Master’s thesis
Particle Physics and Cosmology

Junction conditions in general relativity

Shankar Bhandari

July, 2024

Supervisor(s): Niko Jokela

Censor(s): Mark Hindmarsh
Niko Jokela

UNIVERSITY OF HELSINKI
Particle Physics and Astrophysical Sciences

PL 64 (Gustaf Hällströmin katu 2a)
FI-00014 University of Helsinki

HELSINGIN YLIOPISTO — HELSINGFORS UNIVERSITET — UNIVERSITY OF HELSINKI

Tiedekunta — Fakultet — Faculty	Koulutusohjelma — Utbildningsprogram — Education programme
Faculty of science	Particle Physics and Astrophysical Sciences

Tekijä — Författare — Author

Shankar Bhandari

Työn nimi — Arbetets titel — Title

Junction conditions in general relativity

Opintosuunta — Studieinriktning — Study track

Particle Physics and Cosmology

Työn laji — Arbetets art — Level	Aika — Datum — Month and year	Sivumäärä — Sidoantal — Number of pages
Master's thesis	July, 2024	41 pages

Tiivistelmä — Referat — Abstract

In this thesis, the metric junction conditions are investigated for null and non-null hypersurfaces. Along with the junction conditions we also investigate the thin shell formalism, which arises when the second junction condition is violated. The second junction condition can be violated since violating it leads to only a delta-function singularity in the Riemann tensor. This singularity is allowed since it has a physical explanation via the Einstein equation. The delta-function singularity in the Riemann tensor corresponds to an infinitesimally thin layer of matter at the hypersurface. We also present an example calculation for thin shell formalism by calculating junction conditions for general spherically symmetric spacetimes being joined by stationary spherically symmetric hypersurface. At the end, we also mention the practical usage of the junction conditions.

Avainsanat — Nyckelord — Keywords

Junction conditions, thin shell formalism, general relativity, stationary lightlike shells, null shells

Säilytyspaikka — Förvaringsställe — Where deposited

Muita tietoja — övriga uppgifter — Additional information

Contents

Notation and conventions

We mostly follow the notations used in Poisson (2004) [1]. We use the mostly plus metric signature (−1,1,1,1). Greek letters $(\alpha, \beta, \dots)$ used as indices range from 0 to 3, lower-case Latin indices $(a, b, \dots)$ range from 1 to 3, and upper-case Latin indices $(A,B, \dots)$ range from 2 to 3. Throughout the text we will adopt convention of $K^{\alpha\beta\dots}{}_{\gamma\delta\dots} = K^{\alpha\beta\dots}_{\gamma\delta\dots}$. We use geometrized units where $G = c = 1$.

List of frequently occurring and important symbols (adapted from [1]):

Symbol	**Description**
x^α	Arbitrary coordinates on a manifold
y^a	Arbitrary coordinates on a hypersurface
θ^A	Arbitrary coordinates on a two-surface
$e^\alpha_a = \partial x^\alpha/\partial y^a$, $e^\alpha_A = \partial x^\alpha/\partial\theta^A$	Holonomic basis vectors
$g_{\alpha\beta}$	Metric on a manifold
h_{ab}	Induced metric on a hypersurface
σ_{AB}	Induced metric on a two-surface
$\xi_{,\alpha} = \partial_\alpha\xi$	Partial differentiation with respect to x^α
$\xi_{,a} = \partial_a\xi$	Partial differentiation with respect to y^a
$A^\alpha{}_{;\beta} = \nabla_\beta A^\alpha$	Covariant differentiation in a manifold
$\pounds_w A^\alpha$	Lie derivative of A^α along w^α

1. Introduction

The topic of this thesis is junction conditions in general relativity. The junction conditions arise from the study of discontinuities in matter and spacetime. The discontinuities in the matter and spacetime arise when soldering two spacetimes with each other. The soldering is done such that the boundary between the spacetimes is confined in a hypersurface. This hypersurface is infinitesimally thin to make some calculations manageable. Unlike in the Newtonian theory, where there is flat space and the coordinates are well-defined, the study of discontinuities is not as simple or easy in general relativity [2]. This is because, in general relativity, where we have curved spacetimes, we also have to account for different coordinate systems that might not be smooth everywhere, which means we have to disentangle the discontinuity arising from coordinate bumps from the ones present physically [2]. The discontinuities arise from the fact that the union of two metrics in the two distinct spacetimes does not form a smooth solution to the Einstein field equation [3]. Then the junction conditions are derived by demanding a sufficiently smooth transition from one spacetime to another.

The study of junction conditions in General relativity has been going on since the 1920s. The studies done by Lanczos, Darmois [4], Lichenerowicz [5], and O'Brien and Synge [6] investigated junction conditions for time- and spacelike hypersurfaces but it was in 1966 when W. Israel [2] formulated it in purely geometric way which was also independent of the coordinates of the background spacetime. The case for null geometries (null hypersurface) is more complicated since the induced metric on the hypersurface becomes degenerate. The null case was investigated by Taub [7] and Clarke [8] and the complete formulation was done by Barrabes and Israel [9], where they laid out a general algorithm for calculating the discontinuities across hypersurfaces, which was better suited for practical applications [3]. The junction conditions for singular hypersurfaces are usually called Israel junction conditions while the junction conditions for non-singular hypersurfaces are called Darmois–Israel conditions. For a more detailed historical account see the introduction chapters of [10–12]. The main area of focus in research relating to junction conditions is Israel junction conditions and their wide applications. For example, the Israel junction conditions are useful for calculating phase transition bubbles in cosmology.

I have chosen this topic because of my interest in differential geometry in general relativity and a desire to learn more about it. This thesis aims to serve as a basic introduction to junction conditions in a way that is understandable for students with an introductory level knowledge of general relativity.

The thesis is divided into two parts: in the first part we derive the Israel junction conditions using the distributional method closely following the derivation by Poisson in his book “Relativist’s Toolkit” [1]. More specifically we follow closely chapter 3 of the book. The material that we follow in the book is based on the papers [2,9]. In this thesis, I have provided more detail in some parts of derivations while leaving the non-essential parts that are not required to familiarize with the topic. In the second part, we do an example calculation for general spherically symmetric metrics being joined by a stationary null spherical hypersurface. We also present a concrete example of the Reissner–Nordström exterior joined to the de Sitter interior. This part is based on the calculations of Barrabes and Israel in [13], but we present the calculations using the formulation of Poisson [1].

In Chapter 2 we introduce the basic geometry needed to understand hypersurfaces, while in Chapter 3 we derive the junction conditions for non-null hypersurfaces. We also discuss thin shells that are the result of the Israel junction condition. In Chapter 4 we derive the Junction conditions for null hypersurfaces completing the Israel formalism. In Chapter 5 we present an example calculation based on the examples of Barrabes and Isreal [13] using the formulation and notations of Poisson [1] that are used in this thesis. We present some of the practical applications of the Junction conditions in Chapter 6 and conclude the thesis in Chapter 7.

2. Geometry of hypersurface

Understanding the geometry of hypersurfaces is necessary to understand the physics of soldering two spacetimes together. This is because the soldered spacetimes are separated by a hypersurface whose geometry determines the conditions for the soldering. In this chapter, we introduce all the necessary geometrical concepts.

2.1 Definition

Generally a hypersurface is a $(n-1)$-dimensional submanifold of a n-dimensional manifold [1,14]. In this thesis, we will be working in four-dimensional spacetime. We will use the following definition for a hypersurface: *in a four-dimensional spacetime manifold, a hypersurface is a three-dimensional submanifold that can be either timelike, spacelike, or null* [1].

A specific hypersurface Σ in a manifold $\mathcal{V}$ with coordinates x^α can be defined via the restriction on the coordinates

$$\Phi(x^\alpha) = 0 \quad , \tag{2.1}$$

where $\Phi(x^\alpha)$ is a function that describes the shape of the three-dimensional hypersurface in the four-dimensional space. For example, a two-sphere in a three- or four-dimensional space is described by $\Phi(x^\alpha) = x^2 + y^2 + z^2 - r^2 = 0$. We can also define Σ via parametric equations of the form

$$x^\alpha = x^\alpha(y^a) \quad , \tag{2.2}$$

where y^a $(a = 1, 2, 3)$ are coordinates intrinsic to the hypersurface Σ [1]. An example of parametric equations for two-sphere are $x = r\cos\phi\sin\theta$, $y = r\sin\phi\sin\theta$, and $z = r\cos\theta$, where $\{\theta, \phi\}$ are the intrinsic coordinates of the two-sphere.

For Σ defined via $\Phi = 0$, the vector $\Phi_{,\alpha}$ is normal to it since the derivative of Φ is zero along all tangent directions. For null hypersurfaces the vector $\Phi_{,\alpha}$ is null, which means that it is norm squared $g^{\mu\nu}\Phi_\mu\Phi_\nu$ is zero.

2.2 Normal vector

For non-null Σ we can define a unit normal n^α:

$$n_\alpha = \frac{\varepsilon\Phi_{,\alpha}}{|g^{\mu\nu}\Phi_{,\mu}\Phi_{,\nu}|^{1/2}} \quad , \tag{2.3}$$

which is defined such that n^α point in the direction of increasing Φ: $n^\alpha\Phi_{,\alpha} > 0$, with

$$n^\alpha n_\alpha = \varepsilon \equiv \begin{cases} -1 & \text{if } \Sigma \text{ is spacelike} \\ +1 & \text{if } \Sigma \text{ is timelike} \end{cases} \quad . \tag{2.4}$$

[1]. Because $g^{\mu\nu}\Phi_{,\mu}\Phi_{,\nu}$ is zero for null Σ, the unit normal is not defined for null Σ. Instead in the null case we only define the normal vector

$$k_\alpha = -\Phi_{,\alpha} \tag{2.5}$$

where the sign is chosen for k^α to be future directed when Φ increases toward future [3]. Because k^α is orthogonal to itself it is also tangent to the null Σ. The null normal vector satisfies $k^\alpha_{;\beta}k^\beta = \kappa k^\alpha$, the general form of the geodesic equation [1]. This means that a null Σ is generated by null geodesics as shown in Figure 2.1 and k^α is tangent to the generators. These generators are parameterized by λ such that the displacement along each generator is $dx^\alpha = -k^\alpha d\lambda$.

For a null Σ it is beneficial to use a coordinate system that suits the concept of generators. In particular, it is useful to let the parameter λ be one of the coordinates.

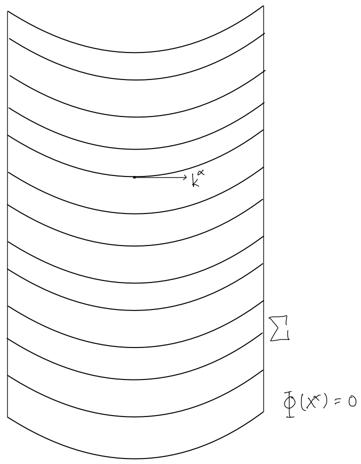

Figure 2.1: A null hypersurface Σ, its generators denoted by the lines running along the hypersurface, and a null tangent vector k^α.

