

Gaussian Fields and Equations of Divergence-form: Continuity, Boundedness, and Approximations

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DOCTORAL DISSERTATION

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空前絶後

くうぜんぜつご
(*kuuzenzetsugo*)

(so marvellous or horrible
that it may be) the first
and probably the last

Abstract

This thesis addresses the renormalised pressure equation on a domain in the d -dimensional Euclidean space, with a particular focus on the periodic setting, where the domain is taken to be the d -dimensional flat torus. The equation under consideration is a second-order, divergence-form partial differential equation with a stochastic coefficient given by ξ , a realisation of a random field.

Motivated by applications in physical models, such as geothermal heating, the random field ξ is taken to be essentially log-normal, having the form $\xi = \exp(\diamond X)$, where X is a log-correlated Gaussian field. This random field, defined only as a generalised function, exhibits a logarithmic-type covariance structure and undefined variance, posing significant challenges in defining solutions to the equation. As the log-correlated Gaussian field can be realised as a scaling limit of suitable Hölder-continuous fields, in the first article we explore the continuity properties of similar fields, and obtain a Kolmogorov–Chentsov-type characterisation for their continuity.

In the stochastic differential equation setting, previous work has utilised the Wick product $\diamond$ to address the challenge posed by the irregularity of the random field ξ . We explore alternative methods in the second and third articles in the thesis. In particular, to deal with the problem of renormalisation, our work draws on regularisation and approximation techniques, in spirit traced back to the seminal papers of Wong and Zakai. These methods are closely related to stochastic (Itô) integration theory, which has seen a resurgence in recent years. Notably in the single-variable setting, in the second article we adopt a mollification-approximation approach, considering both the Wick product $\diamond$ and the standard scalar product in the differential equation, with the multiplicative constant in the renormalisation depending on the structure of the field X . In the third article, dealing with the d -dimensional case, we tackle the questions of existence and uniqueness of solutions with the use of an operator-theoretic framework, while making use of well-known regularity results of degenerate elliptic equations in the deterministic setting. The thesis also presents a detailed analysis of the interplay between these regularisation techniques and the challenges posed by the roughness of the log-correlated Gaussian field.

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Speaking of giving credit where it is due, I will list some important names: Gohar Aleksanyan, Susanna Heikkilä, Jarmo Jääskeläinen, Petri Laarne, Petteri Piiroinen, Joni Puljujärvi, Julia Sanders, Saara Sarsa, Janne Siipola, Jaakko Sinko, Sami Vihko, Aleksis Vuoksenmaa, and Christian Webb – many thanks for interesting and amusing conversations in the lecture halls, coffee rooms, corridors, and so on – you have made my stay in Kumpula much more enjoyable. Lastly, I want to thank my mother and my late father for their unwavering encouragement and support in my endeavours.

I have most probably forgotten to mention someone: unfortunate, but it is only the first of the many likely errors in this dissertation for which I naturally take full responsibility of.

Helsinki, November 2024
Patrik Nummi

List of included articles

- [A] Nummi, P., & Viitasaari, L. (2024). *Necessary and sufficient conditions for continuity of hypercontractive processes and fields*. Statistics and Probability Letters, 208. Published under Creative Commons CC-BY license.
- [B] Avelin, B., Kuusi, T., Nummi, P., Saksman, E., Tölle, J. M., & Viitasaari, L. (2024). *1D stochastic pressure equation with log-correlated Gaussian coefficients*. Manuscript submitted for publication. arXiv:2402.09127
- [C] Avelin, B., Kuusi, T., Nummi, P., Saksman, E., Tölle, J. M., & Viitasaari, L. (2024). *Renormalized stochastic pressure equation with log-correlated Gaussian coefficients*. Manuscript submitted for publication. arXiv:2405.17195v2

In [A], the results were derived by P.N. under supervision of L.V. Both authors participated equally in the writing process. In [B] the first version of the results were derived by P.N. All authors contributed equally to the writing and analysis of the article. In [C] the results were derived by all authors, P.N. wrote the first version of the article, and after that all authors contributed equally to polishing and writing the article.

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1 Introduction

The subject of this thesis is the renormalised pressure equation on a domain $V \subset \mathbb{R}^d$, that is, the following ‘elliptic’ second-order divergence-form stochastic partial differential equation

$$\begin{cases} -\nabla \cdot (\xi \times \nabla U) = f, \\ \text{boundary conditions on } \partial V, \end{cases} \quad (1.0.1)$$

where ξ is a realisation of a random field on V . The product ‘ $\times$ ’ represents an algebraic renormalisation. From a mathematical point of view, the inclusion of this renormalisation is rather straightforward to motivate – a simple one-dimensional formal computation shows that for many of the interesting random fields ξ , a solution U cannot exist unless the weak integrability properties of ξ are countered in some way.

In our case, motivated by the physical modelling of geothermal heating, we will take ξ to be lognormal. In [LRW10] the authors investigate the case $\xi = \exp(Z - \mathbb{E}[Z]/2)$, where $Z(x) = \sum_{k \geq 1} \sigma_k(x) z_k$, for deterministic functions σ_k , and z_k are uncorrelated standard Gaussians. In order to renormalise the equation, many authors have considered the Wick-product ‘ $\diamond$ ’ in place of $\times$, alongside the choice $\xi = \exp(\diamond W) := \exp(W - \mathbb{E}[W^2]/2)$, the Wick exponential of the white noise W , see [HØUZ09] and the references therein. In either case, the solution methods of the authors rely on Gaussian white noise analysis, pioneered by Takeyuki Hida ([Hid03] contains an overview of the techniques of white noise analysis in addition to Hida’s own musings on the structure and function of the white noise analysis paradigm). Thus, due to the roughness of the random field, the solution obtained via the white noise analysis techniques is a distribution-valued object. We remark that a vast literature on these spaces, such as the Hida, Kondratiev, or even larger exponential stochastic distribution spaces, awaits the interested reader; see,

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for instance [Kuo96, PS07, KLP⁺96, Oli]. Other noteworthy approaches to deal with partial differential equations with rough random noise include those of Martin Hairer and his collaborators in the context of regularity structures (see, for example [Hai13, BCCH20]), and rough path theory (à la Terry Lyons) combined with controlled paths and paradifferential calculus as in [GIP12].

Our work specifically concerns with the periodic version of the problem (1.0.1), that is, we take $V = \mathbb{T}^d$, the d -dimensional torus, and the case $\xi = \exp(\diamond X)$, where X is the log-correlated Gaussian field – a generalised random field with a specific logarithmic-type covariance structure (and, in particular, undefined variance). Equivalently, one may define $X = \Delta^{-d/4}W$, where $\Delta^{-d/4}$ is the negative-order fractional Laplacian, and W is the white noise (see, for example [DRSV17]). In fact, it can be shown that X exists as a suitable limit of fractional Gaussian random functions (see [DRSV17, NR17]).

Our methods in solving (1.0.1) are close in spirit to the typical deterministic analysis approach of (integral) transforming the problem in (1.0.1), showing that the solution on the transformation side is as user-friendly as possible, and then inverse-transforming back. Our choice of transformation is the S-transform; it is intimately connected to the Wick product $\diamond$ in the problem. We remark that the use of transformations to treat problems such as (1.0.1) is hardly anything new on its own: for similar techniques using the S-transform or its close cousin, the Hermite transform, see [Pot92, HLØ⁺95]. What is novel in our approach is that by applying the ideas of theory of degenerate elliptic partial differential equations on the S-transform side (or, in fact our extension of the usual S-transform), we are able to skirt the use of stochastic distribution spaces. Instead, we rely on deterministic Sobolev and Fourier theory. In the end, our solution of (1.0.1) is a random L^2 function – or more specifically, a function that belongs to a suitable L^2 subspace which is characterised by our choice of S-transform. This in turn yields also a representation formula for the solution.

When it comes to dealing with the irregularity of the random field ξ in (1.0.1), there are other alternatives to what was outlined above. One of these we can essentially trace back to the seminal work of Eugene Wong and Moshe Zakai [WZ65], which are regularisation and approximation methods. These in turn are closely related to the more traditional stochastic (Itô) integration theory, which has seen renewed interest in the past few years. In particu-

lar, we mention the work of Francesco Russo and Pierre Vallois, and their stochastic calculus via regularisation (see [RV07] for a survey). Indeed, in the one-dimensional case of the problem in (1.0.1), we consider a mollification-approximation approach for both $\times$ being the Wick product $\diamond$ and when $\times$ is the usual scalar product on $\mathbb{R}$. As before, we take $\xi = \exp(\diamond X)$, up to a multiplicative constant, with X being the log-correlated Gaussian field – but the point is that the multiplicative constant depends on the field X , as it has to act as the requisite renormalisation in the absence of the Wick product $\diamond$.

The structure of this dissertation is as follows. In Chapter 2 we will explain the background to the study of continuity of Gaussian (and related) processes and fields. This is in service of Article [A], concerning the continuity of random fields satisfying a certain (Weibull-type) moment condition on their increments. These results are summarised in Section 4.1. In Chapter 3, we move on to discuss the stochastic partial differential equation, and survey its deterministic and stochastic backgrounds in Sections 3.1 and 3.2, respectively. Our key findings in the single-variable (Article [B]) and multivariable case (Article [C]) are summarised in Sections 4.2 and 4.3, respectively – the backdrop to these problems is explored in Section 3.3.

2 Random fields and their continuity

2.1 Stochastic processes and random fields

We start by recalling the concept of a random field defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{T}$ be a set, d a (pseudo-)metric on $\mathcal{T}$, and $(E, \mathcal{E})$ a measurable space. A *random field* (indexed by $\mathcal{T}$) taking its values in $(E, \mathcal{E})$ is a collection $X = \{X_t : t \in \mathcal{T}\}$ of measurable maps X_t from a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to $(E, \mathcal{E})$. When $\mathcal{T}$ is one-dimensional (as in the case $\mathcal{T} \subset \mathbb{R}$) it is customary to refer to (X_t) as a stochastic process. We recall that the probability distribution (or law) of a random element X , taking its values in E , is denoted by F_X (or simply F), and is the induced probability measure given by

$$F_X(A) := (\mathbb{P} \circ X^{-1})(A),$$

for each $A \in \mathcal{E}$. In what follows, an important special case of distributions is the Gaussian distribution; we say a centred random variable X is a (mean-zero) *Gaussian* with variance $\sigma^2 > 0$, if $\mathbb{E}[\exp(itX)] = \exp(-\frac{\sigma^2 t^2}{2})$ for all $t \in \mathbb{R}$. This is denoted by $X \sim N(0, \sigma^2)$.

The finite dimensional distributions of a random field $X = (X_t)_{t \in \mathcal{T}}$ are represented by the probability distributions of the random vectors

$$\{X_{t_1}, \dots, X_{t_k} : t_1, \dots, t_k \in \mathcal{T}, k \in \mathbb{N}\}.$$

We say that two random fields obey the same law, if all of their finite-dimensional distributions agree.

A typical example of a stochastic process is obtained from taking a collection of random variables $(X_i)_{i=1}^n$, taking their values in some vector space E ,

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and scalars $(a_i)_{i=1}^n$, and defining the additive process

$$Y_n = \sum_{i=1}^n a_i X_i. \quad (2.1.1)$$

This type of a random process is called a *random walk*, and processes having this form are ubiquitous in theory and applications alike. Naturally, this example allows for suitable generalisations to fields, and in addition, to vector-valued random elements X_i as well. When the index set is not countable, we interpret the sum in (2.1.1) as an integral; if the constants a_i are arbitrary, we may think of Y being *spanned* by the field $\{X_1, X_2, \dots\}$. For the remainder of this section, we will mainly focus on the case $E = \mathbb{R}$, and take $\mathcal{T}$ to be some compact subset of the real line $\mathbb{R}$.

