# The Pollaczeck-Khinchine Formula with a Recursive Approach to Estimating Ruin Probabilities

**Juan José Cosgaya Arrieta**

A thesis submitted in partial fulfillment of the requirements
for the degree of
Master of Mathematics and Statistics

HELSINGIN YLIOPISTO

Department of Mathematics and Statistics
University of Helsinki
June 8, 2024

Advisor: Jaakko Lehtomaa

## ACKNOWLEDGEMENTS

I would like to express my gratitude to those who have supported and guided me during the completion of my master's thesis.

First and foremost, I am grateful to my advisor, Jaakko Lehtomaa, for his insightful feedback and flexibility. His interest and dedication to this field and his students, have been a source of inspiration for me.

Finally and not the least, I extend my thanks to my parents for their support and encouragement during the writing of this thesis, despite the distance that separates us. I could not be more grateful to have them by my side.

*Juan José Cosgaya*

# Contents

**Notation** **iv**

**1 Introduction** **1**

**2 Fundamentals of Ruin Theory** **2**

1 The main idea . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 2

2 The ladder height decomposition . . . . . . . . . . . . . . . . . . . . 4

**3 The Poisson Model** **7**

1 The model . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 7

2 The Poisson model as a random walk . . . . . . . . . . . . . . . . . 8

3 The ruin probability . . . . . . . . . . . . . . . . . . . . . . . . . . 10

4 The ladder height distribution . . . . . . . . . . . . . . . . . . . . . 11

**4 The Pollaczeck-Khinchine formula** **21**

1 The formula . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 21

2 Simulation method . . . . . . . . . . . . . . . . . . . . . . . . . . . 22

3 Estimations of the ruin probability . . . . . . . . . . . . . . . . . . 25

3.1 Exponential distribution . . . . . . . . . . . . . . . . . . . . 25

3.2 Pareto distribution . . . . . . . . . . . . . . . . . . . . . . . 26

**5 Conclusions** **30**

**Bibliography** **30**

**A Code** **33**

# Notation

**Processes and Variables**

| | |
|---|---|
| $(Y_t)$ | Loss process |
| $(Y_n)^{RW}$ | Random walk associated to $(Y_t)$ |
| $(X_t)$ | Accumulated claims process |
| $(P_t)$ | Premiums process |
| $N_t$ | Number of claims at time $t$ |
| $Z_n, Z$ | Individual claims |
| $T(u)$ | Ruin time for initial capital $u$ |
| $\psi(u)$ | Ruin probability for initial capital $u$ |
| $M$ | Maximum loss |
| $K$ | Number of records |
| $T_n$ | Jump times |
| $\Delta T_n$ | Inter-occurrence times |
| $\tau_k$ | $k$th record time |
| $R_k$ | $k$th record |
| $H_k$ | $k$th ladder height |
| $\zeta_n$ | $Z_n - \Delta T_n$ |

**Distributions and Functions**

| | |
|---|---|
| $B$ | Distribution function of the individual claims |
| $\bar{B}$ | Tail of the individual claims |
| $F(x)$ | $\int_{-\infty}^{x} \bar{B}(y)\mathrm{d}y / \mathbb{E}Z$ |
| $F^{*k}$ | $k$th convolution of $F$ |

**Parameters and Constants**

| | |
|---|---|
| $\lambda$ | Parameter of the Poisson process $N_t$ |
| $p$ | $\mathbb{P}(\tau_1 < \infty)$ |

**Sets and Measures**

| | |
|---|---|
| $\Omega$ | Probability space |
| $\mathcal{P}(A)$ | Power set of $A$ |

| | |
|---|---|
| $\mathbb{N}_0$ | Natural numbers including 0 |
| $\mathbb{R}^+$ | Positive real numbers |
| $1/2^{\mathbb{N}}$ | $\{1/2^n \in \mathbb{Q} \mid n \in \mathbb{N}\}$ |
| $\delta\mathbb{N}$ | $\{\delta n \mid n \in \mathbb{N}\}$ |
| $\mu$ | Lebesgue measure in $\mathbb{R}$ |

**Operators and Relations**

| | |
|---|---|
| $\mathbb{E}[\cdot]$ | Expectation operator |
| $\mathbb{P}(\cdot)$ | Probability of an event |
| $\sup\{\cdot\}$ | Supremum |
| $\inf\{\cdot\}$ | Infimum |
| $\lim\{\cdot\}$ | Limit |

**Abstract**

In the field of insurance mathematics, it is critical to control the solvency of an insurance company. In particular, calculating the probability of ruin, which is the probability that the company's surplus falls below zero. In this thesis a review of the fundamentals of ruin theory, the modelling process and some results and methods for the estimation of ruin probabilities is made. Most of the theorems are taken from different bibliographical sources, but a good amount of the proofs presented are original, in order to provide a more rigorous and detailed explanation.

A central focus of this thesis is the Pollaczek-Khinchine formula. This formula provides a solution for the probability distribution of the maximum potential loss of an insurance company in terms of convolutions of a particular function related to the claim sizes. Apart from the theoretical results that may be derived from it and its elegance, its usefulness lies in the ideas underlying it. Specially, the idea to understand the maximum potential loss of the company as the biggest of the historical records in the loss process.

Using these ideas, a recursive approach to estimating ruin probabilities is explained. This approach results in an easy to program and efficient bounds method which allows for any type of claim sizes (that is, the random variables that model how big are the claims of the insureds). The only restrictions imposed come from the fact that this discussion takes place within the Poisson model. This framework allows for various claim size distributions and models the number of claims as a Poisson process.

Finally, two examples of light and heavy-tailed claim size distributions are simulated using this recursive approach. This shows the applicability of the method and the differences between light and heavy-tailed distributions with regards to the ruin probabilities that emerge from them.

**Keywords:** Risk Theory, Ruin Theory, Poisson Model, Pollaczeck-Khinchine Formula, Ruin Probabilities, Insurance Mathematics, Ladder Height Distribution, Bounds Method

# 1 | Introduction

Insurance companies play a fundamental role by providing protection against various risks both to individuals and organizations. In order to be able to meet the future obligations and claims, the insurance company must be in control of its solvency. A part of this is evaluating the ruin probability which will be properly defined in Chapter 2.

To this end, insurance mathematics are essential as they provide the probabilistic modelling for the insurance companies which allows to finally compute the ruin probabilities. The model studied in this thesis will be the Poisson model which has a natural justification. In this model the amount of claims at time $t$, $N_t$, are given by a Poisson process (see Chapter 3).

After achieving a sensible model for the insurance company, calculating the ruin probability could consist on finding the distribution of the ruin time (2.1). However an alternative approach could guide us to different insights and formulas. Taking such an approach, in which the ruin probability is calculated in terms of the distribution of the maximum loss (2.2), the Pollaczeck-Khinchine formula (4.1) can be obtained.

However, it is not very practical for actual calculations. Therefore, it might be of practical interest to study a recursive approach to estimating these ruin probabilities based on this formula. With formulas (4.4)-(4.7) ruin probabilities can be estimated with a simple python script shown in Appendix A.

# 2 | Fundamentals of Ruin Theory

Within the field of Insurance Mathematics, Ruin Theory is a subbranch that focuses on developing probabilistic models for the ruin event. That is, the event that a company becomes insolvent. This modelling is essential for insurance companies, on one hand to deal with the law and authorities, and on the other hand to understand it's own solvency.

Given an initial capital of the company, say $u$, one of the most important quantities Ruin Theory aims to calculate is the ruin probability, $\psi(u)$. Prior to this, however, defining in mathematical terms all of the important quantities and concepts is needed. This section is devoted to doing such thing.

First, the core quantities from which all of the other ones are defined, are discussed and established as postulates for further development. Next, the important definitions for the following sections are given.

## 1 | The main idea

If one wishes to model an insurance company, there are two main quantities to be concerned about. The income and the outcome cash flows. A total knowledge of these cash flows would mean a complete knowledge of any important quantity or event such as the ruin event or the amount of losses. Furthermore, solvency would be reduced to a binary question.

In practice, accurate knowledge of these cash flows, especially the outcome cash flow, is impossible. Therefore the use of Probability Theory is indispensable. The whole framework must rely on some probability space whose events are directly linked to any possible event in the real world. Only one postulate is needed for this.

**Postulate 1.** The set of all possible future events of the physical world, $\Omega$, exists and forms a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ for some $\sigma$-algebra $\mathcal{F}$ and some probability $\mathbb{P}$.

Since the main interest of this thesis lies in the Pollaczeck-Khinchine formula, only one income and one outcome sources will be considered: the premiums and the compensations. Premiums are the cash flows that the clients agree to pay to the company, while compensations are the cash flows that the company pays to the clients whenever claims occur.

Although simplified, this provides enough framework to develop interesting models and achieve useful conclusions. In fact, adding more complexity is done in an easier way on top of this framework, rather than on its own.

With this in mind, the **premiums collected** at $t > 0$ and **accumulated claims** at $t > 0$ are modelled as stochastic processes on $\Omega$ over time and are denoted by $(P_t)$ and $(X_t)$ respectively.

The accumulated claims come from the **individual claims** $Z_n$ that the company has to pay. Therefore, the $(X_t)$ process is further modelled as a compound random process

$$X_t = Z_1 + \cdots + Z_{N_t}.$$

Where $N_t, Z_1, Z_2, ...$ are independent, $Z_1, Z_2, ...$ are i.i.d. and positive with distribution function $B$ and $(N_t)$ is a counting process. The random variables $Z_n$ represent the **claim amounts**, and $N_t$ the **number of claims** at time $t$.

These are the basic building blocks of the typical model for an insurance company. From them, the rest of definitions are made.

**Definition 2.1.** The **losses** of the company at time $t$ are defined as

$$Y_t = \sum_{n=1}^{N_t} Z_n - P_t.$$

As it was stated earlier, the goal of Ruin Theory is to calculate the probability of ruin. Therefore it is crucial to have a definition for the ruin time.