As such we introduce the following coordinates for null Σ:

$$y^a = \{\lambda, \theta^A\} \quad , \tag{2.6}$$

where $A = 2,3$. The coordinates θ^A label the generators, are constant along each generator, and span the 2d-space transverse to the generators. One can go from one generator to another via a change in the θ^A coordinates. This is demonstrated in Figure 2.1

For the null case, we need to introduce a transversal vector N^α over Σ in addition to and independently of the normal vector k^α [13]. This is required to complete the basis and we will need this vector to calculate the discontinuities in the transverse direction. The transversal null vector is defined such that [1]

$$N_\alpha k^\alpha = -1 \qquad \& \qquad N_\alpha e^\alpha_A = 0 \quad . \tag{2.7}$$

2.3 First fundamental form

The line element in a manifold $\mathcal{V}$ is

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$$

To get a line element and the metric in Σ, we must restrict the line element of the spacetime manifold along Σ. We do this via projecting the metric along Σ with the help of projection vectors $e^\alpha_a \equiv \frac{\partial x^\alpha}{\partial y^a}$, where x^α satisfies the parametric equation (2.2) for the Σ. These are holonomic basis vectors of Σ and are tangent to curves contained in it [1]. Projecting the metric with these we get the induced metric, or the first fundamental form, of Σ

$$h_{ab} = g_{\alpha\beta}e^\alpha_a e^\beta_b \tag{2.8}$$

such that the line element in Σ becomes .

$$ds^2_\Sigma = h_{ab}dy^a dy^b \tag{2.9}$$

The induced metric transforms as a scalar under the transformation $x^\alpha \to x^{\alpha'}$ and as a tensor under the transformation $y^a \to y^{a'}$. Objects that transform like tensors on coordinates of Σ but as scalar in coordinates of $\mathcal{V}$ are referred to as three-tensors. Similarly, if the surface is a two-dimensional subsurface, an object that transforms as a tensor in the two-dimensional coordinates while transforming as a scalar in the coordinates it is embedded in, is called a two-tensor.

Because k^α is null we can simplify the first fundamental form for null Σ. We have $e^\alpha_1 = (\frac{\partial x^\alpha}{\partial \lambda}) \equiv k^\alpha$, which means $h_{11} = g_{\alpha\beta}k^\alpha k^\beta = 0$ and $h_{1A} = g_{\alpha\beta}k^\alpha e^\beta_A = 0$ because e^α_A is orthogonal to k^α. This means that on the null Σ the metric simplifies to

$$ds^2_\Sigma = \sigma_{AB}d\theta^A\theta^B \quad , \tag{2.10}$$

where

$$\sigma_{AB} = g_{\alpha\beta} e^{\alpha}_{A} e^{\beta}_{B} \tag{2.11}$$

is a two-tensor.

We conclude this section by listing the inverse metric completeness relations directly from [1]. These will be useful for later calculations. When Σ is not null, the inverse metric is

$$g^{\alpha\beta} = \varepsilon n^{\alpha} n^{\beta} + h^{ab} e^{\alpha}_{a} e^{\beta}_{b} \quad , \tag{2.12}$$

where h^{ab} is the inverse induced metric. When Σ is null, we have

$$g^{\alpha\beta} = -k^{\alpha} N^{\beta} - N^{\alpha} k^{\beta} + \sigma^{AB} e^{\alpha}_{A} e^{\beta}_{B} \quad , \tag{2.13}$$

where σ^{AB} is the inverse of σ_{AB}.

2.4 Second fundamental form

For non-null Σ, we can define the following symmetric three-tensor

$$K_{ab} \equiv n_{\alpha;\beta} e^{\alpha}_{a} e^{\beta}_{b} \quad , \tag{2.14}$$

which is called the *extrinsic curvature*, or the *second fundamental form* [1]. It can also be written in terms of the Lie derivative of the metric along the unit normal n,

$$K_{ab} = \frac{1}{2} \left(\pounds_{n} g_{\alpha\beta} e^{\alpha}_{a} e^{\beta}_{b} \right) \quad . \tag{2.15}$$

The second fundamental form is a useful intrinsic property of Σ because it carries information about the derivative of the metric in the normal direction [1]. The contraction of the extrinsic curvature is defined as

$$K \equiv h^{ab} K_{ab} = n^{\alpha}_{;\alpha} \quad . \tag{2.16}$$

For null Σ, if we define the extrinsic curvature to be

$$K_{ab} \equiv k_{\alpha;\beta} e^{\alpha}_{a} e^{\beta}_{b} \tag{2.17}$$

$$= \frac{1}{2} \left(\pounds_{k} g_{\alpha\beta} e^{\alpha}_{a} e^{\beta}_{b} \right) \quad , \tag{2.18}$$

we get a tangential derivative $\pounds_{k} g_{\alpha\beta}$ which is not what we desire [1]. What we want is an object that carries information about the derivative of the metric in the transverse direction. For this purpose, we introduce the transverse curvature that is defined as [1]

$$C_{ab} \equiv -N_{\alpha} e^{\alpha}_{a;\beta} e^{\beta}_{b} \tag{2.19}$$

$$= \frac{1}{2} \left(\pounds_{N} g_{\alpha\beta} e^{\alpha}_{a} e^{\beta}_{b} \right) \quad . \tag{2.20}$$

This quantity will be useful for the intrinsic formulation of junction conditions for null shells.

3. Junction conditions for non-null geometries

Since we are aiming to solder two spacetimes, we must first specify the situation and define some notations and quantities. We want to solder two spacetimes V^- and V^+ which together form a spacetime $\mathcal{V}$, such that the boundary between V^+ and V^- is the hypersurface Σ. We want to solder the spacetimes smoothly such that they form a valid solution to the Einstein equation. This gives us some conditions on the metric and its derivative, called the junction conditions [1].

We denote the quantities in the spacetime regions $V^\pm$ with $\pm$ in either the subscript or the superscript, for example, $g^+_{\alpha\beta}$ is the metric in V^+ and the coordinates in V^- are x^α_-. We want the union of the metrics $g^+_{\alpha\beta}$ and $g^-_{\alpha\beta}$ to form a valid solution to the Einstein equation in $\mathcal{V}$ and this is not a simple task, because it might not be possible to directly compare the coordinates in V^- to those in V^+ [1,2]. Because of this, it is best to formulate the junction conditions in a coordinate-independent way, using the intrinsic properties of the hypersurface Σ that sits at the boundary.

In this chapter, we will derive the junction conditions and the thin shell formalism for non-null hypersurfaces and leave the treatment of null hypersurfaces for the next chapter. The null and non-null geometries are distinct, and separating their treatment is natural. However, it is noteworthy that it is possible to formulate the junction conditions for a general hypersurface, that can be null or non-null at different points in the hypersurface as done in [11]. We will not be discussing them as they are beyond the scope of this thesis.

3.1 Preliminaries

3.1.1 Assumptions

These assumptions are directly from Poisson's book [1]. We assume the coordinate system y^a is the same on both sides of the hypersurface Σ. We assume the normal vector always points from V^- to V^+. We assume there is a continuous coordinate system x^α, different from $x^\alpha_\pm$, on both sides of the Σ that overlaps with $x^\alpha_\pm$ in an open region in $V^\pm$ that contains Σ. We will be working with these coordinates to develop the coordinate-independent junction conditions.

3.1.2 Distributions

Suppose we have a variable l such that it is 0 at Σ and $l > 0$ ($l < 0$) in V^+ (V^-). One such variable could be the proper distance (or the proper time) along a geodesics that intersects Σ orthogonally. In the scope of this thesis, it is not important what the l is as long as it satisfies the properties we need it to have. We can use this variable to write the metric as a distribution with the help of Heaveside distribution. The Heaveside distribution $\Theta(l)$ is $+1$ when $l > 0$, 0 when $l < 0$, and indeterminate when $l = 0$. It is also possible to use a Heaviside distribution with $\Theta(0) = 1/2$ to derive the junction conditions like in [8], but we will stick to the distribution where it is indeterminate. The Heaveside distribution has the following properties:

$$\Theta^2(l) = \Theta(l) \tag{3.1}$$

$$\Theta(l)\Theta(-l) = 0 \tag{3.2}$$

$$\frac{d}{dl}\Theta(l) = \delta(l) \quad , \tag{3.3}$$

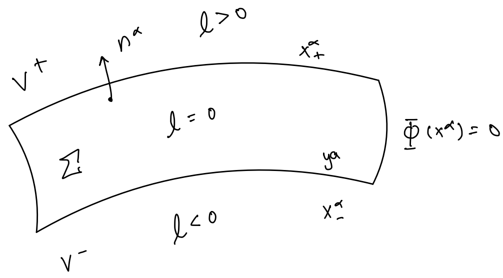

Figure 3.1: Example of two spacetimes being joined by a non-null hypersurface Σ.

where $\delta(l)$ is the Dirac distribution.

We will also be using the following notation,

$$[A] = A\Big|_{\Sigma^+} - A\Big|_{\Sigma^-} \quad , \tag{3.4}$$

which tells us the difference in the quantity A on different sides of the hypersurface, or a jump of the quantity A across the hypersurface.

3.2 The metric in a soldered spacetime

We will proceed with formulating junction conditions with a distributional approach following in the footsteps of Poisson [1]. The distributional formulation is equivalent to the Darmois–Israel formulation that is based on the Gauss–Codazzi decomposition of spacetime [10]. The derivation of the second junction condition is based on exercise 4 from chapter 3 of Poisson's book [1].

The situation is as described at the start of the chapter: we have a spacetime manifold $\mathcal{V}$ that is partitioned into two spacetime regions by Σ. We are working with the metric formulation of general relativity with 4 dimensions and a Levi-Civita connection. In the distributional approach, we demand the solution of the Einstein equation for $\mathcal{V}$ to be,

$$g_{\alpha\beta} = \Theta(l)g^+_{\alpha\beta} + \Theta(-l)g^-_{\alpha\beta} \quad , \tag{3.5}$$

which is a reasonable demand, since the metrics $g^{\pm}_{\alpha\beta}$ are the full solutions in their own spacetimes regions that they characterize. Unfortunately, some problems arise at the boundary of these spacetimes at Σ, and there needs to be some additional constraint on the metric for the distributional metric to be the solution. These constraints come from the demands for the geometry to be well-defined with no singularities. Since $\Theta(l = 0)$ is undefined, we will use the induced quantities on Σ when we have $l = 0$. For example, the metric in (3.5) is not defined at $l = 0$, so we use the induced metric to describe the metric at $l = 0$ at each side of Σ. In the case of the metric, it turns out that the first junction condition demands the induced metric be the same on both sides which means we have a single induced metric on both sides of Σ. However, not all quantities are the same on both sides. This leads to a quantity having a different value or definition at $l = 0$ depending on which side of Σ it is in.

3.3 First junction condition

Now let us look into what the concept of geometry being well-defined means for the metric. One way to think about it is that the requirement for the geometry to be well-defined means that the metric must be continuous everywhere including across the hypersurface Σ. This can be thought of as the first junction condition, albeit not formulated in terms of the intrinsic quantities of Σ. However, since we are looking at this through the distributional solutions to the Einstein equation, it is best to develop the first junction condition from this approach.

