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Quasiregular geometry: from maps to curves

Susanna Heikkilä

Doctoral dissertation

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Abstract

This dissertation concerns mathematical analysis, in particular quasiconformal geometry. The main theme of the dissertation is properties of quasiregular maps, quasiregular curves, and quasiregularly elliptic closed manifolds. The dissertation consists of four articles.

In the first article, we prove a Bonk–Heinonen type growth result for quasiregular curves satisfying additional algebraic assumptions. More precisely, we prove that such curves satisfy a weak reverse Hölder inequality which allows us to establish energy estimates.

In the second article, we prove with Pankka and Prywes a version of Liouville's theorem for quasiregular curves whose target is a product manifold equipped with a form intrinsic to the product structure. We also show that the result has stability with respect to small distortion. Additionally, we give examples of quasiregular curves which are not discrete.

In the third article, we prove with Pankka that, if a closed manifold admits a quasiregular map from a Euclidean space, then there exists an embedding of algebras from the cohomology of the target manifold to the exterior algebra. We also show that the main result implies a topological classification of closed simply connected quasiregularly elliptic four-dimensional manifolds.

In the fourth article, we generalize the above-mentioned embedding result for quasiregular curves. Specifically, we show that, under additional algebraic assumptions, if a closed manifold admits a quasiregular curve from a Euclidean space, then there exists a non-trivial homomorphism from the cohomology of the target manifold to the exterior algebra. We also apply the main result to give examples of closed manifolds which are not quasiregularly elliptic.

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Helsinki, December 2023

Susanna Heikkilä

List of included articles

This dissertation consists of an introductory part and the following four articles.

- [A] S. Heikkilä. Signed quasiregular curves. *J. Anal. Math.*, 150(1): 37–55, 2023.
- [B] S. Heikkilä, P. Pankka, and E. Prywes. Quasiregular Curves of Small Distortion in Product Manifolds. *J. Geom. Anal.*, 33(1):Paper No. 1, 44, 2023.
- [C] S. Heikkilä and P. Pankka. De Rham algebras of closed quasiregularly elliptic manifolds are Euclidean. Preprint, 2023. `https://arxiv.org/abs/2302.11440`
- [D] S. Heikkilä. Quasiregular curves and cohomology. Preprint, 2023. `https://arxiv.org/abs/2312.04347`

In the introductory part, these articles will be referred to as [A], [B], [C], and [D], respectively.

In the physical edition of this dissertation, the included versions of [A] and [B] are the final published versions, reproduced with permission from Springer Nature. The included versions of [C] and [D] are, respectively, the second and first pre-print version on arXiv.

The author of this dissertation made significant contributions to the research and writing of the joint articles [B] and [C].

1. Generalization of holomorphic maps

The study of quasiregular maps stems from the study of holomorphic maps. Since a holomorphic map $f = u+iv\colon \Omega \to \mathbb{C}$ on a domain $\Omega \subset \mathbb{C}$ satisfies the Cauchy–Riemann equations $\partial_1 u = \partial_2 v$ and $\partial_1 v = -\partial_2 u$, it follows that

$$Df(z) = \begin{pmatrix} \partial_1 u(z) & \partial_2 u(z) \\ \partial_1 v(z) & \partial_2 v(z) \end{pmatrix} = \begin{pmatrix} a(z) & -b(z) \\ b(z) & a(z) \end{pmatrix}$$

and hence

$$Df(z)^t Df(z) = \begin{pmatrix} a^2(z) + b^2(z) & 0 \\ 0 & a^2(z) + b^2(z) \end{pmatrix}.$$

We obtain that the operator norm of Df squared equals the Jacobian determinant of f, i.e.,

$$\|Df(z)\|^2 = a^2(z) + b^2(z) = \det Df(z)$$

for each $z \in \Omega$. Thus, holomorphic maps give examples of continuous maps $f \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{C})$, where $\Omega \subset \mathbb{C}$ is a domain, satisfying $\|Df(z)\|^2 = \det Df(z)$ for almost every $z \in \Omega$. In fact, all such maps are holomorphic; see e.g. the discussion in [2, p. 27].

The Cauchy–Riemann equations are defined for maps $\Omega \to \mathbb{R}^2$ on domains $\Omega \subset \mathbb{R}^2$. In contrast, the equality $\|Df\|^n = \det Df$ almost everywhere in Ω is a condition that can be posed for continuous maps $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$, where $\Omega \subset \mathbb{R}^n$ is a domain, for any $n \geq 2$. However, for $n \geq 3$, posing such a restriction yields a very limited collection of maps, namely, restrictions of Möbius transformations. This result is originally attributed to Liouville [15] and the following formulation is due to Reshetnyak [23]; see also [7, Theorem 16] and [12, Theorem 5.1.1].

Theorem 1 (Generalized Liouville's theorem)**.** *Let $n \geq 3$, let $\Omega \subset \mathbb{R}^n$ be a domain, and let $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ be a continuous map for which $\|Df\|^n = \det Df$ almost everywhere in Ω. Then f is a constant or a restriction of a Möbius transformation on $\mathbb{S}^n$.*

We remark that n is the natural integrability exponent here and in what follows since $\det Df \in L^1_{\mathrm{loc}}(\Omega)$ for $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$. However, the Generalized Liouville's theorem (Theorem 1) holds even for maps $f \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$, where $p(n) < p < n$; for details see [12, Theorem 14.4.2] and [12, Theorem 15.3.1].

