The Method of Layer Potentials

Unique Solvability of the Dirichlet Problem
for Laplace's Equation in C^1-domains
with L^p-Boundary Data

Master's Thesis

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for Laplace's Equation in C^1-domains
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Tiivistelmä — Referat — Abstract

In this thesis, we apply the method of layer potentials to prove that there is a unique solution to the Dirichlet problem for Laplace's equation in C^1-domains. We assume that C^1-domains are subsets of $\mathbb{R}^d$, $d \geq 2$, they are bounded and they have connected boundaries. In addition, we assume that the boundary data of the Dirichlet problem belong to the Lebesgue space $L^p(\partial D, \sigma)$ with $1 < p < \infty$.

We will follow the work of E. B. Fabes, M. Jodeit Jr. and N. M. Rivière. In their work, they solved various boundary value problems in domains that were merely C^1 by applying the method of layer potentials. The method of layer potentials is a procedure for solving the boundary value problems in the form of layer potentials. We will use it to solve the Dirichlet problem in the form of the double layer potential. The double layer potential satisfies Laplace's equation and the boundary values of the double layer potential are given by an operator $\frac{1}{2}I + K$. It turns out that in C^1-domains the operator K is compact on $L^p(\partial D)$. Consequently, we can use the Fredholm theory to deduce that the operator $\frac{1}{2}I + K$ is invertible and thus, we obtain a double layer potential solution to the Dirichlet problem. Finally, we will establish the uniqueness of the solution by using the properties of Green's function.

In the end of this thesis, we will also discuss how the method of layer potentials can be applied to the Dirichlet problem for Laplace's equation in Lipschitz domains with L^2-boundary data by following the doctoral dissertation of G. H. Verchota.

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Chapter 1

Introduction

In this thesis, we apply the method of layer potentials to prove that there is a unique solution to the Dirichlet problem for Laplace's equation in C^1-domains. We assume that C^1-domains are subsets of $\mathbb{R}^d$, $d \geq 2$, they are bounded and they have connected boundaries. In addition, we assume that the boundary data of the Dirichlet problem belong to the Lebesgue space $L^p(\partial D, \sigma)$ with $1 < p < \infty$. We will solve the Dirichlet problem by following the work of E. B. Fabes, M. Jodeit Jr. and N. M. Rivière [8]. In the end of this thesis, we will also discuss how the method of layer potentials can be applied to the Dirichlet problem for Laplace's equation in Lipschitz domains with L^2-boundary data by following the work of G. H. Verchota [18].

The Dirichlet problem

The Dirichlet problem is a boundary value problem. To formulate a boundary value problem we need a domain, a partial differential equation and a boundary condition. By a domain, we mean an open and connected subset of $\mathbb{R}^d$, $d \geq 2$, whereas, by a C^1-domain we mean a domain whose boundary is given locally by a graph of a continuously differentiable function. Throughout this thesis, we denote C^1-domains by D and their boundaries by ∂D.

The partial differential equation we have chosen to associate to the Dirichlet problem is Laplace's equation:

$$\Delta u(x) := \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2}(x) = 0, \quad x \in \mathbb{R}^d.$$

Laplace's equation is perhaps the most fundamental partial differential equation. It can be used to model steady-state phenomena in physics, for example in electrostatics and fluid mechanics and it is a special case of other important partial differential equations like Poisson's equation, the heat equation and the wave equation. Moreover, its solutions are harmonic functions that are important in many areas of mathematics. For more information on applications of Laplace's equation, see [10]. For the basic properties of harmonic functions, see [7].

Finally, we need a boundary condition. Suppose that a function g belongs to the Lebesgue space $L^p(\partial D, \sigma)$, where σ denotes the surface measure of ∂D. Formally, the boundary condition can be given as follows

$$u|_{\partial D} = g.$$

However, restricting the values of the solution u onto the boundary ∂D does not make sense, if the function u is not defined on the boundary. This will be the case when we try to find a layer potential solution. Clearly, we have to find a way to interpret the boundary condition meaningfully. For example, solutions can be assumed to be continuous up to the boundary or one can apply the trace theorem [7]. Unfortunately, neither of these approaches do not meet our needs: the former assumption is inconsistent with the L^p-boundary data and the latter does not give us tools for finding solutions to the Dirichlet problem. Instead, we require that solutions achieve boundary values almost everywhere on the boundary in the non-tangential sense. This means that we approach the boundary along certain cones.

Later on, it turns out that we have to make some modifications to the Dirichlet problem to achieve the uniqueness of the solution. Nevertheless, for now, we content ourselves to formulate the Dirichlet problem in the following, incomplete manner.

Problem 1.1. (Dirichlet problem). Let D be a bounded C^1-domain with connected boundary and suppose $g \in L^p(\partial D)$ for some $p \in (1, \infty)$. Find a real-valued function u, defined in D, that satisfies the following conditions:

$$\begin{cases} \Delta u = 0 & \text{in } D \\ u = g & \text{on } \partial D. \end{cases}$$

The method of layer potentials

We will solve the Dirichlet problem by applying the method of layer potentials. In short, the method of layer potentials is a procedure that allows us to reduce a boundary value problem into a problem of solving a certain boundary integral equation. In the case of the Dirichlet problem for Laplace's equation, the first step of the procedure is to take a density $f \in L^p(\partial D)$ and to define a double layer potential

$$\mathcal{K}f(x) = \frac{1}{\omega_d} \int_{\partial D} \frac{\langle \nu(w), x - w \rangle}{|x - w|^d} f(w)\, d\sigma(w), \quad x \in \mathbb{R}^d \setminus \partial D.$$

The terms density and potential originate from physics in which, for example, electric potentials and charge densities are of interest. In the equation above, ω_d denotes the surface area of the unit sphere S^{d-1} and $\nu(w)$ denotes the inward-pointing unit normal vector of the boundary ∂D at a point $w \in \partial D$. Moreover, $\langle \cdot, \cdot \rangle$ is the Euclidian inner product. The double layer potential with an arbitrary density satisfies Laplace's equation inside a domain D. Thus, by defining the double layer potential we have transformed the original problem into finding a suitable density f.

The second step of the procedure is to approach the boundary ∂D non-tangentially and establish the following jump relation for the double layer potential:

$$\lim_{\substack{x \to z \\ x \in \Gamma_\alpha^i(z)}} \mathcal{K} f(x) = \frac{1}{2} f(z) + \frac{1}{\omega_d} \int_{\partial D} \frac{\langle \nu(w), z - w \rangle}{|z - w|^d} f(w)\, d\sigma(w), \quad \text{a.e. } z \in \partial D.$$

On the left-hand side of the above identity, we have approached the boundary non-tangentially along the cones $\Gamma_\alpha^i(z)$. The jump relation together with the boundary condition leads us to a boundary integral equation

$$\frac{1}{2} f(z) + \frac{1}{\omega_d} \int_{\partial D} \frac{\langle \nu(w), z - w \rangle}{|z - w|^d} f(w)\, d\sigma(w) = g(z), \quad \text{a.e. } z \in \partial D.$$

If we were able to solve the density f from the boundary integral equation, we would obtain the double layer potential solution to the Dirichlet problem.

The third step of the procedure is to ensure that the integral in the boundary integral equation is well-defined at least in some sense. This is not a trivial matter in the case of a C^1-domain, because the above integral is strictly singular. Fortunately, it turns out that the integral can be interpreted in the principal value sense. The corresponding operator

$$K : L^p(\partial D) \to L^p(\partial D), \quad K f(z) = \text{p.v.} \frac{1}{\omega_d} \int_{\partial D} \frac{\langle \nu(w), z - w \rangle}{|z - w|^d} f(w)\, d\sigma(w),$$

will be called the boundary integral operator and we can use it to represent the boundary integral equation in the operator form:

$$(\tfrac{1}{2} I + K) f = g.$$

The fourth and the final step of the procedure is to invert the above equation. More precisely, we must prove that the operator $\frac{1}{2} I + K$ is invertible in $L^p(\partial D)$. In C^1-domains, this can be done by applying the Fredholm theory, because it turns out that the boundary integral operator K is compact. In Lipschitz domains, the Fredholm theory is not applicable, because the boundary integral operator K is not necessarily compact. Thus, one has to find a new method for proving the invertibility of the operator $\frac{1}{2}I + K$.

To prove the invertibility of the operator $\frac{1}{2}I+K$ we have to study a single layer potential

$$\mathcal{S}f(x) = \frac{1}{\omega_d(d-2)} \int_{\partial D} \frac{f(w)}{|x-w|^{d-2}}\, d\sigma(w), \quad x \in \mathbb{R}^d \setminus \partial D.$$

The single layer potential satisfies Laplace's equations in $\mathbb{R}^d \setminus \partial D$ but there is no jump on the boundary. Instead, normal derivatives of the single layer potential satisfy a similar jump relation as the double layer potential. For example, we will find out that

$$\lim_{\substack{x \to z \\ x \in \Gamma^i_\alpha(z)}} \langle \nu(z), \nabla \mathcal{S}f(x) \rangle = -\frac{1}{2} f(z) + \frac{1}{\omega_d} \int_{\partial D} \frac{\langle \nu(z), w - z\rangle}{|w-z|^d} f(w)\, d\sigma(w)$$

for almost every $z \in \partial D$ in the non-tangential sense. As before, the integral on the right-hand side of the above identity can be interpreted as a principal value operator

$$K^* : L^p(\partial D) \to L^p(\partial D), \quad K^* f(z) = \text{p.v.} \frac{1}{\omega_d} \int_{\partial D} \frac{\langle \nu(z), w-z\rangle}{|w-z|^d} f(w)\, d\sigma(w).$$

The operator K^* turns out to be the adjoint of K. This is a crucial observation because to prove the invertibility of the operator $\frac{1}{2}I + K$, it suffices to show instead that the operator $\frac{1}{2}I + K^*$ is invertible. This can be done by applying the properties of the single layer potential and the second Green's formula.

Finally, it is worth mentioning that although the method of layer potentials implies the uniqueness of the double layer potential solution for the Dirichlet problem, there may be other solutions that are not in the form of the double layer potential. To rule out these other solutions one has to modify the Dirichlet problem by adding a further condition on it. Then, by using the properties of Green's function, it is possible to prove the uniqueness of the Dirichlet problem.

Overview

The main part of this thesis is divided into seven chapters. All of these chapters share a common feature: they aim at building up arguments which allow us to deduce that the Dirichlet problem has a unique solution and that the solution can be obtained by applying the method of layer potentials.

The back part of this thesis contains the Appendix, the Glossary and the Bibiliography.

In Chapter 2, we go through some preliminary results concerning C^1-domains, bounded linear operators and maximal operators. These results are supposed to help the reader to get into the subject and later on, also convince from the validity of some rather complicated proofs.

In Chapter 3, we look into the vast theory of singular integral operators and pick up what is necessary for the developement of this thesis. We begin with some useful convergence results and then we establish basic properties of the weakly singular integral operators. After these preliminary considerations, we go into the Calderon-Zygmund theory of singular integral operators which we apply to understand certain type of strictly singular integral operators. We will see that these operators are bounded by applying the strong results of R. R. Coifman, A. McIntosh and Y. Meyer concerning the boundedness of the Cauchy integral along Lipschitz curves [3]. The boundedness results will be generalized into higher dimensions by applying the method of rotations.

In Chapter 4, we establish basic properties of the layer potentials. We begin with the single layer potential: we show that the single layer potential is a harmonic function and that there is no jump when crossing the boundary of a domain. After we have dealt with the single layer potential, we continue to study the double layer potential. We also prove that the double layer potential is a harmonic function and we establish a result that gives insight, why there is the one half term in the jump relations.

In Chapter 5, we prove that the boundary integral operators K and K^* are well-defined and compact in $L^p(\partial D)$. To do this, we have to consider the corresponding operators in local coordinates. We prove that these corresponding operators are compact in $L^p(\mathbb{R}^{d-1})$ and then we pass the compactness to the operators K and K^*. In the end of this chapter, we state a further result that says that the boundary integral operator K is compact in the Sobolev-type space $L^p_1(\partial D)$.

In Chapter 6, we establish jump relations for the double layer potential and for the normal derivatives of the single layer potential. In other words, we show that the boundary values of these potentials are given by jump relations that are related to the boundary integral operators. To succeed in this, we introduce a concept of non-tangential maximal functions and prove that the non-tangential maximal functions related to the potentials are bounded. Then, with some further efforts and with the help of the boundedness of the non-tangential maximal functions, we show that the double layer potential and the normal derivative of the single layer potential satisfy the jump relations.

In Chapter 7, we prove the main result of this thesis, which is the unique solvability of the Dirichlet problem in C^1-domains. First, we treat the existence part of the problem by proving that the operator $\frac{1}{2}I + K$ is invertible in $L^p(\partial D)$. After that, we treat the uniqueness part of the problem, which leads us to consider the properties of Green's function. We finish by demonstrating ideas that allow one to generalize the method of layer potentials into Lipschitz domains.

The Appendix contains some results and proofs that support the results in the main text. The reason why these results and proofs are left to the Appendix is that they would have unnecessarily slowed down the proceeding of the text.

The Glossary contains the most essential symbols and notations. For each symbol or a notation, there is a short description of its meaning and the number of the page where the symbol or the notation has first been used or defined.

The Bibliography contains the list of books and articles that are referenced in this thesis. The reader is advised explore at least some of these references: the most important reference is the article [8] of which results this thesis tries to explain. Another important article is [19] or alternatively, the dissertation [18]. The books [7] and [13] are recommended.

Few remarks on notations

The aim has been to create consistent and simple notational conventions. The most important ones are being listed here. Throughout this thesis, D will denote a C^1-domain, except in the last section where D will also denote a Lipschitz domain. The points on the boundary of the domain D will be consistently denoted by z and w. The points in $\mathbb{R}^d \setminus \partial D$ will be denoted by x and y, but sometimes it is more convenient to use notations $\tilde{x}$ and $\tilde{y}$ instead and reserve the symbols x and y for the points in $\mathbb{R}^{d-1}$. For the sake of simplicity, constants will be absorbed into a single constant that will be denoted by C. Exceptions are made only if the constants are important for the outcome or if the calculations become easier to follow.

Chapter 2

Preliminaries

In this chapter, we state some results concerning C^1-domains, bounded linear operators and maximal operators. The purpose of this chapter is to provide background information on the subjects that are necessary for understanding this thesis.

2.1 Properties of C^1-domains

We begin this section by defining a concept of a C^1-domain.

Definition 2.1. We say that a domain D is C^1 if for each point $z \in \partial D$ there is a Cartesian coordinate system, a radius $r > 0$ and a continuously differentiable function $\varphi : \mathbb{R}^{d-1} \to \mathbb{R}$ with a compact support such that

$$D \cap B(z,r) = \{(x,t) : x \in \mathbb{R}^{d-1},\ t > \varphi(x)\,\} \cap B(z,r).$$

To clarify the above definition, a domain is C^1 if it can be represented locally as a graph of a continuously differentiable function. Moreover, for each $z \in \partial D$, the Cartesian coordinate system can be selected such that $z = 0$, $\varphi(0) = 0$ and $\nabla\varphi(0) = 0$. This is because the coordinate system can be translated to the point $z \in \partial D$ and then rotated such that the t-axis points in the direction of the inward-pointing unit normal vector $\nu(z)$. The following lemma states that an inward-pointing unit normal vector exists at every point on the boundary of a C^1-domain.

Lemma 2.1. *Suppose that D is a C^1-domain, let $B \subset \mathbb{R}^d$ be a ball and suppose that $\varphi \in C_0^1(\mathbb{R}^{d-1})$ is the corresponding boundary function. Then the inward pointing unit normal vector at a point $z = (x, \varphi(x)) \in \partial D \cap B$ is given by*

$$\nu(z) = (-\nabla\varphi(x), 1)/\sqrt{1 + |\nabla\varphi(x)|^2}.$$

Proof. Vectors $\tau_j(z) = (\tau_{j,1}(z), \ldots, \tau_{j,d}(z))$, $j = 1, \ldots, d-1$, that are defined by

$$\tau_{j,k}(z) = \begin{cases} 1, & \text{if } k = j \\ \partial_j \varphi(x), & \text{if } k = d \\ 0, & \text{otherwise} \end{cases}$$

form a basis of the tangent plane of ∂D at a point $z = (x, \varphi(x))$ in the coordinate system of B. By calculating, we see that

$$\langle \nu(z), \tau_j(z) \rangle = 0$$

for every $j = 1, \ldots, d-1$. Thus, the vector $\nu(z)$ is perpendicular to the tangent plane of ∂D at a point z. Moreover, if we take a positive number $h > 0$ such that $z + h\nu(z) \in B$, then also

$$z + h\nu(z) \in \{(y, t) : y \in \mathbb{R}^{d-1},\ t > \varphi(x)\}.$$

From the definition of a C^1-domain, we obtain $z + h\nu(z) \in D$. We conclude that $\nu(z)$ is the inward pointing unit normal vector at z. □

One of the reasons why the authors of [8] managed to use the method of layer potentials in C^1-domains was that the boundary of a C^1-domain can be covered with a finite number of balls such that the corresponding boundary functions have arbirarily small Lipschitz norms. Small Lipschitz norms made it possible to apply results of A. P. Calderón [2] for proving properties of boundary integral operators. For more information see Chapters 3 and 5.

Lemma 2.2. *Suppose that D is a bounded C^1-domain and $\gamma \geq 1$ is a constant. Then, for every number $m > 0$, there is a finite cover of balls $\{B_i\}_{i=1}^n$ for ∂D and functions $\varphi_i \in C_0^1(\mathbb{R}^{d-1})$, $i = 1, \ldots, n$, such that*

$$D \cap \gamma B_i = \{(x, t) : x \in \mathbb{R}^{d-1},\ t > \varphi_i(x)\} \cap \gamma B_i \tag{2.1}$$

and

$$\|\nabla \varphi_i\|_\infty := \sup\{|\nabla \varphi_i(x)| : x \in \mathbb{R}^{d-1}\} \leq m \tag{2.2}$$

for every $i = 1, \ldots, n$.

Proof. Let $z \in \partial D$. Then for some radius $R_z > 0$, we have

$$D \cap B(z, R_z) = \{(x, t) : x \in \mathbb{R}^{d-1},\ t > \varphi_z(x)\} \cap B(z, R_z),$$

where $\varphi_z \in C_0^1(\mathbb{R}^{d-1})$. We may assume that the coordinate system is chosen such that $z = 0$ and $\varphi_z(0) = \nabla \varphi_z(0) = 0$. Because $x \mapsto |\nabla \varphi_z(x)|$ is a continuous function, then for some radius $R'_z > 0$, we have

$$\sup_{|x| < R'_z} |\nabla \varphi_z(x)| \leq m.$$

Now, it is possible to choose a function $\varphi_z \in C_0^1(\mathbb{R}^{d-1})$ such that it satisfies a condition $\|\varphi_z\|_\infty \le m$ and

$$D \cap B(z, R'_z) = \{(x,t) : x \in \mathbb{R}^{d-1},\ t > \varphi_z(x)\} \cap B(z, R'_z).$$

In fact, we could have chosen a smaller radius $r_z := R'_z/\gamma$. Consequently, $\|\varphi_z\|_\infty \le m$ and

$$D \cap B(z, \gamma r) = \{(x,t) : x \in \mathbb{R}^{d-1},\ t > \varphi_z(x)\} \cap B(z, \gamma r).$$

To complete the proof, we notice that the boundary ∂D is compact in $\mathbb{R}^d$ and $\{B(z, r_z) : z \in \partial D\}$ is a cover for ∂D. Thus, there exists a finite cover balls $\{B_i\}_{i=1}^n$ and functions $\varphi_i \in C_0^1(\mathbb{R}^{d-1})$, $i = 1, \dots, n$, such that they satisfy the conditions (2.1) and (2.2). □

The above lemma gives us a reason for the following definitions.

Definition 2.2. Suppose that U is an open subset of $\mathbb{R}^{d-1}$. We say that a function $p : U \to \partial D$ is a local parametrization of ∂D, if the function p determines a homeomorphism from U into $p(U)$.

Definition 2.3. Suppose that D is a bounded C^1-domain, $\{B_i\}_{i=1}^n$ is a cover for ∂D and $\varphi_i : \mathbb{R}^{d-1} \to \mathbb{R}$, $i = 1, \dots, n$, are continuously differentiable functions such that

$$B_i \cap \partial D = B_i \cap \{(x,t) : x \in \mathbb{R}^{d-1}, t > \varphi_i(x)\}.$$

Also, assume that functions

$$p_i : U_i \to B_i \cap \partial D, \quad p_i(x) = (x, \varphi_i(x)), \quad i = 1, \dots, n$$

are local parametrizations of ∂D. Then we say that

$$\mathscr{F}_{\partial D} := (\{B_i\}, \{\varphi_i\}, \{p_i\}, \{U_i\})_{i=1}^n,$$

is a family of local characteristics of the boundary ∂D.

When we are dealing with functions that have singularities on the boundary of a domain, we must be careful when we approach the boundary. One way to be careful is to define certain cones on the boundary and then approach the boundary along these cones. We call this a non-tangential approach. Now, let us define the cones.

Definition 2.4. Suppose that D is a C^1-domain. An infinite cone with two components, an aperture $\alpha \in (0,1)$ and vertex at $z \in \partial D$, is defined by

$$\Gamma_\alpha(z) = \{x \in \mathbb{R}^d : |\langle \nu(z), x - z\rangle| > \alpha |x - z|\,\}.$$

A distinction is made between an interior cone

$$\Gamma^i_\alpha(z) = \{x \in \mathbb{R}^d : \langle \nu(z), x - z\rangle > \alpha |x - z|\,\}$$

and an exterior cone

$$\Gamma^e_\alpha(z) = \{x \in \mathbb{R}^d : \langle \nu(z), x - z\rangle < -\alpha |x - z|\,\}.$$

Remark 2.1. Notice that cones satisfy the property: if $\alpha \leq \beta$ then $\Gamma_\beta \subset \Gamma_\alpha$.

If we wish to study boundary values of the layer potentials, we have to approach the boundary non-tangentially. For this reason, we formulate the following lemma that in many situations allows us to apply the dominated convergence theorem and therefore deduce the existence of the non-tangential limits.

Lemma 2.3. *Suppose that D is a C^1-domain, $z \in \partial D$ and let $\Gamma_\alpha(z)$ be a cone with an aperture $\alpha \in (0,1)$. Then there is a number $\delta > 0$ and a constant $C > 0$ such that*

$$|x - w| \geq C|z - w|$$

for every $x \in \Gamma_\alpha(z) \cap B(z,\delta)$ and $w \in \partial D$.

Proof. It suffices to assume that $\Gamma_\alpha(z)$ is an interior cone because the proof for the exterior cone would be similar. For convenience, we denote $\Gamma_\alpha := \Gamma^i_\alpha$ in this proof.

We choose a number $\delta > 0$ and a cone $\Gamma_\beta(z)$ with an aperture $\beta \in (0,\alpha)$ such that $\Gamma_\beta(z) \cap B(z, 2\delta) \subset D$. Then we assume that $x \in \Gamma_\alpha(z) \cap B(z,\delta)$. If $w \in \partial D \cap \Gamma_\beta(z)$, then $|z - w| > 2\delta$ and applying the triangle inequality, we obtain

$$|x - w| \geq |z - w| - |x - z| \geq |z - w| - \delta \geq \frac{1}{2}|z - w|.$$

Thus, we may assume $w \in \partial D \cap (\Gamma_\beta(z))^c$. Let $\xi \in \partial\Gamma_\beta(z)$ be a point that satisfies

$$|x - \xi| = \operatorname{dist}(x, \partial\Gamma_\beta(z)).$$

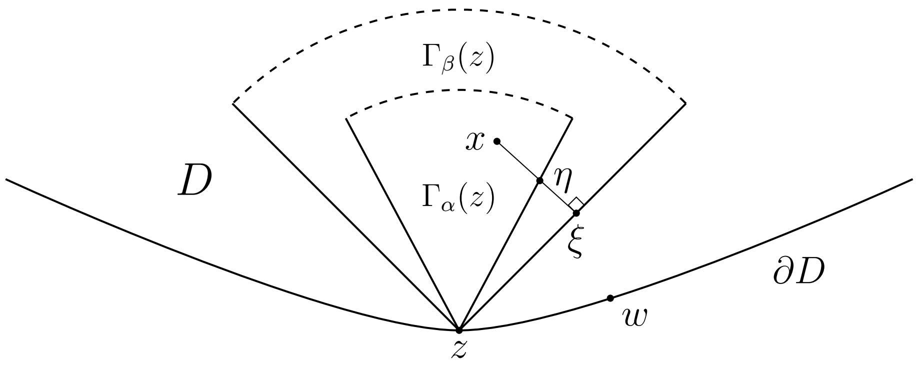

Figure 2.1: Two cones on the boundary

Also, assume that a line passing through the points x and ξ intersects the cone $\Gamma_\alpha(z)$ at a point η. Because w is outside the cone $\Gamma_\beta(z)$, we have

$$|x-\xi| \le |x-w|. \tag{2.3}$$

Moreover, a geometric argument (see Figure 2.1) allows us to deduce

$$\frac{|\xi-z|}{|\xi-\eta|} = \text{constant}.$$

Using the above properties, we get

$$|\xi-z| \le C|\xi-\eta| \le C|x-\xi| \le C|x-w|. \tag{2.4}$$

The triangle inequality and the inequalities (2.3) and (2.4) imply

$$|x-z| \le |x-\xi| + |\xi-z| \le C|x-w|.$$

Once again, by using the triangle inequality, we get

$$|x-w| \ge |z-w| - |x-z| \ge |z-w| - C|x-w|.$$

Finally, we obtain

$$|x-w| \ge C|z-w|,$$

which completes the proof. □

One of the characteristics of C^1-domain is that its boundary straightens locally: the smaller the environment, the more the boundary reminds a plane. This observation is realised below.

Lemma 2.4. *Suppose that D is a C^1-domain. Then, for every $z \in \partial D$, the following conditions hold:*

$$\text{(i)} \lim_{r\to 0} \frac{|B(z,r)\cap D|}{|B(z,r)|} = \frac{1}{2} \quad \textit{and} \quad \text{(ii)} \lim_{r\to 0} \frac{\sigma(\partial B(z,r)\cap D)}{\sigma(\partial B(z,r))} = \frac{1}{2}.$$

Proof. (i) Because D is a C^1-domain, there is a radius $r > 0$ and a function $\varphi \in C_0^1(\mathbb{R}^{d-1})$ such that

$$D \cap B(z,r) = \{(x,t) : x \in \mathbb{R}^{d-1},\ t > \varphi(x)\} \cap B(z,r).$$

Assume that we have selected a coordinate system such that $z = 0$, $\varphi(0) = 0$ and $\nabla\varphi(0) = 0$. Then, for each number $m > 0$, there is a radius $r_m > 0$ such that

$$\sup_{|x|\le r_m} |\nabla\varphi(x)| \le m$$

and r_m tends to zero as m tends to zero. To prove (i), it suffices to show that

$$\lim_{m\to 0} \frac{|B(z,r_m)\cap D|}{|B(z,r_m)|} = \frac{1}{2}.$$

For a number $m > 0$, we choose an aperture $\alpha_m = m/\sqrt{1+m^2}$. According to Lemma A.3, we have

$$|B(z,r_m)\cap \Gamma_{\alpha_m}(z)| = C_{\alpha_m}|B(z,r_m)|,$$

where C_{α_m} tends to one, as m tends to zero. Consequently,

$$\lim_{m\to 0} \frac{|B(z,r_m)\cap \Gamma_{\alpha_m}(z)|}{|B(z,r_m)|} = 1$$

and therefore

$$\lim_{m\to 0} \frac{|B(z,r_m)\cap \Gamma^i_{\alpha_m}(z)|}{|B(z,r_m)|} = \frac{1}{2} = \lim_{m\to 0} \frac{|B(z,r_m)\cap \Gamma^e_{\alpha_m}(z)|}{|B(z,r_m)|}. \tag{2.5}$$

Suppose that $\tilde{x} \in B(z,r_m)\cap \Gamma^i_{\alpha_m}(z)$. Let us denote $\tilde{x} = (x,t)$, where $x \in \mathbb{R}^{d-1}$ and $t > 0$. From the definition of the cone $\Gamma^i_{\alpha_m}(z)$, we get

$$t = \langle \nu(z), \tilde{x} - z\rangle > \alpha_m|\tilde{x} - z| = \alpha_m\sqrt{|x|^2 + t^2}. \tag{2.6}$$

According to the mean value theorem, there is a vector $\xi \in B(z,r_m)$ such that

$$\varphi(x) = \varphi(x) - \varphi(0) = \langle \nabla\varphi(\xi), x\rangle \le m|x|. \tag{2.7}$$

By combining the observations (2.6) and (2.7), we get

$$t > m|x| \geq \varphi(x).$$

Therefore

$$B(z, r_m) \cap \Gamma^i_{\alpha_m}(z) \subset B(z, r_m) \cap D. \tag{2.8}$$

With a similar argument as above, we could have deduced

$$B(z, r_m) \cap \Gamma^e_{\alpha_m}(z) \subset B(z, r_m) \cap D^c. \tag{2.9}$$

Now, using observations (2.5), (2.8) and (2.9), we see that

$$\lim_{m\to 0} \frac{|B(z, r_m) \cap D|}{|B(z, r_m)|} \geq \frac{1}{2} \quad \text{and} \quad \lim_{m\to 0} \frac{|B(z, r_m) \cap D^c|}{|B(z, r_m)|} \geq \frac{1}{2}.$$

On the other hand, we know that

$$\lim_{m\to 0} \frac{|B(z, r_m) \cap D|}{|B(z, r_m)|} + \lim_{m\to 0} \frac{|B(z, r_m) \cap D^c|}{|B(z, r_m)|} = 1.$$

Thus, the only option is

$$\lim_{m\to 0} \frac{|B(z, r_m) \cap D|}{|B(z, r_m)|} = \frac{1}{2}.$$

(ii) Let us define a function $u(y) = |y|^2$. Then, using the first Green's formula (see Appendix), we have

$$2d|B(z,r)| = \int_{B(z,r)} \Delta u(y)\, dy = \int_{\partial B(z,r)} \frac{\partial u}{\partial \nu}(w)\, d\sigma(w) = 2r\sigma(\partial B(z,r)).$$

With a similar argument, we obtain

$$2d|B(z,r) \cap D| = 2r\sigma(\partial B(z,r) \cap D) + 2\int_{B(z,r)\cap\partial D} \langle w, \nu(w)\rangle\, d\sigma(w).$$

Now, using the above identities, we have

$$\frac{|B(z,r) \cap D|}{|B(z,r)|} = \frac{\sigma(\partial B(z,r) \cap D)}{\sigma(\partial B(z,r))} + \frac{1}{\omega_d r^d}\int_{B(z,r)\cap\partial D} \langle w, \nu(w)\rangle\, d\sigma(w).$$

The second term on the right-hand side tends to zero, as r tends to zero. We see this by estimating

$$\begin{aligned}
\left|\int_{B(z,r)\cap\partial D} \langle w, \nu(w)\rangle\, d\sigma(w)\right| &\leq \int_{B(x,r)} |\varphi(y) - \langle \nabla\varphi(y), y\rangle|\, dy \\
&\leq \int_{B(x,r)} |y||\varepsilon(|y|)|\, dy \\
&\leq \omega_d r^d \sup_{0<s<r} |\varepsilon(s)|.
\end{aligned}$$

In the above estimate, $\varepsilon(r)$ is function that tends to zero, as r tends to zero. Finally, the above deduction and the part (i) implies

$$\lim_{r\to 0}\frac{\sigma(\partial B(z,r)\cap D)}{\sigma(\partial B(z,r))}=\lim_{r\to 0}\frac{|B(z,r)\cap D|}{|B(z,r)|}=\frac{1}{2}.$$

This completes the proof. □

We turn our attention to functions that are defined on a boundary of a C^1-domain. To understand properties of such functions, we often have to restrict these functions into a local setting. A tool that allows us to do such a restriction is called a partition of unity. The following theorem is based on [15].

Theorem 2.1. *Suppose that G is compact subset of $\mathbb{R}^d$ and $\{B_i\}_{i=1}^n$ is a finite cover of balls for G. Then there are functions $\zeta_i \in C_0^\infty(\mathbb{R}^d)$, $i=1,\dots,n$, that satisfy properties:*

$$0\le \zeta_i\le 1,\quad \operatorname{supp}(\zeta_i)\subset B_i \quad \text{and} \quad \sum \zeta_i = 1.$$

We say that the family of functions $\{\zeta_i\}_{i=1}^n$ is a partition of unity subordinate to the cover $\{B_i\}_{i=1}^n$.

In forthcoming situations we will need functions defined on ∂D that satisfy a Lipschitz condition and that are dense in $C(\partial D)$ and $L^p(\partial D)$. We define such a class of functions below.

Definition 2.5. Let U denote an open subset of $\mathbb{R}^{d-1}$. The space $C^1(\partial D)$ consists of functions $f:\partial D\to\mathbb{R}$ such that for every local parametrization $p:U\to\partial D$ of ∂D, a composition function $f\circ p$ belongs to $C^1(U)$.

Lemma 2.5. *Suppose that $f\in C^1(\partial D)$. Then there is a constant $M>0$ such that*

$$|f(z)-f(w)|\le M|z-w| \tag{2.10}$$

for every $z,w\in\partial D$ for which $z\ne w$.

Proof. Because D is a C^1-domain, there is a family of local characteristics $(\{B_i\},\{\varphi_i\},\{p_i\},\{U_i\})_{i=1}^n$ of the boundary ∂D. Furthermore, there is a partition of unity $\{\zeta_i\}_{i=1}^n$ subordinate to the cover $\{B_i\}_{i=1}^n$. Consequently, we may write

$$f=\sum_{i=1}^n f\zeta_i. \tag{2.11}$$

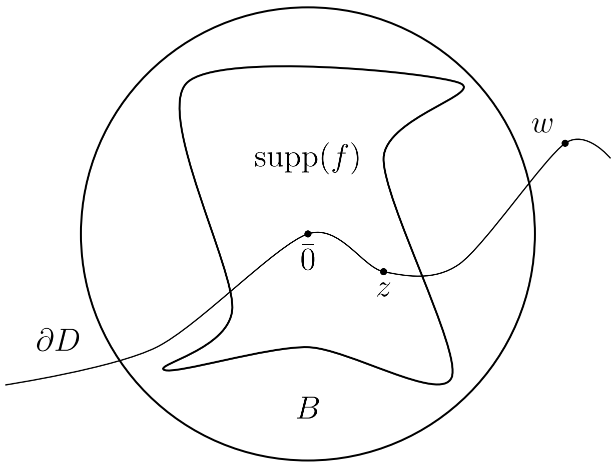

Figure 2.2: Separation of the points z and w

It suffices to fix $i \in \{1, \dots, n\}$, denote $B := B_i$, $\varphi := \varphi_i$, $p := p_i$ and $U := U_i$ and show that a compactly supported function $f := f\zeta_i$ satisfies the property (2.10).

By considering the Figure 2.2, we see that if $z \in B^c \cap \partial D$ or $w \in B^c \cap \partial D$, then f satisfies the property (2.10). Thus, we may assume $z, w \in B \cap \partial D$.

Because $f \in C_0^1(\partial D \cap B)$, then $f \circ p \in C_0^1(U)$. In fact, $f \circ p$ is a Lipschitz function in U. Consequently, there is a number $M > 0$ such that

$$\begin{aligned} |f(z) - f(w)| &= |(f \circ p)(p^{-1}(z)) - (f \circ p)(p^{-1}(w))| \\ &\leq M|p^{-1}(z) - p^{-1}(w)| \\ &\leq M|z - w|. \end{aligned}$$

This completes the proof. □

Finally, we introduce a tool that allows us to approach the boundary of a C^1-domain with smooth domains. Such a tool is called an approximation scheme and it is due to G. H. Verchota [19].

Theorem 2.2. (Approximation scheme). *Let D be a bounded C^1-domain and let $\{\Gamma_\alpha\}$ be a family of cones for ∂D with a fixed aperture $\alpha \in (0, 1)$. Then the following properties hold:*

(i) *There are smooth domains Ω_j, $j \in \mathbb{N}$, such that $\Omega_j \subset D$ for every $j \in \mathbb{N}$ and the sequence $\{\Omega_j\}_{j=1}^\infty$ converges to D in the following sense: there are homeomorphisms $p_j : \partial D \to \partial \Omega_j$, $j \in \mathbb{N}$, such that*

$$\sup_{z \in \partial D} |z - p_j(z)| \to 0,$$

as j tends to infinity and $p_j(z) \in \Gamma^i_\alpha(z)$ for every $z \in \partial D$.

(ii) *There is a cover of coordinate cylinders $Z \subset \mathbb{R}^d$ for ∂D such that*

$$Z \cap \Omega_j = Z \cap \{(x,t) : x \in \mathbb{R}^{d-1},\, t > \varphi_j(x)\}$$

and

$$Z \cap D = Z \cap \{(x,t) : x \in \mathbb{R}^{d-1},\, t > \varphi(x)\},$$

where $\varphi \in C^1_0(\mathbb{R}^{d-1})$ and $\varphi_j \in C^\infty_0(\mathbb{R}^{d-1})$ and the sequence $\{\varphi_j\}_{j=1}^\infty$ converges uniformly to φ.

(iii) *There are positive functions $J_j : \partial D \to \mathbb{R}$, $j \in \mathbb{N}$ and numbers m and M, such that*

$$\inf_{z\in\partial D} |J_j(z)| \geq m > 0 \quad \text{and} \quad \sup_{z\in\partial D} |J_j(z)| \leq M < \infty$$

for all $j \in \mathbb{N}$ and the sequence $\{J_j\}_{j=1}^\infty$ converges to 1 in $L^q(\partial D)$ for every $q \in (1,\infty)$ and pointwise for almost everywhere on ∂D. Furthermore, the change of variables formula

$$\int_{\partial\Omega_j} f(\tilde{w})\, d\sigma_j(\tilde{w}) = \int_{\partial D} (f \circ p_j)(w) J_j(w)\, d\sigma(w)$$

holds for every $f \in L^1(\partial\Omega_j)$.

