M ; then let $Z = (z - z')/T$ and use (T, Z, z') near the interior of td .

4.2.1 The Dirichlet and Neumann heat kernels

Since the heat space here only has boundary faces td , sf , tf , and the side faces, for an index family $\mathcal{F} = (F_{\text{td}}, F_{\text{sf}}, F_{E_{10}}, F_{E_{01}})$, define $\mathcal{A}_h^{\mathcal{F}}$ to be the space of kernels in $\mathcal{A}_h^{\mathcal{F}}(\Omega_h^2)$, as functions of (T, z, z') , which vanish to infinite order at tf . Similarly, we define $\Psi^{a,b,c,d}$ to be the set of operators whose kernels are in $\mathcal{A}_h^{\mathcal{F}}$ for some index family \mathcal{F} which has leading orders a, b, c, d , at the corresponding faces.

Let us begin with the Neumann heat kernel. We follow [35]. We construct first a parametrix:

Proposition 4.2 *There exists an element of $\Psi^{-2,-2,0,0}$ whose Schwartz kernel $H^{(1)}$ satisfies Neumann boundary conditions in the left (unprimed) variable, whose limit as $T \rightarrow 0$ is $\delta(z - z')$, and with*

$$T^2 \mathcal{L}H^{(1)} \in \mathcal{A}_h^{\infty,-1,0,0}.$$

Proof The idea is to solve our model problems to infinite order at td and first order at sf , and to do so in a way that satisfies Neumann boundary conditions.

In the interior of td , we use the ansatz

$$H^{(1)}(T, Z, z') \sim \sum_{j=0}^{\infty} T^{-2+j} \mathcal{H}_{-2+j,\text{td}}(Z, z'). \tag{4.20}$$

As in [35], we formally apply $t\mathcal{L}$ to this expansion and set the result equal to zero. We can solve, inductively, for each coefficient function $\mathcal{H}_{-2+j,\text{td}}$. For example, the equation for $j = 0$ is

$$-T^{-2}\mathcal{H}_{-2,\text{td}} - T^{-2} \left(\partial_{ZZ} + \frac{1}{2}Z\partial_Z \right) \mathcal{H}_{-2,\text{td}} = 0.$$

By direct computation (4.16) letting $\mathcal{H}_{-2,\text{td}}$ be the expression (4.9), namely

$$\mathcal{H}_{-2,\text{td}}(Z, z') = \frac{1}{4\pi} e^{-\frac{1}{4}Z^2},$$

it solves this equation for $j = 0$. For the higher order terms, we have to expand $T^2\mathcal{L}$ in a power series in T , and more terms than just its restriction to td will be involved. Nevertheless, one may show inductively that there exist terms $\mathcal{H}_{-2+j,\text{td}}(Z, z')$ for all j that satisfy the formal ansatz. These terms each decay rapidly in Z . Since Ω is a subset of a smooth manifold M and this construction is uniform over the interior of M , all terms $\mathcal{H}_{-2+j,\text{td}}$ are smooth up to $\text{sf} \cap \text{td}$. The result also satisfies the delta function initial condition. We omit the details, as they may be found in [35, 40], as well as other references.

We verified in Sect. 4.1.1 that $\mathcal{H}_{-2,\text{sf},N}$, defined in (4.6), is compatible with $\mathcal{H}_{-2,\text{td}}$, so it is possible to choose an element of $\Psi^{-2,-2,0,0}$ whose kernel $H^{(1)}$ simultaneously has leading order $T^{-2}\mathcal{H}_{-2,\text{sf},N}$ at sf and has full expansion (4.20) at td , which vanishes to infinite order at tf , and which is smooth down to E_{10} and E_{01} . In fact it is also possible to choose $H^{(1)}$ to

satisfy Neumann boundary conditions. To see this, examine the expansion of $\mathcal{H}_{-2,\text{sf},N}$ as we approach E_{10} and E_{01} . Boundary defining functions for those faces are ξ and ξ' respectively. Indeed, (4.6) is smooth in ξ and ξ' , and it has no order 1 term at $\xi = 0$ or $\xi' = 0$. We may thus choose $H^{(1)}$ so that there is no term of order 1 in its expansions at E_{10} and E_{01} . This $H^{(1)}$ satisfies Neumann boundary conditions, as claimed.

It remains to show that

$$T^2\mathcal{L}H^{(1)} \in \mathcal{A}_h^{\infty,-1,0,0}.$$

First we have to show that $T^2\mathcal{L}H^{(1)}$ is polyhomogeneous. However, the operator $T^2\mathcal{L}$ lifts to one which is tangent to all boundary hypersurfaces except for E_{10} , and at E_{10} it may be written as $\rho_{E_{10}}^{-2}$ times such an operator. Since such operators preserve polyhomogeneity and also preserve the infinite order vanishing at tf , the polyhomogeneity follows.

Now we compute the leading orders. Since $H^{(1)}$ has the full expansion (4.20), which is annihilated by $t\mathcal{L}$, we see that $T^2\mathcal{L}H^{(1)}$ has order ∞ at td .

At sf , we claim that the model problem is the same as for a half-space. We have solved the model problem for a half-space to first order, so we get an improvement of one order, from -2 to -1 . To see this, compute $t\mathcal{L}$ in the coordinates (4.4). We get

$$\begin{aligned} T^2\mathcal{L} &= \frac{1}{2}T\partial_T - \partial_{XX} - \frac{1}{2}X\partial_X - \partial_{\xi\xi} - \frac{1}{2}\xi\partial_\xi - \frac{1}{2}\xi'\partial_{\xi'} \\ &\quad + a_1(XT + x', \xi T)\partial_{XX} + Ta_2(XT + x', \xi T)\partial_X + Ta_3(XT + x', \xi T)\partial_\xi. \end{aligned} \tag{4.21}$$

We apply $T^2\mathcal{L}$ to our pc expansion at sf , namely

$$T^{-2}\mathcal{H}_{-2,\text{sf},N} + T^{-1}\mathcal{H}_{-1,\text{sf},N} + \dots$$

Since $T^2\mathcal{L}$ is tangent to sf , the leading order of the result will be at worst -2 . We claim it is actually -1 . Indeed, as with the half-space, the application of the first six terms in (4.21) to $T^{-2}\mathcal{H}_{-2,\text{sf},N}$ yields zero by (4.16). Moreover, applying $T^2\mathcal{L}$ to only the terms with order T^{-1} or higher yields something of order at most -1 . So the only possible term of order -2 in the expansion of $t\mathcal{L}H^{(1)}$ at sf comes from

$$\begin{aligned} &T^{-2}(a_1(XT + x', \xi T)\partial_{XX} + Ta_2(XT + x', \xi T)\partial_X \\ &\quad + Ta_3(XT + x', \xi T)\partial_\xi)\mathcal{H}_{-2,\text{sf},N}. \end{aligned}$$

However, the coefficients of the last two terms vanish at sf to first order in T , and since $a_1(x', 0) = 0$ identically, so does the coefficient of the first term. In other words, the Laplacian in boundary normal coordinates is the same as that for a half-space up to terms which are lower order at sf . Thus $t\mathcal{L}H^{(1)}$ has order -1 at sf , as desired.

At E_{01} , $T^2\mathcal{L}$ is tangent to E_{01} , so the leading order remains unchanged at 0. Finally, at E_{10} , at any point in the interior of E_{10} , we can use the coordinates (T, x, y, x', y') , in which y is the defining function for E_{10} . So $H^{(1)}$ has a smooth expansion in y down to $y = 0$, with smooth dependence on all other variables. Applying $T^2\mathcal{L}$ would usually turn a term of order y^γ into a term of order $y^{\gamma-2}$, but since $H^{(1)}$ has a smooth expansion, it stays smooth. So the leading order of $T^2\mathcal{L}H^{(1)}$ at E_{10} is 0, completing the proof. \square

As in [35], this can be improved at E_{10} , the analogue of rf :

Proposition 4.3 *There exists an element $H^{(2)} \in \mathcal{A}_h^{-2,-2,0,0}$ which satisfies Neumann boundary conditions in the left variable, with*

$$\lim_{t \rightarrow 0} H^{(2)} = \delta(z - z') \quad \text{and} \quad T^2 \mathcal{L}H^{(2)} \in \mathcal{A}_h^{\infty,-1,\infty,0}.$$

Proof This is a standard argument, as in [33, p. 32] and [35]. We use the fact that $T^2 \mathcal{L}$ is elliptic in the y -direction to iteratively solve away the Taylor expansion of $T^2 \mathcal{L}H^{(1)}$ at E_{10} . To be concrete, let A_2 be a pc kernel on Ω_h^2 , smooth at E_{10} , whose expansion in any coordinate neighborhood (t, x, y, x', y') is

$$\frac{1}{2}y^2 \left(\mathcal{L}H^{(1)} \right) (t, x, 0, x', y') + O(y^3).$$

Then $(T^2 \mathcal{L})A_2$ is pc as well, and its leading order term at E_{10} is

$$\left(T^2 \mathcal{L}H^{(1)} \right) (t, x, 0, x', y').$$

Furthermore, using the coordinates (4.4) it is straightforward to see that A_2 may be chosen to be pc down to sf and have the same order as $y^2 \mathcal{L}H^{(1)}$ at sf, namely $2 - 2 = 0$. So if we consider $H^{(1)} - A_2$, then $(T^2 \mathcal{L})(H^{(1)} - A_2)$ is pc and vanishes to first order, rather than zeroth order, at E_{10} . Moreover $H^{(1)} - A_2$ still satisfies Neumann boundary conditions.

This construction may now be iterated to produce A_3, A_4 , et cetera, so that $H^{(1)} - \sum_{j=2}^k A_j$ vanishes to order $k - 1$ at E_{10} . The A_j may be summed, and then setting

$$H^{(2)} = H^{(1)} - \sum_{j=2}^{\infty} A_j$$

gives us the result. Note that since each A_j has order -1 or better at sf ($\mathcal{L}H^{(1)}$ has order -3 there, but y^2 has order 2), the leading order term of $H^{(2)}$ at sf is still $\mathcal{H}_{-2, sf, N}$. □

Now, as in [35], let $P^{(2)} = (t \mathcal{L})H^{(2)}$. Then, as a kernel,

$$\mathcal{L}H^{(2)} = \frac{1}{T^2} P^{(2)} \in \mathcal{A}_h^{\infty,-3,\infty,0}.$$

Kernels on Ω_h^2 may be naturally identified as convolution operators on $[0, \infty) \times \Omega$, acting in the usual way. In this sense, as in [40, (7.67)] we have

$$\mathcal{L}H^{(2)} = \text{Id} - \left(-\frac{1}{T^2} P^{(2)} \right). \tag{4.22}$$

To see this for any $g \in C^\infty([0, \infty) \times \Omega)$, we have

$$\mathcal{L}H^{(2)} * g(t) = (\partial_t + \Delta) \int_0^t [H^{(2)}g](s)(t - s) ds.$$

By the fundamental theorem of calculus and the definition of $P^{(2)}$, this becomes

$$[H^{(2)}g(t)](0) + \int_0^t \left[\frac{1}{s} [P^{(2)}g](s) \right] (t - s) ds.$$

Since $[H^{(2)}g(t)](0) = g(0)$ by the delta function initial condition, we have (4.22).

To invert the right-hand side of (4.22), we use the Neumann series

$$\left(\text{Id} + \frac{1}{T^2} P^{(2)}\right)^{-1} = \text{Id} - \sum_{j=1}^{\infty} \left(-\frac{1}{T^2} P^{(2)}\right)^j =: \text{Id} + P^{(3)}.$$

By our composition Theorem 3.16, since $-T^{-2} P^{(2)} \in \mathcal{A}_h^{\infty, -3, \infty, 0}$, we obtain that for each j ,

$$\left(-\frac{1}{T^2} P^{(2)}\right)^j \in \mathcal{A}_h^{\infty, -4+j, \infty, 0}.$$

This series may therefore be asymptotically summed, and we obtain that $P^{(3)} \in \mathcal{A}_h^{\infty, -3, \infty, 0}$.

In fact this asymptotic sum is a legitimately convergent sum. This is asserted in [35] in the edge case and proven in [40, p. 270] for compact manifolds; the same applies here as well.