We start from the requirement that a well-defined geometry must have a well-defined metric connection across Σ [1]. The metric is already well-defined in equation (3.5). We also want to define the connection similarly,

$$\Gamma^{\alpha}_{\beta\gamma} = \Theta(l)\Gamma^{\alpha+}_{\beta\gamma} + \Theta(-l)\Gamma^{\alpha-}_{\beta\gamma} \quad . \tag{3.6}$$

But this is not what we get when we calculate the connection from the metric (3.5) using the standard definition of the Christoffel symbols,

$$\Gamma^{\alpha}_{\beta\gamma} \equiv \frac{1}{2} g^{\alpha\lambda} \left(g_{\lambda\beta,\gamma} + g_{\lambda\gamma,\beta} - g_{\beta\gamma,\lambda} \right), \tag{3.7}$$

because the derivative of the metric (3.5) is

$$g_{\alpha\beta,\gamma} = \Theta(l) g^{+}_{\alpha\beta,\gamma} + \Theta(-l) g^{-}_{\alpha\beta,\gamma} + \varepsilon \delta(l) [g_{\alpha\beta}] n_{\gamma} \quad . \tag{3.8}$$

We see that the derivative of the metric contains a term proportional to $\delta(l)$. This means that the Christoffel symbols (3.7) will have a term proportional to $\delta(l)\Theta(l)$ which is not defined distributionally. As such, we demand that this term must disappear. This is the same as demanding that the $\delta(l)$-function part of the derivative of the metric must disappear. So we demand

$$[g_{\alpha_\beta}] = 0 \quad , \tag{3.9}$$

for a joining of two spacetimes to be possible. This equation is formulated in the coordinates x^{α} and to get it in terms of the properties of Σ we make use of the holonomic basis vectors to project it [1]:

$$[g_{\alpha_\beta}] = 0 \tag{3.10}$$

$$[g_{\alpha_\beta}] e^{\alpha}_{a} e^{\beta}_{b} = 0 \tag{3.11}$$

$$[h_{ab}] = 0 \quad . \tag{3.12}$$

(3.12) is the first junction condition: the jump of the induced metric must be zero across the hypersurface. This is also called the first Darmois condition or the first Darmois–Israel junction condition. The condition (3.9), before the projection with the holonomic basis vectors, can be called the first Lichnerowicz condition [10]. The first Lichnerowicz condition implies the first Darmois–Israel condition (3.12) and they are equivalent when the coordinates $x^{\alpha}_{\pm}$ are continuous at Σ [10]. Since one of our assumptions is the continuity of the coordinates $x^\alpha_\pm$ at Σ, these conditions are equivalent to us and we call them the first junction condition.

When the first junction condition is satisfied the connection becomes well-defined and has the form we wanted it to have,

$$\Gamma^\alpha_{\beta\gamma} = \Theta(l)\Gamma^{\alpha+}_{\beta\gamma} + \Theta(-l)\Gamma^{\alpha-}_{\beta\gamma} \quad . \tag{3.13}$$

So when the first junction condition is satisfied the full connection of the spacetime $\mathcal{V}$ is the distributional connection with only Heaveside distribution terms. Note that this does not eliminate a possible discontinuity in the connection at Σ, which means that the first junction condition makes only the metric continuous everywhere. The possibility of the discontinuity of connection means a possible discontinuity in the first derivative of the metric. The curvature of the spacetime depends on the first derivative of the connection. Thus to get a good picture we must compute the Riemann curvature tensor and see how it behaves when calculated from the distributional Christoffel symbols.

3.4 Distributional Riemann curvature tensor

Since the first junction condition is critical for any spacetime to be soldered smoothly, we will now discuss only the situations where it holds. When the first junction condition is satisfied we have the connection as written in (3.13). To construct the Riemann (curvature) tensor we need the derivative of $\Gamma^\alpha_{\beta\gamma}$,

$$\Gamma^\alpha_{\beta\gamma,\delta} = \Theta(l)\Gamma^{\alpha+}_{\beta\gamma,\delta} + \Theta(-l)\Gamma^{\alpha-}_{\beta\gamma,\delta} + \varepsilon\delta(l)[\Gamma^\alpha_{\beta\gamma}]n_\delta \quad . \tag{3.14}$$

The Riemann tensor for the Levi-Civita connection is defined as

$$R^\alpha_{\beta\gamma\delta} = \left(\Gamma^\alpha_{\delta\beta,\gamma} - \Gamma^\alpha_{\gamma\beta,\delta} + \Gamma^\alpha_{\gamma\mu}\Gamma^\mu_{\delta\beta} - \Gamma^\alpha_{\delta\mu}\Gamma^\mu_{\gamma\beta}\right) \quad . \tag{3.15}$$

Calculating this by plugging in the equations (3.13) and (3.14) we get

$$
\begin{aligned}
R^{\alpha}_{\beta\gamma\delta} &= \Theta(l)\{\Gamma^{\alpha+}_{\delta\beta,\gamma} - \Gamma^{\alpha+}_{\gamma\beta,\delta} + \Gamma^{\alpha+}_{\gamma\mu}\Gamma^{\mu+}_{\delta\beta} - \Gamma^{\alpha+}_{\delta\mu}\Gamma^{\mu+}_{\gamma\beta}\} \\
&+ \Theta(-l)\{\Gamma^{\alpha-}_{\delta\beta,\gamma} - \Gamma^{\alpha-}_{\gamma\beta,\delta} + \Gamma^{\alpha-}_{\gamma\mu}\Gamma^{\mu-}_{\delta\beta} - \Gamma^{\alpha-}_{\delta\mu}\Gamma^{\mu-}_{\gamma\beta}\} \qquad (3.16) \\
&+ \varepsilon\delta(l)[\Gamma^{\alpha}_{\delta\beta}]n_\gamma - \varepsilon\delta(l)[\Gamma^{\alpha}_{\gamma\beta}]n_\delta \\
&= \Theta(l)R^{\alpha+}_{\beta\gamma\delta} + \Theta(-l)R^{\alpha-}_{\beta\gamma\delta} + \delta(l)A^{\alpha}_{\beta\gamma\delta}, \qquad (3.17)
\end{aligned}
$$

where,

$$
A^{\alpha}_{\beta\gamma\delta} \equiv \varepsilon\left([\Gamma^{\alpha}_{\delta\beta}]n_\gamma - [\Gamma^{\alpha}_{\gamma\beta}]n_\delta\right), \qquad (3.18)
$$

is the δ-function part of the Riemann tensor. As we can see, the Riemann tensor is well-defined distributionally but there is a δ-function singularity part. The question then remains on how to handle this singularity. It turns out there are two ways to go about it. The first is to demand that there is no singularity and that the full Riemann tensor must be a distribution of the Riemann tensors in their respective spacetime regions similar to the metric. This would lead to the second junction condition. The second option is to give a physical explanation to this singularity, which leads to interpreting the δ-function part of the Riemann tensor coming from a thin shell of matter at Σ [1,2].

3.5 Second junction condition

As noted in the previous section, requiring the $\delta(l)$ part of the Riemann tensor to vanish leads to the second junction condition. The constraints $A^{\alpha}_{\beta\gamma\delta} = 0$ or $[\Gamma^{\alpha}_{\beta\gamma}] = 0$ make the δ-function part of the Riemann tensor vanish, but these are not the constraints we are looking for since they are only valid in the coordinates x^α. We are looking for coordinate-independent formulation in terms of intrinsic properties of Σ. To get to such formulation we start with the condition $[\Gamma^{\gamma}_{\alpha\beta}] = 0$ and project it onto Σ,

$$[\Gamma^\gamma_{\alpha\beta}]e^\alpha_a e^\beta_b = 0 \tag{3.19}$$

$$[\Gamma^\gamma_{ab}] = 0 \quad . \tag{3.20}$$

The constraint

$$[\Gamma^\gamma_{ab}] = 0 \tag{3.21}$$

is enough to make the Riemann tensor regular for non-null cases, the proof for this is given in the appendix A.1. We now try to relate this to the extrinsic curvature. Using the following relations from chapter 3.7.5 from Poisson's book [1],

$$[n_{\alpha;\beta}] = -[\Gamma^\gamma_{\alpha\beta}]n_\gamma \qquad \& \qquad [K_{ab}] = [n_{\alpha;\beta}]e^\alpha_a e^\beta_b \quad , \tag{3.22}$$

we get

$$[K_{ab}] = -[\Gamma^\gamma_{ab}]n_\gamma \quad . \tag{3.23}$$

Since the discontinuity of Γ^γ_{ab} is directed along the normal n^γ—this follows directly from equations (3.27) and (3.28)—we can also write

$$[K_{ab}]n^\gamma = -\varepsilon[\Gamma^\gamma_{ab}] \quad . \tag{3.24}$$

The equations (3.23) and (3.24) tell us that

$$[K_{ab}] = 0 \Leftrightarrow [\Gamma^\gamma_{ab}] = 0 \quad . \tag{3.25}$$

This means that $[K_{ab}] = 0$ implies $A^\alpha_{\beta\gamma\delta} = 0$ in (3.18). Thus our second junction condition is

$$[K_{ab}] = 0 \quad . \tag{3.26}$$

3.6 Surface energy-momentum tensor

As mentioned earlier, it is not strictly necessary for all physical systems to satisfy the second junction condition, if we can find an interpretation for the δ-function singularity of the Riemann tensor. We can interpret the δ-function part of the Riemann tensor as coming from the δ-function part of the energy-momentum tensor via the Einstein equation. This δ-function term in the energy-momentum tensor would mean that there is a singular thin layer of matter, or a thin shell, at Σ [1,2]. We can show this by computing the δ-function part of the Ricci tensor and Ricci scalar and using the Einstein equation to relate this to the δ-function part of the energy-momentum tensor. Before doing that, however, it is helpful to write the discontinuity of the derivative of the metric in terms of a tensor [1],

$$[g_{\alpha\beta,\gamma}] = \kappa_{\alpha\beta} n_\gamma \tag{3.27}$$

$$\Rightarrow \kappa_{\alpha\beta} = \varepsilon [g_{\alpha\beta,\gamma}] n^\gamma \quad . \tag{3.28}$$

The second equation follows from the discontinuity of the derivative of the metric only being directed along the normal vector, see appendix A.2 for proof.