(iv) *There are inward-pointing unit normal vectors $\nu_j : \Omega_j \to \mathbb{R}^d$ such that for each $q \in (1,\infty)$ holds*

$$\|\nu - \nu_j \circ p_j\|_{L^q(\partial D)} \to 0,$$

as j tends to infinity.

The approximations scheme described above is denoted by $\Omega_j \nearrow D$. The domain D can also be approximated from the outside by smooth domains $\Omega_j \supset D$. Such an approximation scheme is denoted by $\Omega_j \searrow D$.

2.2 Bounded linear operators

In this section we study bounded linear operators and especially, compact operators. We establish notations, definitions and theorems that we need in further chapters. For a detailed discussion on bounded linear operators we refer to [16].

Throughout this section, we assume that X and Y are Banach spaces and that the operators will be linear. We begin by defining bounded operators and we introduce the concept of the operator norm.

Definition 2.6. An operator $T : X \to Y$ is bounded if there is a constant $C > 0$ such that

$$\|Tf\|_Y \leq C\|f\|_X \tag{2.12}$$

for every $f \in X$. The set of bounded operators is denoted by $B(X, Y)$.

Example 2.1. The identity operator $I : X \to X$, $I(x) = x$ is bounded.

Theorem 2.3. *A set of bounded linear operators $B(X, Y)$ associated with the operator norm*

$$\|T\| = \sup\{\|Tf\|_Y : \|f\|_X \leq 1\}$$

is a normed vector space. Furthermore, the norm $\|T\|$ is the smallest constant for which the inequality (2.12) *holds.*

In the context of boundary value problems, we are interested in solving problems of the type $Tf = g$. It is natural to ask, whether the solution f depends continuously on the data g. For this reason, we need a concept of invertibility.

Definition 2.7. An operator $T \in B(X, Y)$ is invertible, if there exists an operator $S \in B(Y, X)$ such that $TS = I_Y$ and $ST = I_X$. The operator S is called an inverse of the operator T and it will be denoted by T^{-1}.

The following theorems are useful when one wants to deduce, whether an operator is invertible.

Theorem 2.4. *Suppose $T \in B(X)$ and $\|T\| < 1$. Then an operator $I + T$ is invertible on X.*

Theorem 2.5. *If the operator $T \in B(X, Y)$ is bijective, then it is invertible.*

Next, let us define compact operators and state some of their properties.

Definition 2.8. A linear operator $T : X \to Y$ is compact if for every bounded sequence $\{x_j\}_{j=1}^{\infty}$ in X, the sequence $\{Tx_j\}_{j=1}^{\infty}$ contains a convergent subsequence in Y.

Remark 2.2. It is an immediate consequence of the above definition that compact operators are bounded. Nevertheless, there are bounded operators which are not compact. For example, the identity operator I is not always compact.

Theorem 2.6. *Suppose that operators $T_1, T_2 \in B(X, Y)$ are compact and $f, g \in X$. Then the operator $fT_1 + gT_2$ is also compact.*

Theorem 2.7. *Suppose that $T \in B(X,Y)$. If there is a sequence of compact operators $\{T_i\}_{i=1}^{\infty} \subset B(X,Y)$ that converges to T in the operator norm, then the operator T is compact.*

Let G be a closed subset of $\mathbb{R}^d$ and let $p \in (1,\infty)$. We are interested especially in bounded linear operators that are mappings from Lebesgue space $L^p(G)$ to itself. We define a concept of an adjoint in this setting:

Definition 2.9. Let $p,q \in (1,\infty)$ be conjugate exponents, i.e. $p^{-1}+q^{-1}=1$, let $f \in L^p(G)$, $g \in L^q(G)$ and let $\langle \cdot \mid \cdot \rangle : L^p(G) \times L^q(G) \to \mathbb{R}$ be a bilinear form defined by

$$\langle f \mid g \rangle = \int_G f(x)g(x)\,dx.$$

If the operators $T : L^p(G) \to L^p(G)$ and $T^* : L^q(G) \to L^q(G)$ satisfy the condition

$$\langle Tf \mid g \rangle = \langle f \mid T^* g \rangle$$

for every $f \in L^p(G)$ and $g \in L^q(G)$, then the operators T and T^* are called adjoints.

Theorem 2.8. *Suppose that $p,q \in (1,\infty)$ are conjugate exponents and suppose that $T : L^p(G) \to L^p(G)$ is a compact operator. Then the adjoint operator $T^* : L^q(G) \to L^q(G)$ is also compact.*

Now, we are ready to introduce a theorem that is known as the Fredholm alternative. The Fredholm alternative is a powerful tool for solving equations of the type $(I-T)f = g$.

Theorem 2.9. *Let $p,q \in (1,\infty)$ be dual exponents and assume that the operators $T : L^p(G) \to L^p(G)$ and $T^* : L^q(G) \to L^q(G)$ are compact adjoints. Then only one of the following alternatives holds:*

(i) *The operators $I-T$ and $I-T^*$ are bijective.*

(ii) *Equations $(I-T)f=0$ and $(I-T^*)f=0$ have non-trivial solutions.*

As a consequence of the Fredholm alternative and Theorem 2.5, we formulate a corollary that will be useful when we find solutions to the Dirichlet problem via layer potential approach.

Corollary 2.1. *Suppose that $T : L^p(G) \to L^p(G)$ is a compact operator. If either of the equations*

$$(I-T^*)f = 0 \quad \text{or} \quad (I-T)f = 0$$

has only the trivial solution $f=0$, then the operators $I-T$ and $I-T^*$ are invertible.

Remark 2.3. The Fredholm alternative and its corollary can be generalized to concern bounded linear operators that map between general Banach spaces X and Y. See for example [4].

2.3 Maximal operators

In this section, we present some properties of certain maximal operators. We begin by stating the boundedness property of the Hardy-Littlewood maximal operator that we define by

$$\mathcal{M}f(x) = \sup_{r>0} \frac{1}{r^d} \int_{B(x,r)} |f(y)|\, dy, \quad f \in L^p(\mathbb{R}^d).$$

Theorem 2.10. *Suppose that $p \in (1,\infty)$. The Hardy-Littlewood maximal operator $\mathcal{M}$ satisfies an estimate*

$$\|\mathcal{M}f\|_{L^p(\mathbb{R}^d)} \leq C\|f\|_{L^p(\mathbb{R}^d)}$$

for every $f \in L^p(\mathbb{R}^d)$ and for some constant $C > 0$.

We are also interested in maximal operators that can be majorized by the Hardy-Littlewood maximal operator. Suppose that N is a function defined in $\mathbb{R}^d$ and $\{N_\varepsilon\}_{\varepsilon>0}$ is a family of functions that are defined by

$$N_\varepsilon(x) = \frac{1}{\varepsilon^d} N\Big(\frac{x}{\varepsilon}\Big).$$

Then we define a maximal operator $\mathcal{M}_N$ by writing

$$\mathcal{M}_N f(x) := \sup_{\varepsilon>0} |(N_\varepsilon * f)(x)| = \sup_{\varepsilon>0} \left| \int_{\mathbb{R}^d} N_\varepsilon(y-x) f(y)\, dy \right|.$$

It depends on the function N, whether a maximal operator $\mathcal{M}_N$ is majorized by the Hardy-Littlewood maximal operator. To be able to formulate sufficient conditions on N which allow us to majorize $\mathcal{M}_N$ by $\mathcal{M}$, we first make the following definition.

Definition 2.10. Suppose N is a function defined on $\mathbb{R}^d$ and $x, y \in \mathbb{R}^d$. We say that the function N has a radially decreasing majorant, if there is a function N_0 such that the following properties hold:

(i) $N_0(x) = N_0(y)$ whenever $|x| = |y|$.

(ii) $N_0(x) \leq N_0(y)$ whenever $|x| \geq |y|$.

(iii) $|N(x)| \leq N_0(x)$ for almost every $x \in \mathbb{R}^d$.

By using the above definition, we formulate the following theorem that can be found from [13].

Theorem 2.11. *Let $p \in (1, \infty)$. Suppose that a function N has a radially decreasing majorant N_0 that is continuous and integrable in $\mathbb{R}^d$. Then the estimate*

$$\mathcal{M}_N f(x) := \sup_{\varepsilon > 0} |(N_\varepsilon * f)(x)| \leq C \mathcal{M} f(x)$$

holds for every $f \in L^p(\mathbb{R}^d)$ and for some constant $C > 0$.

The properties of the above maximal operators will be needed in Chapters 5 and 6. In addition, we will also need other types of maximal operators. In Chapter 3, we will define a maximal operator for the singular integral operators and in Chapter 6 we will need the concept of non-tangential maximal operator. These other types of maximal operators will be discussed in detail as we proceed.

Chapter 3

Singular Integral Operators

In this chapter, we study singular integral operators. We are interested in singular integral operators because we come across them in the study of boundary values of layer potentials. Especially, the boundary integral operators K and K^* are singular integral operators.

Let us state what we mean by a singular integral operator. Suppose that G is a subset of $\mathbb{R}^d$ and μ is a measure in G. Also, assume that $f \in L^p(G)$ for some $p \in (1, \infty)$. We say that an operator T is a singular integral operator if it can be represented in the form

$$Tf(x) = \int_G k(x, y) f(y)\, d\mu(y)$$

and the kernel $k : G \times G \to \mathbb{R}$ has singularities on a diagonal $x = y$. Furthermore, we say that a singular integral operator is either weakly singular or strictly singular. An operator is weakly singular if the rate of singularity does not reach certain critical level, whereas an operator is strictly singular if it is not weakly singular. For example, the kernel Ψ of the boundary integral operator K defined on a smooth domain satisfies an estimate

$$|\Psi(z, w)| \leq C|z - w|^{2-d}.$$

Here the rate of singularity does not reach the critical level and therefore understanding the behaviour of such operators is relatively easy. Instead, the kernel of the boundary integral operator K defined on a purely C^1-domain satisfies only an estimate

$$|\Psi(z, w)| \leq C|z - w|^{1-d}.$$

Here the rate of singularity reaches the critical level precisely and therefore understanding the behaviour of such operators is difficult.

The definition of singular integral operators is somewhat vague because in practice we do not immediately know, whether the integral exists and if it does, then in what sense? To overcome this difficulty, we introduce the concept of truncated operators

$$T_\varepsilon f(x) = \int_G k_\varepsilon(x,y) f(y)\, dy, \quad \varepsilon > 0,$$

where the truncated kernels are defined by

$$k_\varepsilon(x,y) = k(x,y)\mathbf{1}_{B(x,\varepsilon)^c}(y).$$

Truncated operators are well-defined because their kernels do not contain singularities. By using the truncated operators we can ask, whether the limit exists if we let ε tend to zero. The following definitions give us two ways of interpreting the convergence of a singular integral operator.

Definition 3.1. Let $p \in (1,\infty)$. We say that an operator T exists pointwise for almost everywhere in G, if for each $f \in L^p(G)$, the operator T satisfies the conditions

$$\lim_{\varepsilon \to 0} |Tf(x) - T_\varepsilon f(x)| = 0 \quad \text{and} \quad |Tf(x)| < \infty$$

for almost every $x \in G$.

Definition 3.2. Let $p \in (1,\infty)$. We say that a pointwise convergent operator T exists in $L^p(G)$, if for each $f \in L^p(G)$, the operator T satisfies condition

$$\lim_{\varepsilon \to 0} \|Tf - T_\varepsilon f\|_{L^p(G)} = 0.$$

If an operator T exists pointwise for almost everywhere, then we call it a principal value operator and we denote

$$Tf(x) := \text{p.v.} \int_G k(x,y) f(y)\, d\mu(y).$$

However, we are not usually able to tell immediately, whether a singular integral operator is defined in the principal value sense. A tool that helps to do this is the maximal operator T_* defined by

$$T_* f(x) = \sup_{\varepsilon > 0} |T_\varepsilon f(x)|.$$

Unlike the principal value operator, the maximal operator is immediately well-defined. Therefore, it is convenient to establish properties for the maximal operator first and after that, deduce properties for the principal value operator.

This chapter obeys the following structure. In Section 3.1, we establish conditions that allow us to deduce convergence of a singular integral operator in the sense of Definition 3.2.

In Section 3.2, we study weakly singular integral operators. We need weakly singular integral operators because they help us to understand strictly singular integral operators. For example, in Chapter 5 we manage to prove that certain strictly singular integral operators are compact by approximating them with a sequence of weakly singular integral operators. Finally, this leads us to conclude that the boundary integral operators K and K^* are compact.

In Section 3.3, we begin to study strictly singular integral operators. At this stage, we restrict our considerations into singular integral operators that map functions whose domain of definition is $\mathbb{R}$. Such operators will be called one-dimensional strictly singular integral operators. Our goal is to establish boundedness result for a certain type of one-dimensional maximal singular integral operator. Obtaining this result will be the most crucial step towards understanding properties of the boundary integral operators K and K^*.

In Section 3.4, we continue to study strictly singular integral operators. This time, we consider operators that map functions whose domain of definition is $\mathbb{R}^{d-1}$. Such operators will be called multi-dimensional strictly singular integral operators. The goal of this section is to establish boundedness results for certain maximal singular integral operators that are closely related to the boundary integral operators K and K^*. The boundedness results will be established by the method of rotations. The method of rotations is a procedure that reduces the problem of the boundedness of a multi-dimensional operator into problem of the boundedness of a one-dimensional operator. By applying the method of rotations we are able to use results obtained for one-dimensional strictly singular integral operators.

3.1 Convergence results

Let G be a domain in $\mathbb{R}^{d-1}$ or a boundary of a bounded domain in $\mathbb{R}^d$ and let μ be a measure in G. Also, assume that $p \in (1, \infty)$.

The purpose of this section is to establish sufficient conditions that allow us to show that a singular integral operator $T : L^p(G) \to L^p(G)$ converges in $L^p(G)$. A simple observation below helps us to establish these conditions.

Lemma 3.1. *Suppose that the maximal operator T_* satisfies the condition*

$$\|T_* f\|_{L^p(G)} \leq Cm\|f\|_{L^p(G)}$$

for some constants $C > 0$ and $m > 0$ and for every $f \in L^p(G)$. Then, for *every $\lambda > 0$, the following estimate holds*

$$\mu(\{x \in G : T_* f(x) > \lambda\}) \leq \frac{Cm^p}{\lambda^p} \|f\|_{L^p(G)}^p.$$

Proof. Let us denote $A = \{x \in G : T_* f(x) > \lambda\}$. Then

$$\mu(A) \leq \frac{1}{\lambda^p} \int_A |T_* f(x)|^p \, d\mu(x) \leq \frac{1}{\lambda^p} \|T_* f\|_{L^p(G)}^p \leq \frac{Cm^p}{\lambda^p} \|f\|_{L^p(G)}^p.$$

This completes the proof. □

Next, we show that the boundedness of the maximal operator T_* and pointwise convergence of T are sufficient conditions that imply convergence of the operator T in $L^p(G)$.

Lemma 3.2. *Suppose that $T : L^p(G) \to L^p(G)$ is a singular integral operator that satisfies the following conditions:*

(i) *The maximal operator satisfies an estimate*

$$\|T_* f\|_{L^p(G)} \leq C \|f\|_{L^p(G)}$$

for every $f \in L^p(G)$ and for some constant $C > 0$.

(ii) *The operator T converges pointwise for almost everywhere in G.*

Then the operator T converges in $L^p(G)$.

Proof. Suppose that $f \in L^p(G)$. We notice that the condition (i) implies that the truncations T_ε belong to $L^p(G)$, because

$$\|T_\varepsilon f\|_{L^p(G)} \leq \|T_* f\|_{L^p(G)} \leq C \|f\|_{L^p(G)} < \infty.$$

Let us write

$$\|Tf - T_\varepsilon f\|_{L^p(G)}^p = \int_G |Tf(x) - T_\varepsilon f(x)|^p \, d\mu(x).$$

To use the dominated convergence theorem, we estimate that

$$|Tf(x) - T_\varepsilon f(x)|^2 \leq 2^p |T_* f(x)|^p$$

for almost every $x \in G$. The function on the right-hand side is integrable because

$$\int_G |T_* f(x)|^p \, d\mu(x) = \|T_* f\|_{L^p(G)}^p \leq C \|f\|_{L^p(G)}^p < \infty.$$

Now, due to the dominated convergence theorem, we may pass the limit inside the integral sign and therefore

$$\lim_{\varepsilon \to 0} \|Tf - T_\varepsilon f\|_{L^p(G)} = 0,$$

which completes the proof. □

It is possible to weaken the conditions even more.

Theorem 3.1. *Suppose $T : L^p(G) \to L^p(G)$ is a singular integral operator that satisfies the following conditions:*

(i) *For every $f \in L^p(G)$ the maximal operator satisfies the inequality*

$$\|T_* f\|_{L^p(G)} \leq C\|f\|_{L^p(G)}.$$

(ii) *For every $g \in C^1(G)$ the limit*

$$\lim_{\varepsilon \to 0} T_\varepsilon g(x)$$

exists for almost every $x \in G$.

Then the operator T exists in $L^p(G)$ and pointwise for almost everywhere.

Proof. Suppose $f \in L^p(G)$ and $g \in C^1(G)$. According to Lemma 3.2, it suffices to prove that the operator T converges pointwise for almost everywhere in G. To prove this, we introduce an operator

$$\Lambda(Tf)(x) = \limsup_{\varepsilon \to 0} T_\varepsilon f(x) - \liminf_{\varepsilon \to 0} T_\varepsilon f(x)$$

and we show that $\Lambda(Tf)(x) = 0$ for almost every $x \in G$.

The condition (ii) implies that $\Lambda(Tg)(x) = 0$ for almost every $x \in G$. Therefore, we may estimate

$$\Lambda(Tf)(x) = \Lambda(Tf)(x) - \Lambda(Tg)(x) \leq 2\,T_*(f-g)(x). \tag{3.1}$$

Suppose $\lambda > 0$. The estimate (3.1) implies that

$$\{x \in G : \Lambda(Tf)(x) > \lambda\,\} \subset \{x \in G : T_*(f-g)(x) > \lambda/2\,\}.$$

Then we use Lemma 3.1 to obtain

$$\mu(\{x \in G : \Lambda(Tf)(x) > \lambda\,\}) \leq \frac{C}{\lambda^p}\|f-g\|^p_{L^p(G)}.$$

There exists a sequence $\{g_j\}_{j=1}^{\infty}$ of functions in $C^1(G)$ that converges to f, because $C^1(G)$ is a dense subspace of $L^p(G)$. As a result, we have

$$\mu(\{x \in G : \Lambda(Tf)(x) > \lambda\,\}) = 0.$$

Let us take a sequence $\{\lambda_j\}_{j=1}^{\infty}$ of positive numbers that converges to 0. Then, we see that

$$\mu(\{x \in G : \Lambda(Tf)(x) > 0\,\}) \leq \bigcup_{n=1}^{\infty} \mu(\{x \in G : \Lambda(Tf)(x) > \lambda_n\,\}) = 0.$$

Thus, for almost every $x \in G$, we have

$$\Lambda(Tf)(x) = 0.$$

Finally, we note that

$$\int_G |Tf(x)|^p \, d\mu(x) \leq \|T_* f\|_{L^p(G)} \leq C\|f\|_{L^p(G)} < \infty,$$

which implies that $|Tf(x)| < \infty$ for almost every $x \in G$. □

It is helpful that it suffices to show the boundedness of the maximal operator T_* instead of actual operator T. Moreover, we are able to assume densities to be C^1-functions instead of L^p-functions. This makes it possible to use the dominated convergence theorem in the forthcoming proofs.

3.2 Weakly singular integral operators

In this section G denotes either a closure of a bounded domain in $\mathbb{R}^{d-1}$ or a boundary of some bounded domain in $\mathbb{R}^d$. We also assume that $p \in (1, \infty)$.

We begin by defining weakly singular integral operators.

Definition 3.3. A singular integral operator $T : L^p(G) \to L^p(G)$ determined by a continuous kernel k is called weakly singular, if there exists a number $0 < \alpha \leq d - 1$ such that the kernel satisfies

$$|k(x, y)| \leq C|x - y|^{1+\alpha-d} \tag{3.2}$$

for every $x, y \in G$ for which $x \neq y$.

With the help of the following lemma we are able to prove that weakly singular integral operators are bounded.

Lemma 3.3. *Let $T : L^p(G) \to L^p(G)$ be a weakly singular integral operator. Then the kernel k of the operator T satisfies the following conditions*

$$\int_G |k(x, y)| \, d\mu(x) \leq C \quad \textit{and} \quad \int_G |k(x, y)| \, d\mu(y) \leq C$$

for some constant $C > 0$.

Proof. It suffices to show that at least one of the above conditions is true. Suppose that $x \in G$ and $\varepsilon > 0$. Let us denote

$$k_\varepsilon(x, y) := k(x, y)\mathbf{1}_{B(x,\varepsilon)^c}(y) \quad \text{and} \quad \tilde{k}_\varepsilon(x, y) = k(x, y)\mathbf{1}_{B(x,\varepsilon)}(y).$$

Then, we may estimate

$$\int_G |k(x,y)|\, d\mu(y) \le \int_G |k_\varepsilon(x,y)|\, d\mu(y) + \int_G |\tilde{k}_\varepsilon(x,y)|\, d\mu(y).$$

The first term on the right-hand side is bounded by some constant because G is bounded. We estimate the second term as follows:

$$\begin{aligned}
\int_G |\tilde{k}(x,y)|\, d\mu(y) &\le \int_{B(x,\varepsilon)} C|x-y|^{1+\alpha-d}\, d\mu(y) \\
&\le C\int_0^\varepsilon \left(\int_{\partial B(x,t)} |x-y|^{1+\alpha-d} d\sigma_t(y) \right) dt \\
&\le C\int_0^\varepsilon \sigma(\partial B(x,t)) t^{1+\alpha-d}\, dt \\
&\le C\int_0^\varepsilon t^{\alpha-1}\, dt \\
&\le C\varepsilon^\alpha.
\end{aligned}$$

The above observations complete the proof. □

Theorem 3.2. *Suppose that $T : L^p(G) \to L^p(G)$ is a weakly singular integral operator. Then T is bounded on $L^p(G)$ and converges pointwise for almost everywhere in G.*

Proof. Suppose that $f \in L^p(G)$ and $\varepsilon > 0$. To show the existence of the pointwise limit

$$Tf(x) = \lim_{\varepsilon \to 0} \int_G k_\varepsilon(x,y) f(y)\, d\mu(y),$$

we must show that the function $k(x,\cdot)f$ is integrable for almost every $x \in G$. To do this, we write

$$|k(x,y)f(y)| = |k(x,y)|^{1/q}(|k(x,y)|^{1/p}|f(y)|),$$

where $p, q \in (1,\infty)$ are conjugate exponents. Then we use Hölder's inequality and Lemma 3.3 to obtain

$$\int_G |k(x,y)f(y)|\, d\mu(y) \le C\left(\int_G |k(x,y)||f(y)|^p\, d\mu(y) \right)^{1/p}.$$

Now, Fubini's theorem and Lemma 3.3 imply that

$$\int_G \left(\int_G |k(x,y)f(y)|\, d\mu(y) \right)^p d\mu(x) \le C\|f\|^p_{L^p(G)} < \infty.$$

From the above estimate, we deduce that $k(x,\cdot)f$ is integrable for almost every $x \in G$.

Because $k_\varepsilon(x,\cdot)f$ has an integrable majorant for almost every $x \in G$, the dominated convergence theorem implies that the operator T exists for almost everywhere in G. Furthermore, the operator T is bounded on $L^p(G)$ because there is a constant $C > 0$ such that

$$\|Tf\|_{L^p(G)} \leq \left[\int_G \left(\int_G |k(x,y)f(y)|\,d\mu(y)\right)^p d\mu(x)\right]^{1/p} \leq C\|f\|_{L^p(G)}.$$

These observations complete the proof. □

We wish to show that weakly singular integral operators are compact. For this reason, we introduce a concept of Hilbert-Schmidt integral operators that are known to be compact. For a proof, we refer to [11].

Definition 3.4. We say that an integral operator $T : L^2(G) \to L^2(G)$ determined by a kernel k is a Hilbert-Schmidt if the kernel k satisfies the condition

$$\int_G \int_G |k(x,y)|^2\,d\mu(y)\,d\mu(x) < \infty.$$

Theorem 3.3. *Hilbert-Schmidt operators $T : L^2(G) \to L^2(G)$ are compact.*

Next, let us consider singular integral operators with bounded kernels.

Example 3.1. Suppose $T : L^2(G) \to L^2(G)$ is a singular integral operator determined by a kernel k that satisfies the condition

$$|k(x,y)| \leq M < \infty \tag{3.3}$$

for every $x, y \in G$ and for some constant $M > 0$. Then T is a Hilbert-Schmidt operator.

Lemma 3.4. *Suppose that $T : L^p(G) \to L^p(G)$ is a singular integral operator determined by a kernel k that satisfies a condition* (3.3)*. Then the operator T is compact.*

Proof. Suppose that $q \in (1,2)$ and let $r \in (2,\infty)$ be the conjugate exponent of q. Let $\varepsilon > 0$ and assume $\{f_j\}_{j=1}^{\infty}$ is a bounded sequence in $L^q(G)$. Let us select a number $\lambda := \lambda_\varepsilon$ and define functions

$$g_j(x) := \begin{cases} f_j(x), & \text{if } |f_j(x)| > \lambda \\ 0, & \text{if } |f_j(x)| \leq \lambda \end{cases}$$

and

$$h_j(x) := \begin{cases} f_j(x), & \text{if } |f_j(x)| \le \lambda \\ 0, & \text{if } |f_j(x)| < \lambda. \end{cases}$$

Then we may write

$$f_j = g_j + h_j.$$

The sequence $\{h_j\}_{j=1}^{\infty}$ is bounded on $L^2(G)$. Thus, the sequence $\{Th_j\}_{j=1}^{\infty}$ has a convergent subsequence in $L^2(G)$. We choose a corresponding subsequence $\{f_j\}_{j=1}^{\infty}$, but for simplicity, we do not distinguish between sequences and subsequences.

If $g \in L^1(G)$, then

$$\begin{aligned} \|Tg\|_{L^q(G)}^q &\le \int_G \left| \int_G k(x,y)g(y)\,d\mu(y) \right|^q d\mu(x) \\ &\le \int_G \left(\int_G |k(x,y)||g(y)|\,d\mu(y) \right)^q d\mu(x) \\ &\le C\|g\|_{L^1(G)}^q. \end{aligned}$$

Let us denote

$$A_j := \{x \in G : |f_j(x)| > \lambda\}.$$

By choosing a suitable λ, using Hölder's inequality, Chebyshev's inequality and the boundedness of the sequence $\{f_j\}_{j=1}^{\infty}$, we get

$$\|g_j\|_{L^1(G)} \le \|\mathbf{1}_{A_j}\|_{L^r(G)}\|f_j\|_{L^q(G)} \le C\mu(A_j) \le C\lambda_\varepsilon^{-q} < \varepsilon/3.$$

Finally, by choosing a large enough $N > 0$, we see that

$$\|Tf_m - Tf_n\|_{L^q(G)} \le C\|g_m - g_n\|_{L^1(G)} + C\|Th_m - Th_n\|_{L^2(G)} < \varepsilon,$$

whenever $m, n \ge N$. Because $\{Tf_j\}_{j=1}^{\infty}$ is a Cauchy sequence in a Banach space $L^q(G)$, we know that $\{Tf_j\}_{j=1}^{\infty}$ converges. Thus, T is compact on $L^q(G)$.

With the above arguments, we could have deduced that the operator

$$\widetilde{T} : L^q(G) \to L^q(G), \quad \widetilde{T}f(x) = \int_G k(y,x)f(y)\,d\mu(y)$$

is compact. Thus, according to Theorem 2.8, the operator T is compact on $L^r(G)$, because $T : L^r(G) \to L^r(G)$ is the adjoint of $\widetilde{T}$. □

By applying the above lemma, we are able to prove compactness of the weakly singular integral operators.

Theorem 3.4. *A weakly singular integral operator $T : L^p(G) \to L^p(G)$ is compact.*

Proof. Let us use notations introduced in the proof of Lemma 3.3. Suppose that T is determined by a weakly singular kernel k. We define truncated operators

$$T_\varepsilon f(x) := \int_G k_\varepsilon(x, y) f(y)\, dy, \quad \varepsilon > 0.$$

The kernel k_ε satisfies condition (3.3) because

$$|k_\varepsilon(x, y)| \le C|x-y|^{1+\alpha-d} \mathbf{1}_{B(x,\varepsilon)^c}(y) \le C\varepsilon^{1+\alpha-d} < \infty.$$

Thus, according to Lemma 3.4, the operators $\{T_\varepsilon\}_{\varepsilon>0}$ are compact on $L^p(G)$.

Now, let us denote

$$\widetilde{T}_\varepsilon f(x) := \int_G \tilde{k}_\varepsilon(x, y) f(y)\, d\mu(y) = (T - T_\varepsilon) f(x).$$

With a similar arguments as in Lemma 3.3, we estimate

$$\int_G |\tilde{k}_\varepsilon(x, y)|\, d\mu(y) \le C\varepsilon^\alpha \quad \text{and} \quad \int_G |\tilde{k}_\varepsilon(x, y)|\, dx \le C.$$

Consequently, by applying the proof of the Theorem 3.2, we get

$$\|(T - T_\varepsilon) f\|_{L^p(G)} = \|\widetilde{T}_\varepsilon f\|_{L^p(G)} \le C\varepsilon^{\alpha/q} \|f\|_{L^p(G)},$$

where q is a conjugate exponent of p. By letting ε tend to zero, we observe that T is a limit of compact operators T_ε in the operator norm. Thus, the operator T is compact. □

3.3 One-dimensional strictly singular integral operators

In this section, we apply theory of Calderon-Zygmund operators to prove results concerning a singular integral operator $\mathcal{C}$ defined by

$$\mathcal{C} f(s) = \text{p.v.} \int_{\mathbb{R}} \frac{1}{s-t} \left(\frac{\rho(s) - \rho(t)}{s-t} \right) F \left(\frac{\varphi(s) - \varphi(t)}{s-t} \right) f(t)\, dt.$$

In the above formula, ρ and φ are Lipschitz functions on $\mathbb{R}$ and a function $F : \mathbb{R} \to \mathbb{R}$ is defined by

$$F(s) = (1 + s^2)^{-d/2}.$$

Our goal is to prove that the maximal operator $\mathcal{C}_*$ is bounded and its operator norm depends on the Lipschitz norm of a function ρ. Especially, we prove that the operator norm of $\mathcal{C}_*$ tends to zero as the Lipschitz norm of a function ρ tends to zero. This property is crucial when we show that boundary integral operators K and K^* are compact on $L^p(\partial D)$. For more information see Chapter 5.

For our needs, it suffices to restrict the study of Calderon-Zygmund operators into one dimension. By this we mean that we study operators that are mappings from the Schwarz class $\mathscr{S}(\mathbb{R})$ into class of tempered distributions $\mathscr{S}'(\mathbb{R})$. After we have managed to show that the operators $\mathcal{C}$ and $\mathcal{C}_*$ are bounded, we may treat these operators as mappings from $L^p(\mathbb{R})$ to itself in the sense of extensions.

This section obeys the following pattern. First, we define standard kernels and we show that the kernel of the operator $\mathcal{C}$ is a standard kernel. With the help of standard kernels we define Calderon-Zygmund operators and then we ask, whether $\mathcal{C}$ is a Calderon-Zygmund operator? To answer to this question, we apply results concerning the boundedness of the Cauchy integral along Lipschitz curves. Finally, by using standard results of Calderon-Zygmund operators, we conclude that the maximal operator $\mathcal{C}_*$ is bounded and its operator norm is majorized by a constant multiple of $\|\rho'\|_\infty$.

Many parts of this section are influenced by the book of L. Grafakos [13]. Thus, the reader is advised to explore the book for the deeper understanding of this section.

Definition 3.5. Suppose that k is a function defined on $\mathbb{R}^2 \setminus \{(s,s) : s \in \mathbb{R}\}$. The function k is a standard kernel if there are numbers $\delta > 0$ and $A > 0$ such that the following properties hold:

(i) The function k satisfies

$$|k(s,t)| \leq A|s-t|^{-1}. \tag{3.4}$$

(ii) If $|s-s'| \leq \frac{1}{2}\max\{|s-t|, |s'-t|\}$, then

$$|k(s,t) - k(s',t)| \leq \frac{A|s-s'|^\delta}{(|s-t| + |s'-t|)^\delta}. \tag{3.5}$$

(iii) If $|t-t'| \leq \frac{1}{2}\max\{|s-t|, |s-t'|\}$, then

$$|k(s,t) - k(s,t')| \leq \frac{A|t-t'|^\delta}{(|s-t| + |s-t'|)^\delta}. \tag{3.6}$$

Remark 3.1. If k is a standard kernel with parameters $\delta > 0$ and $A > 0$, we denote $k \in SK(\delta, A)$.

It is laborious to check whether a kernel k satisfies the conditions (3.5) and (3.6). Fortunately, it turns out that it suffices to check two easier conditions. The first condition is that the kernel k is antisymmetric, which means that it satisfies

$$k(s,t) = -k(t,s).$$

The other condition is that there is a constant $A' > 0$ such that

$$|\partial_s k(s,t)| \leq A'|s-t|^{-2}. \tag{3.7}$$

We formulate and prove a test for standard kernels precisely below, but first we need to consider the following lemma.

Lemma 3.5. *If the numbers s, s' and t satisfy the condition*

$$|s-s'| \leq \frac{1}{2}\max\{|s-t|, |s'-t|\},$$

then they also satisfy

$$\max\{|s-t|, |s'-t|\} \leq 2\min\{|s-t|, |s'-t|\}. \tag{3.8}$$

Furthermore, if a number ξ is between s and s', then

$$|\xi - t| \geq \frac{1}{4}(|s-t| + |s'-t|). \tag{3.9}$$

Proof. We begin by proving the inequality (3.8). First, let $|s-t| \leq |s'-t|$. Then it suffices to show that

$$|s'-t| \leq 2|s-t|.$$

Using the assumptions, we get

$$|s-s'| \leq \frac{1}{2}|s'-t|.$$

Then the triangle inequality and the above estimate imply that

$$|s-t| \geq |s'-t| - |s-s'| \geq \frac{1}{2}|s'-t|.$$

Second, let $|s'-t| \leq |s-t|$. Now, by using similar arguments as before, we deduce that

$$|s-t| \leq 2|s'-t|.$$

By combining the above observations, we obtain the inequality (3.8).

Next, we prove the inequality (3.9). We use the triangle inequality twice to estimate

$$|\xi - t| \geq \frac{1}{2}(|s-t| - |s-\xi|) + \frac{1}{2}(|s'-y| - |s'-\xi|).$$

We know that the number ξ is between s and s'. Thus, the above inequality can written in the form

$$|\xi - t| \geq \frac{1}{2}(|s-t| + |s'-t| - |s-s'|).$$

Furthermore, we use the inequality (3.8) to estimate

$$\begin{aligned} |s-s'| &\leq \frac{1}{2}\max\{|s-t|, |s'-t|\} \leq \min\{|s-t|, |s'-t|\} \\ &\leq \frac{1}{2}(|s-t| + |s'-t|). \end{aligned}$$

The inequality (3.9) follows from the above estimates. □

Now, we are ready to formulate and prove the test for standard kernels.

Lemma 3.6. *Suppose that a kernel k is antisymmetric and it satisfies conditions* (3.4) *and* (3.7)*. Then k is a standard kernel with parameters $\delta = 1$ and $A = 16A'$.*

Proof. According to the mean value theorem, there exists a number ξ between s and s' such that

$$|k(s,t) - k(s',t)| \leq |\partial_s k(\xi, t)||s-s'|.$$

By applying condition (3.7) and Lemma 3.5, we see that

$$|k(s,t) - k(s',t)| \leq \frac{A'|s-s'|}{|\xi - t|^2} \leq \frac{16A'|s-s'|}{(|s-t| + |s'-t|)^2}.$$

Furthermore, we see that the condition (3.6) holds because the kernel k is antisymmetric. □

Next, we apply the above test to the kernel of the singular integral operator $\mathcal{C}$.

Lemma 3.7. *Suppose that $\rho, \varphi : \mathbb{R} \to \mathbb{R}$ are Lipschitz functions and suppose that $F : \mathbb{R} \to \mathbb{R}$ is a function defined by*

$$F(s) = (1 + s^2)^{-d/2}.$$

Then the kernel k defined by

$$k(s,t) = \frac{1}{s-t}\left(\frac{\rho(s)-\rho(t)}{s-t}\right) F\left(\frac{\varphi(s)-\varphi(t)}{s-t}\right)$$

is a standard kernel with parameters $\delta = 1$ and $A = 16(3 + 2d\|\varphi'\|_\infty^2)\|\rho'\|_\infty$.

Proof. The kernel k is antisymmetric and it satisfies the estimate

$$|k(s,t)| \leq \|\rho'\|_\infty |s-t|^{-1}.$$

Therefore, it suffices to check the condition (3.7). Let us make some observations. First, we notice that $|F(s)| \leq 1$ and $|F'(s)| \leq d\|\varphi'\|_\infty$. We can also calculate

$$\left|\frac{\partial}{\partial s}\left(\frac{\rho(s)-\rho(t)}{s-t}\right)\right| \leq 2\|\rho'\|_\infty |s-t|^{-1}$$

and

$$\left|\frac{\partial}{\partial s}\left[\frac{1}{s-t} F\left(\frac{\varphi(s)-\varphi(t)}{s-t}\right)\right]\right| \leq (1 + 2d\|\varphi'\|_\infty^2)|s-t|^{-2}.$$

These observations imply that

$$|\partial_s k(s,t)| \leq (3 + 2d\|\varphi'\|_\infty^2)\|\rho'\|_\infty |s-t|^{-2},$$

which completes the proof. □

Let us define Calderon-Zygmund operators.