Finally, set

$$H^{(3)} = H^{(2)} \left(\text{Id} + P^{(3)}\right).$$

By the definition of convolution, the Neumann boundary conditions, being satisfied by $H^{(2)}$, are also satisfied by $H^{(3)}$. Since $H^{(2)} \in \mathcal{A}_h^{-2, -2, 0, 0}$ and $P^{(3)} \in \mathcal{A}_h^{\infty, -3, \infty, 0}$, our composition Theorem 3.16 tells us that

$$H^{(3)} \in \mathcal{A}_h^{-2, -2, 0, 0} + \mathcal{A}_h^{\infty, -1, 0, 0}.$$

Since $\mathcal{L}H^{(3)} = \text{Id}$, $H^{(3)}$ satisfies the delta function initial condition. By uniqueness for the Neumann heat kernel on a manifold with boundary, $H^{(3)}$ must therefore be the true heat kernel.

Theorem 4.4 *The Neumann heat kernel on Ω is pc on Ω_h^2 , and is an element of $\mathcal{A}_h^{-2, -2, 0, 0}$. Its expansion at td is given by (4.20), and its leading term at sf is given by $\mathcal{H}_{-2, sf, N}$. Moreover it is smooth down to both E_{j_0} and E_{0j} , and its expansion at sf is T^{-2} times a smooth expansion.*

Proof The heat kernel is smooth down to E_{j_0} because $H^{(2)}$ is smooth by construction, and the composition theorem implies $H^{(3)}$ is smooth. The required statement at sf follows from the same logic. Finally, it is smooth down to E_{0j} because it is symmetric. For the leading term statements, the leading terms of $H^{(2)}$ have the claimed properties, and $H^{(2)} P^{(3)}$ vanishes rapidly at td and is lower order than $H^{(2)}$ at sf. This completes the proof. \square

An analogous theorem, proved in an identical fashion, holds for the Dirichlet heat kernel. Since the Dirichlet boundary condition is the Friedrichs extension for a one-dimensional cone, this is in fact a special case of [35]. Note that the Dirichlet boundary condition implies that the heat kernel vanishes to first order at E_{10} and E_{01} :

Theorem 4.5 [35] *The Dirichlet heat kernel on Ω is pc on Ω_h^2 , and is an element of $\mathcal{A}_h^{-2, -2, 1, 1}$. Its expansion at td is given by (4.20), and its leading term at sf is given by $\mathcal{H}_{-2, sf, D}$. Moreover it is smooth down to both E_{j_0} and E_{0j} , and is T^{-2} times a smooth expansion at sf.*

4.2.2 The Robin heat kernel

We now construct the Robin heat kernel on Ω as a correction, or perturbation, of the Neumann heat kernel on Ω . To distinguish it, let $H_{\text{Neumann}}(t, z, z')$ be the Neumann heat kernel, which

is pc on Ω_h^2 by the previous section. Our Robin boundary condition is

$$\frac{\partial u(x, y)}{\partial y} \Big|_{y=0} = \kappa(x)u(x, 0).$$

Our model will be the Robin heat kernel for a half-space with constant κ . With that in mind, define

$$H_{Robin}^{(0)} := H_{Neumann} - \frac{\kappa(XT + x')}{2\sqrt{\pi}T} \exp\left[-\frac{1}{4}X^2\right] \operatorname{erfc}\left(\frac{1}{2}(\xi + \xi')\right).$$

The distinction here is that now κ depends on the variable x , which can be expressed in terms of the coordinates X, T , and x' as $x = XT + x'$.

Both terms are pc. The first term is an element of $\mathcal{A}_h^{-2,-2,0,0}$ and the second term is an element of $\mathcal{A}_h^{\infty,-1,0,0}$. Now we compute, using $\operatorname{erfc}'(s) = -2e^{-s^2}/\sqrt{\pi}$, that

$$\begin{aligned} & \left(\frac{\partial}{\partial y} - \kappa\right) H_{Robin}^{(0)} \Big|_{y=0} \\ &= \frac{1}{T} \frac{\partial}{\partial \xi} \Big|_{\xi=0} \left(-\frac{\kappa(XT + x')}{2\sqrt{\pi}T} \exp\left[-\frac{1}{4}X^2\right] \operatorname{erfc}\left(\frac{1}{2}(\xi + \xi')\right) \right) \\ & \quad - \kappa H_{Robin}^{(0)} \Big|_{y=0} \\ &= \frac{\kappa(XT + x')}{2\pi T^2} \exp\left[-\frac{1}{4}X^2\right] \exp\left[-\frac{1}{4}\xi'^2\right] \\ & \quad - \kappa(XT + x') H_{Neumann} \Big|_{y=0} \\ & \quad + \frac{(\kappa(XT + x'))^2}{2\sqrt{\pi}T} \exp\left[-\frac{1}{4}X^2\right] \operatorname{erfc}\left(\frac{1}{2}\xi'\right). \end{aligned} \tag{4.23}$$

Let this right-hand side be $c(T, X, x', \xi')$. To put it politely, this is not zero. We correct this defect by defining

$$H_{Robin}^{(1)} := H_{Robin}^{(0)} - ye^{-(y/T)^2} c(T, X, x', \xi').$$

This fixes the Robin defect: the derivative in y at $y = 0$ of the second term above is precisely $-c(T, X, x', \xi')$, and κ times this term is zero at $y = 0$, so we have

$$\left(\frac{\partial}{\partial y} - \kappa(x)\right) H_{Robin}^{(1)} \Big|_{y=0} = 0.$$

Lemma 4.6 *The function $H_{Robin}^{(1)}$ is an element of $\mathcal{A}_h^{-2,-2,0,0}$, satisfying Robin boundary conditions in the left variable. At *td*, it has the same expansion as $H_{Neumann}$. At *sf*, the first two terms of its asymptotic expansion are the first two terms for $H_{Neumann}$ plus $T^{-1}\mathcal{H}_{-1,sf,R}$, where*

$$\mathcal{H}_{-1,sf,R} := -\frac{\kappa(x')}{2\sqrt{\pi}} \exp\left[-\frac{1}{4}X^2\right] \operatorname{erfc}\left(\frac{1}{2}(\xi + \xi')\right).$$

(Comparing to (4.18), the only change is that κ is now a function of x' rather than a constant.) Furthermore,

$$t\mathcal{L}H_{Robin}^{(1)} \in \mathcal{A}_h^{\infty,0,0,0}.$$

Remark 4.7 Note that $H_{Robin}^{(1)}$ is a slightly better parametrized than $H^{(1)}$ was for the Neumann and Dirichlet problems. This is because we have solved the model problem to two orders at sf, rather than just to first order. This is necessary because we want to identify the sub-leading term of the true Robin heat kernel at sf.

Proof We just showed $H^{(1)}$ does satisfy the Robin boundary condition. Now we claim it is pc. At first it appears there are problems with the second term at $y = T = 0$ away from the diagonal, and that we might need to blow up the intersection of E_{10} and tf. However, at this intersection, $c(T, X, x', \xi')$ decays rapidly, so in fact $H_{Robin}^{(1)}$ is already pc. It decays to infinite order at tf because both terms in its definition do. We also have

$$H_{Robin}^{(1)} - H_{Neumann} = -ye^{-(y/T)^2}c(T, X, x', \xi') - \frac{\kappa(XT + x')}{2\sqrt{\pi}T} \exp\left[-\frac{1}{4}X^2\right] \operatorname{erfc}\left(\frac{1}{2}(\xi + \xi')\right).$$

It is immediate that the leading order of the second term at sf is in fact $T^{-1}\mathcal{H}_{-1,\text{sf},R}$, and that the second term vanishes to infinite order at td and is smooth up to all other boundary hypersurfaces. We claim that

$$ye^{-(y/T)^2}c(T, X, x', \xi') \in \mathcal{A}_h^{\infty,0,0,0}, \tag{4.24}$$

which immediately implies the statements concerning the asymptotic expansions of $H_{Robin}^{(1)}$ at td and sf.

To prove (4.24), we need to check decay. The statements at E_{10} and E_{01} are obvious, and the presence of $ye^{-(y/T)^2}$ implies the requisite infinite-order decay at td. The trickier face is sf. Since y decays to first order at sf, and $e^{-(y/T)^2}$ is smooth, we need to examine $c(T, X, x', \xi')$ and show it has order at worst -1 . The third term in (4.23) is already order -1 . However, the first and second terms have order -2 , but we shall compute that their difference has order -1 . As $T \rightarrow 0$, because $H_{Neumann}$ is polyhomogeneous, its restriction to $y = 0$ has an expansion at sf, and the leading term is (4.6), so

$$\begin{aligned} H_{Neumann}|_{y=0} &= \frac{1}{T^2}\mathcal{H}_{-2,\text{sf},N}\Big|_{\xi=0} + O\left(\frac{1}{T}\right) \\ &= \frac{1}{2\pi T^2} \exp\left[-\frac{1}{4}X^2\right] \exp\left[-\frac{1}{4}(\xi')^2\right] + O\left(\frac{1}{T}\right). \end{aligned}$$

Examining (4.23), this T^{-2} term here cancels the first T^{-2} term, and thus $c(T, X, x', \xi')$ is $O(\frac{1}{T})$ at sf. Therefore the order of $ye^{-(y/T)^2}c(T, X, x', \xi')$ is at worst $1 + (-1) = 0$ at sf, proving the claim (4.24).

Now consider $(t\mathcal{L})H_{Robin}^{(1)}$. Since $\mathcal{L}H_{Neumann} = 0$ because it is the heat kernel, we have

$$\begin{aligned} (t\mathcal{L})H_{Robin}^{(1)} &= (t\mathcal{L})(H_{Robin}^{(1)} - H_{Neumann}) \\ &= (t\mathcal{L})(-ye^{-(y/T)^2}c(T, X, x', \xi')) \\ &\quad - (t\mathcal{L})\left(\frac{\kappa(XT + x')}{2\sqrt{\pi}T} \exp\left[-\frac{1}{4}X^2\right] \operatorname{erfc}\left(\frac{1}{2}(\xi + \xi')\right)\right). \end{aligned} \tag{4.25}$$

The first term is an element of $\mathcal{A}_h^{\infty,0,0,0}$ before applying $t\mathcal{L}$. Since $t\mathcal{L}$ is tangent to all boundary hypersurfaces except E_{10} , it preserves the orders, and at E_{10} , $t\mathcal{L}$ takes a smooth expansion to a smooth expansion. The second term is an element of $\mathcal{A}_h^{\infty,-1,0,0}$, but we can write the

Taylor expansion of $\kappa(XT + x')$ in powers of T . The T^0 term is just $\kappa(x')$, yielding $\mathcal{H}_{-1, sf, R}$, which $t\mathcal{L}$ annihilates by (4.19). All terms except the T^0 term are elements of $\mathcal{A}_h^{\infty, 0, 0, 0}$, and thus remain so after the application of $t\mathcal{L}$. This completes the proof of Lemma 4.6. \square

From here the argument is very similar to the Neumann and Dirichlet arguments.

Proposition 4.8 *There exists an element $H_{Robin}^{(2)} \in \mathcal{A}_h^{-2, -2, 0, 0}$ which satisfies Robin boundary conditions in the left factor, with $\lim_{t \rightarrow 0} H^{(2)} = \delta(z - z')$, and with*

$$t\mathcal{L}H^{(2)} \in \mathcal{A}_h^{\infty, 0, \infty, 0}.$$

Moreover the expansions of $H_{Robin}^{(2)}$ and $H_{Robin}^{(1)}$ are identical for all terms at td and for the terms of order -2 and -1 at sf .

Proof We add terms at order y^2 and up at the face E_{10} , as in Proposition 4.3. Note that each of these terms is y^k for $k \geq 2$ times a term which has order -2 at sf , and thus each of these terms has order greater than or equal to zero at sf , so there is no effect on the first two terms of the expansion there. \square

Now we let

$$P_{Robin}^{(2)} := t\mathcal{L}H_{Robin}^{(2)}; \quad P_{Robin}^{(3)} := - \sum_{j=1}^{\infty} \left(-\frac{1}{T^2} P_{Robin}^{(2)} \right)^j.$$

We have $T^{-2}P_{Robin}^{(2)} \in \mathcal{A}_h^{\infty, -2, \infty, 0}$, so its j th power is an element of $\mathcal{A}_h^{\infty, -4+2j, \infty, 0}$ by the composition theorem, and thus $P_{Robin}^{(3)} \in \mathcal{A}_h^{\infty, -2, \infty, 0}$. As before the sum is convergent, not just asymptotically convergent. Then set

$$H_{Robin}^{(3)} = H_{Robin}^{(2)} \left(\text{Id} + P_{Robin}^{(3)} \right).$$

This satisfies the Robin boundary conditions and the initial condition, so by uniqueness it is the true Robin heat kernel. By composition, $H_{Robin}^{(2)}P_{Robin}^{(3)} \in \mathcal{A}_h^{\infty, 0, 0, 0}$, so $H_{Robin}^{(3)}$ has the same first two terms at sf and same full expansion at td as $H_{Robin}^{(2)}$. We have now proved:

Theorem 4.9 *The Robin heat kernel on Ω , with smooth non-negative Robin parameter $\kappa(x)$, is pc on Ω_h^2 , and is an element of $\mathcal{A}_h^{-2, -2, 0, 0}$, smooth down to both E_{10} and E_{01} and equal to T^{-2} times a smooth expansion at sf . It is equal to the Neumann heat kernel on Ω plus a correction term which is an element of $\mathcal{A}_h^{\infty, -1, 0, 0}$ and which has leading order $\mathcal{H}_{-1, sf, R}$ at sf .*

Indeed, the only part of this we have not addressed is the smoothness, and it follows as in the Neumann case.