Using (3.28) allows us to write the discontinuity of the Christoffel symbol as

$$[\Gamma^\alpha_{\beta\gamma}] = \frac{1}{2}\left(\kappa^\alpha_\beta n_\gamma + \kappa^\alpha_\gamma n_\beta - \kappa_{\beta\gamma} n^\alpha\right), \tag{3.29}$$

which, using (3.18), gives the following equations for the δ-function parts of the Riemann tensor $A^\alpha_{\beta\gamma\delta}$, the Ricci tensor $A_{\alpha\beta}$, and the Ricci scalar A,

$$A^\alpha_{\beta\gamma\delta} = \frac{\varepsilon}{2}\left(\kappa^\alpha_\delta n_\beta n_\gamma - \kappa^\alpha_\gamma n_\beta n_\delta - \kappa_{\beta\delta} n^\alpha n_\gamma + \kappa_{\beta\gamma} n^\alpha n_\delta\right) \tag{3.30}$$

$$A_{\alpha\beta} \equiv A^\mu_{\alpha\mu\beta} = \frac{\varepsilon}{2}\left(\kappa_{\mu\alpha} n^\mu n_\beta + \kappa_{\mu\beta} n^\mu n_\alpha - \kappa n_\alpha n_\beta - \varepsilon \kappa_{\alpha\beta}\right) \tag{3.31}$$

$$A \equiv A^\alpha_\alpha = \varepsilon\left(\kappa_{\mu\nu} n^\mu n^\nu - \varepsilon\kappa\right) \quad , \tag{3.32}$$

where $\kappa \equiv \kappa^\alpha_\alpha$ [1]. The raising and lowering of the indices of $A^\alpha_{\beta\gamma\delta}$ can be done by using the Lichnerowicz condition (3.9). For example, $A^{\alpha\beta}_{\gamma\delta} = \varepsilon\left([\Gamma^\alpha_{\delta\nu} g^{\nu\beta}] n_\gamma - [\Gamma^\alpha_{\gamma\nu} g^{\nu\beta}] n_\delta\right)$.

The δ-function part of the Einstein tensor $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R$ is then

$$E_{\alpha\beta} = A_{\alpha\beta} - \frac{1}{2}Ag_{\alpha\beta} \quad . \tag{3.33}$$

Then the Einstein tensor as a whole is

$$G_{\alpha\beta} = \Theta(l)G^{+}_{\alpha\beta} + \Theta(-l)G^{-}_{\alpha\beta} + \delta(l)E_{\alpha\beta} \quad . \tag{3.34}$$

The Einstein equation is

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta} \quad . \tag{3.35}$$

We can then infer that the energy-momentum tensor can be written as the following distribution,

$$T_{\alpha\beta} = \Theta(l)T^{+}_{\alpha\beta} + \Theta(-l)T^{-}_{\alpha\beta} + \delta(l)S_{\alpha\beta} \quad , \tag{3.36}$$

with

$$S_{\alpha\beta} \equiv \frac{1}{8\pi}\left(A_{\alpha\beta} - \frac{1}{2}Ag_{\alpha\beta}\right) \tag{3.37}$$

being the δ-function part of the energy-momentum tensor. We interpret the δ-function terms with the presence of a thin distribution of matter at Σ with the energy-momentum tensor $S_{\alpha\beta}$ [1]. This surface layer is usually called a thin shell or a bubble wall. We can also write the δ-function part of the energy-momentum tensor in terms of the extrinsic curvature of Σ. $S_{\alpha\beta}$ is tangent to the hypersurface Σ and admits the decomposition [1],

$$S^{\alpha\beta} = S^{ab}e^{\alpha}_{a}e^{\beta}_{b} \tag{3.38}$$

$$\Rightarrow S_{ab} = S_{\alpha\beta}e^{\alpha}_{a}e^{\beta}_{b} \tag{3.39}$$

$$S_{ab} = -\kappa_{\alpha\beta}e^{\alpha}_{a}e^{\beta}_{b} + h^{mn}\kappa_{\mu\nu}e^{\mu}_{m}e^{\nu}_{n}h_{ab} \quad . \tag{3.40}$$

Using the relations (3.23) and (3.29) we get

$$[K_{ab}] = \frac{\varepsilon}{2}\kappa_{\alpha\beta}e^{\alpha}_{a}e^{\beta}_{b} \quad . \tag{3.41}$$

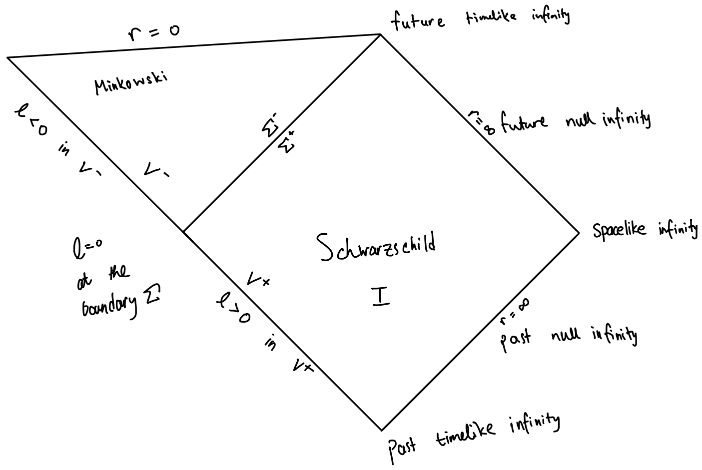

Figure 3.2: Penrose diagram of Minkowski spacetime interior joined to Schwarzschild spacetime exterior via singular timelike hypersurface Σ. The different regions of the full Schwarzschild spacetime have been omitted.

This gives us the Lanczos equation [1,2,10],

$$S_{ab} = -\frac{\varepsilon}{8\pi}\left([K_{ab}] - [K]h_{ab}\right) \quad . \tag{3.42}$$

The complete surface energy-momentum tensor of Σ is then

$$T_{\Sigma}^{\alpha\beta} = \delta(l) S^{ab} e_a^{\alpha} e_b^{\beta} \tag{3.43}$$

and it is zero if $[K_{ab}] = 0$ as expected [1]. This means that when our second junction condition is violated we have a spacetime that is singular at Σ, but one that is not unphysical. This singularity comes from a thin surface layer with energy-momentum tensor $T_{\Sigma}^{\alpha\beta}$. An example of a thin shell is denoted by a Penrose diagram in Figure 3.2, where a Minkowski spacetime is joined to a Schwarzschild spacetime via singular timelike hypersurface. For an example calculation regarding collapsing thin shell, see section 3.9 of Poisson's book [1].

Summary We have now derived the junction conditions for timelike and spacelike hypersurfaces. The first junction condition is

$$[h_{ab}] = 0 \quad , \tag{3.44}$$

and the second junction condition is

$$[K_{ab}] = 0 \quad . \tag{3.45}$$

In the case where the second junction condition is violated, we get a description of a thin shell on the hypersurface. This formulation of the junction conditions is called the Darmois–Israel formalism/junction conditions or Israel junction conditions. When the second junction condition is violated it is called the thin shell formalism or the Israel formalism. In the next chapter, we will derive the junction conditions and the thin shell formalism for null hypersurfaces.

4. Junction conditions for null geometries

We now consider a null hypersurface Σ that partitions the spacetime into two regions $V^{\pm}$ with metrics $g^{\pm}_{\alpha\beta}$ in the coordinates $x^{\alpha}_{\pm}$. We use the convention where V^- (V^+) is in the past (future) of Σ. We also use the variable τ, instead of l as in the non-null case, to describe the distributions defined in section 3.1.2 of the previous chapter, such that Σ is located at $\tau = 0$. We use the coordinates that we defined in section 2.2,

$$y^a = (\lambda, \theta^A) \quad , \tag{4.1}$$

on Σ and assume them to be the same on both sides of Σ. λ is taken to be an arbitrary variable in the null generators of Σ while θ^A are used to label the generators.

We have the following tangent vectors $e^{\alpha}_{\pm a} = \partial x^{\alpha}_{\pm}/\partial y^a$ on each side of the

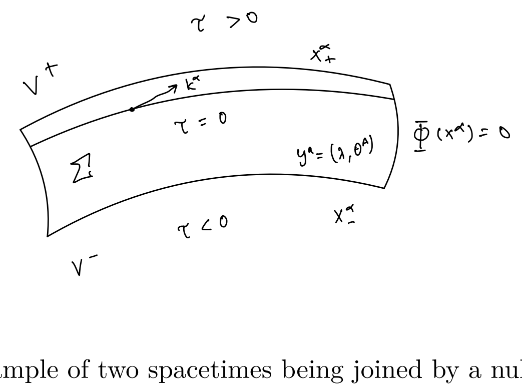

Figure 4.1: Example of two spacetimes being joined by a null hypersurface Σ.

hypersurface:

$$k^\alpha = \left(\frac{\partial x^\alpha}{\partial \lambda}\right)_{\theta^A} = e^\alpha_\lambda \quad \& \quad e^\alpha_A = \left(\frac{\partial x^\alpha}{\partial \theta^A}\right)_\lambda \quad , \tag{4.2}$$

where k^α is a null vector tangent to the generators and e^α_A are two spacelike vectors. We have suppressed $\pm$ to simplify the notation. These vectors satisfy

$$k_\alpha k^\alpha = 0 = k_\alpha e^\alpha_A \quad . \tag{4.3}$$

The metric is degenerate in a three-dimensional null hypersurface so we use the non-degenerate metric in a two-dimensional subsurface S of Σ to describe Σ

$$\sigma_{AB} = g_{\alpha\beta} e^\alpha_A e^\beta_B \quad . \tag{4.4}$$

4.1 Junction conditions

The junction conditions for null hypersurfaces are similar to the ones for non-null hypersurfaces. In the null case, we get

$$[g_{\alpha\beta}] = 0 \Leftrightarrow [\sigma_{AB}] = 0 \tag{4.5}$$

from requiring that the metric be continuous and the geometry be well-defined, just like in the non-null case. The second condition is also similar to the non-null case, but this time we have the transverse curvature C_{ab} instead of the curvature K_{ab}. The second junction condition for null hypersurfaces is

$$[C_{ab}] = 0 \quad . \tag{4.6}$$

The difference with the non-null case is that in the non-null case, $\kappa_{ab} = 0$—defined in the equation (3.28)—is enough to make the Riemann tensor regular, while in the null case, the tensor γ_{ab}—defined in the same way as κ_{ab} via the equation (4.7)—being zero is not enough to make the Riemann tensor regular [8].

Now that we have established the first and the second junction conditions, we will derive the thin shell formalism for null shells in the next section.

4.2 Thin shell formalism

We can describe the discontinuity of the derivative of the metric in terms of a tensor $\gamma_{\alpha\beta}$ defined by

$$[g_{\alpha\beta,\gamma}] = -\gamma_{\alpha\beta}k_\gamma \quad . \tag{4.7}$$

Using (2.7) we get

$$\gamma_{\alpha\beta} = [g_{\alpha\beta,\gamma}]N^\gamma \quad . \tag{4.8}$$

This means that any derivative of the metric tangent to the subsurface S is zero, which means that the discontinuity can only be along the normal vector just like in the non-null case. We use

$$[\Gamma^\gamma_{\alpha\beta}] = [N_{\alpha;\beta}]k^\gamma \quad \Rightarrow \quad [N_{\alpha;\beta}] = -[\Gamma^\gamma_{\alpha\beta}]N_\gamma \tag{4.9}$$

to get

$$[C_{ab}] = [N_{\alpha;\beta}]e^\alpha_a e^\beta_b \tag{4.10}$$

$$= -[\Gamma^\gamma_{\alpha\beta}]N_\gamma e^\alpha_a e^\beta_b \tag{4.11}$$

$$= -[\Gamma^\gamma_{ab}]N_\gamma \quad . \tag{4.12}$$

Because $S^{\alpha\beta}$ is tangential to Σ, we get

$$[C_{ab}] = 0 \Rightarrow S^{\alpha\beta} = 0 \quad . \tag{4.13}$$

This was the junction condition we had presented for the null Σ.