Definition 3.6. Suppose that k is a standard kernel with parameters $\delta > 0$ and $A > 0$. A continuous linear operator $T : \mathscr{S}(\mathbb{R}) \to \mathscr{S}'(\mathbb{R})$ is a Calderon-Zygmund operator associated with a kernel k if the following conditions hold:

(i) For each $f \in C_0^\infty(\mathbb{R})$, whose support does not contain x, the operator T satisfies

$$Tf(x) = \int_{\mathbb{R}} k(x,y) f(y)\, dy.$$

(ii) For each $f \in \mathscr{S}(\mathbb{R})$ and for some constant $B > 0$, the operator T satisfies the estimate

$$\|Tf\|_{L^2(\mathbb{R})} \leq B\|f\|_{L^2(\mathbb{R})}.$$

Remark 3.2. If the operator T is a Calderon-Zygmund operator with parameters $\delta > 0$, $A > 0$ and $B > 0$, we denote $T \in CZO(\delta, A, B)$.

We wish to show that the singular integral operator $\mathcal{C}$ is a Calderon-Zygmund operator. To prove this, we need the following theorem, first proven by R. R. Coifman, A. McIntosh and Y. Meyer in their paper [3].

Theorem 3.5. *Suppose that $\rho, \varphi : \mathbb{R} \to \mathbb{R}$ are Lipschitz functions and let $F : \mathbb{R} \to \mathbb{R}$ be a function defined by*

$$F(s) = (1 + s^2)^{-d/2}.$$

Then the singular integral operator $\mathcal{C}$ defined by

$$\mathcal{C}f(s) = \text{p.v.} \int_{\mathbb{R}} \frac{1}{s-t} \left(\frac{\rho(s) - \rho(t)}{s-t} \right) F \left(\frac{\varphi(s) - \varphi(t)}{s-t} \right) f(t)\, dt$$

is bounded on $L^2(\mathbb{R})$ and it satisfies the estimate

$$\|\mathcal{C}f\|_{L^2(\mathbb{R})} \leq C\|\rho'\|_\infty \|f\|_{L^2(\mathbb{R})}. \tag{3.10}$$

With the help of Lemma 3.7 and Theorem 3.5, we conclude that the singular integral operator $\mathcal{C} : \mathscr{S}(\mathbb{R}) \to \mathscr{S}'(\mathbb{R})$ is a Calderon-Zygmund operator. In fact, we can interpret $\mathcal{C}$ as a singular integral operator from $L^2(\mathbb{R})$ to itself in the sense of the extension operator. We know even more: standard arguments of the Calderon-Zygmund theory imply that an operator $T \in CZO(\delta, A, B)$ has a bounded extension that maps $L^p(\mathbb{R})$ to itself and whose operator norm satisfies

$$\|T\| \leq C \max\{p, (p-1)^{-1}\}(A + B). \tag{3.11}$$

Here we have assumed that $p \in (1, \infty)$ and denoted the extension operator by T. Calderon-Zygmund theory implies further that also the maximal operator T_* is bounded and its operator norm satisfies the estimate (3.11). For the detailed discussion on this subject see [13].

Because we know that $\mathcal{C} \in CZO(\delta, A, B)$, where the parameters are $\delta = 1$, $A = 16(3 + 2d\|\varphi'\|_\infty^2)\|\rho'\|_\infty$ and $B = C\|\rho'\|_\infty$, the above discussion allows us to formulate the following theorems.

Theorem 3.6. *Let $p \in (1, \infty)$. The singular integral operator $\mathcal{C}$ admits an bounded extension from $L^p(\mathbb{R})$ to itself. The bounded extension, which is also denoted by $\mathcal{C}$, satisfies the estimate*

$$\|\mathcal{C}f\|_{L^p(\mathbb{R})} \leq C\|\rho'\|_\infty \|f\|_{L^p(\mathbb{R})}$$

for every $f \in L^p(\mathbb{R})$ and for some constant $C > 0$.

Theorem 3.7. *The maximal operator $\mathcal{C}_* : L^p(\mathbb{R}) \to L^p(\mathbb{R})$ is bounded and it satifies the estimate*

$$\|\mathcal{C}_* f\|_{L^p(\mathbb{R})} \leq C\|\rho'\|_\infty \|f\|_{L^p(\mathbb{R})}$$

for every $f \in L^p(\mathbb{R})$ and for some constant $C > 0$.

3.4 Multi-dimensional strictly singular integral operators

In this section, we study certain singular integral operators that map functions from $L^p(\mathbb{R}^{d-1})$ to itself. Such operators are called multi-dimensional strictly singular integral operators. The goal of this section is to establish boundedness results for maximal singular integral operators whose kernels are essentially of the form

$$k(x, y) = \frac{A(x) - A(y)}{[|x-y|^2 + (\varphi(x) - \varphi(y))^2]^{d/2}}, \tag{3.12}$$

where A and φ are Lipschitz functions from $\mathbb{R}^{d-1}$ into $\mathbb{R}$. For example, one of these kernels is defined by

$$k(x, y) = \frac{\rho(x) - \rho(y) - \langle \nabla\rho(y), x - y\rangle}{[|x-y|^2 + (\varphi(x) - \varphi(y))^2]^{d/2}}, \tag{3.13}$$

where $\rho : \mathbb{R}^{d-1} \to \mathbb{R}$ is a Lipschitz function.

To be able to establish such boundedness results, we apply the method of rotations. The method of rotations is a procedure that allows us to reduce the boundedness of the multi-dimensional singular integral operator into the boundedness of the corresponding one-dimensional singular integral operator. Consequently, we are able to apply the results obtained in the previous section. The method of rotations is formulated and proved below.

Theorem 3.8. (Method of rotations). *Let $p \in (1, \infty)$, $\omega \in \partial B(0, 1)$ and let $\eta \in \mathbb{R}^{d-1}$ be a vector that satisfies $\langle \eta, \omega\rangle = 0$. Suppose that T is a multi-dimensional singular integral operator defined by the kernel k and suppose that $\mathcal{T}$ is a one-dimensional singular integral operator defined by the kernel*

$$\tilde{k}(s, t) = k(\eta + s\omega, \eta + t\omega)(s - t)^{d-2}.$$

If there is a constant $C > 0$ that does not depend on either ω or η and such that the maximal operator $\mathcal{T}_*$ satisfies the estimate

$$\|\mathcal{T}_* g\|_{L^p(\mathbb{R})} \leq C\|g\|_{L^p(\mathbb{R})}$$

for every $g \in L^p(\mathbb{R})$, then the maximal operator T_* is bounded on $L^p(\mathbb{R}^{d-1})$.

Furthermore, if there is a number $m_0 > 0$ such that the maximal operator $\mathcal{T}_*$ satisfies an estimate

$$\|\mathcal{T}_* g\|_{L^p(\mathbb{R})} \le C m_0 \|g\|_{L^p(\mathbb{R})}$$

for every $g \in L^p(\mathbb{R})$, then the maximal operator T_* satisfies an estimate

$$\|T_* f\|_{L^p(\mathbb{R}^{d-1})} \le C m_0 \|f\|_{L^p(\mathbb{R}^{d-1})}$$

for every $f \in L^p(\mathbb{R}^{d-1})$.

Proof. First, we represent the truncated operators T_ε in the form

$$T_\varepsilon f(x) = \int_{\partial B(0,1)} \left(\int_\varepsilon^\infty k(x, x+t\omega) f(x+t\omega) t^{d-2}\, dt \right) d\sigma(\omega).$$

On the other hand, we can write

$$T_\varepsilon f(x) = \int_{\partial B(0,1)} \left(\int_{-\infty}^{-\varepsilon} k(x, x+t\omega) f(x+t\omega) t^{d-2}\, dt \right) d\sigma(\omega).$$

By combining these representations, we obtain

$$T_\varepsilon f(x) = \frac{1}{2} \int_{\partial B(0,1)} \left(\int_{|t|>\varepsilon} k(x, x+t\omega) f(x+t\omega) t^{d-2}\, dt \right) d\sigma(\omega).$$

Then, for each $\omega \in \partial B(0,1)$, we define truncated operators

$$R_{\omega,\varepsilon} f(x) := \int_{|t|>\varepsilon} k(x, x+t\omega) f(x+t\omega) t^{d-2}\, dt$$

and the maximal operator will be denoted by R_ω. We use elementary estimates to obtain

$$T_* f(x) \le \frac{1}{2} \int_{\partial B(0,1)} |R_\omega f(x)|\, d\sigma(\omega).$$

We estimate further by applying Minkowski's integral inequality [12]

$$\begin{aligned}
\|T_* f\|_{L^p(\mathbb{R}^{d-1})} &= \left(\int_{\mathbb{R}^{d-1}} |T_* f(x)|^p\, dx \right)^{1/p} \\
&\le \left(\int_{\mathbb{R}^{d-1}} \left| \frac{1}{2} \int_{\partial B(0,1)} |R_{\omega,*} f(x)|\, d\sigma(\omega) \right|^p dx \right)^{1/p} \\
&\le \frac{1}{2} \int_{\partial B(0,1)} \left(\int_{\mathbb{R}^{d-1}} |R_\omega f(x)|^p\, dx \right)^{1/p} d\sigma(\omega) \\
&= \frac{1}{2} \int_{\partial B(0,1)} \|R_\omega f\|_{L^p(\mathbb{R}^{d-1})}\, d\sigma(\omega).
\end{aligned}$$

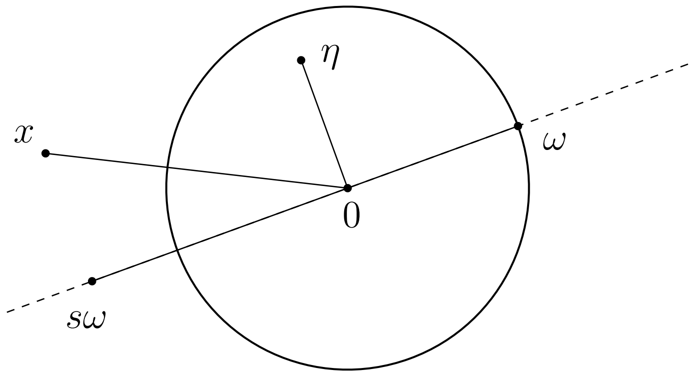

Figure 3.1: The decomposition of the point x

If $\omega \in \partial B(0,1)$ is fixed, then for each point $x \in \mathbb{R}^{d-1}$ there is a number $s \in \mathbb{R}$ and a vector $\eta \in \mathbb{R}^{d-1}$ such that $x = \eta + s\omega$ and $\langle \eta, \omega \rangle = 0$. This can be seen by choosing $s = \langle x, \omega \rangle$ and $\eta = x - s\omega$. The decomposition of a vector x allows us to write

$$\|R_\omega f\|^p_{L^p(\mathbb{R}^{d-1})} = \int_{\mathbb{R}^{d-1}} |R_\omega f(x)|^p \, dx = \int \left(\int_{-\infty}^{\infty} |R_\omega f(\eta + s\omega)|^p \, ds \right) d\eta.$$

On the right-hand side of the above identity the outer integral is taken over planes that are perpendicular to the vector ω. Let us define a function $g(t) := f(\eta + t\omega)$. Then we observe that

$$\begin{aligned} R_{\omega,\varepsilon} f(\eta + s\omega) &= \int_{|t-r|>\varepsilon} k(\eta + s\omega, \eta + t\omega) f(\eta + t\omega)(s-t)^{d-2} \, dt \\ &= \mathcal{T}_\varepsilon g(s). \end{aligned}$$

Now, the boundedness of the operator $\mathcal{T}_*$ implies that

$$\int_{-\infty}^{\infty} |R_\omega f(\eta + s\omega)|^p \, ds = \|\mathcal{T}_* g\|^p_{L^p(\mathbb{R})} \le C \|g\|^p_{L^p(\mathbb{R})} = C \int_{-\infty}^{\infty} |f(\eta + s\omega)|^p \, ds.$$

According to the assumption, the constant $C > 0$ does not depend on the vector η. Therefore,

$$\|R_\omega f\|^p_{L^p(\mathbb{R}^{d-1})} \le C \int \left(\int_{-\infty}^{\infty} |f(\eta + s\omega)|^p \, ds \right) d\eta \le C \|f\|^p_{L^p(\mathbb{R}^{d-1})}.$$

The constant $C > 0$ does not depend on the vector ω either. Therefore, we get

$$\|T_* f\|_{L^p(\mathbb{R}^{d-1})} \le \frac{1}{2} \int_{\partial B(0,1)} \|R_\omega f\|_{L^p(\mathbb{R}^{d-1})} \, d\sigma(\omega) \le C \|f\|_{L^p(\mathbb{R}^{d-1})}.$$

In addition, if there is a constant $m_0 > 0$ such that

$$\|\mathcal{T}_* g\|_{L^p(\mathbb{R})} \leq C m_0 \|g\|_{L^p(\mathbb{R})}$$

for every $g \in L^p(\mathbb{R})$, then by following the above deduction, we obtain

$$\|T_* f\|_{L^p(\mathbb{R}^{d-1})} \leq C m_0 \|f\|_{L^p(\mathbb{R}^{d-1})}$$

for every $f \in L^p(\mathbb{R}^{d-1})$. □

Next, we apply the method of rotations to multi-dimensional strictly singular integral operators whose kernels are of the form (3.12).

Theorem 3.9. *Let $p \in (1, \infty)$. Suppose that $A, \varphi : \mathbb{R}^{d-1} \to \mathbb{R}$ are Lipschitz functions and let $T : \mathscr{S}(\mathbb{R}^{d-1}) \to \mathscr{S}'(\mathbb{R}^{d-1})$ be a singular integral operator defined by the kernel*

$$k(x, y) = \frac{A(x) - A(y)}{[|x - y|^2 + (\varphi(x) - \varphi(y))^2]^{d/2}}.$$

Then the maximal operator T_* satisfies the estimate

$$\|T_* f\|_{L^p(\mathbb{R}^{d-1})} \leq C \|\nabla A\|_\infty \|f\|_{L^p(\mathbb{R}^{d-1})}$$

for some constant $C > 0$ and for every $f \in L^p(\mathbb{R}^{d-1})$.

Proof. Let $\omega \in \partial B(0, 1)$ and let $\eta \in \mathbb{R}^{d-1}$ be perpendicular to ω. We define Lipschitz functions $\tilde{A}(s) = A(\eta + s\omega)$ and $\tilde{\varphi}(s) = \varphi(\eta + s\omega)$. Also, we recall the function $F(s) = (1 + s^2)^{-d/2}$ and the singular integral operator $\mathcal{T}$ that was introduced in Theorem 3.8. Then we write the kernel of the operator $\mathcal{T}$ in the form:

$$\begin{aligned}
\tilde{k}(s, t) &= \frac{(A(\eta + s\omega) - A(\eta + t\omega))(s - t)^{d-2}}{[|s - t|^2 + (\varphi(\eta + s\omega) - \varphi(\eta + t\omega))^2]^{d/2}} \\
&= \frac{1}{s - t} \left(\frac{\tilde{A}(s) - \tilde{A}(t)}{s - t} \right) \left[1 + \left(\frac{\tilde{\varphi}(s) - \tilde{\varphi}(t)}{s - t} \right)^2 \right]^{-d/2} \\
&= \frac{1}{s - t} \left(\frac{\tilde{A}(s) - \tilde{A}(t)}{s - t} \right) F\left(\frac{\tilde{\varphi}(s) - \tilde{\varphi}(t)}{s - t} \right).
\end{aligned}$$

According to Rademacher's theorem, Lipschitz functions are almost everywhere differentiable [7]. Therefore, we have

$$\tilde{A}'(s) = \langle \omega, \nabla A(\eta + s\omega) \rangle \quad \text{and} \quad \tilde{\varphi}'(s) = \langle \omega, \nabla \varphi(\eta + s\omega) \rangle$$

for almost every $s \in \mathbb{R}$. These observations allow us to estimate

$$\|\tilde{A}'\|_\infty \leq \|\nabla A\|_\infty \quad \text{and} \quad \|\tilde{\varphi}'\|_\infty \leq \|\nabla \varphi\|_\infty.$$

Theorem 3.7 implies that the maximal operator $\mathcal{T}_*$ satisfies the estimate

$$\|\mathcal{T}_* g\|_{L^p(\mathbb{R})} \leq C\|\nabla A\|_\infty \|g\|_{L^p(\mathbb{R})}$$

for every $g \in L^p(\mathbb{R})$. Especially, the constant $C > 0$ is independent of the vectors ω and η. Finally, we apply Theorem 3.8 to obtain

$$\|T_* f\|_{L^p(\mathbb{R}^{d-1})} \leq C\|\nabla A\|_\infty \|f\|_{L^p(\mathbb{R}^{d-1})}$$

for every $f \in L^p(\mathbb{R}^{d-1})$ and for some constant $C > 0$. □

We end this section by establishing two boundedness results concerning multi-dimensional strictly singular integral operators that are related to the boundary integral operators K and K^*.

Theorem 3.10. *Let $p \in (1, \infty)$. Suppose that $\rho, \varphi : \mathbb{R}^{d-1} \to \mathbb{R}$ are Lipschitz functions and let $T : \mathscr{S}(\mathbb{R}^{d-1}) \to \mathscr{S}'(\mathbb{R}^{d-1})$ be a singular integral operator defined by the kernel*

$$k(x, y) = \frac{\rho(x) - \rho(y) - \langle \nabla \rho(y), x - y \rangle}{[|x - y|^2 + (\varphi(x) - \varphi(y))^2]^{d/2}}.$$

Also, let us denote $m := \|\nabla \rho\|_\infty$. Then the maximal operator T_* satisfies the estimate

$$\|T_* f\|_{L^p(\mathbb{R}^{d-1})} \leq C_m \|f\|_{L^p(\mathbb{R}^{d-1})}$$

for every $f \in L^p(\mathbb{R}^{d-1})$ and for some constant C_m that tends to zero as m tends to zero.

Proof. Let $f \in L^p(\mathbb{R}^{d-1})$. The idea of the proof is to represent the kernel (3.13) using kernels of the type (3.12). To succeed in this, we make some definitions. First, let us define Lipschitz functions A_i, $i = 1, \ldots, d$, by writing

$$A_i(x) = \begin{cases} x_i, & \text{if } i = 1, \ldots, d-1 \\ \rho(x), & \text{if } i = d. \end{cases}$$

Second, we define functions v_i, $i = 1, \ldots, d$, by writing

$$v_i(y) = \begin{cases} -\partial_{x_i} \rho(y), & \text{if } i = 1, \ldots, d-1 \\ 1, & \text{if } i = d. \end{cases}$$

Third, we define singular integral operators T_i, $i = 1, \dots, d$, that are determined by the kernels

$$k_i(x,y) = \frac{A_i(x) - A_i(y)}{[|x-y|^2 + (\varphi(x) - \varphi(y))^2]^{d/2}}.$$

Using these definitions we are able to write

$$k(x,y) = \sum_{i=1}^{d} k_i(x,y) v_i(y)$$

and then estimate

$$T_* f(x) \le \sum_{i=1}^{d} T_{i,*}(v_i f)(x).$$

Now, by applying Minkowski's inequality and Theorem 3.9, we obtain

$$\begin{aligned} \|T_* f\|_{L^p(\mathbb{R}^{d-1})} &\le \sum_{i=1}^{d} \|T_{i,*}(v_i f)\|_{L^p(\mathbb{R}^{d-1})} \\ &\le \sum_{i=1}^{d} C \|\nabla A_i\|_\infty \|v_i\|_\infty \|f\|_{L^p(\mathbb{R}^{d-1})} \\ &\le C \|\nabla \rho\|_\infty \|f\|_{L^p(\mathbb{R}^{d-1})} \\ &=: C_m \|f\|_{L^p(\mathbb{R}^{d-1})}. \end{aligned}$$

The constant $C > 0$ is independent of m. Thus, it is clear that the constant C_m tends to zero as m tends to zero. □

Theorem 3.11. *Suppose that* $\rho, \varphi : \mathbb{R}^{d-1} \to \mathbb{R}$ *are Lipschitz functions and let* $T : \mathscr{S}(\mathbb{R}^{d-1}) \to \mathscr{S}'(\mathbb{R}^{d-1})$ *be a singular integral operator defined by the kernel*

$$k(x,y) = \frac{\rho(x) - \rho(y) - \langle \nabla \rho(x), x - y \rangle}{[|x-y|^2 + (\varphi(x) - \varphi(y))^2]^{d/2}}. \tag{3.14}$$

Also, let us denote $m := \|\nabla \rho\|_\infty$. *Then the maximal operator* T_* *satisfies the estimate*

$$\|T_* f\|_{L^p(\mathbb{R}^{d-1})} \le C_m \|f\|_{L^p(\mathbb{R}^{d-1})}$$

for every $f \in L^p(\mathbb{R}^{d-1})$ *and for some constant* C_m *that tends to zero as* m *tends to zero.*

Proof. Let $f \in L^p(\mathbb{R}^{d-1})$. Let us use notations of Theorem 3.10. Then we may write

$$k(x,y) = \sum_{i=1}^{d} k_i(x,y) v_i(x).$$

Using the above identity, we estimate

$$T_*f(x) \leq \sum_{i=1}^{d} \|v_i\|_\infty T_{i,*}f(x).$$

Finally, we get

$$\|T_*f\|_{L^p(\mathbb{R}^{d-1})} \leq \sum_{i=1}^{d} \|v_i\|_\infty \|T_{i,*}f\|_{L^p(\mathbb{R}^{d-1})} \leq C_m \|f\|_{L^p(\mathbb{R}^{d-1})}.$$

Because C_m tends to zero as m tends to zero, the proof is completed. □

Chapter 4

Layer Potentials

In this chapter, we study properties of layer potentials for Laplace's equation. We first study single layer potentials and then we proceed to study double layer potentials. The reason why we are interested in layer potentials is that they are good candidates for being solutions to the boundary value problems for Laplace's equation: they are harmonic and they obey certain jump relations on the boundary.

4.1 Single layer potential

We begin this section by defining the single layer potential for Laplace's equation.

Definition 4.1. Let $p \in (1, \infty)$, $x \in \mathbb{R}^d \setminus \partial D$ and $w \in \partial D$. The single layer potential $\mathcal{S}$ with a density $f \in L^p(\partial D)$ is defined by

$$\mathcal{S}f(x) = \frac{1}{\omega_d(d-2)} \int_{\partial D} \frac{f(w)}{|x-w|^{d-2}}\, d\sigma(w), \quad \text{when } d \geq 3.$$

To simplify notations, we denote the kernel of the single layer potential by

$$\Phi(x, w) = \frac{1}{\omega_d(d-2)} \frac{1}{|x-w|^{d-2}}, \quad \text{when } d \geq 3.$$

In the special case $d = 2$, the single layer potential is defined by

$$\mathcal{S}f(x) = \frac{1}{2\pi} \int_{\partial D} \log(|x-w|) f(w)\, d\sigma(w).$$

Then we denote the kernel by

$$\Phi(x, w) = \frac{1}{2\pi} \log(|x-w|).$$

Example 4.1. Let us consider the single layer potential $\mathcal{S}f$ defined on a sphere in $\mathbb{R}^3$ with a radius $r > 0$ and with a constant density $f = \rho$. Then the single layer potential can be written in the form:

$$\mathcal{S}f(x) = \frac{\rho}{4\pi}\int_{\partial B(0,r)} \frac{d\sigma(w)}{|x-w|} = \frac{\rho}{4\pi}\int_0^{2\pi}\int_0^{\pi} \frac{r^2 \sin\theta \, d\theta \, d\vartheta}{(|x|^2 + r^2 - 2|x|r\cos\theta)^{1/2}}.$$

With simple calculations, we get

$$\mathcal{S}f(x) = \begin{cases} \rho r, & \text{if } |x| \le r \\ \rho r(r/|x|), & \text{if } |x| > r \end{cases}$$

and

$$\nabla\mathcal{S}f(x) = \begin{cases} 0, & \text{if } |x| < r \\ -\rho r^2(x/|x|^3), & \text{if } |x| > r. \end{cases}$$

Now, let us make some observations. First, we notice that the single layer potential is harmonic outside the boundary:

$$\Delta\mathcal{S}f(x) = \operatorname{div}(\nabla\mathcal{S}f(x)) = 0, \quad x \in \mathbb{R}^3 \setminus \partial B(0,r).$$

Second, the single layer potential is continuous across the boundary. Third, the single layer potential and its gradient decrease to zero outside the sphere. More precisely, we have

$$|\mathcal{S}f(x)| \le C|x|^{-1} \quad \text{and} \quad |\nabla\mathcal{S}f(x)| \le C|x|^{-2},$$

whenever $|x| \ge r$.

In general, the single layer potential satisfies the first two properties mentioned above. Furthermore, the third property can be generalized when $d \ge 3$. However, the third property does not hold when $d = 2$. In the following, we will show that these claims are true. We begin by proving that the single layer potential is harmonic.

Theorem 4.1. *Let $p \in (1,\infty)$. The single layer potential $\mathcal{S}$ with a density $f \in L^p(\partial D)$ is harmonic in $\mathbb{R}^d \setminus \partial D$.*

Proof. The structure of the proof is as follows: first we observe that the kernel of the single layer potential is harmonic. Second, we show that we can pass the differentiation inside the integral sign.

Let $x \in \mathbb{R}^d \setminus \partial D$ and $w \in \partial D$. By differentiation we observe that

$$\nabla_x\Phi(x,w) = -\frac{1}{\omega_d}\frac{x-w}{|x-w|^d}.$$

Consequently, we get

$$\begin{aligned}\Delta_x \Phi(x,w) &= \operatorname{div}\left(\nabla_x \Phi(x,w)\right)\\ &= -\frac{1}{\omega_d}\left(\left\langle x-w, \nabla_x\left(\frac{1}{|x-w|^d}\right)\right\rangle + \frac{\operatorname{div}(x-w)}{|x-w|^d}\right)\\ &= -\frac{1}{\omega_d}\left(\left\langle x-w, \frac{-d(x-w)}{|x-w|^{d+2}}\right\rangle + \frac{d}{|x-w|^d}\right)\\ &= 0.\end{aligned}$$

Because $\mathbb{R}^d \setminus \partial D$ is open, there is a radius $r > 0$ such that the ball $B(x,r)$ is contained in $\mathbb{R}^d \setminus \partial D$. Let $\{h_j\}_{j=1}^{\infty}$ be a sequence of real numbers that satisfies $|h_j| < r$. Then for all $j \in \mathbb{N}$, we have

$$\frac{\mathcal{S}f(x+h_je_i) - \mathcal{S}f(x)}{h_j} = \int_{\partial D}\left(\frac{\Phi(x+h_je_i,w) - \Phi(x,w)}{h_n}\right) f(w)\,d\sigma(w),$$

where $e_i \in \mathbb{R}^d$, $i = 1\ldots,d$, are elements of the standard basis of $\mathbb{R}^d$. The mean value theorem implies that for each $j \in \mathbb{N}$, there is a vector $\xi_j \in B(x,r)$ such that

$$\left|\frac{\Phi(x+h_je_i,w) - \Phi(x,w)}{h_j}\right| \leq \left|\frac{\partial \Phi(\xi_j,w)}{\partial x_i}\right|. \tag{4.1}$$

Furthermore, the function $(x,w) \mapsto \partial_{x_i}\Phi(x,w)$ is continuous on a compact set $\overline{B(x,r)} \times \partial D$, which together with the inequality (4.1) imply that

$$\left|\frac{\Phi(x+h_je_i,w) - \Phi(x,w)}{h_j}\right| |f(w)| \leq C|f(w)|. \tag{4.2}$$

The term on the right-hand side of the inequality (4.2) is integrable because ∂D is finite and $f \in L^p(\partial D)$. Now, due to the dominated convergence theorem, we can pass the limit inside the integral sign:

$$\frac{\partial}{\partial x_i}\mathcal{S}f(x) = \lim_{j\to\infty}\frac{\mathcal{S}f(x+h_je_i) - \mathcal{S}f(x)}{h_j} = \int_{\partial D}\frac{\partial \Phi(x,w)}{\partial x_i} f(w)\,d\sigma(w).$$

With a similar argument as above we see that

$$\frac{\partial^2}{\partial x_i^2}\mathcal{S}f(x) = \int_{\partial D}\frac{\partial^2 \Phi(x,w)}{\partial x_i^2} f(w)\,d\sigma(w).$$

Finally, by combining the above observations, we obtain

$$\Delta \mathcal{S}f(x) = \int_{\partial D}\Delta_x \Phi(x,y) f(y)\,d\sigma(y) = 0.$$

This completes the proof. □

The singularity inside the integral prevents us from defining the single layer potential on the boundary. However, we are able to define the boundary integral operator $S : L^p(\partial D) \to L^p(\partial D)$ in the principal value sense

$$Sf(z) = \text{p.v.} \int_{\partial D} \Phi(z,w) f(w)\, d\sigma(w).$$

We can define the above operator in the principal value sense because the kernel $\Phi(z,w)$ satisfies the conditions of Lemma 3.3. These conditions are satisfied because the kernel $\Phi(z,w)$ is weakly singular apart from the two dimensional case. The two dimensional case is justified in Lemma A.4.

The following theorem states that the boundary values of the single layer potential are obtained continuously from the outside values by a non-tangential approach.

Theorem 4.2. *Suppose that D is a C^1-domain in $\mathbb{R}^d$ and $\{\Gamma_\alpha\}$ is a family of cones for ∂D with a fixed aperture $\alpha \in (0,1)$. Also, let $p \in (1,\infty)$ and $f \in L^p(\partial D)$. Then the single layer potential satisfies*

$$\lim_{\substack{x \to z \\ x \in \Gamma_\alpha(z)}} \mathcal{S}f(x) = Sf(z)$$

for almost every $z \in \partial D$.

Proof. First, let us assume that $d \geq 3$. Let $\{x_j\}_{j=1}^\infty$ be a sequence that converges to $z \in \partial D$ and whose elements belong to $\Gamma_\alpha(z)$. If $j \in \mathbb{N}$ is large enough and if $z \neq w$, then we may use Lemma 2.3 to estimate

$$|\Phi(x_j,w) f(w)| \leq \frac{C|f(w)|}{|x_j - w|^{d-2}} \leq \frac{C|f(w)|}{|z-w|^{d-2}} = C|\Phi(z,w) f(w)|.$$

According to the proof of Theorem 3.2, the function $\Phi(z,\cdot) f$ is integrable for almost every $z \in \partial D$. Therefore, by the dominated convergence theorem, we have

$$\begin{aligned}
\lim_{j\to\infty} \mathcal{S}f(x_j) &= \lim_{j\to\infty} \int_{\partial D} \Phi(x_j,w) f(w)\, d\sigma(w) \\
&= \int_{\partial D} \lim_{j\to\infty} \Phi(x_j,w) f(w)\, d\sigma(w) \\
&= \int_{\partial D} \Phi(z,w) f(w)\, d\sigma(w) \\
&= Sf(z)
\end{aligned}$$

for almost every $z \in \partial D$.

Second, let us assume that $d = 2$. We write

$$\Phi(x_j, w) = \Phi(x_j, w)\mathbf{1}_{\{|x_j - w| > 1\}}(w) + \Phi(x_j, w)\mathbf{1}_{\{|x_j - w| < 1\}}(w).$$

Then we apply Lemma 2.3 to estimate

$$|\Phi(x_j, w)\mathbf{1}_{\{|x_j - w| < 1\}}(w)f(w)| \leq C(|\log(|z - w|)| + 1)|f(w)|.$$

Furthermore, we have

$$|\Phi(x_j, w)\mathbf{1}_{\{|x_j - w| > 1\}}(w)f(w)| \leq C|f(w)|.$$

By combining the above estimates, we get

$$|\Phi(x_j, w)f(w)| \leq C(|\log(|z - w|)| + 1)|f(w)|.$$

With the help of Lemma A.4 and the arguments of Theorem 3.2, we can deduce that the right-hand side of the above estimate is integrable. Therefore, the dominated convergence theorem allows us to pass the limit inside the integral sign in a similar manner as before. Thus, the proof is complete. □

The rate of which the single layer potential and its gradient decreases outside the boundary is given in the following lemma. In Chapter 7, we will find this property useful when we consider the invertibility of certain integral operators.

Lemma 4.1. *Let $p \in (1, \infty)$, $f \in L^p(\partial D)$ and $d \geq 3$. There is a radius $R > 0$ and a constant $C > 0$ such that the single layer potential $\mathcal{S}$ satisfies*

$$|\mathcal{S}f(x)| \leq C|x|^{2-d} \quad \textit{and} \quad |\nabla \mathcal{S}f(x)| \leq C|x|^{1-d},$$

whenever $|x| \geq R$.

Proof. There exists $R > 0$ such that $D \subset B(\bar{0}, R)$. Suppose that $|x| > 2R$ and $w \in \partial D$. Then we may deduce

$$\frac{1}{2}|x| \leq |x| - R \leq |x - w| + (|w| - R) \leq |x - w|,$$

which implies that

$$|\Phi(x, w)| \leq C|x|^{2-d} \quad \text{and} \quad |\nabla \Phi(x, w)| \leq C|x|^{1-d}.$$

Finally, using the above estimates, we see that

$$|\mathcal{S}f(x)| = \int_{\partial D} |\Phi(x, w)||f(w)|\, d\sigma(w) \leq \frac{C\|f\|_{L^p(\partial D)}}{|x|^{d-2}}$$

and

$$|\nabla \mathcal{S}f(x)| = \int_{\partial D} |\nabla \Phi(x, w)||f(w)|\, d\sigma(w) \leq \frac{C\|f\|_{L^p(\partial D)}}{|x|^{d-1}}.$$

This completes the proof. □

Notice that the two dimensional case was not included in the above lemma. However, it is possible to show that the single layer potential vanishes at infinity also in the two dimensional case for certain densities. For details, see Lemma A.5.

We end this section by considering the normal derivative of the single layer potential.

Definition 4.2. Let D be a C^1-domain, let $\{\Gamma_\alpha\}$ be a family of cones with a fixed aperture $\alpha \in (0,1)$ and let $\nu(z)$ be the inward-pointing unit normal vector of a domain D at a point $z \in \partial D$. The normal derivative of the single layer potential is defined by

$$\frac{\partial}{\partial \nu^i}\mathcal{S}f(z) := \lim_{\substack{x \to z \\ x \in \Gamma^i_\alpha(z)}} \langle \nu(z), \nabla \mathcal{S}f(x)\rangle$$

on the condition that the limit exists.

Remark 4.1. It is also possible to define the normal derivative of the single layer potential by approaching the boundary ∂D from the exterior of the domain D. In such a situation, we denote

$$\frac{\partial}{\partial \nu^e}\mathcal{S}f(z) := \lim_{\substack{x \to z \\ x \in \Gamma^e_\alpha(z)}} \langle \nu(z), \nabla \mathcal{S}f(x)\rangle$$

and we assume that $\nu(z)$ are inward-pointing unit normal vectors with respect to the exterior of the domain D.

The existence of the above limits is not a trivial matter, because by arguing as in Theorem 4.1, we observe

$$\langle \nu(z), \nabla \mathcal{S}f(x)\rangle = \frac{1}{\omega_d}\int_{\partial D} \frac{\langle \nu(z), w - x\rangle}{|w-x|^d} f(w)\, d\sigma(w).$$

Due to strict singularity inside the integral, we cannot approach the boundary ∂D without further considerations. It turns out that the normal derivative of the single layer potential exists and it is related through a certain jump relation to the boundary integral operator

$$K^* f(z) = \text{p.v.}\frac{1}{\omega_d}\int_{\partial D} \frac{\langle \nu(z), w - z\rangle}{|w-z|^d} f(w)\, d\sigma(w), \quad f \in L^p(\partial D).$$

The jump relations will be established in Chapter 6, where we study boundary values of the layer potentials more closely. Nevertheless, for now, we may consider a jump relation in a simple example.

Example 4.2. Suppose that $\mathcal{S}f$ is the single layer potential defined on sphere in $\mathbb{R}^3$ with a radius $r > 0$ and with a constant density $f = \rho$. Also, suppose that $z \in \partial B(0, r)$. Then, by using the calculations of the previous example, we may deduce that

$$\lim_{\substack{x \to z \\ x \in \Gamma_\alpha^i(z)}} \langle \nu(z), \nabla \mathcal{S}f(x) \rangle = 0 \quad \text{and} \quad \lim_{\substack{x \to z \\ x \in \Gamma_\alpha^e(z)}} \langle \nu(z), \nabla \mathcal{S}f(x) \rangle = \rho.$$

Furthermore, we may calculate

$$K^* f(z) = \frac{\rho}{4\pi} \int_{\partial B(0,r)} \frac{\langle \nu(z), w - z \rangle}{|w - z|^3} \, d\sigma(w) = \frac{\rho}{8\pi r} \int_{\partial B(0,r)} \frac{d\sigma(w)}{|w - z|} = \frac{1}{2}\rho.$$

By combining the above observations, we get the jump relations

$$\frac{\partial}{\partial \nu^i} \mathcal{S}f(z) = -(\tfrac{1}{2}I - K^*)f(z) \quad \text{and} \quad \frac{\partial}{\partial \nu^e} \mathcal{S}f(z) = -(\tfrac{1}{2}I + K^*)f(z).$$

For the jump relations in the general case, see Theorem 6.6.

4.2 Double layer potential

We begin this section by defining the double layer potential for Laplace's equation.

Definition 4.3. Let $x \in \mathbb{R}^d \setminus \partial D$ and $w \in \partial D$. The double layer potential $\mathcal{K}$ is defined by

$$(\mathcal{K}f)(x) = \frac{1}{\omega_d} \int_{\partial D} \frac{\langle \nu(w), x - w \rangle}{|x - w|^d} f(w) \, d\sigma(w).$$

To simplify notations, we denote the kernel of the double layer potential by

$$\Psi(x, w) = \frac{1}{\omega_d} \frac{\langle \nu(w), x - w \rangle}{|w - x|^d}.$$

In Chapter 7, we will look for a solution to the Dirichlet problem as a double layer potential. This is possible because the double layer potential is harmonic function.