5 The heat kernel on a curvilinear polygonal domain

Consider a two-dimensional sector S_γ with angle $\gamma \in (0, 2\pi)$. We investigate heat kernels on S_γ to serve as models for our eventual construction of the heat kernel on curvilinear polygonal domains.

5.1 Properties of the heat kernel for an infinite sector

We will consider the D–D, N–N, and D–N heat kernels. Recall the expression from [6, p. 592 (3.42)], which in our setting is simply

$$H(t, r, \theta, r', \theta') = \frac{1}{2t} \exp\left[-\frac{r^2 + (r')^2}{4t}\right] \sum_{j=1}^{\infty} I_{\mu_j}\left(\frac{rr'}{2t}\right) \phi_j(\theta)\phi_j(\theta'). \quad (5.1)$$

Here I_{μ_j} are the modified Bessel functions, and (ϕ_j, μ_j) are the eigenfunctions, and corresponding eigenvalues, of the appropriate eigenvalue problem (D–D, N–N, or D–N) on the interval $[0, \gamma]$.

The pc properties of (5.1) are not obvious from the expression alone. They are equally non-obvious from the equivalent expression given by the inverse Laplace transform of the Green's function. However, we claim:

Lemma 5.1 *In each of the three settings, D–D, N–N, and D–N, the heat kernel (5.1) is pc on our double space $(S_\gamma)_h^2$.*

Proof The proof is based on the reflection argument in [2, Sect. 3]. Consider the D–D case for the moment. The sector S_γ doubles to an infinite flat cone $C_{2\gamma}$, and if we let L cut this cone in half, then we claim that the D–D heat kernel on S_γ is

$$H_{S_\gamma}(t, r, \theta, r', \theta') = H_{C_{2\gamma}}(t, r, \theta, r', \theta') - H_{C_{2\gamma}}(t, r, \theta, \text{ref}_L(r', \theta')). \quad (5.2)$$

Above $H_{C_{2\gamma}}$ is the Friedrichs heat kernel on $C_{2\gamma}$, and ref_L is reflection across L . Indeed it is clear that the difference of heat kernels satisfies the heat equation and the initial condition on S_γ , as well as the Dirichlet boundary condition. By uniqueness of the heat kernel, we have (5.2).

The pc properties of H_{S_γ} may now be deduced from those of $H_{C_{2\gamma}}$, as in [2]. By [41], [35], $H_{C_{2\gamma}}$ is pc on a double heat space. In the notation of [35], the x -coordinate is r , there is no y -coordinate, and the z -coordinate is θ . The Mazzeo–Vertman heat space is not exactly the same as our heat space $(C_{2\gamma})_h^2$, as [35] do not create a face hvff, but nevertheless: \square

Proposition 5.2 *The Mazzeo–Vertman heat space is a blow-down of $(C_{2\gamma})_h^2$, and therefore $H_{C_{2\gamma}}$ is pc on $(C_{2\gamma})_h^2$.*

Proof Begin with the manifold with corners $[0, \infty) \times (C_{2\gamma})_0 \times (C_{2\gamma})_0$. The Mazzeo–Vertman heat space is created by blowing up:

- $\{0\} \times \tilde{V} \times \tilde{V}$; and
- the lift of the interior $T = 0$ diagonal.

From this space, we make a further blow-up at $[0, \infty) \times \tilde{V} \times \tilde{V}$. We claim that the resulting space is $(C_{2\gamma})_h^2$, which is all we need. Indeed, this further blow-up is disjoint from the lift of the interior $T = 0$ diagonal and thus may be done second instead of third, by Proposition 3.8. It may then be done first instead of second, since nested blow-ups commute (again by Proposition 3.8). Hence our heat space $(C_{2\gamma})_h^2$ is a blow-up (indeed, an overblown version) of the Mazzeo–Vertman heat space. \square

This takes care of the direct term, which is the first term in the right side (5.2). The reflected term [the second term in the right side of (5.2)] is pc on a nearly identical space, the only difference being that we blow up the $T = 0$ anti-diagonal $\{T = 0, r = r', \theta = \text{ref}_L(\theta')\}$

in the last step rather than the $t = 0$ diagonal. The lifts of the diagonal and anti-diagonal are not disjoint. They intersect at the lift of $\{T = 0, r = r', \theta = \theta' \in L\}$. So in order to obtain a space on which both the direct and reflected terms are pc we blow up that lift before dealing with the diagonal and anti-diagonal. We do this: first blow up that lift, then blow up the diagonal and anti-diagonal, and we have obtained a space on which the direct and reflected terms are both pc.

When we restrict our spatial arguments to lie in S_γ , we claim that this space is the double heat space $(S_\gamma)_h^2$. Indeed, blowing up the lift of $\{T = 0, r = r', \theta = \theta' \in L\}$ is precisely what is needed to create the face sf. We are doing it after blowing up hvff, hvlf, and hvrf, rather than before, but these blow-ups are disjoint (since we have already created ff) and therefore commute. The anti-diagonal does not appear once we have restricted our arguments to lie in S_γ . Therefore H_{S_γ} is pc on $(S_\gamma)_h^2$, as desired.

The argument for the N–N heat kernel is identical; there is a plus sign instead of a minus sign in (5.2). For the D–N heat kernel, we double twice, to the cone $C_{4\gamma}$, and use the method of images with four terms rather than two. The details are very similar and we omit them here. This proves Lemma (5.1). □

Having proven that H_{S_γ} is pc on the double space, we may write down its leading order models at the various boundary hypersurfaces. We begin at ff. In the interior of ff, good coordinates are given by

$$T = \sqrt{t}, \quad R := \frac{r}{T}, \quad R' := \frac{r'}{T}, \quad \theta, \quad \theta'. \tag{5.3}$$

In fact these coordinates are good uniformly down to hvlf and hvrf in the Mazzeo–Vertman double space, but to create $(S_\gamma)_h^2$ we have made an additional blowup at hvff, which in these coordinates is $\{R = R' = 0\}$. Fortunately this is not important for our present concerns. Writing (5.1) in the coordinates (5.3) gives

$$H_{S_\gamma} = \frac{1}{2} T^{-2} \exp \left[-\frac{1}{4} (R^2 + (R')^2) \right] \sum_{j=1}^{\infty} I_{\mu_j} \left(\frac{1}{2} R R' \right) \phi_j(\theta) \phi_j(\theta'). \tag{5.4}$$

This motivates the definition of the models

$$\begin{aligned} &\mathcal{H}_{-2, \text{ff}, DD}, \text{ resp. } H_{-2, \text{ff}, DN}, \text{ resp. } H_{-2, \text{ff}, NN} \\ &:= \frac{1}{2} \exp \left[-\frac{1}{4} (R^2 + (R')^2) \right] \sum_{j=1}^{\infty} I_{\mu_j} \left(\frac{1}{2} R R' \right) \phi_j(\theta) \phi_j(\theta'), \end{aligned} \tag{5.5}$$

where (ϕ_j, μ_j) are the eigenfunctions and eigenvalues of the appropriate problems on $[0, \gamma]$. It is then true that the leading term of the expansion of H_{S_γ} at ff, with D–D, D–N, or N–N boundary conditions, is $T^{-2} \mathcal{H}_{-2, \text{ff}, DD}$, $T^{-2} \mathcal{H}_{-2, \text{ff}, DN}$, or $T^{-2} \mathcal{H}_{-2, \text{ff}, NN}$, respectively.

As usual, the heat kernel is decaying to infinite order at tf. We claim that its models at sf and at td are familiar:

Proposition 5.3 *The leading order models at sf and td of H_{S_γ} are the same as the models (4.11), (4.6), and (4.9) for a manifold with boundary, namely $\mathcal{H}_{-2, \text{sf}, D}$ or $\mathcal{H}_{-2, \text{sf}, N}$ at each of the two components of sf (depending on the boundary condition) and $\mathcal{H}_{-2, \text{td}}$ at td.*

Proof This is true because of locality; in fact, all the models are the same, not just the leading order. In any patch of sf away from ff, the spatial variables are near the boundary of S_γ but bounded away from the corner. Since we are looking at short-time asymptotics, Kac’s principle holds: the heat kernel can be approximated to infinite order in T by the heat kernel

on a half-plane. To make this precise, we quote [32, Theorem 3]; see also [43]. Since this works on any patch of sf away from ff , and the models themselves are pc on sf , they must agree on all of sf , including down to ff . \square

Since H_{S_y} is pc on the double heat space, its leading order models must be compatible with each other at the intersections. We will use this in the construction of the heat kernel for curvilinear polygonal domains. In particular:

Corollary 5.4 *We have the following compatibility conditions for the D – D heat kernel:*

$$\mathcal{H}_{-2,ff,DD}|_{ff \cap sf} = \mathcal{H}_{-2,sf,D}|_{ff \cap sf}; \quad \mathcal{H}_{-2,ff,DD}|_{ff \cap td} = \mathcal{H}_{-2,td}|_{ff \cap td}.$$

An analogous result holds for the N – N heat kernel at both intersections, and for the D – N heat kernel at $ff \cap td$. Moreover, if sf_1 is the Dirichlet component of sf and sf_2 is the Neumann component of sf , we have the appropriate compatibility conditions for the D – N heat kernel at $ff \cap sf$:

$$\mathcal{H}_{-2,ff,DN}|_{ff \cap sf_1} = \mathcal{H}_{-2,sf_1,D}|_{ff \cap sf_1}; \quad \mathcal{H}_{-2,ff,DN}|_{ff \cap sf_2} = \mathcal{H}_{-2,sf_2,N}|_{ff \cap sf_2}.$$

Remark 5.5 The preceding two results may be of independent interest, as it is not obvious from the explicit expressions of the leading order models that they satisfy these compatibility conditions.

5.2 Construction of the heat kernel

As before, let Ω be a curvilinear polygonal domain, a subdomain of a larger surface M . Label its edges E_j and its vertices V_j , with V_j connecting E_j and E_{j+1} (with the appropriate generalization for multiple connected boundary components, which are allowed). For each j , let Ω_j be a surface with smooth boundary, also a subdomain of M , such that E_j is a subset of the boundary of Ω_j . Such a surface may always be created. In fact, using the tubular neighborhood theorem, Ω_j may be chosen to be contained within a small neighborhood (in M) of E_j .

5.3 Dirichlet and Neumann boundary conditions

We now construct the heat kernel in the setting where the boundary conditions on each side are either Dirichlet or Neumann, rather than Robin. Consider the heat space Ω_h^2 . We define a kernel $H^{(1)}$ on this heat space by specifying its leading order behavior at various boundary hypersurfaces. In Fig. 6, we show the faces of the double heat space whose boundaries have non-empty intersection, and in Fig. 7, we zoom in on the double heat space near the intersection of \bar{V} and E . In fact, we will define $H^{(1)}$ on a blown-down version of the heat space, without the faces $hvff_{jk}$. Call this space $\tilde{\Omega}_h^2$. By the proof of Proposition 5.2, the $hvff$ blowup may be done last, so $\tilde{\Omega}_h^2$ is in fact a blow-down of Ω_h^2 . The reason is that the blowup at $hvff$ is not necessary for the heat kernel for an exact cone, and is not done in Mazzeo–Vertman [35]. Here we only require this blowup to obtain the composition formula in Theorem 3.16.

First, at td , we require $H^{(1)}$ to have the usual local asymptotic expansion (4.20), namely

$$H^{(1)} \sim \sum_{j=0}^{\infty} T^{-2+j} \mathcal{H}_{-2+j,td}. \tag{5.6}$$

Naturally we also ask that $H^{(1)}$ decay to infinite order at tf .