We recall that by the definition of a random field, for each fixed random parameter $\omega \in \Omega$, the mapping $t \mapsto X_t(\omega)$ defines a random element with values in E . We call this mapping $t \mapsto X_t(\omega)$ the *sample path* of the field X . We follow the usual convention of often suppressing the dependence on ω ; hence we write $t \mapsto X_t$ for the sample path of X .

We now take $\mathcal{T} = [0, T]$ for a fixed time horizon $T > 0$, and start from the archetypal random process in defined on the continuum, namely the Brownian motion:

Definition 2.1.1 (Brownian motion). A real-valued stochastic process $B = \{B_t : t \in [0, T]\}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a *Brownian motion* if B is a mean-zero Gaussian process with almost surely continuous sample paths and has a covariance structure given by

$$\mathbb{E} B_s B_t = s \wedge t,$$

for all $s, t \in [0, T]$.

One defines a d -dimensional Brownian motion based on the one-dimensional one: namely $\bar{B} = \{\bar{B}_t : t \in [0, T]\} = (B_t^1, \dots, B_t^d)$ is a d -dimensional Brownian motion if each (B_t^k) for $1 \leq k \leq d$ is an independent one-dimensional Brownian motion for all t . Hence the vector-valued generalisation of a Brownian motion is in some ways trivial: due to independence, many properties transfer over in a natural way immediately from the one-dimensional case.

It is well-known that there are several ways to show the existence of a Brownian motion process. Perhaps the most concrete is provided by Donsker's theorem, which states that if the independent and identically distributed random variables X_i have mean-zero and unit variance, then choosing a $\sqrt{n}$ -scaling in (2.1.1) for $Y_{[nt]}, t \in [0, 1]$ yields a process that converges in distribution to the Brownian motion at the limit as $n \rightarrow \infty$.

An important generalisation of the Brownian motion, the fractional Brownian motion (fBM), dates its roots back to the seminal work of Mandelbrot and Van Ness [MN68]. Iterating on the earlier ideas of Paul Lévy, they defined fBM as a process admitting a certain kind of a fractional integral representation with respect to the Brownian motion and showed that such a process is well-defined. In addition, Mandelbrot and van Ness established some of the key properties of fBM. Among the most important of these is the fact that while the increments of the Brownian motion are independent, for fBM this need not be so. Instead of the fractional integral approach, we will here exploit the Gaussian structure of fBM and thus define the process via its covariance:

Definition 2.1.2 (Fractional Brownian motion). Let $H \in (0, 1)$ be a parameter. A real-valued stochastic process $B^H = \{B_t^H : t \in [0, T]\}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called a *fractional Brownian motion* (or fBM for short) if B^H is a mean-zero Gaussian process and has covariance

$$\mathbb{E} B_s B_t = \frac{1}{2} \left(|t|^{2H} + |s|^{2H} - |t - s|^{2H} \right), \quad (2.1.2)$$

for all $s, t \in [0, T]$. The parameter H is called the *Hurst index* (or Hurst parameter) associated with the fractional Brownian motion.

We remark that the Hurst index H alters the properties of the process B^H dramatically:

1. Stochastic properties: $H = 1/2$ corresponds to the Brownian motion; hence for $H = 1/2$ the increments $B_t^H - B_s^H$, for $t > s$ are independent. For $H > 1/2$ these increments are positively correlated, and for $H < 1/2$ they are negatively correlated.
2. Regularity: as we will see (Example 2.2.3), the Hurst index H effectively acts as a smoothness parameter for the sample paths of B^H . More specif-

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ically, the sample paths of B^H are Hölder continuous up to index $H - \varepsilon$ for any $\varepsilon > 0$.

2.2 Sample path continuity

As we mentioned in the introduction, for an $\mathbb{R}^n$ -valued random field X the mapping $t \mapsto X_t(\omega)$ on $\mathcal{T}$ defines the sample path. Here we recall that $\mathcal{T}$ is some index set, equipped with a (pseudo-)metric d . Investigating the properties of such mappings is a natural question in several ways. For one, random fields often arise from the modelling of dynamical phenomena. In these situations, one would like to have as much smoothness (and/or integrability) as possible to make the problem more tractable. More generally, regularity properties of mappings between metric spaces are a classical question from a pure mathematics perspective as well.

We start by recalling some terminology related to random fields. Let X and Y be two random fields indexed over a set $\mathcal{T}$. We say that X is a *version* (or a *modification*) of Y , if for all t in $\mathcal{T}$ we have $\mathbb{P}(\omega : X_t(\omega) = Y_t(\omega)) = 1$. It should be clear that if X is a version of Y , then the finite-dimensional distributions of X and Y agree. We also recall that X and Y are said to be *indistinguishable* if $\mathbb{P}(\omega : \forall t \in \mathcal{T}, X_t(\omega) = Y_t(\omega)) = 1$. Clearly if X and Y are indistinguishable, they are also modifications of one another – but generally not vice versa. However, it is a straightforward real analysis exercise to show that if X and Y are both modifications of one another and continuous, and their index set $\mathcal{T}$ contains a dense countable subset (for example, if $\mathcal{T} = \mathbb{R}^n$), then X and Y are indistinguishable. This motivates the concept of a *separable random field*¹; this is a technical assumption that resolves some potential measurability problems. Henceforth we will assume that all our random fields are separable. In addition, this assumption guarantees that $\sup_{t \in \mathcal{T}} X_t$ is measurable.

¹A $\mathbb{R}^d$ -valued random field X , on a topological space $\mathcal{T}$ is called *separable* if there exists a countable dense subset $D \subset \mathcal{T}$ and a fixed event N with $\mathbb{P}(N) = 0$ such that, for any closed $B \subset \mathbb{R}^d$ and open $I \subset \mathcal{T}$,

$$\{\omega : X(t, \omega) \in B \forall t \in I\} \Delta \{\omega : X(t, \omega) \in B \forall t \in I \cap D\} \subset N.$$

Here Δ denotes the symmetric difference of sets, that is $A \Delta B = (A \cap B^c) \cup (A^c \cap B)$.

2.2.1 The classical Gaussian continuity theory

Turning now to the sample path continuity of a stochastic field, we recall some elementary concepts which will be useful in the sequel. For the sake of clarity, we begin by distinguishing between different types of continuity. Let $t_0, t \in \mathcal{T}$, and assume that $d(t, t_0) \rightarrow 0$. If this implies that

- $\mathbb{E} |X_t - X_{t_0}|^2 \rightarrow 0$, we say that X is L^2 -continuous at t_0 .
- $\mathbb{P}(\omega : X_t(\omega)) \rightarrow \mathbb{P}(\omega : X_{t_0}(\omega))$, we say that X is *continuous in probability* (or *stochastically continuous*) at t_0 .
- $X_t(\omega) \rightarrow X_{t_0}(\omega)$ for $\mathbb{P}$ -almost all $\omega \in \Omega$, we say that X is *almost surely continuous* at t_0 .

As usual, whenever these conditions hold everywhere in $\mathcal{T}$, we simply say that X is stochastically continuous. In the sequel, an important subclass of continuous fields will consist of those which have Hölder continuous sample paths:

Definition 2.2.1 (Hölder continuity). Let $\mathcal{T}$ be a set, and d a (pseudo-)metric on $\mathcal{T}$. Let $X = (X_t)_{t \in \mathcal{T}}$ be a random field taking its values in $\mathbb{R}^n$, with $n \geq 1$. We say that X is *Hölder continuous of order $\alpha \in (0, 1]$* (or α -Hölder) if there exists a random variable $K = K(\omega)$ such that

$$\|X_s - X_t\| \leq K d(s, t)^\alpha$$

for all $s, t \in \mathcal{T}$. If X is α -Hölder with $\alpha = 1$, then X is called *Lipschitz continuous*.

On the question of continuity of a random field $(X_t)_{t \in \mathcal{T}}$, a classical tool at our disposal is the renowned Kolmogorov–Chentsov theorem, which gives a straightforward criterion to establish sample path continuity. Below we present a well-known refinement of this standard result, where we obtain Hölder continuous paths as a consequence.

Theorem 2.2.2 (Kolmogorov–Chentsov). *Let X be a random field on $[a, b]^n$, taking its values in $\mathbb{R}^n$. If there exist positive constants α, β, C such that, for every $s, t \in [a, b]^n$*

$$\mathbb{E} \|X_t - X_s\|^\alpha \leq C \|t - s\|^{n+\beta},$$

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then there exists a γ -Hölder continuous modification of X , denoted by $\tilde{X}$, for every $\gamma < \frac{\beta}{\alpha}$.

In fact, much of the overarching theme of this section is exemplified by the Kolmogorov–Chentsov theorem. It is the textbook example of a result telling how analytical properties (e.g. continuity) of a stochastic field can be inferred from its statistical attributes (e.g. moments of increments). In addition, we note that no assumptions on the distribution of the random field X are imposed by this theorem. For example, this flexibility is applied in Article [A] to establish sample path continuity. Now, to conclude this general treatment, we present a typical application of Theorem 2.2.2, by showing that the fractional Brownian motion has Hölder continuous paths.

Example 2.2.3 (Sample path continuity of fBM). The fractional Brownian motion of Hurst index $H \in (0, 1)$ admits a version whose paths are Hölder continuous of order $H - \varepsilon$, for all $\varepsilon > 0$.

We argue this as follows: indeed, recall that for a centred Gaussian random variable ξ we have for all $0 < p < \infty$

$$\mathbb{E}[|\xi|^p] = C(p) \left(\mathbb{E}[|\xi|^2] \right)^{p/2},$$

where $C(p)$ is a positive constant depending only on p . Applying this to the Gaussian random variable $\xi = B_t^H - B_s^H$ and using the covariance of the fBM, namely (2.1.2), we obtain

$$\mathbb{E} |B_t^H - B_s^H|^p = C(p) \left(\mathbb{E} |B_t^H - B_s^H|^2 \right)^{p/2} = C(p) |t - s|^{Hp}.$$

The claim now follows from Theorem 2.2.2. ♣

Having given this brief introduction to the general theory of continuity of random fields, we will next move on to discuss real-valued fields which are Gaussian or ‘close’ to being Gaussian. As ubiquitous as Gaussians are in applications and theory alike, it should not come as a surprise that their continuity is very much a classical topic in stochastic analysis, and one that was already remarkably well understood by the early 1970s. Later developments have mainly concerned themselves with refining and extending these results to fields which share only some of the many useful characteristics of Gaussians. In what follows, we will attempt only a quick summary of the relevant

results. We refer the interested reader to [Lif95] and the references therein for the complete picture.