For a continuous map $f \in W^{1,n}_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$, where $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ is a domain, the identity $\|Df\|^n = \det Df$ almost everywhere in Ω means heuristically that f preserves oriented angles almost everywhere in Ω. Hence, the Generalized Liouville's theorem (Theorem 1) can be summarized as follows: *maps preserving oriented angles yield a rich theory in the complex plane and a meagre theory in higher dimensions.*

1.1. **Quasiregular maps and their properties.** Quasiregular maps are obtained by relaxing preservation of oriented angles to their bounded distortion; see Figure 1 for an illustration of infinitesimal behaviour of quasiregular maps. Formally, a continuous map $f\colon M \to N$ between oriented Riemannian n-manifolds, $n \geq 2$, is *K-quasiregular for $K \geq 1$* if $f \in W^{1,n}_{\mathrm{loc}}(M,N)$ and

$$\|Df(x)\|^n \leq K J_f(x)$$

for almost every $x \in M$; here $Df(x)\colon T_xM \to T_{f(x)}N$ is the weak differential of f. We note that, since M and N are equidimensional oriented Riemannian manifolds, the operator norm $\|Df\|$ and the Jacobian determinant J_f are well-defined almost everywhere.

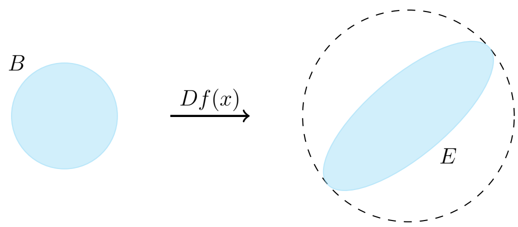

FIGURE 1. At almost every point $x \in M$, the weak differential $Df(x)$ of a K-quasiregular map $f\colon M \to N$ maps balls $B \subset T_xM$ to ellipsoids $E \subset T_{f(x)}N$ satisfying $\mathrm{vol}(B') \leq K\mathrm{vol}(E)$, where B' is the dashed ball.

In the complex plane, preservation of oriented angles corresponds precisely to holomorphicity. Even if we allow bounded distortion of oriented angles, we still have the following connection to holomorphicity in two dimensions; see e.g. [2, Theorem 5.5.1].

Theorem 2 (Stoïlow factorization)**.** *Let $\Omega \subset \mathbb{C}$ be a domain and let $f\colon \Omega \to \mathbb{C}$ be a K-quasiregular map. Then there exist a K-quasiregular homeomorphism $g\colon \Omega \to \Omega'$ and a holomorphic map $h\colon \Omega' \to \mathbb{C}$ for which $f = h \circ g$.*

Bounded distortion of oriented angles is flexible enough to admit varied examples also in higher dimensions; see e.g. [30, Section I.3]. On the other hand, bounded distortion of oriented angles is restrictive enough to obtain interesting results. The following seminal results are due to Reshetnyak; see [22], [24], [25], [26], and [27].

Theorem 3. *Let M and N be oriented Riemannian n-manifolds and let $f\colon M \to N$ be a non-constant K-quasiregular map. Then*

(1) *f is open and discrete,*
(2) *$J_f > 0$ almost everywhere in M,*
(3) *f is differentiable almost everywhere in M,*
(4) *f is locally $1/K'$-Hölder continuous for each $K' > K$.*

We remark that, in fact, a K-quasiregular map $\Omega \to \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^n$ is a domain, is $1/K$-Hölder continuous.

We assume a priori that a quasiregular map $M \to N$ between oriented Riemannian n-manifolds belongs to $W^{1,n}_{\mathrm{loc}}(M,N)$ since n is the natural integrability exponent. However, the differentials of quasiregular maps have, in fact, higher integrability, where the exponent depends only on the dimension and the distortion constant. This higher integrability is originally attributed to Gehring [8] and the following formulation is due to Elcrat and Meyers [16]; see also [3, Theorem 5.1].

Theorem 4 (Higher integrability)**.** *Let M and N be oriented Riemannian n-manifolds and let $f\colon M \to N$ be a non-constant K-quasiregular map. Then $f \in W^{1,p}_{\mathrm{loc}}(M,N)$, where $p = p(n,K) > n$.*

A celebrated result due to Rickman [28] is that quasiregular maps admit a variation of Picard theorem; recall that the classical Picard theorem states that, if $f\colon \mathbb{C} \to \mathbb{C}$ is holomorphic and $\mathbb{C} \setminus f(\mathbb{C})$ contains two distinct points, then f is constant.

Theorem 5 (Rickman's Picard Theorem)**.** *If $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is K-quasiregular and $\mathbb{R}^n \setminus f(\mathbb{R}^n)$ contains q distinct points, where $q = q(n,K) > 0$ is an integer, then f is constant.*

We note that, in contrast to the classical Picard theorem, it is possible that $q(n,K) > 2$ if $n \geq 3$; see [29] and [6].

1.2. **Quasiregular ellipticity.** A connected and oriented Riemannian n-manifold N is *K-quasiregularly elliptic* if there exists a non-constant K-quasiregular map $\mathbb{R}^n \to N$. We also say that N is *quasiregularly elliptic* if it is K-quasiregularly elliptic for some $K \geq 1$.

As positive examples, the n-sphere $\mathbb{S}^n$ and the n-torus $\mathbb{T}^n = (\mathbb{S}^1)^n$ are 1-quasiregularly elliptic. Namely, the stereographic projection

$$\iota\colon \mathbb{R}^n \to \mathbb{S}^n,\ x = (x_1,\ldots,x_n) \mapsto \frac{1}{1+\|x\|^2}\left(1-\|x\|^2, 2x_1, \ldots, 2x_n\right),$$

and the standard covering map

$$\pi\colon \mathbb{R}^n \to \mathbb{T}^n,\ (x_1,\ldots,x_n) \mapsto (e^{2\pi i x_1},\ldots,e^{2\pi i x_n}),$$

are 1-quasiregular. In fact, ι is a smooth embedding and π is a smooth local isometry. We remark that, for $n, m \geq 2$, the map

$$\iota \times \iota\colon \mathbb{R}^{n+m} \to \mathbb{S}^n \times \mathbb{S}^m, \quad (x,y) \mapsto (\iota(x), \iota(y)),$$

is not quasiregular, while

$$\pi \times \pi\colon \mathbb{R}^{n+m} \to \mathbb{T}^{n+m}, \quad (x,y) \mapsto (\pi(x), \pi(y)),$$

is the map $\pi\colon \mathbb{R}^{n+m} \to \mathbb{T}^{n+m}$ and thus 1-quasiregular.