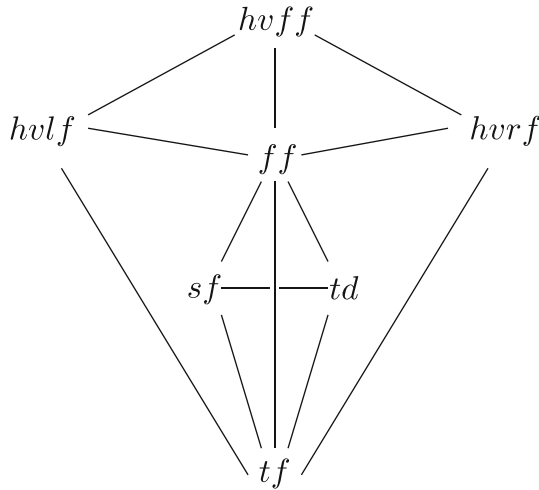


Fig. 6 The lines between boundary faces of the double heat space indicate faces whose boundaries have non-empty intersection. The E_{j0} and E_{0j} faces are omitted for the sake of simplicity. We note that E_{j0} has non-empty intersection with $hvr f$, sf , and tf , whereas E_{0j} has non-empty intersection with $hvl f$, sf , and tf

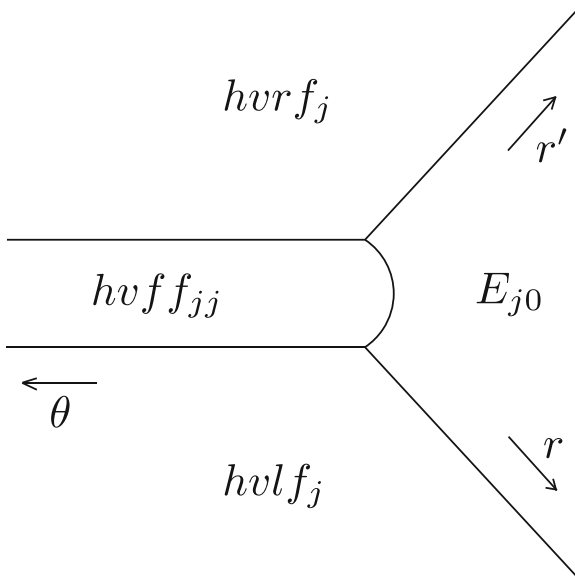


Fig. 7 This is a schematic illustration of the intersection between the side and front faces

Observe that since Ω is a subdomain of a smooth manifold M , each model $\mathcal{H}_{-2+j,td}$ is smooth up to the boundary of Ω , so in particular smooth up to each sf_j and each ff_j .

At each side face sf_j , we use the heat kernel on Ω_j , in boundary normal coordinates, as a model, where Ω_j is the surface with boundary defined above. By Theorems 4.4 and 4.5, the

heat kernel H^{Ω_j} is pc on $(\Omega_j)_h^2$, and its expansion at the face sf of $(\Omega_j)_h^2$ may be written

$$\sum_{k=0}^{\infty} T^{-2+k} \mathcal{H}_{-2+k,\text{sf}}^{\Omega_j}$$

for models $\mathcal{H}_{-2+k,\text{sf}}^{\Omega_j}$ which are pc on sf. Note that the leading order $\mathcal{H}_{-2,\text{sf}}^{\Omega_j}$ is either $\mathcal{H}_{-2,\text{sf},D}$ or $\mathcal{H}_{-2,\text{sf},N}$, defined in (4.11) and (4.6), as appropriate. Now our face sf_j is simply a subdomain of the face sf of $(\Omega_j)_h^2$, namely the restriction of sf to the region where both spatial variables are elements of $E_j \subseteq \partial\Omega_j$. So we simply set

$$H^{(1)} \sim \sum_{k=0}^{\infty} T^{-2+k} \mathcal{H}_{-2+k,\text{sf}}^{\Omega_j}. \tag{5.7}$$

The requirements (5.7) and (5.6) are compatible since the heat kernel H^{Ω_j} is pc and has the same models. They are also both compatible with infinite-order decay at tf, and with the appropriate boundary conditions on E_{j0} , for the same reason.

Near each face ff_j it is necessary to pick local polar coordinates. Define r and θ to be the polar-coordinate version of boundary normal coordinates along E_j , so that i.e. $x = r \cos \theta$ and $y = r \sin \theta$. In these coordinates, E_j is given by $\theta = 0$, and E_{j+1} is given by a curve with angle $\theta = \alpha_j$ at the origin. Of course, we could also have chosen polar coordinates from the boundary normal coordinates along E_{j+1} , so that E_{j+1} is precisely $\theta = \alpha_j$ and E_j is a curve with angle $\theta = 0$ at the origin. These two choices of polar coordinate systems agree to second order in a neighborhood of $r = 0$. With this described, coordinates valid in the interior of ff are (5.3). Moreover, in these coordinates, we have $R = 0$ at hvf_j , $R' = 0$ at hvr_f , $\theta = 0$ at E_{j0} , $\theta = \alpha_j$ at $E_{j+1,0} \cap \text{ff}_j$, $\theta' = 0$ at E_{0j} , and $\theta' = \alpha_j$ at $E_{0,j+1} \cap \text{ff}_j$. The face hvf_j is an extra, overblown face at $R = R' = 0$.

The point is that at ff_j , we can just use one of the models $\mathcal{H}_{-2,\text{ff},DD}$, $\mathcal{H}_{-2,\text{ff},DN}$ (or its flipped variant $\mathcal{H}_{-2,\text{ff},ND}$) or $\mathcal{H}_{-2,\text{ff},NN}$, depending on which boundary conditions we are imposing on E_j and E_{j+1} . These models are defined in (5.5). So we require that at each ff_j ,

$$H^{(1)} \sim T^{-2} \mathcal{H}_{-2,\text{ff},DD}(R, \theta, R', \theta'), \tag{5.8}$$

with DD replaced by DN, ND, or NN depending on the boundary conditions. Since the Laplacian is equal to the Laplacian for a straight sector to leading order at ff, these models solve the model problem at each ff_j .

By Corollary 5.4, the requirement (5.8) is compatible with (5.6) and (5.7) at td and at sf_j . At sf_{j+1} , it also follows from Corollary 5.4, because even though we no longer have an exact cone and thus the boundary normal coordinates for E_{j+1} do not agree with the polar coordinates (r, θ) everywhere, these two coordinate systems do agree to second order in ρ_{ff_j} . A similar argument, from the exact cone condition, shows that (5.8) is compatible with the appropriate boundary conditions at E_{j0} and $E_{j+1,0}$.

Since (5.8) is identical to the model for the exact sectorial heat kernel, this is also compatible with $H^{(1)}$ being pc on $\widehat{\Omega}_h^2$ rather than Ω_h^2 .

The point of checking compatibility is that as a result, we know that there exists a kernel $H^{(1)}$, pc on $\widehat{\Omega}_h^2$, with the expansions (5.6), (5.7), and (5.8) at the boundary hypersurfaces td, sf_j , and ff_j respectively, and which decays to infinite order at tf and satisfies the appropriate boundary conditions at each E_{j0} . If we let $\nu_{0,j}$ be the smallest eigenvalue of the appropriate cross-sectional Laplacian in θ at each component ff_j (note that in the Neumann–Neumann case we have $\nu_{0,j} = 0$, and otherwise $\nu_{0,j} > 0$), then the leading orders of $H^{(1)}$ may be chosen as follows:

- -2 at td , sf_j , and ff_j , with only integer powers in the expansion;
- 0 at E_{j0} and E_{0j} for each j , with only integer powers in the expansion; and
- $v_{0,j}$ at $hvlf_j$ and $hvrff_j$, with other fractional powers.

As in the case of manifolds with boundary, we consider $P^{(1)} := t\mathcal{L}H^{(1)}$. It is pc on $\widetilde{\Omega}_h^2$. Since both $\Delta H^{(1)}$ and $\partial_t H^{(1)}$ satisfy the boundary conditions (by the eigenfunction expansion, the Laplacian preserves the boundary conditions when it is applied), so does $P^{(1)}$. The leading orders of $P^{(1)}$ are as follows:

- ∞ at td and at each sf_j , since we have solved the model problem to all orders;
- -1 at ff_j , since we have solved the model problem to one order;
- 0 at E_{0j} , and $v_{0,j}$ at $hvrff_j$, since the lift of $t\mathcal{L}$ is tangent to these hypersurfaces;
- 0 at E_{j0} , since $t\mathcal{L}$ decreases the index set by 2, but not when applied to a *smooth* expansion; and
- $v_{0,j} - 1$ at $hvlf_j$, since $t\mathcal{L}$ decreases the index set by 2 at this face but the leading term is killed, as in Mazzeo–Vertman [35]. The leading order term of the Laplacian is the same as for the flat Laplacian, and our model is the flat heat kernel there. Technically this requires us to choose $H^{(1)}$ to be equal to (5.8) in a neighborhood of $hvlf_j$, which we can do. Note this is compatible with the boundary condition at E_{j0} as well.

We now construct an improved parametrix which has error decaying to infinite order at both $hvlf$ and E_{j0} . We do this in two steps, first eliminating the error at $hvlf$. This proceeds exactly as in Mazzeo–Vertman [35], as $hvlf$ is the analogue of their face rf . In the interior of $hvlf$, r is a boundary defining function for $hvlf$; the other variables are $\theta \in [0, \alpha_j]$, T , and $z' \in \Omega$. To remove a term $r^\gamma a(\theta, t, z')$ in the expansion of $P^{(1)}$ at $hvlf_j$, we need to solve the indicial equation on the cone $C([0, \alpha_j])$ in (r, θ) , with t and z' as parameters:

$$\left(-\partial_{rr} - \frac{1}{r^2}(\partial_{\theta\theta} + \frac{1}{4})\right)u(r, \theta, T, z') = r^\gamma T^{-2}a(\theta, T, z'),$$

with the appropriate boundary conditions at $\theta = 0$ and $\theta = \alpha_j$. Since a is a term in the expansion of $P^{(1)}$, it, itself, satisfies those boundary conditions. Therefore, as in [33, 35] a solution u exists with asymptotic behavior at $r = 0$ given by either $r^{\gamma+2}$ or possibly $r^{\gamma+2} \log r$ in case of an unlucky indicial root coincidence. The dependence in t and z' is purely parametric, so $u(r, \theta, T, z')$ is pc in a neighborhood of $hvlf_j$. We multiply $u(r, \theta, T, z')$ by a cutoff function equal to 1 on a neighborhood of $hvlf_j$, choosing the cutoff function so that its gradient is parallel to each edge E_{j0} and thus preserves the boundary conditions at $\theta = 0$ and $\theta = \alpha_j$. Then subtracting this product from $H^{(1)}$ eliminates the term $r^\gamma a(\theta, T, z')$ in the expansion at $hvlf$ and does not change the leading order of $H^{(1)}$ at any other boundary hypersurface. In particular, since $hvlf$ does not intersect td or sf , the expansion of $H^{(1)}$ there is unchanged. Moreover, r vanishes at ff , so u actually decays to the same order as a at ff , and thus the leading order of the expansion of $H^{(1)}$ at ff is unchanged.

Iterating this process produces a parametrix $H^{(2a)}$ and an error $P^{(2a)}$ with all the same properties as $H^{(1)}$ and $P^{(1)}$, except for two differences. First, there may be logarithmic terms at ff_j (as well as at $hvlf_j$) once we go one order down in the expansion. Second, $P^{(2a)}$ now vanishes to infinite order at $hvlf_j$ for each j .

To remove the error at E_{j0} , we follow the same template as for manifolds with boundary, using boundary ellipticity. Namely, if y is the boundary normal coordinate for the side Ω_j , add a kernel which is equal to

$$\frac{1}{2}y^2 \left(\mathcal{L}H^{(2a)}\right)(T, x, 0, x', y') + O(y^3)$$

and supported in a neighborhood of E_{j0} . This improves the order of the error at E_{j0} from 0 to 1. Note also that $\mathcal{L}H^{(2a)}$ vanishes to infinite order at sf_j , so there is no effect on the expansion at sf_j . Iterating this process and taking an asymptotic sum, as for manifolds with boundary, we obtain the following.