Theorem 4.3. *Let $p \in (1, \infty)$. The double layer potential $\mathcal{K}$ with a density $f \in L^p(\partial D)$ is harmonic in $\mathbb{R}^d \setminus \partial D$.*

Proof. Let $x \in \mathbb{R}^d \setminus \partial D$ and $w \in \partial D$. By differentiation, we get

$$\nabla_x \Psi(x,w) = \frac{1}{\omega_d} \frac{\nu(w)}{|x-w|^d} - d \frac{x-w}{|x-w|^2} \Psi(x,w).$$

Then, with simple but somewhat laborious calculations, we obtain

$$\begin{aligned}\Delta_x \Psi(x,w) &= \operatorname{div}(\nabla_x \Psi(x,w)) \\ &= \frac{-d}{|x-w|^2} \Psi(x,w) - d \left(\frac{d-2}{|x-w|^2} \Psi(x,w) + \frac{1-d}{|x-w|^2} \Psi(x,w) \right) \\ &= 0.\end{aligned}$$

Because the kernel $\Psi(\cdot, w)$ of the double layer potential is harmonic and because the arguments of Theorem 4.1 allow us to pass the differentiation inside the integral, we see that

$$\Delta \mathcal{K} f(x) = \int_{\partial D} \Delta_x \Psi(x,w) f(w) \, d\sigma(w) = 0.$$

This completes the proof. □

In the previous section, we found out that a strict singularity made it problematic to define the normal derivative of the single layer potential on the boundary ∂D. The same applies to the boundary values of the double layer potential, because the kernel of the double layer potential is also strictly singular. However, we can define a boundary integral operator

$$Kf(z) = \text{p.v.} \frac{1}{\omega_d} \int_{\partial D} \frac{\langle \nu(w), z-w \rangle}{|z-w|^d} f(w) \, d\sigma(w), \quad f \in L^p(\partial D),$$

in the principal value sense (see Chapter 3). In chapter 5, the boundary integral operator K turns out to be well-defined and in Chapter 6, we will find a connection between K and the boundary values of the double layer potential. To establish this connection, we need the following lemma.

Lemma 4.2. *The kernel of the double layer potential satisfies the property*

$$\lim_{r \to 0} \int_{|x-w|>r} \Psi(x,w) \, d\sigma(w) = \begin{cases} 1, & \text{if } x \in D \\ 1/2, & \text{if } x \in \partial D \\ 0, & \text{if } x \in \mathbb{R}^d \setminus \overline{D}. \end{cases}$$

Proof. First, suppose that $x \in \mathbb{R}^d \setminus \overline{D}$. Similar arguments as in Theorem 4.1 imply that the function $y \mapsto \Phi(x,y)$ is harmonic in D. Therefore, the first Green's formula implies

$$\int_{\partial D} \Psi(x,w) \, d\sigma(w) = \int_{\partial D} \frac{\partial \Phi(x,w)}{\partial \nu(w)} \, d\sigma(w) = - \int_D \Delta_y \Phi(x,y) \, dy = 0.$$

Second, suppose that $x \in D$ and let us take a radius $r > 0$ such that $B(x,r)$ is contained in D. Let us denote $D_r = D \setminus B(x,r)$. The function $y \mapsto \Phi(x,y)$ is harmonic in D_r. Again, the first Green's formula implies

$$0 = \int_{\partial D_r} \Psi(x,w)\, d\sigma(w) = \int_{\partial D} \Psi(x,w)\, d\sigma(w) + \int_{\partial B(x,r)} \Psi(x,w)\, d\sigma(w).$$

If $w \in \partial B(x,r)$, then $\nu(z)$ is an outward pointing unit normal vector and

$$\Psi(x,w) = \frac{1}{\omega_d} \frac{\langle \nu(w), x-w \rangle}{|x-w|^d} = -\frac{1}{\omega_d r^{d-1}}.$$

Thus, we have

$$\int_{\partial D} \Psi(x,w)\, d\sigma(w) = 1.$$

Third, suppose $x \in \partial D$. Let us write

$$\int_{|x-w|>r} \Psi(x,w)\, d\sigma(w) = \int_{\partial D_r} \Psi(x,w)\, d\sigma(w) - \int_{\partial B(x,r)\cap D} \Psi(x,w)\, d\sigma(w).$$

Once again, the first Green's formula implies

$$\int_{D_r} \Psi(x,w)\, d\sigma(w) = 0.$$

Finally, we use Lemma 2.4 to obtain

$$\lim_{r\to 0} \int_{|x-w|>r} \Psi(x,w)\, d\sigma(w) = \lim_{r\to 0} \frac{\sigma(\partial B(x,r)\cap D)}{\sigma(\partial B(x,r))} = \frac{1}{2}.$$

This completes the proof. □

Chapter 5

Boundary Integral Operators

In this chapter, our goal is to establish certain boundedness, existence and compactness properties for the boundary integral operators K and K^* which are formally defined by

$$Kf(z) = \text{p.v.}\frac{1}{\omega_d}\int_{\partial D}\frac{\langle \nu(w), z-w\rangle}{|z-w|^d}f(w)\,d\sigma(w), \quad z \in \partial D$$

and

$$K^*f(z) = \text{p.v.}\frac{1}{\omega_d}\int_{\partial D}\frac{\langle \nu(z), w-z\rangle}{|w-z|^d}f(w)\,d\sigma(w), \quad z \in \partial D.$$

To be precise, we will show that the corresponding maximal operators K_* and $(K^*)_*$ are bounded on $L^p(\partial D)$. Then, by applying this observation, we show that K and K^* exist in $L^p(\partial D)$ and pointwise for almost everywhere on ∂D. Finally, we show that the operators K and K^* are compact on $L^p(\partial D)$. Throughout this chapter, we assume that D is a C^1-domain and $p \in (1, \infty)$.

This chapter is divided into two sections. In section 5.1, we consider the operators K and K^* in a local setting. This means that we define and study the corresponding operators in local coordinates. In fact, we will prove that such operators satisfy the same boundedness, existence and compactness properties that we wish to prove for the operators K and K^*. Because the locally defined operators are mappings from $L^p(\mathbb{R}^{d-1})$ to itself, we may apply results obtained for the multi-dimensional singular integral operators.

In Section 5.2, we pass the properties of the locally defined operators to the boundary integral operators K and K^*. Furthermore, we finally show that K and K^* are adjoints and we state a further result concerning compactness of the operator K on a Sobolev-type space $L_1^p(\partial D)$.

5.1 Operators in local coordinates

In this section, we study operators that are closely related to the boundary integral operators K and K^*. Before getting into details, let us take a closer look at truncated operators K_ε and K_ε^* in local coordinates.

Because D is a C^1-domain we may take a ball B such that

$$B \cap D = B \cap \{(x,t) : x \in \mathbb{R}^{d-1},\, t > \varphi(x)\}$$

for some function $\varphi \in C_0^1(\mathbb{R}^{d-1})$. If we take two points $z, w \in B \cap \partial D$ for which $z \neq w$, then in the local coordinates $x, y \in \mathbb{R}^{d-1}$, we have representations $z = (x, \varphi(x))$ and $w = (y, \varphi(y))$. Now we are able to represent the kernels of the boundary integral operators K and K^* in the local coordinates as follows:

$$\Psi(z,w) = \frac{1}{\omega_d} \frac{1}{\sqrt{1 + |\nabla\varphi(y)|^2}} \frac{\varphi(x) - \varphi(y) - \langle \nabla\varphi(y), x - y\rangle}{[|x-y|^2 + (\varphi(x) - \varphi(y))^2]^{d/2}}$$

and

$$\Psi(w,z) = \frac{1}{\omega_d} \frac{1}{\sqrt{1 + |\nabla\varphi(x)|^2}} \frac{\varphi(y) - \varphi(x) - \langle \nabla\varphi(x), y - x\rangle}{[|y-x|^2 + (\varphi(y) - \varphi(x))^2]^{d/2}}.$$

These representations give us a reason to define kernels

$$k(x,y) := \frac{\varphi(x) - \varphi(y) - \langle \nabla\varphi(y), x - y\rangle}{[|x-y|^2 + (\varphi(x) - \varphi(y))^2]^{d/2}} \tag{5.1}$$

and

$$k^*(x,y) = \frac{\varphi(y) - \varphi(x) - \langle \nabla\varphi(x), y - x\rangle}{[|y-x|^2 + (\varphi(y) - \varphi(x))^2]^{d/2}}. \tag{5.2}$$

Moreover, we define a set

$$\mathcal{E}(x,\varepsilon) = \{y \in \mathbb{R}^{d-1} : |x-y|^2 + (\varphi(x) - \varphi(y))^2 > \varepsilon^2\}.$$

If the function $f \in L^p(\partial D)$ were compactly supported in $B \cap \partial D$, then we could represent the boundary integral operator K in the local coordinates:

$$Kf(x, \varphi(x)) = \frac{1}{\omega_d} \int_{\mathbb{R}^{d-1}} k(x,y) \mathbf{1}_{\mathcal{E}(x,\varepsilon)}(y) f(y, \varphi(y))\, dy.$$

By denoting

$$g(y) := f(y, \varphi(y))(1 + |\nabla\varphi(y)|^2)^{1/2},$$

we would obtain a representation also for the boundary integral operator K^* in the local coordinates:

$$K^* f(x, \varphi(x)) = \omega_d^{-1}(1 + |\nabla\varphi(x)|^2)^{-1/2} \int_{\mathbb{R}^{d-1}} k^*(x, y)\mathbf{1}_{\mathcal{E}(x,\varepsilon)}(y)g(y)\, dy.$$

The local representations for the boundary integral operators K and K^* hint that to understand K and K^*, we have to study operators $\widetilde{K}$ and $\widetilde{K}^*$ determined by the truncations

$$\widetilde{K}_\varepsilon f(x) := \int_{\mathbb{R}^{d-1}} k(x, y)\mathbf{1}_{\mathcal{E}(x,\varepsilon)}(y)f(y)\, dy$$

and

$$\widetilde{K}^*_\varepsilon f(x) := \int_{\mathbb{R}^{d-1}} k^*(x, y)\mathbf{1}_{\mathcal{E}(x,\varepsilon)}(y)f(y)\, dy.$$

In fact, we will show that the operators $\widetilde{K}$ and $\widetilde{K}^*$ satisfy similar boundedness, existence and compactness properties that we would like to establish for the boundary integral operators K and K^*. In the next section, we will show how these properties can be passed from the local setting into global setting.

To study the operators $\widetilde{K}$ and $\widetilde{K}^*$, we introduce some notations. Suppose $\rho, \varphi \in C^1_0(\mathbb{R}^{d-1})$ and let us denote $\widetilde{m} := \|\nabla\varphi\|_\infty$ and $m := \|\nabla\rho\|_\infty$. Also, assume $x \in \mathbb{R}^{d-1}$, $\varepsilon > 0$ and let us define sets

$$\mathcal{F}(x, \varepsilon) = \{y \in \mathbb{R}^{d-1} : |x - y| > \varepsilon\}$$

and

$$\mathcal{F}_{\widetilde{m}}(x, \varepsilon) := \{y \in \mathbb{R}^{d-1} : |x - y| > \varepsilon/\sqrt{1 + \widetilde{m}^2}\}.$$

We will be interested in operators having kernels of the type

$$\tau(x, y) := \frac{\rho(x) - \rho(y) - \langle\nabla\rho(y), x - y\rangle}{[|x - y|^2 + (\varphi(x) - \varphi(y))^2]^{d/2}} =: \tau^*(y, x). \tag{5.3}$$

Especially, due to Theorems 3.10 and 3.11, we already know that the maximal operators $\widetilde{T}_*$ and $(\widetilde{T}^*)_*$ determined by truncations

$$\widetilde{T}_\varepsilon f(x) = \int_{\mathbb{R}^{d-1}} \tau(x, y)\mathbf{1}_{\mathcal{F}(x,\varepsilon)}(y)f(y)\, dy \tag{5.4}$$

and

$$\widetilde{T}^*_\varepsilon f(x) = \int_{\mathbb{R}^{d-1}} \tau^*(x, y)\mathbf{1}_{\mathcal{F}(x,\varepsilon)}(y)f(y)\, dy, \tag{5.5}$$

are bounded on $L^p(\mathbb{R}^{d-1})$. We even know that the operator norms of $\widetilde{T}_*$ and $(\widetilde{T}^*)_*$ depend on m such that the operator norm tends to zero as m tends to zero. We formulate this observation below.

Lemma 5.1. *Suppose that the maximal operator $\widetilde{T}_*$ is determined by either* (5.4) *or* (5.5). *Then*

$$\|\widetilde{T}_* f\|_{L^p(\mathbb{R}^{d-1})} \leq C_m \|f\|_{L^p(\mathbb{R}^{d-1})}$$

for every $f \in L^p(\mathbb{R}^{d-1})$ and for a constant $C_m > 0$ that tends to zero as m tends to zero.

We would like to prove a similar boundedness result concerning maximal operators T_* and $(T^*)_*$ determined by truncations

$$T_\varepsilon f(x) = \int_{\mathbb{R}^{d-1}} \tau(x,y) \mathbf{1}_{\mathcal{E}(x,\varepsilon)}(y) f(y)\, dy \tag{5.6}$$

and

$$T_\varepsilon^* f(x) = \int_{\mathbb{R}^{d-1}} \tau^*(x,y) \mathbf{1}_{\mathcal{E}(x,\varepsilon)}(y) f(y)\, dy. \tag{5.7}$$

This is done in the following lemma.

Lemma 5.2. *Suppose that the maximal operator T_* is determined by either* (5.6) *or* (5.7). *Then*

$$\|T_* f\|_{L^p(\mathbb{R}^{d-1})} \leq C_m \|f\|_{L^p(\mathbb{R}^{d-1})}$$

for every $f \in L^p(\mathbb{R}^{d-1})$ and for a constant $C_m > 0$ that tends to zero as m tends to zero.

Proof. Suppose that $f \in L^p(\mathbb{R}^{d-1})$. For notational convinience, let us denote $\mathcal{E} := \mathcal{E}(x,\varepsilon)$, $\mathcal{F} := \mathcal{F}(x,\varepsilon)$ and $\mathcal{F}_{\widetilde{m}} := \mathcal{F}_{\widetilde{m}}(x,\varepsilon)$. If we take $y \in \mathcal{E}$, then we may estimate

$$\varepsilon^2 < |x-y|^2 + (\varphi(x) - \varphi(y))^2 \leq (1 + \widetilde{m}^2)|x-y|^2,$$

which implies $\mathcal{E} \subset \mathcal{F}_{\widetilde{m}}$. This allows us to write

$$\mathcal{E} = (\mathcal{F}_{\widetilde{m}} \cap \mathcal{F}_{\widetilde{m}}^c) \cup (\mathcal{F}_{\widetilde{m}} \cap (\mathcal{E}^c)^c) = \mathcal{F}_{\widetilde{m}} \setminus (\mathcal{F}_{\widetilde{m}} \cap \mathcal{E}^c).$$

We use the triangle inequality to obtain

$$T_* f(x) \leq \sup_{\varepsilon>0} \left| \int_{\mathcal{F}} \tau(x,y) f(y)\, dy \right| + \sup_{\varepsilon>0} \left| \int_{\mathcal{F}_{\widetilde{m}} \cap \mathcal{E}^c} \tau(x,y) f(y)\, dy \right|.$$

The first term on the right-hand side of the above inequality equals to $\widetilde{T}_* f(x)$. Furthermore, we may estimate

$$\left| \int_{\mathcal{F}_{\widetilde{m}} \cap \mathcal{E}^c} \tau(x,y) f(y)\, dy \right| \leq 2m \left(\frac{1}{\varepsilon^{d-1}} \int_{B(x,\varepsilon)} |f(y)|\, dy \right).$$

As a result, we have

$$T_*f(x) \le \widetilde{T}_*f(x) + Cm\mathcal{M}f(x),$$

where $\mathcal{M}$ is the Hardy-Littlewood maximal operator. Therefore, Minkowski's inequality, the boundedness of the Hardy-Littlewood maximal operator and Lemma 5.1 imply that

$$\|T_*f\|_{L^p(\mathbb{R}^{d-1})} \le \|\widetilde{T}_*f\|_{L^p(\mathbb{R}^{d-1})} + Cm\|\mathcal{M}f\|_{L^p(\mathbb{R}^{d-1})} \le C_m\|f\|_{L^p(\mathbb{R}^{d-1})},$$

where the constant $C_m > 0$ tends to zero as m tends to zero. □

From the above lemma, we see that the maximal operators $\widetilde{K}_*$ and $(\widetilde{K}^*)_*$ are bounded on $L^p(\mathbb{R}^{d-1})$. Still, we wish to establish existence and compactness properties for the operators $\widetilde{K}$ and $\widetilde{K}^*$. To succeed in this, we formulate and prove the following lemma.

Lemma 5.3. *Let G be a domain in $\mathbb{R}^{d-1}$. Suppose that T is a singular integral operator and $\{T_j\}_{j=1}^{\infty}$ is a sequence of singular integral operators that satisfies the following conditions:*

(i) *Operators T_j, $j \in \mathbb{N}$, exist pointwise for almost everywhere in G.*

(ii) *There is a constant $C_j > 0$ such that*

$$\|(T - T_j)_*f\|_{L^p(G)} \le C_j\|f\|_{L^p(G)}$$

for every $f \in L^p(G)$ and the constant C_j tends to zero as j tends to infinity.

Then the operator T exists in $L^p(G)$ and pointwise for almost everywhere in G. Furthermore, if operators T_j, $j \in \mathbb{N}$, are compact on $L^p(G)$, then the operator T is also compact on $L^p(G)$.

Proof. Suppose that $x \in G$ and $f \in L^p(G)$. According to assumption (i), we have

$$\Lambda(T_jf)(x) := \limsup_{\varepsilon\to 0} T_{j,\varepsilon}f(x) - \liminf_{\varepsilon\to 0} T_{j,\varepsilon}f(x) = 0$$

for almost every $x \in G$. Then, we estimate

$$\Lambda(Tf)(x) = \Lambda((T - T_j)f)(x) \le 2(T - T_j)_*f(x).$$

Let $\lambda > 0$. Then, by arguing as in the proof of Theorem 3.1, we observe that

$$|\{x \in G : \Lambda(Tf)(x) > \lambda\}| \le C_j\|f\|^p_{L^p(G)}.$$

Letting j tend to infinity, we get

$$\Lambda(Tf)(x) = 0$$

for almost every $x \in G$, which implies that the operator T converges pointwise for almost everywhere in G. Moreover, Lemma 3.2 implies that the operator T exists in $L^p(G)$.

Finally, assume that $\{T_j\}_{j=1}^{\infty}$ is a sequence of compact operators in $L^p(G)$. We estimate

$$\|(T - T_j)f\|_{L^p(G)} \leq \|(T - T_j)_* f\|_{L^p(G)} \leq C_j \|f\|_{L^p(G)}.$$

Consequently, we have

$$\|T - T_j\|_{L^p(G)} \leq C_j \to 0,$$

as j tends to infinity. Now, Theorem 2.7 implies that the operator T is compact on $L^p(G)$. □

According to the above lemma, to prove existence and compactness properties for the operators $\widetilde{K}$ and $\widetilde{K}^*$, we have to construct sequences of compact operators that converge respectively to $\widetilde{K}$ and $\widetilde{K}^*$ in the operator norm. To construct such sequences, we introduce a lemma proven in Appendix.

Lemma 5.4. *Let $\varphi : \mathbb{R}^{d-1} \to \mathbb{R}$ be a continuously differentiable function with a compact support. Then there are functions $\psi_j \in C_0^{\infty}(\mathbb{R}^{d-1})$, $j \in \mathbb{N}$, such that the sequence $\{\psi_j\}_{j=1}^{\infty}$ converges uniformly to φ and the sequence $\{\nabla\psi_j\}_{j=1}^{\infty}$ converges uniformly to $\nabla\varphi$.*

Remark 5.1. If the function φ is merely Lipschitz, then the above Lemma does not hold. For this reason we fail to prove the compactness of the boundary integral operators on Lipschitz domains that are not C^1-domains. To see that the above lemma does not hold for purely Lipschitz functions, consider a function that is defined locally by $\varphi(x) = |x|$.

Now, suppose $\{\psi_j\}_{j=1}^{\infty}$ is as in Lemma 5.4, fix $j \in \mathbb{N}$ and define kernels

$$k_j(x, y) := \frac{\psi_j(x) - \psi_j(y) - \langle \nabla\psi_j(y), x - y\rangle}{[|x - y|^2 + (\varphi(x) - \varphi(y))^2]^{d/2}} =: k_j^*(y, x).$$

The kernels k_j and k_j^* are weakly singular (see Appendix). Therefore, we know that the operators

$$\widetilde{K}_j f(x) = \lim_{\varepsilon \to 0} \int_G k_j(x, y) \mathbf{1}_{\mathcal{E}(x,\varepsilon)}(y) f(y)\, dy$$

and

$$\widetilde{K}_j^* f(x) = \lim_{\varepsilon \to 0} \int_G k_j^*(x,y) \mathbf{1}_{\mathcal{E}(x,\varepsilon)}(y) f(y)\, dy$$

exist pointwise for almost everywhere in a bounded domain $G \subset \mathbb{R}^{d-1}$ and they are compact on $L^p(\overline{G})$. By denoting $\rho := \varphi - \psi_j$ and recalling kernels τ and τ^* that were defined by (5.3), we obtain

$$(\widetilde{K}_\varepsilon - \widetilde{K}_{j,\varepsilon}) f(x) = \int_G \tau(x,y) \mathbf{1}_{\mathcal{E}(x,\varepsilon)}(y) f(y)\, dy$$

and

$$(\widetilde{K}_\varepsilon^* - \widetilde{K}_{j,\varepsilon}^*) f(x) = \int_G \tau^*(x,y) \mathbf{1}_{\mathcal{E}(x,\varepsilon)}(y) f(y)\, dy.$$

Then Lemma 5.2 implies

$$\|(\widetilde{K} - \widetilde{K}_j)_* f\|_{L^p(G)} \le \|(\widetilde{K} - \widetilde{K}_j)_* f\|_{L^p(\mathbb{R}^{d-1})} \le C_j \|f\|_{L^p(G)}$$

and

$$\|(\widetilde{K}^* - \widetilde{K}_j^*)_* f\|_{L^p(G)} \le \|(\widetilde{K}^* - \widetilde{K}_j^*)_* f\|_{L^p(\mathbb{R}^{d-1})} \le C_j \|f\|_{L^p(G)}$$

for every $f \in L^p(G)$ and the constant $C_j > 0$ tends to zero as j tends to infinity. Thus, we are ready to state the main results of this section.

Theorem 5.1. *Let G be a bounded domain in $\mathbb{R}^{d-1}$ and let $p \in (1, \infty)$. Then the operator $\widetilde{K} : L^p(G) \to L^p(G)$, determined by truncations*

$$\widetilde{K}_\varepsilon f(x) := \int_G k(x,y) \mathbf{1}_{\mathcal{E}(x,\varepsilon)}(y) f(y)\, dy, \quad \varepsilon > 0,$$

where the kernel k is defined by (5.1), satisfies the following conditions:

(i) *The maximal operator $\widetilde{K}_*$ is bounded on $L^p(G)$.*

(ii) *The operator $\widetilde{K}$ exists in $L^p(G)$ and pointwise for a.e. in G.*

(iii) *The operator $\widetilde{K}$ is compact on $L^p(\overline{G})$.*

Theorem 5.2. *Let G be a bounded domain in $\mathbb{R}^{d-1}$ and let $p \in (1, \infty)$. Then the operator $\widetilde{K}^* : L^p(G) \to L^p(G)$, determined by truncations*

$$\widetilde{K}_\varepsilon^* f(x) := \int_G k^*(x,y) \mathbf{1}_{\mathcal{E}(x,\varepsilon)}(y) f(y)\, dy, \quad \varepsilon > 0,$$

where the kernel k^* is defined by (5.2), satisfies the following conditions:

(i) *The maximal operator $(\widetilde{K}^*)_*$ is bounded on $L^p(G)$.*

(ii) *The operator $\widetilde{K}^*$ exists in $L^p(G)$ and pointwise for a.e. in G.*

(iii) *The operator $\widetilde{K}^*$ is compact on $L^p(\overline{G})$.*

5.2 Properties of boundary integral operators

In this section, we show that the boundary integral operators K and K^* satisfy boundedness, existence and compactness properties. To do this, we formulate and prove a lemma that allows us to pass the properties of the operators $\widetilde{K}$ and $\widetilde{K}^*$ to the boundary integral operators K and K^*. Also, we will find out that K and K^* are adjoints and the operator K is compact on the Sobolev-type space $L_1^p(\partial D)$.

Before proceeding to the following lemma, the reader is advised to recall Definition 2.3.

Lemma 5.5. *Suppose that D is a C^1-domain, $p \in (1, \infty)$ and let T be a singular integral operator defined by*

$$Tf(z) = \int_{\partial D} \Theta(z, w) f(w)\, d\sigma(w), \quad z \in \partial D,\ f \in L^p(\partial D),$$

where the kernel Θ satisfies the estimate

$$|\Theta(z, w)| \leq C|z - w|^{-d} \tag{5.8}$$

for every $z, w \in \partial D$ for which $z \neq w$. Suppose that

$$\mathscr{F}_{\partial D} = (\{B_i\}, \{\varphi_i\}, \{p_i\}, \{U_i\})_{i=1}^n$$

is a family of local characteristics of the boundary ∂D and let $\mathcal{T}_i$, $i = 1, \ldots, n$, be singular integral operators defined by

$$\mathcal{T}_i g(x) = \int_{U_i} \Theta(p_i(x), p_i(y)) g(y)\, dy, \quad x \in U_i,\ g \in L^p(U_i).$$

Then the following statements hold:

(i) *If for each $i = 1, \ldots, n$, the maximal operator $\mathcal{T}_{i,*}$ satisfies the estimate*

$$\|\mathcal{T}_{i,*} g\|_{L^p(U_i)} \leq C\|g\|_{L^p(U_i)}$$

for every $g \in L^p(U_i)$ and for some constant $C > 0$, then the maximal operator T_* is bounded on $L^p(\partial D)$.

(ii) *If for each $i = 1, \ldots, n$, the operator $\mathcal{T}_i$ exists in $L^p(U_i)$ and pointwise for almost everywhere in U_i, then the operator T exists in $L^p(\partial D)$ and pointwise for almost everywhere on ∂D.*

(iii) *If for each $i = 1, \dots, n$, the operator $\mathcal{T}_i$ is compact on $L^p(\overline{U}_i)$, then the operator T is compact on $L^p(\partial D)$.*

Proof. First, let us make some observations that concern all the parts (i),(ii) and (iii). There is a partition of unity $\{\zeta_i\}_{i=1}^n$ subordinate to the cover $\{B_i\}_{i=1}^n$. Therefore, we may write

$$T_\varepsilon f(z) = \sum_{i=1}^n \int_{\partial D} \Theta_\varepsilon(z,w)(f\zeta_i)(w)\, d\sigma(w).$$

Let $(\{B\}, \{\varphi\}, \{\tilde{p}\}, \{U\})$ be one of the members of the family $\mathscr{F}_{\partial D}$. By observing that

$$\operatorname{supp}(f\zeta_i) \subset B_i \cap \partial D,$$

it suffices to assume f is compactly supported on $B \cap \partial D$ and $w \in \partial D \cap B$. Let us write

$$T_\varepsilon f(z) = T_\varepsilon f(z)\mathbf{1}_{B^c \cap \partial D}(z) + T_\varepsilon f(z)\mathbf{1}_{B \cap \partial D}(z) := T_{1,\varepsilon} f(z) + T_{2,\varepsilon} f(z)$$

If $z \in B^c \cap \partial D$, then there is a number $m > 0$ such that $|z - w| > m$ for every $w \in \operatorname{supp}(f)$. As a result,

$$|\Theta(z,w)\mathbf{1}_{B^c \cap \partial D}(z)| \le C < \infty,$$

which means that T_1 is a weakly singular integral operator. According to results in Section 3.2, we know that $T_{1,*}$ is bounded on $L^p(\partial D)$, T_1 exists in $L^p(\partial D)$ and pointwise for almost everywhere on ∂D. Furthermore, T_1 is compact on $L^p(\partial D)$. Therefore, it suffices to establish the corresponding properties for the operator T_2. This will be done below.

(i) If $z \in B \cap \partial D$, then we may write $z = \tilde{p}(x)$. Let us define a function

$$g(y) := (f \circ \tilde{p})(y)(1 + |\nabla\varphi(y)|^2)^{1/2}.$$

The function g satisfies an estimate

$$\begin{aligned} \|g\|_{L^p(U)}^p &\le C \int_U |(f \circ \tilde{p})(y)|^p (1 + |\nabla\varphi(y)|^2)^{1/2}\, dy \\ &= C \int_{B \cap \partial D} |f(w)|^p\, d\sigma(w) \\ &= C\|f\|_{L^p(\partial D)}^p. \end{aligned}$$

Also, we use the function g to write

$$\begin{aligned}
T_{2,\varepsilon}f(z) &= \int_{\partial D} \Theta_\varepsilon(z,w)f(w)\,d\sigma(w) \\
&= \int_U \Theta_\varepsilon(\tilde{p}(x),\tilde{p}(y))(f\circ\tilde{p})(y)(1+|\nabla\varphi(y)|^2)^{1/2}\,dy \\
&= \int_U \Theta_\varepsilon(\tilde{p}(x),\tilde{p}(y))g(y)\,dy \\
&= \mathcal{T}_\varepsilon(g\circ\tilde{p}^{-1})(z).
\end{aligned}$$

Then, we are able to estimate

$$\begin{aligned}
\|T_{2,*}f\|^p_{L^p(\partial D)} &= \int_{B\cap\partial D} |\mathcal{T}_*(g\circ\tilde{p}^{-1})(z)|^p\,d\sigma(z) \\
&= \int_U |\mathcal{T}_*g(x)|^p(1+|\nabla\varphi(x)|^2)^{1/2}\,dx \\
&\le C\|\mathcal{T}_*g\|^p_{L^p(U)} \\
&\le C\|g\|^p_{L^p(U)} \\
&\le C\|f\|^p_{L^p(\partial D)}.
\end{aligned}$$

Finally, by applying Minkowski's inequality, we get

$$\|T_*f\|_{L^p(\partial D)} \le \|T_{1,*}f\|_{L^p(\partial D)} + \|T_{2,*}f\|_{L^p(\partial D)} \le C\|f\|_{L^p(\partial D)}.$$

(ii) Using the notations of the part (i), we may write

$$T_{2,\varepsilon}f(z) = \begin{cases} \mathcal{T}_{2,\varepsilon}g(x), & \text{if } z\in B\cap\partial D \\ 0, & \text{if } z\in B^c\cap\partial D. \end{cases}$$

Because the operator $\mathcal{T}$ exists pointwise for almost everywhere on U, then the operator T_2 exists pointwise for almost everywhere on ∂D. According to part (i) and Lemma 3.2, the operator T_2 exists also in $L^p(\partial D)$. From these observations, we see that the operator T exists in $L^p(\partial D)$ and pointwise for almost everywhere in ∂D.

(iii) Suppose $\{f_j\}_{j=1}^\infty$ is a bounded sequence of functions in $L^p(\partial D)$ with compact support in $B\cap\partial D$. Using the notations of the part (i), we write

$$T_2f_j(z) = \begin{cases} \mathcal{T}g_j(x), & \text{if } z\in B\cap\partial D \\ 0, & \text{if } z\in B^c\cap\partial D. \end{cases}$$

We know that the operator $\mathcal{T}$ is compact on $L^p(\bar{U})$. Therefore, there is a subsequence $\{g_{j_m}\}_{m=1}^\infty$ and function $\tilde{h}$ in $L^p(\bar{U})$ such that

$$\|\mathcal{T}g_{j_m} - \tilde{h}\|_{L^2(\bar{U})} \to 0,$$

as m tends to infinity. Then we define a function

$$h(z) = \begin{cases} \tilde{h}(x), & \text{if } z \in B \cap \partial D \\ 0, & \text{if } z \in B^c \cap \partial D. \end{cases}$$

Finally, we see that there is a subsequence $\{f_{j_m}\}_{m=1}^{\infty}$ such that

$$\|T_2 f_{j_m} - h\|_{L^p(\partial D)} \to 0,$$

as m tends to infinity. We conclude that T_2 is compact on $L^p(\partial D)$ and as a result T is compact on $L^p(\partial D)$. □

The above lemma was formulated with the operators K and K^* in mind: the kernels of K and K^* satisfy the condition (5.8) and we see that

$$\begin{aligned} \Psi_\varepsilon(\tilde{p}(x), \tilde{p}(y)) &= \frac{\langle (\nu \circ \tilde{p})(y), \tilde{p}(x) - \tilde{p}(y) \rangle}{|\tilde{p}(x) - \tilde{p}(y)|^d} \mathbf{1}_{\{|\tilde{p}(x) - \tilde{p}(y)| > \varepsilon\}}(\tilde{p}(y)) \\ &= \frac{\varphi(x) - \varphi(y) - \langle \nabla \varphi(y), x - y \rangle}{[|x-y|^2 + (\varphi(x) - \varphi(y))^2]^{d/2}} \mathbf{1}_{\mathcal{E}(x,\varepsilon)}(y) \\ &= k(x,y) \mathbf{1}_{\mathcal{E}(x,\varepsilon)}(y). \end{aligned}$$

In a similar manner, we also see that

$$\Psi_\varepsilon(\tilde{p}(y), \tilde{p}(x)) = k^*(x,y) \mathbf{1}_{\mathcal{E}(x,\varepsilon)}(y).$$

According to Theorems 5.1 and 5.2, we know that the operators $\widetilde{K}$ and $\widetilde{K}^*$ determined by truncations

$$\widetilde{K}_\varepsilon g(x) = \int_U \Psi_\varepsilon(\tilde{p}(x), \tilde{p}(y)) g(y)\, dy$$

and

$$\widetilde{K}^*_\varepsilon g(x) = \int_U \Psi_\varepsilon(\tilde{p}(y), \tilde{p}(x)) g(y)\, dy$$

satisfy boundedness, existence and compactness properties. Thus, Lemma 5.5 allows us to state the main results of this chapter.

Theorem 5.3. *Let D be a C^1-domain and let $p \in (1, \infty)$. Then the boundary integral operator $K : L^p(\partial D) \to L^p(\partial D)$ defined by*

$$Kf(z) = \text{p.v.} \frac{1}{\omega_d} \int_{\partial D} \frac{\langle \nu(w), z - w \rangle}{|z-w|^d} f(w)\, d\sigma(w)$$

satisfies the following conditions:

(i) *The maximal operator* K_* *is bounded on* $L^p(\partial D)$.

(ii) *The operator* K *exists in* $L^p(\partial D)$ *and pointwise for a.e. on* ∂D.

(iii) *The operator* K *is compact on* $L^p(\partial D)$.

Theorem 5.4. *Let* D *be a* C^1*-domain and let* $p \in (1, \infty)$. *Then the boundary integral operator* $K^* : L^p(\partial D) \to L^p(\partial D)$ *defined by*

$$K^* f(z) = \text{p.v.} \frac{1}{\omega_d} \int_{\partial D} \frac{\langle \nu(z), w - z \rangle}{|w - z|^d} f(w)\, d\sigma(w)$$

satisfies the following conditions:

(i) *The maximal operator* $(K^*)_*$ *is bounded on* $L^p(\partial D)$.

(ii) *The operator* K^* *exists in* $L^p(\partial D)$ *and pointwise for a.e. on* ∂D.

(iii) *The operator* K^* *is compact on* $L^p(\partial D)$.

With the help of the above theorems it can be seen that the boundary integral operators K and K^* are adjoints. This is proven below.

Theorem 5.5. *Suppose that the numbers* $p, q \in (1, \infty)$ *are conjugate exponents. Then* $K : L^p(\partial D) \to L^p(\partial D)$ *and* $K^* : L^q(\partial D) \to L^q(\partial D)$ *are adjoint operators.*

Proof. Suppose $f \in L^p(\partial D)$ and $g \in L^q(\partial D)$. We use Fubini's theorem to see that

$$\begin{aligned}
\langle K_\varepsilon f \mid g \rangle &= \int_{\partial D} K_\varepsilon f(z) g(z)\, d\sigma(z) \\
&= \int_{\partial D} \left(\int_{\partial D} \Psi_\varepsilon(z, w) f(w)\, d\sigma(w) \right) g(z)\, d\sigma(z) \\
&= \int_{\partial D} f(w) \left(\int_{\partial D} \Psi_\varepsilon(z, w) g(z)\, d\sigma(z) \right) d\sigma(w) \\
&= \int_{\partial D} f(w) K_\varepsilon^* g(w)\, d\sigma(w) \\
&= \langle f \mid K_\varepsilon^* g \rangle.
\end{aligned}$$

Because the operator K is bounded on $L^p(\partial D)$, by applying Hölder's inequality, we get

$$\|(Kf)g\|_{L^1(\partial D)} \le \|Kf\|_{L^p(\partial D)} \|g\|_{L^q(\partial D)} \le C \|f\|_{L^p(\partial D)} \|g\|_{L^q(\partial D)} < \infty.$$

As a result, the dominated convergence implies that

$$\begin{aligned}\langle Kf \mid g\rangle &= \int_{\partial D} Kf(z)g(z)\,d\sigma(z)\\ &= \lim_{\varepsilon\to 0}\int_{\partial D} K_\varepsilon f(z)g(z)\,d\sigma(z)\\ &= \lim_{\varepsilon\to 0}\langle K_\varepsilon f \mid g\rangle.\end{aligned}$$

In a similar manner, we deduce that

$$\langle f \mid K^* g\rangle = \lim_{\varepsilon\to 0}\langle f \mid K_\varepsilon^* g\rangle.$$

Finally, by combining the above observations, we get

$$\langle Kf \mid g\rangle = \langle f \mid K^* g\rangle,$$

which completes the proof. □

In the end of this section, we state a result concerning the compactness of the boundary integral operator K on a Sobolev-type space $L_1^p(\partial D)$. We need this result in Chapter 7, where we consider the uniqueness of the Dirichlet problem.

Before we are able to state the result, we have to define the space $L_1^p(\partial D)$. The definition of $L_1^p(\partial D)$ requires a concept of weak derivatives.