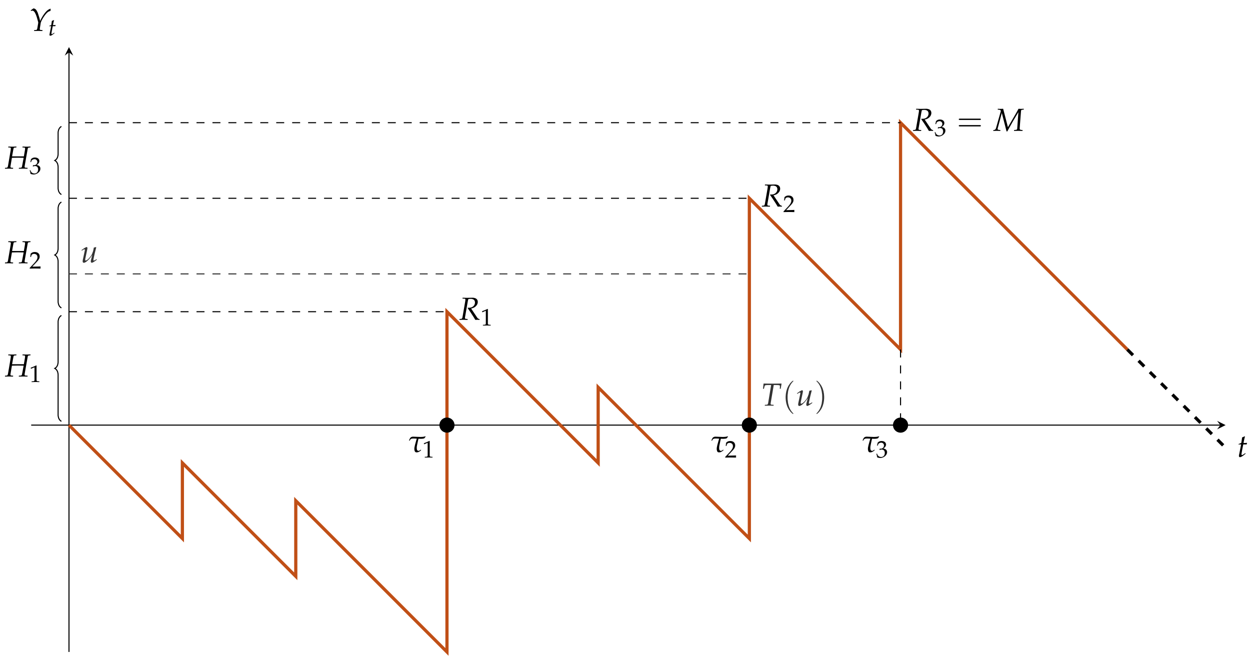


Figure 2.1: Sample path for $Y_t$ (taking $P_t = t$) displaying the ruin time $T(u)$, record times $\tau_k$, records $R_k$, ladder heights $H_k$ and maximum loss $M$. It is assumed that in this sample path the maximum is attained at $\tau_3$, so there are no more records after this time.

**Definition 2.2.** Let the initial capital of the company be $u \geq 0$. Then we define the **ruin time** as the following random variable

$$T(u) = \begin{cases} \inf\{t \geq 0 | Y_t > u\} & \text{if the infimum exists} \\ \infty & \text{otherwise.} \end{cases} \tag{2.1}$$

In the figure 2.1 a sample path for $(Y_t)$ along with the ruin time can be seen graphically. This allows to define the ruin probability in mathematical terms.

**Definition 2.3.** Let the initial capital of the company be $u \geq 0$. Then the **ruin probability** is defined as

$$\psi(u) = \mathbb{P}(T(u) < \infty).$$

## 2 | The ladder height decomposition

There is however an alternative way to calculate the same probability. Taking this is other approach is in fact how the Pollaczeck-Khinchine formula is obtained.

**Definition 2.4.** The **maximum loss** is defined as

$$M = \sup\{Y_t | t \geq 0\}. \tag{2.2}$$

Note that $\psi(u) = \mathbb{P}(M > u)$. Therefore the distribution of $M$ determines the ruin probability.

This maximum loss can be viewed as a sum of the overshoots of each historical record. Looking at figure 2.1, it can be seen that $M = H_1 + H_2 + H_3$. To formalize this idea the following definitions must be made:

**Definition 2.5.** The **record times**, $\tau_k$ for $k = 0, 1, 2, ...$, of the stochastic process $(Y_t)$ are defined by $\tau_0 = 0$ and for $k = 1, 2, \ldots$

$$\tau_k = \begin{cases} \inf\{t | Y_t > Y_{\tau_{k-1}}\} & \text{if the infimum exists} \\ \infty & \text{if } \tau_{k-1} = \infty \text{ or the infimum doesn't exist.} \end{cases}$$

**Definition 2.6.** The **records**, $R_k$ for $k = 1, 2, ...$, of the stochastic process $Y_t$ are defined by

$$R_k = \begin{cases} Y_{\tau_k} & \text{if } \tau_k < \infty \\ \infty & \text{if } \tau_k = \infty. \end{cases}$$

**Definition 2.7.** The **ladder heights**, $H_k$ for $k = 1, 2, ...$, of the stochastic process $Y_t$ are defined by

$$H_k = \begin{cases} Y_{\tau_k} - Y_{\tau_{k-1}} & \text{if } \tau_k < \infty \\ \infty & \text{if } \tau_k = \infty. \end{cases}$$

**Definition 2.8.** The **number of records** $K$ of the stochastic process $Y_t$ is

$$K = \sup\{k \mid \tau_k < \infty\}.$$

From these definitions it follows that almost surely

$$M = \sum_{k=0}^{K} H_k = R_K.$$

This is just formalizing the idea that the maximum loss is the sum of the ladder heights (see 2.1).

It's possible to define the jump times, together with the inter-occurence times. These will be crucial for some of the proofs in the Poisson model.

**Definition 2.9.** The **jump times** $T_0, T_1, T_2, \ldots$ are defined by $T_0 = 0$ and:

$$T_n = \inf\{t \in \mathbb{R}^+ \mid N_t > N_{T_{n-1}}\} \qquad n = 0, 1, 2, \ldots$$

**Definition 2.10.** The **inter-occurence times** of claims $\Delta T_1, \Delta T_2 \ldots$ are defined by:

$$\Delta T_n = T_n - T_{n-1} \qquad n = 1, 2, \ldots$$

# 3 | The Poisson Model

Given the framework developed in chapter 2 which enables the study of the ruin problem, a concrete model is needed in order to derive interesting results and estimate probabilities in practice. In particular, it is necessary to assign a sensible distribution to the claim amount $Z$, the number of claims $N_t$ and the premiums $P_t$.

In this chapter a specific model is presented, the Poisson model. Then, following the structure of lemmas and theorems given in Chapter 4 of [1] an expression for the ruin probability and for the first ladder height is found. The proofs presented are however original. From these, the Pollaczeck-Khinchine formula will follow as a corollary in the next chapter.

## 1 | The model

In the Poisson model the premiums are set to be given by $P_t = pt$. That is, a continuous payment with rate $p > 0$. This means that every year a total of $p$ units of money are collected in premiums. This is a good approximation, since there can be some degree of control on the premiums. In this thesis for simplicity, and without loss of generality we will assume that $p = 1$.

The number of claims $N_t$ is going to be modelled as a homogeneous Poisson process with rate $\lambda > 0$. As it is well known, the Poisson distribution can be interpreted as a limit of a binomial distribution. That is, a Poisson distribution with parameter $\lambda$ is describing the amount of events that occur in one unit of time, if events occur independently and at a constant average rate of $\lambda$ events/unit of time. The homogeneous Poisson process is just an extension of this idea to a time continuum.

It is indeed a very natural choice if we take into account that every counting process with independent and stationary increments is in fact a Poisson process [9, Theorem 5.2.1]. The number of claims must be a counting process and furthermore, in many different contexts it is sensible to assume that the increments are independent and stationary. That is, that the number of accidents occurring at some point in time do not affect the number of accidents at some other point in the future.

The Poisson model doesn't specify any conditions for the claim sizes.[1] Depending on the choice made for these, one particular model or another will arise. This is completely reasonable, as the homogeneous Poisson process assumption to model the number of claims should be good in a great variety of contexts, whereas the claim size is clearly dependent on the insured goods.

Summarizing, in this model $Y_t = \sum_{n=1}^{N_t} Z_n - P_t$ where:

- $N_t$ is a homogeneous Poisson process with rate $\lambda$
- $Z_n$ are i.i.d. and independent of $N_t$ with probability distribution function $B$
- $P_t = t$

It is not perfect, as the independence between the occurrence of the events and the constancy of the average rate $\lambda$ are questionable assumptions. But it constitutes a good approximation to reality in some contexts nonetheless.

*Remark 3.1*. The main feature of this model is that it repeats itself continually. Understanding this heuristic argument, many of the theorems that are going to be presented feel very natural and intuitive. For any time $q > 0$, $Y_{q+t} - Y_q$ is identically distributed as $Y_t$. This is due to the fact that $N_{q+t} - N_q$ has the same distribution as $N_t$ because $N_t$ is a Poisson process, and all the $Z_n$ are i.i.d. and independent of $N_t$.

## 2 | The Poisson model as a random walk

The Poisson model is a model in continuous time. However, generally when dealing with random variables, it is easier to describe and work with discrete processes. The losses $Y_t = \sum_{n=1}^{N_t} Z_n - P_t$ in the Poisson model look very much like a random walk. So a natural question to ask is: Can the losses stochastic process in the Poisson model be viewed as a random walk?

The answer is (almost) yes, and this is the basis of the Sparre-Andersen model for solvency in continuous time. The assumptions of that model are slightly different, but it is possible to arrive to the same conclusions in this setting.

[1] And perhaps we should consider it as a more specific framework more than an actual model.

The key thing to notice is that $M$ and all the ladder heights $H_k$ are determined by the claim sizes $Z_n$ and the inter-occurence times $\Delta T_n$.

**Definition 3.1.** The **random walk associated to** $(Y_t)$ is defined by

$$Y_l^{RW} = \sum_{n=1}^{l} \zeta_n \quad l = 1, 2, \ldots$$

where

$$\zeta_n = Z_n - \Delta T_n.$$

*Remark 3.2*. Of course, the $\zeta_n$ random variables are i.i.d. This is seen by taking into consideration that after each jump of $(Y_t)$, the process repeats itself, making the $\Delta T_n$ i.i.d. (a Poisson process is a renewal process, i.e. the interarrival times are i.i.d. See [10]) Even more intuitively, since the Poisson process $N_t$ has independence of increments and it determines completely $\Delta T_n$, the information about some increment doesn't affect the probabilities of the other increments.