We let $Z = (Z_t)_{t \in \mathcal{T}}$ be a separable, centred, real-valued Gaussian field on an index set $\mathcal{T}$, with a canonical (pseudo-)metric d_Z defined by $d_Z(s, t) = \sqrt{\mathbb{E} |Z(s) - Z(t)|^2}$. We remark that since a Gaussian process is entirely characterised by its covariance, the index set $\mathcal{T}$ is deliberately chosen to be as generic as possible. This ensures that the obtained continuity results are not dependent in any way on the special structure of $\mathbb{R}^n$, for instance. We will write $N_d(\varepsilon)$ for the *metric entropy* for Z , that is, the smallest number of balls of radius ε (with respect to d_Z) needed to cover $\mathcal{T}$. Set $Y := \sup_{t \in \mathcal{T}} Z_t$. Then we have:

1. Dudley upper bound [Dud67]:

$$\mathbb{E}[Y] \leq K \int_0^\infty \sqrt{\log N_{d_Z}(\varepsilon)} d\varepsilon,$$

where $K > 0$ is a universal constant and the integral on the right-hand side is called the *metric entropy integral*. Thus this integral being finite implies boundedness of the process, and as a corollary, its uniform continuity. In fact, this result can be expressed also as a tail bound: for $u \geq 0$

$$\begin{aligned} \mathbb{P} \left(\sup_{s, t \in \mathcal{T}} |X_s - X_t| \leq K \int_0^\infty \sqrt{\log N_{d_Z}(\varepsilon)} d\varepsilon + u \cdot \text{diam}(\mathcal{T}) \right) \\ \leq 1 - \exp(-u^2). \end{aligned}$$

2. Borell–TIS inequality: assuming in addition that Y is almost surely finite, we have:

$$\mathbb{P}(|Y - \mathbb{E} Y| > u) \leq 2 \exp(-u^2/2\sigma^2), \quad (2.2.1)$$

where $\sigma^2 := \max_{t \in \mathcal{T}} \text{Var}[Z_t]$. Thus the tails of the process $Y - \mathbb{E}[Y]$ decay at a rate faster than an exponential, which is no worse than any of the Gaussians Y is a supremum of; moreover, $2 \max_{t \in \mathcal{T}} \text{Var}[Z_t]$ acts as the scale of this decay.

3. Talagrand’s majorizing measure theorem [Tal87]: there exists a proba-

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bility measure m on $(\mathcal{T}, d)$ such that

$$\sup_{x \in \mathcal{T}} \int_0^\infty g(m(B_\varepsilon(x))) d\varepsilon \leq K \mathbb{E}[Y],$$

where $g(t) = (\log(1/t))^{1/2}$ for $0 \leq t \leq 1$, and $B_\varepsilon(x) = \{y \in \mathcal{T} : d(x, y) < \varepsilon\}$ is an open ball of radius ε centred at x . The moral is that the metric entropy integral acts as a lower bound for $Y = \sup_{t \in \mathcal{T}} Z_t$.

2.2.2 Beyond Gaussian continuity; joint Hölder continuity

Viens and Vizcarra showed in [VV07] that the above upper bound-type results remain essentially valid also in the context of so-called sub-Gaussian processes, which we will introduce next. Recall that if Z is a centred, real-valued Gaussian random variable with variance $\sigma^2 > 0$, its moment generating function $M_Z(t) := \mathbb{E}[\exp(tZ)]$ always exists on $\mathbb{R}$, and moreover we have $M_Z(t) = \exp(t^2\sigma^2/2)$ for all $t \in \mathbb{R}$. Motivated by this property of the Gaussians, we say that a real-valued centred process $X = (X_t)_{t \in \mathcal{T}}$ is *sub-Gaussian* (with respect to the (pseudo-)metric d) if, for all $s, t \in \mathcal{T}$ and every $\lambda \in \mathbb{R}$,

$$\mathbb{E}[\exp(\lambda(X_s - X_t))] \leq \exp\left(\frac{\lambda^2}{2}d(s, t)^2\right).$$

In addition to Gaussians, the class of sub-Gaussian processes includes, for example Rademacher processes, uniformly distributed processes, and, in general, any process with a compactly supported distribution.

Before proceeding to examine a further generalisation on sub-Gaussian processes, we remark that sub-Gaussianity can be equivalently expressed via a particular exponential decay of the tail probabilities. More specifically, it follows from a straightforward application of the Markov inequality that if $(X_t)_{t \in \mathcal{T}}$ is a sub-Gaussian process relative to the (pseudo-)metric d , then for all $u \in \mathbb{R}$ we have

$$\mathbb{P}(|X_s - X_t| > u) \leq \exp\left(-\frac{1}{2}\left(\frac{u}{d(s, t)}\right)^2\right). \quad (2.2.2)$$

Equivalently, it is possible to define sub-Gaussian processes as such that a certain moment growth condition for the increments of the process is satis-

fied; or, one may define sub-Gaussian processes as those living in a suitable Orlicz space. For a complete description of sub-Gaussians and proofs that the aforementioned definitions are indeed equivalent, see [Ver18], and for a more thorough treatment on general Banach space-valued sub-Gaussian processes and their continuity, we refer the reader to the monograph by Ledoux and Talagrand [LT91].

As we previously stated, the Gaussian upper-bound regularity results summarised above remain essentially valid for processes satisfying (2.2.2). To investigate this further, Viens and Vizcarra in [VV08] introduced the following sub-Nth chaos property, which acts as a natural extension of (2.2.2). Their aim was to investigate the majorizing measure conditions of Talagrand, as in [Tal87], in the context of non-Gaussian processes. Indeed, we introduce the following:

Definition 2.2.4 (Sub-Nth chaos property). Let $\mathcal{T}$ be a set, equipped with a (pseudo-)metric d . Let $X = (X_t)_{t \in \mathcal{T}}$ be a centred stochastic process. If there exists a constant $c > 0$ such that

$$\mathbb{P}(|X_s - X_t| \geq u) \leq 2 \exp \left(-c \left(\frac{|u|}{d(s, t)} \right)^{2/N} \right)$$

holds for all $u > 0$ and $s, t \in \mathcal{T}$, then we say that X has the sub-Nth chaos property, relative to the scale $d(s, t)$.

With the choices of $N = 1, c = 1/2$, it is apparent from (2.2.2) that the class of processes having the sub-Nth chaos property extends the concepts of a sub-Gaussian process. For our discussion the key result in [VV08] is the following extension of the Borell-TIS inequality:

Theorem 2.2.5 (Viens & Vizcarra (2008)). *For each positive integer N , there exists universal constants C_N and C'_N , depending only on N , such that if X defined on $\mathcal{T}$ is a sub-Nth-Gaussian chaos field with respect to the (pseudo-)metric d , then for each $t_0 \in \mathcal{T}$ and $s \geq 0$,*

$$\mathbb{P} \left(\sup_{t \in \mathcal{T}} |X_t - X_{t_0}| > C_N M + s C'_N D \right) \leq 2 \exp \left(-\frac{s^{\frac{2}{N}}}{2} \right),$$

where $M = \int_0^\infty (\log N_d(\varepsilon))^{N/2} d\varepsilon$ and $D = \text{diam}_d(I)$ is the diameter of $\mathcal{T}$ under d . Here $N_d(\cdot)$ denotes the metric entropy for X .

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In addition, if D and M are finite, then X is almost surely bounded.

A similar result for processes satisfying a certain moment growth condition for the increments is proved in Article [A]; see Section 4.1. We note that although many of the continuity results in [VV08] rely on methods from Malliavin calculus or treatment of Gaussian functionals in general – and as such it is not obvious how these can be generalised beyond the Gaussian scope – the proof of Theorem 2.2.5 is essentially a generic chaining argument à la Michel Talagrand, and therefore not reliant on the ‘Gaussian’ connection.

Speaking of general approaches to establish continuity of random processes and fields, we will not discuss generic chaining techniques further here (we refer the interested reader to the monograph [Tal05]), but move on to introduce an important result, the classical Garsia–Rodemich–Rumsey (GRR) lemma, which was proved by the aforementioned authors in [GRR70]. The setup is the following: let $\psi: [0, \infty) \rightarrow [0, \infty)$ be non-decreasing with $\lim_{u \rightarrow \infty} \psi(u) = \infty$ and let $p: [0, 1] \rightarrow [0, 1]$ be continuous and non-decreasing, with $p(0) = 0$. Set

$$\begin{aligned}\psi^{-1}(u) &= \sup_{\psi(v) \leq u} v, & \text{if } \psi(0) \leq u < \infty \\ p^{-1}(u) &= \max_{p(v) \leq u} v, & \text{if } 0 \leq u \leq p(1).\end{aligned}$$

Then we have:

Lemma 2.2.6 (Garsia–Rodemich–Rumsey, 1970). *Let f be a continuous function on $[0, 1]$ and suppose that*

$$\int_0^1 \int_0^1 \psi \left(\frac{|f(x) - f(y)|}{p(x - y)} \right) dx dy \leq B < \infty.$$

Then, for all $s, t \in [0, 1]$ we have

$$|f(t) - f(s)| \leq 8 \int_0^{|s-t|} \psi^{-1} \left(\frac{4B}{u^2} \right) dp(u).$$

We remark that the GRR lemma implies in particular the following useful fractional Sobolev embedding, valid for any continuous function f on $[0, 1]$:

let $p \geq 1$ and $\alpha p > 1$, and then

$$|f(t) - f(s)| \leq C_{\alpha,p} |t - s|^{\alpha - \frac{1}{p}} \left(\int_0^1 \int_0^1 \frac{|f(x) - f(y)|^p}{|x - y|^{\alpha p + 1}} dx dy \right)^{1/p}, \quad (2.2.3)$$

for all $s, t \in [0, 1]$. Here we remark that it is precisely (2.2.3) (and its extensions to several dimensions) we apply to establish some of our continuity results in Article [A]; see Section 4.1.

Interestingly, the GRR lemma does not a priori assume that the sample paths of the stochastic process obey any particular distribution, and in fact the lemma and its proof on their own are completely devoid of probabilistic content. The authors applied the lemma in the original paper to study the sample paths of Gaussian processes in particular, which is somewhat ironic in retrospect given the powerful generality of the lemma. Later, the GRR lemma and its variations have found much use beyond the Gaussian scope: improvement of the well-known Burkholder–Davis–Gundy inequalities for L^p -martingales [BY82] (see also [GS85]); showing growth rates for moment inequalities for stochastic flows on $\mathbb{R}^d$ in [IS99]; establishing almost sure Hölder continuity of the Schramm–Loewner evolution as in [FTY21], to name a few. A classical application of the GRR lemma is to infer Theorem 2.2.2 (Kolmogorov–Chentsov) above, see for instance [SV79] for a proof.

The GRR lemma can be extended also to higher dimensions. Garsia himself proved a variant of the lemma in the multivariable case in [Gar72], and a metric space version of the lemma can be found in [AI96], for example. We will next discuss the multi-parameter case, but with a twist: instead of looking at increments $|X_s - X_t|$ for $s, t \in [0, 1]^n$, we will consider rectangular increments, and introduce a concept called *joint Hölder continuity* of stochastic processes on $\mathbb{R}^n$. Our presentation and notation in this passage are mainly based on [Hu16]. To illustrate the idea, let $F: [0, T]^2 \rightarrow \mathbb{R}$ be a deterministic function, and consider investigating the existence of some constant $C > 0$, such that for any $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$

$$|F(y_1, y_2) - F(x_1, y_2) - F(y_1, x_2) + F(x_1, x_2)| \leq C |x_1 - y_1|^{\alpha_1} |x_2 - y_2|^{\alpha_2}$$

holds, for $\alpha_1, \alpha_2 \in (0, 1]$. Notice that above, unlike in the usual multivariable case, the right-hand side $|x_1 - y_1|^{\alpha_1} |x_2 - y_2|^{\alpha_2}$ does not define a distance on

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$\mathbb{R}^2$; rather, it is a volume-like metric.