Definition 5.1. Suppose that $U \subset \mathbb{R}^{d-1}$ is a domain and $p \in (1,\infty)$. A function $f \in L^p(U)$ has weak derivatives, if for each $j = 1,\dots,d-1$, there exists a function $g_j \in L^p(\partial D)$ such that the following property holds

$$\int_U \phi_{x_j} f\,dx = -\int_U \phi g_j\,dx$$

for every $\phi \in C_0^\infty(\mathbb{R}^{d-1})$.

The above definition does not allow us to define weak derivatives of functions that are defined on the boundary ∂D. However, if we take a cover $\{B_i\}_{i=1}^n$ for ∂D as described in Lemma 2.2 and a partition of unity $\{\zeta_i\}_{i=1}^n$ subordinate to the cover $\{B_i\}_{i=1}^n$, then for every $f \in L^p(\partial D)$, we can define functions

$$(\zeta f)_i(x) := \zeta_i(x,\varphi_i(x))f(x,\varphi_i(x)).$$

Then we may test, whether these functions have weak derivatives in $\mathbb{R}^{d-1}$.

Now, we define the space $L_1^p(\partial D)$ that turns out to be a Banach space [8].

Definition 5.2. Suppose that D is a C^1-domain, let $\{B_i\}_{i=1}^n$ be a finite cover of balls for ∂D, let $\{\zeta_i\}_{i=1}^n$ be a partition of unity subordinate to the cover $\{B_i\}_{i=1}^n$ and let $p \in (1, \infty)$. The space $L_1^p(\partial D)$ consists of functions f that belong to $L^p(\partial D)$ and for which the partial derivatives of $(\zeta f)_i$ exists in the weak sense and $\nabla(\zeta f)_i$ belongs to $L^p(\mathbb{R}^{d-1})$ for every $i = 1, \ldots, n$.

Theorem 5.6. *The space $L_1^p(\partial D)$ associated with a norm*

$$\|f\|_{L_1^p(\partial D)} = \|f\|_{L^p(\partial D)} + \sum_{i=1}^{n} \|\nabla(\widetilde{\zeta_i f})\|_{L^p(\mathbb{R}^{d-1})}$$

is a Banach space. Furthermore, the norms associated to the different coverings and partition of unities are equivalent.

Finally, we state the desired result. A proof can be found on [8].

Theorem 5.7. *Let D be a C^1-domain and let $p \in (1, \infty)$. The boundary integral operator $K : L_1^p(\partial D) \to L_1^p(\partial D)$,*

$$Kf(z) = \text{p.v.}\frac{1}{\omega_d} \int_{\partial D} \frac{\langle \nu(w), z - w \rangle}{|z - w|^d} f(w)\, d\sigma(w)$$

is well-defined and compact.

Chapter 6

Boundary Values of the Layer Potentials

In this chapter, we study boundary values of the layer potentials. To be precise, we study boundary values of the double layer potential and the normal derivative of the single layer potential. The boundary values of the single layer potential were already discussed in Chapter 4.

The reason why we discuss the double layer potential and the normal derivative of the single layer potential in the same context is that both of them satisfy certain jump relations. We are interested in these jump relations because they determine boundary values of the double layer potential and the normal derivative of the single layer potential. To be able to establish the jump relations, we need the concept of the non-tangential maximal function.

Definition 6.1. Suppose that D is a C^1-domain and $\{\Gamma_\alpha\}$ is a family of cones for ∂D with a fixed aperture $\alpha \in (0,1)$. Also, suppose that $\delta > 0$ and let u be a function defined in a domain D. The interior non-tangential maximal function of u is defined by

$$u_*^i(z) = \sup\{|u(x)| : x \in D,\, |x-z| < \delta,\, x \in \Gamma_\alpha^i(z)\,\}, \quad z \in \partial D.$$

Similarly, the exterior non-tangential maximal function of u is defined by

$$u_*^e(z) = \sup\{|u(x)| : x \in \mathbb{R}^d \setminus \overline{D},\, |x-z| < \delta,\, x \in \Gamma_\alpha^e(z)\,\}, \quad z \in \partial D.$$

The boundedness of the non-tangential maximal functions for the double layer potential and for the gradient of the single layer potential play a crucial role when we establish jump relations. Consequently, the following sections will obey the following pattern: we first establish the boundedness result for the non-tangential maximal functions. After that we apply the boundedness result to obtain the jump relations.

6.1 Boundary values of the double layer potential

We begin this section by showing that the non-tangential maximal function of the double layer potential is bounded.

Theorem 6.1. *Suppose that D is a C^1-domain and $\{\Gamma_\alpha\}$ is a family of cones for ∂D with a fixed aperture $\alpha \in (0,1)$. Also, let $p \in (1,\infty)$ and $f \in L^p(\partial D)$. Then there is a number $\delta > 0$ such that the non-tangential maximal function*

$$(\mathcal{K}f)^i_*(z) = \{|\mathcal{K}f(\tilde{x})| : \tilde{x} \in D,\, |\tilde{x} - z| < \delta,\, \tilde{x} \in \Gamma^i_\alpha(z)\,\}$$

is bounded on $L^p(\partial D)$. In other words, the non-tangential maximal function satisfies the estimate

$$\|(\mathcal{K}f)^i_*\|_{L^p(\partial D)} \leq C\|f\|_{L^p(\partial D)}$$

for some constant $C > 0$ that is independent of the function f.

Proof. Suppose that $z \in \partial D$ and $r > 0$. For notational convenience, we define a non-tangential approach region

$$\mathcal{R}_\alpha(z,r) := \{\tilde{x} \in \mathbb{R}^{d-1} : \tilde{x} \in D,\, |\tilde{x} - z| < r,\, \tilde{x} \in \Gamma_\alpha(z)\,\}.$$

Because D is a C^1-domain, then due to Lemma 2.2, there is a finite cover of balls $\{B(z_i,\delta_i)\}_{i=1}^n$ and functions $\varphi_i \in C^1_0(\mathbb{R}^{d-1})$, $i = 1,\ldots,n$, such that

$$B(z_i, 4\delta_i) \cap D = B(z_i, 4\delta_i) \cap \{(x,t) : x \in \mathbb{R}^{d-1},\, t > \varphi_i(x)\,\}$$

and

$$\|\nabla\varphi_i\|_\infty < \frac{\alpha}{6}$$

for every $i = 1,\ldots,n$. Furthermore, due to Theorem 2.1, there is a partition of unity $\{\zeta_i\}_{i=1}^n$ subordinate to the cover $\{B(z_i,\delta_i)\}_{i=1}^n$, which allows us to estimate

$$(\mathcal{K}f)_*(z) \leq \sum_{i=1}^n (\mathcal{K}(\zeta_i f))_*(z). \tag{6.1}$$

From the estimate (6.1), we see that it suffices to consider the boundedness of the non-tangential maximal function for a fixed $i \in \{1,\ldots,n\,\}$. Therefore, we denote $\varphi := \varphi_i$ and $B := B(z_i,\delta_i)$. We can also assume that the function f is compactly supported in B because ζ_i is compactly supported in B. Now, we select $\delta = \min\{\,\delta_i : i = 1,\ldots,n\,\}$ and then we write

$$(\mathcal{K}f)_*(z) = (\mathcal{K}f)_*(z)\mathbf{1}_{(3B)^c\cap\partial D}(z) + (\mathcal{K}f)_*(z)\mathbf{1}_{3B\cap\partial D}(z).$$

Suppose that $z \in (3B)^c \cap \partial D$ and $\tilde{x} \in \mathcal{R}_\alpha(z,\delta)$. If $w \in B$, then we know that $|\tilde{x} - w| > \delta$, which implies

$$|\mathcal{K}f(\tilde{x})| \leq \frac{1}{\omega_d} \int_{B \cap \partial D} \left| \frac{\langle \nu(w), \tilde{x} - w \rangle}{|\tilde{x} - w|^d} \right| |f(w)| \, d\sigma(w) \leq C \|f\|_{L^p(\partial D)}.$$

As a result, we have

$$\|(\mathcal{K}f)_* \mathbf{1}_{(3B)^c \cap \partial D}\|_{L^p(\partial D)} \leq C \|f\|_{L^p(\partial D)}. \tag{6.2}$$

Suppose that $z \in 3B \cap \partial D$ and $\tilde{x} \in \mathcal{R}_\alpha(z,\delta)$. Then $z = (x_0, \varphi(x_0))$ for some $x_0 \in \mathbb{R}^{d-1}$. Also, we know that $\tilde{x} \in 4B \cap D$, which means we have a representation $\tilde{x} = (x,t)$, where $x \in \mathbb{R}^{d-1}$ and $t > \varphi(x)$. The condition $\tilde{x} \in \Gamma_\alpha(z)$ implies that

$$\frac{t - \varphi(x_0) - \langle \nabla\varphi(x_0), x - x_0 \rangle}{(1 + |\nabla\varphi(x_0)|^2)^{1/2}} > \alpha \left[|x - x_0|^2 + (t - \varphi(x_0))^2 \right]^{1/2}.$$

From this we see that

$$\begin{aligned} t - \varphi(x_0) &> \alpha \left(1 + |\nabla\varphi(x_0)|^2\right)^{1/2} \left[|x - x_0|^2 + (t - \varphi(x_0))^2 \right]^{1/2} \\ &\quad + \langle \nabla\varphi(x_0), x - x_0 \rangle \\ &\geq \alpha |x - x_0| - |\langle \nabla\varphi(x_0), x - x_0 \rangle| \\ &> \tfrac{5}{6} \alpha |x - x_0|. \end{aligned}$$

The above observations give us a reason to define a set

$$\widetilde{\mathcal{R}}_\alpha(x_0) := \{ (x,t) : x \in \mathbb{R}^{d-1},\ t > \varphi(x),\ t - \varphi(x_0) > \tfrac{5}{6}\alpha |x - x_0| \}$$

because then we can write $\mathcal{R}_\alpha(z,\delta) \subset \widetilde{\mathcal{R}}_\alpha(x_0)$.

Let us write the double layer potential in local coordinates:

$$\begin{aligned} \mathcal{K}f(\tilde{x}) &= \frac{1}{\omega_d} \int_{\partial D} \frac{\langle \nu(w), \tilde{x} - w \rangle}{|\tilde{x} - w|^d} f(w) \, d\sigma(w) \\ &= \frac{1}{\omega_d} \int_{\mathbb{R}^{d-1}} \frac{t - \varphi(y) - \langle \nabla\varphi(y), x - y \rangle}{[|x - y|^2 + (t - \varphi(y))^2]^{d/2}} f(y, \varphi(y)) \, dy. \end{aligned}$$

By adopting notations

$$k(x,t;y) := \frac{t - \varphi(y) - \langle \nabla\varphi(y), x - y \rangle}{[|x - y|^2 + (t - \varphi(y))^2]^{d/2}} \quad \text{and} \quad g(y) := \omega_d^{-1} f(y, \varphi(y))$$

we can write

$$\mathcal{K}f(\tilde{x}) = \int_{\mathbb{R}^{d-1}} k(x,t;y) g(y) \, dy =: Tg(x,t).$$

Furthermore, we estimate

$$\begin{aligned}(\mathcal{K}f)_*(z) &= \sup\{|\mathcal{K}f(\tilde{x})| : \tilde{x} \in \mathcal{R}_\alpha(z,\delta)\,\} \\ &\leq \sup\{|Tg(x,t)| : (x,t) \in \widetilde{\mathcal{R}}_\alpha(x_0)\} \\ &=: T_*g(x_0).\end{aligned}$$

The above estimate hints that the boundedness of the maximal operator T_* might imply the boundedness of the non-tangential operator $(\mathcal{K}f)_*$.

We show that the operator T_* satisfies the estimate

$$\|T_*g\|_{L^p(\mathbb{R}^{d-1})} \leq C\|g\|_{L^p(\mathbb{R}^{d-1})} we$$

for every locally integrable function $g \in L^p(\mathbb{R}^{d-1})$ and for some $C > 0$. Let us denote $\lambda := \max\{3|x - x_0|, t - \varphi(x_0)\}$ and suppose $(x,t) \in \widetilde{\mathcal{R}}_\alpha(x_0)$. Then we write

$$\begin{aligned}Tg(x,t) &= \int_{|x_0-y|\leq\lambda} k(x,t;y)g(y)\,dy + \int_{|x_0-y|>\lambda} k(x,t;y)g(y)\,dy \\ &=: I_1 + I_2.\end{aligned}$$

First, we estimate the integral I_1. Therefore, assume $|x_0-y| \leq \lambda$. Because $(x,t) \in \widetilde{\mathcal{R}}_\alpha(x_0)$, we have an estimate

$$|x - x_0| \leq \frac{6}{5\alpha}(t - \varphi(x_0)).$$

Then, because $\alpha \in (0,1)$, we see that

$$\lambda = \max\{3|x - x_0|, t - \varphi(x_0)\} \leq \frac{18}{5\alpha}(t - \varphi(x_0)),$$

which implies

$$t - \varphi(x_0) \geq \frac{5\alpha}{18}\lambda. \tag{6.3}$$

By applying the triangle inequality, the estimate (6.3) and assumptions, we get

$$t - \varphi(y) \geq (t - \varphi(x_0)) - |\varphi(x_0) - \varphi(y)| \geq \frac{5\alpha}{18}\lambda - \frac{\alpha}{6}|x_0 - y| \geq \frac{\alpha}{9}\lambda.$$

Furthermore, we estimate

$$|t - \varphi(y)| \leq (t - \varphi(x_0)) + |\varphi(x_0) - \varphi(y)| \leq \left(1 + \frac{\alpha}{6}\right)\lambda$$

and

$$|\langle \nabla\varphi(y), x - z\rangle| \leq \frac{\alpha}{6}\left(|x - x_0| + |x_0 - z|\right) \leq \frac{\alpha}{3}\lambda.$$

Now, by using the above observations, we see that

$$|k(x, t; y)| = \frac{|t - \varphi(y) - \langle \nabla\varphi(y), x - y\rangle|}{[|x - y|^2 + (t - \varphi(y))^2]^{d/2}.} \leq \frac{C\lambda}{(t - \varphi(y))^d} \leq C\lambda^{1-d}.$$

Consequently, we have

$$|I_1| \leq \int_{B(x_0,\lambda)} |k(x, t; y)||g(y)|\, dy \leq C\mathcal{M}g(x_0).$$

Second, we estimate the integral I_2. Let us write

$$\begin{aligned} I_2 = &\int_{|x_0-y|>\lambda} (k(x, t; y) - k(x_0, t; y))g(y)\, dy \\ &+ \int_{|x_0-y|>\lambda} (k(x_0, t; y) - k(x_0, \varphi(x_0); y))g(y)\, dy \\ &+ \int_{|x_0-y|>\lambda} k(x_0, \varphi(x_0); y)g(y)\, dy \\ =:&\ J_1 + J_2 + J_3. \end{aligned}$$

To estimate the integral J_1, we first apply the mean value theorem and then we use the properties of the maximal operators that were introduced in Section 2.3. According to the mean value theorem, there exists a vector

$$\xi = x_0 + s(x - x_0), \quad 0 < s < 1,$$

such that

$$|k(x, t; y) - k(x_0, t; y)| = |\nabla_x k(\xi, t; y)||x - x_0|. \tag{6.4}$$

For this reason, let us differentiate

$$\begin{aligned} \nabla_x k(x, t; y) = &\frac{\nabla\varphi(y)}{[|x - y|^2 + (t - \varphi(y))^2]^{d/2}} \\ &- \frac{d(t - \varphi(y))(x - y)}{[|x - y|^2 + (t - \varphi(y))^2]^{d/2+1}} \\ &+ \frac{d\langle \nabla\varphi(y), x - y\rangle(x - y)}{[|x - y|^2 + (t - \varphi(y))^2]^{d/2+1}}. \end{aligned}$$

With simple arguments, we estimate

$$|\nabla_x k(x, t; y)| \leq C\left[|x - y|^2 + (t - \varphi(y))^2\right]^{-d/2} \leq C|x - y|^{-d}. \tag{6.5}$$

Furthermore, we notice

$$|x_0 - y| = |\xi - y - s(x - x_0)| \leq |\xi - y| + |x - x_0| \leq |\xi - y| + \frac{1}{3}|x_0 - y|,$$

which implies

$$|\xi - y| \geq \frac{2}{3}|x_0 - y|. \tag{6.6}$$

Then, by using estimates (6.4), (6.5) and (6.6), we obtain

$$|k(x, t; y) - k(x_0, t; y)| = \frac{C|x - x_0|}{|\xi - y|^d} \leq \frac{C|x - x_0|}{|x_0 - y|^d}.$$

Before proceeding, recall notations and definitions concerning maximal operators in Section 2.3. Assume $\hat{x} \in \mathbb{R}^{d-1}$ and let us define a kernel

$$N(\hat{x}) = |\hat{x}|^{-d}\mathbf{1}_{B(0,1)^c}(\hat{x}).$$

The kernel N has a radially decreasing majorant

$$N_0(\hat{x}) = N(\hat{x}) + \mathbf{1}_{B(0,1)}(\hat{x})$$

that is continuous and integrable in $\mathbb{R}^{d-1}$. Thus, according to Theorem 2.11, we have an estimate

$$\sup_{\varepsilon>0} (N_\varepsilon * |g|)(\hat{x}) \leq C\mathcal{M}g(\hat{x}).$$

Before estimating the absolute value of the integral J_1, we make one more observation, that is

$$\mathbf{1}_{B(x_0,\lambda)^c}(y) = \mathbf{1}_{B(0,1)^c}\Big(\frac{y - x_0}{\lambda}\Big).$$

Now, we are ready to estimate the integral J_1:

$$\begin{aligned}
|J_1| &\leq \int_{B(x_0,\lambda)^c} |k(x, t; y) - k(x_0, t, y)||g(y)|\, dy \\
&\leq C|x - x_0| \int_{\mathbb{R}^{d-1}} |y - x_0|^{-d}\mathbf{1}_{B(x_0,\lambda)^c}(y)|g(y)|\, dy \\
&= C\left(\frac{|x - x_0|}{\lambda}\right) \int_{\mathbb{R}^{d-1}} \frac{1}{\lambda^{d-1}} \left|\frac{y - x_0}{\lambda}\right|^{-d} \mathbf{1}_{B(0,1)^c}\Big(\frac{y - x_0}{\lambda}\Big)|g(y)|\, dy \\
&\leq C \int_{\mathbb{R}^{d-1}} N_\lambda(y - x_0)|g(y)|\, dy \\
&\leq C \sup_{\lambda>0} (N_\lambda * |g|)(x_0) \\
&\leq C\mathcal{M}g(x_0).
\end{aligned}$$

To estimate the integral J_2 we deduce similarly as above: according to the mean value theorem, there exists a number η between t and $\varphi(x_0)$ such that

$$|k(x_0,t;y)-k(x_0,\varphi(x_0);y)| \le |\partial_t k(x_0,\eta;y)|(t-\varphi(x_0)). \tag{6.7}$$

By differentiating and then by estimating, we find out that

$$|\partial_t k(x_0,t;y)| \le C\left[|x_0-y|^2+(t-\varphi(y))^2\right]^{-d/2} \le C|x_0-y|^{-d}. \tag{6.8}$$

Then we combine estimates (6.7) and (6.8) to obtain

$$|k(x_0,t;y)-k(x_0,\varphi(x_0);y)| \le C\frac{t-\varphi(x_0)}{|x_0-y|^d}.$$

We are ready to estimate the integral J_2:

$$\begin{aligned}
|J_2| &\le \int_{B(x_0,\lambda)^c} |k(x_0,t;y)-k(x_0,\varphi(x_0);y)||g(y)|\,dy \\
&\le C(t-\varphi(x_0))\int_{\mathbb{R}^{d-1}} |y-x_0|^{-d}\mathbf{1}_{B(x_0,\lambda)^c}(y)|g(y)|\,dy \\
&= C\left(\frac{t-\varphi(x_0)}{\lambda}\right)(N_\lambda * |g|)(x_0) \\
&\le C\sup_{\lambda>0}(N_\lambda * |g|)(x_0) \\
&\le C\mathcal{M}g(x_0).
\end{aligned}$$

To estimate the integral J_3, we write

$$J_3 = \int_{\mathbb{R}^{d-1}} \tau(x_0,y)\mathbf{1}_{\mathcal{F}(x_0,\lambda)}(y)g(y)\,dy =: \widehat{T}_\lambda g(x_0).$$

Here, we have used notations from Chapter 5 and we have denoted

$$\tau(x_0,y) := \frac{\varphi(x_0)-\varphi(y)-\langle\nabla\varphi(y),x_0-y\rangle}{[|x_0-y|^2+(\varphi(x_0)-\varphi(y))^2]^{d/2}}.$$

Hence, according to Lemma 5.1, the maximal operator $\widehat{T}_*$ is bounded and moreover, the integral J_3 is majorized by $\widehat{T}_*$.

By combining the estimates for the integrals J_1, J_2 and J_3, we are able to estimate the integral I_2 as follows

$$|I_2| \le C(\mathcal{M}g(x_0)+\widehat{T}_*g(x_0)).$$

Likewise, by combining estimates for the integrals I_1 and I_2, we obtain

$$|Tg(x,t)| \le |I_1|+|I_2| \le C(\mathcal{M}g(x_0)+\widehat{T}_*g(x_0)),$$

which implies

$$T_* g(x_0) \le C(\mathcal{M} g(x_0) + \widehat{T}_* g(x_0)).$$

By applying Minkowski's inequality and the boundedness of the maximal operators $\mathcal{M}$ and $\widehat{T}_*$, we see that

$$\|T_* g\|_{L^p(\mathbb{R}^{d-1})} \le C\|g\|_{L^p(\mathbb{R}^{d-1})}$$

for some constant $C > 0$.

We have nearly finished the proof because the boundedness of the maximal operator T_* allows us to deduce:

$$\begin{aligned}
\|(\mathcal{K} f)_* \mathbf{1}_{3B\cap\partial D}\|^p_{L^p(\partial D)} &= \int_{\partial D} |(\mathcal{K} f)_*(z)|^p \mathbf{1}_{3B\cap\partial D}(z)\, d\sigma(z) \\
&\le C \int_{\mathbb{R}^{d-1}} |T_* g(x_0)|^p\, dx_0 \\
&= C\|T_* g\|^p_{L^p(\mathbb{R}^{d-1})} \\
&\le C\|g\|^p_{L^p(\mathbb{R}^{d-1})} \\
&\le C\|f\|^p_{L^p(\partial D)}.
\end{aligned}$$

Finally, by combining the above estimate and the estimate (6.2), we conclude that there is a constant $C > 0$ such that

$$\|(\mathcal{K} f)_*\|_{L^p(\partial D)} \le C\|f\|_{L^p(\partial D)}.$$

This completes the proof. □

We note that the above proof applies also for the non-tangential maximal functions of the single layer potential. Thus, we get the following theorem.

Theorem 6.2. *Suppose that D is a C^1-domain and $\{\Gamma_\alpha\}$ is a family of cones for ∂D with a fixed aperture $\alpha \in (0, 1)$. Also, let $p \in (1, \infty)$ and $f \in L^p(\partial D)$. Then there is a number $\delta > 0$ such that the interior non-tangential maximal function*

$$(\mathcal{S} f)^i_*(z) = \{|\mathcal{S} f(\tilde{x})| : \tilde{x} \in D,\, |\tilde{x} - z| < \delta,\, \tilde{x} \in \Gamma^i_\alpha(z)\}$$

and the exterior non-tangential maximal function

$$(\mathcal{S} f)^e_*(z) = \{|\mathcal{S} f(\tilde{x})| : \tilde{x} \in \mathbb{R}^d \setminus \overline{D},\, |\tilde{x} - z| < \delta,\, \tilde{x} \in \Gamma^e_\alpha(z)\}$$

are bounded on $L^p(\partial D)$. In other words, the non-tangential maximal functions satisfy estimates

$$\|(\mathcal{S} f)^i_*\|_{L^p(\partial D)} \le C\|f\|_{L^p(\partial D)} \quad \text{and} \quad \|(\mathcal{S} f)^e_*\|_{L^p(\partial D)} \le C\|f\|_{L^p(\partial D)}$$

for some constant $C > 0$ that is independent of the function f.

For the next proof, we establish some notations.

Notation 6.1. Suppose that $\{\Gamma_\alpha\}$ is a family of cones for ∂D with a fixed aperture $\alpha \in (0,1)$ and suppose that u is a function defined in $\mathbb{R}^d \setminus \partial D$. If $z \in \partial D$, then we use the following notations:

$$\limsup_{\substack{x\to z\\ x\in\Gamma^i_\alpha(z)}} u(x) := \lim_{\delta\to 0}\big(\sup\{u(x) : x \in D,\, |x-z| < \delta,\, x \in \Gamma^i_\alpha(z)\,\}\big);$$

$$\limsup_{\substack{x\to z\\ x\in\Gamma^e_\alpha(z)}} u(x) := \lim_{\delta\to 0}\big(\sup\{u(x) : x \in \mathbb{R}^d \setminus \overline{D},\, |x-z| < \delta,\, x \in \Gamma^e_\alpha(z)\,\}\big).$$

Notations for infimum are defined similarly.

With the help of Theorem 6.1, we prove the jump relation for the double layer potential.

Theorem 6.3. *Suppose D is a C^1-domain and $\{\Gamma_\alpha\}$ is a family of cones for ∂D with a fixed aperture $\alpha \in (0,1)$. Also, let $p \in (1,\infty)$ and $f \in L^p(\partial D)$. Then the double layer potential satisfies the jump relation*

$$\lim_{\substack{x\to z\\ x\in\Gamma^i_\alpha(z)}} \mathcal{K}f(x) = (\tfrac{1}{2}I + K)f(z) \tag{6.9}$$

for almost every $z \in \partial D$.

Proof. We prove this theorem in three stages. First, let us assume that $f \in C^1(\partial D)$. Let $\{x_j\}_{j=1}^\infty$ be a sequence in $\Gamma^i_\alpha(z)$ converging to $z \in \partial D$. According to Lemma 4.2, we may write

$$\mathcal{K}f(x_j) = f(z) + \int_{\partial D} \Psi(x_j,w)(f(w)-f(z))\,d\sigma(w).$$

Because $f \in C^1(\partial D)$, there is a constant $C > 0$ such that

$$|f(w) - f(z)| \le C|w-z|.$$

If we assume that j is large, then we may apply Lemma 2.3 and we obtain

$$|\Psi(x_j,w)||f(w)-f(z)| \le C\frac{|w-z|}{|x_j-w|^{d-1}} \le C|w-z|^{2-d}.$$

Now, due to the dominated convergence theorem, we have

$$\lim_{\substack{x\to z\\ x\in\Gamma^i_\alpha(z)}} \mathcal{K}f(x) = f(z) + \int_{\partial D} \Psi(z,w)(f(w)-f(z))\,d\sigma(w)$$

for almost every $z \in \partial D$. On the other hand, the dominated convergence theorem can be used to deduce

$$\int_{\partial D} \Psi(z,w)(f(w)-f(z))\,d\sigma(w) = \lim_{\varepsilon\to 0}\int_{\partial D} \Psi_\varepsilon(z,w)(f(w)-f(z))\,d\sigma(w).$$

By recalling the definition of the operator K and Lemma 4.2 and then using the above identities, we get

$$\lim_{\substack{x\to z\\ x\in\Gamma_\alpha^i(z)}} \mathcal{K}f(x) = \left(\tfrac{1}{2}I + K\right) f(z).$$

Second, assume $f \in L^p(\partial D)$. Because $C^1(\partial D)$ is a dense subspace of $L^p(\partial D)$, there is a sequence of functions $\{f_j\}_{j=1}^\infty$ in $C^1(\partial D)$ such that it converges to f in $L^p(\partial D)$. We already know that functions $\mathcal{K}f_j$ have pointwise limits for almost everywhere in ∂D. Thus,

$$\Lambda(\mathcal{K}f_j)(z) = \limsup_{\substack{x\to z\\ x\in\Gamma_\alpha^i(z)}} (\mathcal{K}f_j)(x) - \liminf_{\substack{x\to z\\ x\in\Gamma_\alpha^i(z)}} (\mathcal{K}f_j)(x) = 0$$

for almost every $z \in \partial D$. By using the above observation, we estimate

$$\Lambda(\mathcal{K}f)(z) \le 2(\mathcal{K}(f-f_j))_*(z).$$

Now, the boundedness of the non-tangential maximal function of $\mathcal{K}f$ and similar arguments as in Theorem 3.1 imply that non-tangential limits of $\mathcal{K}f$ exist for almost everywhere on ∂D.

Third, we ensure that (6.9) holds also when $f \in L^p(\partial D)$. For this reason, let us define a sequence of function $\{u_j\}_{j=1}^\infty$ on ∂D by writing

$$\begin{aligned} u_j(z) =& \frac{1}{2}(f_j(z)-f(z)) + (Kf_j(z)-Kf(z)) \\ &+ \lim_{\substack{x\to z\\ x\in\Gamma_\alpha^i(z)}} (\mathcal{K}f(x)-\mathcal{K}f_j(x)). \end{aligned}$$

The operators K and $(\mathcal{K}f)_*$ are bounded on $L^p(\partial D)$. Therefore,

$$\|u_j\|_{L^p(\partial D)} \le C\|f-f_j\|_{L^p(\partial D)} \to 0,$$

as j tends to infinity. Hence, there is a subsequence $\{u_{n_k}\}_{k=1}^\infty$ that converges to zero for almost every $z \in \partial D$. The functions u_j were defined so that we could write

$$\lim_{\substack{x\to z\\ x\in\Gamma_\alpha^i(z)}} \mathcal{K}f(x) = \frac{1}{2}f(z) + Kf(z) + u_{j_k}(z).$$

Finally, letting k tend to infinity, we have

$$\lim_{\substack{x\to z\\ x\in\Gamma^i_\alpha(z)}} \mathcal{K}f(x) = \frac{1}{2}f(z) + Kf(z)$$

for almost every $z \in \partial D$. □

To the end of this section, we state a further result that we need in order to prove the uniqueness of the Dirichlet problem. For a proof, we refer to [8].

Theorem 6.4. *Suppose that D is a C^1-domain and $\{\Gamma_\alpha\}$ is a family of cones for ∂D with a fixed aperture $\alpha \in (0,1)$. Also, let $p \in (1,\infty)$ and $f \in L^p_1(\partial D)$. Then then there is a number $\delta > 0$ such that the non-tangential maximal function*

$$(\nabla\mathcal{K}f)^i_*(z) = \sup\{|\nabla\mathcal{K}f(x)| : x \in D,\ |x-z| < \delta,\ x \in \Gamma^i_\alpha(z)\,\}$$

satisfies the estimate

$$\|(\nabla\mathcal{K}f)^i_*\|_{L^p(\partial D)} \le C\|f\|_{L^p_1(\partial D)}$$

for some constant $C > 0$ that is independent of the function f.

6.2 Normal derivative of the single layer potential

We begin by stating a boundedness result for the non-tangential maximal functions of the gradient of the single layer potential. We omit the proof because it is similar to that of Theorem 6.1.

Theorem 6.5. *Suppose that D is a C^1-domain and $\{\Gamma_\alpha\}$ is a family of cones for ∂D with a fixed aperture $\alpha \in (0,1)$. Also, let $p \in (1,\infty)$ and $f \in L^p(\partial D)$. Then there is a number $\delta > 0$ such that that the interior non-tangential maximal function*

$$(\nabla\mathcal{S}f)^i_*(z) = \sup\{|\nabla\mathcal{S}f(z)| : x \in D,\ |x-z| < \delta,\ x \in \Gamma^i_\alpha(z)\,\}$$

and the exterior non-tangential maximal function

$$(\nabla\mathcal{S}f)^e_*(z) = \sup\{|\nabla\mathcal{S}f(z)| : x \in \mathbb{R}^d \setminus \overline{D},\ |x-z| < \delta,\ x \in \Gamma^e_\alpha(z)\,\}$$

are bounded on $L^p(\partial D)$. In other words, the non-tangential maximal functions satisfy the estimates

$$\|(\nabla Sf)^i_*\|_{L^p(\partial D)} \le C\|f\|_{L^p(\partial D)} \quad \textit{and} \quad \|(\nabla Sf)^e_*\|_{L^p(\partial D)} \le C\|f\|_{L^p(\partial D)}$$

for some constant $C > 0$ that is independent of the function f.

We will use Theorem 6.5 to establish jump relations for the normal derivative of the single layer potential. However, we first consider the following lemma, in which we will use notations of Theorem 6.1. Especially, recall the notation

$$\mathcal{R}_\alpha(z,r) := \{\tilde{x} \in \mathbb{R}^{d-1} : \tilde{x} \in D,\, |\tilde{x} - z| < r,\, \tilde{x} \in \Gamma^i_\alpha(z)\,\}.$$

Lemma 6.1. *Suppose that D is a C^1-domain and $\{\Gamma_\alpha\}$ is a family of cones for ∂D with a fixed aperture $\alpha \in (0,1)$. Also, let $p \in (1,\infty)$, suppose that a function ϱ belongs to $L^\infty(\partial D, \mathbb{R}^d)$ and $\mathcal{T}$ is an integral operator defined by*

$$\mathcal{T}\varrho(\tilde{x}) := \int_{\partial D} \frac{\langle \varrho(w), \tilde{x} - w\rangle}{|\tilde{x} - w|^d}\, d\sigma(w), \quad \tilde{x} \in D.$$

Then there is a number $\delta > 0$ such that the non-tangential maximal function

$$(\mathcal{T}\varrho)^i_*(z) = \sup\{|(\mathcal{T}\varrho)(\tilde{x})| : \tilde{x} \in \mathcal{R}_\alpha(z,\delta)\}$$

satisfies the estimate

$$\|(\mathcal{T}\varrho)_*\|_{L^p(\partial D)} \le C\|\varrho\|_\infty$$

for some constant $C > 0$ that is independent of the function ϱ.

Proof. Suppose that $z \in \partial D$. There is a finite cover of balls $\{B(z_i,\delta_i)\}_{i=1}^n$ for ∂D and functions $\varphi_i \in C_0^\infty(\mathbb{R}^{d-1})$, $i = 1,\dots,n$, such that

$$B(z_i, 4\delta_i) \cap D = B(z_i, 4\delta_i) \cap \{(x,t) : x \in \mathbb{R}^{d-1},\, t > \varphi_i(x)\,\}$$

and

$$\|\nabla\varphi_i\|_\infty < \frac{\alpha}{6}$$

for every $i = 1,\dots,n$. Furthermore, there is a partition of unity $\{\zeta_i\}_{i=1}^n$ subordinate to the cover $\{B(z_i,\delta_i)\}_{i=1}^n$, which allows us to estimate

$$(\mathcal{T}\varrho)_*(z) \le \sum_{i=1}^\infty (\mathcal{T}(\varrho\zeta_i))_*(z).$$

We fix $i \in \{1,\dots,n\}$ and denote $B := B(z_i,\delta_i)$ and $\varphi := \varphi_i$. Then we may assume that the function ϱ is compactly supported in B. We select $\delta := \min\{\delta_i : i = 1,\dots,n\}$.

Suppose that $z \in (3B)^c \cap \partial D$ and $\tilde{x} \in \mathcal{R}_\alpha(z,\delta)$. Then

$$|(\mathcal{T}\varrho)(\tilde{x})| \le \int_{\partial D} \left|\frac{\langle \varrho(w), \tilde{x} - w\rangle}{|\tilde{x} - w|^d}\right| d\sigma(w) \le \frac{1}{\delta^{d-1}} \int_{\partial D} |\varrho(w)|\, d\sigma(w) \le C\|\varrho\|_\infty.$$

As a result

$$\|(\mathcal{T}\varrho)_*\mathbf{1}_{(3B)^c\cap\partial D}\|_{L^p(\partial D)} \le C\|\varrho\|_\infty.$$

Suppose that $z \in 3B \cap \partial D$ and $\tilde{x} \in \mathcal{R}_\alpha(z, \delta)$. Then we are able to denote $z = (x_0, \varphi(x_0))$ and $\tilde{x} = (x, t)$, where $x \in \mathbb{R}^{d-1}$ and $t > \varphi(x)$. Also, let us denote $\varrho := (\varrho_1, \dots, \varrho_d)$ and

$$A_m(x, t; y) := \begin{cases} x_m - y_m, & \text{if } m = 1, \dots, d-1 \\ t - \varphi(y), & \text{if } m = d. \end{cases}$$

By using the above notations, we define kernels

$$k_m(x, t; y) = \frac{A_m(x, t; y)}{\left[|x-y|^2 + (t - \varphi(y))^2\right]^{d/2}}$$

and functions

$$g_m(y) := \varrho_m(y, \varphi(y))(1 + |\nabla\varphi(y)|^2)^{1/2}$$

for every $m = 1, \dots, d$. These definitions allow us to write

$$\begin{aligned} \mathcal{T}\varrho(\tilde{x}) &= \int_{\partial D} \frac{\langle \varrho(w), \tilde{x} - w\rangle}{|\tilde{x} - w|^d}\, d\sigma(w) \\ &= \sum_{m=1}^{d} \int_{\mathbb{R}^{d-1}} k_m(x, t; y) g_m(y)\, dy \\ &=: \sum_{m=1}^{d} T_m g_m(x, t). \end{aligned}$$

Furthermore, we have

$$\begin{aligned} (\mathcal{T}\varrho)_*(z) &\le \sum_{m=1}^{d} \sup\{|T_m g_m(x, t)| : (x, t) \in \widetilde{\mathcal{R}}_\alpha(x_0)\} \\ &=: \sum_{m=1}^{d} \widetilde{T}_m g_m(x_0). \end{aligned}$$

Assume that $g \in L^p(\mathbb{R}^{d-1})$ and g is compactly supported in $\mathbb{R}^{d-1}$. We show that

$$\|\widetilde{T}_m g\|_{L^p(\mathbb{R}^{d-1})} \le C\|g\|_{L^p(\mathbb{R}^{d-1})}$$

for every $m = 1, \dots, d$ for some constant $C > 0$. By arguing in a similar manner as in the proof of Theorem 6.1, we deduce

$$\widetilde{T}_m g(x_0) \le C\mathcal{M}g(x_0) + \sup_{\lambda > 0}\left|\int_{|y - x_0| > \lambda} k_m(x_0, \varphi(x_0); y) g(y)\, dy\right|.$$

By denoting

$$B_m(x) = \begin{cases} x_m, & \text{if } m = 1, \dots, d-1 \\ \varphi(x), & \text{if } m = d, \end{cases}$$

we see that

$$k_m(x_0, \varphi(x_0); y) = \frac{B_m(x_0) - B_m(y)}{[|x_0 - y|^2 + (\varphi(x_0) - \varphi(y))^2]^{d/2}}.$$

According to Theorem 3.10, maximal operators defined by

$$\widehat{T}_{m,*} g(x_0) := \sup_{\lambda > 0} \left| \int_{|y - x_0| > \lambda} k_m(x_0, \varphi(x_0); y) g(y) \, dy \right|$$

are bounded on $L^p(\mathbb{R}^{d-1})$. Consequently, we get

$$\|\widetilde{T}_m g\|_{L^p(\mathbb{R}^{d-1})} \le C \|\mathcal{M} g\|_{L^p(\mathbb{R}^{d-1})} + \|\widehat{T}_{m,*} g\|_{L^p(\mathbb{R}^{d-1})} \le C \|g\|_{L^p(\mathbb{R}^{d-1})}.$$

Finally, we see that

$$\|(\mathcal{T} \varrho)_* \mathbf{1}_{3B \cap \partial D}\|_{L^p(\partial D)} \le C \sum_{m=1}^{d} \|\widetilde{T}_m g_m\|_{L^p(\mathbb{R}^{d-1})} \le C \|\varrho\|_\infty.$$

This completes the proof. □

With the help of the boundedness of the non-tangential maximal functions and the above lemma, we prove the jump relations for the normal derivative of the single layer potential.