We emphasize that the quasiregular ellipticity of an n-manifold N is a property of N alone; if N is quasiregularly elliptic, it may admit dissimilar non-constant quasiregular maps $\mathbb{R}^n \to N$. For example,

n-spheres admit Alexander maps $\mathbb{R}^n \to \mathbb{S}^n$ [1] in addition to stereographic projections $\iota\colon \mathbb{R}^n \to \mathbb{S}^n$; see Figure 2 for the two-dimensional Alexander map. In fact, Alexander maps yield that finite products of spheres of arbitrary dimension are quasiregularly elliptic. On the other hand, it is worth noting that a finite product of spheres admits a 1-quasiregular map from a Euclidean space only if either each sphere in the product is one-dimensional or the product has only one factor; see [4, Proposition 1.4] for details.

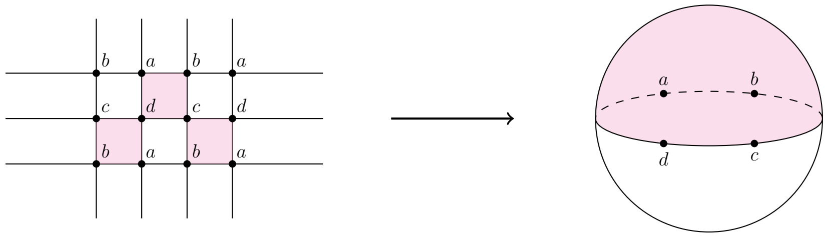

FIGURE 2. Alexander map $\mathbb{R}^2 \to \mathbb{S}^2$.

As discussed above, $\mathbb{S}^n$ and $\mathbb{T}^n$ are quasiregularly elliptic for each $n \geq 2$. As a matter of fact, up to a diffeomorphism, the only closed quasiregularly elliptic 2-manifolds are $\mathbb{S}^2$ and $\mathbb{T}^2$ by Stoïlow factorization (Theorem 2) and the Uniformization theorem. Thus, for example, the genus 2 surface $\mathbb{T}^2 \# \mathbb{T}^2$ is not quasiregularly elliptic. In three dimensions, up to a diffeomorphism, the only closed quasiregularly elliptic manifolds are $\mathbb{S}^3$, $\mathbb{T}^3$, $\mathbb{S}^2 \times \mathbb{S}^1$, and their quotients by Jormakka [14] and Perelman's solution to the Geometrization conjecture. Hence, for example, the connected sum $\mathbb{T}^3 \# \mathbb{T}^3$ is not quasiregularly elliptic.

In Jormakka's proof, the obstruction for quasiregular ellipticity is related to the growth of the fundamental group of the target. In higher dimensions, we have the following result proven by Varopoulous in 1992.

Theorem 6 ([32, Theorem X.5.1])**.** *Let N be a closed quasiregularly elliptic n-manifold. Then the fundamental group of N is virtually nilpotent and has polynomial growth of degree at most n.*

Theorem 6 does not yield further classifications but it does give examples of non-ellipticity in higher dimensions, e.g. the connected sum $\mathbb{T}^n \# \mathbb{T}^n$ for $n \geq 4$. As a matter of fact, connected sums $\mathbb{T}^n \# N$, where N is a closed, connected, and oriented Riemannian n-manifold, are not quasiregularly elliptic if $H^k_{\mathrm{dR}}(N) \neq 0$ for some $1 \leq k \leq n-1$ by a result due to Peltonen [19].

A natural question arising from these results is whether obstructions for quasiregular ellipticity require non-trivial fundamental group. This question was first posed by Gromov [9, p. 200] already in 1981.

Question 7 (Gromov's question)**.** *Are all closed, simply connected, and oriented Riemannian manifolds quasiregularly elliptic?*

In 2001, Bonk and Heinonen proved a qualitative obstruction for quasiregular ellipticity related to the size of the de Rham algebra.

Theorem 8 ([4, Theorem 1.1])**.** *Let N be a closed K-quasiregularly elliptic n-manifold. Then* $\dim H^k_{\mathrm{dR}}(N) \le C(n,k,K)$ *for* $k = 0, \dots, n$.

Almost two decades later, Prywes improved Theorem 8 by obtaining precise and sharp dimension bounds; the sharpness is given by $\mathbb{T}^n$.

Theorem 9 ([21, Theorem 1.1])**.** *Let N be a closed quasiregularly elliptic n-manifold. Then* $\dim H^k_{\mathrm{dR}}(N) \le \binom{n}{k}$ *for* $k = 0, \dots, n$.

As a consequence of Theorem 9, Prywes answered Gromov's question (Question 7). Namely, Prywes gave the following examples of closed, simply connected, and oriented Riemannian 4-manifolds, which are not quasiregularly elliptic.

Corollary 10 ([21, Corollary 1.2])**.** *Let $k \ge 4$. Then the connected sum $\#^k(\mathbb{S}^2 \times \mathbb{S}^2)$ is not quasiregularly elliptic.*

With Pankka, we prove in [C] that, in fact, the de Rham algebra of a closed quasiregularly elliptic n-manifold is a subalgebra of the exterior algebra $\bigwedge^* \mathbb{R}^n$.

Theorem 11 ([C, Theorem 1.1])**.** *Let N be a closed quasiregularly elliptic n-manifold. Then there exists an embedding of graded algebras* $H^*_{\mathrm{dR}}(N) \to \bigwedge^* \mathbb{R}^n$.

Observe that, since $\dim \bigwedge^k \mathbb{R}^n = \binom{n}{k}$ for $k = 0, \dots, n$, Theorem 9 is an immediate corollary of Theorem 11.

Theorem 11 yields the following topological classification of closed simply connected quasiregularly elliptic 4-manifolds.