We next consider the general case on $\mathbb{R}^n$ and introduce some notation. Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ be two vectors in $\mathbb{R}^n$. We denote $x' = (x_1, \dots, x_{n-1})$ and $y' = (y_1, \dots, y_{n-1})$. For each integer $k = 1, 2, \dots, n$, we define

$$V_{k,y}x = (x_1, \dots, x_{k-1}, y_k, x_{k+1}, \dots, x_n).$$

Let f be a function from $\mathbb{R}^n$ to $\mathbb{R}^m$. We define the operator $V_{k,y}$ acting on f in the following way:

$$(V_{k,y}f)(x) = f(V_{k,y}x).$$

Next, we define the joint (rectangular) increment of a function f on an n -dimensional rectangle as

$$\square_y^n f(x) = \prod_{k=1}^n (I - V_{k,y})f(x),$$

where I denotes the identity operator on $\mathbb{R}^n$. On the plane ($n = 2$), this rectangular increment is simply, for $x = (x_1, x_2)$ and $y = (y_1, y_2)$,

$$\square_y^2 f(x) = f(x_1, x_2) - f(x_1, y_2) - f(y_1, x_2) + f(y_1, y_2).$$

Hu and Le proved in [HL13] (see also [Hu16]) the following multi-parametric extension of the GRR lemma for the rectangular increments:

Theorem 2.2.7 (Multiparametric GRR; Hu & Le, 2013). *Let f be a continuous function on $[0, 1]^n$ and suppose that*

$$\int_{[0,1]^n} \int_{[0,1]^n} \psi \left(\frac{|\square_y^n f(x)|}{\prod_{k=1}^n p_k(|x_k - y_k|)} \right) dx dy \leq B < \infty.$$

Then for all $s, t \in [0, 1]^n$ we have

$$|\square_s^n f(t)| \leq 8 \int_0^{|s_1 - t_1|} \dots \int_0^{|s_n - t_n|} \psi^{-1} \left(\frac{4^n B}{u_1^2 \dots u_n^2} \right) dp_1(u_1) \dots dp_n(u_n).$$

We point out that a two-parameter version of this result was already estab-

lished in [Ral08], where the authors applied it to obtain a regularity result for fractional Brownian fields.

As a consequence of the multiparametric GRR lemma for rectangular increments, Hu and Le present in [HL13] the following joint Hölder continuity criterion result, which is akin to a Kolmogorov–Chentsov theorem for joint rectangular increments.

Theorem 2.2.8. *Let W be a continuous random field on $\mathbb{R}^n$. Suppose that there exists constants $\alpha, \beta_k, 1 \leq k \leq n$ and K such that for every $x, y \in [0, 1]^n$,*

$$\mathbb{E} \left[\left| \square_y^n W(x) \right|^\alpha \right] \leq K \prod_{k=1}^n |x_k - y_k|^{1+\beta_k}.$$

Then, for every $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ with $0 < \varepsilon_k \alpha < \beta_k$ for all $1 \leq k \leq n$, there exists a random variable η with $\mathbb{E}[\eta^\alpha] \leq K$, such that the following inequality holds almost surely

$$|\square_t^n W(s)| \leq C \eta(\omega) \prod_{k=1}^n |t_k - s_k|^{\beta_k \alpha^{-1} - \varepsilon_k}$$

for all $s, t \in [0, 1]^n$ where C is a constant defined by

$$C = 8^n 4^{n/\alpha} \prod_{k=1}^n \left(1 + \frac{2}{\beta_k - \alpha \varepsilon_k} \right).$$

We remark that a similar result, proved under a specific growth condition on the moments of the (rectangular) increments of the field, can be found in Article [A] (see Section 4.1).

3 Stochastic partial differential equations with fractional Gaussian noise

In the previous chapter we investigated the Hölder continuity of random fields satisfying a specific type of growth condition on the moments of their increments. In this chapter we will introduce a random object which arises as a suitable scaling limit of certain Hölder continuous random fields – namely, the Gaussian log-correlated field. Indeed, this is a generalised Gaussian field on $\mathbb{T}^d$, the d -dimensional unit torus, and it gives rise to an associated random measure, called the Gaussian multiplicative chaos measure, or GMC measure for short. Our end goal is to investigate certain types of stochastic partial differential equations and their solutions, for which the GMC measure arises naturally. Before tackling this SPDE theory, we will briefly introduce the main cast of characters, starting from the cornerstone of modern PDE theory, the Sobolev spaces.

3.1 Real analysis facts

3.1.1 Sobolev spaces

In this subsection we will briefly recall some notation and general facts about Sobolev spaces on the d -dimensional torus $\mathbb{T}^d$. There is a vast literature on these spaces in general; the standard references for this section are [AF03, DPV12].

We start by recalling that if $f, g \in L^p(\mathbb{T}^d)$ for $p \geq 1$ and $\alpha = (\alpha_1, \dots, \alpha_n) \in$

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$\mathbb{N}_0^n$ is a multi-index, then g is said to be the *weak derivative* of f if

$$\int_{\mathbb{T}^d} f D^\alpha \varphi = (-1)^{|\alpha|} \int_{\mathbb{T}^d} g \varphi$$

holds for all $\varphi \in C^\infty(\mathbb{T}^d)$. Here $D^\alpha \varphi = \frac{\partial^{|\alpha|} \varphi}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$, where $|\alpha| = \sum_i \alpha_i$ denotes the length of the multi-index.

For $k \in \mathbb{N}$ and $p \geq 1$, the classical Sobolev space $W^{k,p}(\mathbb{T}^d)$ is defined as

$$W^{k,p}(\mathbb{T}^d) = \left\{ f \in L^p(\mathbb{T}^d) : D^\alpha f \in L^p(\mathbb{T}^d) \text{ for all } \alpha \in \mathbb{N}_0^n, |\alpha| \leq k \right\},$$

and $W^{k,p}(\mathbb{T}^d)$ equipped with the norm

$$\|f\|_{W^{k,p}(\mathbb{T}^d)} = \left(\sum_{|\alpha| \leq k} \|D^\alpha f\|_{L^p(\mathbb{T}^d)}^p \right)^{1/p},$$

is a Banach space. When $p = 2$, we denote $W^{k,2}(\mathbb{T}^d)$ by $H^k(\mathbb{T}^d)$; these spaces, equipped with the scalar product

$$\langle f, g \rangle_{H^k} = \sum_{|\alpha| \leq k} \int_{\mathbb{T}^d} D^\alpha f D^\alpha g,$$

are Hilbert spaces.

Next, we consider Sobolev spaces of fractional order, and start by fixing the exponent $s \in (0, 1)$. For any $p \geq 1$, the fractional-order Sobolev space $W^{s,p}(\mathbb{T}^d)$ is defined as

$$W^{s,p}(\mathbb{T}^d) = \left\{ u \in L^p(\mathbb{T}^d) : \frac{|u(x) - u(y)|}{|x - y|^{\frac{n}{p} + s}} \in L^p(\mathbb{T}^d \times \mathbb{T}^d) \right\}.$$

This intermediate space between L^p and $W^{1,p}$ has a natural norm given by

$$\|u\|_{W^{s,p}(\mathbb{T}^d)} = \left(\int_{\mathbb{T}^d} |u(x)|^p dx + [u]_{W^{s,p}(\mathbb{T}^d)}^p \right)^{1/p},$$

where

$$[u]_{W^{s,p}(\mathbb{T}^d)}^p := \int_{\mathbb{T}^d \times \mathbb{T}^d} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dx dy$$

is called the *Gagliardo seminorm*. For $s > 1$, we define

$$W^{s,p}(\mathbb{T}^d) = \left\{ f \in W^{k,p}(\mathbb{T}^d) : \partial^\alpha f \in W^{\sigma,p}(\mathbb{T}^d), \forall \alpha \in \mathbb{N}_0^n, |\alpha| = m \right\}$$

where $m = \lfloor s \rfloor$ and $\sigma = s - m$. We recall that if $s > 0$ and $p \in [1, \infty)$, then a variant of the famous result by Meyers and Serrin states that $C^\infty(\mathbb{T}^d) \cap W^{s,p}(\mathbb{T}^d)$ is dense in $W^{s,p}(\mathbb{T}^d)$ (see, for instance [Leo23, Theorem 6.65]). Here $C^\infty(\mathbb{T}^d)$ denotes the space of infinitely differentiable functions on $\mathbb{T}^d$.

We remark that on the torus it is sometimes more convenient to work with an equivalent definition of the fractional Sobolev space which makes use of the Fourier transform $\mathcal{F}$. Let $f \in L^1(\mathbb{T}^d)$. Then the Fourier coefficients of f are given by, for $n \in \mathbb{Z}^d$,

$$\hat{f}(n) = \int_{\mathbb{T}^d} f(x) e^{-ix \cdot n} dx.$$

Let $s \in \mathbb{R}$, then $W^{s,2}(\mathbb{T}^d)$ is defined as

$$W^{s,2}(\mathbb{T}^d) = \{f \in S'(\mathbb{T}^d) \text{ s.t. } \|f\|_{W^{s,2}(\mathbb{T}^d)}^2 = \sum_{n \in \mathbb{Z}^d} \langle n \rangle^{2s} |\hat{f}(n)|^2 < \infty\},$$

where $S'(\mathbb{T}^d)$ is the space of tempered distributions on $\mathbb{T}^d$, and $\hat{f}(n)$ denotes the Fourier coefficients of f , and $\langle n \rangle^2 = 1 + |n|^2 = 1 + \sum_{k=1}^d n_k^2$. One can show that $\widehat{W}^{s,2}(\mathbb{T}^d)$ is a Hilbert space, and moreover that $\widehat{W}^{s,2}(\mathbb{T}^d) = W^{s,2}(\mathbb{T}^d)$ (see, for example [Tri10] and references therein).

In addition, for any $s \in (0, \infty)$, $p \in (1, \infty)$ such that $1/p + 1/q = 1$, we denote by $(W^{s,p}(\mathbb{T}^d))^* = W^{-s,q}(\mathbb{T}^d)$ the topological dual of $W^{s,p}(\mathbb{T}^d)$, a space of generalised functions (in the sense of Schwartz), equipped with a natural norm given by

$$\|f\|_{W^{-s,q}(\mathbb{T}^d)} = \sup_{\substack{g \in W^{s,p}(\mathbb{T}^d), \\ \|g\|_{W^{s,p}(\mathbb{T}^d)} \leq 1}} |\langle f, g \rangle|.$$

Above $\langle f, g \rangle$ denotes the action of the generalised function f on the Sobolev function g .

There are many interesting embedding results between Sobolev spaces and other spaces of interest, such as the space of Hölder continuous functions. For

our purposes, the following embedding theorem will be useful; it is a special case of a more general Besov space embedding. The proof of the general result can be found in [Tri10].

Theorem 3.1.1. *Let $s_1 < s_0$. If $s_1 - d/p_1 \leq s_0 - d/p_0$, then*

$$W^{s_0, p_0}(\mathbb{T}^d) \hookrightarrow W^{s_1, p_1}(\mathbb{T}^d).$$

In addition, if $p_1 \leq p_0$, then

$$W^{s, p_0}(\mathbb{T}^d) \hookrightarrow W^{s, p_1}(\mathbb{T}^d).$$

Above $\hookrightarrow$ denotes continuous embedding.

3.1.2 Highlights from theory of degenerate elliptic equations

In this section we will introduce some useful facts about divergence-form partial differential equations with degenerate coefficients, with a view towards the regularity part of Article [C]. The presentation in this section is mainly based on [FJK82, FKS82]. For more details on variational problems, see [KS80]. In addition, a very useful resource for Muckenhoupt weights as they relate to degenerate elliptic equations is the monograph [HKM06].

We start by introducing a suitable class of weights, and the associated weighted Sobolev spaces. We say that non-negative, measurable function w on $\mathbb{T}^d$ is a Muckenhoupt A_2 weight if w satisfies the A_2 condition: that is, there exists $C > 0$ such that for all Euclidean balls $B \subset \mathbb{T}^d$ we have

$$\left(\frac{1}{|B|} \int_B w \, dx \right) \left(\frac{1}{|B|} \int_B w^{-1} \, dx \right) \leq C < \infty. \quad (3.1.1)$$

Above $|B|$ denotes the Lebesgue measure of B . For set $E \subset \mathbb{T}^d$, we denote $w(E) = \int_E w \, dx$.