Theorem 6.6. *Suppose that D is a C^1-domain and $\{\Gamma_\alpha\}$ is a family of cones for ∂D with a fixed aperture $\alpha \in (0, 1)$. Also, let $p \in (1, \infty)$ and $f \in L^p(\partial D)$. Then the normal derivative of the single layer potential satisfies the following jump relations:*

(i) *For almost every $z \in \partial D$*

$$\lim_{\substack{x \to z \\ x \in \Gamma_\alpha^i(z)}} \langle \nu(z), \nabla \mathcal{S} f(x) \rangle = -(\tfrac{1}{2} I - K^*) f(z).$$

(ii) *For almost every $z \in \partial D$*

$$\lim_{\substack{x \to z \\ x \in \Gamma_\alpha^e(z)}} \langle \nu(z), \nabla \mathcal{S} f(x) \rangle = -(\tfrac{1}{2} I + K^*) f(z).$$

Proof. Proofs of the parts (i) and (ii) are similar [8]. Therefore, we consider only the part (i). Furthermore, it suffices to assume that a density f belongs to $C^1(\partial D)$ because we can argue as in the proof of Theorem 6.3.

Suppose that $x \in D$, $z \in \partial D$ and recall from the Chapter 4 that

$$\langle \nu(z), \nabla \mathcal{S} f(x) \rangle = \frac{1}{\omega_d} \int_{\partial D} \frac{\langle \nu(z), w - x \rangle}{|w - x|^d} f(w)\, d\sigma(w).$$

Let us define functions

$$u_1(x) = \frac{1}{\omega_d} \int_{\partial D} \frac{\langle \nu(z), w - x \rangle}{|w - x|^d} (f(w) - f(z))\, d\sigma(w)$$

and

$$u_2(x) = \frac{1}{\omega_d} \int_{\partial D} \frac{\langle \nu(z) - \nu(w), w - x \rangle}{|w - x|^d}\, d\sigma(w).$$

With the help of these functions and Lemma 4.2, we write

$$\langle \nu(z), \nabla \mathcal{S} f(x) \rangle = u_1(x) + f(z) u_2(x) - f(z). \tag{6.10}$$

Similar arguments as in the proof of Theorem 6.3 yield

$$\lim_{\substack{x \to z \\ x \in \Gamma^i_\alpha(z)}} u_1(x) = K^* f(z) - f(z) \left(\lim_{\varepsilon \to 0} \int_{\partial D} \Psi_\varepsilon(w, z)\, d\sigma(w) \right). \tag{6.11}$$

The function $z \mapsto \nu(z)$ is continuous on ∂D. Therefore, there is a sequence $\{\nu_j\}_{j=1}^\infty$ in $C^1(\partial D, \mathbb{R}^d)$ that converges uniformly to ν. Now, we write the function u_2 in the form:

$$\begin{aligned} u_2(x) &= \frac{1}{\omega_d} \left\langle \nu(z) - \nu_j(z), \int_{\partial D} \frac{w - x}{|w - x|^d}\, d\sigma(w) \right\rangle \\ &\quad - \frac{1}{\omega_d} \int_{\partial D} \frac{\langle \nu(w) - \nu_j(w), x - w \rangle}{|x - w|^d}\, d\sigma(w) \\ &\quad + \frac{1}{\omega_d} \int_{\partial D} \frac{\langle \nu_j(z) - \nu_j(w), w - x \rangle}{|w - x|^d}\, d\sigma(w) \\ &=: v_{1,j}(x) + v_{2,j}(x) + v_{3,j}(x). \end{aligned}$$

First, let us establish pointwise convergence of the last term for a fixed $j \in \mathbb{N}$. We know that vector ν_j satisfies Lipschitz condition and we can also apply Lemma 2.3. Hence, we may estimate

$$\left| \frac{\langle \nu_j(z) - \nu_j(w), w - x \rangle}{|w - x|^d} \right| \le C|x - w|^{2-d} \le C|z - w|^{2-d}.$$

The above estimate justifies the use of the dominated convergence theorem. As a consequence, we get

$$\lim_{\substack{x\to z\\ x\in\Gamma^i_\alpha(z)}} v_{3,j}(x) = \frac{1}{\omega_d}\int_{\partial D} \frac{\langle \nu_j(z)-\nu_j(w), w-z\rangle}{|w-z|^d}\, d\sigma(w)$$

for almost every $z \in \partial D$. Moreover, the existence of the pointwise limits imply that

$$\Lambda(v_{3,j})(z) = \limsup_{\substack{x\to z\\ x\in\Gamma^i_\alpha(z)}} v_{3,j}(x) - \liminf_{\substack{x\to z\\ x\in\Gamma^i_\alpha(z)}} v_{3,j}(x) = 0$$

for almost every $z \in \partial D$.

Next, we prove pointwise convergence of the function u_2. Let us denote $g \equiv 1$ on ∂D. Then, by applying Theorem 6.5, we estimate

$$\|(\nu-\nu_j)(\nabla \mathcal{S}g)_*\|_{L^p(\partial D)} \le \|\nu-\nu_j\|_\infty \|(\nabla\mathcal{S}g)_*\|_{L^p(\partial D)} \le C\|\nu-\nu_j\|_\infty.$$

Consequently, the sequence $\{|\nu-\nu_j|(\nabla\mathcal{S}g)_*\}_{j=1}^\infty$ converges to zero in $L^p(\partial D)$. In a similar manner, using Lemma 6.1, we are able to estimate

$$\|(v_{2,j})_*\|_{L^p(\partial D)} \le C\|\nu-\nu_j\|_\infty.$$

These observations imply the existence of the subsequences

$$\{|\nu-\nu_{j_k}|(\nabla\mathcal{S}g)_*\}_{k=1}^\infty \quad \text{and} \quad \{(v_{2,j_k})_*\}_{k=1}^\infty \tag{6.12}$$

which converge pointwise to zero for almost everywhere on ∂D. Then, by estimating

$$\begin{aligned}\Lambda(u_2)(z) &\le \Lambda(v_{1,j_k})(z) + \Lambda(v_{2,j_k})(z)\\ &\le |\nu(z)-\nu_{j_k}(z)|(\nabla\mathcal{S}g)_*(z) + (v_{2,j_k})_*(z)\end{aligned}$$

and using the pointwise convergence of sequences (6.12), we deduce that

$$\Lambda(u_2)(z) = 0$$

for almost every $z \in \partial D$.

We know even more, because for almost every $z \in \partial D$, we have

$$\Lambda(v_{1,j}+v_{2,j})(z) = \Lambda(u_2 - v_{3,j})(z) \le \Lambda(u_2)(z) + \Lambda(v_{3,j})(z) = 0.$$

Therefore, we can define a function

$$V_j(z) := \lim_{\substack{x\to z\\ x\in\Gamma^i_\alpha(z)}} (v_{1,j}+v_{2,j})(x)$$

and then estimate by using Minkowski's inequality

$$\|V_j\|_{L^p(\partial D)} \leq \|(\nu - \nu_j)(\nabla Sg)_* + (v_{2,j})_*\|_{L^p(\partial D)} \leq C\|\nu - \nu_j\|_\infty.$$

With similar arguments as before, we are able to find a subsequence $\{V_{j_k}\}_{k=1}^\infty$ that converges to zero for almost everywhere on ∂D.

Now, we are ready to approach the boundary:

$$\begin{aligned}\lim_{\substack{x\to z\\ x\in\Gamma^i_\alpha(z)}} u_2(x) &= V_{j_k}(z) + \lim_{\substack{x\to z\\ x\in\Gamma^i_\alpha(z)}} v_{3,j_k}(x)\\ &= V_{j_k}(z) + \frac{1}{\omega_d}\int_{\partial D} \frac{\langle \nu_{j_k}(z) - \nu_{j_k}(w), w - z\rangle}{|w-z|^d}\, d\sigma(w).\end{aligned}$$

By letting k tend to infinity and then using the dominated convergence theorem and Lemma 4.2, we get

$$\lim_{\substack{x\to z\\ x\in\Gamma^i_\alpha(z)}} u_2(x) = \frac{1}{2} + \lim_{\varepsilon\to 0}\int_{\partial D} \Psi_\varepsilon(w,z)\, d\sigma(w). \tag{6.13}$$

Finally, by combining results (6.10), (6.11) and (6.13), we obtain

$$\lim_{\substack{x\to z\\ x\in\Gamma^i_\alpha(z)}} \langle \nu(z), \nabla \mathcal{S}f(x)\rangle = -(\tfrac{1}{2}I - K^*)f(z).$$

This completes the proof. □

Chapter 7

Unique Solvability of the Dirichlet Problem

Our initial goal was to prove that there is a unique solution to the Dirichlet problem for Laplace's equation with L^p-boundary data in a bounded C^1-domain with a connected boundary. In other words, we set ourselves a task to prove that for each bounded C^1-domain D with a connected boundary ∂D and for each $g \in L^p(\partial D)$, where $p \in (1, \infty)$, there is a unique function u, defined in D, which solves the problem

$$\begin{cases} \Delta u = 0 & \text{in } D \\ u = g & \text{on } \partial D. \end{cases} \tag{7.1}$$

Unfortunately, we cannot solve the problem (7.1) because it is formulated too vaguely. To obtain a unique solution, we must modify the problem: the boundary condition must be formulated precisely and the behaviour of the solution has to be restricted near the boundary. Below, we will reformulate the problem (7.1) in such a way that it will be meaningful also when the solutions are not assumed to be continuous up to the boundary.

Naturally, the problem is divided into two parts: for the existence and for the uniqueness of the solution. Our strategy for proving the existence part is to choose the double layer potential as a candidate for the solution. This is a good choice because the double layer potential satisfies Laplace's equation and its non-tangential boundary values are given by an operator $\frac{1}{2}I + K$, where the operator K is known to be compact. Thus, if we interpret the boundary condition of the problem (7.1) in the non-tangential sense, then we have reduced the problem of the existence of the solution into the problem of the invertibility of the Fredholm-type operator $\frac{1}{2}I + K$. The invertibility of the operator $\frac{1}{2}I + K$ will be proven in Section 7.1.

To be able to prove the uniqueness part, we must add an auxiliary condition to the problem (7.1). It suffices to require that the L^p-norm of the non-tangential maximal function of the solution is finite. However, it turns out that the solution satisfies even stronger condition: the L^p-norm of the non-tangential maximal function is majorized by the L^p-norm of the boundary data. Thus, we end up adding this condition to the problem (7.1).

It is important to notice that we are not allowed to use the maximum principle to prove the uniqueness of the solution. The use of the maximum principle requires that the solution obtains a maximum in the closure of the domain D. Due to L^p-boundary data, we are not allowed to make such an assumption. Instead, our strategy for proving the uniqueness of the solution will be to use the properties of the Green's function. The result that implies the uniqueness of the solution to the Dirichlet problem will be proven in Section 7.2.

In Section 7.3, we will state and prove the main result of this thesis. For clarity, we formulate the main result also here as a proposition.

Proposition 7.1. *Let D be a C^1-domain with a connected boundary ∂D, suppose $\{\Gamma_\alpha\}$ is a family of cones for ∂D with a fixed aperture $\alpha \in (0,1)$. Furthermore, let $p \in (1,\infty)$ and suppose $g \in L^p(\partial D)$. Then there is a unique harmonic function u such that the following conditions hold:*

(i) *For almost every $z \in \partial D$*

$$\lim_{\substack{x \to z \\ x \in \Gamma^i_\alpha(z)}} u(x) = g(z).$$

(ii) *There is a number $\delta > 0$ such that the interior non-tangential maximal function*

$$u^i_*(z) = \sup\{|u(x)| : x \in \partial D,\ |x - z| < \delta,\ x \in \Gamma^i_\alpha(z)\}$$

satisfies the condition

$$\|u^i_*\|_{L^p(\partial D)} \le C\|g\|_{L^p(\partial D)}.$$

Furthermore, the solution has the form of the double layer potential:

$$u(x) = \frac{1}{\omega_d}\int_{\partial D} \frac{\langle \nu(w), x - w\rangle}{|x - w|^d} f(w)\, d\sigma(w).$$

7.1 Existence of the solution

In this section, we prove that the Fredholm-type operator $\frac{1}{2}I + K$ is invertible on spaces $L^p(\partial D)$ and $L^p_1(\partial D)$. The actual proof is quite a simple consequence of the Fredholm alternative and the second Green's formula. However, to justify the use of the second Green's formula requires laborious considerations. These considerations are done in Lemmas 7.1 and 7.2. The reader is advised to omit these lemmas in the first reading and come back after finished reading the proof of Theorem 7.1.

Lemma 7.1. *Assume that $p \in (1,\infty)$, $f \in L^p(\partial D)$ and $(\frac{1}{2}I + K^*)f = 0$. Then $f \in L^q(\partial D)$ for every $q \in (1,\infty)$.*

Proof. At first, suppose that $f \in L^r(\partial D)$ for some $r \in (1,\infty)$. Due to Lemma 2.2, for every number $m > 0$, there is a finite cover of balls $\{B(z_i,\delta_i)\}_{i=1}^n$ for ∂D and functions $\varphi_i \in C_0^1(\mathbb{R}^{d-1})$, $i = 1,\dots,n$, such that

$$3B(z_i,\delta_i) \cap \partial D = \{(x,t) : x \in \mathbb{R}^{d-1},\, t > \varphi_i(x)\} \quad \text{and} \quad \|\varphi_i\|_\infty < m$$

for every $i = 1,\dots,n$. Then we select functions $\theta_i, \psi_i \in C_0^\infty(\mathbb{R}^d)$ with properties

$$\theta_i(x) = \begin{cases} 1, & \text{if } x \in B(z_i,\delta_i) \\ 0, & \text{if } x \notin B(z_i,2\delta_i) \end{cases} \qquad \text{and} \qquad \psi_i \equiv 1 \quad \text{on } \operatorname{supp}(\theta_i).$$

For simplicity, we will not write subscripts: for example we write $\theta := \theta_i$ and $\psi := \psi_i$. Especially, we will use notation $z_0 := z_i$. We observe that

$$\psi\theta = \theta \quad \text{and} \quad f = -2K^*f.$$

By using these observations, we get

$$(I + (2\psi K^*\psi))(\theta f) = -2\psi\theta K^* f + 2\psi K^*(\theta f) = -2\psi(\theta K^* - K^*\theta)f =: g.$$

Let $s \in (1,\infty)$ and $h \in L^s(\partial D)$. Also, let us denote $T := 2\psi K^*\psi$. By applying Lemma 5.2, it can be seen that

$$\|Th\|_{L^s(\partial D)} = C_m \|h\|_{L^s(\partial D)},$$

where the constant $C_m < 1$ when m is small enough. Therefore, according to Theorem 2.4, we know that the operator $I + T$ is invertible on $L^s(\partial D)$. Consequently, if we are able to show that $g \in L^s(\partial D)$, then $\theta f \in L^s(\partial D)$. In fact, then we have $f \in L^s(\partial D)$ because we may estimate

$$|f| \le \sum_{i=1}^n \theta_i |f| \quad \text{on } \partial D$$

and then apply Minkowski's inequality. We summarize the above observations as follows:

$$s \in (1,\infty) \text{ and } g \in L^s(\partial D) \quad \Longrightarrow \quad f \in L^s(\partial D). \tag{7.2}$$

Suppose that $r > d-1$. The function g can be written in the form

$$g(z) = -2\psi(z)\, \text{p.v.} \int_{\partial D} \Psi(w,z)(\theta(z)-\theta(w)) f(w)\, d\sigma(w),$$

which allows us to estimate

$$|g(z)| \le C \int_{\partial D} \frac{|f(w)|}{|z-w|^{d-2}}\, d\sigma(w).$$

If t is the conjugate exponent of r, then $t(d-2) < d-1$. By applying Hölder's inequality, we estimate further:

$$|g(z)| \le \left(\int_{\partial D} \frac{d\sigma(w)}{|z-w|^{t(d-2)}} \right)^{1/t} \|f\|_{L^r(\partial D)} \le C\|f\|_{L^r(\partial D)}.$$

Then $g \in L^q(\partial D)$ for every $q \in (1,\infty)$.

Next, suppose $r = d-1$. We wish to show that $g \in L^q(\partial D)$ for every $q \in (1,\infty)$. First, by applying Theorem 5.4, we notice that

$$\|g\|_{L^q(\partial D)} \le C\|(\theta K^* - K^*\theta) f\|_{L^q(\partial D)} \le C\|f\|_{L^q(\partial D)} \le C\|f\|_{L^r(\partial D)} < \infty,$$

when $q \in (1, r]$. In other words, we have

$$g \in L^q(\partial D) \quad \forall q \in (1, r]. \tag{7.3}$$

Second, we write

$$\begin{aligned} g(z) &= -2\psi(z) \int_{\partial D} \Psi(w,z)(\theta(z)-\theta(w)) \mathbf{1}_{B(z_0,3\delta)^c}(w) f(w)\, d\sigma(w) \\ &\quad - 2\psi(z) \int_{\partial D} \Psi(w,z)(\theta(z)-\theta(w)) \mathbf{1}_{B(z_0,3\delta)}(w) f(w)\, d\sigma(w) \\ &=: g_1(z) + g_2(z). \end{aligned}$$

Because ψ is compactly supported in $B(z_0, 2\delta)$, we see that

$$|g_1(z)| \le \frac{C}{\delta^{d-2}} \int_{\partial D} |f(w)|\, d\sigma(w) \le C\|f\|_{L^r(\partial D)},$$

which implies that

$$\|g_1\|_{L^q(\partial D)} \le C\|f\|_{L^r(\partial D)} < \infty \quad \forall q \in (r,\infty). \tag{7.4}$$

If $|z - z_0| \geq 3\delta$, then we have $g_2(z) = 0$. Thus, we assume $|z - z_0| < 3\delta$, which gives us a representation $z = (x, \varphi(x))$, where $x \in \mathbb{R}^{d-1}$. Now, we are able to estimate

$$\begin{aligned} |g_2(z)| &\leq C\psi(z) \int_{\partial D} \frac{|f(w)|\mathbf{1}_{B(z_0,3\delta)}(w)}{|z-w|^{d-2}}\, d\sigma(w) \\ &\leq C \int_{\mathbb{R}^{d-1}} k(x,y)\tilde{\psi}(x)\tilde{f}(y)\, dy. \end{aligned}$$

In the above estimate, $\tilde{f}$ is a compactly supported function that belongs to $L^r(\mathbb{R}^{d-1})$. Moreover, we have denoted $\tilde{\psi}(x) = \psi(x, \varphi(x))$ and

$$k(x,y) = \left[|x-y|^2 + (\varphi(x) - \varphi(y))^2\right]^{-d/2+1}.$$

Let us define a function

$$\tilde{h}(x) := \int_{\mathbb{R}^{d-1}} k(x,y)\tilde{\psi}(x)\tilde{f}(y)\, dy.$$

The kernel of the above integral is weakly singular. Thus, we may estimate

$$\|\tilde{h}\|_{L^r(\mathbb{R}^{d-1})} \leq C\|\tilde{f}\|_{L^r(\mathbb{R}^{d-1})} < \infty. \tag{7.5}$$

By differentiating, we get

$$\nabla_x k(x,y) = (2-d)\frac{x - y + (\varphi(x) - \varphi(y))\nabla\varphi(x)}{[|x-y|^2 + (\varphi(x) - \varphi(y))^2]^{d/2}}$$

and therefore, we have

$$\begin{aligned} \nabla\tilde{h}(x) = (2-d)\tilde{\psi}(x) \sum_{i=1}^{d-1} \int_{\mathbb{R}^{d-1}} &\frac{x_i - y_i}{[|x-y|^2 + (\varphi(x) - \varphi(y))^2]^{d/2}}\tilde{f}(y)\, dy \\ + (2-d)\tilde{\psi}(x)\nabla\varphi(x) &\int_{\mathbb{R}^{d-1}} \frac{\varphi(x) - \varphi(y)}{[|x-y|^2 + (\varphi(x) - \varphi(y))^2]^{d/2}}\tilde{f}(y)\, dy. \\ + \int_{\mathbb{R}^{d-1}} &k(x,y)\nabla\tilde{\psi}(x)\tilde{f}(y)\, dy. \end{aligned}$$

By applying Theorem 3.9 to the integrals in the first and the second term in the above equation and by observing that the kernel of the last integral is weakly singular, we obtain

$$\|\nabla\tilde{h}\|_{L^r(\mathbb{R}^{d-1})} \leq C\|\tilde{f}\|_{L^r(\mathbb{R}^{d-1})} < \infty. \tag{7.6}$$

The estimates (7.5) and (7.6) imply that the function h belongs to a Sobolev space $W^{1,r}(\mathbb{R}^{d-1})$. In fact, the Sobolev embedding theorem implies that $\tilde{h}$ belongs to $L^q(\mathbb{R}^{d-1})$ for every $q \in (r, \infty)$. For a detailed discussion on Sobolev spaces and embedding theorems we refer to [1].

Because the function $\tilde{h}$ majorizes the function g_2, we have

$$\|g_2\|_{L^q(\partial D)} \leq C\|h\|_{L^q(\mathbb{R}^{d-1})} < \infty \quad \forall q \in (p, \infty). \tag{7.7}$$

By combining the observations (7.3), (7.4) and (7.7), we finally see that $g \in L^q(\partial D)$ for every $q \in (1, \infty)$.

We summarize the above observations as follows:

$$r \geq d-1 \text{ and } f \in L^r(\partial D) \quad \Longrightarrow \quad g \in L^q(\partial D) \;\; \forall q \in (1, \infty). \tag{7.8}$$

In fact, we can combine observations (7.2) and (7.8) to get

$$r \geq d-1 \text{ and } f \in L^r(\partial D) \quad \Longrightarrow \quad f \in L^q(\partial D) \;\; \forall q \in (1, \infty). \tag{7.9}$$

Now, suppose that $p \in (1, \infty)$ and $f \in L^p(\partial D)$. According to (7.9) it suffices to show that $f \in L^r(\partial D)$ for some $r \geq d-1$. If $p \geq d-1$, then the proof would be complete. However, if $p < d-1$, then we select a number $p_1 \in (p, \infty)$ that satisfies

$$\frac{1}{p_1} = \frac{1}{p} - \frac{1}{d-1}$$

and then we show that $g \in L^{p_1}(\partial D)$ because then the observation (7.2) would imply that $f \in L^{p_1}(\partial D)$. To show that $g \in L^{p_1}(\partial D)$, we represent the function g as a sum of two functions exactly as before:

$$g = g_1 + g_2.$$

The same argument as before implies

$$\|g_1\|_{L^{p_1}(\partial D)} \leq C\|f\|_{L^p(\partial D)} < \infty.$$

Therefore, it suffices to show that $g_2 \in L^{p_1}(\partial D)$. If $|z - z_0| \geq 3\delta$, then we have $g_2(z) = 0$. Thus, we assume $|z - z_0| < 3\delta$, which gives us a representation $z = (x, \varphi(x))$, where $x \in \mathbb{R}^{d-1}$. Now, we are able to estimate

$$|g_2(x, \varphi(x))| \leq C \int_{\mathbb{R}^{d-1}} \frac{\tilde{f}(x)}{|x-y|^{d-2}}\, dy \leq C(I_s\tilde{f})(x),$$

where $\tilde{f}$ belongs to $L^p(\mathbb{R}^{d-1})$ and I_s is a Riesz potential with a parameter $s = 1$. The Riesz potential improves integrability of functions, which in our case means that we have

$$\|g_2\|_{L^{p_1}(\partial D)} \leq C\|I_s\tilde{f}\|_{L^{p_1}(\mathbb{R}^{d-1})} \leq C\|\tilde{f}\|_{L^p(\mathbb{R}^{d-1})} < \infty.$$

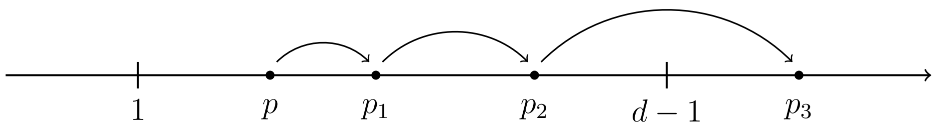

Figure 7.1: Iteration

For a detailed discussion on Riesz potentials we refer to [13].

Once again, if $p_1 \geq d-1$, then the proof would be complete. However, if $p_1 < d-1$, then we select a number $p_2 \in (p_1, \infty)$ that satisfies

$$\frac{1}{p_2} = \frac{1}{p_1} - \frac{1}{d-1}.$$

By arguing as above, we see that $g \in L^{p_2}(\partial D)$ and therefore $f \in L^{p_2}(\partial D)$. If we continue like this, we will find a number $m \in \mathbb{N}$ such that $p_m \geq d-1$ and $f \in L^{p_m}(\partial D)$. Finally, we deduce that $f \in L^q(\partial D)$ for every $q \in (1, \infty)$. □

Next, we justify the use of the second Green's formula for the single layer potential.

Lemma 7.2. *Let D be a bounded C^1-domain and suppose that $f \in L^q(\partial D)$ for every $q \in (1, \infty)$. Then*

(i) $\int_D |\nabla \mathcal{S} f|^2 \, dy = \int_{\partial D} \mathcal{S} f \left[(\frac{1}{2} I - K^*) f \right] d\sigma.$

(ii) $\int_{\mathbb{R}^d \setminus D} |\nabla \mathcal{S} f|^2 \, dy = \int_{\partial D} \mathcal{S} f \left[(\frac{1}{2} I + K^*) f \right] d\sigma.$

Proof. (i) For notational convenience, we denote $u = \mathcal{S} f$. The idea of the proof is to apply the divergence theorem. Unfortunately, we cannot apply it immediately on a domain D. However, we may take an approximation scheme $\Omega_j \nearrow D$ that was introduced in Section 2.1, then apply the divergence theorem on smooth domains Ω_j and finally let Ω_j converge to D.

Let us write

$$\int_D |\nabla u(y)|^2 \, dy = \int_{\Omega_j} |\nabla u(y)|^2 \, dy + \int_{D \setminus \Omega_j} |\nabla u(y)|^2 \, dy.$$

Now, we can apply the divergence theorem to first term on the right-hand side of the above identity:

$$\begin{aligned}\int_{\Omega_j} |\nabla u(y)|^2\, dy &= -\int_{\partial\Omega_j} u(\tilde{w})\langle \nu_j(\tilde{w}), \nabla u(\tilde{w})\rangle\, d\sigma_j(\tilde{w}) \\ &= \int_{\partial\Omega_j} u(\tilde{w})\langle \nu \circ p_j^{-1}(\tilde{w}) - \nu_j(\tilde{w}), \nabla u(\tilde{w})\rangle\, d\sigma_j(\tilde{w}) \\ &\quad - \int_{\partial\Omega_j} u(\tilde{w})\langle \nu \circ p_j^{-1}(\tilde{w}), \nabla u(\tilde{w})\rangle\, d\sigma_j(\tilde{w}) \\ &=: I_{1,j} + I_{2,j}.\end{aligned}$$

To integrate over ∂D instead of $\partial\Omega_j$, we use the change of variables formula:

$$I_{1,j} = \int_{\partial D} (u \circ p_j(w))\langle \nu(w) - \nu_j \circ p_j(w), \nabla u \circ p_j(w)\rangle J_j(w)\, d\sigma(w)$$

and

$$I_{2,j} = -\int_{\partial D} (u \circ p_j(w))\langle \nu(w), \nabla u \circ p_j(w)\rangle J_j(w)\, d\sigma(w).$$

To estimate the integral $I_{1,j}$, we recall the non-tangential approach region

$$\mathcal{R}_\alpha(w, \delta) = \{\tilde{x} \in \mathbb{R}^d : \tilde{x} \in D,\ |\tilde{x} - w| < \delta,\ \tilde{x} \in \Gamma_\alpha^i(w)\,\}$$

and we note that we may assume $p_j(w) \in \mathcal{R}_\alpha(w, \delta)$ when $\delta > 0$ is sufficiently small. Then by applying the above observation, generalized Hölder's inequality and Theorems 6.2 and 6.5, we get

$$\begin{aligned}|I_{1,j}| &\le \int_{\partial D} |u \circ p_j(w)||\nu(w) - \nu_j \circ p_j(w)||\nabla u \circ p_j(w)| J_j(w)\, d\sigma(w) \\ &\le C \int_{\partial D} |\nu(w) - \nu_j \circ p_j(w)||u_*(w)||(\nabla u)_*^i(w)|\, d\sigma(w) \\ &\le C\|\nu - \nu_j\|_{L^r(\partial D)}\|u_*\|_{L^s(\partial D)}\|(\nabla u)_*^i\|_{L^q(\partial D)} \\ &\le C\|\nu - \nu_j\|_{L^r(\partial D)}\|f\|_{L^q(\partial D)}.\end{aligned}$$

The integral $I_{1,j}$ tends to zero as j tends to zero, because from the assumption we know that $f \in L^q(\partial D)$ and moreover, ν_j converges to ν in $L^r(\partial D)$.

Let us define functions

$$v_j(w) := (u \circ p_j(w))\langle \nu(w), \nabla u \circ p_j(w)\rangle J_j(w).$$

The functions v_j are majorized by an integrable function on ∂D, because with a similar arguments as before, we may estimate

$$\begin{aligned}\int_{\partial D} |v_j(w)|\, d\sigma(w) &\leq \int_{\partial D} |u \circ p_j(w)||\nabla u \circ p_j(w)| d\sigma(w)\\ &\leq C \int_{\partial D} |u_*(w)||(\nabla u)^i_*(w)|\, d\sigma(w)\\ &\leq C\|u_*\|_{L^s(\partial D)}\|(\nabla u)^i_*\|_{L^q(\partial D)}\\ &\leq C\|f\|_{L^q(\partial D)}\\ &< \infty.\end{aligned}$$

Furthermore, we have

$$\begin{aligned}\lim_{j\to\infty} v_j(w) &= \left(\lim_{\substack{x\to w\\ x\in\Gamma^i_\alpha(w)}} u(x)\right)\cdot\left(\lim_{\substack{x\to w\\ x\in\Gamma^i_\alpha(w)}} \langle \nu(w), \nabla u(x)\rangle\right)\\ &= Sf(w)\left[-(\tfrac{1}{2}I - K^*)f(w)\right].\end{aligned}$$

Now, the dominated convergence theorem implies

$$\lim_{j\to\infty}\int_{\Omega_j} |\nabla u(y)|^2\, dy = \int_{\partial D} Sf(w)\left[(\tfrac{1}{2}I - K^*)f(w)\right] d\sigma(w).$$

To complete the proof of the part (i), we show that the integral

$$I_{3,j} := \int_{D\setminus\Omega_j} |\nabla u(y)|^2\, dy$$

tends to zero as j tends to infinity. By using the partition of unity, we may assume that u is compactly supported in $Z \cap (D \setminus \Omega_j)$, where $Z \subset \mathbb{R}^d$ is a coordinate cylinder. Thus, there is a neighbourhood $U \subset \mathbb{R}^{d-1}$ such that we may estimate

$$\begin{aligned}\int_{D\setminus\Omega_j} |\nabla u(y)|^2\, dy &\leq \int_U \int_{\varphi(x)}^{\varphi_j(x)} |\nabla u(x,t)|^2\, dt\, dx\\ &\leq \int_U \left[(\nabla u)^i_*(x,\varphi(x))\right]^2 (\varphi_j(x) - \varphi(x))\, dx\\ &\leq \|\varphi - \varphi_j\|_\infty \|(\nabla u)^i_*\|^2_{L^2(\partial D)}\\ &\leq C\|\varphi - \varphi_j\|_\infty \|f\|^2_{L^2(\partial D)}.\end{aligned}$$

Here φ_j denotes a local boundary function of a domain Ω_j that converges uniformly to φ. Thus, the proof of the part (i) is completed.

(ii) Once again, we cannot apply the divergence theorem immediately on the domain $\mathbb{R}^d \setminus D$. For this reason, we take an approximation scheme $\Omega_j \searrow D$ and then we write

$$\int_{\mathbb{R}^d \setminus D} |\nabla u(y)|^2 \, dy = \int_{\Omega_j \setminus D} |\nabla u(y)|^2 \, dy + \int_{\mathbb{R}^d \setminus \Omega_j} |\nabla u(y)|^2 \, dy.$$

With similar arguments as in the part (i), we deduce that

$$\lim_{j \to 0} \int_{\Omega_j \setminus D} |\nabla u(y)|^2 \, dy = 0.$$

Suppose $R > 0$ is such that $\Omega_j \subset B(0, R)$ for all $j \in \mathbb{N}$. For notational convenience, we write $B_R := B(0, R)$. Then we apply the divergence theorem to obtain

$$\begin{aligned}
\int_{\mathbb{R}^d \setminus \Omega_j} |\nabla u(y)|^2 \, dy &= \int_{\mathbb{R}^d \setminus B_R} |\nabla u(y)|^2 \, dy + \int_{B_R \setminus \Omega_j} |\nabla u(y)|^2 \, dy \\
&= \int_{\mathbb{R}^d \setminus B_R} |\nabla u(y)|^2 \, dy - \int_{\partial B_R} u(w) \langle \nu(w), \nabla u(w) \rangle \, d\sigma(w) \\
&\quad - \int_{\partial \Omega_j} u(\tilde{w}) \langle \nu(\tilde{w}), \nabla u(\tilde{w}) \rangle \, d\sigma_j(\tilde{w}) \\
&=: I_{1,R} + I_{2,R} + I_{3,j}.
\end{aligned}$$

By using Lemma 4.1, we estimate

$$|I_{1,R}| \leq C \int_{\mathbb{R}^d \setminus B_R} |y|^{2-2d} \, dy \leq C_R,$$

where the constant C_R tends to zero as R tends to infinity. Also, by applying Lemma 4.1, we get

$$|I_{2,R}| \leq \int_{\partial B_R} |u(w)| |\nabla u(w)| \, d\sigma(w) \leq C \int_{B_R} |w|^{3-2d} \, d\sigma(w) \leq C R^{2-d}.$$

Similar arguments as in the part (i) imply that

$$\lim_{j \to \infty} I_{3,j} = \int_{\partial D} Sf(w) \left[(\tfrac{1}{2} I + K^*) f(w) \right] d\sigma(w).$$

Finally, by letting R and j tend to infinity, we see that

$$\lim_{j \to 0} \int_{\mathbb{R}^d \setminus \Omega_j} |\nabla u(y)|^2 \, dy = \int_{\partial D} Sf(w) \left[(\tfrac{1}{2} I + K^*) f(w) \right] d\sigma(w),$$

which completes the proof. □

Theorem 7.1. *Let $p \in (1, \infty)$ and suppose that D is a bounded C^1-domain with a connected boundary ∂D. Then the operators $\frac{1}{2}I + K$ and $\frac{1}{2}I + K^*$ are invertible on $L^p(\partial D)$.*

Proof. According to Corollary 2.1, to prove the invertibility of the operators $\frac{1}{2}I + K$ and $\frac{1}{2}I + K^*$, it suffices to show that the operator $\frac{1}{2}I + K^*$ is injective. Therefore, we assume

$$(\tfrac{1}{2}I + K^*)f = 0 \tag{7.10}$$

and we will show that $f = 0$. To do this, we use Lemmas 7.1 and 7.2 and the properties of the single layer potential Sf. For notational convenience, we write $u = Sf$.