Corollary 12 ([C, Corollary 1.2])**.** *A closed and simply connected* 4-*manifold is quasiregularly elliptic if and only if it is homeomorphic to either $\#^k(\mathbb{S}^2 \times \mathbb{S}^2)$ or $\#^i\mathbb{C}P^2\#^j\overline{\mathbb{C}P^2}$ for some $k, i, j \in \{0,1,2,3\}$; here we interpret $\#^0(\mathbb{S}^2 \times \mathbb{S}^2) = \#^0\mathbb{C}P^2\#^0\overline{\mathbb{C}P^2} = \mathbb{S}^4$.*

Indeed, Theorem 11 implies that, if N is a closed simply connected quasiregularly elliptic 4-manifold, then definite subspaces of the intersection form of N have dimension at most three. Combining with a classical 4-dimensional classification result (see [C, Appendix A] for further discussion), we obtain a table on homeomorphism classes of closed simply connected quasiregularly elliptic 4-manifolds; see Figure 3. The quasiregular ellipticity of the manifolds in question follows from piecewise linear constructions by Piergallini and Zuddas [20, Theorem 1.2].

We finish this section by discussing the idea behind the proof of Theorem 11. We recall the following energy growth estimate for quasiregular maps due to Bonk and Heinonen.

$-\backslash+$	0	1	2	3
0	$\mathbb{S}^4$	$\mathbb{C}P^2$	$\#^2\mathbb{C}P^2$	$\#^3\mathbb{C}P^2$
1	$\overline{\mathbb{C}P^2}$	$\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$, $\mathbb{S}^2\times\mathbb{S}^2$	$\#^2\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$	$\#^3\mathbb{C}P^2\#\overline{\mathbb{C}P^2}$
2	$\#^2\overline{\mathbb{C}P^2}$	$\mathbb{C}P^2\#^2\overline{\mathbb{C}P^2}$	$\#^2\mathbb{C}P^2\#^2\overline{\mathbb{C}P^2}$, $\#^2\mathbb{S}^2\times\mathbb{S}^2$	$\#^3\mathbb{C}P^2\#^2\overline{\mathbb{C}P^2}$
3	$\#^3\overline{\mathbb{C}P^2}$	$\mathbb{C}P^2\#^3\overline{\mathbb{C}P^2}$	$\#^2\mathbb{C}P^2\#^3\overline{\mathbb{C}P^2}$	$\#^3\mathbb{C}P^2\#^3\overline{\mathbb{C}P^2}$, $\#^3\mathbb{S}^2\times\mathbb{S}^2$

FIGURE 3. Smooth 4-manifolds, which are closed, simply connected, and oriented, and whose intersection forms do not have definite subspaces of dimension four or higher.

Theorem 13 ([4, Theorem 1.11])**.** *Let N be a closed, connected, and oriented Riemannian n-manifold satisfying $H^k_{\mathrm{dR}}(N)\neq 0$ for some $1\leq k\leq n-1$. Let $f\colon\mathbb{R}^n\to N$ be a non-constant K-quasiregular map. Then*

$$\liminf_{r\to\infty} r^{-\varepsilon}\int_{B^n_r} J_f > 0,$$

where $\varepsilon=\varepsilon(n,K)>0$; here B^n_r denotes the Euclidean ball in $\mathbb{R}^n$ of radius r centered at the origin.

The idea of the proof of Theorem 11 is as follows. We may assume that $H^\ell_{\mathrm{dR}}(N)\neq 0$ for some $1\leq\ell\leq n-1$ since otherwise $H^*_{\mathrm{dR}}(N)$ embeds trivially into $\bigwedge^*\mathbb{R}^n$. Thus there exists a non-constant quasiregular map $f\colon\mathbb{R}^n\to N$ for which $\int_{\mathbb{R}^n} J_f=\infty$ by the quasiregular ellipticity of N and Theorem 13. Hence, we may apply Rickman's Hunting Lemma (see [28, Lemma 5.1] or [5, Lemma 2.1]) to obtain a sequence of balls $B^n(a_j,r_j)\subset\mathbb{R}^n$ satisfying

$$j\leq\int_{B^n(a_j,2r_j)} J_f\leq D\int_{B^n(a_j,r_j)} J_f$$

for $j\geq 1$, where $D=D(n)>1$ is a constant.

We define quasiregular maps $f_j\colon B^n_2\to N$ by $x\mapsto f(r_jx+a_j)$ and normalized pull-backs $f_j^\#\colon H^*_{\mathrm{dR}}(N)\to\mathscr{W}^*(B^n_2)$ by

$$H^k_{\mathrm{dR}}(N)\ni c\mapsto\left(\int_{B^n} J_{f_j}\right)^{-\frac{k}{n}} F^*(h(c)),$$

where $h\colon H^*_{\mathrm{dR}}(N) \to \Omega^*(N)$ denotes the choice of harmonic representative. Estimates for the normalized pull-backs $f_j^{\#}$ and weak compactness arguments yield that, after passing to a subsequence if necessary, there exists a weak limit $L\colon \oplus_{k=1}^{n-1} H^k_{\mathrm{dR}}(N) \to \oplus_{k=1}^{n-1} L^{\frac{n}{k}}(B_2^n; \bigwedge^k \mathbb{R}^n)$ of $(f_j^{\#})$ restricted to $\oplus_{k=1}^{n-1} H^k_{\mathrm{dR}}(N)$.

By Poincaré duality, there exist $c \in H^\ell_{\mathrm{dR}}(N)$ and $c' \in H^{n-\ell}_{\mathrm{dR}}(N)$ satisfying $c \wedge c' = [\mathrm{vol}_N]_{\mathrm{dR}}$. Hence, we obtain a natural linear map $H^n_{\mathrm{dR}}(N) \to L^1(B_2^n; \bigwedge^n \mathbb{R}^n)$ determined by $[\mathrm{vol}_N]_{\mathrm{dR}} \mapsto Lc \wedge Lc'$. We show that the weak limit L with this extension on $H^n_{\mathrm{dR}}(N)$ is, in fact, an embedding of graded algebras $L\colon H^*_{\mathrm{dR}}(N) \to \mathscr{W}^*(B_2^n)$. The embedding $H^*_{\mathrm{dR}}(N) \to \bigwedge^* \mathbb{R}^n$ is then obtained as the evaluation $c \mapsto (Lc)(x_0)$ with a suitable choice $x_0 \in B_2^n$.