Proposition 5.6 *There exists a kernel $H^{(2)}$ pc on $\tilde{\Omega}_h^2$, satisfying the appropriate combination of Dirichlet and Neumann boundary conditions, with $\lim_{t \rightarrow 0} H^{(2)} = \delta(z - z')$, where if we let $P^{(2)} = t\mathcal{L}H^{(2)}$,*

- $H^{(2)}$ vanishes to infinite order at tf and has the full expansions (5.6) and (5.7) at td and each sf_j respectively;
- $H^{(2)}$ has leading term given by (5.8) at each ff_j , with the next term being one full order lower (possibly logarithmic);
- $P^{(2)}$ vanishes to infinite order at tf , td , each sf_j , each $h\text{vlf}_j$, and each E_{j0} ;
- $P^{(2)}$ has leading order -1 at each ff_j , 0 at each E_{0j} , and $\nu_{0,j}$ at each $h\text{vrf}_j$.

We will need to compose, so we will blow up to pass to Ω_h^2 by creating $h\text{vff}_{jk}$.

Corollary 5.7 *The kernels $H^{(2)}$ and $P^{(2)}$ also lift to be pc on Ω_h^2 , with leading orders $2\nu_{j,0}$ and ∞ respectively at $h\text{vff}_{jk}$.*

Now we eliminate the last error by forming the formal Neumann series

$$\text{Id} + P^{(3)} := \text{Id} - \sum_{k=1}^{\infty} \left(-\frac{1}{T^2} P^{(2)} \right)^k .$$

Note that $T^{-2}P^{(2)}$ vanishes to infinite order at all faces except for ff_j , E_{0j} , and $h\text{vrf}_j$, where it has leading orders -3 , 0 , and $\nu_{0,j}$ respectively. We use Theorem 3.16 to analyze the power $(-T^{-2}P^{(2)})^k$. We see immediately that it also vanishes to infinite order at all other faces and has leading order $-4 + k$ at ff_j . At $h\text{vrf}_j$, the index sets have an inductive relationship: the index set for the k^{th} power is the extended union of the index set for the $(k - 1)^{\text{st}}$ power with k plus the index set for the 0^{th} power. The union of all of these is indeed a legitimate index set. In particular, there are only a finite number of extended unions involved at order less than s for each value of s . The leading order is $\nu_{0,j}$, and there is no logarithmic term at that leading order. At E_{0j} , the index set is the same as that for $P^{(2)}$, with leading order zero.

All of this allows us to asymptotically sum the Neumann series, and as before, the sum is convergent. The sum $P^{(3)}$ has the same leading orders as $P^{(2)}$ at each boundary hypersurface. As before, we let

$$H^{(3)} = H^{(2)} \left(\text{Id} + P^{(3)} \right) ,$$

and deduce that $H^{(3)}$ is the true heat kernel.

By Theorem 3.16, the term $H^{(2)}P^{(3)}$ vanishes to infinite order at tf , td and sf , with leading order -1 at ff_j , so it does not affect the expansion at td and sf and does not affect the first term at ff_j . It has leading order greater than or equal to zero everywhere else, with no logarithmic terms. This tells us the following:

Theorem 5.8 *The heat kernel for Ω , with Dirichlet or Neumann boundary conditions on each side E_j , is pc on Ω_h^2 , vanishing to infinite order at tf and continuous down to all boundary hypersurfaces except for td , sf_j , and ff_j .*

Its full expansions at td and each sf_j are (5.6) and (5.7), which are the same as those for a closed manifold, and a manifold with boundary and the appropriate boundary condition, respectively.

Its expansion at ff_j has leading term (5.8) and no other terms within one order.

Remark 5.9 It is certainly possible to push through the composition formula and compute the index sets of the heat kernel for Ω at other faces (hvff, hvrf, hvlf). However, we do not think that the results obtained in this fashion are optimal—there are quite a lot of log terms which may not actually exist—so we omit the statements. In fact, it is possible that the off-diagonal faces for positive time, namely hvff_{jk} for $j \neq k$, are not necessary at all, but our results are easier and likely quicker to prove this way.

The following corollary is a version of Kac’s principle of not feeling the boundary for the Dirichlet boundary condition [22]; see also [43] for the Neumann and Robin boundary conditions.

Corollary 5.10 *The full expansions at td , sf_j , and ff_j are local, in the sense that if two domains with corners Ω and Ω' are isometric in a region R , then the expansions at the corresponding faces of the heat spaces Ω_h^2 and $(\Omega')_h^2$ agree to all orders when the spatial variables are restricted to lie within the interior of R .*

Note that these are all the faces in the lift of $\{t = 0\}$ where the heat kernel has nontrivial behavior, so the statement implies that any global contribution to the heat kernel at $t = 0$ is $O(t^\infty)$.

Proof The corollary follows immediately for td and sf_j from the construction, since the expansions there are the same as for $H^{(2)}$. For ff_j , it is also true: although the powers $(T^{-2}P^{(2)})^j$ are compositions and thus not local, by the composition theorem, their expansions at ff_j only depend on the expansion of $P^{(2)}$ itself at ff_j , which is local. Thus the expansion of $P^{(3)}$ at ff_j is local, and using the composition theorem again, so is the expansion of the true heat kernel. □

5.4 Robin boundary conditions

The construction of the Robin heat kernel proceeds very similarly to that of the Neumann heat kernel, though the boundary condition is somewhat more complicated. For each edge E_j , let $\kappa_j(x)$ be a smooth function on E_j . The key lemma is as follows.

Lemma 5.11 *There exists a kernel $H_{Robin}^{(1)}$, pc on $\tilde{\Omega}_h^2$ and with $\lim_{t \rightarrow 0} H^{(2)} = \delta(z - z')$, such that, letting $P_{Robin}^{(1)} = H_{Robin}^{(1)}$,*

- $H_{Robin}^{(1)}$ satisfies Robin boundary conditions with parameter $\kappa_j(x)$ on each edge E_j ;
- $H_{Robin}^{(1)}$ vanishes to infinite order at tf and has the full expansion (5.6) at td ;
- At each sf_j , $H_{Robin}^{(1)}$ has the same full expansion as the Robin heat kernel on Ω_j , with a parameter agreeing with $\kappa_j(x)$ upon restriction to E_j ;
- At each ff_j , $H_{Robin}^{(1)}$ has the same leading term (5.8) as in the Dirichlet, Neumann, or mixed cases, using the Neumann model at every Robin edge; and
- $P_{Robin}^{(1)}$ has leading orders ∞ at td and at sf_j , -1 at ff_j , 0 at E_{j0} and E_{j0} , $v_{j,0}$ at $hvrff_j$, and $v_{j,0} - 1$ at $hvlff_j$.

Proof The issue is compatibility of all these requirements, noting that Robin boundary conditions are more complicated than Dirichlet or Neumann boundary conditions at the intersections of E_{j0} with sf_j and ff_j . However, it turns out that Robin boundary conditions only affect the *sub-leading* terms of the expansion of $H^{(1)}$ at ff . This is why the Robin heat kernel may be viewed as a correction of the Neumann heat kernel.

We require the full expansion (5.6) at td , and observe that this is compatible with the expansion at sf_j , whose form is guaranteed by Theorem 4.9, and the leading term (5.8) at ff_j . Indeed the compatibility between td and sf_j follows from the fact that the Robin heat kernel on Ω_j is pc . The compatibility between td and ff_j follows from the fact that the Neumann heat kernel on Ω is pc . The compatibility between sf_j and the leading term at ff_j follows from the fact that the leading term at sf_j is the same as for the Neumann problem on Ω , so we can use Corollary 5.4 as in the previous section. It remains only to show that we can find such a kernel which also satisfies Robin boundary conditions.

To do this, note that Robin boundary conditions imply that if u is the leading order term (zeroth order) of the expansion of $H_{Robin}^{(1)}$ at ff , then the next term must be

$$\kappa(x)y \cdot u.$$

Since y vanishes at ff_j and sf_j as well as E_{0j} , this term vanishes to an order at ff_j and sf_j which is one higher than the order of u there. Hence any compatibility requirements only affect the lower order terms.

In order to dissect the compatibility requirements imposed by Robin boundary conditions, we zoom in near a triple intersection $E_{j0} \cap sf_j \cap ff_j$. Let boundary defining functions ρ_E , ρ_{sf} , and ρ_{ff} be chosen so that the product of all three is y ; we use these three coordinates and suppress the (parametric) dependence in all other coordinates. We write out the (previously specified) expansion at sf_j as well as the (unknown save for the first term) expansion at ff_j , doing both for $tH^{(1)}$ rather than $H^{(1)}$ to keep notation simple:

$$\begin{aligned} tH_{Robin}^{(1)} &\cong \sum_{i=0}^{\infty} \rho_{sf}^i g_i(\rho_E, \rho_{ff}) \text{ at } sf_j; \\ tH_{Robin}^{(1)} &\cong \sum_{j=0}^{\infty} \rho_{ff}^j h_j(\rho_E, \rho_{sf}) \text{ at } ff_j. \end{aligned} \tag{5.9}$$

We also write the expansion of each g_i at $\rho_{ff} = 0$:

$$g_i(\rho_E, \rho_{ff}) \cong \sum_{k=0}^{\infty} \rho_{ff}^k a_{ik}(\rho_E) + O(\rho_{ff}^{\infty}). \tag{5.10}$$

In order for the expansions (5.9) to be compatible with each other, for each j , we need

$$h_j(\rho_E, \rho_{sf}) \cong \sum_{i=0}^{\infty} \rho_{sf}^i a_{ij}(\rho_E) + O(\rho_{sf}^{\infty}). \tag{5.11}$$

On the other hand, in these coordinates, our Robin boundary condition becomes

$$\begin{aligned} &\left(\frac{1}{\rho_{sf}\rho_{ff}} \frac{\partial}{\partial \rho_E} - \kappa(\rho_{sf}, \rho_{ff}) \right) H_{Robin}^{(1)} = 0, \\ \text{i.e. } &\left(\frac{\partial}{\partial \rho_E} - \rho_{sf}\rho_{ff}\kappa(\rho_{sf}, \rho_{ff}) \right) tH_{Robin}^{(1)} = 0. \end{aligned}$$

Plugging in (5.9), organizing, and equating the coefficients of the ρ_{sf}^i terms tells us that the compatibility condition at $sf_j \cap E_{j0}$ is, for each $i \geq 1$:

$$(g_i)_{\rho_E}(0, \rho_{ff}) = \rho_{ff} \cdot \left(\text{the coefficient of } \rho_{sf}^i \text{ in } \sum_{\ell=0}^i \kappa(\rho_{sf}, \rho_{ff}) g_{\ell-1}(0, \rho_{ff}) \rho_{sf}^\ell \right), \quad (5.12)$$

and that this derivative is zero when $i = 0$. Similarly, the compatibility condition at $ff_j \cap E_{j0}$ is, for each $j \geq 1$,

$$(h_j)_{\rho_E}(0, \rho_{sf}) = \rho_{sf} \cdot \left(\text{the coefficient of } \rho_{ff}^j \text{ in } \sum_{m=0}^j \kappa(\rho_{sf}, \rho_{ff}) h_{m-1}(0, \rho_{sf}) \rho_{ff}^m \right), \quad (5.13)$$

and that the derivative is zero when $j = 0$.

Recall that the full expansion of $H_{Robin}^{(1)}$ is specified at sf_j ; since that heat kernel satisfies Robin conditions, we assume the compatibility condition (5.12). We have also specified the first term $h_0(\rho_E, \rho_{sf})$ at ff_j . Since it satisfies a Neumann boundary condition, its ρ_E derivative at E_{0j} is indeed zero, as required. We need to show that lower-order terms $h_j, j \geq 1$, may be chosen to simultaneously guarantee (5.11) and (5.13). Working one j at a time, (5.11) prescribes the full expansion of $h_j(\rho_E, \rho_{sf})$ at $\rho_{sf} = 0$, and (5.13) prescribes the order 1 term of h_j at $\rho_E = 0$ in terms of the order-0 term of h_{j-1} . As long as these two requirements are consistent we are fine.