The Riemann tensor can be singular since we can interpret the singularity as being a thin shell of matter. This matter can be radiation, such as gravitational waves, or neutrino bursts from a supernova (when approximating the neutrino mass as zero), or matter at the event horizon [9]. Let us investigate the more interesting case where the energy-momentum tensor is singular at the junction. The δ-function part of the Riemann tensor is [1]

$$R(\Sigma)^{\alpha}_{\beta\gamma\delta} = -(-k_\mu u^\mu)^{-1}([\Gamma^{\alpha}_{\delta\beta}]k_\gamma - [\Gamma^{\alpha}_{\gamma\beta}]k_\delta)\delta(\tau) \quad . \tag{4.14}$$

The difference with the equations (3.17) and (3.18) for the non-null case comes from the term $-k_\mu u^\mu$, where u^μ is the velocity of the observer, which depends on the choice of observers. Since the term $k_\mu u^\mu$ is a scalar that depends on the observers, it does not change under the contractions required to get the energy-momentum tensor. We can define the δ-function of part of the energy-momentum tensor as

$$T^{\alpha\beta}_{\Sigma} = (-k_\mu u^\mu)^{-1} S^{\alpha\beta}\delta(\tau) \quad . \tag{4.15}$$

As such we can write

$$R(\Sigma)^{\alpha}_{\beta\gamma\delta} = (-k_\mu u^\mu)^{-1} A^{\alpha}_{\beta\gamma\delta}\delta(\tau) \tag{4.16}$$

$$\Rightarrow A^{\alpha}_{\beta\gamma\delta} = -([\Gamma^{\alpha}_{\delta\beta}]k_\gamma - [\Gamma^{\alpha}_{\gamma\beta}]k_\delta) \tag{4.17}$$

$$\Rightarrow S^{\alpha\beta} = \frac{1}{8\pi}(A^{\alpha\beta} - \frac{1}{2}Ag^{\alpha\beta}) \quad , \tag{4.18}$$

where $A_{\alpha\beta} = A^{\mu}_{\alpha\mu\beta}$ and $A = A^{\mu}_{\mu}$. When contracting, we raise or lower the indices of $A^{\alpha}_{\beta\gamma\delta}$ just like in the non-null case by raising or lowering the indices before calculating the jump in connection across Σ. Using the definition of the Christoffel-symbol (3.7) and (4.7), we get

$$[\Gamma^{\alpha}_{\beta\gamma}] = -\frac{1}{2}(\gamma^{\alpha}_{\beta}k_\gamma + \gamma^{\alpha}_{\gamma}k_\beta - \gamma_{\beta\gamma}k^\alpha) \quad . \tag{4.19}$$

Substituting (4.19) in (4.17), we get

$$A^{\alpha}_{\beta\gamma\delta} = \frac{1}{2}\left(\gamma^{\alpha}_{\delta}k_\beta k_\gamma - \gamma_{\delta\beta}k^\alpha k_\gamma - \gamma^{\alpha}_{\gamma}k_\beta k_\delta + \gamma_{\gamma\beta}k^\alpha k_\delta\right) \quad . \tag{4.20}$$

Contracting the first and the 3rd indices, we get

$$A_{\alpha\beta} \equiv A^{\mu}_{\alpha\mu\beta} = \frac{1}{2}(k_\alpha \gamma^{\mu}_{\beta}k_\mu + k_\beta\gamma_{\mu\alpha}k^\mu - \gamma^{\mu}_{\mu}k_\alpha k_\beta) \quad . \tag{4.21}$$

With further contraction we get

$$A \equiv A^{\nu}_{\nu} = \frac{1}{2}(2k^{\nu}\gamma^{\mu}_{\nu}k_{\mu} - \gamma^{\mu}_{\mu}k^{\nu}k_{\nu}) \tag{4.22}$$

$$A = \gamma_{\mu\nu}k^{\nu}k^{\mu} \quad . \tag{4.23}$$

Using the contractions above we get the energy-momentum tensor of Σ (4.15), up to a factor of $(-k_{\mu}u^{\mu})^{-1}$,

$$S^{\alpha\beta} = \frac{1}{8\pi}(A^{\alpha\beta} - \frac{1}{2}Ag^{\alpha\beta}) \tag{4.24}$$

$$= \frac{1}{16\pi}(k^{\alpha}\gamma_{\mu}{}^{\beta}k^{\mu} + k^{\beta}\gamma_{\mu}{}^{\alpha}k^{\mu} - \gamma^{\mu}_{\mu}k^{\alpha}k^{\beta} - \gamma_{\mu\nu}k^{\nu}k^{\mu}g^{\alpha\beta}) \quad . \tag{4.25}$$

To simplify $S^{\alpha\beta}$, we first do the following expansion using the completeness relation (2.13):

$$\gamma_{\mu}{}^{\alpha}k^{\mu} = \gamma_{\mu\nu}k^{\mu}g^{\alpha\nu} \tag{4.26}$$

$$= -(\gamma_{\mu\nu}k^{\mu}N^{\nu})k^{\alpha} - (\gamma_{\mu\nu}k^{\mu}k^{\nu})N^{\alpha} + (\sigma^{AB}\gamma_{\mu\nu}k^{\mu}e^{\nu}_{B})e^{\alpha}_{A} \quad . \tag{4.27}$$

The completeness relation also gives us

$$-2\gamma_{\mu\nu}k^{\mu}N^{\nu} = \gamma^{\mu}_{\mu} - \sigma^{AB}\gamma_{\mu\nu}e^{\mu}_{A}e^{\nu}_{B} \quad . \tag{4.28}$$

Defining $\gamma_A \equiv \gamma_{\alpha\beta}e^{\alpha}_{A}k^{\beta}$ and $\gamma_{AB} \equiv \gamma_{\mu\nu}e^{\mu}_{A}e^{\nu}_{B}$ and substituting (4.28) into (4.27), we get

$$\gamma_{\mu}{}^{\alpha}k^{\mu} = \frac{1}{2}(\gamma^{\mu}_{\mu} - \sigma^{AB}\gamma_{AB})k^{\alpha} + (\sigma^{AB}\gamma_{B})e^{\alpha}_{A} - (\gamma_{\mu\nu}k^{\mu}k^{\nu})N^{\alpha} \quad . \tag{4.29}$$

Using (4.29) we can simplify (4.25),

$$S^{\alpha\beta} = \frac{1}{16\pi}\left(-\sigma^{AB}\gamma_{AB}k^{\alpha}k^{\beta} + \sigma^{AB}\gamma_{B}(k^{\alpha}e^{\beta}_{A} - e^{\alpha}_{A}k^{\beta}) - \gamma_{\mu\nu}k^{\mu}k^{\nu}\sigma^{AB}e^{\alpha}_{A}e^{\beta}_{B}\right) \tag{4.30}$$

$$= \mu k^{\alpha}k^{\beta} + j^{A}(k^{\alpha}e^{\beta}_{A} + e^{\alpha}_{A}k^{\beta}) + p\sigma^{AB}e^{\alpha}_{A}e^{\beta}_{B} \quad , \tag{4.31}$$

where

$$\mu \equiv -\frac{1}{16\pi}(\sigma^{AB}\gamma_{AB}) \tag{4.32}$$

can be interpreted as the surface density of the shell,

$$j^A \equiv \frac{1}{16\pi}(\sigma^{AB}\gamma_B) \tag{4.33}$$

as the surface current, and

$$p \equiv -\frac{1}{16\pi}(\gamma_{\alpha\beta}k^\alpha k^\beta) \tag{4.34}$$

as the pressure [1]. While the energy-momentum tensor of the shell is tangent to Σ it is not worthwhile to decompose it like in the non-null case (3.38). This is because we already have the energy-momentum tensor of the surface in terms of the quantities we can measure: the surface density, the surface current, and the pressure. The physically observable quantities are given by multiplying the base quantities by $(-k_\mu u^\mu)^{-1}$, for example

$$\mu_{\text{physical}} = (-k_\mu u^\mu)^{-1}\mu \quad . \tag{4.35}$$

We can write the quantities μ, j^A, and p in terms of the jump in transverse curvature

$$[C_{ab}] = [-\Gamma^\gamma_{\alpha\beta}]N_\gamma e^\alpha_a e^\beta_b \quad , \tag{4.36}$$

which using (2.7) and (4.19) becomes

$$[C_{ab}] = \frac{1}{2}\gamma_{\alpha\beta}e^\alpha_a e^\beta_b \quad , \tag{4.37}$$

giving us

$$[C_{AB}] = \frac{1}{2}\gamma_{\alpha\beta}e^\alpha_A e^\beta_B = \frac{1}{2}\gamma_{AB} \qquad \Rightarrow \mu = -\frac{1}{8\pi}\sigma^{AB}[C_{AB}] \tag{4.38}$$

$$[C_{\lambda B}] = \frac{1}{2}\gamma_{\alpha\beta}k^\alpha e^\beta_B = \frac{1}{2}\gamma_B \qquad \Rightarrow j^A = \frac{1}{8\pi}\sigma^{AB}[C_{\lambda B}] \tag{4.39}$$

$$[C_{\lambda\lambda}] = \frac{1}{2}\gamma_{\alpha\beta}k^\alpha k^\beta \qquad \Rightarrow p = -\frac{1}{8\pi}[C_{\lambda\lambda}] \quad . \tag{4.40}$$

By writing the surface quantities in terms of the jump in transverse curvature we have completed the thin shell formalism for null shells.

Having completed the thin shell formalism, there is one thing left to do: apply it to calculations. In the next chapter, we will apply the thin shell formalism to calculate the junction conditions for two spherically symmetric spacetimes joined by a singular null hypersurface.

5. Stationary lightlike shells in spherical geometries

As an example of the thin shell formalism developed in the section 4, we will calculate the junction conditions for stationary spherical null shells in general spherically symmetric geometries. As a concrete example, we will use these results to calculate the junction conditions for the stationary null shell soldering interior de Sitter spacetime with exterior Reissner–Nordström spacetime.

The stationary null shell is interesting in that having a continuous parameter across the spacetime (usually the radius r) is an initial condition and does not tell us if the soldering of the spacetime is affine or not [13]. This is in contrast to non-stationary null shells, for which a continuous parameter across the hypersurface means that the connection is affine. For stationary null shells, we can define different kinds of solderings corresponding to shells with different physical characteristics, such as different surface quantities.

5.1 Junction conditions for the general spherically symmetric metrics

We will be using the Eddington–Finkelstein coordinates, that has the advanced time u and the retarded time v. These are defined as $u = t + r^*$ and $v = t - r^*$, with

$r^* = \int \frac{dr}{f(r)}$, where $f(r)$ is the function that appears in the spherically symmetric metric of the form $ds^2 = f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega^2$ [15]. For our use case in this section, it is enough to use only the advanced time u.