We note that the assumption $H^\ell_{\mathrm{dR}}(N) \neq 0$, where $1 \le \ell \le n-1$, is sufficient and necessary for our approach. Indeed, if $N = \mathbb{S}^n$, we cannot guarantee the existence of a limiting operator $L\colon H^*_{\mathrm{dR}}(N) \to \mathscr{W}^*(B_2^n)$ for normalized pull-backs $f_j^{\#}$ of doubling quasiregular maps $f_j\colon B_2^n \to N$; see Figure 4 for an example, where the measures associated to normalized Jacobians collapse to a 1-dimensional Hausdorff measure.

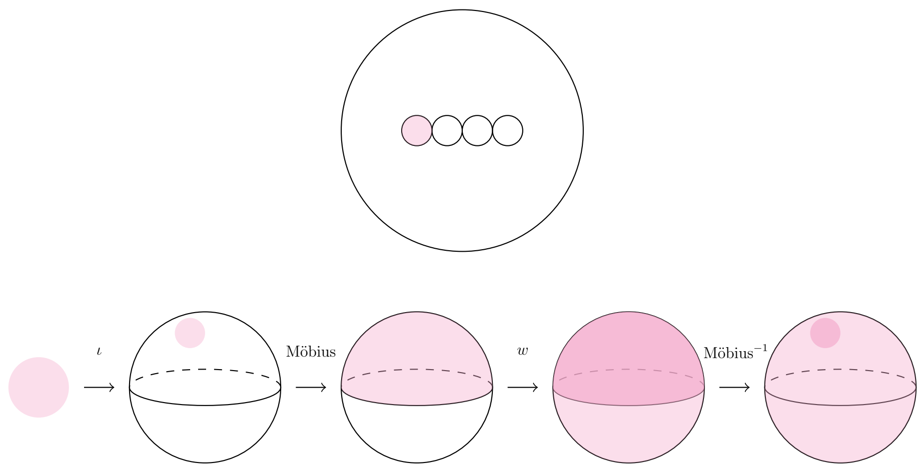

Figure 4. Each map $f_j\colon B_2^n \to \mathbb{S}^n$ is defined in the following way. We cover the line segment $[-1,1] \times \{0\}^{n-1}$ with j balls of radius $1/j$. Outside the balls of radius $1/j$, the map f_j is the stereographic projection $\iota\colon \mathbb{R}^n \to \mathbb{S}^n$. In each ball of radius $1/j$, the map f_j maps onto $\mathbb{S}^n$ surjectively as depicted above, where $w\colon \mathbb{R}^{n-1} \times \mathbb{C} \to \mathbb{R}^{n-1} \times \mathbb{C}$ is the map $(x, re^{i\theta}) \mapsto (x, re^{i3\theta})$.

2. Generalization of quasiregular maps

Equidimensionality of domain and target manifolds is an inbuilt property of quasiregular maps since the determinant of a linear map $(V, \mathrm{vol}_V) \to (W, \mathrm{vol}_W)$ between oriented vector spaces is well-defined only when $\dim V = \dim W$. In what follows, we introduce a generalization of quasiregular maps suitable for maps into higher dimensional targets, e.g., $\mathbb{R}^2 \to \mathbb{R}^3$.

2.1. **Quasiregular curves and their properties.** For the definition, let N be an oriented Riemannian manifold and let $\omega \in \Omega^n(N)$, $2 \leq n \leq \dim N$, be a closed non-vanishing form, i.e., $d\omega = 0$ and $\omega(x) \neq 0$ for each $x \in N$. A continuous map $F\colon M \to N$ from an oriented Riemannian n-manifold into N is a *K-quasiregular ω-curve for $K \geq 1$* if $F \in W^{1,n}_{\mathrm{loc}}(M, N)$ and

$$(\|\omega\| \circ F)\|DF\|^n \leq K(\star F^*\omega)$$

almost everywhere in M; here the *comass norm of* ω is defined pointwise by

$$\|\omega(x)\| = \max\{\omega_x(v_1, \ldots, v_n) \colon v_1, \ldots, v_n \in T_xM \text{ unit vectors}\}$$

and $\star F^*\omega$ is the Hodge star dual of $F^*\omega$, i.e., the measurable function satisfying $F^*\omega = (\star F^*\omega)\mathrm{vol}_M$ almost everywhere in M.

We remark that K-quasiregular vol_N-curves are precisely K-quasiregular maps since $\|\mathrm{vol}_N\| \equiv 1$ and $\star f^*\mathrm{vol}_N = J_f$ for a continuous map $f \in W^{1,n}_{\mathrm{loc}}(M, N)$ between oriented Riemannian n-manifolds.

Holomorphic curves are also examples of quasiregular curves.

Example 14. Let $k \geq 1$ and let $F = (f_1, \ldots, f_k)\colon \Omega \to \mathbb{C}^k$, where $\Omega \subset \mathbb{C}$ is a domain. Then the following are equivalent:

(1) F is a holomorphic curve, i.e., each coordinate map $f_i\colon \Omega \to \mathbb{C}$ is holomorphic and
(2) F is a 1-quasiregular ω_{sym}-curve, where ω_{sym} is the standard symplectic form $\omega_{\mathrm{sym}} = \sum_{i=1}^k dx_i \wedge dy_i \in \Omega^2(\mathbb{C}^k)$.

The implication from (1) to (2) follows by a simple computation. Since planar 1-quasiregular maps are holomorphic, the other implication follows e.g. by [B, Lemma 4.2].