Given a measurable set $E \subset \mathbb{T}^d$ we will denote by $L^p(E, w)$, $1 \leq p < \infty$ the Banach space of all measurable functions defined on E for which

$$\|u\|_{L^p(E, w)} := \left(\int_E |u(x)|^p w(x) \, dx \right)^{1/p}$$

is finite. For $1 \leq p < \infty$ we will denote by $W^{k, p}(E, w)$ the weighted Sobolev

space, which by definition is the completion of $C^\infty(E)$ under the norm

$$\|u\|_{W^{k,p}(E)} := \|u\|_{L^p(E,w)} + \sum_{|\alpha|=1}^k \|\partial^\alpha u\|_{L^p(E,w)},$$

where α is a multi-index. When $p = 2$, we use the shorthand $H^k(E, w)$ for these spaces. We remark that weighted Sobolev spaces, similar to their ‘unweighted’ counterparts, admit all kinds of useful embedding inequalities. Here we content ourselves with the weighted Poincaré inequality:

Theorem 3.1.2 (Weighted Poincaré inequality). *Let $B = \{y \in \mathbb{T}^d : |y - x| < r\} \subset \mathbb{T}^d$. Then for all $f \in C^\infty(B)$ such that $\int_B f w dx = 0$, there exists a constant $C > 0$ such that*

$$\left(\frac{1}{w(B)} \int_B |f|^2 w dx \right)^{1/2} \leq Cr \left(\frac{1}{w(B)} \int_B |\nabla f|^2 w dx \right)^{1/2}.$$

We will next discuss divergence-form PDEs in weighted Sobolev spaces. For any $w \in A_2$, we introduce the second-order differential operator L by setting

$$Lu := \nabla \cdot (w \nabla u),$$

and we may consider the associated PDE

$$Lu = \mu, \tag{3.1.2}$$

for some finite measure μ supported in $\mathbb{T}^d$. We understand the solution to (3.1.2) in the weighted Sobolev space $H^1(\mathbb{T}^d, w)$ in an analogous way to the usual ‘unweighted’ case, via a suitable variational formulation:

Definition 3.1.3. Let $f = (f_1, \dots, f_n)$ on $\mathbb{T}^d$ be such that $|f|/w \in L^2(\mathbb{T}^d, w)$. We say that $u \in H^1(\mathbb{T}^d, w)$ is a solution of $Lu = -\nabla \cdot f$, if for every $\varphi \in C^\infty(\mathbb{T}^d)$ we have

$$\int_{\mathbb{T}^d} w \nabla u \cdot \nabla \varphi dx = \int_{\mathbb{T}^d} f \cdot \nabla \varphi dx.$$

We have the following existence and uniqueness result, which is essentially just a restatement of the classical Lax-Milgram theorem:

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Theorem 3.1.4. *Given $f = (f_1, \dots, f_d)$ such that $|f|/w \in L^2(\mathbb{T}^d, w)$, and $\int_{\mathbb{T}^d} f_i = 0$, there exists a unique periodic solution u in $H^1(\mathbb{T}^d, w)$ of $Lu = -\nabla \cdot f$.*

For degenerate elliptic equations, one may state different maximum principles for the weighted Sobolev spaces (as these are essentially modifications of the usual maximum principles). Here we will state a version of the maximum principle allowing us to obtain bounds on the maximum with respect to the right-hand side of the equation, or the weighted L^2 norm of the solution itself, depending on the setup:

Theorem 3.1.5 (Maximum principles). *1. Let $k > d/(d-1) + \varepsilon$, $\varepsilon > 0$, and assume that $f = (f_1, \dots, f_d)$ is such that $|f|/w \in L^p(\mathbb{T}^d, w)$ with $p > 2d - \varepsilon$. Let $\beta = k(1 - 2/p)$. Then, if $u \in H^1(\mathbb{T}^d, w)$ solves $Lu = -\nabla \cdot f$, we have*

$$\max_{x \in \mathbb{T}^d} |u(x)| \leq C(d) w(\mathbb{T}^d)^{\beta-1/2k} \| |f|/w \|_{L^p(\mathbb{T}^d, w)}.$$

2. Denote $B_1 = \{y : |y - x| < r\}$ and $B_2 = \{y : |y - x| < 2r\}$. Let u be a solution of $Lu = 0$ in $H^1(B_2)$. Then u satisfies

$$\max_{B_1} |u| \leq C \left(\frac{1}{w(B_2)} \int_{B_2} u^2 w dx \right)^{1/2}.$$

Finally, we state an estimate for the size of the Green function, which is an important starting point when establishing regularity of the solution of the deterministic equation in Article [C].

Theorem 3.1.6. *Fix a large ball $\Sigma \subset \mathbb{T}^d$ of radius R . Let G be the Green function for (3.1.2). Then*

$$G(x, y) \approx \int_{|x-y|}^R \frac{s^2}{w(B(x, s))} \frac{ds}{s}, \quad x, y \in \frac{1}{4}\Sigma,$$

where $B(x, s) = \{x_0 \in \mathbb{T}^d : |x_0 - x| < s\}$, and by $\approx$ we mean that up to a constant, G obeys some lower and upper bound with respect to the right-hand side of the expression.

The proof can be found in [FJK82]. It is interesting that locally the Green function G for (3.1.2) has the same radial characteristics as the classical Green

function. Moreover, the consequence of this theorem is that controlling $w(B(x, s))$ essentially leads to controlling the Green function itself.

3.2 Stochastics facts

3.2.1 Log-correlated Gaussian field

We now move on to stochastics, and discuss the log-correlated Gaussian field on the unit torus $\mathbb{T}^d$. As we noted at the start of this chapter, this can be viewed as the scaling limit of certain Hölder continuous random fields. We remark that the results of this section are fundamentally not at all reliant on the structure of $\mathbb{T}^d$; it is our domain of choice for the simple reason that we will consider a periodic problem in the sequel, see Sections 4.2 and 4.3. Here, with some natural modifications, we could have the field live in some open, connected set $U \subset \mathbb{R}^d$ just as well. In this section we will only focus on the very basics of log-correlated fields; an interested reader will find many further properties of log-correlated fields in the survey article [DRSV17] and the references therein.

Definition 3.2.1 (Log-correlated Gaussian field). A centred Gaussian field X defined on $\mathbb{T}^d$ is called a *Gaussian log-correlated field* if its covariance kernel R_X satisfies

$$R(y, z) := R_X(y, z) := \mathbb{E}[X(y)X(z)] = \log \frac{1}{|z - y|} + g(z, y), \quad (3.2.1)$$

where $g \in C^\infty(\mathbb{T}^d \times \mathbb{T}^d)$.

A centred Gaussian field with a covariance given by (3.2.1) cannot be a function. Indeed, it is well-known that a Gaussian random vector X has all moments finite, while a Gaussian X with covariance (3.2.1) has undefined variance. Therefore, as these pointwise evaluations of a log-correlated Gaussian field are not bounded, (c.f. (3.2.2)), we interpret the covariance R yielding the integral kernel of the covariance operator associated to the Gaussian field X :

$$\mathbb{E}\langle X, \psi_1 \rangle \langle X, \psi_2 \rangle = \int_{\mathbb{T}^d \times \mathbb{T}^d} R(y, z) \psi_1(y) \psi_2(z) dz dy$$

for any $\psi_1, \psi_2 \in C^\infty(\mathbb{T}^d)$.

Remark 3.2.2. We point out that the chief role of the function g in Definition 3.2.1 is to guarantee that the function R in (3.2.1) results in a well-defined covariance function; for instance, R is required to be positive definite. By this we mean that for all integers $n \geq 1$, any numbers $\xi_1, \dots, \xi_n$, and any $x_1, \dots, x_n \in \mathbb{T}^d$ we have

$$\sum_{i,j=1}^n \xi_i R(x_i, x_j) \xi_j \geq 0.$$

Example 3.2.3. When $d = 1$, we are on $\mathbb{T}$, the unit torus; for $t \in [0, 2\pi)$, we define the formal random Fourier series

$$X(t) = \sum_{n=1}^{\infty} A_n \frac{\cos(nt)}{n^{1/2}} + B_n \frac{\sin(nt)}{n^{1/2}}, \quad (3.2.2)$$

where A_n, B_n are independent, standard normal random variables. As all partial sums of X are Gaussian, X is a Gaussian process; however, as the factor $n^{-1/2}$ is rather weak, X does not define a function. Nevertheless, a formal computation of the covariance of (3.2.2) results precisely in a expression of the form (3.2.1), with $g \equiv 0$.

Remark 3.2.4. We already noted that being a generalised stochastic field, the pointwise values of the log-correlated Gaussian field are not well-defined. However, the log-correlated field could be described as being borderline to being a function: more specifically, as the authors show in [JSW20], the log-correlated field is almost surely an element of $H^{-\varepsilon}(\mathbb{T}^d)$ for $\varepsilon > 0$, the negative-order Sobolev space of order ε . An alternative way to look at the Gaussian log-correlated field is to see it as a limit of certain Gaussian Hölder-continuous fields, as alluded to in the beginning of this section. For example, in the case $d = 1$, recall the fractional Brownian motion (B_t^H) with Hurst parameter $H \in (0, 1)$, as in Definition 2.1.2. It can be shown, as the authors do in [NR17], that (B_t^H) , when appropriately normalised, converges as $H \rightarrow 0$ to a Gaussian generalised field with covariance given by (3.2.1). This result can essentially be extended to the general d -dimensional case as well, see [HN22, DRSV17].

3.2.2 Gaussian multiplicative chaos measure

In this section we recall some basic facts on the Gaussian multiplicative chaos measures. For details, we refer to survey articles [DRSV17, RV13, Pow21] and references therein.

Definition 3.2.5 (Wick exponential). Let Z be a centred Gaussian random variable. We define the Wick exponential of Z by

$$e^{\diamond Z} := e^{Z - \mathbb{E}[Z^2]/2}.$$

Remark 3.2.6. The Wick exponential can also be expressed as an $L^2(\Omega)$ convergent series

$$e^{\diamond Z} = \sum_{n=0}^{\infty} \frac{Z^{\diamond n}}{n!},$$

where $Z^{\diamond n}$ are Wick powers defined via Wick products, see Definition 3.2.9.

The family of Gaussian multiplicative chaos (GMC) measures $d\mu_\beta$ on $\mathbb{T}^d$, depending on a parameter $\beta \in (0, \sqrt{2d})$, are random measures constructed from Wick exponentials of the log-correlated field X . More specifically, the GMC measure on the torus is formally defined for all Borel sets $A \in \mathcal{B}(\mathbb{T}^d)$ as the random measure μ_β given by

$$\mu_\beta(A) = \int_A e^{\beta X(y) - \frac{\beta^2}{2} \mathbb{E}[X(y)^2]} dy, \quad (3.2.3)$$

where X is a log-correlated Gaussian field as in Definition 3.2.1. The fact that such a measure exists is non-trivial. For a proof in the general d -dimensional setting, see for example [RV13].