According to Lemma 7.1, the assumption (7.10) implies that $f \in L^q(\partial D)$ for each $q \in (1, \infty)$. Thus, we may use Lemma 7.2 to obtain

$$\int_{\mathbb{R}^d \setminus \overline{D}} |\nabla u(x)|^2 \, dx = \int_{\partial D} u(z) \left[(\tfrac{1}{2}I + K^*) f(z) \right] d\sigma(z) = 0.$$

The above identity implies that the gradient of u vanishes in $\mathbb{R}^d \setminus \overline{D}$. Due to the Jordan-Brouwer separation theorem [14], our assumptions on the domain D imply that $\mathbb{R}^d \setminus \overline{D}$ is connected. Hence, according to Lemma A.8, we see that the function u is constant in $\mathbb{R}^d \setminus \overline{D}$. In fact, the function u is zero because Lemmas 4.1 and A.5 state that

$$\lim_{|x| \to \infty} u(x) = 0.$$

Now, by using Theorem 4.2, we observe that

$$Sf(z) = \lim_{\substack{x \to z \\ x \in \Gamma^i_\alpha(z)}} u(x) = \lim_{\substack{x \to z \\ x \in \Gamma^e_\alpha(z)}} u(x) = 0$$

for almost every $z \in \partial D$. Using the above identity and Lemma 7.2, we get

$$\int_D |\nabla u(x)|^2 \, dx = \int_{\partial D} u(z) \left[(\tfrac{1}{2}I - K^*) f(z) \right] d\sigma(z) = 0.$$

The above identity implies that the gradient of the function u vanishes in the domain D. As a result, by using Theorem 6.6, we obtain

$$(\tfrac{1}{2}I - K^*) f(z) = \lim_{\substack{x \to z \\ x \in \Gamma^i_\alpha(z)}} \langle \nu(z), \nabla u(x) \rangle = 0$$

for every $z \in \partial D$. Finally, we have

$$f = (\tfrac{1}{2}I - K^*)f + (\tfrac{1}{2}I + K^*)f = 0.$$

This completes the proof. □

As a consequence of the above theorem and some earlier results, we find out that the operator $\frac{1}{2}I + K$ is invertible also on a Sobolev-type space $L_1^p(\partial D)$.

Corollary 7.1. *Let $p \in (1, \infty)$. The operator $\frac{1}{2}I + K : L_1^p(\partial D) \to L_1^p(\partial D)$ is invertible.*

Proof. We know that $L_1^p(\partial D) \subset L^p(\partial D)$ and the operator $\frac{1}{2}I + K$ from $L^p(\partial D)$ to itself is injective. Thus, the operator $\frac{1}{2}I + K$ from $L_1^p(\partial D)$ to itself is also injective. Now, Theorems 5.6 and 5.7 and Corollary 2.1 with Remark 2.3 imply that the operator $\frac{1}{2}I + K : L_1^p(\partial D) \to L_1^p(\partial D)$ is invertible. □

7.2 Uniqueness of the solution

In this section, we prove a result that implies the uniqueness of the Dirichlet problem. As we mentioned earlier, we are not able to apply the maximum principle, because the solution is not assumed to obtain a maximum. Instead, we can prove the uniqueness by using the properties of Green's function. Let us write its definition [7].

Definition 7.1. Suppose that the function $y \mapsto \phi^x(y)$ with a fixed $x \in D$ is a solution to the boundary value problem

$$\begin{cases} \Delta\phi^x = 0, & \text{in } D \\ \phi^x = \Phi(x, y), & \text{on } \partial D. \end{cases} \tag{7.11}$$

Then Green's function for the domain D is defined by

$$G(x, y) = \Phi(x, y) - \phi^x(y)$$

for every $x, y \in D$ for which $x \neq y$.

Green's function allows us to obtain representations for the solutions of the boundary value problems.

Example 7.1. Let $h \in C(\overline{D})$ and assume that a function $u \in C^2(\overline{D})$ solves the boundary value problem

$$\begin{cases} \Delta u = h & \text{in } D \\ u = 0 & \text{on } \partial D. \end{cases}$$

Then the solution has a representation

$$u(x) = -\int_D G(x, y)h(y)\, dx.$$

For details we refer to [7].

At first, the use Green's function does not seem practical because to determine Green's function, we must find a solution to the boundary value problem (7.11) for every $x \in D$. However, we are able to find such a solution using the method of layer potentials. By denoting

$$g_x(y) = \Phi(x, y) \quad \text{and} \quad T = (\tfrac{1}{2}I + K)^{-1},$$

the double layer potential approach to the boundary value problem (7.11) leads us to the solution

$$\phi^x(y) = \mathcal{K}(Tg_x)(y). \tag{7.12}$$

Thus, we have Green's function for a domain D in the form

$$G(x, y) = g_x(y) - \mathcal{K}(Tg_x)(y). \tag{7.13}$$

To be precise, we note that the function ϕ^x defined in (7.12) obtains boundary values in the non-tangential sense for almost everywhere on the boundary ∂D. Therefore, it is not clear, whether Example 7.1 works in this situation. Though, if we assume that the functions h and u are compactly supported in D, then we shall have a similar result as in Example 7.1.

Lemma 7.3. *Suppose that $h \in C_0(D)$ and a function $u \in C_0^2(D)$ satisfies Poisson's equation*

$$\Delta u = h \quad \text{in } D.$$

Then the function u has a representation

$$u(x) = -\int_D G(x, y)h(y)\, dy,$$

where Green's function G is defined by the formula (7.13).

Proof. Functions h and u are compactly supported in D. Hence, we may assume that they are zero outside a C^1-domain D_0 that is contained in D. Then, the third Green's formula implies that

$$\int_D \phi^x(y)h(y)\, dy = \int_{D_0} \phi^x(y)\Delta u(y)\, dy = 0.$$

Finally, using a representation for u in [7] on the page 34, we have

$$u(x) = -\int_D g_x(y)h(y)\, dy = -\int_D G(x, y)h(y)\, dy.$$

This completes the proof. □

The form (7.13) of Green's function allows us to deduce some properties that we need for proving the uniqueness of the Dirichlet problem. These properties are gathered in the following lemma.

Lemma 7.4. *Let D be a C^1-domain and $\{\Gamma_\alpha\}$ is a family of cones for ∂D with a fixed aperture $\alpha \in (0,1)$. Let $p \in (1,\infty)$, suppose that $x \in D$ is fixed and G is Green's function defined in* (7.13). *Then there is a number $\delta > 0$ such that the non-tangential maximal functions*

$$(G(x))^i_*(z) = \sup\{|G(x,y)| : y \in D,\, |y-z| < \delta,\, y \in \Gamma^i_\alpha(z)\,\}$$

and

$$(\nabla_y G(x))^i_*(z) = \sup\{|\nabla_y G(x,y)| : y \in D,\, |y-z| < \delta,\, y \in \Gamma^i_\alpha(z)\,\}$$

belong to $L^p(\partial D)$.

Proof. Suppose that $z \in \partial D$. Let us estimate

$$(G(x))^i_*(z) \le (g_x)^i_*(z) + (\mathcal{K}(Tg_x))^i_*(z).$$

According to Theorem 6.1, there is a number $\delta > 0$ such that the non-tangential maximal function of $\mathcal{K}$ is bounded on $L^p(\partial D)$. Furthermore, Theorem 7.1 implies that T is also bounded on $L^p(\partial D)$. Consequently, we estimate

$$\|(\mathcal{K}(Tg_x))^i_*\|_{L^p(\partial D)} \le C\|Tg_x\|_{L^p(\partial D)} \le C\|g_x\|_{L^p(\partial D)}.$$

With a fixed $x \in D$ and with a sufficiently small $\delta > 0$, functions $g_x|_{\partial D}$ and $(g_x)^i_*$ belong to $L^p(\partial D)$. Therefore, we have

$$\|(G(x))^i_*\|_{L^p(\partial D)} \le \|(g_x)^i_*\|_{L^p(\partial D)} + C\|g_x\|_{L^p(\partial D)} < \infty.$$

As before, we estimate

$$(\nabla_y G(x))^i_*(z) \le (\nabla_y g_x)^i_*(z) + (\nabla_y \mathcal{K}(Tg_x))^i_*(z)$$

and we can assume that $\delta > 0$ is as small as necessary. Thus, according to Theorem 6.4 and Corollary 7.1, we have

$$\|(\nabla_y \mathcal{K}(Tg_x))^i_*\|_{L^p(\partial D)} \le C\|Tg_x\|_{L^p_1(\partial D)} \le C\|g_x\|_{L^p_1(\partial D)}.$$

With a fixed $x \in \partial D$, we have $g_x|_{\partial D} \in L^p_1(\partial D)$ and $(\nabla_y g_x)^i_* \in L^p(\partial D)$. Therefore, we may estimate

$$\|(\nabla_y G(x))^i_*\|_{L^p(\partial D)} \le \|(\nabla_y g_x)^i_*\|_{L^p(\partial D)} + \|g_x\|_{L^p_1(\partial D)} < \infty.$$

This completes the proof. □

We are ready to prove the result that implies the uniqueness of the solution for the Dirichlet problem.

Theorem 7.2. *Suppose that D is a C^1-domain and $\{\Gamma_\alpha\}$ is a family of cones for ∂D with a fixed aperture $\alpha \in (0,1)$. Also, let $p \in (1,\infty)$ and assume that a function u satisfies conditions:*

$$\Delta u = 0 \quad \text{in } D, \qquad \lim_{\substack{x\to z\\ x\in\Gamma^i_\alpha(z)}} u(x) = 0 \quad \text{for a.e. } z \in \partial D, \qquad u_* \in L^p(\partial D).$$

Then the function u is zero in D.

Proof. Suppose that $\varepsilon > 0$ and $x \in D$. Let us define a set

$$D_\varepsilon = \{y \in D : \operatorname{dist}(y, \partial D) \geq \varepsilon\},$$

and a function $\psi_\varepsilon \in C_0^\infty(D)$ that satisfies the following conditions:

$$0 \leq \psi_\varepsilon \leq 1; \quad \psi_\varepsilon \equiv 1 \text{ in } D_\varepsilon; \quad |\nabla\psi_\varepsilon| \leq C\varepsilon^{-1} \quad \text{and} \quad |\Delta\psi_\varepsilon| \leq C\varepsilon^{-2}.$$

Such function exists according to [8]. Furthermore, let us define functions $v = u\psi_\varepsilon$ and $h = u\Delta\psi_\varepsilon + 2\langle\nabla u, \nabla\psi_\varepsilon\rangle$. These functions satisfy assumptions of Lemma 7.3. Consequently, we have a representation

$$v(x) = -\int_D G(x,y)h(y)\,dy,$$

where G is Green's function defined in (7.13). In fact, for a sufficiently small $\varepsilon > 0$, we have

$$u(x) = -\int_D G(x,y)\Delta\psi_\varepsilon(y)u(y)\,dy - 2\int_D G(x,y)\langle\nabla\psi_\varepsilon(y), \nabla u(y)\rangle\,dy.$$

We apply Lemma A.2 to the second term on the right-hand side. As a result, we get

$$u(x) = \int_D G(x,y)\Delta\psi_\varepsilon(y)u(y)\,dy + 2\int_D \langle\nabla_y G(x,y), \nabla\psi_\varepsilon(y)\rangle u(y)\,dy.$$

To complete the proof, it suffices to show that both terms on the right-hand side in the above identity tend to zero as ε tends to zero.

We begin with the second term. According to Lemma 2.2 there is a finite cover $\{B_i\}_{i=1}^n$ for ∂D such that

$$D \cap B_i = \{(\eta, t) : \eta \in \mathbb{R}^{d-1},\, t > \varphi_i(\eta)\,\} \cap B_i$$

and such that the boundary functions $\{\varphi_i\}_{i=1}^n$ satisfy

$$m_i := \|\nabla\varphi_i\|_\infty < \frac{1}{\alpha}\sqrt{1-\alpha^2}$$

for every $i = 1, \ldots, n$. Assume that $\delta > 0$ is a number such that all the non-tangential maximal functions that will be introduced in this proof belong to $L^q(\partial D)$ for every $q \in (1, \infty)$. Then, let us define a set

$$E_\delta = \{y \in \mathbb{R}^d : \operatorname{dist}(y, \partial D) \leq \delta\}$$

and functions $\{\psi_i\}_{i=1}^n$ that satisfy the following properties:

$$0 \leq \psi_i \leq 1; \quad \sum \psi_i \equiv 1 \text{ in } E_\delta \quad \text{and} \quad \operatorname{supp}(\psi_i) \subset B_i.$$

When $\varepsilon > 0$ is sufficiently small, we have $\nabla\psi_\varepsilon = 0$ in D_ε and $D \setminus D_\varepsilon \subset E_\delta$. Then we may estimate

$$\left|\int_D \langle \nabla_y G(x,y), \nabla\psi_\varepsilon(y)\rangle u(y)\, dy\right| \leq \sum_{i=1}^n \underbrace{\frac{C}{\varepsilon}\int_{D\setminus D_\varepsilon} |\nabla_y G(x,y)||u(y)|\psi_i(y)\, dy.}_{=:I_i}$$

Because $\operatorname{supp}(\psi_i) \subset B_i$, there exists a number $0 < c_i \leq r_i$ such that $\psi_i(\eta, \varphi(\eta)) = 0$, whenever $|\eta| > c_i$. We continue to estimate:

$$\begin{aligned} I_i &\leq \frac{C}{\varepsilon}\int_{|\eta|\leq c_i}\int_0^\varepsilon |\nabla_y G(x;\eta, t+\varphi_i(\eta))||u(\eta, t+\varphi_i(\eta))|\, dt\, d\eta \\ &\leq C\int_{|\eta|\leq c_i}\left(\sup_{0\leq r\leq\varepsilon}|\nabla_y G(x;\eta, r+\varphi_i(\eta))|\right)\left(\frac{1}{\varepsilon}\int_0^\varepsilon |u(\eta, t+\varphi_i(\eta))|\, dt\right) d\eta. \end{aligned}$$

To be precise, we should have integrated over an interval $(0, (1+m_i)\varepsilon)$ with respect to t. The reason for this can be seen by considering Figure 7.2 and then estimating

$$|\varphi_i(\eta) - \varphi_\varepsilon(\eta)| \leq |\varphi_i(\eta) - \varphi_i(\xi)| + |\varphi_i(\xi) - \varphi_\varepsilon(\eta)| \leq m_i|\eta - \xi| + \varepsilon \leq (1+m_i)\varepsilon.$$

However, the above integration is justified because we can choose $\varepsilon > 0$ to be as small as necessary. Let us denote $y = (\eta, t+\varphi_i(\eta))$ and $w = (\eta, \varphi_i(\eta))$. In the coordinate system of B_i, we have $y = w + t\nu(z_i)$, where z_i is the center of the ball B_i. Then, we notice that $y \in \Gamma_\alpha^i(w)$ because y satisfies

$$\langle \nu(w), y - w\rangle = t\langle \nu(w), \nu(z_i)\rangle \geq t(1+m_i^2)^{-1/2} > \alpha|y - w|.$$

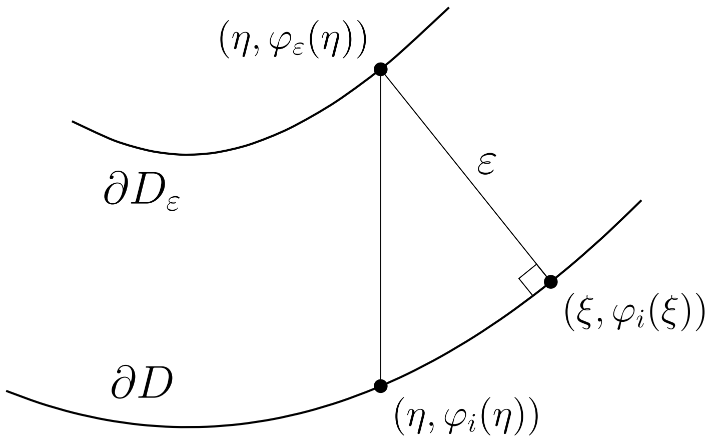

Figure 7.2: Parallel surfaces

Now, using the second assumption, we observe that

$$\lim_{t\to 0+} u(\eta, t+\varphi_i(\eta)) = 0. \tag{7.14}$$

We wish to use the dominated convergence theorem to show that the terms I_i tend to zero, as ε tends to zero. Using the third assumption, we have

$$\frac{1}{\varepsilon}\int_0^\varepsilon |u(\eta, t+\varphi_i(\eta))|\, dt \le u_*(\eta, \varphi_i(\eta)) \in L^p(\{\eta : |\eta| \le c_i\}).$$

Also, when $\varepsilon > 0$ is small enough, due to Lemma 7.4, we have

$$\sup_{0\le r\le\varepsilon} |\nabla_y G(x;\eta, r+\varphi_i(\eta))| \le (\nabla_y G(x))_*(\eta, \varphi_i(\eta)) \in L^q(\{\eta : |\eta| \le c_i\}),$$

where q can be selected to be the conjugate exponent of p. We use Hölder's inequality to see that

$$u_*(\nabla_y G(x))_* \in L^1(\{\eta : |\eta| \le c_i\}).$$

Now, the dominated convergence theorem together with the observation (7.14) allows us to deduce that

$$\lim_{\varepsilon\to 0}\int_D \langle \nabla_y G(x,y), \nabla\psi_\varepsilon(y)\rangle u(y)\, dy = 0. \tag{7.15}$$

According to [8], a similar deduction as above implies that

$$\lim_{\varepsilon\to 0}\int_D G(x,y)\Delta\psi_\varepsilon(y) u(y)\, dy = 0. \tag{7.16}$$

Finally, by combining the results (7.15) and (7.16), we see that the function u is zero in D. $\square$

7.3 Conclusion and further discussion

Let us now reformulate our main result and complete its proof.

Theorem 7.3. *Let D be a bounded C^1-domain with a connected boundary ∂D, suppose that $\{\Gamma_\alpha\}$ is a family of cones for ∂D with a fixed aperture $\alpha \in (0,1)$ and suppose that $g \in L^p(\partial D)$ for some $p \in (1,\infty)$. Then there is a unique harmonic function u, defined in D, such that the following conditions hold:*

(i) *For almost every $z \in \partial D$*

$$\lim_{\substack{x \to z \\ x \in \Gamma^i_\alpha(z)}} u(x) = g(z).$$

(ii) *There exists a number $\delta > 0$ such that the interior non-tangential maximal function*

$$u^i_*(z) = \sup\{|u(x)| : x \in D,\ |x - z| < \delta,\ x \in \Gamma^i_\alpha(z)\}$$

satisfies the condition

$$\|u^i_*\|_{L^p(\partial D)} \le C\|g\|_{L^p(\partial D)}.$$

Furthermore, the solution has the form of the double layer potential:

$$u(x) = \frac{1}{\omega_d}\int_{\partial D} \frac{\langle \nu(w), x - w\rangle}{|x - w|^d} f(w)\, d\sigma(w), \quad f \in L^p(\partial D).$$

Proof. For the existence of the solution, we select a function

$$f = \left(\tfrac{1}{2}I + K\right)^{-1} g.$$

Then we observe that the double layer potential

$$u(x) = \frac{1}{\omega_d}\int_{\partial D} \frac{\langle \nu(w), x - w\rangle}{|x - w|^d} f(w)\, d\sigma(w),$$

satisfies the conditions (i) and (ii).

For the uniqueness of the solution, we suppose that harmonic functions u_1 and u_2 satisfy the conditions (i) and (ii). Then, if we define a function $u = u_2 - u_1$, we observe that u satisfies the conditions of the Theorem 7.2. As a result, we have $u_1 = u_2$. This completes the proof. □

Now that we know the Dirichlet problem for Laplace's equation is uniquely solvable in C^1-domains with L^p-boundary data, one might be curious to know, whether it possible to apply the method of layer potentials in more general domains, for example in Lipschitz domains. The answer is positive: the use of the method of layer potentials was first generalized to Lipschitz domains by G. H. Verchota in his dissertation [18]. In the following, we shall outline his work briefly. Also, we will divert from the original convention and assume that D is a Lipschitz domain. For a precise definition of a Lipschitz domain we refer to [19].

The most fundamental problem that prevented the use of the method of layer potentials on Lipschitz domains was the open question of the boundedness of the Cauchy integral along Lipschitz curves. This major barrier was removed by the authors of the paper [3] in which they gave a positive answer to the open question. Indeed, the Cauchy integral was bounded along Lipschitz curves. Still, there were problems to be resolved.

The first problem was that Lipschitz domains are locally determined by Lipschitz functions that are only almost everywhere differentiable. As a result, to establish jump relations for the layer potentials, Verchota had to construct a suitable family cones that allowed him to approach the boundary of a Lipschitz domain. Clearly, the family of cones whose members open in the direction of the inward-pointing normal vector and have a fixed aperture, is not meaningful in the case of Lipschitz domains.

The second problem was that the boundary integral operators K and K^* are not necessarily compact on Lipschitz domains. Therefore, the Fredholm theory is not applicable and thus Verchota had to find new arguments for proving the invertibility of the operator $\frac{1}{2}I + K$. He overcame the lack of Fredholm theory by applying so-called Rellich identities [19]. The Rellich identities allowed him to establish certain operator inequalities that worked as substitutes for the compactness of the boundary integral operators. More precicely, Verchota proved that the inequality

$$\|(\tfrac{1}{2}I - K^*)f\|_{L^2(\partial D)} \leq C\left(\|(\tfrac{1}{2}I + K^*)f\|_{L^2(\partial D)} + \left|\int_{\partial D} Sf\,d\sigma\right|\right)$$

holds for every $f \in L^2(\partial D)$ and for some constant $C > 0$. In fact, he managed to prove that by taking an approximation scheme $\Omega_j \searrow D$, the corresponding operator inequalities

$$\|(\tfrac{1}{2}I - K^*)f_j\|_{L^2(\partial\Omega_j)} \leq C\left(\|(\tfrac{1}{2}I + K^*)f_j\|_{L^2(\partial\Omega_j)} + \left|\int_{\partial\Omega_j} Sf_j\,d\sigma_j\right|\right)$$

hold for every $f_j \in L^2(\partial\Omega_j)$ and for a constant $C > 0$ that depends only on the Lipschitz domain D.

Next, we will follow the ideas of Verchota and we will demonstrate how the above operator inequalities can be applied to prove invertibility of the operator $\frac{1}{2}I + K$, where the boundary integral operator K is associated to a Lipschitz domain. The following lemma forms a structure for the forthcoming proof.

Lemma 7.5. *Suppose that $\mathcal{H}$ is a Hilbert space, let $T : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator, let T^* denote the adjoint of T and assume that the following conditions hold:*

(i) $\mathrm{Im}(\frac{1}{2}I + T)$ *is dense in* $\mathcal{H}$.

(ii) $\mathrm{Im}(\frac{1}{2}I + T^*)$ *is closed in* $\mathcal{H}$.

(iii) $\mathrm{Im}(\frac{1}{2}I + T^*)$ *is dense in* $\mathcal{H}$.

Then the operator $\frac{1}{2}I + T$ is invertible on $\mathcal{H}$.

Proof. According to Theorem 2.5, it suffices to show that the operator $\frac{1}{2}I+T$ is bijective. First, we check that the operator $\frac{1}{2}I + T$ is surjective. Because $\mathrm{Im}(\frac{1}{2}I + T^*)$ is assumed to be closed in $\mathcal{H}$ and we know that

$$\tfrac{1}{2}I + T^* = (\tfrac{1}{2}I + T)^*,$$

the closed range theorem [20] implies that $\mathrm{Im}(\frac{1}{2}I+T)$ is closed in $\mathcal{H}$. Because $\mathrm{Im}(\frac{1}{2}I + T)$ is dense in $\mathcal{H}$, the operator $\frac{1}{2}I + T$ is surjective.

Second, we check that the operator $\frac{1}{2}I+T$ is injective. We use properties of Hilbert spaces [16] and the assumptions (ii) and (iii) to see that

$$\mathrm{Ker}(\tfrac{1}{2}I + T) = (\mathrm{Im}(\tfrac{1}{2}I + T^*))^\perp = \mathcal{H}^\perp = \{0\}.$$

Thus, the operator $\frac{1}{2}I + T$ is injective. □

Theorem 7.4. *Let D be a Lipschitz domain and let $K : L^2(\partial D) \to L^2(\partial D)$ be a boundary integral operator defined as usual:*

$$Kf(z) = \text{p.v.}\frac{1}{\omega_d}\int_{\partial D} \frac{\langle \nu(w), x - w\rangle}{|x - w|^d} f(y)\, dy.$$

Then the operator $\frac{1}{2}I + K$ is invertible on $L^2(\partial D)$.

Proof. The idea of the proof is to apply Lemma 7.5 to the operator K and check that the corresponding conditions (i), (ii) and (iii) are satisfied. Consequently, the proof is divided into three parts.

(i) We prove that $\mathrm{Im}(\frac{1}{2}I+K)$ is dense in $L^2(\partial D)$. It suffices to prove that the operator $\frac{1}{2}I+K^*$ is injective because then we have

$$\big[\mathrm{Im}(\tfrac{1}{2}I+K)\big]^{\perp}=\mathrm{Ker}(\tfrac{1}{2}I+K^*)=\{0\}$$

and furthermore

$$\overline{\mathrm{Im}(\tfrac{1}{2}I+K)}=\big[\mathrm{Im}(\tfrac{1}{2}I+K)\big]^{\perp\perp}=\{0\}^{\perp}=L^2(\partial D).$$

The injectivity of the operator $\frac{1}{2}I+K^*$ can be proven by similar arguments that were used in the proof of Theorem 7.1. We omit the details.

(ii) We prove that $\mathrm{Im}(\frac{1}{2}I+K^*)$ is closed in $L^2(\partial D)$. To do this, we assume that a sequence $\{h_j\}_{j=1}^{\infty}\subset\mathrm{Im}(\frac{1}{2}I+K^*)$ converges to $g\in L^2(\partial D)$ and then we show that g belongs to $\mathrm{Im}(\frac{1}{2}I+K^*)$. There is a sequence $\{f_j\}_{j=1}^{\infty}\subset L^2(\partial D)$ such that

$$h_j=(\tfrac{1}{2}I+K^*)f_j,\quad j\in\mathbb{N}.$$

If the sequence $\{f_j\}_{j=1}^{\infty}$ is bounded on $L^2(\partial D)$, then it contains a weakly convergent subsequence [16]. For notational convenience, we do not distinguish between sequences and subsequences and therefore we say that $\{f_j\}_{j=1}^{\infty}$ converges to $f\in L^2(\partial D)$. Now, suppose that $h\in L^2(\partial D)$. Then

$$\begin{aligned}\langle g,h\rangle&=\lim_{j\to\infty}\big\langle(\tfrac{1}{2}I+K^*)f_j,h\big\rangle=\lim_{j\to\infty}\big\langle f_j,(\tfrac{1}{2}I+K)h\big\rangle=\big\langle f,(\tfrac{1}{2}I+K)h\big\rangle\\&=\big\langle(\tfrac{1}{2}I+K^*)f,h\big\rangle.\end{aligned}$$

Because h was arbitrary, we get $g=(\frac{1}{2}I+K^*)f$ and therefore g belongs to $\mathrm{Im}(\frac{1}{2}I+K^*)$.

If the sequence $\{f_j\}_{j=1}^{\infty}$ is unbounded on $L^2(\partial D)$, then we define sequences $\{\tilde f_j\}_{j=1}^{\infty}$ and $\{\tilde h_j\}_{j=1}^{\infty}$ by writing

$$\tilde f_j=f_j\|f_j\|_{L^2(\partial D)}^{-1}\quad\text{and}\quad\tilde h_j=(\tfrac{1}{2}I+K^*)\tilde f_j,\quad j\in\mathbb{N}.$$

The sequence $\{\tilde h_j\}_{j=1}^{\infty}$ converges to zero in $L^2(\partial D)$ and the sequence $\{\tilde f_j\}_{j=1}^{\infty}$ is bounded. Then, by using the same deduction as in the above paragraph, we see that the sequence $\{\tilde f_j\}_{j=1}^{\infty}$ converges weakly to a function $\tilde f\in L^2(\partial D)$ that satisfies

$$(\tfrac{1}{2}I+K^*)\tilde f=0.$$

In the beginning of this proof, it was shown that the operator $\frac{1}{2}I+K^*$ is injective. Thus, we have $\tilde f=0$, which means that the sequence $\{\tilde f_j\}_{j=1}^{\infty}$ converges weakly to zero. Now, recall the operator inequality:

$$\|(\tfrac{1}{2}I-K^*)\tilde f_j\|_{L^2(\partial D)}\le C\left(\|(\tfrac{1}{2}I+K^*)\tilde f_j\|_{L^2(\partial D)}+\left|\int_{\partial D}S\tilde f_j\,d\sigma\right|\right).$$

The right-hand side of the operator inequality tends to zero as j tends to zero, because by applying Fubini's theorem and the weak convergence of the sequence $\{\tilde{f}_j\}_{j=1}^{\infty}$, we see that

$$\int_{\partial D} S\tilde{f}_j \, d\sigma \to 0,$$

as j tends to infinity. Finally, we estimate

$$\|\tilde{f}_j\|_{L^2(\partial D)} \leq \|(\tfrac{1}{2}I + K^*)\tilde{f}_j\|_{L^2(\partial D)} + \|(\tfrac{1}{2}I - K^*)\tilde{f}_j\|_{L^2(\partial D)}$$

and we see that the sequence $\{\tilde{f}_j\}_{j=1}^{\infty}$ tends to zero as j tends to infinity. However, this is impossible because the sequence $\{\tilde{f}_j\}_{j=1}^{\infty}$ satisfies

$$\|\tilde{f}_j\|_{L^2(\partial D)} = 1$$

for all $j \in \mathbb{N}$. This contradiction implies that the sequence $\{f_j\}_{j=1}^{\infty}$ has to be bounded in $L^2(\partial D)$ and therefore the function g belongs to $\mathrm{Im}(\frac{1}{2}I + K^*)$.

(iii) We prove that $\mathrm{Im}(\frac{1}{2}I + K^*)$ is dense in $L^2(\partial D)$. To do this, it suffices to show that a dense subspace of $L^2(\partial D)$, that is

$$\{\, g|_{\partial D} : \, g \in C_0^{\infty}(\mathbb{R}^d) \,\},$$

is contained in $\mathrm{Im}(\frac{1}{2}I + K^*)$. Hence, we take $g \in C_0^{\infty}(\mathbb{R}^d)$ and we show that the restriction function $g|_{\partial D}$ belongs to $\mathrm{Im}(\frac{1}{2}I + K^*)$. Also, we take an approximation scheme $\Omega_j \searrow D$ for a Lipschitz domain and then we define layer potentials S_j and K_j in a usual way on $\partial\Omega_j$ for all $j \in \mathbb{N}$. According to Theorem 7.1, operators $\frac{1}{2}I + K_j^*$, $j \in \mathbb{N}$, are invertible on $L^2(\partial\Omega_j)$. Thus, there is a sequence of functions $\{f_j\}_{j=1}^{\infty}$ such that

$$(\tfrac{1}{2}I + K_j^*)f_j = g|_{\partial\Omega_j}, \quad j \in \mathbb{N}. \tag{7.17}$$

Let us define a sequence $\{F_j\}_{j=1}^{\infty}$ of functions on ∂D that are defined by

$$F_j = (f_j \circ p_j)\, J_j, \quad j \in \mathbb{N}.$$

Note that then we can write

$$f_j = (F_j \circ p_j^{-1})(J_j \circ p_j^{-1})^{-1}, \quad j \in \mathbb{N}.$$

If there is a number $M > 0$ such that

$$\|f_j\|_{L^2(\partial\Omega_j)} \leq M < \infty, \tag{7.18}$$

then the sequence $\{F_j\}_{j=1}^{\infty}$ is bounded on $L^2(\partial D)$ because we may estimate:

$$\begin{aligned}\|F_j\|^2_{L^2(\partial D)} &\le C\int_{\partial D} |f_j\circ p_j(w)|^2 J_j(w)\,d\sigma(w)\\ &= C\int_{\partial\Omega_j} |f_j(\tilde{w})|^2\,d\sigma_j(\tilde{w})\\ &= C\|f_j\|^2_{L^2(\partial\Omega_j)}.\end{aligned}$$

Therefore, the sequence $\{F_j\}_{j=1}^{\infty}$ contains a subsequence that converges to some function $F\in L^2(\partial D)$ in the weak sense.

For simplicity, we will abbreviate our notations in the following calculations. Suppose $h\in C_0^\infty(\mathbb{R}^d)$. Then we have

$$\begin{aligned}\int_{\partial\Omega_j}\left[(\tfrac{1}{2}I+K_j^*)f_j\right]h\,d\sigma_j &= \int_{\partial\Omega_j} f_j\left[(\tfrac{1}{2}I+K_j)h\right]d\sigma_j\\ &= \int_{\partial\Omega_j}(F_j\circ p_j^{-1})(J_j\circ p_j^{-1})^{-1}\left[(\tfrac{1}{2}I+K_j)h\right]d\sigma_j\\ &= \int_{\partial D} F_j\left[(\tfrac{1}{2}I+K_j)h\circ p_j\right]d\sigma.\end{aligned}$$

We continue to write

$$\begin{aligned}\int_{\partial\Omega_j}\left[(\tfrac{1}{2}I+K_j^*)f_j\right]h\,d\sigma_j = &\int_{\partial D} F_j\left[(\tfrac{1}{2}I+K_j)h\circ p_j-(\tfrac{1}{2}I+K)h\right]d\sigma\\ &+\int_{\partial D}F_j\left[(\tfrac{1}{2}I+K)h\right]d\sigma.\end{aligned}$$

Because functions g and h are smooth, we may apply the dominated converge theorem to see that

$$\int_{\partial\Omega_j}\left[(\tfrac{1}{2}I+K_j^*)f_j\right]h\,d\sigma_j=\int_{\partial D}((gh)\circ p_j)J_j\,d\sigma\to\int_{\partial D} gh\,d\sigma,$$

as j tends to infinity. Furthermore, the weak convergene of the sequence $\{F_j\}_{j=1}^{\infty}$ implies

$$\begin{aligned}\lim_{j\to\infty}\int_{\partial D}F_j\left[(\tfrac{1}{2}I+K)h\right]d\sigma &= \lim_{j\to\infty}\langle F_j,(\tfrac{1}{2}I+K)h\rangle=\langle F,(\tfrac{1}{2}I+K)h\rangle\\ &= \langle(\tfrac{1}{2}I+K^*)F,h\rangle.\end{aligned}$$

If we were able to show that

$$\lim_{j\to\infty}\int_{\partial D}F_j\left[(\tfrac{1}{2}I+K_j)h\circ p_j-(\tfrac{1}{2}I+K)h\right]d\sigma=0, \tag{7.19}$$

then we would have

$$\langle g, h\rangle_{L^2(\partial D)} = \langle(\tfrac{1}{2}I + K^*)F, h\rangle_{L^2(\partial D)}$$

for every $h \in C_0^\infty(\mathbb{R}^d)$. Consequently, the function $(\frac{1}{2}I + K^*)F - g|_{\partial D}$ would belong to $C_0^\infty(\mathbb{R}^d)^\perp$ and hence, by choosing sequence $\{h_j\}_{j=1}^\infty \subset C_0^\infty(\mathbb{R}^d)$ that converges to $(\frac{1}{2}I+K^*)F-g|_{\partial D}$ and using an elementary property of an inner product space (see page 67 on [16]), we would see that

$$\|(\tfrac{1}{2}I + K^*)F - g\|_{L^2(\partial D)} \leq \|(\tfrac{1}{2}I + K^*)F - g - h_j\|_{L^2(\partial D)} \to 0,$$

as j tends to infinity. Therefore, we would obtain

$$g|_{\partial D} = (\tfrac{1}{2}I + K)F.$$

By applying Hölder's inequality and noticing that we may replace sequences with subsequences, we see that to prove (7.19), it suffices to show that

$$\|(\tfrac{1}{2}I + K_j)h \circ p_j - (\tfrac{1}{2}I + K)h\|_{L^2(\partial D)} \to 0, \tag{7.20}$$

as j tends to infinity. With the help of Lemma 4.2, we write

$$\begin{aligned} G_j(z) &:= (\tfrac{1}{2}I + K_j)h \circ p_j(z) - (\tfrac{1}{2}I + K)h(z) \\ &= \frac{1}{\omega_d}\int_{\partial\Omega_j} \frac{\langle \nu_j(\tilde{w}), p_j(z) - \tilde{w}\rangle}{|p_j(z) - \tilde{w}|^d}(h(\tilde{w}) - h \circ p_j(z))\, d\sigma_j(\tilde{w}) \\ &\quad - \frac{1}{\omega_d}\int_{\partial D} \frac{\langle \nu(w), z - w\rangle}{|z - w|^d}(h(w) - h(z))\, d\sigma(w) \\ &\quad + (h \circ p_j(z) - h(z)). \end{aligned}$$

By adopting a notation

$$\Theta(z, w) := \frac{1}{\omega_d}\frac{z - w}{|z - w|^d}(h(w) - h(z)),$$

we are able to write

$$\begin{aligned} G_j(z) = \bigg(&\int_{\partial D} \langle \nu_j \circ p_j(w), \Theta(p_j(z), p_j(w))\rangle J_j(w)\, d\sigma(w) \\ &- \int_{\partial D} \langle \nu(w), \Theta(z, w)\rangle\, d\sigma(w)\bigg) \\ &+ (h \circ p_j(z) - h(z)). \end{aligned}$$

Note that we do not have means to use the dominated convergence theorem. For example, we cannot assume that functions p_j, $j \in \mathbb{N}$, are bi-Lipschitz. Therefore, let us fix $\varepsilon > 0$ and then we continue to write

$$\begin{aligned} G_j(z) = \bigg(&\int_{\partial D} \langle \nu_j \circ p_j(w), \Theta(p_j(z), p_j(w)) \rangle \mathbf{1}_{B(p_j(z),\varepsilon)^c}(p_j(w)) J_j(w)\, d\sigma(w) \\ &- \int_{\partial D} \langle \nu(w), \Theta(z,w) \rangle \mathbf{1}_{B(z,\varepsilon)^c}(w)\, d\sigma(w) \bigg) \\ &+ \int_{\partial D} \langle \nu_j \circ p_j(w), \Theta(p_j(z), p_j(w)) \rangle \mathbf{1}_{B(p_j(z),\varepsilon)}(p_j(w)) J_j(w)\, d\sigma(w) \\ &- \int_{\partial D} \langle \nu(w), \Theta(z,w) \rangle \mathbf{1}_{B(z,\varepsilon)}(w)\, d\sigma(w) \\ &+ (h \circ p_j(z) - h(z)) \\ =:\ & G_{1,j}(z) + G_{2,j}(z) + G_{3,j}(z) + G_{4,j}(z). \end{aligned}$$

We estimate

$$\begin{aligned} |G_{1,j}(z)| &\le C \int_{\partial D} \frac{\mathbf{1}_{B(p_j(z),\varepsilon)^c}(p_j(w))}{|p_j(z) - p_j(w)|^{d-2}}\, d\sigma(w) + C \int_{\partial D} \frac{\mathbf{1}_{B(z,\varepsilon)^c}(w)}{|z-w|^{d-2}}\, d\sigma(w) \\ &\le C\varepsilon^{2-d}. \end{aligned}$$

The dominated convergence theorem implies that the sequence $\{G_{1,j}\}_{j=1}^{\infty}$ converges to zero pointwise for almost everywhere on ∂D. As a result, the sequence $\{G_{1,j}\}_{j=1}^{\infty}$ converges to zero also in $L^2(\partial D)$. Also, we may estimate

$$\begin{aligned} |G_{2,j}(z)| &\le C \int_{\partial D} \frac{\mathbf{1}_{B(p_j(z),\varepsilon)^c}(p_j(w))}{|p_j(z) - p_j(w)|^{d-2}}\, d\sigma(w) \\ &\le C \int_{\partial \Omega_j} \frac{\mathbf{1}_{B(\tilde{z},\varepsilon)}(\tilde{w})}{|\tilde{z} - \tilde{w}|^{d-2}} [J_j \circ p_j^{-1}(\tilde{w})]^{-1}\, d\sigma_j(\tilde{w}) \\ &\le C \int_{\partial \Omega_j} \frac{\mathbf{1}_{B(\tilde{z},\varepsilon)}(\tilde{w})}{|\tilde{z} - \tilde{w}|^{d-2}}\, d\sigma_j(\tilde{w}). \end{aligned}$$

By moving into local coordinates, we estimate further:

$$\begin{aligned} |G_{2,j}(z)| &\le C \int_{\mathbb{R}^{d-1}} \frac{\mathbf{1}_{B(x,\varepsilon)}(y)}{|x-y|^{d-2}}\, dy \\ &\le C\varepsilon^{2-d} \int_{\mathbb{R}^{d-1}} \left| \frac{y-x}{\varepsilon} \right|^{2-d} \mathbf{1}_{B(0,1)}\Big(\frac{y-x}{\varepsilon} \Big)\, dy \\ &\le C\varepsilon \int_{B(0,1)} |y|^{2-d}\, dy \\ &\le C\varepsilon. \end{aligned}$$

By arguing in a similar manner as above, we would get

$$|G_{3,j}(z)| \leq C\varepsilon.$$

Furthemore, the dominated convergence theorem implies that

$$\|G_{4,j}\|_{L^2(\partial D)} = \int_{\partial D} |h \circ p_j(z) - h(z)|^2 \, d\sigma(w) \to 0,$$

as j tends to infinity. By combining the above observations we see that the sequence $\{G_j\}_{j=1}^{\infty}$ converges to zero in $L^2(\partial D)$, because we could have selected $\varepsilon > 0$ to be arbirarily small. Hence the condition (7.20) is true and hence, we have shown that $\mathrm{Im}(\frac{1}{2}I + K^*)$ is dense in $L^2(\partial D)$, assuming that the sequence $\{f_j\}_{j=1}^{\infty}$ satisfies the condition (7.18).