**Theorem 3.1.** *Let $(Y_t)$ be a stochastic process as defined in the Poisson model. Then ruin occurs if and only if ruin occurs for $(Y_l^{RW})$.*

*Proof.* This result is clear as ruin occurs at a jump, and $(Y_l^{RW})$ takes precisely the values that $(Y_t)$ takes at each jump. ■

**Theorem 3.2.** *Let $(Y_t)$ be a stochastic process as defined in the Poisson model. Then $Y_t \to -\infty$ as $t \to \infty$ almost surely if and only if $Y_n^{RW} \to -\infty$ as $n \to \infty$ almost surely.*

*Proof.* Suppose $Y_t \to -\infty$ a.s. Then, for a.e. $\omega$, $Y_t(\omega) \to -\infty$. That is, for a.e. $\omega$, $\forall x \in \mathbb{R} : \exists s : \forall t > s : Y_t(\omega) < x$. Taking $N$ to be the first natural number such that $T_N(\omega) > s$, for a.e. $\omega$, $\forall x \in \mathbb{R} : \exists N : \forall n > N : Y_n^{RW}(\omega) = Y_{T_n} < x$. That is, $Y_n^{RW} \to -\infty$.

The other implication follows the same argument, taking $s = T_N$. ■

*Remark 3.3*. The idea is that the random walk $Y_n^{RW}$ and the process $Y_t$ are not equal, but behave similarly. In particular, events such as ruin are shared and they have the same behaviour asymptotically.

# 3 | The ruin probability

As it was stated earlier, the idea behind the Pollaczeck-Khinchine formula is to calculate the distribution of $M$ by means of the ladder heights. However, they are not i.i.d. as the set $\{H_n \leq x\}$ includes the set $\{\tau_n < \infty\}$. Therefore, $\mathbb{P}(H_n \leq x \mid H_{n-1} = \infty) = 0$, and obviously $\mathbb{P}(H_n \leq x) \neq 0$ but if they were independent, $\mathbb{P}(H_n \leq x \mid H_{n-1} = \infty) = \mathbb{P}(H_n \leq x)$. Nonetheless, the following theorem holds true:

**Theorem 3.3.** *Let $k \in \mathbb{N}$ and $x_1, \ldots, x_k \in \mathbb{R}$. Then the joint distribution of $\{H_1, \ldots, H_k\}$ given $K = k$ is the same as the joint distribution of $n$ i.i.d. random variables distributed as $H_1$ conditioned to $\tau_1 < \infty$. That is,*

$$\mathbb{P}(H_1 \leq x_1, \ldots, H_k \leq x_k \mid K = k), = F(x_1) \ldots F(x_k)$$

*where $F(x) = \mathbb{P}(H_1 \leq x_1 \mid \tau_1 < \infty)$.*

*Proof.* Let $k_1, \ldots, k_n \in \mathbb{N}$ and consider the set

$$A = \left\{N_{\tau_1} - N_{\tau_0} = n_1, \ldots N_{\tau_k} - N_{\tau_{k-1}} = n_k, K = k, H_1 \leq x_1, \ldots, H_k \leq x_k\right\}.$$

Then for some sets $B_1, \ldots, B_n$,

$$A = \left\{(\zeta_1, \ldots \zeta_{n_1}) \in B_1, \ldots, (\zeta_{n_{k-1}+1}, \ldots \zeta_{n_k}) \in B_k, \zeta_{n_k+1} + \ldots + \zeta_{n_k+l} \leq 0 \; \forall l \in \mathbb{N}\right\}.$$

Since the values of $\zeta_n$ determine at which jumps records occur. Also, the condition $\zeta_{n_k+1} + \ldots + \zeta_{n_k+l} \leq 0 \; \forall l \in \mathbb{N}$ ensures that the last record is the $k$th one.

These are independent random variables, so

$$\mathbb{P}(A) = \\ \mathbb{P}((\zeta_1, \ldots \zeta_{n_1}) \in B_1) \cdots \mathbb{P}((\zeta_{n_{k-1}+1}, \ldots \zeta_{n_k}) \in B_k) \mathbb{P}(\zeta_{n_k+1} + \ldots + \zeta_{n_k+l} \leq 0 \; \forall l \in \mathbb{N}).$$

Which is the same as

$$\mathbb{P}(A) = \mathbb{P}(N_{\tau_1} - N_{\tau_0} = n_1, H_1 \leq x_1) \cdots \mathbb{P}(N_{\tau_1} - N_{\tau_0} = n_k, H_1 \leq x_k) \mathbb{P}(\tau_1 = \infty).$$

Since the $\zeta_n$ are i.i.d. Summing over all possible $n_1, \ldots, n_k$

$$\mathbb{P}(K = k, H_1 \leq x_1, \ldots, H_k \leq x_k) = \\ \mathbb{P}(\tau_1 < \infty, H_1 \leq x) \cdots \mathbb{P}(\tau_1 < \infty, H_1 \leq x_k)\mathbb{P}(\tau_1 = \infty). \tag{3.1}$$

Taking the limit when $x_1, \ldots x_n \to \infty$

$$\mathbb{P}(K = k) = \mathbb{P}(\tau_1 < \infty)^k \mathbb{P}(\tau_1 = \infty). \tag{3.2}$$

Finally, from (3.1) and (3.2)

$$\mathbb{P}(H_1 \leq x_1, \ldots, H_k \leq x_k \mid K = k) = F(x_1) \ldots F(x_k).$$

■

In this proof the distribution of $K$ was also found:

> **Theorem 3.4.** *Let $k \in \mathbb{N}$, and define $p = \mathbb{P}(\tau_1 < \infty)$. Then*
>
> $$\mathbb{P}(K = k) = (1 - p)p^k.$$

The ladder heights behave conditionally as if they were independent with distribution $F$. This means, in particular that $\mathbb{P}(H_1 + \cdots + H_k \leq u \mid K = k) = F^{*k}(u)$. The probability distribution function of $M$ can be written as

$$\begin{aligned} \mathbb{P}(M \leq u) &= \sum_{k=0}^{\infty} \mathbb{P}(M \leq u \mid K = k)\mathbb{P}(K = k) && (3.3) \\ &= \sum_{k=0}^{\infty} \mathbb{P}(H_1 + \cdots + H_k \leq u \mid K = k)\mathbb{P}(K = k) && (3.4) \\ &= (1 - p) \sum_{k=0}^{\infty} p^k F^{*k}(u). && (3.5) \end{aligned}$$

The ruin probability is just $\psi(u) = 1 - \mathbb{P}(M \leq u)$. The only thing left to be calculated at this point is the ladder height distribution.

## 4 | The ladder height distribution

First some lemmas are stated and proved.

**Lemma 3.5.** *For the Poisson model, if $\lambda \mathbb{E}Z < 1$, then $Y_t \to -\infty$ as $t \to \infty$ a.s.*

*Proof.* The proof relies on the fact that $Y_t \to -\infty \iff Y_n^{RW} \to -\infty$.

By the law of large numbers

$$\frac{1}{n} Y_n^{RW} \to \mathbb{E}\zeta.$$

Therefore, $\mathbb{E}\zeta < 0$ implies that $Y_n^{RW} \to -\infty$. This is proved by contradiction, if $Y_n^{RW} \not\to -\infty$ then $\lim_{n\to\infty} 1/n Y_n^{RW} \geq 0$.

Once proven that $\mathbb{E}\zeta < 0$, the result follows. For that matter, notice that

$$\mathbb{P}(\Delta T_n > t) = \mathbb{P}(N_{T_{n-1}+t} - N_{T_{n-1}} = 0)$$

and

$$\mathbb{P}(N_{T_{n-1}+t} - N_{T_{n-1}} = 0) = \mathbb{P}(N_t = 0) = e^{-\lambda t}.$$

By the integrated tail probability expectation formula:

$$\mathbb{E}\Delta T_n = \int_0^\infty e^- \lambda t = 1/\lambda.$$

Therefore $\mathbb{E}\zeta = \mathbb{E}Z - 1/\lambda$. If $\lambda \mathbb{E}Z < 1$, then $\mathbb{E}\zeta < 0$.

■

**Lemma 3.6.** *Let $q \geq 0$, then the process $(Y_t^*)$ given by $Y_t^* = Y_q - Y_{q-t}$ for $0 \leq t \leq q$, is also a Poisson process of parameter $\lambda$.*

*Proof.* Since $(Y_t)$ is a Poisson process, given $0 \leq u \leq t \leq q$ the distribution of

$$Y_t^* - Y_u^* = (Y_q - Y_{q-t}) - (Y_q - Y_{q-u}) = Y_{q-u} - Y_{q-t}.$$

is a Poisson distribution with parameter $\lambda((q-u) - (q-t)) = \lambda(t-u)$.

Also, since $(Y_t)$ is a Poisson process, these increments are independent.

Therefore, $(Y_t^*)$ is a Poisson process of parameter $\lambda$. ■

**Lemma 3.7.** *Let $A \in \mathcal{P}((-\infty, 0])$ be a measurable set and $\mu$ the Lebesgue measure. Then if $\lambda \mathbb{E} Z < 1$*

$$\mu(A) = \mathbb{E} \int_0^\infty \mathbb{1}(Y_t \in A, \tau_1 > t) \mathrm{d}t =: R(A). \tag{3.6}$$

*That is, the Lebesgue measure of $A$ is the expected time $Y_t$ spends in $A$ before reaching the first ladder height.*

*Proof.* Let $q \geq 0$,

$$\begin{aligned}
\mathbb{P}(Y_q \in A, \tau_1 > q) &= \mathbb{P}(Y_q \in A, Y_t \leq 0 \; \forall t \in [0, q]) \\
&= \mathbb{P}(Y_q - Y_0 \in A, Y_t - Y_0 \leq 0 \; \forall t \in [0, q]) \\
&= \mathbb{P}(Y_q - Y_0 \in A, Y_{q-t} - Y_0 \leq 0 \; \forall t \in [0, q]) \\
&= \mathbb{P}(Y_q^* \in A, Y_q^* - Y_t^* \leq 0 \; \forall t \in [0, q]) \\
&= \mathbb{P}(Y_q \in A, Y_q \leq Y_t \; \forall t \in [0, q]).
\end{aligned}$$

That is, the probability that a path lies inside $A$ at time $q$ and does not exceed 0 before $q$ is the same as the probability that it lies inside $A$ and the value at $q$ is a historical minimum. Notice that although the probabilities are the same, the sets are different and contain different paths.

Justifications for the equalities:

1. For the first one, notice that if $\tau_1 > q$, $Y_t$ can not reach above 0 before $q$. Since $\tau_1$ is the first time where this happens. The same happens in the other direction.