In fact, more generally, products of quasiregular maps are quasiregular curves with respect to the form intrinsic to the product structure.

Example 15. Let M be an oriented Riemannian n-manifold. Let $k \geq 1$ and let $N = N_1 \times \cdots \times N_k$ be a Riemannian product of oriented Riemannian n-manifolds N_i. Let $K \geq 1$ and let $F = (f_1, \ldots, f_k)\colon M \to N$ be a map for which each coordinate map $f_i\colon M \to N_i$ is K-quasiregular. Then F is a (k^nK)-quasiregular $\mathrm{vol}_N^\times$-curve, where

$$\mathrm{vol}_N^\times = \sum_{i=1}^k \pi_i^*\mathrm{vol}_{N_i} \in \Omega^n(N)$$

and $\pi_i\colon N \to N_i$ are the projections.

Whereas we could heuristically describe quasiregular maps as maps which distort oriented angles by a bounded amount, it is considerably more difficult to give intuition for quasiregular curves in general; see e.g. Section 2.3 for examples of quasiregular curves with intricate behaviour.

Since quasiregular curves generalize quasiregular maps, a natural question arises:

Question 16. *Which characteristic properties of quasiregular maps are inherited by quasiregular curves and which are not?*

We observe immediately that, if $F\colon M \to N$ is a quasiregular curve, where $\dim M < \dim N$, then F is not an open map; there exist even elementary examples which are not interior, see e.g. the example in [18, p. 980].

2.2. **Analytic properties.** On a positive note, some characteristic properties of quasiregular maps are indeed inherited by quasiregular curves. For example, Onninen and Pankka proved that quasiregular curves have higher integrability and are differentiable almost everywhere.

Theorem 17 ([17, Corollary 1.4])**.** *Let N be an oriented Riemannian manifold and let $\omega \in \Omega^n(N)$ a closed non-vanishing form. Let M be an oriented Riemannian n-manifold and let $F\colon M \to N$ be a K-quasiregular ω-curve. Then $F \in W^{1,p}_{\mathrm{loc}}(M, N)$ for some $p = p(n, K) > n$ and F is differentiable almost everywhere in M.*

Onninen and Pankka also showed that K-quasiregular ω-curves between manifold are locally $\alpha(K', \omega)$-Hölder continuous for each $K' > K$, where $\alpha(K', \omega) \leq 1/K'$; see [17, Corollary 1.2]. The optimal Hölder exponent was recovered by Ikonen.

Theorem 18 ([11, Theorem 1.5])**.** *Let N be an oriented Riemannian manifold and let $\omega \in \Omega^n(N)$ a closed non-vanishing form. Let M be an oriented Riemannian n-manifold and let $F\colon M \to N$ be a K-quasiregular ω-curve. Then F is locally $1/K'$-Hölder continuous for each $K' > K$.*

We remark that a K-quasiregular ω-curve $\Omega \to \mathbb{R}^m$, where $\Omega \subset \mathbb{R}^n$ is a domain and $\omega \in \bigwedge^n \mathbb{R}^m$, is $1/K$-Hölder continuous; recall that Euclidean K-quasiregular maps are $1/K$-Hölder continuous.

2.3. **Examples of non-discreteness.** On a negative note, the class of quasiregular curves includes non-constant curves which are not discrete. The following result is due to Iwaniec, Verchota, and Vogel.

Lemma 19 ([13, Lemma 5])**.** *There exists a non-constant quasiregular $(dx_1 \wedge dy_1 + dx_2 \wedge dy_2)$-curve $\mathbb{C} \to \mathbb{C}^2$ which is constant in the lower half space $\mathbb{R} \times (-\infty, 0]$.*

We note that $\mathbb{C}^2 = (\mathbb{R}^2)^2$ and $dx_1 \wedge dy_1 + dx_2 \wedge dy_2 = \mathrm{vol}^{\times}_{(\mathbb{R}^2)^2}$. In the same spirit, we prove the following result with Pankka and Prywes in [B].

Theorem 20 ([B, Theorem 2.1])**.** *Let $n \geq 3$. Then there exists $k \geq 1$ and a non-constant quasiregular $\mathrm{vol}^{\times}_{(\mathbb{R}^n)^k}$-curve $\mathbb{R}^n \to (\mathbb{R}^n)^k$ which is constant in the lower half-space $\mathbb{R}^{n-1} \times (-\infty, 0]$.*

The quasiregular curves in Lemma 19 and Theorem 20 have large distortion. The following result states that there exist non-constant quasiregular $\mathrm{vol}^{\times}_{(\mathbb{R}^2)^k}$-curves $\mathbb{R}^2 \to (\mathbb{R}^2)^k$, which are not discrete, with arbitrarily small distortion for each $k \geq 2$. The result is based on Rosay's map [31, Section 4] and we give the following formulation in [B, Theorem 3.1].

Theorem 21 (Rosay)**.** *For each $K > 1$ and $k \geq 2$, there exists a non-constant K-quasiregular $\mathrm{vol}^{\times}_{(\mathbb{R}^2)^k}$-curve $\mathbb{R}^2 \to (\mathbb{R}^2)^k$ which is not discrete.*

2.4. **Generalized Liouville's theorem.** Returning to the positive side, we have the following variant of Liouville's theorem for quasiregular curves.

Theorem 22 ([B, Theorem 5.2])**.** *Let $n \geq 3$, let $\Omega \subset \mathbb{R}^n$ be a domain, and let $F = (f_1, \ldots, f_k)\colon \Omega \to (\mathbb{R}^n)^k$ be a non-constant 1-quasiregular $\mathrm{vol}^{\times}_{(\mathbb{R}^n)^k}$-curve. Then there exists an index $i_0 \in \{1, \ldots, k\}$ for which the coordinate map $f_{i_0}\colon \Omega \to \mathbb{R}^n$ is a restriction of a Möbius transformation on $\mathbb{S}^n$ and each $f_i\colon \Omega \to \mathbb{R}^n$, for $i \neq i_0$, is a constant map.*

More generally, small distortion quasiregular $\mathrm{vol}^{\times}_N$-curves into a product manifold N are carried by quasiregular maps.