However, if the sequence $\{f_j\}_{j=1}^{\infty}$ does not satisfy the condition (7.18), then we define sequences $\{\tilde{f}_j\}_{j=1}^{\infty}$ and $\{\tilde{F}_j\}_{j=1}^{\infty}$ by writing

$$\tilde{f}_j = f_j \|f_j\|_{L^2(\partial\Omega_j)}^{-1} \quad \text{and} \quad \tilde{F}_j = (\tilde{f}_j \circ p_j) J_j, \quad j \in \mathbb{N}.$$

The sequence $\{\tilde{f}_j\}_{j=1}^{\infty}$ satisfies the condition (7.18), which implies, as before, that the sequence $\{\tilde{F}_j\}_{j=1}^{\infty}$ converges to some function $\tilde{F} \in L^2(\partial D)$ in the weak sense. Furthermore, we have

$$(\tfrac{1}{2}I + K_j^*)\tilde{f}_j = g|_{\partial\Omega_j} \|f_j\|_{L^2(\partial\Omega_j)}^{-1}, \quad j \in \mathbb{N}.$$

The above condition corresponds to the condition (7.17) and therefore, by repeating the earlier arguments of the part (iii), we obtain

$$(\tfrac{1}{2}I + K^*)\tilde{F} = 0.$$

Because the operator $\frac{1}{2}I + K^*$ was shown to be injective, we deduce that the sequence $\{\tilde{F}_j\}_{j=1}^{\infty}$ converges to zero in the weak sense.

We apply the operator inequality, to get

$$\|\tilde{f}_j\|_{L^2(\partial\Omega_j)} \leq C \left(\|(\tfrac{1}{2}I + K^*)\tilde{f}_j\|_{L^2(\partial\Omega_j)} + \left| \int_{\partial\Omega_j} S_j \tilde{f}_j \, d\sigma_j \right| \right),$$

where the constant $C > 0$ is independent of j. Recall that the approximation scheme $\Omega_j \searrow D$ is such that the constants in the operator inequalities depend only on the domain D. Now, because we know that

$$\|\tilde{f}_j\|_{L^2(\partial\Omega_j)} = 1$$

for all $j \in \mathbb{N}$ and

$$\|(\tfrac{1}{2}I + K_j^*)\tilde{f}_j\|_{L^2(\partial\Omega_j)} \to 0,$$

as j tends to infinity, the proof will be complete after we have shown that

$$\left| \int_{\partial\Omega_j} S_j \tilde{f}_j \, d\sigma_j \right| \to 0, \tag{7.21}$$

as j tends to infinity. Let us write

$$\begin{aligned}
H_j(z) &:= \omega_d(d-2) \int_{\partial\Omega_j} S_j \tilde{f}_j(\tilde{z}) \, d\sigma_j(\tilde{z}) \\
&= \omega_d(d-2) \int_{\partial D} \left[S_j \tilde{f}_j \circ p_j(z) \right] J_j(z) \, d\sigma(z) \\
&= \int_{\partial D} \int_{\partial\Omega_j} \frac{\tilde{f}(\tilde{w}) \, d\sigma_j(\tilde{w})}{|p_j(z) - \tilde{w}|^{d-2}} J_j(z) \, d\sigma(z) \\
&= \int_{\partial D} \int_{\partial D} \frac{[(\tilde{f} \circ p_j) J_j](w) \, d\sigma(w)}{|p_j(z) - p_j(w)|^{d-2}} J_j(z) \, d\sigma(z) \\
&= \int_{\partial D} \int_{\partial D} \frac{\tilde{F}_j(w) \, d\sigma(w)}{|p_j(z) - p_j(w)|^{d-2}} J_j(z) \, d\sigma(z).
\end{aligned}$$

We continue to write

$$\begin{aligned}
H_j(z) = &\int_{\partial D} \int_{\partial D} (|p_j(z) - p_j(w)|^{2-d} - |z-w|^{2-d}) \tilde{F}_j(w) J_j(z) \, d\sigma(w) \, d\sigma(z) \\
&+ \int_{\partial D} \int_{\partial D} \frac{\tilde{F}_j(w)}{|z-w|^{d-2}} (J_j(z) - 1) \, d\sigma(w) \, d\sigma(z) \\
&+ \int_{\partial D} \int_{\partial D} \frac{\tilde{F}_j(w)}{|z-w|^{d-2}} \, d\sigma(w) \, d\sigma(z) \\
=: &\, H_{1,j}(z) + H_{2,j}(z) + H_{3,j}(z).
\end{aligned}$$

Once again, let us fix $\varepsilon > 0$ and denote

$$\Upsilon(z,w) = |z-w|^{2-d}.$$

Moreover, let us adopt notations

$$\Upsilon_\varepsilon(z,w) = \Upsilon(z,w) \mathbf{1}_{B(z,\varepsilon)}(w) \quad \text{and} \quad \Upsilon_\varepsilon^c(z,w) = \Upsilon(z,w) \mathbf{1}_{B(z,\varepsilon)^c}(w).$$

By using the above notations we may write

$$\begin{aligned}
H_{1,j}(z) &= \int_{\partial D}\int_{\partial D} (\Upsilon^c_\varepsilon(p_j(z), p_j(w)) - \Upsilon^c_\varepsilon(z,w))\tilde{F}_j(w) J_j(w)\, d\sigma(w)\, d\sigma(z) \\
&\quad + \int_{\partial D}\int_{\partial D} \Upsilon_\varepsilon(p_j(z), p_j(w))\tilde{F}_j(w) J_j(w)\, d\sigma(w)\, d\sigma(z) \\
&\quad - \int_{\partial D}\int_{\partial D} \Upsilon_\varepsilon(z,w)\tilde{F}_j(w) J_j(w)\, d\sigma(w)\, d\sigma(z) \\
&=: \tilde{H}_{1,j}(z) + \tilde{H}_{2,j}(z) + \tilde{H}_{3,j}(z).
\end{aligned}$$

By applying Hölder's inequality, we estimate

$$|\tilde{H}_{1,j}(z)| \le C \int_{\partial D} \left(\int_{\partial D} |\Upsilon^c_\varepsilon(p_j(z), p_j(w)) - \Upsilon^c_\varepsilon(z,w)|^2\, d\sigma(w)\right)^{1/2} d\sigma(z).$$

By letting j tend to infinity, we see that $\{\tilde{H}_{1,j}\}_{j=1}^\infty$ converges to zero pointwise for almost everywhere on ∂D. Fubini's theorem and earlier observations allow us to estimate

$$\begin{aligned}
|\tilde{H}_{2,j}(z)| &= C \int_{\partial D} |\tilde{F}_j(w)| \left(\int_{\partial D} |\Upsilon_\varepsilon(p_j(w), p_j(z))|\, d\sigma(z) \right) d\sigma(w) \\
&\le C\varepsilon \int_{\partial D} |\tilde{F}_j(w)|\, d\sigma(w) \\
&\le C\varepsilon.
\end{aligned}$$

Similarly, we see that

$$|\tilde{H}_{3,j}(z)| \le C\varepsilon.$$

Because $\varepsilon > 0$ was selected to be arbitrarily small, we conclude that the sequence $\{H_{1,j}\}_{j=1}^\infty$ converges pointwise to zero for almost everywhere on ∂D. By applying Hölder's inequality, we find out that

$$H_{2,j}(z) = \int_{\partial D} S\tilde{F}_j(z)(J_j(z) - 1)\, d\sigma(z) \le C\|J_j - 1\|_{L^2(\partial D)} \to 0,$$

as j tends to infinity. Because the sequence $\{\tilde{F}_j\}_{j=1}^\infty$ converges zero in the weak sense, we see that

$$H_{3,j}(z) = \int_{\partial D} \tilde{F}_j(w) \left(\int_{\partial D} \frac{d\sigma(z)}{|z-w|^{d-2}} \right) d\sigma(w) \to 0,$$

as j tends to infinity.

By combining the above observations, we conclude that the sequence $\{H_j\}_{j=1}^\infty$ converges pointwise to zero for almost everywhere on ∂D. Thus, the condition (7.21) is satisfied and the proof is complete. □

The invertibility of the operator $\frac{1}{2}I + K$ and the jump relation for the double layer potential in Lipschitz domains imply that there is a solution to the Dirichlet problem in the non-tangential sense. The uniqueness of the solution can be achieved by adding a boundedness condition for the non-tangential maximal function into the Dirichlet problem. This is because the result of B. E. J. Dahlberg [5] states that the non-tangential maximal function of the solution is majorized by the L^p-norm of the boundary data. Moreover, according to Verchota, the proof of Theorem 7.2 works also on Lipschitz domains. Thus, we get the following result.

Theorem 7.5. *Let D be a Lipschitz domain with a connected boundary ∂D, suppose that $\{\Gamma\}$ is a regular family of cones for ∂D described in* [19] *and assume that $g \in L^2(\partial D)$. Then there is a unique harmonic function u, defined in D, such that the following conditions hold:*

(i) *For almost every $z \in \partial D$*

$$\lim_{\substack{x \to z \\ x \in \Gamma^i(z)}} u(x) = g(z).$$

(ii) *The interior non-tangential maximal function*

$$u_*^i(z) = \sup\{|u(x)| : x \in \Gamma^i(z)\}$$

satisfies a condition

$$\|u_*^i\|_{L^p(\partial D)} \leq C\|g\|_{L^p(\partial D)}.$$

Furthermore, the solution has the form of the double layer potential:

$$u(x) = \frac{1}{\omega_d} \int_{\partial D} \frac{\langle \nu(w), x - w \rangle}{|x - w|^d} f(w)\, d\sigma(w), \quad f \in L^2(\partial D).$$

Inspired by the work of E. B. Fabes, M. Jodeit Jr. and N. M. Rivière in [8] the method of layer potentials has been applied successfully to various boundary value problems in domains that are C^1 or more general. As a consequence, there are several directions where one could continue. Maybe the most natural choice would be to study the Neumann problem for Laplace's equation, first in C^1-domains with L_0^p-boundary data assuming that $1 < p < \infty$ and then in Lipschitz domains with L_0^2-boundary data. The former is considered in [8] and the latter in [18].

A consistent step from the above results would be to consider unique solvability of the Dirichlet and the Neumann problems for Laplace's equation in Lipschitz domains with L^p-boundary data by applying the method of layer potentials. This was done by B. E. J. Dahlberg and C. E. Kenig in their work [6]. They were able to prove that for each bounded Lipschitz domain there is a number $\varepsilon > 0$, depending on the Lipschitz domain, such that the corresponding Dirichlet problem is uniquely solvable with L^p-boundary data, where $2 - \varepsilon < p < \infty$, and the solution has a double layer potential representation. Furthermore, they proved that a similar result holds also for the Neumann problem, when $1 < p < 2 + \varepsilon$.

A final direction to be introduced here is the use of the method of layer potentials for boundary value problems that are not associated to Laplace's equation but to other partial differential equations. For example, we could follow the work [9] and consider the initial boundary value problem in a C^1-domain $D \subset \mathbb{R}^d$ for the heat equation

$$\begin{cases} u_t - \Delta u = 0 & \text{in } D_T \\ u = g & \text{on } \partial D_T \\ u = 0 & \text{in } D \times \{t = 0\}. \end{cases}$$

Here we have denoted $D_T = D \times (0, T)$ and $\partial D_T = \partial D \times (0, T)$ for some $T > 0$ and we have assumed that the boundary data g belong to $L^p(\partial D_T)$ with $p \in (1, \infty)$. To solve this problem, we would proceed almost in a similar manner as in the case of the Dirichlet problem for Laplace's equation. The difference is that we would try to find a solution in the form of the double layer heat potential:

$$u(x,t) = \int_0^t \int_{\partial D} \frac{\langle \nu(w), x - w \rangle}{(t-s)^{d/2+1}} \exp\left(-\frac{|x-w|^2}{4(t-s)} \right) f(w,s)\, d\sigma(w)\, ds.$$

Then we would establish a jump relation

$$\lim_{\substack{x \to z \\ x \in \Gamma_\alpha^i(z)}} u(x,t) = (c_d I + J) f(z,t)$$

for almost every $(z,t) \in \partial D_T$, where the constant c_d depends only on the dimension d and the where J is the corresponding boundary integral operator defined by

$$Jf(z,t) = \lim_{\varepsilon \to 0} \int_0^{t-\varepsilon} \int_{\partial D} \frac{\langle \nu(w), z - w \rangle}{(t-s)^{d/2+1}} \exp\left(-\frac{|z-w|^2}{4(t-s)} \right) f(w,s)\, d\sigma(w)\, ds.$$

In this way, we would have reduced the initial boundary value problem into the problem of the invertibility of the operator $c_d I + J$ in $L^p(\partial D_T)$.

However, the preceding subjects are not in the scope of this thesis and therefore their detailed treatment is left as a challenge for the reader.

Appendix

The purpose of this appendix is to fill in the gaps of the main text. We begin by considering implications of the divergence theorem and after that, we prove some miscellaneous lemmas. Throughout this appendix, we assume that D is a bounded C^1-domain.

Theorem A.1. (Divergence theorem). *Suppose that* $u \in C^1(\overline{D})$. *Then*

$$\int_D \operatorname{div}(u)\,dx = -\int_{\partial D} \langle u, \nu \rangle\, d\sigma.$$

As an immediate consequence, we get Green's formulas [7].

Theorem A.2. (Green's formulas). *Suppose that* $u, v \in C^2(\overline{D})$. *Then*

(i) $\int_D \Delta u\, dx = -\int_{\partial D} \frac{\partial u}{\partial \nu}\, d\sigma.$

(ii) $\int_D \langle \nabla u, \nabla v \rangle\, dx = -\int_D u \Delta v\, dx - \int_{\partial D} u \frac{\partial v}{\partial \nu}\, d\sigma.$

(iii) $\int_D (u\Delta v - v \Delta u)\, dx = \int_D \left(v \frac{\partial u}{\partial \nu} - u \frac{\partial v}{\partial \nu} \right)\, d\sigma.$

Lemma A.1. *Suppose that functions* u, v *and* w *belong to* $C^2(\overline{D})$. *Then*

$$\int_D w \langle \nabla v, \nabla u \rangle\, dx = -\int_D \langle \nabla w, \nabla v \rangle u\, dx - \int_D w(\Delta v) u\, dx - \int_{\partial D} w \frac{\partial v}{\partial \nu} u\, d\sigma.$$

Proof. We apply the divergence theorem to obtain

$$\int_D \operatorname{div}(wu\nabla v)\, dx = -\int_{\partial D} w \frac{\partial v}{\partial \nu} u\, d\sigma.$$

On the other hand, we can calculate

$$\operatorname{div}(wu\nabla v) = w\langle \nabla v, \nabla u \rangle + \langle \nabla w, \nabla v \rangle u + w(\Delta v) u.$$

Finally, by using these observations, we obtain the desired identity. □

Lemma A.2. *Let $w, u \in C^1(D)$ and $v \in C^2(D)$. Also, assume that the support of the function v is contained in $D_\varepsilon := \{x \in D : \mathrm{dist}(x, \partial D) \geq \varepsilon\}$ for some $\varepsilon > 0$. Then*

$$\int_D w\langle \nabla v, \nabla u\rangle \, dy = -\int_D \langle \nabla w, \nabla v\rangle u \, dx - \int_D w(\Delta v) u \, dx.$$

Proof. We take an approximation scheme $\Omega_j \nearrow D$ such that $\partial\Omega_j \subset D \setminus D_\varepsilon$ for all $j \in \mathbb{N}$. Then, by Lemma A.1, we have

$$\int_{\Omega_j} w\langle \nabla v, \nabla u\rangle \, dx = -\int_{\Omega_j} \langle \nabla w, \nabla v\rangle u \, dx - \int_{\Omega_j} w(\Delta v) u \, dx.$$

We estimate

$$|w(x)\langle \nabla v(x), \nabla u(x)\rangle| \leq |w(x)\nabla u(x)| \mathbf{1}_{D_\varepsilon}(x),$$

and observe that function on the right-hand side of the above inequality is integrable in D, because $w\nabla u \in C(\overline{D}_\varepsilon)$. Similar estimates imply that we are able to use the dominated convergence theorem. Therefore, we let j tend to infinity and as a consequence, we obtain the desired identity. $\square$

Lemma A.3. *Suppose that $z \in \mathbb{R}^d$, $r > 0$ and $\alpha \in (0, 1)$. Then*

$$|B(z, r) \cap \Gamma_\alpha(z)| = C_\alpha |B(z, r)|,$$

where $C_\alpha > 0$ is a constant that tends to one, as α tends to zero.

Proof. We will denote $z = (x, t)$, where $x \in \mathbb{R}^{d-1}$ and $t \in \mathbb{R}$. Let us define a truncated cone

$$U(\alpha, r) := \{(x, t) \in \mathbb{R}^d : t > \alpha\sqrt{|x|^2 + t^2},\, t < r\alpha\}$$

and a segment of a ball

$$V(\alpha, r) = \{(x, t) \in \mathbb{R}^d : \sqrt{|x|^2 + t^2} < r,\, t > r\alpha\}.$$

These sets satisfy the condition:

$$|B(z, r) \cap \Gamma_\alpha(z)| = 2|U(\alpha, r)| + 2|V(\alpha, r)|.$$

Let us denote $\gamma_\alpha = \sqrt{1 - \alpha^2}/\alpha$. If $(x, t) \in U(\alpha, r)$, then $|x| < \gamma_\alpha t$. By recalling that

$$|B(z, r)| = \frac{\omega_d}{d} r^d$$

and then integrating along the t-axis, we obtain

$$|U(\alpha,r)| = \int_0^{r\alpha} |B(x,\gamma_\alpha t)|dt = \frac{\omega_{d-1}}{(d-1)\omega_d}\gamma_\alpha^{d-1}\alpha^d|B(z,r)|.$$

If $(x,t) \in V(\alpha,r)$, then $|x| < \sqrt{r^2-t^2}$. Once again, integration along the t-axis and some elementary calculations allow us to deduce that

$$|V(z,r)| = \left(\frac{d}{\omega_d}\frac{\omega_{d-1}}{d-1}\int_{\arcsin(\alpha)}^{\frac{\pi}{2}} \cos^d(t)\,dt\right)|B(z,r)|.$$

Now, we have

$$|B(z,r)\cap\Gamma_\alpha(z)| = C_\alpha|B(z,r)|,$$

where the constant $C_\alpha > 0$ is given by

$$C_\alpha = \frac{2\,\omega_{d-1}}{(d-1)\omega_d}\gamma_\alpha^{d-1}\alpha^d + 2\frac{d}{\omega_d}\frac{\omega_{d-1}}{d-1}\int_{\arcsin(\alpha)}^{\frac{\pi}{2}} \cos^d(t)\,dt.$$

The first term on the right-hand side converges to zero, as α tends to zero. The second term converges to one, as α tends to zero. We see this by using the dominated convergence theorem and the identity

$$\int_0^{\frac{\pi}{2}} \cos^d(t)\,dt = \frac{1}{2}\frac{\omega_d}{d}\frac{d-1}{\omega_{d-1}}. \tag{7.22}$$

The identity (7.22) follows easily after we notice that

$$|B(0,1)| = \int_{-1}^{1}\left(\int_{|x|<\sqrt{1-t^2}} dx\right)dt = 2\frac{\omega_{d-1}}{d-1}\int_0^{\frac{\pi}{2}} \cos^d(t)\,dt.$$

This completes the proof. □

Lemma A.4. *Suppose that $D \subset \mathbb{R}^2$ is a bounded C^1-domain. Then the kernel*

$$\Phi(z,w) = \frac{1}{2\pi}\log(|z-w|)$$

of the single layer potential satisfies the following conditions

$$\int_{\partial D} |\Phi(z,w)|\,d\sigma(z) \le C \quad \text{and} \quad \int_{\partial D} |\Phi(z,w)|\,d\sigma(w) \le C$$

for some constant $C > 0$.

Proof. Suppose that $\varepsilon > 0$ is sufficiently small. Then we may estimate

$$\begin{aligned}\int_{\partial D} |\Phi(z,w)| \mathbf{1}_{B(z,\varepsilon)}(w)\, d\sigma(w) &\le C \int_{x-\varepsilon}^{x+\varepsilon} |\log(|x-y|)|\; dy \\ &\le -C \int_0^{\varepsilon} \log(t)\, dt \\ &= C\varepsilon(1-\log(\varepsilon)).\end{aligned}$$

Also, we notice that

$$\lim_{\varepsilon\to 0} \varepsilon \log(\varepsilon) = 0.$$

These observations and the arguments of Lemma 3.3 complete the proof. □

Lemma A.5. *Let D be a bounded C^1-domain in $\mathbb{R}^2$. Suppose that a density f belongs to $L^p(\partial D)$ for some $p \in (1,\infty)$ and that it satisfies the equation*

$$(\tfrac{1}{2}I + K^*)f = 0.$$

Then the single layer potential $\mathcal{S}$, related to the density f, vanishes at infinity. In other words, the single layer potential satisfies the condition

$$\lim_{|x|\to\infty} \mathcal{S}f(x) = 0.$$

Proof. Fubini's theorem allows us to change the order of the integration:

$$\begin{aligned}\int_{\partial D} K_\varepsilon^* f(z)\, d\sigma(z) &= \int_{\partial D} \left(\int_{\partial D} \Psi_\varepsilon(w,z) f(w)\, d\sigma(w) \right) d\sigma(z) \\ &= \int_{\partial D} f(w) \left(\int_{\partial D} \Psi_\varepsilon(w,z)\, d\sigma(z) \right) d\sigma(w).\end{aligned}$$

By applying the above identity, the dominated convergence theorem and Lemma 4.2, we get

$$\begin{aligned}\int_{\partial D} K^* f(z)\, d\sigma(z) &= \int_{\partial D} f(w) \left(\lim_{\varepsilon\to 0} \int_{\partial D} \Psi_\varepsilon(w,z)\, d\sigma(z) \right) d\sigma(w) \\ &= \frac{1}{2} \int_{\partial D} f(w)\, d\sigma(w).\end{aligned}$$

The above identity and the assumption imply that

$$\int_{\partial D} f\, d\sigma = \frac{1}{2}\int_{\partial D} f\, d\sigma + \int_{\partial D} K^* f\, d\sigma = \int_{\partial D} (\tfrac{1}{2}I + K^*) f\, d\sigma = 0.$$

Now, we may represent the single layer potential in the form

$$\begin{aligned}\mathcal{S}f(x) &= \frac{1}{2\pi}\int_{\partial D}(\log|x-w| - \log|x|)\, f(w)\, d\sigma(w) + \frac{\log|x|}{2\pi}\int_{\partial D} f(w)\, d\sigma(w)\\ &= \frac{1}{2\pi}\int_{\partial D}\log\left(|x-w|/|x|\right) f(w)\, d\sigma(w).\end{aligned}$$

Let us denote

$$w_0 := \sup\{|w| : w \in \partial D\}$$

and assume that $|x| > w_0$. Then, by applying the triangle inequality, we can estimate

$$|\log\left(|x-w|/|x|\right)| \le \max\left\{-\log\left(1 - w_0/|x|\right), \log\left(1 + w_0/|x|\right)\right\} =: m(|x|).$$

The above estimate implies that

$$|\mathcal{S}f(x)| \le C m(|x|)\|f\|_{L^p(\partial D)}.$$

Because the function m converges to zero as $|x|$ tends to infinity, we know that $\mathcal{S}f(x)$ converges to zero as $|x|$ tends to infinity. □

Lemma A.6. *Let $\varphi : \mathbb{R}^{d-1} \to \mathbb{R}$ be a continuously differentiable function with compact support. Then there are functions $\psi_j \in C_0^\infty(\mathbb{R}^{d-1})$ such that the sequence $\{\psi_j\}_{j=1}^\infty$ converges uniformly to φ and the sequence $\{\nabla\psi_j\}_{j=1}^\infty$ converges uniformly to $\nabla\varphi$.*

Proof. We use mollifiers η_ε that are defined in [7] on the page 713. For every $j \in \mathbb{N}$, we define functions

$$\psi_j = \eta_{\varepsilon_j} * \varphi \in C_0^\infty(\mathbb{R}^d).$$

Suppose that $x \in \mathbb{R}^d$. Then, we have

$$|\varphi(x) - \psi_j(x)| \le \int_{B(x,\varepsilon)} |\varphi(x) - \varphi(y)|\eta_{\varepsilon_j}(x-y)\, dy \le C\varepsilon_j.$$

This implies that

$$\|\varphi - \psi_j\|_\infty \le C\varepsilon_j \to 0,$$

as j tends to infinity.

With a similar argument, we obtain

$$\begin{aligned}|\nabla\varphi(x) - \nabla\psi_j(x)| &\le \int_{B(x,\varepsilon)} |\nabla\varphi(x) - \nabla\varphi(y)|\eta_{\varepsilon_j}(x-y)\, dy\\ &\le \sup_{y\in B(x,\varepsilon_j)} |\nabla\varphi(x) - \nabla\varphi(y)|.\end{aligned}$$

We observe that partial derivatives $\partial_{x_i}\varphi$ are uniformly continuous in $\mathbb{R}^d$ because they are continuous and compactly supported. Therefore, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\sup_{y\in B(x,\varepsilon_j)} |\nabla\varphi(x) - \nabla\varphi(y)| \leq \sum_{i=1}^{d} \sup_{y\in B(x,\varepsilon_j)} |\partial_{x_i}\varphi(x) - \partial_{x_i}\varphi(y)| \leq \varepsilon,$$

when $j \geq N$. Finally, we let $\varepsilon > 0$ tend to zero. As a consequence

$$\|\nabla\varphi - \nabla\psi_j\|_\infty \to 0,$$

when j tends to infinity. □

Lemma A.7. *Let $\varphi \in C_0^2(\mathbb{R}^d)$. Then*

$$|\varphi(x) - \varphi(y) - \langle\nabla\varphi(y), x-y\rangle| \leq C|x-y|^2.$$

Proof. Let us define a function $g(t) = \varphi(y + t(x-y))$ and let us denote $z(t) = y + t(x-y)$. Taylor's theorem implies that there is a number $\xi \in]s,t[$ such that

$$g(t) = g(s) + g'(s)(t-s) + \frac{1}{2}g''(\xi)(t-s)^2.$$

By differentiating, we get

$$g'(t) = \langle\nabla\varphi(z(t)), x-y\rangle \quad \text{and} \quad g''(t) = \left\langle \frac{d}{dt}\left(\nabla\varphi(z(t))\right), x-y \right\rangle.$$

The Schwartz inequality implies

$$|g''(t)| = \left|\left\langle \frac{d}{dt}\left(\nabla\varphi(z(t))\right), x-y \right\rangle\right| \leq \left|\frac{d}{dt}\left(\nabla\varphi(z(t))\right)\right| |x-y|.$$

To estimate the right-hand side term in the above inequality, we write

$$\frac{d}{dt}\left(\nabla\varphi(z(t))\right) = \sum_i \frac{d}{dt}(\partial_i\varphi(z(t)))e_i = \sum_i \left(\sum_j \langle\partial_{ij}\varphi(z(t)), x-y\rangle e_j\right) e_i.$$

Because the second partial derivatives of φ are continuous and compactly supported, we may estimate

$$\left|\frac{d}{dt}\left(\nabla\varphi(z(t))\right)\right| \leq \left(\sum_i \sum_j |\partial_{ij}\varphi(z(t))|\right) |x-y| \leq C|x-y|.$$

Finally, we obtain

$$|\varphi(x) - \varphi(y) - \langle\nabla\varphi(y), x-y\rangle| = \frac{1}{2}|g''(\xi)| \leq C|x-y|^2,$$

which completes the proof. □

Lemma A.8. *Let U be a connected subset of $\mathbb{R}^d$ and suppose that a function $u \in C^1(D)$ satisfies $\nabla u = 0$ in U. Then u is constant in U.*

Proof. Suppose $x, y \in U$. Because U is a connected subset of $\mathbb{R}^d$, it is possible to connect the points x and y with a polygonal path such that it is contained in U [17]. Suppose that the polygonal path is defined by a sequence of points $\{z_i\}_{i=1}^n$, in which $z_1 = x$ and $z_n = y$. As a consequence of the mean value theorem there are points $\xi_i \in U$, such that

$$|u(z_{i+1}) - u(z_i)| = |\nabla u(\xi_i)||z_{i+1} - z_i| = 0,$$

for every $i = 1, \ldots, n-1$. Finally, the use of the triangle inequality would imply $u(x) = u(y)$. Thus, the function u is constant in U. □

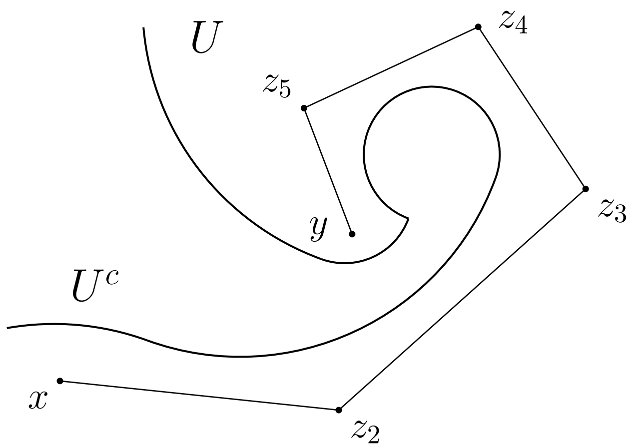

Figure A.1: Polygonal path

Glossary

$A^{\perp}$	orthogonal complement of a set $A \subset X$, $A^{\perp} = \{x \in X : \langle x, a \rangle = 0 \text{ for all } a \in A\}$	104
A^c	complement of a set $A \subset X$, $A^c = X \setminus A$	10
$B(X,Y)$	space of bounded linear operators $T : X \to Y$	17
$B(x,r)$	ball with a radius $r > 0$ centered at a point $x \in \mathbb{R}^d$	7
CZO	class of Calderon-Zygmund operators	35
C_0^{∞}	space of compactly supported smooth functions	14
C_0^k	space of compactly supported, k times continuously differentiable functions	7
$G(x,y)$	Green's function	96
I	identity operator, $I : X \to X$, $I(x) = x$	17
J_j	Jacobian related to $\Omega_j \nearrow D$ or to $\Omega_j \searrow D$	16
K	boundary integral operator	3
K^*	adjoint of the boundary integral operator K	4
$L^p(X)$	Lebesgue space of functions $f : X \to \mathbb{R}$	2
$L_0^p(X)$	subspace of $L^p(X)$ containing functions that satisfy $\int_X f\,dx = 0$	113
$L_1^p(\partial D)$	Sobolev-type space of functions $f : \partial D \to \mathbb{R}$	66
S	boundary integral operator related to $\mathcal{S}$	46
SK	class of standard kernels	32
S^{d-1}	unit sphere in $\mathbb{R}^d$	3
T^{-1}	inverse of an operator T	17
T^*	adjoint of an operator T	18
T_*	maximal operator of T, $T_* f(x) = \sup\{\lvert T_\varepsilon f(x)\rvert : \varepsilon > 0\}$	22
T_ε	truncated operator of T, $T_\varepsilon f(x) = \int k_\varepsilon(x,y) f(y)\,dy$	22
Z	coordinate cylinder	16
Δ	Laplace operator, $\Delta = \sum \frac{\partial^2}{\partial x_i^2}$	1

$\Gamma_\alpha(z)$	infinite cone with two components, an aperture $\alpha \in (0,1)$ and a vertex at a point $z \in \partial D$	10
$\Gamma^e_\alpha(z)$	exterior cone	10
$\Gamma^i_\alpha(z)$	interior cone	10
$\mathbb{N}$	set of natural numbers, $\mathbb{N} = \{1, 2, \dots\}$	15
Φ	kernel of the single layer potential	43
Ψ	kernel of the double layer potential	49
$*$	convolution, $f * g(x) = \int_{\mathbb{R}^{d-1}} f(x-y)g(y)\,dy$	20
$\circ$	composition, $f \circ g(x) = f(g(x))$	14
$\overline{A}$	closure of a set A	45
$\langle \cdot, \cdot \rangle$	Euclidian inner product, $\langle x, y \rangle = \sum x_i y_i$	3
$\langle \cdot \,\vert\, \cdot \rangle$	inner product related to $L^p(X)$, $\langle f \,\vert\, g \rangle = \int_X fg\,dx$	18
log	natural logarithm	43
$\mathcal{K}$	double layer potential	2
$\mathcal{M}$	Hardy-Littlewood maximal operator	19
$\mathcal{R}_\alpha(z,\delta)$	non-tangential approach region	68
$\mathcal{S}$	single layer potential	4
$\mathscr{F}_{\partial D}$	family of local characteristics for a boundary ∂D	9
$\mathscr{S}$	space of Schwarz functions	31
$\mathscr{S}'$	space of tempered distributions	31
∇	gradient, $\nabla f = (\frac{\partial f}{\partial x_1}, \dots \frac{\partial f}{\partial x_d})$	7
$\nearrow$	approximation scheme from inside, $\Omega_j \nearrow D$	16
$\nu(z)$	inward-pointing unit normal vector of a domain D at a point $z \in \partial D$	3
ω_d	surface area of a unit sphere in $\mathbb{R}^d$	3
∂A	boundary of a set A	1
$\searrow$	approximation scheme from outside, $\Omega_j \searrow D$	16
σ	surface measure on ∂D	2
σ_j	surface measure on $\partial \Omega_j$	16
$\mathbf{1}_A$	characteristic function of a set A, if $x \in A$, then $\mathbf{1}_A(x) = 1$; if $x \notin A$, then $\mathbf{1}_A(x) = 0$	22
$\mathrm{Im}(T)$	image of the operator $T : X \to Y$, $\mathrm{Im}(T) = \{Tf : f \in X\}$	104
$\mathrm{Ker}(T)$	kernel of the operator $T : X \to Y$, $\mathrm{Ker}(T) = \{f \in X : Tf = 0\}$	104
$\mathrm{div}(v)$	divergence of a vector v, $\mathrm{div}(v) = \sum \frac{\partial v_i}{\partial x_i}$	45
$\mathrm{supp}(f)$	support of a function f	14
$\vert\vert T \vert\vert$	operator norm of $T : X \to Y$, $\vert\vert T\vert\vert = \sup\{\vert\vert Tf\vert\vert_Y : \vert\vert f\vert\vert_X \le 1\}$	17

$\|\|\cdot\|\|_{L^p(X)}$	norm related to the Lebesgue space $L^p(X)$, $\|\|f\|\|_{L^p(X)} = \big(\int_X \|f\|^p \, dx\big)^{1/p}$	19
$\|\|f\|\|_\infty$	supremum norm of a function $f : X \to Y$, $\|\|f\|\|_\infty = \sup\{\|f(x)\| : x \in X\}$	8
e_i	i:th element of the standard basis of $\mathbb{R}^d$	45
$f\|_A$	restriction of a function f to a set A	2
$k_\varepsilon(x, y)$	truncated kernel, $k_\varepsilon(x, y) = k(x, y)\mathbf{1}_{B(x,\varepsilon)^c}(y)$	22
u_*^e	exterior non-tangential maximal function of u	67
u_*^i	interior non-tangential maximal function of u	67
$\|\cdot\|$	Euclidian norm, $\|x\| = \sqrt{x_1^2 + \cdots + x_d^2}$	2

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