The general spherically symmetric metric is, in terms of advanced time u,

$$ds^2 = -e^{\psi} du(f e^{\psi} du + 2\zeta dr) + r^2 d\Omega^2 \quad , \tag{5.1}$$

where ψ and f are functions of u and r [13]. ζ is a sign factor which is +1 if r increases towards the future along a ray $u = \text{const}$ [13]. For example the function $f(u{,}r)$ for static Reissner–Nordström spacetime is $f_{RN}(r) = 1 - \frac{2m}{r} + \frac{e^2}{r^2}$ and for static de Sitter spacetime it is $f_{dS}(r) = 1 - \frac{8}{3}\pi\rho_0 r^2$.

We now consider a stationary shell Σ, for which we have $f^{\pm}(u,r) = 0$ at the soldering radius of $r = r_0$ common to both geometries of the form (5.1). The induced metric of Σ is

$$\sigma_{AB} d\theta^A \otimes d\theta^B = r^2 d\Omega^2 \quad , \tag{5.2}$$

where σ_{AB} is the metric of Σ from (4.4). Since Σ is null, we get the conditions $ds^2 = 0$ and $d\Omega^2 = 0$. This gives us the relation between u and r at Σ,

$$\Big(f(u,r)e^{\psi(u,r)}du\Big)\Big|_{r=r_0} = (-2\zeta dr)\Big|_{r=r_0} \tag{5.3}$$

$$\Rightarrow \frac{\partial r}{\partial u}\Big|_{r=r_0} = -\frac{\zeta}{2}(f(u,r)e^{\psi(u,r)})\Big|_{r=r_0} \quad . \tag{5.4}$$

Note that while the function $f(u{,}r)$ is zero at $r = r_0$, the derivatives of $f(u{,}r)$ aren't necessarily zero. This is why we only set $r = r_0$ at the end of the calculations. The integral of the equation (5.4) gives us the equation $\Phi(x^\alpha) = 0$ that defines the hypersurface in the coordinates x^α. We do not need to integrate this to get the junction conditions since it is easier to get them using the parametric equations that describe Σ. The intrinsic coordinates of Σ are $y^a = (\lambda, \theta, \phi)$, with the parametric equations

$$u = \lambda \quad \& \quad \theta = \theta \quad \& \quad \phi = \phi \quad . \tag{5.5}$$

The null vector is then

$$k^\alpha = \frac{\partial x^\alpha(y^a)}{\partial \lambda}\Big|_{r=r_0} \tag{5.6}$$
$$= (1, \frac{\partial r}{\partial u}\Big|_{r=r_0}, 0, 0) \quad . \tag{5.7}$$

The transverse null vector, given by the condition $N^\alpha N_\alpha = 0$ and $k^\alpha N_\alpha = -1$, is

$$N^\alpha = (0,1,0,0)\zeta e^{-\psi}\Big|_{r=r_0} \quad . \tag{5.8}$$

The spacelike tangent basis vectors of Σ are

$$e_2^\alpha = (0,0,1,0) \quad \& \quad e_3^\alpha = (0,0,0,1) \quad . \tag{5.9}$$

Now that we have the necessary vectors along with the metric, we can calculate the junction conditions using the equation (4.37): $[C_{ab}] = \frac{1}{2}\gamma_{\alpha\beta}e_a^\alpha e_b^\beta$, where $\gamma_{\alpha\beta} = [g_{\alpha\beta,\eta}]N^\eta$ as defined in the equation (4.7). This means that we can write the second fundamental forms on their respective sides as

$$\Rightarrow C_{ab}^{\pm} = \frac{1}{2}(g^{\pm}_{\alpha\beta,\eta}N^\eta_\pm e_a^\alpha e_b^\beta)\Big|_{r=r_0} \quad . \tag{5.10}$$

Since from now on we will only be calculating on the shell Σ with $r = r_0$, let's suppress the notation $\Big|_{r=r_0}$ until the end of the section. Looking at the metric (5.1) and the null vectors (5.7) and (5.8) we see that the non-zero components of $\gamma_{\alpha\beta}$ are γ_{00}, γ_{01}, γ_{10}, γ_{22} and γ_{00}. Let's start by calculating the components of $C_{ab}^{\pm}$, the component $C_{\lambda\lambda}^{\pm}$ is

$$C_{\lambda\lambda}^{\pm} = \frac{1}{2}g^{\pm}_{\alpha\beta,\eta}N^\eta_\pm k^\alpha k^\beta \tag{5.11}$$
$$= \frac{1}{2}g^{\pm}_{00,1}N^1_\pm k^0 k^0 + g^{\pm}_{01,1}N^1_\pm k^0 k^1 \tag{5.12}$$
$$= -\frac{1}{2}(e^{2\psi^\pm}f^\pm)_{,r}\zeta^\pm e^{-\psi^\pm} + (-e^{\psi^\pm}\zeta^\pm)_{,r}\zeta^\pm e^{-\psi^\pm}\frac{\partial r}{\partial u} \tag{5.13}$$
$$= -\frac{1}{2}(f^\pm_{,r}e^{\psi^\pm} + 2f^\pm e^{\psi^\pm}\frac{\partial \psi^\pm}{\partial r})\zeta^\pm - \frac{\partial \psi^\pm}{\partial r}\frac{\partial r}{\partial u} \tag{5.14}$$
$$= -\frac{1}{2}f^\pm_{,r}\zeta^\pm e^{\psi^\pm} + \frac{\partial \psi^\pm}{\partial u} \quad , \tag{5.15}$$

where in the last step we used (5.4) to get $-f^{\pm}e^{\psi^{\pm}}\frac{\partial\psi^{\pm}}{\partial r}\zeta^{\pm} = 2\frac{\partial\psi^{\pm}}{\partial u}$, the components of $C^{\pm}_{AB}$ are

$$C^{\pm}_{AB} = \frac{1}{2}g^{\pm}_{\alpha\beta,\eta}N^{\eta}_{\pm}e^{\alpha}_{A}e^{\beta}_{B} = \frac{1}{2}g^{\pm}_{AB,1}N^{1}_{\pm} \tag{5.16}$$

$$\Rightarrow C^{\pm}_{22} = \frac{1}{2}(r^2)_{,r}\zeta^{\pm}e^{-\psi^{\pm}} \tag{5.17}$$

$$\Rightarrow C^{\pm}_{33} = \frac{1}{2}(r^2\sin^2\theta)_{,r}\zeta^{\pm}e^{-\psi^{\pm}} \quad , \tag{5.18}$$

$$\tag{5.19}$$

and the component $C_{\lambda B}$ is

$$C^{\pm}_{\lambda B} = \frac{1}{2}g^{\pm}_{\alpha\beta,\eta}N^{\eta}_{\pm}k^{\alpha}e^{\beta}_{B} \tag{5.20}$$

$$= \frac{1}{2}g^{\pm}_{\alpha B,1}N^{1}_{\pm}k^{\alpha} = 0 \quad . \tag{5.21}$$

These give the junction conditions

$$[C_{\lambda\lambda}] = \left[-\frac{1}{2}f_{,r}\zeta e^{\psi} + \frac{\partial\psi}{\partial u}\right] \tag{5.22}$$

$$[C_{22}] = r_0[\zeta e^{-\psi}] \tag{5.23}$$

$$[C_{33}] = r_0\sin^2\theta[\zeta e^{-\psi}] \tag{5.24}$$

$$[C_{\lambda B}] = 0 \quad . \tag{5.25}$$

Using equations (4.38)–(4.40) we get the surface quantities of the null shell: the surface density is

$$\mu = -\frac{1}{8\pi}\sigma^{AB}[C_{AB}] \tag{5.26}$$

$$\Rightarrow 4\pi r_0\mu = -[\zeta e^{-\psi}] \quad , \tag{5.27}$$

the pressure is

$$p = -\frac{1}{8\pi}[C_{\lambda\lambda}] \tag{5.28}$$

$$\Rightarrow 8\pi p = -\left[-\frac{1}{2}f_{,r}\zeta e^{\psi} + \frac{\partial\psi}{\partial u}\right] \quad , \tag{5.29}$$

and the current is

$$j^A = 0 \quad . \tag{5.30}$$

We have arrived at exactly the same result as Barrabes and Israel in their paper [13]. From this result, we can see that the surface current is always zero for a null stationary shell with spherical geometry and that the junction conditions depend on the soldering parameter $\psi(u,r)$, the radial derivative of the function $f(u,r)$, and the sign factor ζ.

For everywhere-static spacetimes, where there is no radial energy flow, we can simplify with $\psi = \psi(u)$ and $f = f(r)$ [13]. From here on out we assume our examples to be with everywhere-static spacetimes.

5.1.1 Different types of solderings

As mentioned earlier, we can freely choose the soldering type via the function $\psi(u)$ and these solderings correspond to different kinds of shells. Two examples are static soldering, where we choose $\psi^{\pm}(u) = 0$, and affine soldering, where the soldering parameter u is an affine parameter such that the surface pressure vanishes. Using the equations (5.27) and (5.29), we get the surface quantities

$$4\pi r_0 \mu = -[\zeta] \quad \& \quad 8\pi p = \frac{1}{2}[\zeta f_{,r}] \tag{5.31}$$

for static soldering [13]. For affine soldering we have $e^{-\psi} = -\kappa_0 \zeta (u - u_0)$—by integrating equation (5.29) with the condition that the pressure vanish—with $\kappa_0 = \frac{1}{2}\frac{\partial f}{\partial r}\Big|_{r=r_0}$ and $u_0 = \text{const}$, which, using the equations (5.27) and (5.29), gives us the conditions [13]

$$4\pi r_0 \mu = [\kappa_0 (u - u_0)] \quad \& \quad 8\pi p = 0 \quad . \tag{5.32}$$

We see that the junction conditions look very different for the two soldering types and they indeed describe shells with different physical characteristics. The difference in physical characteristics comes from the shells having different surface quantities. We can see from equations (5.31) that the statically soldered shell has non-zero surface density and non-zero pressure. In contrast, equations (5.32) tell us that the affinely soldered shell has zero pressure, and surface density that is different from the surface density of the statically soldered shell. The pressure and the surface density of the statically soldered shell are time-independent while the affinely soldered shell has time-dependent surface density [13].

Now that we have derived the surface quantities of the stationary null shell soldering two general spherically symmetric spacetimes, we are left with presenting an example.

5.2 De Sitter interior in Reissner–Nordström exterior

For a concrete example, let us examine a Reissner–Nordström (RN) exterior with a de Sitter (dS) interior shown in Figure 5.1. The null shell Σ separating the dS interior with RN exterior is along ABC in Figure 5.1. Let us first investigate the segment AB, where we have the dS interior being in the future of the RN exterior. Since the dS interior is to the future of the segment AB, we have dS spacetime as V^+ and the Region II of RN spacetime to the past of segment AB as V^-.