Theorem 23 ([B, Theorem 1.7])**.** *Let $n \geq 3$ and $k \geq 1$. Then there exists $\varepsilon = \varepsilon(n, k) > 0$ for the following. Let M be a connected and oriented Riemannian n-manifold and let $N = N_1 \times \cdots \times N_k$ be a Riemannian product of oriented Riemannian n-manifolds. Then, for every non-constant $(1 + \varepsilon)$-quasiregular $\mathrm{vol}^{\times}_N$-curve $F = (f_1, \ldots, f_k)\colon M \to N$, there exists a unique index $i_0 \in \{1, \ldots, k\}$ for which*

1. *the coordinate map f_{i_0} is a quasiregular local homeomorphism and*
2. *$\|DF(x)\| \leq \frac{4}{3}\|Df_{i_0}(x)\|$ for almost every $x \in M$.*

2.5. **Energy growth estimate.** In [A], we prove an analogue of Theorem 13 for a subclass of quasiregular curves. To state the result, we recall the following notation.

Let N be a connected and oriented Riemannian manifold. We denote by $C^{\infty}_b(N)$ the space of smooth and bounded functions on N and by $\mathcal{Z}^k_b(N)$ the space of smooth, bounded, and closed k-forms on N for $k = 1, \ldots, \dim N$. For $2 \leq k \leq \dim N$, $\mathcal{A}^k_b(N)$ denotes the $C^{\infty}_b(N)$-algebra generated by products $\alpha \wedge \beta$ of forms $\alpha \in \mathcal{Z}_b^\ell(N)$ and $\beta \in \mathcal{Z}_b^{k-\ell}(N)$ with $1 \le \ell \le k-1$, i.e.,

$$\mathcal{A}_b^k(N) = \mathrm{span}_{C_b^\infty(N)}\left(\bigcup_{\ell=1}^{k-1} \mathcal{Z}_b^\ell(N) \wedge \mathcal{Z}_b^{k-\ell}(N)\right).$$

For example, $\mathcal{A}_b^k(\mathbb{R}^n) = \Omega_b^k(\mathbb{R}^n)$ for each $2 \le k \le n$. Indeed, $\mathcal{A}_b^k(\mathbb{R}^n) \subset \Omega_b^k(\mathbb{R}^n)$ since $\mathcal{Z}_b^\ell(\mathbb{R}^n) \wedge \mathcal{Z}_b^{k-\ell}(\mathbb{R}^n) \subset \Omega_b^k(\mathbb{R}^n)$ and $\Omega_b^k(\mathbb{R}^n)$ is a $C_b^\infty(\mathbb{R}^n)$-module. On the other hand, $\Omega_b^k(\mathbb{R}^n) \subset \mathcal{A}_b^k(\mathbb{R}^n)$ since $dx_{i_1} \wedge \cdots \wedge dx_{i_k} \in \mathcal{A}_b^k(\mathbb{R}^n)$ for each multi-index $(i_1, \ldots, i_k) \in \{1, \ldots, n\}^k$.

As another example, $\mathrm{vol}_N \in \mathcal{A}_b^n(N)$ if N is a closed, connected, and oriented Riemannian n-manifold satisfying $H_{\mathrm{dR}}^k(N) \neq 0$ for some $1 \le k \le n-1$; see [10, Proposition 2.13].

Our growth theorem for quasiregular ω-curves $\mathbb{R}^n \to N$ with $\omega \in \mathcal{A}_b^n(N)$ now reads as follows.

Theorem 24 ([A, Theorem 1.1])**.** *Let N be a connected and oriented Riemannian manifold and let $\omega \in \mathcal{A}_b^n(N)$ be a non-vanishing closed form satisfying $\inf_N \|\omega\| > 0$. Let $F\colon \mathbb{R}^n \to N$ be a non-constant K-quasiregular ω-curve. Suppose that there exist $\varphi_1, \ldots, \varphi_j \in C_b^\infty(N)$ and*

$$\alpha_1 \wedge \beta_1, \ldots, \alpha_j \wedge \beta_j \in \bigcup_{\ell=1}^{k-1} \mathcal{Z}_b^\ell(N) \wedge \mathcal{Z}_b^{k-\ell}(N)$$

for which $\omega = \sum_{i=1}^j \varphi_i \alpha_i \wedge \beta_i$ and either

1. *the almost everywhere defined measurable functions $\star F^*(\alpha_i \wedge \beta_i)$ do not change sign or*
2. *each φ_i is a constant function.*

Then

$$\liminf_{r \to \infty} r^{-\varepsilon} \int_{B_r^n} F^*\omega > 0,$$

where $\varepsilon = \varepsilon(n, K, \omega) > 0$.

The idea behind the proof of Theorem 24 is to show that the function $\star F^*\omega \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ satisfies a weak reverse Hölder inequality which implies the desired growth estimate.

2.6. **Quasiregular ellipticity.** In the setting of quasiregular curves, there are two natural notions of quasiregular ellipticity. Let N be a connected and oriented Riemannian manifold and let $\omega \in \Omega^n(N)$ be a non-vanishing closed form.

We say that a pair (N, ω) is *quasiregularly elliptic* if there exists a non-constant quasiregular ω-curve $\mathbb{R}^n \to N$. We also say that a pair (N, ω) is *infinite energy quasiregularly elliptic* if there exists a quasiregular ω-curve $F\colon \mathbb{R}^n \to N$ for which $\int_{\mathbb{R}^n} F^*\omega = \infty$.

We remark that, by Theorem 13, a pair (N, vol_N), where N is a closed n-manifold with $H^k_{\mathrm{dR}}(N) \neq 0$ for some $1 \leq k \leq n-1$, is infinite energy quasiregularly elliptic if and only if N is quasiregularly elliptic.