To check this, we just plug (5.11) into (5.13). After rearrangement and equating like terms, we see that we need for each i and $j > 1$,

$$a'_{ij}(0) = \text{the coefficient of } \rho_{sf}^i \rho_{ff}^j \text{ in } \sum_{\ell=0}^{i-1} \sum_{m=0}^{j-1} \kappa(\rho_{sf}, \rho_{ff}) a_{\ell,m}(0) \rho_{sf}^{\ell+1} \rho_{ff}^{m+1}. \quad (5.14)$$

This, in turn, is guaranteed by plugging (5.10) into (5.12), completing the proof of Lemma 5.11. □

The construction of the Robin heat kernel is now analogous to the Dirichlet and Neumann cases. We solve away the error at hvf_j and then at E_{j0} . When solving away the error at hvf_j , we need to remove a term $r^\gamma a(\theta, t, z')$. Since $\partial_\theta = r \partial_y$, the coefficient $a(\theta, t, z')$ actually solves Neumann conditions, rather than Robin conditions, at $\theta = 0$ and $\theta = \alpha_j$. So as in the Neumann construction, the indicial equation may be solved and the solution, which has leading order $\gamma + 2$ at hvf_j , may be added to our parametrix in a neighborhood of hvf_j . Of course this does not preserve the Robin condition at E_{j0} . However, the error has leading order $\gamma + 2$ at hvf_j , and y has order 1 there. If we just add back κy times this Robin error in a neighborhood of hvf_j , the result satisfies the Robin boundary condition. Moreover, after applying $t\mathcal{L}$, the result has error at worst $(\gamma + 2) + 1 - 2 = \gamma + 1$ there. So this construction may be iterated to remove the error at hvf_j .

The error at E_{j0} may be eliminated in the same way as before, since adding terms at order 2 at E_{j0} does not affect the Robin boundary condition there. The construction of the formal Neumann series proceeds precisely as before, and yields:

Theorem 5.12 *The heat kernel for Ω , with Dirichlet, Neumann, or Robin boundary conditions on each side E_j , is pc on Ω_h^2 , vanishing to infinite order at tf and continuous down to all boundary hypersurfaces except for td, sf_j , and ff_j .*

Its full expansion at td is (5.6), which is the same as that for a closed manifold.

Its full expansion at each sf_j is (5.7), which is identical to that for the manifold with boundary Ω_j and the appropriate (Dirichlet/Neumann/Robin) boundary condition. Note that by Theorem 4.9, at any Robin component of sf_j , the leading term is equal to the leading term for the heat kernel on Ω_j with Neumann boundary conditions, the second term is $\mathcal{H}_{-1, sf, R}$, and all other terms are at order $T = t^{1/2}$.

Its expansion at each ff_j has leading term (5.8), with Neumann conditions at any Neumann OR Robin component, and Dirichlet conditions at any other Dirichlet component. There are no other terms within one order in $T = t^{1/2}$.

6 Heat trace on a curvilinear polygonal domain

Let Ω be a curvilinear polygonal domain as defined previously, with a Dirichlet, Neumann, or Robin condition along each side. Assume any Robin parameters $\kappa(x)$ are smooth along each side. In the previous section we have constructed the heat kernel for Ω and shown that it is pc on Ω_h^2 . We now pass to the heat trace.

The first thing to do is to restrict to the diagonal. The lifted diagonal in Ω_h^2 is a p-submanifold and is diffeomorphic to Ω_h via the lift of the map $(t, z, z) \rightarrow (t, z)$. The identification of faces is $hvf \rightarrow sv, ff \rightarrow pv, sf \rightarrow pe, td \rightarrow tf$. Therefore, by restriction:

Proposition 6.1 *The diagonal heat kernel $H_\Omega(t, z, z)$ is pc on Ω_h , with leading order -2 at tf , each pv_j , and each pe_j , as well as non-negative leading orders at all other boundary hypersurfaces.*

Remark 6.2 Naturally, all locality statements about the kernel still hold when it is restricted to the diagonal. For example, the expansion at tf is the same as that for a closed manifold. The expansion at pe_j is the same as that for a manifold with boundary. If there are any Robin edges, the expansion at the corresponding pe_j is the same as the Neumann expansion, plus the restriction to the diagonal of $\mathcal{H}_{-1, sf, R}$, plus terms of order zero.

Let π_1 be the lift of the projection map from $\Omega_0 \times [0, 1)_T$ to $[0, 1)_T$ to a map from Ω_h to $[0, 1)_T$. This map is the composition of a projection map and a blow-down map and therefore is a b-map which is a b-submersion. Since the image space has no corners it is automatically b-normal, and therefore π_1 is a b-fibration. Thus, from the pushforward theorem:

Theorem 6.3 *The heat trace $\text{Tr } H_\Omega(t)$ has a pc expansion in $T = t^{1/2}$.*

We can say substantially more, and in fact can explicitly identify all terms in this expansion up to and including the t^0 term, by carefully analyzing push-forward by this integration map. The integration is with respect to the usual measure dz on Ω . Multiplying both sides by the canonical density dT , we get

$$\int_\Omega H_\Omega(T^2, z, z) dz dT = \text{Tr } H_\Omega(T^2) dT.$$

The density $dz dT$ is $\nu(\Omega \times [0, 1)_T)$, but it is not $\nu(\Omega_h)$. Using an analogous process to the proof of Proposition 3.15,

$$(\beta)^*(dz dT) = \rho_{ff}^2 \rho_{sf} \nu(\Omega_h).$$

So, writing integration as a push-forward by π_1 , we obtain

$$(\pi_1)_*(H^\Omega(T^2, z, z) \rho_{pv}^2 \rho_{pe} \cdot \nu(\Omega_h)) = \text{Tr } H_\Omega(T^2) \cdot \nu([0, 1)_T).$$

From this we see that we really need to understand $H^\Omega(T^2, z, z)\rho_{pv}^2\rho_{pe}$.

Remark 6.4 This transformation to canonical densities explains why the leading terms at pe and pv , though they both have order -2 , only contribute at orders -1 and 0 respectively to the heat trace.

Consider the function $H^\Omega(T^2, z, z)\rho_{pv}^2\rho_{pe}$. Its expansions are as follows:

- At tf , there is an expansion in integer powers of T beginning with T^{-2} .
- At pe , there is an expansion in integer powers of T beginning with T^{-1} .
- At pv , there is an expansion with leading term at T^0 which may have logarithmic terms beginning at $T \log T$.

We may say more about these expansions. Each of them is inherited from the expansion at the corresponding face in the double space. From that analysis, each term of the expansion of $H^\Omega(T^2, z, z)$ at the face tf is T^j times a smooth function of z . Since $T = \rho_{tf}\rho_{pe}\rho_{pv}$ for suitable boundary defining functions, the coefficient of the term of order ρ_{tf}^j at tf has leading order j at pe and at pv . When multiplying by $\rho_{pv}^2\rho_{pe}$, though, this coefficient has leading order $j + 1$ at pe and $j + 2$ at pv . Similarly, the order ρ_{pe}^j term at pe has leading order $j + 1$ at pv .

What this means is that no extended unions appear in the pushforward theorem. Recall that an extended union only occurs when the coefficient of a term of order j at one boundary hypersurface in the preimage of $\{t = 0\}$ itself has leading order at most j at an adjacent such boundary hypersurface, which may produce a term $t^j \log t$ (or $t^j (\log t)^2$ if all three boundary hypersurfaces are involved). The preceding discussion shows that this does not happen. So any logarithmic terms in the heat trace expansion must come from logarithmic terms at the face ff (i.e. pv), and thus arise at order $T^{1/2}$ at the earliest. Therefore

$$\text{Tr } H_\Omega(t) = a_{-1}t^{-1} + a_{-1/2}t^{-1/2} + a_0t^0 + O(t^{1/2} \log t).$$

Moreover, the coefficients a_{-1} , $a_{-1/2}$, and a_0 are the sum of the contributions from each of the three faces tf , pe , and pv .

These contributions are easy to evaluate. At tf , the expansion is just the usual heat trace expansion from the interior of a manifold (as the coefficients are all the same), giving a contribution of

$$\frac{A(\Omega)}{4\pi t} + \frac{1}{12\pi} \int_\Omega K(z) dz + O(t).$$

At pe_j , for the same reason, the expansion is the heat trace expansion for a manifold with boundary, giving a contribution for each edge E_j . The McKean–Singer asymptotics [37] tell us what this term must be in the Dirichlet and Neumann settings. In the Robin setting, there is an extra contribution at t^0 coming from the integral of $\mathcal{H}_{-1, sf, Robin}$, and it is easy to see that it will be an integral of $\kappa(x)$ over the boundary times a constant. From [14, Theorem 5.2], we know what the constant must be.⁸ All in all, the contribution from pe_j is, where $k_g(x)$ is

⁸ An expression for this term also appears in [54]. However, it differs by an overall sign from the expression in [14]. A direct computation due to Félix Houde [20] indicates that the reference [14] has the correct sign.

the geodesic curvature on the boundary,

$$\begin{aligned}
 &-\frac{\ell(E_j)}{8\sqrt{\pi}}t^{-1/2} + \frac{1}{12\pi} \int_{E_j} k_g(x) dx + O(t^{1/2}) \text{ in the Dirichlet setting;} \\
 &\frac{\ell(E_j)}{8\sqrt{\pi}}t^{-1/2} + \frac{1}{12\pi} \int_{E_j} k_g(x) dx + O(t^{1/2}) \text{ in the Neumann setting;} \\
 &\text{and } \frac{\ell(E_j)}{8\sqrt{\pi}}t^{-1/2} + \frac{1}{12\pi} \int_{E_j} k_g(x) dx - \frac{1}{2\pi} \int_{E_j} \kappa(x) dx + O(t^{1/2})
 \end{aligned}$$

in the Robin setting. As discussed previously, at pv , the leading order contribution to the heat trace is at T^0 . This reflects the fact that rdr lifts to T^2RdR , thereby canceling the factor of T^{-2} . The leading order term in the expansion of the diagonal heat kernel at pv is the same as it is for the heat kernel on an exact sector of the same angle, and therefore may be calculated by studying the model heat kernel on that sector.

6.1 Vertex contributions

We recall our explicit calculations of the Green’s kernels for infinite circular sectors to compute the “vertex contribution” to the short time asymptotic expansion of the heat trace. For this purpose it is convenient to define:

$$\begin{aligned}
 A &:= \int_0^\infty K_{i\mu}(r\sqrt{s})K_{i\mu}(r_0\sqrt{s}) \cosh(\pi - |\phi_0 - \phi|)\mu d\mu, \\
 B &:= \int_0^\infty K_{i\mu}(r\sqrt{s})K_{i\mu}(r_0\sqrt{s}) \frac{\sinh \pi \mu}{\sinh \gamma \mu} \cosh(\phi + \phi_0 - \gamma)\mu d\mu \\
 C &:= \int_0^\infty K_{i\mu}(r\sqrt{s})K_{i\mu}(r_0\sqrt{s}) \frac{\sinh(\pi - \gamma)\mu}{\sinh \gamma \mu} \cosh(\phi - \phi_0)\mu d\mu, \\
 F &:= \int_0^\infty K_{i\mu}(r\sqrt{s})K_{i\mu}(r_0\sqrt{s}) \frac{\sinh(\pi\mu)}{\cosh \gamma \mu} \sinh((\phi + \phi_0 - \gamma)\mu)d\mu
 \end{aligned}$$

and

$$E := - \int_0^\infty K_{i\mu}(r\sqrt{s})K_{i\mu}(r_0\sqrt{s}) \frac{\cosh(\pi - \gamma)\mu}{\cosh \gamma \mu} \cosh((\phi - \phi_0)\mu)d\mu.$$

The Dirichlet and Neumann Green’s functions are, respectively,

$$G_D = \frac{1}{\pi^2} (A - B + C), \quad G_N = \frac{1}{\pi^2} (A + B + C).$$

For the Dirichlet condition at $\phi = 0$ and Neumann condition at $\phi = \gamma$, the Green’s function is

$$\frac{1}{\pi^2} (A + F + E).$$

In [43, Sect. 3] we have computed the contributions of the terms A , B , and C to the heat trace; see also [51] for an earlier computation along similar lines. In particular, we computed the integral of each of these expressions, along the diagonal $r = r_0$ and $\phi = \phi_0$ over the region $[0, R]_r \times [0, \gamma]_\phi$ with respect to polar coordinates (r, ϕ) . The vertex contribution comes solely from the C term in the D–D and N–N cases. There we see that the C term contributes

to the heat trace [43, Sect. 3.1.3]

$$\frac{\pi^2 - \gamma^2}{24\pi\gamma}. \tag{6.1}$$

In the D–N case, the vertex contribution arises from the terms F and E.