Next, in service of the forthcoming Section 3.3.1, we briefly consider the case $d = 1$, and state a result guaranteeing that a GMC measure with some additional useful properties exists. If θ is a smooth mollifier, we denote by X_ε the mollified field $X_\varepsilon = X * \theta_\varepsilon$, where $\theta_\varepsilon(y) = \varepsilon^{-1} \theta(\frac{y}{\varepsilon})$ for $\varepsilon > 0$. Then X_ε defines a centred and smooth Gaussian field. A proof of the following useful technical result can be found in Article [B] (we remark that a similar result can be found elsewhere as well, for instance in [JSW20]):

Lemma 3.2.7 (Approximation lemma). *Let X be a log-correlated Gaussian field on $(0, T)$, and $\beta \in (0, 1)$ be a parameter. Let $0 < \varepsilon < \varepsilon'$, $x, y \in (0, T)$. Then the*

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convolution approximation X_ϵ of X satisfies:

1. X_ϵ has continuous realizations and $\mathbb{E}[X_\epsilon(z)^2]$ is finite and independent of z .
2. The sequence of random measures

$$d\mu_{\beta,\epsilon}(y) := \mathbb{1}_{(0,T)}(y) \exp \left(\beta X_\epsilon(y) - \frac{\beta^2}{2} \mathbb{E} [X_\epsilon^2] \right) dy, \quad (3.2.4)$$

converges in probability (in the space of Radon measures equipped with the topology of weak convergence) towards a random measure $d\mu_\beta(y)$, supported on $[0, T]$, called the Gaussian multiplicative chaos measure.

3. $\lim_{\epsilon' \rightarrow 0} \mathbb{E} [X_\epsilon(x) X_{\epsilon'}(y)] = R(x, y)$, for $x \neq y$ fixed.

We remark that there is nothing particular to the one-dimensional torus in Lemma 3.2.7; it can be shown that the lemma holds for GMC measures on $\mathbb{T}^d$ for $0 < \beta < \sqrt{d}$.

For the reader's convenience, we record the fact that the GMC measure μ_β has positive p -moments up to order $2/\beta^2$ [RV13, Theorem 2.11]:

Theorem 3.2.8. *Let X be the log-correlated Gaussian field on $\mathbb{T}^d$, and let $d\mu_\beta$ the associated Gaussian multiplicative chaos measure, with parameter $0 < \beta < \sqrt{2d}$. Then the measure μ_β admits finite p -moments for all $p \in (0, \frac{2}{\beta^2})$. In particular, $\mathbb{E}[\mu_\beta(A)] = |A|$ for any $A \in \mathcal{B}(\mathbb{T}^d)$, where $|A|$ denotes the Lebesgue measure of A .*

As our solutions of stochastic differential equations (see Section 4.2) will be representable in terms of the GMC measure, heuristically this result gives us the expected integrability for the solution.

3.2.3 Wick products and S-transforms

In this section we will introduce one of the main tools of Article [B] and Article [C], the S-transform. Since our definition is a generalisation of the usual S-transform on a Gaussian Hilbert space, we will start by recalling some of the relevant Gaussian Hilbert space theory of S-transforms and related objects. The standard reference for this part of the discussion is the monograph [Jan97].

We start by recalling what is a *Gaussian linear space*: it is a linear space of real-valued random variables, defined on some common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that each variable in the space is a centred Gaussian. A Gaussian has all moments; hence this is an L^2 subspace. A *Gaussian Hilbert space* is a Gaussian linear space which is complete; that is, it is a closed subspace of $L^2(\Omega, \mathcal{F}, \mathbb{P})$, consisting of centred Gaussian random variables. It is a general fact that a Gaussian linear space can always be completed: if $G \subset L^2(\Omega)$ is a Gaussian linear space, then its closure $\overline{G}$ in $L^2(\Omega)$ is a Gaussian Hilbert space.

We next recall the basic Wiener-Itô chaos decomposition of $L^2(\Omega, \mathcal{F}(H), \mathbb{P})$. Here H is assumed to be a Gaussian Hilbert space, and $\mathcal{F}(H)$ denotes the σ -algebra generated by H . For $n \geq 0$, let $\mathcal{P}_n(H)$ be the closure in $L^2(\Omega, \mathcal{F}(H), \mathbb{P})$ of the linear space

$$\mathcal{P}_n(H) = \{p(\xi_1, \dots, \xi_m) : p \text{ is a polynomial of degree } \leq n; \xi_1, \dots, \xi_m \in H; m < \infty\}.$$

Setting $\mathcal{H}^n := \overline{\mathcal{P}_n(H)} \cap \overline{\mathcal{P}_{n-1}(H)}^\perp$, with $\mathcal{H}^0 = \overline{\mathcal{P}_0(H)}$, we have the orthogonal decomposition of L^2 , called the Wiener-Itô chaos decomposition:

$$\bigoplus_{n=0}^{\infty} \mathcal{H}^n = L^2(\Omega, \mathcal{F}(H), \mathbb{P}).$$

Via the Wiener-Itô chaos decomposition we will define some of the typical objects needed in the sequel. We start from the Wick product and the Wick powers:

Definition 3.2.9 (Wick product). Let $Z \in \mathcal{H}^m, Y \in \mathcal{H}^n$. We define the *Wick product* of Z and Y by

$$Z \diamond Y = \pi_{m+n}(ZY),$$

where π_n denotes the orthogonal projection of $L^2(\Omega)$ onto $\mathcal{H}^n$. We extend $\diamond$ by bilinearity to a binary operator on the space of all elements of $L^2(\Omega)$ having finite chaos decomposition.

We define *Wick powers* of $Z^{\diamond n}$, $n = 0, 1, 2, \dots$ inductively as

$$\begin{cases} Z^{\diamond 0} &= 1, \\ Z^{\diamond n} &= Z \diamond Z^{\diamond(n-1)}, \quad n = 1, 2, \dots \end{cases}$$

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We note that in defining the Wick product, the finiteness assumption on the chaos decompositions is essential; L^p spaces in general are not closed with respect to the Wick product (see, for example [GHL⁺92, HY09]).

One of the more important objects in our analysis will be the Wick exponential, essentially an exponential of a Gaussian, which we already introduced in Definition 3.2.5, but now extend to a general Hilbert setting:

Definition 3.2.10 (Wick exponential). Let H be a Gaussian Hilbert space, and let $Z \in H$. We define the *Wick exponential* of Z by

$$e^{\diamond Z} := e^{Z - \mathbb{E}[Z^2/2]} = \sum_{n=0}^{\infty} \frac{Z^{\diamond n}}{n!}, \quad (3.2.5)$$

where the convergence is in $L^2(\Omega)$.

We note that a very useful property of Wick exponentials $e^{\diamond Z}$, $Z \in H$, is that their span is a dense subspace of $L^2(\Omega, \mathcal{F}(H), \mathbb{P})$.

We now define the S-transform for $Z \in L^2(\Omega, \mathcal{F}(H), \mathbb{P})$ as the weighted expectation of Z with respect to the weight given by a Wick exponential. More specifically:

Definition 3.2.11 (S-transform). Let $Z \in L^2(\Omega, \mathcal{F}(H), \mathbb{P})$. Then the S-transform $(SZ) : H \mapsto \mathbb{R}$ is defined through

$$(SZ)(h) = \mathbb{E} \left[Z e^{\diamond h} \right], \quad h \in H. \quad (3.2.6)$$

We noted above that the Wick exponentials are dense in L^2 – from this we infer that the S-transform is injective on L^2 . That is, if $(SZ_1)(h) = (SZ_2)(h)$ for all $h \in H$ (or for all h from a dense subset of H), then $Z_1 = Z_2$ almost surely.

Example 3.2.12. Let $Z, Y \in H$. Then

$$(S e^{\diamond Z})(Y) = \mathbb{E} \left[e^{\diamond Z} e^{\diamond Y} \right] = e^{\mathbb{E}[ZY]}.$$

In fact, if $Z \in \mathcal{H}^m$, $Y \in \mathcal{H}^n$, then it can be shown that we have

$$(S(Z \diamond Y))(h) = (SZ)(h) \cdot (SY)(h), \quad (3.2.7)$$

for all $h \in H$. These examples show that the S-transform is arguably the natural (integral) transform to consider when dealing with the Wick exponential or the Wick product. That is, Wick multiplication translates to the usual scalar product on the S-transform side.

As we remarked above, L^p spaces are not closed with respect to the Wick product in general. However, by using the S-transform, we may extend the definition of the Wick product in the L^p -setting:

Definition 3.2.13. Let H be a Gaussian Hilbert space. Let $Z, Y \in \bigcup_{q \geq 1} L^q(\Omega)$, and $p > 1$. We say that the Wick product of Z and Y exists in $L^p(\Omega, \mathcal{F}(H), \mathbb{P})$ if there exists an element $V \in L^p(\Omega, \mathcal{F}(H), \mathbb{P})$ such that

$$(SZ)(h) \cdot (SY)(h) = (SV)(h),$$

for all $h \in H$, and we denote V by $Z \diamond Y$.

We remark that from the injectivity of the S-transform we see that the definition of $Z \diamond Y$ is unique, and, together with equation (3.2.7), it follows that Definition 3.2.13 indeed extends that of Definition 3.2.9.

We now move on to the more general setting of dealing with a Gaussian space generated by the log-correlated field, or more precisely, by the Gaussian multiplicative chaos measure. Above it was possible to use the S-transform to extend the definition of the Wick product; this possibility hinges on the fact that we are dealing with L^p objects¹. For the log-correlated field, it is not possible to make sense of any of the pointwise values $X_\beta(z)$, $z \in \mathbb{T}^d$. Thus we want to find a suitable definition for the S-transform which mimics the ‘classical’ definition in (3.2.6). We remark that we are going to state several results without justifications in the sequel; these can be found in Article [B].

We start by defining the Gaussian Hilbert space analogue in the present case. We recall that by X_β we denote βX , and $d\mu_\beta$ is a shorthand for the Gaussian multiplicative chaos (GMC) measure, formally given in (3.2.3).

Definition 3.2.14 ($L^2_\beta(\Omega)$ space generated by GMC). Let $\beta \in (0, \sqrt{d})$. The space $L^2_\beta(\Omega) := L^2(\Omega, \mathcal{F}_{d\mu_\beta}, \mathbb{P})$ generated by the GMC measure $d\mu_\beta$ is the

¹A more accurate statement is that it is possible to extend the definition of Wick products beyond the L^p setting, to Kondratiev distribution spaces; for details on these stochastic distribution spaces, see the Appendix in Article [B], or [KLS96].

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linear space spanned by random variables of form

$$\int_{\mathbb{T}^d} \varphi(z) d\mu_\beta(z), \quad \varphi \in C^\infty(\mathbb{T}^d) \quad (3.2.8)$$

and closed in $L^2(\Omega)$.