2. $Y_0$ is equal to 0, so nothing changes.

3. Note that $[0, q] = [q, q - q]$.

4. By definition of $Y_t^*$.

5. By lemma 3.6.

Now integrating with respect to d$q$ we get

$$\int_0^\infty \mathbb{E}\left[\mathbb{1}(Y_q \in A, \tau_1 > q)\right] \mathrm{d}q = \int_0^\infty \mathbb{E}\left[\mathbb{1}(Y_q \in A, Y_q \leq Y_t \; \forall t \in [0, q])\right] \mathrm{d}q,$$

applying Fubini's theorem

$$\mathbb{E}\int_0^\infty \mathbb{1}(Y_q \in A, \tau_1 > q)\mathrm{d}q = \mathbb{E}\int_0^\infty \mathbb{1}(Y_q \in A, Y_q \leq Y_t\ \forall t \in [0,q])\mathrm{d}q.$$

This is the expected time that $Y_t$ spends in $A$ while being in a historical minimum at the same time. For the sample path in figure 3.1, the integral inside the expectation would be the measure of the set $A_1 \cup A_2 \cup A_3$ which coincides with the measure of $A$. Therefore $\mathbb{E}\int_0^\infty \mathbb{1}(Y_q \in A, Y_q \leq Y_t\ \forall t \in [0,q])\mathrm{d}q = \mu(A)$.

To prove this formally, define $t_0 = 0$, $q_i = \inf\{t > t_i | Y_t < \lim_{u \nearrow t_{i-1}} Y_q \text{ and } Y_t \in A\}$ and $t_i = \inf\{t > q_i | N_t > N_{q_i}\}$ for $i = 1, 2, ...$ The $q_i$ are the times that $Y_t$ reaches a minimum while being in $A$, and the $t_i$ are the times at which it has a jump that moves it away of the minimum. Also, let $r = \inf\{t > 0 | Y_t = \inf A\}$ and $l = \max\{i \in \mathbb{N} | q_i < r\}$.

$$\mathbb{E}\int_0^\infty \mathbb{1}(Y_q \in A, Y_q \leq Y_t\ \forall t \in [0,q])\mathrm{d}q = \mathbb{E}\left[\sum_{i=1}^{l-1}(t_i - q_i) + r - q_l\right].$$

Now notice that $\mu(A) = \sum_{i=1}^{l-1}(Y_{t_i} - Y_{q_i}) + (Y_r - Y_{q_l})$ almost surely, since $Y_t \to -\infty$ a.s. (which implies that $r$ exists a.s.) and $\mu$ is a measure. But by the definition of $Y_t$:

$$\sum_{i=1}^{l-1}(Y_{t_i} - Y_{q_i}) + (Y_r - Y_{q_l}) = \sum_{i=1}^{l-1}(t_i - q_i) + r - q_l.$$

■

Now we are ready to calculate the distribution of $H_1$ in terms of the distribution function $B$ of the claim sizes.

**Theorem 3.8.** *Let $y \in \mathbb{R}^+$. Suppose $\lambda \mathbb{E} Z < 1$. Then*

$$\mathbb{P}(H_1 \leq y) = \lambda \int_{-\infty}^0 B(y - x)\mathrm{d}\mu(x).$$

*Proof.* The goal is to rewrite the set $\{H_1 \leq y\}$ in order to obtain an integral expresion for the probability. Note that if $\omega$ is in this set, at $\tau_1(\omega)$ there is a jump

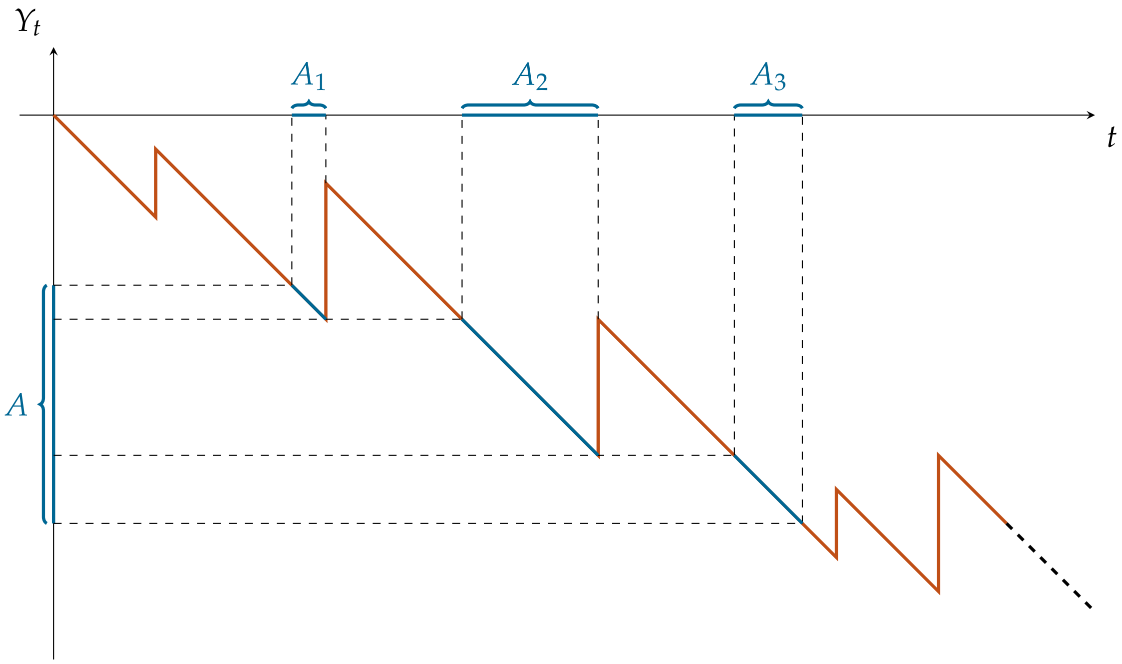


Figure 3.1: Sample path for $Y_t$ and projections of the set $A$ onto $Y_t$, projected again onto the vertical axis.

which gets $Y_t$ to be somewhere in the set $(0, y]$. Mathematically speaking we get

$$\{H_1 \leq y\} = \bigcup_{t>0} \left\{ N_t > \lim_{u \nearrow t} N_u, \tau_1 \geq t, Z_{N_t} + \lim_{u \nearrow t} Y_u \in (0, y] \right\} \qquad =: A.$$

That is, the union for every $t > 0$ of the set of $\omega$ such that there is a jump at $t$, $Y_t$ has not reached the first ladder step before $t$ and after this jump $Y_t \in (0, y]$.

However, we can not use the properties of $\mathbb{P}$ with this kind of set right away, as it consists of an uncountable union of sets, whose measure is actually 0 (since the probability of a jump occurring at a particular time is 0). The key idea of the proof is to divide $\mathbb{R}^+$ into small intervals of size $\delta > 0$ so that we can compute the probabilities of these sets and sum over every interval. Formally, we define the set

$$B := \bigcap_{\delta \in 1/2^{\mathbb{N}}} \bigcup_{t \in \delta\mathbb{N}} \left[ \{N_t = N_{t-\delta} + 1, \tau_1 \geq t - \delta, Z_{N_t} + Y_{t-\delta} \in (0, y + \delta]\} \cup \right.$$
$$\left. \{N_t > N_{t-\delta} + 1, \tau_1 \geq t - \delta, \exists q \in (t - \delta, t] : Y_q \in (0, y]\} \right],$$

where $1/2^{\mathbb{N}} = \{1/2^n \in \mathbb{Q} | n \in \mathbb{N}\}$ and $\delta\mathbb{N} = \{\delta n | n \in \mathbb{N}\}$.

Now we prove that indeed, $A = B$:

$\subset$ Suppose $\omega \in A$. Then there is a $t > 0$ such that if $\omega$ occurs, there is a jump at $t$, $\tau_1(\omega) \geq t$ and at this jump $Y_t(\omega) \in (0, y]$. For any given $\delta \in 1/2^{\mathbb{N}}$, there exists some $q \in \delta\mathbb{N}$ such that $t \in (q - \delta, q]$. Now, since there is a jump at $t$, either $\omega \in \{N_q = N_{q-\delta} + 1\}$ or $\omega \in \{N_q > N_{q-\delta} + 1\}$. In any case, also $\tau_1(\omega) \geq t \geq q - \delta$. In the first case, since $Y_t(\omega) = Y_{q-\delta}(\omega) - (t - q + \delta) + Z_{N_t}(\omega) \in (0, y]$ and by assumption $N_t(\omega) = N_q(\omega)$, necessarily $Y_{q-\delta}(\omega) + Z_{N_q}(\omega) \in (0, y + \delta]$. Finally, in the second case case we can take the time $s \in (q - \delta, q]$ as $t$, so that $Y_t(\omega) \in (0, y]$. That is, $\omega \in B$.

$\supset$ For the other inclusion, assume $\omega \in B$. Since $N_t$ is a counting process, we can choose $\delta \in 1/2^{\mathbb{N}}$ sufficiently small so that $\forall t \in \delta\mathbb{N} : N_t(\omega) \leq N_{t-\delta}(\omega) + 1$. To see this, note that $N_t$ can only take a countable amount of values, namely, those in $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, and it is increasing. This means that there is only a countable amount of jumps. If $\forall t > 0 : \exists u > t : N_u(\omega) > N_t(\omega)$ (that is, the jumps do not stop occurring at a certain time), then the infimum of the set of (time) distances between each jump is greater than 0. Otherwise there would be an accumulation point and $N_u(\omega)$ wouldn't be defined after that point, since it would have to be greater than every natural number. If this is the case then we can definitely find a small enough $\delta$ by taking a fraction of the infimum distance. If however, $\exists t > 0 : \forall u > t : N_u(\omega) = N_t(\omega)$, we only have a finite amount of jumps, namely $N_t(\omega)$. And we can find the mentioned $\delta$. After this analysis, we know that for a small enough $\delta \in 1/2^{\mathbb{N}}$, there exists some $t \in \delta\mathbb{N}$ such that there is exactly one jump in $(t - \delta, t]$, $\tau_1(\omega) \geq t - \delta$ and $Z_{N_t}(\omega) + Y_{t-\delta}(\omega) \in (0, y + \delta]$. This means that $\tau_1(\omega) \in (t - \delta, t]$. Since there are no more jumps in this interval, for $q = \tau_1(\omega)$ we have

$$N_q(\omega) > \lim_{u \nearrow q} N_u(\omega), \tau_1(\omega) \geq q, Z_{N_q}(\omega) + \lim_{u \nearrow q} Y_u \in (0, y].$$

That is, $\omega \in A$.