6.1.1 Contribution from the F term

Let us make some manipulations

$$\begin{aligned} & \frac{\sinh(\pi\mu)}{\cosh\gamma\mu} \sinh((\phi + \phi_0 - \gamma)\mu) \\ &= \frac{\sinh(\pi\mu)}{\cosh\gamma\mu} \sinh((\phi + \phi_0 - \gamma)\mu) - \frac{\sinh(\pi\mu)}{\sinh\gamma\mu} \cosh((\phi + \phi_0 - \gamma)\mu) \\ & \quad + \frac{\sinh(\pi\mu)}{\sinh\gamma\mu} \cosh((\phi + \phi_0 - \gamma)\mu) \\ &= \frac{\sinh((\phi + \phi_0 - \gamma)\mu) \sinh\gamma\mu - \cosh((\phi + \phi_0 - \gamma)\mu) \cosh\gamma\mu}{\sinh\gamma\mu \cosh\gamma\mu} \sinh(\pi\mu) \\ & \quad + \frac{\sinh(\pi\mu)}{\sinh\gamma\mu} \cosh((\phi + \phi_0 - \gamma)\mu). \end{aligned}$$

This expression simplifies to:

$$- \frac{2 \sinh(\pi\mu)}{\sinh(2\gamma\mu)} \cosh((\phi + \phi_0 - 2\gamma)\mu) + \frac{\sinh(\pi\mu)}{\sinh\gamma\mu} \cosh((\phi + \phi_0 - \gamma)\mu) =: -2B_1 + B_2.$$

By the calculation of the trace of the B term in [43, Sect. 3], the contribution of B₂ is $\frac{R}{4\sqrt{\pi t}} + O(\sqrt{t})$. Next we note that, for $\phi = \phi_0$,

$$\int_0^\gamma B_1 d\phi = \frac{\sinh \pi\mu}{2\mu} = \int_0^\gamma B_2 d\phi.$$

Hence the contributions of B₁ and B₂ are the same, so that F contributes

$$-2 \frac{R}{4\sqrt{\pi t}} + \frac{R}{4\sqrt{\pi t}} + O(\sqrt{t}) = -\frac{R}{4\sqrt{\pi t}} + O(\sqrt{t}). \tag{6.2}$$

Consequently, this gives no contribution because the coefficient of t^0 vanishes.

6.1.2 Contribution from the E term

Finally, we study the term E. We need to compute

$$-\frac{1}{\pi^2} \int_0^\infty K_{i\mu}(r\sqrt{s}) K_{i\mu}(r_0\sqrt{s}) \frac{\cosh(\pi - \gamma)\mu}{\cosh\gamma\mu} \cosh((\phi - \phi_0)\mu) d\mu$$

This is similar to the computation of the C term, which we would like to recycle. Hence, we add and subtract:

$$\begin{aligned} & -\frac{\cosh(\pi - \gamma)\mu}{\cosh \gamma\mu} \cosh((\phi - \phi_0)\mu) + \frac{\sinh(\pi - \gamma)\mu}{\sinh \gamma\mu} \cosh((\phi - \phi_0)\mu) \\ & -\frac{\sinh(\pi - \gamma)\mu}{\sinh \gamma\mu} \cosh((\phi - \phi_0)\mu) \\ & = \frac{-\cosh(\pi - \gamma)\mu \sinh \gamma\mu + \sinh(\pi - \gamma)\mu \cosh \gamma\mu}{\sinh \gamma\mu \cosh \gamma\mu} \cosh((\phi - \phi_0)\mu) \\ & \quad -\frac{\sinh(\pi - \gamma)\mu}{\sinh \gamma\mu} \cosh((\phi - \phi_0)\mu). \end{aligned}$$

This reduces to:

$$\frac{2 \sinh(\pi - 2\gamma)\mu}{\sinh(2\gamma\mu)} \cosh((\phi - \phi_0)\mu) - \frac{\sinh(\pi - \gamma)\mu}{\sinh \gamma\mu} \cosh((\phi - \phi_0)\mu) =: 2C_1 - C_2.$$

We recognize the term

$$\int_0^\infty K_{i\mu}(r\sqrt{s})K_{i\mu}(r_0\sqrt{s})C_2d\mu = C.$$

Consequently, we already know the contribution to the trace from C_2 , because it is the same as that which we computed for C

$$\frac{\gamma}{2\pi} \cdot \frac{\pi^2 - \gamma^2}{12\gamma^2} + O(t^\infty), \quad t \downarrow 0.$$

The reason we write it in this way is to recall the differences between the contributions of C_1 and C_2 . The factor of γ in $\frac{\gamma}{2\pi}$ comes from the trace calculation in which we integrate the angular coordinate over $(0, \gamma)$. This factor is therefore the same in C_1 . Hence when we consider C_1 , we just need to change γ to 2γ in the second factor only. The contribution of C_1 is

$$\frac{\gamma}{2\pi} \cdot \frac{\pi^2 - (2\gamma)^2}{12(2\gamma)^2},$$

and hence the trace contribution of E is

$$\frac{\pi^2 - 4\gamma^2}{48\pi\gamma} - \frac{\pi^2 - \gamma^2}{24\pi\gamma} = -\frac{\pi^2 + 2\gamma^2}{48\pi\gamma} + O(t^\infty). \tag{6.3}$$

We have now computed the contribution of the vertex to the t^0 term. Any Robin–Dirichlet, or Robin–Neumann, or Robin–Robin corner is treated as if the Robin conditions were Neumann conditions, as the corresponding models at $ff_{diag,j}$ are the same.

The vertex contribution for an interior angle of γ is therefore:

$$\frac{\pi^2 - \gamma^2}{24\pi\gamma} \text{ for D–D, N–N, R – –R, and N–R boundary conditions} \tag{6.4}$$

or

$$-\frac{\pi^2 + 2\gamma^2}{48\pi\gamma} \text{ for D–N and D–R mixed boundary conditions.} \tag{6.5}$$

The vertex contribution (6.5) appears to be new and may be of independent interest. In §B we show how, given the D–D corner contribution, one may also use the more familiar series

expression for the heat kernel as in [6] to compute the N–N and D–N corner contribution. The result is of course the same as we have computed here. In summary, we have Theorem 1.2.

Remark 6.5 The Gauss–Bonnet theorem dictates that

$$2\pi \chi(\Omega) = \int_{\Omega} K(z) dz + \int_{\partial\Omega} k_g(x) dx + \sum_{j=1}^n (\pi - \alpha_j),$$

where $\chi(\Omega)$ is the Euler characteristic of Ω . This yields an alternate expression for a_0 :

$$\begin{aligned} a_0 &= \frac{1}{6} \chi(\Omega) - \frac{1}{12\pi} \sum_{j=1}^n (\pi - \alpha_j) - \frac{1}{2\pi} \sum_{j \in \mathcal{E}_R} \int_{E_j} \kappa_j(x) dx \\ &\quad + \sum_{j \in V_{=}} \frac{\pi^2 - \alpha_j^2}{24\pi\alpha_j} + \sum_{j \in V_{\neq}} \frac{-\pi^2 - 2\alpha_j^2}{48\pi\alpha_j}. \end{aligned}$$

Remark 6.6 It is straightforward to allow for surfaces which may also have isolated conical singularities. An isolated conical singularity with opening angle 2α will give contribute to the heat trace:

$$\frac{\pi^2 - \alpha^2}{12\pi\alpha}.$$

6.2 Vertices as spectral invariants

Here we apply our results, presenting several contexts in which the presence, or lack, of vertices is spectrally determined. We also show that a jump in boundary condition is spectrally determined.

Theorem 6.7 *Let Σ be a surface with at least one vertex with interior angle not equal to π and either the Dirichlet boundary condition or the Neumann boundary condition. Let Ω be a smoothly bounded surface with either the Dirichlet boundary condition or the Neumann boundary condition such that $\chi(\Omega) \leq \chi(\Sigma)$. Then Σ and Ω are not isospectral.*

Proof It suffices to compare the short time asymptotic expansion of the heat traces and demonstrate that the coefficients cannot be the same for Σ and Ω . The coefficient a_0 for Σ is:

$$a_0(\Sigma) = \frac{\chi(\Sigma)}{6} - \frac{1}{12\pi} \sum_{j=1}^n (\pi - \alpha_j) + \sum_{j=1}^n \frac{\pi^2 - \alpha_j^2}{24\pi\alpha_j},$$

where Σ has n vertices with interior angles α_j . This expression simplifies to:

$$a_0(\Sigma) = \frac{\chi(\Sigma)}{6} - \frac{n}{12} + \sum_{j=1}^n \frac{\pi^2 + \alpha_j^2}{24\pi\alpha_j}.$$

On the other hand,

$$a_0(\Omega) = \frac{\chi(\Omega)}{6} \leq \frac{\chi(\Sigma)}{6}.$$

Since at least one $\alpha_j \neq \pi$, it is a straightforward exercise in multivariable analysis [31] to demonstrate the strict inequality

$$a_0(\Sigma) > \frac{\chi(\Sigma)}{6} \geq \frac{\chi(\Omega)}{6} = a_0(\Omega).$$

□

We obtain a similar result for the Robin boundary condition. Recall the Robin boundary condition is,

$$u = \kappa \frac{\partial u}{\partial \nu}, \quad \text{on all smooth boundary components, } \kappa \geq 0.$$

Above, $\frac{\partial u}{\partial \nu}$ is the inward pointing unit normal, as in (4.17).

Theorem 6.8 *Let Σ be a surface with at least one vertex with interior angle not equal to π with the Robin boundary condition as above, with constant Robin parameter. Let Ω be a smoothly bounded surface with $\chi(\Omega) \leq \chi(\Sigma)$. Assume the same Robin boundary condition on $\partial\Omega$. Then Σ and Ω are not isospectral.*

Proof We argue by contradiction. Assume that Σ and Ω are isospectral. Then, they must have the same heat trace coefficients. The terms $a_{-1/2}(\Sigma)$ and $a_{-1/2}(\Omega)$ show that the boundaries of Ω and Σ have the same length. Hence, since at least one of the angles α_j is not equal to π , we have

$$a_0(\Sigma) = \frac{\chi(\Sigma)}{6} - \frac{n}{12} + \sum_{j=1}^n \frac{\pi^2 + \alpha_j^2}{24\pi\alpha_j} - \frac{\kappa|\partial\Sigma|}{2\pi} > \frac{\chi(\Sigma)}{6} - \frac{\kappa|\partial\Sigma|}{2\pi}.$$

Above, $|\partial\Sigma|$ is the length of the boundary of Σ , n is the number of vertices, and α_j is the interior angle at the j^{th} vertex. On the other hand

$$a_0(\Omega) = \frac{\chi(\Omega)}{6} - \frac{\kappa|\partial\Omega|}{2\pi} = \frac{\chi(\Omega)}{6} - \frac{\kappa|\partial\Sigma|}{2\pi} < a_0(\Sigma).$$

This is the desired contradiction. □

For the case of smoothly bounded surfaces, the spectrum also detects a jump in the boundary condition, even without vertices. This is depicted in Fig. 8.

Theorem 6.9 *Let Σ be a smoothly bounded surface which has Dirichlet boundary condition and Neumann boundary condition on a single boundary component (that is, a nontrivial Zaremba boundary condition), with a Friedrichs extension at the interface. Let Ω be a smoothly bounded surface which has either Neumann or Dirichlet boundary condition (not mixed). Assume that*

$$\chi(\Omega) \geq \chi(\Sigma).$$

Then Σ and Ω are not isospectral.

Proof For Σ , the heat trace coefficient

$$a_0(\Sigma) = \frac{\chi(\Sigma)}{6} - \frac{n}{16},$$

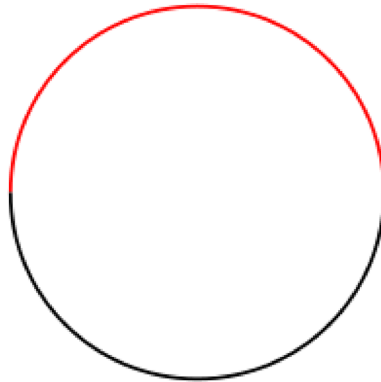


Fig. 8 For a circular domain, impose the Dirichlet boundary on the red arc and the Neumann boundary on the black arc, taking the Friedrichs extension at the interface. Such a domain is not isospectral to any simply connected smoothly bounded domain which has either the Dirichlet or Neumann condition (but not mixed). In fact, one may take the red and black pieces of the boundary to be of any proportions, not necessarily equal

where n is the number of times the boundary condition jumps between Dirichlet and Neumann. We obtain this because the boundary is smooth, and hence the angle at the “vertex” where the boundary condition jumps is equal to π . On the other hand,

$$a_0(\Omega) \geq \frac{\chi(\Omega)}{6} \geq \frac{\chi(\Sigma)}{6} > a_0(\Sigma),$$

since $n \geq 1$. □

In conclusion, we determine contexts in which entirely mixed Dirichlet–Neumann vertices are spectrally determined. In particular, this shows that we may distinguish between the presence of mixed-boundary condition vertices versus vertices with the same boundary condition on both sides; see Fig. 9.