We start by listing the function $f(r)$ in the metric (5.1) for both the dS interior and the RN exterior. For dS spacetime we have [13]

$$f_{dS}(r) = 1 - \frac{8}{3}\pi\rho_0 r^2 = 1 - \frac{r^2}{r_0^2} \quad . \tag{5.33}$$

For RN spacetime we have [13]

$$f_{RN}(r) = 1 - \frac{2m}{r} + \frac{e^2}{r^2} \quad . \tag{5.34}$$

Since we want to look at a stationary null shell, we have $f^{\pm}(r_0) = 0$, which gives us the relation [13]

$$2mr_0 = r_0^2 + e^2 \quad . \tag{5.35}$$

Along the segment AB in Figure 5.1, we have $f_{dS}(r) = f^+(r)$, $f_{RN}(r) = f^-(r)$, $\zeta^+ = -1$, and $\zeta^- = +1$ [13]. Since we have a choice in the type of soldering, let's compute the junction conditions for static and affine solderings.

5.2.1 Static soldering

In the case of static soldering, using the first equation from (5.31), we get the surface density of Σ along the segment AB:

$$4\pi r_0 \mu = -(-1-1) \tag{5.36}$$

$$\mu = (2\pi r_0)^{-1} \quad . \tag{5.37}$$

Using the second equation from (5.31) and the relation (5.35), we get the pressure along the segment AB:

$$8\pi p = \frac{1}{2}(-f^+_{,r} - f^-_{,r})\Big|_{r=r_0} \tag{5.38}$$

$$= \frac{1}{2}\left(-\left(-2\frac{r}{r_0^2}\right) - \left(\frac{2m}{r^2} - 2\frac{e^2}{r^3}\right)\right)\Bigg|_{r=r_0} \tag{5.39}$$

$$= \left(\frac{1}{r_0} - \frac{m}{r_0^2} + \frac{e^2}{r_0^3}\right) = \frac{m}{r_0^2} \quad . \tag{5.40}$$

These are the results for the segment AB. Along the segments BC, we have the dS interior being to the past of Σ and the RN exterior being to the future of Σ. Because of this, both the surface density μ and the surface pressure p change sign at the point B and are negative along the segment BC [13]. This result is unphysical since it produces negative energies. The negative energies can be prevented, but it requires an ad hoc intervention by a momentary infinite surface tension at point B and this still leads to Σ being unphysical [13].

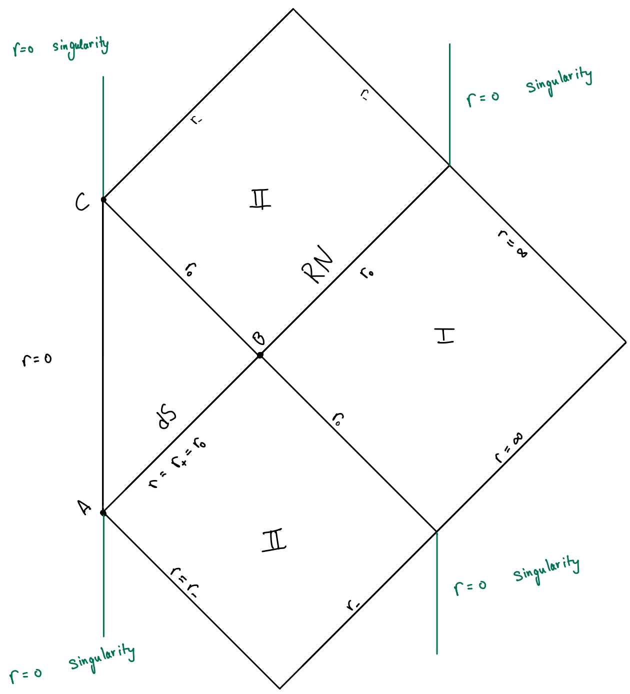

Figure 5.1: Penrose diagram for the de Sitter interior (dS) joined with Reissner–Nordström (RN) exterior. The diagram for the RN exterior is part of the maximally extended diagram. The coordinates $r_{\pm}$ are solutions to $f_{RN}(r) = 0$, such that $r_{\pm} = m \pm (m^2 - e^2)^{\frac{1}{2}}$, and are not to be confused as being coordinates on either side of a hypersurface. $r_+ = r_0$, because of the way we constructed the spacetime and since $f(r_0) = 0$. The dS interior is to the future of Region II of the RN exterior connected via the segment AB of the null shell ABC and it is to the past of Region II of the RN exterior connected via the segment BC of the null shell ABC. For more detail regarding the Reissner–Nordström spacetime see [15].

5.2.2 Affine soldering

While the static soldering leads to an unphysical result, the affine soldering is different since the surface density is time-dependent. Using the junction conditions for the affine soldering from (5.32), we get the following pressure and the surface density along the segment AB in Figure 5.1:

$$p = 0 \tag{5.41}$$

$$4\pi r_0 \mu = \left(\frac{1}{2} f^+_{,r}(u - u_0^+) - \frac{1}{2} f^-_{,r}(u - u_0^-)\right)\Big|_{r=r_0} \tag{5.42}$$

$$= -\frac{1}{r_0}(u - u_0^+) - \frac{1}{r_0}\left(1 - \frac{m}{r_0}\right)(u - u_0^-) \tag{5.43}$$

$$\Rightarrow 4\pi r_0^3 \mu = -(2r_0 - m)u + r_0 u_0^+ + (r_0 - m)u_0^- \quad , \tag{5.44}$$

where the variables u_0^+ and u_0^- are constants that depend on the initial conditions. Along the segment BC, we again have the dS interior to the past of Σ and the RN exterior to the future. This makes dS interior V^- and RN exterior V^+ along the segment BC. Doing a calculation identical to the one for segment AB using the equations (5.32), we get, along the segment BC,

$$p = 0 \tag{5.45}$$

$$4\pi r_0^3 \mu = (2r_0 - m)u - r_0 u_1^- + (r_0 - m)u_1^+ \quad , \tag{5.46}$$

where $u_1^\pm$ are constants different from the constants $u_0^\pm$ and also depend on the initial conditions. This result doesn't make Σ unphysical, since we can have $\mu = 0$ at point B, and due to initial conditions, it is possible to have $\mu \geq 0$ along the whole segment ABC of the Σ [13].

While the statically soldered Σ couldn't join a dS interior with an RN exterior, as shown in Figure 5.1, the affinely soldered Σ could join these two spacetimes together. This concludes our example of how to use the thin shell formalism to calculate the surface quantities for different shells. In the next chapter, we list some applications of the thin shell formalism.

6. Applications of thin shell formalism

So far we have formulated the junction conditions and seen an example of how to use thin shell formalism to calculate the properties of a thin shell. In this chapter, we will look at some of the applications of thin shell formalism.

6.1 Cosmological phase transitions

The thin shell formalism is useful when studying phase transitions in the early universe, for example, in inflation [13]. When two phases exist, the wall separating them can be approximated as a thin shell, or a bubble, to a first approximation [13,16]. These bubbles can be timelike, spacelike, or null depending on the situation. The junction conditions can also be used to study the collision of the bubbles, which could leave a detectable impact on the cosmic microwave background radiation [17–23]. For a review of bubble collisions see [24].

6.2 Black hole dynamics

The thin shell formalism can be used to do calculations regarding black hole dynamics. For example, the Oppenheimer Snyder collapse [25] can be modeled using the junction conditions for a timelike hypersurface with the exterior metric being

Schwarzschild and the interior metric being Friedmann–Lemaître–Robertson–Walker (FLRW) [1]. We can also model the changes in black holes with thin shell formalism, for an example calculation regarding accreting black hole see Chapter 3.11.7 of [1].

Another interesting but speculative use of the junction conditions is the idea of a black hole universe. According to this idea, our universe could be inside a black hole, since it is possible to construct spacetimes with exterior Schwarzschild metric with interior FLRW metric [26].

6.3 Brane world Cosmology

The Israel junction conditions are also used in Brane world cosmology [27,28]. In Brane world cosmology the observable universe is on a hypersurface (called brane) which is embedded in a higher dimensional spacetime (called bulk) [28]. In this model the standard model particles are confined in the brane [28]. For a review of the brane world cosmology see the paper [28].

7. Conclusions

This thesis has presented the junction conditions for soldering two different spacetimes together and the thin shell formalism associated with it. We have learned that demanding the distributional connection to be well-defined leads to the first junction condition and demanding the Riemann tensor to be regular leads to the second junction condition. We can also let the Riemann tensor be singular since it describes a case where the hypersurface separating the two spacetimes has an infinitesimally thin layer of matter. The junction conditions and the thin shell formalism were derived separately for null and non-null hypersurfaces.

After deriving the thin shell formalism we put it to use by deriving the junction conditions for two general spherically symmetric metrics being joined by a stationary null hypersurface and showed a concrete example by calculating soldering of interior de Sitter spacetime with exterior Reissner–Nordström spacetime. We also briefly mentioned the practical applications of the thin shell formalism.

While we have derived the junction conditions of two spacetimes being soldered, we only considered the situation with the metric field. We have not covered the junction conditions for the other fields such as scalar fields and gauge fields, since they are out of the scope of the thesis. For those interested to read more about them the following resources can be consulted [29,30].

A. Appendices

A.1 Appendix 1

We want to prove that for non-null hypersurface $[\Gamma^{\gamma}_{ab}] = 0$ is enough for the Riemann tensor to be regular. Since $[\Gamma^{\gamma}_{\alpha\beta}] = 0 \Rightarrow A^{\gamma}_{\beta\alpha\delta} = 0$, it is enough to show that there exists a gauge where $[\Gamma^{\gamma}_{ab}] = 0 \Rightarrow [\Gamma^{\gamma}_{\alpha\beta}] = 0$ and that $A^{\gamma}_{\beta\alpha\delta}$ is gauge invariant.

From (3.29) we have

$$[\Gamma^{\gamma}_{\alpha\beta}] = \frac{1}{2}(\kappa^{\gamma}_{\alpha} n_{\beta} + \kappa^{\gamma}_{\beta} n_{\alpha} - \kappa_{\alpha\beta} n^{\gamma}) \tag{A.1}$$

$$\Rightarrow [\Gamma^{\gamma}_{ab}] = -\frac{1}{2}\kappa_{ab} n^{\gamma} \quad . \tag{A.2}$$

It follows that $[\Gamma^{\gamma}_{ab}] = 0 \Leftrightarrow \kappa_{ab} = 0$, so it is enough to prove $\kappa_{ab} = 0 \Rightarrow \kappa_{\alpha\beta} = 0$. It turns out that for non-null shells the components of $\kappa_{\alpha\beta}$ that are not κ_{ab} can always be set to zero via gauge transformation, since $\kappa_{\alpha\beta}$ has gauge degree of freedom [13]. Therefore we can say that there exists a gauge where $\kappa_{\alpha\beta} = 0 \Leftrightarrow \kappa_{ab} = 0$. In this gauge, we have $[\Gamma^{\gamma}_{\alpha\beta}] = 0 \Leftrightarrow [\Gamma^{\gamma}_{ab}] = 0$. Since

$$A^{\alpha}_{\beta\gamma\delta} = \varepsilon\left([\Gamma^{\alpha}_{\delta\beta}] n_{\gamma} - [\Gamma^{\alpha}_{\gamma\beta}] n_{\delta}\right) \tag{A.3}$$

is gauge-independent, we can say $[\Gamma^{\gamma}_{ab}] = 0$ is enough for the Riemann tensor to be regular. To confirm this we will show that $A^{\alpha}_{\beta\gamma\delta}$ is indeed gauge-independent.