We also note that the quasiregular ellipticity of a pair (N, ω) depends on both N and ω. For example, a pair $(M \times \mathbb{C}P^2, \pi_2^*\omega_{\mathrm{sym}})$, where M is a connected and oriented Riemannian manifold and $\omega_{\mathrm{sym}} \in \Omega^2(\mathbb{C}P^2)$ is the standard symplectic form, is always (infinite energy) quasiregularly elliptic, while a pair $(M \times \mathbb{C}P^2, \pi_1^*\mathrm{vol}_M \wedge \pi_2^*\omega_{\mathrm{sym}})$ is not necessarily quasiregularly elliptic; see Corollary 26.

In [D], we prove a variation of Theorem 11 which also covers quasiregular ellipticity of pairs (N, ω), where ω is not a top-dimensional form. For the statement of the result, we recall first the classical Künneth theorem: *Let $N = N_1 \times N_2$ be a product of closed smooth manifolds. Then the map*

$$\bigoplus_{\ell=0}^{k} H^\ell_{\mathrm{dR}}(N_1) \otimes H^{k-\ell}_{\mathrm{dR}}(N_2) \to H^k_{\mathrm{dR}}(N),\ c_1 \otimes c_2 \mapsto \pi_1^* c_1 \wedge \pi_2^* c_2,$$

is an isomorphism for each k.

For a smooth manifold N, we say that the *Künneth ideal of N* is the ideal $K^*(N) = \oplus_{k=2}^{\dim N} K^k(N)$ of $H^*_{\mathrm{dR}}(N)$, where the kth layer $K^k(N)$ is the vector space generated by products $c \wedge c'$ of de Rham classes $c \in H^\ell_{\mathrm{dR}}(N)$ and $c' \in H^{k-\ell}_{\mathrm{dR}}(N)$ with $1 \leq \ell \leq k-1$, i.e.,

$$K^k(N) = \mathrm{span}_{\mathbb{R}} \left(\bigcup_{\ell=1}^{k-1} H^\ell_{\mathrm{dR}}(N) \wedge H^{k-\ell}_{\mathrm{dR}}(N) \right).$$

The extension of Theorem 11 for quasiregular curves reads now as follows.

Theorem 25 ([D, Theorem 1.1])**.** *Let (N, ω) be an infinite energy quasiregularly elliptic pair, where N is closed and $0 \neq [\omega]_{\mathrm{dR}} \in K^n(N)$. Then there exists a graded algebra homomorphism $\Phi\colon H^*_{\mathrm{dR}}(N) \to \bigwedge^* \mathbb{R}^n$ for which $\Phi[\omega]_{\mathrm{dR}} \neq 0$.*

In the proof of Theorem 11, we utilize the fact that $[\mathrm{vol}_N]_{\mathrm{dR}}$ splits into a product by Poincaré duality. The intuition behind Theorem 25 is that a similar approach works for quasiregular ω-curves $\mathbb{R}^n \to N$ with infinite energy as long as $[\omega]_{\mathrm{dR}}$ is a finite sum of products.

Comparing Theorem 25 to Theorem 11, the main differences are that we need to add the assumption $[\omega]_{\mathrm{dR}} \in K^n(N)$ and that the homomorphism $H^*_{\mathrm{dR}}(N) \to \bigwedge^* \mathbb{R}^n$ is not necessarily injective - these are both related to the fact that we cannot use Poincaré duality to identify $H^k_{\mathrm{dR}}(N)$ and $H^{n-k}_{\mathrm{dR}}(N)$ if $n < \dim N$.

As a consequence of Theorem 25, we obtain new examples of non-ellipticity in the spirit of Gromov's question (Question 7).

Corollary 26 ([D, Corollary 1.6]). *Let $k \geq 8$ and $S_k = \#^k(\mathbb{S}^2 \times \mathbb{S}^2)$. Let $\omega_{\mathrm{sym}} \in \Omega^2(\mathbb{C}P^2)$ be the standard symplectic form. Then the pair $(S_k \times \mathbb{C}P^2, \pi_1^*\mathrm{vol}_{S_k} \wedge \pi_2^*\omega_{\mathrm{sym}})$ is not quasiregularly elliptic.*

3. Erratum

We thank Toni Ikonen for pointing out the following error on p. 18 in [B] and for providing the alternate argument.

The inequality on page 18, line 2, should be

$$\star L^*\mathrm{vol}_W \leq \frac{1}{n^{n/2}}\|L\|_{\mathrm{HS}}^n,$$

which follows from the inequality of arithmetic and geometric means, and it should be applied in the pointwise estimate on page 18, line -9, as follows.

For almost every $x \in G$, we have

$$(Dv)_x = \iota_x \circ L_x,$$

where $L_x \colon T_xG \to \mathrm{Im}((Dv)_x)$ is the corestriction of $(Dv)_x \colon T_xG \to T_{v(x)}\mathbb{R}^m$ and $\iota_x \colon \mathrm{Im}((Dv)_x) \to T_{v(x)}\mathbb{R}^m$ is the inclusion. For such x satisfying $\star(v^*\omega)_x > 0$, let $V_x \in \wedge^n \mathrm{Im}((Dv)_x)$ be an orientation form of unit comass for which $\iota_x^*\omega_{v(x)} = \lambda_x V_x$, where $\lambda_x > 0$. Since ω is a calibration, it follows that $\lambda_x \leq 1$. Thus

$$(v^*\omega)_x = ((Dv)_x)^*\omega_{v(x)} = (\iota_x \circ L_x)^*\omega_{v(x)} = L_x^*\iota_x^*\omega_{v(x)} = \lambda_x L_x^* V_x$$

and

$$\star(v^*\omega)_x = \star\lambda_x L_x^* V_x \leq \star L_x^* V_x \leq \frac{1}{n^{n/2}}\|L_x\|_{\mathrm{HS}}^n = \frac{1}{n^{n/2}}\|(Dv)_x\|_{\mathrm{HS}}^n.$$

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