Theorem 6.10 *Assume that Σ is a surface with vertices with mixed Dirichlet and Neumann boundary condition such that each vertex has Dirichlet on one side and Neumann on the other side. Moreover, assume that all interior angles are less than $\frac{\pi}{\sqrt{2}}$. Let Ω be any surface which is either:*

1. *smoothly bounded and with either the Dirichlet or Neumann, but not mixed, boundary condition;*
2. *a surface with vertices with either the Dirichlet or Neumann, but not mixed, boundary condition.*

Assume further that $\chi(\Sigma) \leq \chi(\Omega)$. Then Σ and Ω are not isospectral.

Proof We compute the heat trace coefficient for Σ ,

$$a_0(\Sigma) = \frac{\chi(\Sigma)}{6} - \frac{n}{12} + \sum_{j=1}^n \frac{-\pi^2 + 2\alpha_j^2}{48\pi\alpha_j}.$$

Above, n is the number of vertices, and α_j is the interior angle at the j^{th} vertex. By the assumption that $\alpha_j < \frac{\pi}{\sqrt{2}}$ for all j we have

$$a_0(\Sigma) < \frac{\chi(\Sigma)}{6} - \frac{n}{12}.$$



Fig. 9 For a rectangular domain we impose the Dirichlet boundary condition on the red sides and Neumann boundary condition on the black sides

On the other hand, if Ω has smooth boundary and Dirichlet or Neumann boundary condition (not mixed), we have

$$a_0(\Omega) \geq \frac{\chi(\Omega)}{6} \geq \frac{\chi(\Sigma)}{6} > a_0(\Sigma).$$

This shows that Σ and Ω are not isospectral.

In case Ω has m vertices, and a single fixed boundary condition then

$$a_0(\Omega) \geq \frac{\chi(\Omega)}{6} - \frac{n}{12} + \sum_{j=1}^m \frac{\pi^2 + \beta_j^2}{24\pi\beta_j}.$$

Here the interior angle at the j^{th} vertex is β_j . By the assumption that Ω has vertices, at least one $\beta_j \neq \pi$, and therefore

$$a_0(\Omega) > \frac{\chi(\Omega)}{6} \geq a_0(\Sigma).$$

Consequently, Σ and Ω are not isospectral. \square

We conclude with a familiar example which satisfies the hypotheses of the preceding theorem. Let us consider domains in the plane which do not have holes. Let Σ be a rectangular domain with the Dirichlet boundary condition on two opposite sides, and Neumann boundary condition on the other two sides; see Fig. 9. Then, the interior angles are all equal to $\frac{\pi}{2} < \frac{\pi}{\sqrt{2}}$. Consequently, the theorem shows that such a domain is not isospectral to *any* smoothly bounded domain with either Dirichlet or Neumann (but not mixed) boundary condition, nor is it isospectral to *any* domain with corners but which has a single fixed boundary condition, either Dirichlet or Neumann. An analogous result holds for any polygonal domain which has an even number of sides and alternating Dirichlet and Neumann boundary conditions, such that the interior angles do not exceed $\frac{\pi}{\sqrt{2}}$.

Appendix A. Calculation of the Green’s function for Dirichlet, Neumann, and mixed Dirichlet–Neumann boundary conditions

Here we will explain how to obtain the explicit expression for the Green’s functions introduced in Sect. 2.1. Inspired by Fedosov [10], we consider the Kontorovich–Lebedev transform

$$F(x) = \int_0^\infty K_{ix}(z) f(z) \frac{dz}{z}$$

and its inverse transform

$$f(y) = \frac{2}{\pi^2} \int_0^\infty x \sinh(\pi x) K_{ix}(y) F(x) dx.$$

Above, K_ν is the modified Bessel function of second kind. At least formally

$$f(r_0\sqrt{s}) = \int_0^\infty \frac{2}{\pi^2 r} \int_0^\infty x \sinh(\pi x) K_{ix}(r\sqrt{s}) K_{ix}(r_0\sqrt{s}) dx \cdot f(r\sqrt{s}) dr.$$

Hence, in the distributional sense we obtain

$$\frac{2}{\pi^2 r} \int_0^\infty x \sinh(\pi x) K_{ix}(r\sqrt{s}) K_{ix}(r_0\sqrt{s}) dx = \delta(r - r_0). \tag{A.1}$$

We will search for the Green’s function of the following form $G(s, r, \phi, r_0, \phi_0) =$

$$\frac{2}{\pi^2} \int_0^\infty K_{i\mu}(r\sqrt{s}) K_{i\mu}(r_0\sqrt{s}) \mu \sinh(\pi \mu) \Phi(\mu, \phi, \phi_0) d\mu. \tag{A.2}$$

Inserting (A.2) into (2.1) and using the definition of K_ν , we want to solve:

$$\begin{aligned} & \frac{2}{\pi^2} \int_0^\infty K_{i\mu}(r\sqrt{s}) K_{i\mu}(r_0\sqrt{s}) \mu \sinh(\pi \mu) \frac{1}{r^2} [- (i\mu)^2 \Phi(\mu, \phi, \phi_0) - \Phi''(\mu, \phi, \phi_0)] d\mu \\ & = \frac{1}{r} \delta(r - r_0) \delta(\phi - \phi_0). \end{aligned}$$

By (A.1), it will suffice to find Φ such that

$$\begin{aligned} & - (i\mu)^2 \Phi(\mu, \phi, \phi_0) - \Phi''(\mu, \phi, \phi_0) = \delta(\phi - \phi_0), \\ & \alpha \Phi(0) + \beta \Phi'(0) = 0, \quad \alpha \Phi(\gamma) + \beta \Phi'(\gamma) = 0, \end{aligned} \tag{A.3}$$

with either $\alpha = 1$ and $\beta = 0$ or $\alpha = 0$ and $\beta = 1$. Solving the first case will yield (2.2), and the second case will yield (2.3). For this purpose, we note that

$$\Phi_1(\phi) := \alpha \sinh \phi \mu - \mu \beta \cosh \phi \mu, \quad \Phi_2(\phi) := \alpha \sinh(\phi - \gamma) \mu - \mu \beta \cosh(\phi - \gamma) \mu$$

are solutions of (A.3) and satisfy the first and second boundary conditions, respectively. Hence, the Green’s function is obtained by inserting

$$\Phi := - \frac{\Phi_1(\phi) \Phi_2(\phi_0)}{W(\Phi_1, \Phi_2)}, \quad \text{for } \phi < \phi_0, \tag{A.4}$$

into (A.2) where above $W(\Phi_1, \Phi_2)$ is the Wronskian of Φ_1 and Φ_2 .

Similarly, with the Dirichlet–Neumann mixed boundary condition, the Green’s function is of the form (A.2), but in this case Φ solves

$$-\Phi'' + \mu^2 \Phi = 0, \quad \Phi(0) = 0, \quad \Phi'(\gamma) = 0.$$

Now defining

$$\Phi_1(\phi) := \sinh(\phi\mu), \quad \Phi_2(\phi) := \cosh((\phi - \gamma)\mu)$$

the Green’s function is obtained from (A.4) inserted into (A.2).

Appendix B. The corner contribution using Cheeger’s series expression for the heat kernel on an infinite sector

Here we show how, given the contribution for a D–D corner, which has been computed in [43, 51], we may use the series expression of the heat kernel from [6] to compute the contribution for both N–N and D–N corners. For a corner of angle α , the corner contribution is obtained by computing the renormalized integral, (see also [18, 19])

$$\text{f.p.}_{\epsilon=0} \int_{R=0}^{1/\epsilon} \int_0^\alpha \frac{1}{2} R \exp\left[-\frac{1}{2}R^2\right] \sum_{j=1}^\infty I_{\mu_j}\left(\frac{1}{2}R^2\right) |\phi_j(\theta)|^2 d\theta dR.$$

Since the cross-sectional eigenfunctions, ϕ_j , have unit \mathcal{L}^2 norm, this simplifies to

$$\text{f.p.}_{\epsilon=0} \int_{R=0}^{1/\epsilon} \frac{1}{2} R e^{-\frac{1}{2}R^2} \sum_{j=1}^\infty I_{\mu_j}\left(\frac{1}{2}R^2\right) dR. \tag{B.1}$$

In the Dirichlet–Dirichlet case, $\mu_j = j\pi/\alpha$ and (B.1) becomes

$$\text{f.p.}_{\epsilon=0} \int_{R=0}^{1/\epsilon} \frac{1}{2} R e^{-\frac{1}{2}R^2} \sum_{j=1}^\infty I_{j\pi/\alpha}\left(\frac{1}{2}R^2\right) dR. \tag{B.2}$$

By [43, 51], (B.2) is equal to

$$\frac{\pi^2 - \alpha^2}{24\pi\alpha}. \tag{B.3}$$

In the Neumann–Neumann case, the only difference is that there is a zero eigenvalue. So the difference of the corner contributions in the D–D and N–N cases is

$$\text{f.p.}_{\epsilon=0} \int_0^{1/\epsilon} \frac{1}{2} R e^{-\frac{1}{2}R^2} I_0\left(\frac{1}{2}R^2\right) dR. \tag{B.4}$$

This integral may be evaluated directly. First make a substitution in the integral setting $u = \frac{1}{2}R^2$, so that it becomes

$$\text{f.p.}_{\epsilon=0} \int_0^{\frac{1}{2\epsilon^2}} \frac{1}{2} e^{-u} I_0(u) du.$$

In [2, 5.5], a primitive for the integrand is obtained,

$$g(u) := e^{-u}u(I_0(u) + I_1(u)) \implies g'(u) = e^{-u}I_0(u).$$

Since $I_0(0) = I_1(0) = 0$, the integral above is therefore

$$\text{f.p.}_{\epsilon=0} \frac{1}{2} \left[\frac{1}{2\epsilon^2} e^{-\frac{1}{2\epsilon^2}} \left(I_0\left(\frac{1}{2\epsilon^2}\right) + I_1\left(\frac{1}{2\epsilon^2}\right) \right) \right]. \tag{B.5}$$

However, both $I_0(z)$ and $I_1(z)$ have expansions of the form

$$e^z z^{-1/2} (C_0 + C_1 z^{-1} + C_2 z^{-2} + \dots)$$

as $z \rightarrow \infty$. Substituting these expansions for the Bessel functions above, there are only odd powers of ϵ in the expansion. Therefore the coefficient of the ϵ^0 power is zero, and the finite part is zero. This shows that (B.4) equals zero and the Neumann–Neumann corner contribution is (B.3), the same as the Dirichlet–Dirichlet contribution.

In the Dirichlet–Neumann cases, $\mu_j = (j + 1/2)\pi/\alpha$, starting at $j = 0$, so we get

$$\text{f.p.}_{\epsilon=0} \int_0^{1/\epsilon} \frac{1}{2} R e^{-\frac{1}{2}R^2} \sum_{j=0}^{\infty} I_{(j+1/2)\pi/\alpha} \left(\frac{1}{2} R^2 \right) dR. \tag{B.6}$$

Observe that, since renormalized integrals are linear, this equals

$$\begin{aligned} &\text{f.p.}_{\epsilon=0} \int_0^{1/\epsilon} \frac{1}{2} R e^{-\frac{1}{2}R^2} \sum_{j=1}^{\infty} I_{j\pi/(2\alpha)} \left(\frac{1}{2} R^2 \right) dR \\ &- \text{f.p.}_{\epsilon=0} \int_0^{1/\epsilon} \frac{1}{2} R e^{-\frac{1}{2}R^2} \sum_{j=1}^{\infty} I_{j\pi/\alpha} \left(\frac{1}{2} R^2 \right) dR. \end{aligned} \tag{B.7}$$

We recognize this to be (B.2) for angle 2α minus (B.2) for angle α . We therefore obtain the Dirichlet–Neumann mixed corner contribution is

$$\frac{\pi^2 - (2\alpha)^2}{48\pi\alpha} - \frac{\pi^2 - \alpha^2}{24\pi\alpha} = \frac{-\pi^2 - 2\alpha^2}{48\pi\alpha}. \tag{B.8}$$

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