Now we wish to use some properties of $\mathbb{P}$. For that matter, note that the set $B$ is an intersection of a union of disjoint sets. Also, denoting

$$B_\delta := \bigcup_{t \in \delta\mathbb{N}} \left[ \{N_t = N_{t-\delta} + 1, \tau_1 \geq t - \delta, Z_{N_t} + Y_{t-\delta} \in (0, y + \delta]\} \cup \right.$$
$$\left. \{N_t > N_{t-\delta} + 1, \tau_1 \geq t - \delta, \exists q \in (t - \delta, t] : Y_q \in (0, y]\} \right].$$

it can be proven that $B_\delta \supset B_\gamma$ for $\gamma < \delta$. Just notice that the set $B_\delta$ is the set of events where the value of the first record is less than $y + \delta$ if there are no jumps between $\tau_1 - \delta$ and $\tau_1$, and those where the value of the first record is less than $y$ if there is 1 or more jumps. Taking $\gamma < \delta$ then just restricts the set but doesn't add any new events.

By means of upper continuity and countable additivity of the measure $\mathbb{P}$

$$\mathbb{P}(H_1 \leq y) = \lim_{\delta \to 0} \sum_{t \in \delta\mathbb{N}} \left[\mathbb{P}\left(N_t = N_{t-\delta} + 1, \tau_1 \geq t - \delta, Z_{N_t} + Y_{t-\delta} \in (0, y+\delta]\right) \right.$$
$$\left. + \mathbb{P}\left(N_t > N_{t-\delta} + 1, \tau_1 \geq t - \delta, \exists q \in (t-\delta, t] : Y_q \in (0, y]\right)\right].$$

At the end of this proof it will be proven that the second summand is indeed 0. So we are left with the following

$$\mathbb{P}(H_1 \leq y) = \lim_{\delta \to 0} \sum_{t \in \delta\mathbb{N}} \mathbb{P}\left(N_t = N_{t-\delta} + 1, \tau_1 \geq t - \delta, Z_{N_t} + Y_{t-\delta} \in (0, y+\delta]\right).$$

Let $Z$ be an i.i.d. copy of the claim sizes random variables. Since $Z$ and $Z_{N_t}$ are i.i.d. and $N_t - N_{t-\delta}$ is independent from the event $\tau_1 \geq t - \delta$ and from $Z + Y_{t-\delta}$ we can write

$$\begin{aligned}
&= \lim_{\delta \to 0} \sum_{t \in \delta\mathbb{N}} \mathbb{P}\left(N_t = N_{t-\delta} + 1, \tau_1 \geq t - \delta, Z + Y_{t-\delta} \in (0, y+\delta]\right) \\
&= \lim_{\delta \to 0} \sum_{t \in \delta\mathbb{N}} \mathbb{P}\left(N_t = N_{t-\delta} + 1\right) \mathbb{P}\left(\tau_1 \geq t - \delta, Z + Y_{t-\delta} \in (0, y+\delta]\right).
\end{aligned}$$

Now since $N_t - N_{t-\delta}$ is Poisson distributed with parameter $\lambda\delta$, we get

$$\begin{aligned}
&= \lim_{\delta \to 0} \sum_{t \in \delta\mathbb{N}} e^{-\lambda\delta} \lambda\delta \mathbb{P}\left(\tau_1 \geq t - \delta, Z + Y_{t-\delta} \in (0, y+\delta]\right) \\
&= \lim_{\delta \to 0} e^{-\lambda\delta} \sum_{t \in \delta\mathbb{N}} \lambda\delta \mathbb{P}\left(\tau_1 \geq t - \delta, Z + Y_{t-\delta} \in (0, y+\delta]\right) \\
&= \lim_{\delta \to 0} \sum_{t \in \delta\mathbb{N}} \lambda \mathbb{P}\left(\tau_1 \geq t - \delta, Z + Y_{t-\delta} \in (0, y+\delta]\right) \delta.
\end{aligned}$$

To get rid of some $\delta$ in the formula, we can rewrite it by using the explicit form of $t$

$$\begin{aligned}
&= \lim_{\delta \to 0} \sum_{n=1}^{\infty} \lambda \mathbb{P}\left(\tau_1 \geq \delta(n-1), Z + Y_{\delta(n-1)} \in (0, y+\delta]\right) \delta \\
&= \lim_{\delta \to 0} \sum_{n=0}^{\infty} \lambda \mathbb{P}\left(\tau_1 \geq \delta n, Z + Y_{\delta n} \in (0, y+\delta]\right) \delta \\
&= \lim_{\delta \to 0} \sum_{t \in \delta\mathbb{N}_0} \lambda \mathbb{P}\left(\tau_1 \geq t, Z + Y_t \in (0, y+\delta]\right) \delta.
\end{aligned}$$

And we can divide the set inside $\mathbb{P}$ in two as follows

$$\begin{aligned}\mathbb{P}(H_1 \leq y) = &\lim_{\delta\to 0}\sum_{t\in\delta\mathbb{N}_0} \lambda \mathbb{P}\left(\tau_1 \geq t, Z+Y_t \in (0,y]\right)\delta \\ &+ \lim_{\delta\to 0}\sum_{t\in\delta\mathbb{N}_0} \lambda \mathbb{P}\left(\tau_1 \geq t, Z+Y_t \in (y,y+\delta]\right)\delta.\end{aligned}$$

This second term is 0. To see this, note that for $\gamma > \delta$

$$\mathbb{P}\left(\tau_1 \geq t, Z+Y_t \in (y,y+\delta]\right) \leq \mathbb{P}\left(\tau_1 \geq t, Z+Y_t \in (y,y+\gamma]\right).$$

Therefore:

$$\begin{aligned}\lim_{\delta\to 0}\sum_{t\in\delta\mathbb{N}_0} \lambda \mathbb{P}\left(\tau_1 \geq t, Z+Y_t \in (y,y+\delta]\right)\delta &\leq \lim_{\delta\to 0}\sum_{t\in\delta\mathbb{N}_0} \lambda \mathbb{P}\left(\tau_1 \geq t, Z+Y_t \in (y,y+\gamma]\right)\delta \\ &= \int_0^\infty \lambda \mathbb{P}\left(\tau_1 \geq t, Z+Y_t \in (y,y+\gamma]\right)\mathrm{d}t.\end{aligned}$$

Since the right hand side is a Riemann sum. This is true for any $\gamma > 0$, so it is also true for the limit. Also, by dominated convergence we can exchange the integral and limit to get the desired result

$$\begin{aligned}\lim_{\delta\to 0}\sum_{t\in\delta\mathbb{N}_0} \lambda \mathbb{P}\left(\tau_1 \geq t, Z+Y_t \in (y,y+\delta]\right)\delta &\leq \lim_{\gamma\to 0}\int_0^\infty \lambda \mathbb{P}\left(\tau_1 \geq t, Z+Y_t \in (y,y+\gamma]\right)\mathrm{d}t \\ &= \int_0^\infty \lim_{\gamma\to 0} \lambda \mathbb{P}\left(\tau_1 \geq t, Z+Y_t \in (y,y+\gamma]\right)\mathrm{d}t \\ &= \int_0^\infty \lambda \mathbb{P}\left(\tau_1 \geq t, Z+Y_t \in (y,y]\right)\mathrm{d}t = 0.\end{aligned}$$

We only have one term left

$$\mathbb{P}(H_1 \leq y) = \int_0^\infty \lambda \mathbb{P}\left(\tau_1 \geq t, Z+Y_t \in (0,y]\right)\mathrm{d}t.$$

Using Fubini's theorem we can find an expression with diferent integral limits

$$\begin{aligned}&= \int_0^\infty \lambda \mathbb{E}\mathbb{1}\left(\tau_1 \geq t, Z+Y_t \in (0,y]\right)\mathrm{d}t \\ &= \lambda \mathbb{E}\int_0^{\tau_1} \mathbb{1}\left(Z+Y_t \in (0,y]\right)\mathrm{d}t \\ &= \lambda \int_0^{\tau_1} \mathbb{E}\left[\mathbb{1}\left(Z+Y_t \in (0,y]\right)\right]\mathrm{d}t.\end{aligned}$$

And applying the properties of the conditional expectation we can finally write this in terms of the distribution function of $Z$

$$
\begin{aligned}
&= \lambda \int_0^{\tau_1} \mathbb{E}\left[\mathbb{E}\left[\mathbb{1}\left(Z + Y_t \in (0, y]\right) \mid Y_t\right]\right] \mathrm{d}t \\
&= \lambda \mathbb{E} \int_0^{\tau_1} \mathbb{E}\left[\mathbb{1}\left(Z + Y_t \in (0, y]\right) \mid Y_t\right] \mathrm{d}t \\
&= \lambda \mathbb{E} \int_0^{\tau_1} B(y - Y_t) \mathrm{d}t.
\end{aligned}
$$

By approximation with step functions using equation (3.6), it follows that

$$
\mathbb{P}(H_1 \leq y) = \lambda \int_{-\infty}^{0} B(y - x) \mathrm{d}R(x).
$$

By lemma 3.7

$$
\mathbb{P}(H_1 \leq y) = \lambda \int_{-\infty}^{0} B(y - x) \mathrm{d}\mu(x).
$$

Now the only thing left to prove is that

$$
\lim_{\delta \to 0} \sum_{t \in \delta \mathbb{N}} \mathbb{P}\left(N_t > N_{t-\delta} + 1, \tau_1 \geq t - \delta, \exists q \in (t - \delta, t] : Y_q \in (0, y]\right)\right] = 0.
$$

We can divide this in the cases $N_t - N_{t-\delta} = 2, 3, \ldots$ since that will produce disjoint sets

$$
\begin{aligned}
&\lim_{\delta \to 0} \sum_{t \in \delta \mathbb{N}} \mathbb{P}\left(N_t > N_{t-\delta} + 1, \tau_1 \geq t - \delta, \exists q \in (t - \delta, t] : Y_q \in (0, y]\right) \\
&= \lim_{\delta \to 0} \sum_{t \in \delta \mathbb{N}} \sum_{k=2}^{\infty} \mathbb{P}\left(N_t - N_{t-\delta} = k, \tau_1 \geq t - \delta, \exists q \in (t - \delta, t] : Y_q \in (0, y]\right) \\
&= \sum_{k=2}^{\infty} \lim_{\delta \to 0} \sum_{t \in \delta \mathbb{N}} \mathbb{P}\left(N_t - N_{t-\delta} = k, \tau_1 \geq t - \delta, \exists q \in (t - \delta, t] : Y_q \in (0, y]\right).
\end{aligned}
$$

The two sums can be exchanged because the summands are non-negative.