The gauge transformation of of $\kappa_{\alpha\beta}$ is $\kappa_{\alpha\beta} \to \kappa'_{\alpha\beta} = \kappa_{\alpha\beta} + \lambda_{(\alpha} n_{\beta)}$, where $\lambda_{(\alpha} n_{\beta)} = \lambda_{\beta} n_{\alpha} + \lambda_{\alpha} n_{\beta}$ and λ_{α} is some four-vector on the hypersurface [13]. This gives us the gauge transformation of $[\Gamma^\gamma_{\alpha\beta}]$,

$$[\Gamma^\gamma_{\alpha\beta}]' = \frac{1}{2}(\kappa'_{\alpha\delta}g^{\delta\gamma}n_\beta + \kappa'_{\beta\delta}g^{\delta\gamma}n_\alpha - \kappa'_{\alpha\beta}n^\gamma) \tag{A.4}$$

$$= [\Gamma^\gamma_{\alpha\beta}] + (\lambda_{(\alpha}n_{\delta)}g^{\gamma\delta}n_\beta + \lambda_{(\beta}n_{\delta)}g^{\gamma\delta}n_\alpha - \lambda_{(\alpha}n_{\beta)}n^\gamma) \tag{A.5}$$

$$= [\Gamma^\gamma_{\alpha\beta}] + \lambda^\gamma n_{(\alpha}n_{\beta)} \quad . \tag{A.6}$$

This gives us the gauge transformation of (A.3),

$$A'^\alpha_{\beta\gamma\delta} = \varepsilon\left([\Gamma^\alpha_{\delta\beta}]'n_\gamma - [\Gamma^\alpha_{\gamma\beta}]'n_\delta\right) \tag{A.7}$$

$$= A^\alpha_{\beta\gamma\delta} + \varepsilon\lambda^\alpha(n_\gamma n_{(\delta}n_{\beta)} - n_\delta n_{(\gamma}n_{\beta)}) \tag{A.8}$$

$$= A^\alpha_{\beta\gamma\delta} \quad . \tag{A.9}$$

As we can see $A^\alpha_{\beta\gamma\delta}$ is gauge invariant. This means that if $[\Gamma^\gamma_{ab}] = 0$, the Riemann tensor is regular.

A.2 Appendix 2

We want to prove that $[g_{\alpha\beta,\gamma}]e^\gamma_a = 0$, i.e. $[g_{\alpha\beta,\gamma}]$ is only directed along n_γ.

We start from the first junction conditions and the definition of the derivative for a function $f(x)$,

$$f'(x) \equiv \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} \equiv \lim_{h\to 0}\frac{\Delta f(x)}{h} \quad . \tag{A.10}$$

The first junction condition says $[g_{\alpha\beta}] = 0$, i.e. the value of the metric on the hypersurface Σ is the same on both sides. We now look at the hypersurface Σ in Figure (A.1). At point A we have $g^+_{\alpha\beta}\big|_{\Sigma A} = g^-_{\alpha\beta}\big|_{\Sigma A}$. And at point B, such that the distance between A and B is infinitesimally small h, we have $g^+_{\alpha\beta}\big|_{\Sigma B} = g^-_{\alpha\beta}\big|_{\Sigma B}$. Because of this, we can write

$$g^+_{\alpha\beta}\big|_{\Sigma A} - g^+_{\alpha\beta}\big|_{\Sigma B} = g^-_{\alpha\beta}\big|_{\Sigma A} - g^-_{\alpha\beta}\big|_{\Sigma B} \tag{A.11}$$

$$\Delta g^+_{\alpha\beta}\big|_{\Sigma} = \Delta g^-_{\alpha\beta}\big|_{\Sigma} \quad . \tag{A.12}$$

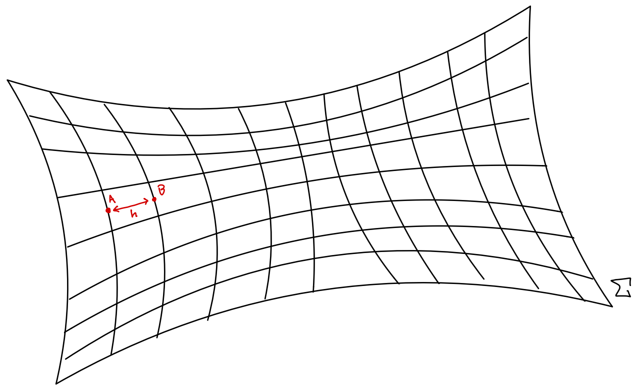

Figure A.1: Two points A and B in a hypersurface Σ. The distance between A and B is h and it is infinitesimal.

The tangential derivative of the metric is

$$g'^{+}_{\alpha\beta}\Big|_{\Sigma} = \lim_{h\to 0} \frac{\Delta g^{+}_{\alpha\beta}\Big|_{\Sigma}}{h} \quad . \tag{A.13}$$

Now the difference in the tangential derivative across Σ, using (A.12), is

$$\left[g'^{+}_{\alpha\beta}\Big|_{\Sigma}\right] = \lim_{h\to 0} \frac{\Delta g^{+}_{\alpha\beta}\Big|_{\Sigma}}{h} - \lim_{h\to 0} \frac{\Delta g^{-}_{\alpha\beta}\Big|_{\Sigma}}{h} \tag{A.14}$$

$$= \lim_{h\to 0} \frac{\Delta g^{+}_{\alpha\beta}\Big|_{\Sigma} - \Delta g^{-}_{\alpha\beta}\Big|_{\Sigma}}{h} \tag{A.15}$$

$$= 0 \quad . \tag{A.16}$$

So the tangential derivative is always continuous due to the first junction condition. The only discontinuity can be along the normal direction, which means that we can write

$$[g_{\alpha\beta,\gamma}] = \kappa_{\alpha\beta} n_{\gamma} \quad . \tag{A.17}$$

Bibliography

[1]E. Poisson, *A Relativist's Toolkit: The Mathematics of Black-Hole Mechanics* (Cambridge University Press, Dec. 2009).

[2]W. Israel, "Singular hypersurfaces and thin shells in general relativity", Nuovo Cim. B **44S10**, [Erratum: Nuovo Cim.B 48, 463 (1967)], 1 (1966).

[3]E. Poisson, *A reformulation of the barrabes-israel null-shell formalism*, 2002.

[4]G. Darmois, "Mémorial des sciences mathématiques (gauthier-villars, paris)", Vol. XXV (1927).

[5]A. Lichnerowicz and T. Teichmann, "Théories relativistes de la gravitation et de l'électromagnétisme", Physics Today **8**, 24 (1955).

[6]S. OBrien and J. L. Synge, "Jump conditions at discontinuities in general relativity", Communications of the Dublin Institute for Advanced Studies. ISSN Series A (Theoretical Physics) 0070-7414 (1952).

[7]A. H. Taub, "Space–times with distribution valued curvature tensors", Journal of Mathematical Physics **21**, 1423–1431 (2008).

[8]C. J. S. Clarke and T. Dray, "Junction conditions for null hypersurfaces", Classical and Quantum Gravity **4**, 265 (1987).

[9]C. Barrabès and P. A. Hogan, "Lightlike signals in general relativity and cosmology", Physical Review D **58**, `10.1103/physrevd.58.044013` (1998).

[10]R. Mansouri and M. Khorrami, "The equivalence of darmois-israel and distributional method for thin shells in general relativity", Journal of Mathematical Physics **37**, 5672–5683 (1996).

[11]M. Mars and J. M. M. Senovilla, "Geometry of general hypersurfaces in space-time: Junction conditions", Class. Quant. Grav. **10**, 1865–1897 (1993).

[12]R. Burghardt, "Linking conditions for models with geometrical basis", Journal of Modern Physics **11**, 355–364 (2020).

[13]C. Barrabès and W. Israel, "Thin shells in general relativity and cosmology: the lightlike limit", Phys. Rev. D **43**, 1129–1142 (1991).

[14]J. Lee, *Manifolds and differential geometry*, Graduate studies in mathematics (American Mathematical Society, 2009).

[15]S. W. Hawking and G. F. R. Ellis, *The large scale structure of space-time*, Cambridge Monographs on Mathematical Physics (Cambridge University Press, 1973).

[16]V. A. Berezin, V. A. Kuzmin, and I. I. Tkachev, "Dynamics of Bubbles in General Relativity", Phys. Rev. D **36**, 2919 (1987).

[17]S. W. Hawking, I. G. Moss, and J. M. Stewart, "Bubble collisions in the very early universe", Phys. Rev. D **26**, 2681–2693 (1982).

[18]W. Z. Chao, "Gravitational effects in bubble collisions", Phys. Rev. D **28**, 1898–1906 (1983).

[19]B. Freivogel, G. T. Horowitz, and S. Shenker, "Colliding with a crunching bubble", Journal of High Energy Physics **2007**, 090–090 (2007).

[20]A. Aguirre and M. C. Johnson, "Towards observable signatures of other bubble universes. ii. exact solutions for thin-wall bubble collisions", Physical Review D **77**, `10.1103/physrevd.77.123536` (2008).

[21]S. Chang, M. Kleban, and T. S. Levi, "When worlds collide", Journal of Cosmology and Astroparticle Physics **2008**, 034 (2008).

[22]M. C. Johnson and I.-S. Yang, "Escaping the crunch: gravitational effects in classical transitions", Physical Review D **82**, 10.1103/physrevd.82.065023 (2010).

[23]M. C. Johnson, H. V. Peiris, and L. Lehner, "Determining the outcome of cosmic bubble collisions in full general relativity", Physical Review D **85**, 10.1103/physrevd.85.083516 (2012).

[24]M. Kleban, "Cosmic bubble collisions", Classical and Quantum Gravity **28**, 204008 (2011).

[25]J. R. Oppenheimer and H. Snyder, "On continued gravitational contraction", Phys. Rev. **56**, 455–459 (1939).

[26]E. Gaztanaga, "How the big bang ends up inside a black hole", Universe **8**, 10.3390/universe8050257 (2022).

[27]H. Kodama, A. Ishibashi, and O. Seto, "Brane world cosmology: gauge-invariant formalism for perturbation", Physical Review D **62**, 10.1103/physrevd.62.064022 (2000).

[28]P. Brax and C. v. d. Bruck, "Cosmology and brane worlds: a review", Classical and Quantum Gravity **20**, R201–R232 (2003).

[29]B. Racskó, "Junction conditions in a general field theory", Classical and Quantum Gravity **41**, 015020 (2023).

[30]L. Avilés, H. Maeda, and C. Martínez, "Junction conditions in scalar–tensor theories", Classical and Quantum Gravity **37**, 075022 (2020).