Example 3.2.15. In order to better understand the space $L_\beta^2(\Omega)$, and motivate the definitions that follow, let us compute formally the covariance between two elements $Z_1 = \int_{\mathbb{T}^d} \varphi_1(z) d\mu_\beta(z)$ and $Z_2 = \int_{\mathbb{T}^d} \varphi_2(z) d\mu_\beta(z)$. Let $R(y, z)$ denote the covariance between $X(y)$ and $X(z)$. We obtain

$$\begin{aligned} \mathbb{E}[Z_1 Z_2] &= \mathbb{E} \left[\int_{\mathbb{T}^d} \varphi_1(z) d\mu_\beta(z) \int_{\mathbb{T}^d} \varphi_2(z) d\mu_\beta(z) \right] \\ &= \mathbb{E} \left[\int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \varphi_1(z) \varphi_2(y) e^{\diamond X_\beta(z)} e^{\diamond X_\beta(y)} dy dz \right] \\ &= \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \varphi_1(z) \varphi_2(y) e^{\beta^2 R(z, y)} dy dz \\ &= \int_{\mathbb{T}^d} \left(\int_{\mathbb{T}^d} \varphi_1(z) e^{\beta^2 R(z, y)} \right) \varphi_2(y) dy \\ &= \langle \overline{\varphi_1}(z), \varphi_2(y) \rangle_{L^2(\mathbb{T}^d)}, \end{aligned}$$

where we denoted $\overline{\varphi_1}(z) := \left(\int_{\mathbb{T}^d} \varphi_1(z) e^{\beta^2 R(z, y)} \right)$. ♣

Motivated by the previous computation, and with a view towards defining the S-transform at $X_\beta(s)$, we introduce the operator G by setting

$$\begin{aligned} G\phi(z) &= G_{\beta^2}\phi(z) := \int_U \exp(\beta^2 \mathbb{E}[X(z)X(y)]) \phi(y) dy \\ &= \int_U |z - y|^{-\beta^2} e^{\beta^2 g(z, y)} \phi(y) dy, \end{aligned} \quad (3.2.9)$$

where $\beta \in (0, \sqrt{d})$. Note that if $g \equiv 0$ in (3.2.1), then the operator G reduces to the Riesz potential on $\mathbb{T}^d$.

Under some technical assumptions on the log-correlated field X_β , we show that

$$G : W^{-\frac{d-\beta^2}{2}, 2}(\mathbb{T}^d) \rightarrow W^{\frac{d-\beta^2}{2}, 2}(\mathbb{T}^d).$$

is a bijective map. As a consequence we obtain the following result, which characterises random variables in $L_\beta^2(\Omega)$.

Corollary 3.2.16. *Let $\beta \in (0, \sqrt{d})$. Suppose that the log-correlated field X is a β -non-degenerate² and let $Z \in L^2_\beta(\Omega)$. Then Z has the form*

$$Z = \int_{\mathbb{T}^d} \varphi(z) d\mu_\beta(z),$$

for some $\varphi \in W_0^{-s_\beta, 2}(\mathbb{T}^d)$, where $s_\beta = \frac{d-\beta^2}{2}$.

We now define the S-transform of L^2_β random variables at $X_\beta(z)$. Indeed, observe first that if $V \in L^2_\beta(\Omega)$, that is, $V = \int_{\mathbb{T}^d} \psi(z) d\mu_\beta(z)$, where $\psi \in W_0^{-s_\beta, 2}(\mathbb{T}^d)$ with $s_\beta = \frac{d-\beta^2}{2}$, then a formal computation yields that

$$ZV = \int_{\mathbb{T}^d} \psi(z) Z e^{\diamond X_\beta(z)} dz$$

for any $Z \in L^2(\Omega)$. Moreover, taking expectation gives, under the formal identification $(SZ)(X_\beta(z)) = \mathbb{E} \left[Z e^{\diamond X_\beta(z)} \right]$, that

$$\mathbb{E}(ZV) = \int_{\mathbb{T}^d} \psi(z) (SZ)(X_\beta(z)) dz.$$

We infer that the mapping $\psi \mapsto \mathbb{E}(ZV)$ is a continuous functional on $W^{-s_\beta, 2}(\mathbb{T}^d)$. By duality, there exists $\zeta \in W^{s_\beta, 2}(\mathbb{T}^d)$ such that

$$\mathbb{E}(ZV) = \int_{\mathbb{T}^d} \psi(z) \zeta(z) dz.$$

This leads to the following definition:

Definition 3.2.17 (S-transform in $L^2_\beta(\Omega)$). Let $Z \in L^2(\Omega)$ and set $s_\beta = \frac{d-\beta^2}{2}$. For $z \in \mathbb{T}^d$ we identify $(SZ)(X_\beta(z))$ as the (unique) element of $W^{s_\beta, 2}(\mathbb{T}^d)$ defined through the duality formula

$$\mathbb{E}(ZV) = \int_{\mathbb{T}^d} \psi(z) (SZ)(X_\beta(z)) dz$$

for all $V = \int_{\mathbb{T}^d} \psi(z) d\mu_\beta(z) \in L^2_\beta(\Omega)$.

We note that this definition makes sense for almost every $z \in \mathbb{T}^d$ which will be sufficient for our purposes.

²This is the aforementioned technical assumption on the log-correlated field X_β ; nothing to be too worried about at this point.

3.3 Stochastic PDEs

Stochastic partial differential equations, or SPDEs for short, have been a very widely studied topic in several sub-disciplines of mathematics during the past few decades. We will not even attempt to survey the impressive body of literature on the subject (nor the interesting connections of deterministic PDEs to SPDEs), but we aim to review some very elementary concepts related to SPDEs in order to situate our contributions to the subject matter at large in a satisfactory manner. Indeed, given that the field of PDEs is already well-populated by all kinds of equations and ways to understand their solutions, it should not come as a surprise that the inclusion of some random element into the mix results in a gargantuan amount of research literature encompassing pure stochastics and mathematical analysis, mathematical physics, mathematical finance, harmonic analysis, population biology, et cetera. For a general review on various aspects on SPDEs, we refer the reader to [LR15, LR17, Zam17, Kry99].

As is typical in PDE theory, there are many ways to classify the equations (linear vs non-linear; constant coefficient vs variable coefficient, elliptic vs parabolic vs ..., etc), but here we will mainly focus on how the random noise enters the picture. Broadly speaking, the noise can be additive or multiplicative (or a combination of both); for simplicity, here our examples are linear equations, but these can be easily generalised to the non-linear realm as well. Indeed, consider first the case of *additive noise*, in which case one often interprets the SPDE as an integral equation. For simplicity, take as an example a wave equation of the type

$$\frac{\partial^2 U}{\partial t^2}(x, t) = \sum_{i=1}^d \frac{\partial^2 U}{\partial x_i^2}(x, t) + \frac{\partial W}{\partial t}(x, t)$$

where W represents the random noise, in both time and space, and U is the unknown (generalised) function. The random noise W is typically non-differentiable, and hence the equation cannot be understood in a pointwise sense. Thus this approach is tightly connected to stochastic integration theory, as illustrated in [Wal86], but also to more modern functional analytic methods, and is historically among the oldest branches of the SPDE theory, its ideas dating back to the SDE work of Kiyosi Itô.

On the other hand, let us consider a general second-order linear partial differential operator L , given by

$$Lu(x) = \sum_{i,j=1}^d a_{ij}(x) \partial_{ij} u(x) + \sum_{i=1}^d b_i(x) \partial_i u(x) + c(x)u(x) + d(x), \quad (3.3.1)$$

where the coefficients a_{ij}, b_i, c and d are some given deterministic functions. Now, if the coefficients a_{ij}, b_i and c are allowed to also depend on the random parameter, then $Lu = 0$ defines an SPDE, where the noise is *multiplicative*. As was the case in the additive noise, most of the interesting coefficients one might consider in applications have rather weak regularity properties. This lack of regularity rules out the standard PDE methods in order to establish the existence of solutions, for example. It is precisely operators of the form (3.3.1) we will investigate next, where formally the ellipticity constant of L is equal to $+\infty$. To make matters slightly easier to approach, we will start from the one-dimensional setting, namely that of Article [B].

3.3.1 One-dimensional considerations

In this section we illustrate some of the theoretical considerations for the one-dimensional problem

$$\begin{cases} -e^{\mathbb{E}[X_\beta(t)^2]} (e^{\diamond(-X_\beta)} \cdot U')' = f & t \in (0, T) \\ \text{boundary data for } U. \end{cases} \quad (3.3.2)$$

Above $X_\beta = \beta X$ is the log-correlated Gaussian field X with a scaling parameter $\beta < 1$ and $f: [0, T] \rightarrow \mathbb{R}$ is a deterministic, integrable function. This explicitly one-dimensional problem has been considered in the literature with X_β taken to be the (smoothed) white noise, see [LØU91, HLØ⁺95]. In [LØU91] the authors show, using the Hermite transform (which is closely related to S-transform defined in Section 3.2.3) that the solution to the problem exists as a *functional process*, that is, a generalised function-valued process. Our results for this problem can be found Article [B], see Section 4.2 for a summary.

We note that in problem (3.3.2) the symbol $\cdot$ denotes the usual scalar product on $\mathbb{R}$; one can also consider a version of the problem where this scalar product is replaced with the Wick product $\diamond$, as defined in Definition 3.2.13

above. This has been done for example in [HØUZ09] when the field X is the white noise, where the authors solve the problem by formal Wick calculus; we consider the similar problem for the log-correlated field in Article [B].

As X_β is a generalised random field with undefined variance, our approach is to proceed via a convolution approximation to obtain a smooth random function. In particular, as per Lemma 3.2.7, we may take the approximation Z_β of X_β so that the variance of $Z_\beta(t)$ is independent of t ; this implies that $\exp(\mathbb{E}[Z_\beta(t)^2]) = \exp(\mathbb{E}[Z_\beta^2])$ for all $t \in (0, 2\pi)$. We illustrate the necessity of the inclusion of the $\exp(-\mathbb{E}[Z_\beta^2])$ term in (3.3.2) for the solution to be well-defined with the following computation. A computation in the same vein can be found in [HØUZ09].

Example 3.3.1. Consider solving formally

$$-(e^{\diamond(-Z(t))} \cdot U'(t))' = f(t), \quad t \in (0, T) \quad (3.3.3)$$

coupled with some deterministic boundary data, where Z is a nice, centred Gaussian field; in particular we assume that $\exp(\mathbb{E}[Z(t)^2]) = \exp(\mathbb{E}[Z^2])$ holds for all $t \in (0, T)$.

Since everything is assumed to be smooth, direct integration yields

$$U'(t) = \left(e^{\diamond(-Z(t))}\right)^{-1} (h(t) + C_1), \quad t \in (0, T)$$

where C_1 is a random constant, and we denote $h(t) := -\int_0^t f(s) ds$. From the definition of the Wick exponential in Definition 3.2.5 it is straightforward to compute that

$$\left(e^{\diamond(-Z(t))}\right)^{-1} = e^{\diamond Z(t)} e^{\mathbb{E} Z^2}.$$

Here we made use of the assumption that the variance $\mathbb{E}[Z^2]$ is independent of t . Therefore, integrating once more, we obtain

$$U(t) = C_2 + e^{\mathbb{E} Z^2} \int_0^t (h(s) + C_1) e^{\diamond Z(s)} ds, \quad t \in (0, T) \quad (3.3.4)$$

where C_1, C_2 are constants. Now, when Z is nice enough, $s \mapsto e^{\diamond Z(s)} ds$ can be viewed as a well-defined (random) measure, given that the Wick exponential $e^{\diamond Z}$ is essentially an exponential of Z . However, the idea that (3.3.4) defines a well-defined solution hinges on the fact that Z is a random field with finite

variance; if this is not satisfied, we need to introduce a suitable scaling to the equation, and that is precisely what we have done in the left-hand-side of (3.3.2). ♣

As the above computation in Example 3.3.1 shows, we need to be careful when discussing in what sense we consider the problem (3.3.2), and also what do we mean when we say that some mathematical object is its solution. On the other hand, we point out that the one-dimensional computation in Example 3.3.1 shows also that provided we can somehow make sense of $e^{\diamond Z}$, the field Z could be any (Gaussian) random field; put differently, there is nothing a priori ‘Gaussian’ in (3.3.3). A similar observation is also made by the authors in [HØUZ09]. We will return to this point of renormalisation in the next section, where we consider what is essentially the (d -dimensional) version of (3.3.2), with the Wick product $\diamond$ replacing the usual scalar product $\cdot$ of $\mathbb{R}$.