We will prove it for $k = 2$, from which an extension to a general value of $k$ is straightforward. Let $Z, Z'$ be i.i.d. copies of the claim size. By similar arguments to the ones used for the other summand

$$
\begin{aligned}
&\lim_{\delta \to 0} \sum_{t \in \delta \mathbb{N}} \mathbb{P}\left(N_t - N_{t-\delta} = 2, \tau_1 \geq t - \delta, \exists q \in (t - \delta, t] : Y_q \in (0, y]\right) \\
&= \lim_{\delta \to 0} \sum_{t \in \delta \mathbb{N}} \mathbb{P}\left(N_t - N_{t-\delta} = 2, \tau_1 \geq t - \delta, Y_{t-\delta} + Z \in (0, y + \delta] \text{ or } Y_{t-\delta} + Z + Z' \in (0, y + \delta]\right).
\end{aligned}
$$

And following analogous steps we arrive at

$$= \lim_{\delta \to 0} \sum_{t \in \delta \mathbb{N}_0} \lambda^2 \mathbb{P}(\tau_1 \geq t, Y_t + Z \in (0, y] \text{ or } Y_t + Z + Z' \in (0, y])\delta^2.$$

Again, taking $\gamma > \delta$ we get an inequality.

$$\begin{aligned}
&\leq \lim_{\delta \to 0} \sum_{t \in \delta \mathbb{N}_0} \lambda^2 \mathbb{P}(\tau_1 \geq t, Y_t + Z \in (0, y] \text{ or } Y_t + Z + Z' \in (0, y])\delta\gamma \\
&= \int_0^\infty \lambda^2 \mathbb{P}(\tau_1 \geq t, Y_t + Z \in (0, y] \text{ or } Y_t + Z + Z' \in (0, y])\gamma \mathrm{d}t.
\end{aligned}$$

Taking limits and using dominated convergence we conclude that this is 0. ■

**Theorem 3.9.** *Suppose Then if $\lambda \mathbb{E} Z < 1$. The distribution of $H_1$ is given by the defective density $\lambda \bar{B}$. Where $\bar{B}$ is defined by $\bar{B} = 1 - B$.*

*Proof.* To find the density we have to differentiate the distribution function

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}y} \mathbb{P}(H_1 \leq y) &= \frac{\mathrm{d}}{\mathrm{d}y} \lambda \int_{-\infty}^0 B(y - x) \mathrm{d}\mu(x) \\
&= \lambda \int_{-\infty}^0 B'(y - x) \mathrm{d}\mu(x) \\
&= -\lambda(B(y) - \lim_{z \to \infty} B(z)) \\
&= \lambda \bar{B}(y).
\end{aligned}$$

■

# 4 | The Pollaczeck-Khinchine formula

## 1 | The formula

**Theorem 4.1.** *Let $Y_t, Z_i$ and $P_t$ be as in the Poisson model. Then if $\lambda \mathbb{E}Z < 1$ the distribution of $M = \sup\{Y_t \mid 0 \leq t < \infty\}$ for $u \in \mathbb{R}^+$ is given by the Pollaczeck-Khinchine formula*

$$\mathbb{P}(M \leq u) = (1-p)\sum_{k=0}^{\infty} p^k F^{*k}(u), \tag{4.1}$$

*where $p = \mathbb{P}(\tau_1 < \infty) = \lambda \mathbb{E}Z$ and $F(x) = \int_{-\infty}^{x} \bar{B}(y)\mathrm{d}y / \mathbb{E}Z$.*

*Proof.* The expression was already given in equation (3.5). Now, the values of $p$ and $F(x)$ are calculated.

We can calculate $p$ given the distribution of $H_1$ as follows

$$\begin{aligned} p &= \mathbb{P}(\tau_1 < \infty) \\ &= \mathbb{P}(H_1 < \infty). \end{aligned}$$

Using the defective density found for $H_1$ in theorem 3.9 and the integrated tail expectation formula

$$\begin{aligned} p &= \int_0^{\infty} \lambda \bar{B}(x)\mathrm{d}x \\ &= \lambda \mathbb{E}Z. \end{aligned}$$

On the other hand, in equation (3.5), $F(x) = \mathbb{P}(H_1 \leq x \mid \tau_1 < \infty)$, (notice that

$\{\tau_1 < \infty\} \subset \{H_1 \leq x\}$) that is

$$\begin{aligned}\mathbb{P}(H_1 \leq x \mid \tau_1 < \infty) &= \frac{\mathbb{P}(H_1 \leq x)}{\mathbb{P}(\tau_1 < \infty)} \\ &= \frac{\lambda \int_0^x \bar{B}(y)\mathrm{d}y}{\lambda \mathbb{E}Z} \\ &= \frac{\int_0^x \bar{B}(y)\mathrm{d}y}{\mathbb{E}Z}.\end{aligned}$$

■

## 2 | Simulation method

Equation (4.1) looks a bit cumbersome for simulation purposes. To calculate the ruin probability for some initial capital $u$ computing a series whose summands contain the $n$th convolution of $F$ would be needed.

Without an analytical expression it might be preferable to use an alternative method. In fact, using the ideas developed to attain formula (4.1) it is possible to calculate some good bounds.

The plan followed here is to approximate the ladder heights from below and from above with discrete random variables. Let $\delta \in \mathbb{R}^+$. This will be our discretization step size. Now define the new random variables

$$\begin{aligned}H'_k &= \max\{r \in \delta\mathbb{N} \mid r \leq H_k\} \\ H''_k &= \max\{r \in \delta\mathbb{N} \mid r \leq H_k + \delta\}.\end{aligned}$$

Aswell as

$$\begin{aligned}M' &= H'_1 + \ldots + H'_K \\ M'' &= H''_1 + \ldots + H''_K.\end{aligned}$$

Clearly $M' \leq M \leq M''$, which implies $\mathbb{P}(M' > u) \leq \mathbb{P}(M > u) \leq \mathbb{P}(M'' > u)$. The goal now is to obtain a sensible formula for these probabilities. In particular, a recursive formula. Notice that no assumption on the distribution of the claim sizes is made. And in fact, this method will be applicable to both light and heavy-tailed claim sizes.

If an expression for the distribution of $M'$ and $M''$ is obtained, $\mathbb{P}(M' > u)$ can be written as a finite sum (and analogously for $M''$)

$$\mathbb{P}(M' > u) = 1 - \sum_{a=0}^{[u/\delta]} \mathbb{P}(M' = a\delta) \tag{4.2}$$
$$\mathbb{P}(M'' > u) = 1 - \sum_{a=0}^{[u/\delta]} \mathbb{P}(M'' = a\delta). \tag{4.3}$$

<u>*Notation*</u>. Let $a \in \mathbb{N}_0$. For simplicity and clarity the following notations are defined

$$f'_a = F((a+1)\delta) - F(a\delta) \qquad f''_a = F(a\delta) - F((a-1)\delta)$$
$$m'_a = \mathbb{P}(M' = a\delta) \qquad m''_a = \mathbb{P}(M'' = a\delta).$$

**Theorem 4.2.** *Let $a \in \mathbb{N}$, the probability $m'_a = \mathbb{P}(M' = a\delta)$ can be calculated recursively with the formulas*

$$m'_0 = \frac{1-p}{1-pf'_0} \tag{4.4}$$
$$m'_a = \frac{p}{1-pf'_0} \sum_{b=1}^{a} f'_b m'_{a-b}. \tag{4.5}$$

*Similarly, for $m''_a$ the following formulas hold*

$$m''_0 = 1 - p \tag{4.6}$$
$$m''_a = p \sum_{b=1}^{a} f''_b m''_{a-b}. \tag{4.7}$$

*Proof.* Let $a \in \mathbb{N}$, then

$$\begin{aligned} \mathbb{P}(M' = a\delta) &= \mathbb{P}(M' = a\delta \mid K = 0)\mathbb{P}(K = 0) \\ &\quad + \sum_{b=0}^{a} \sum_{k=1}^{\infty} \mathbb{P}(M' = a\delta, H'_1 = b\delta \mid K = k)\mathbb{P}(K = k). \end{aligned} \tag{4.8}$$

It's possible to write

$$\begin{aligned} \mathbb{P}(M' = a\delta, H'_1 = b\delta \mid K = k) &= \mathbb{P}(H'_1 + \cdots + H'_k = a\delta, H'_1 = b\delta \mid K = k) \\ &= \mathbb{P}(H'_2 + \cdots + H'_k = (a-b)\delta, H'_1 = b\delta \mid K = k) \end{aligned}$$

And by Theorem 3.3

$$\begin{aligned}\mathbb{P}(M' = a\delta, H_1' = b\delta \mid K = k) &= \left[F^{*k-1}((a-b+1)\delta) - F^{*k-1}((a-b)\delta)\right] f_b' \\ &= \mathbb{P}(M' = (a-b)\delta \mid K = k-1) f_b'.\end{aligned}$$

Inserting these expressions inside the sum in (4.8), together with the fact that $\mathbb{P}(K = k) = \mathbb{P}(K = k-1)p$

$$\begin{aligned}&\sum_{k=1}^{\infty} \mathbb{P}(M' = a\delta, H_1' = b\delta \mid K = k)\mathbb{P}(K = k) \\ &= \sum_{k=1}^{\infty} \mathbb{P}(M' = (a-b)\delta \mid K = k-1) f_b' \mathbb{P}(K = k-1)p \\ &= p f_b' \sum_{k=1}^{\infty} \mathbb{P}(M' = (a-b)\delta \mid K = k-1)\mathbb{P}(K = k-1) \\ &= p f_b' m_{a-b}'.\end{aligned}$$

Now the cases $a = 0$ and $a > 0$ are treated separately

$a = 0$ In this case $\mathbb{P}(M' = 0 \mid K = 0) = 1$, since for $K = 0$, $M' = 0$. Also $\mathbb{P}(K = 0) = 1 - p$.

From (4.8)

$$m_0' = 1 - p + p f_0' m_0'.$$

From which (4.4) follows.

$a > 0$ Now $\mathbb{P}(M' = a\delta \mid K = 0) = 0$, and from (4.8)

$$m_a' = \sum_{b=0}^{a} p f_b' m_{a-b}'.$$

From which equation (4.5) follows.

The proof for the upper bounds can be performed in an analogous manner.

■

It is worth noting that although the method becomes very expensive computationally speaking for large values of $u$, in practice this is not a problem. The premiums received in a year was taken as the monetary unit (essentially it was stated that $P = 1 \cdot t$). Therefore, a value of $u = 100$ is equivalent to having as initial capital 100 times the amount of money received in premiums in a year. Looking at actual figures within the insurance industry in Finland it is clear that being able to calculate ruin probabilities for much bigger values than this is not useful as it is not reasonable. Luckily, for the quantities managed in reality this method is exceptionally fast.

## 3 | Estimations of the ruin probability

The equations derived for the upper and lower bounds of the ruin probability are exceptional. They allow to calculate bounds as precise as desired, and work for any kind of claim size, both light and heavy tailed ones. The only assumptions made are those of the Poisson model (see Section 1 of Chapter 3).

In this section, they are implemented in python (see code in Appendix A) and used to calculate the ruin probabilities for different claim sizes. To use them it is necessary to calculate first the function $F$ from the claim size distribution $B$.

### 3.1 Exponential distribution

Let $Z$ have an exponential distribution with parameter $\alpha$. Then

$$\bar{B}(x) = e^{-\alpha x}\mathbb{1}(x \geq 0).$$

Therefore, for $x > 0$,

$$\begin{aligned} F(x) &= \alpha \int_{-\infty}^{x} e^{-\alpha y}\mathbb{1}(y \geq 0)\mathrm{d}y \\ &= \alpha \int_{0}^{x} e^{-\alpha y}\mathrm{d}y \\ &= \alpha \frac{1-e^{-\alpha x}}{\alpha} \\ &= 1 - e^{-\alpha x}. \end{aligned}$$

The exponential distribution is one of the easiest to work with, and in fact the exact ruin probability can be calculated theoretically. The probability of ruin in this case is $\psi(u) = pe^{-(\alpha-\lambda)u}$ (see Chapter 4 of [1]).

For the simulation, $\alpha$ is taken to be 1 and $\lambda$ to be 0.5, giving a value of $p = 0.5 < 1$. In figure 4.1 the estimation using the formulas derived is plotted together with the theoretical value of $\psi(u)$. As expected, the theoretical value lies inside the region defined by the bounds.

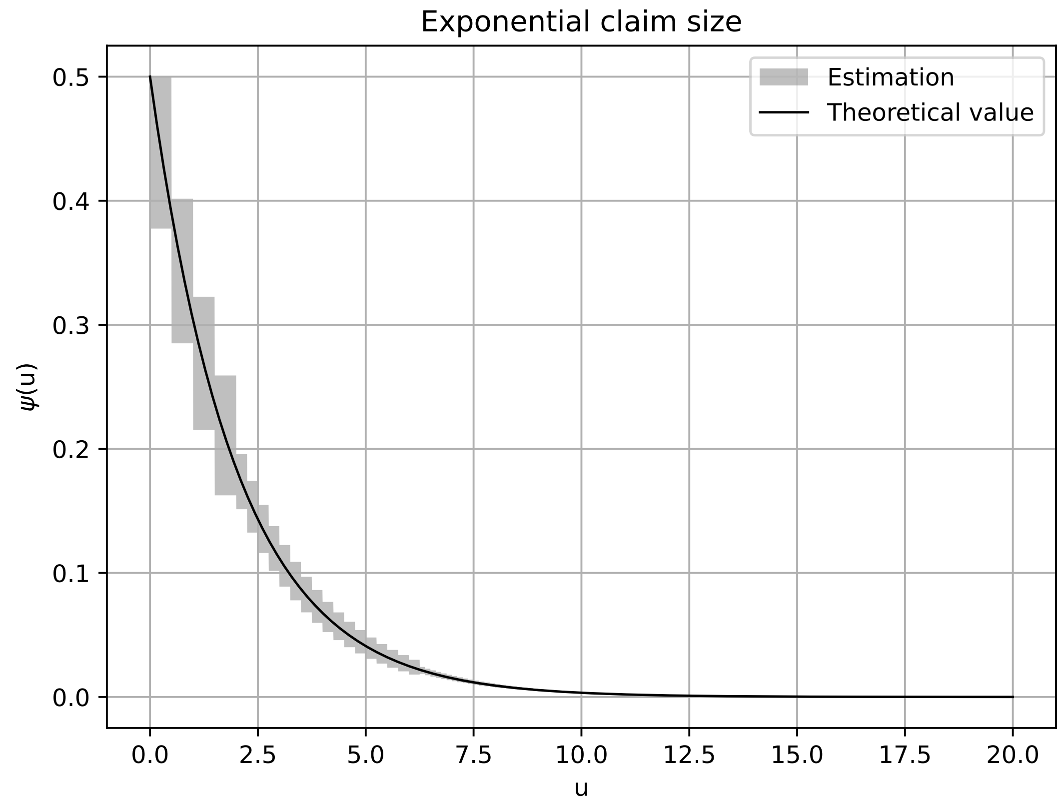


Figure 4.1: In this graph the estimation for the ruin probability in the Poisson model with exponential claim sizes of parameter $\alpha = 1$ and $\lambda = 0.5$ is plotted against the initial capital $u$, together with the theoretical ruin probability. The estimation is calculated with precision 0.5, this one being defined as in the code shown in A.1.

The bounds show a step like behaviour. The reason for this behaviour is that the number of summed terms in equations (4.2) and (4.3) is $[u/\delta] + 1$. Therefore, whenever $u$ increases enough to make $[u/\delta]$ increase in 1 unit, another term is added to the bound calculation, making it jump vertically to a lower value.

### 3.2 Pareto distribution

Let $Z$ have a Pareto distribution with parameter $\alpha$. Then

$$\bar{B}(x) = x^{-\alpha}\mathbb{1}(x \geq 1).$$

And for $\alpha > 1$ the expected value is $\mathbb{E}Z = \alpha/\alpha - 1$. So for $x > 1$

$$F(x) = \frac{\alpha - 1}{\alpha} \int_1^x y^{-\alpha} \mathrm{d}y$$
$$= \frac{1 - x^{1-\alpha}}{\alpha}.$$

In this case $\alpha$ is taken to be 3 and $\lambda$ is set to $1/3$, giving a value of $p = 0.5 < 1$. This was made on purpose, in order to get a comparable situation to the exponential case.

Looking at figures 4.1 and 4.2 it is possible to see the different behaviour in light and heavy tailed distributions. Whereas for the exponential distribution the ruin probability falls to zero exponentially, the decrease int he Pareto case is clearly slower. After around $u = 12.5$, the ruin probability is still above 0.4 and an increase in $u$ almost does not alter this value. Eventually it approaches 0, since the ruin probability tends to 0 as $u$ increases.[1]

Practically speaking, increasing the initial capital doesn't seem to be a good solution in this case. The other parameter that can be controlled in an insurance company is the premium size. In this discussion all of the quantities were normalized to the premium size, that is, we took as the unit of money the premium size. An increase in the premium would correspond to decreasing $\mathbb{E}Z$ in the model. In this case, this is achieved by increasing $\alpha$.

In figure 4.3 the ruin probability with respect to $\alpha$ is plotted, for an initial capital of $u = 15$. The same conclusions as before can be derived from this graph.

[1] This can be seen from the Pollaczeck-Khinchine formula, by taking the limit when $u \to \infty$. The sum and the limit can be exchanged by monotone convergence.

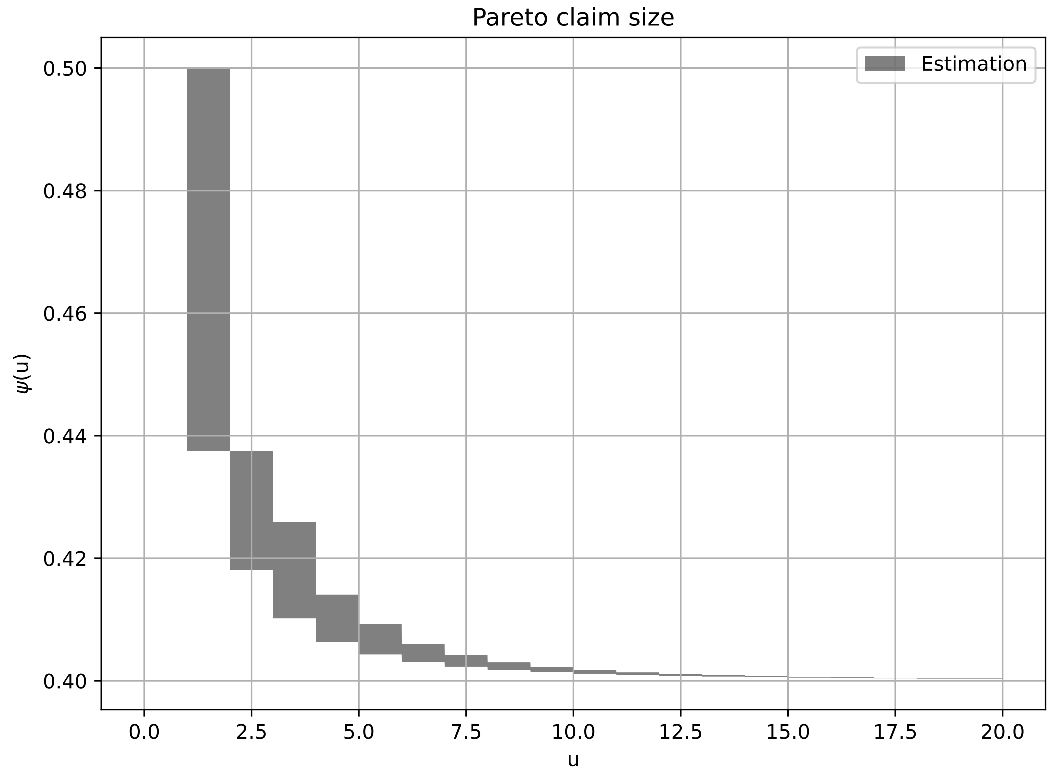


Figure 4.2: In this graph the estimation for the ruin probability in the Poisson model with Pareto claim sizes of parameter $\alpha = 3$ and $\lambda = 1/3$ is plotted against the initial capital $u$. The estimation is calculated with precision 0.5, this one being defined as in the code shown in A.1.

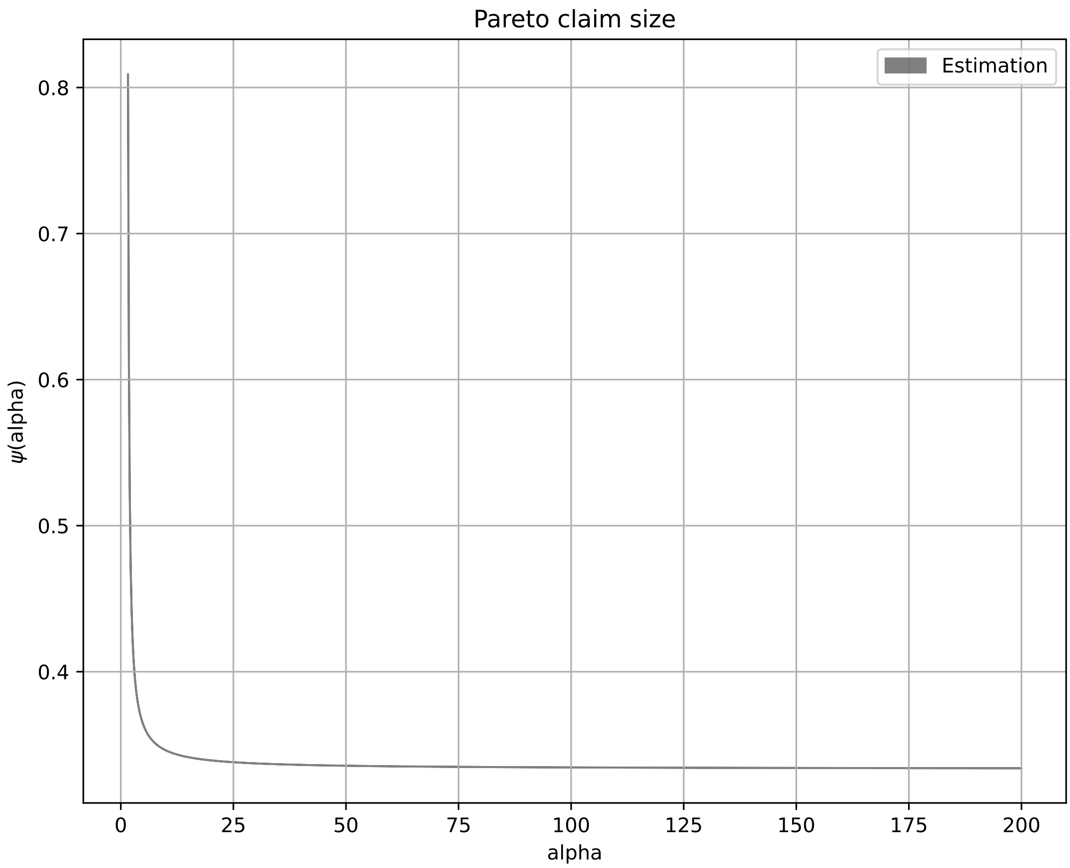


Figure 4.3: In this graph the estimation for the ruin probability in the Poisson model with Pareto claim sizes and $\lambda = 1/3$ for an initial capital $u = 15$ is plotted against the Pareto parameter $\alpha$. The estimation is calculated with precision 0.5, this one being defined as in the code shown in A.1.

# 5 | Conclusions

As was discussed in Chapter 3, the Poisson model is a very versatile framework for computing estimates of ruin probabilities. It allows for all kinds of claim size distributions and has some very general assumptions that apply to many situations. Various calculation methods for the ruin probabilities can be derived from it, but this thesis was centered around those involving the ladder height distribution. In particular, the Pollaczeck-Khinchine formula was derived in Chapters 3 and 4, and although it is an elegant solution, its practical implementation is hindered by the inifnite sum of convolutions it relies on.

In order to have a reliable computational method to implement in a programming language, the well known bounds method was explained. This method uses the results which were derived for the proof of the Pollaczeck-Khinchine formula, as well as the key concepts underlying it. This approach resulted in formulas (4.4), (4.5), (4.6) and (4.7) which were implemented using the code in A.1.

In Section 3 of Chapter 4 the difference between light and heavy tailed distributions is shown. Proving as well the usefulness of this bounds method to study and reveal how different parameters of the model affect the ruin probability. Furthermore, the bounds method is simple to code and fast to compute for reasonable initial capital values.

In conclusion, this thesis has provided a comprehensive overview of the key concepts behind the ladder height decomposition and Pollaczeck-Khinchine formula, explaining one potential of calculating ruin probabilities. Altogether these insights, formulas and methods can be used to understand the basis of ruin theory and construct upon them more complicated and refined models. Which are essential in the broad field of insurance mathematics, in order to develop risk management strategies.

# Bibliography


[1] Soren Asmussen and Hansjorg Albrecher. *Ruin probabilities*. Vol. 14. World scientific, 2010.

[2] John A Beekman. "Collective risk results". In: *Transactions of the Society of Actuaries* 20.182-199 (1968), p. 0.

[3] François Dufresne and Hans U Gerber. "Three methods to calculate the probability of ruin". In: *ASTIN Bulletin: The Journal of the IAA* 19.1 (1989), pp. 71–90.

[4] Riccardo Gatto and Michael Mosimann. "Four approaches to compute the probability of ruin in the compound Poisson risk process with diffusion". In: *Mathematical and Computer Modelling* 55.3 (2012), pp. 1169–1185. ISSN: 0895-7177. DOI: https://doi.org/10.1016/j.mcm.2011.09.041. URL: https://www.sciencedirect.com/science/article/pii/S0895717711005899.

[5] Rob Kaas. "Beekman's convolution formula". In: *Wiley StatsRef: Statistics Reference Online* (2014).

[6] Vladimir Kalashnikov and Vladimir Kalashnikov. "Ruin probability". In: *Geometric Sums: Bounds for Rare Events with Applications: Risk Analysis, Reliability, Queueing* (1997), pp. 171–200.

[7] Ambrose Lo. "Demystifying the Integrated Tail Probability Expectation Formula". In: *The American Statistician* 73.4 (2019), pp. 367–374. DOI: 10.1080/00031305.2018.1497541. eprint: https://doi.org/10.1080/00031305.2018.1497541. URL: https://doi.org/10.1080/00031305.2018.1497541.

[8] Masakiyo Miyazawa and Volker Schmidt. "On ladder height distributions of general risk processes". In: *The Annals of Applied Probability* 3.3 (1993), pp. 763–776.

[9] Tomasz Rolski et al. *Stochastic Processes for Insurance and Finance*. Wiley Series in Probability and Statistics. Chichester, England: Wiley, 2008. ISBN: 9780471997825. DOI: 10.1002/9780470317044.

[10] K. Sriram. *Poisson Processes*. Accessed: 2024-06-08. 2008. URL: http://www.columbia.edu/~ks20/stochastic-I/stochastic-I-PP.pdf.

# A | Code

In this appendix the code written to implement the recursive method of Chapter 4 is presented. It is written in python.

```
# First the formulas used to calculate f_a and m_a ' and '' are defined
def calc_f_lower(k, delta, F):
    return F((k+1)*delta)-F(k*delta)

def calc_m_lower(k, m, f, p):
    if k==0:
        return (1- p )/(1- p*f[0])
    else:
        suma = 0
        for l in range(1,k+1):
            suma = suma + f[l]*m[k-l]
        return (p/(1- p*f[0]))*suma

def calc_f_upper(k, delta, F):
    return F(k*delta)-F((k-1)*delta)

def calc_m_upper(k, m, f, p):
    if k==0:
        return 1- p
    else:
        suma = 0
        for l in range(1,k+1):
            suma = suma + f[l]*m[k-l]
        return p*suma

# Here the previous functions are used to calculate the lower and upper bounds of the probability ruin
def lower_bound(U, delta, p, F):
    f=[]
    m=[]
    for k in range(0,int(U//delta)+1):
        f.append(calc_f_lower(k,delta, F))
```

```
            m.append(calc_m_lower(k,m,f,p))
    suma = 0
    for k in range(0,int(U//delta)+1):
        suma = suma + m[k]
    return 1-suma

def upper_bound(U, delta, p, F):
    f=[]
    m=[]
    for k in range(0,int(U//delta)+1):
        f.append(calc_f_upper(k,delta, F))
        m.append(calc_m_upper(k,m,f,p))
    suma = 0
    for k in range(0,int(U//delta)+1):
        suma = suma + m[k]
    return 1-suma

# Given the initial capital, the bounds are calulated with
    progressively smaller
# discretization sizes until their relative difference is below the
    specified precision
# Finally, the function returns the lower and upper bounds together
    with the discretization size
def bounds(U, p, precision, F):
    ratio = 100
    delta = 2
    while ratio>precision:
        delta = delta/2
        lower = lower_bound(U, delta, p, F)
        upper = upper_bound(U, delta, p, F)
        mean = (lower + upper)*.5
        ratio = (upper - lower)/mean

    return lower, upper, delta
```

Listing A.1: Definition of the functions needed to calculate the bounds

```
import math
import matplotlib.pyplot as plt
import numpy as np
import functions as fs

# Here the conditional first ladder height distribution is defined for
    whichever claim size distribution is chosen
def F(x):
    return 1-math.exp(-x)
```

```


# Here the important parameters are given or a value or calculated
# Those are, the Poisson process parameter lambda, and the probability of the first record to occur p
lamb = .5
p = lamb

# The range of values of initial capital for which to calculate the ruin probability is defined
u = np.linspace(0, 20, 5000)

lower = list(range(len(u)))
upper = list(range(len(u)))
delta = list(range(len(u)))
ruin_prob = list(range(len(u)))

# The ruin_prob variable is used to calculate the exact value, because it is known for the exponential
for k in range(len(u)):
    lower[k], upper[k], delta[k] = fs.bounds(u[k], p, 0.5, F)
    ruin_prob[k] = p*math.exp(-0.5*u[k])

# Plot
fig, ax = plt.subplots()

ax.fill_between(u, lower, upper, alpha=.5, linewidth=0, color = 'gray', label = "Estimation")
ax.plot(u, ruin_prob , linewidth=1, color = 'black', label = "Theoretical value")

# Axes
plt.xlabel('u')
plt.ylabel('$\psi$(u)')

# Title
plt.title('Exponential claim size')

# Grid
plt.grid(True)

# Legend
plt.legend()

plt.show()
```

Listing A.2: Plot of $\psi(u)$ for the exponential claim size