3.3.2 D-dimensional case

Let $V \subset \mathbb{R}^d$. In this section we will consider the questions of solvability and representation of solutions to a second-order divergence-form stochastic PDE of the type

$$\begin{cases} -\nabla \cdot (K \diamond \nabla U) = f, & \text{in } V \\ \text{boundary conditions on } V \end{cases} \quad (3.3.5)$$

where f is some nice function on V , and $K \geq 0$ is a random (generalised) field on V ; in our case, we will take K to be equal to $e^{\diamond(-X_\beta)}$, with $X_\beta = \beta X$ being a log-correlated Gaussian field with scaling parameter $\beta < \sqrt{d}$; see Definition 3.2.1.

In the literature, equation (3.3.5) is referred to as a (stochastic) pressure equation, (see [HLØ⁺95]); under this physical interpretation, when fluid is injected into the region U at a density rate f , the pressure u of the fluid satisfies the problem (3.3.5). Thus K represents the permeability (or porosity) of the medium, and in many applications, such as water flow in porous rocks, the permeability is a rapidly oscillating object to such a high degree that K actually cannot be a function. This gives rise to the Wick product $\diamond$ in (3.3.5) – its role is to counterbalance, that is, to ‘re-normalise’ the wildly oscillating

behaviour of K so that there exists a well-defined solution to (3.3.5).³ We remark that this idea of using the Wick product as essentially a regularisation tool is hardly anything new; it dates back to the work of Kiyosi Itô, see, for instance [Itô44]. In the one-dimensional case (without the Wick product), we offered some further heuristics by formally computing the explicit solution in Example 3.3.1 above.

Typically (3.3.5) is not directly amenable to standard Hilbert space methods (e.g. the Lax–Milgram theorem) in PDE theory, given the lack of uniform ellipticity of the bilinear form induced by the equation (see, however [Våg96]). The problem (3.3.5) has been widely studied in the case of K being the Wick exponential of white noise, in the context of White noise analysis. In [HLØ⁺95] the authors showed that under Dirichlet boundary conditions, the problem admits a solution in the Hida-Kondratiev stochastic distribution space $(\mathcal{S})^{-1}$ (see [KLS96]) under the assumption that f is a Hölder continuous function (see also [LØU91] for a one-dimensional example). They also obtained a representation formula for the solution, and we note that while the authors use more standard methods of stochastic analysis such as Itô integration theory involving auxiliary Brownian motion processes, they also employ the Hermite transform (see [HLØ⁺93]), which is closely related to the S-transform defined in Section 3.2.3 above. For an application of S-transform in solving problems of the type (3.3.5) in stochastic distribution spaces, see [Pot92]. A possible alternative (or a complement) to pure white noise analysis methods is to consider suitable weighted (Hilbert) spaces: that is, seek solutions U such that $\int_V \|u(x, \omega)\|_F^2 \xi(\omega) d\gamma(\omega)$ is finite, where F is a Banach space. Here γ denotes the Gaussian measure and ξ is a positive function, designed to counterbalance the lack of uniform ellipticity of the induced bilinear form. For details on this type of approach, in addition to some numerical considerations, see [LRW10] and the references therein. The case of the ordinary scalar product with log-normal noise is treated in [GS12]. Finally, we remark that a time-dependent variant of (3.3.5) was considered in [Øks90], with a so-called moving boundary condition. Under the assumption that K

³An alternative viewpoint to the matter is given by the authors in [HLØ⁺95], where they note that if $\xi, \eta \in (\mathcal{S})^{-1}$, the Hida–Kondratiev stochastic distribution space (see, for instance [KLS96]), then $\xi \diamond \eta \in (\mathcal{S})^{-1}$; hence the Wick product $\diamond$ can be viewed as the natural product on $(\mathcal{S})^{-1}$. In other words, if K belongs to $(\mathcal{S})^{-1}$, then $\diamond$ is the most convenient product to insert into (3.3.5) to make the problem a priori mathematically tractable in $(\mathcal{S})^{-1}$.

is a Muckenhoupt A_2 weight (cf. Section 3.1.2), the author established that the problem is solvable in the Sobolev space H_0^1 . We note that this was done using the more traditional arguments of stochastic analysis, that is, stopping times, Markov properties and so on, in contrast to our methods in Article [C] which have more of a PDE flavour to them.

We now move on to discuss solving (3.3.5) with periodic boundary conditions; here the periodic boundary conditions are chosen mainly to make the problem mathematically more tractable. The discussion of precise details is relegated to Article [C]; here we will discuss heuristics instead. Our approach is grounded in a particular form of the S-transform, introduced in Section 3.2.3. We will start by motivating our definition for the solution and illustrating some of the challenges in our approach. We consider formally the stochastic PDE

$$\begin{aligned} -\nabla \cdot (e^{\diamond(-X_\beta)} \diamond \nabla U) &= \nabla \cdot (e^{\diamond(-X_\beta)} \diamond F), \quad \text{in } \mathbb{T}^d \\ y \mapsto U(y) &\text{ is } \mathbb{T}^d\text{-periodic,} \end{aligned} \quad (3.3.6)$$

where X_β is the log-correlated Gaussian field with scaling parameter $\beta < \sqrt{d}$, and F is a random function.

Now fix $y \in \mathbb{T}^d$, and take the S-transform of (3.3.6) on Gaussian Z , which does not depend on y . Hence the equation becomes, using Fubini and $S(X \diamond Y)(z) = (SX)(z) \cdot (SY)(z)$,

$$-\nabla \cdot (Se^{\diamond(-X_\beta(y))}(Z) \cdot (S\nabla U(y))(Z)) = \nabla \cdot ((Se^{\diamond(-X_\beta(y))})(Z) \cdot (SF(y))(Z)) \quad (3.3.7)$$

Now, recall that since

$$e^{\diamond(X+Y)} = e^{-\mathbb{E}XY} e^{\diamond X} e^{\diamond Y},$$

and therefore $(Se^{\diamond(-X_\beta(y))})(Z) = e^{-\mathbb{E}[X_\beta(y)Z]}$, so that (3.3.7) becomes

$$-\nabla \cdot (e^{-\mathbb{E}[X_\beta(y)Z]} (S\nabla U(y))(Z)) = \nabla \cdot (e^{-\mathbb{E}[X_\beta(y)Z]} (SF(y))(Z)).$$

Now, choosing $Z = X_\beta(z)$, for fixed $z \in \mathbb{T}^d$, this becomes

$$-\nabla \cdot (e^{-\beta^2 R(y;z)} \nabla u(y;z)) = \nabla \cdot (e^{-\beta^2 R(y;z)} f(y;z)), \quad (3.3.8)$$

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where we have denoted $R(y; z) := \mathbb{E}[X(y)X(z)]$, and $u(y; z) = (S\nabla U(y))(X_\beta(z)) = \mathbb{E}[\nabla U(y)e^{\odot X_\beta(z)}]$, which is well-defined by Definition 3.2.17, and $f(y; z) = (SF(y))(X_\beta(z))$.

Hence we see that the problem in (3.3.6) has a deterministic counterpart on the S -transform side, parametrised by z , given by

$$\begin{aligned} -\nabla \cdot (w(y; z) \nabla u(y; z)) &= \nabla \cdot (w(y; z) f(y; z)), \quad \text{in } \mathbb{T}^d \\ y \mapsto u(y; z) &\text{ is } \mathbb{T}^d\text{-periodic,} \end{aligned} \tag{3.3.9}$$

where $w(y; z) = |y - z|^{\beta^2} e^{-\beta^2 g(y-z)}$, with g smooth and bounded. To borrow a phrasing from [LRW10], in statistical terms we would say that a solution of (3.3.6) is an *unbiased* random perturbation of the solution of (3.3.9).

We remark that in Article [C] we show that the solution to the problem (3.3.9) can be ‘inverse S -transformed’ back. For this, we need as much regularity in the z variable of $u(y; z)$ as possible. As it is a general fact that a solution of an elliptic equation inherits its regularity from the coefficients of the equation (and the domain), here the challenges within $w(y; z) = |y - z|^{\beta^2} e^{-\beta^2 g(y-z)}$ become more apparent. In a general domain, no matter how smooth, as we go closer to the boundary, the m th-order derivative of w , due to $|y - z|^{\beta^2}$, will give insufficient integrability for the Sobolev embedding to smooth functions to work. To rectify the issue, we postpone solving the problem in general smooth domains to further endeavours, and here content ourselves with solving the periodic problem in (3.3.6).

Remark 3.3.2. In the above formal computation, we chose $Z = X_\beta(z)$, for $z \in \mathbb{T}^d$. As X_β is not even a function, this pointwise value is not well-defined. To make sense of this, one might want to consider the distributional pairing $\int \psi(y, z) X_\beta(y) dy$, where ψ is a test function – but for this to yield $X_\beta(z)$ we cannot expect ψ to be even a function, but the Dirac measure at z . One way forward is to consider $Z_\varepsilon = \int \varphi_\varepsilon(y, z) X_\beta(y) dy$, where $y \mapsto \varphi_\varepsilon(y, z)$ is a smooth approximation to the Dirac delta (e.g. the periodic heat kernel on $\mathbb{R}^d$) at z . Note that as $\varepsilon \rightarrow 0$, then formally $\int \varphi_\varepsilon(y, z) X_\beta(y) dy \rightarrow X_\beta(z)$ weakly. Obviously with this approximation the corresponding approximated covariance in (3.3.8) is slightly different, but again formally at the limit as $\varepsilon \rightarrow 0$ we get precisely (3.3.9). Moreover, this approximated version of the problem in (3.3.9) is parametrised by $z \mapsto \varphi(y, z)$, where φ is any smooth approxi-

mation of the Dirac delta. In addition, observe that for (3.3.9) in view of φ_ε being a smooth approximation to the Dirac delta, this implies that we ought to understand the z -parametrisation as being concentrated around z .

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4.1 Article [A]

In this section we will summarise the key results of Article [A]. In the case of real-valued stochastic processes $X = (X_t)_{t \in [0,1]}$, we assume that the increments of X satisfy for all $p \geq 1$

$$(\mathbb{E} |X_t - X_s|^p)^{1/p} \leq C_0 p^\iota \left[\mathbb{E} |X_t - X_s|^2 \right]^{\frac{1}{2}}, \quad (4.1.1)$$

for all $s, t \in [0, 1]$. Here $C_0 > 0$ is a generic fixed constant and $\iota \geq 0$ is a given parameter. We remark that here the unit interval is chosen out of notational convenience; any compact subset of $\mathbb{R}$ would do.

In Article [A] we establish modulus of continuity-type results. As a consequence, under the additional assumption that $\mathbb{E}(X_t - X_s)^2 \leq |t - s|^{2\alpha}$, for $\alpha \in (0, 1]$, we show that for any interval $I \subset [0, 1]$ and any $s \in I$ we have

$$\mathbb{P} \left(\sup_{t \in I} |X_t - X_s| \geq u |I|^\alpha + C_d e^{-\alpha \iota} \iota^\iota \right) \leq C(\beta_0) e^{-\beta_0 u^{1/\iota}},$$

where $C(\beta_0)$ is a finite quantity for sufficiently small β_0 . We note that a similar result for sub-Nth chaos processes, Theorem 2.2.5, was obtained in [VV08].

Our main theorem is a Hölder continuity characterisation result for processes satisfying (4.1.1): X is Hölder continuous (with respect to the usual distance on $\mathbb{R}$) of any order $\gamma < \alpha$ if and only if for any $\epsilon > 0$ we have

$$\mathbb{E}(X_t - X_s)^2 \leq C_\epsilon |t - s|^{2\alpha - \epsilon}. \quad (4.1.2)$$

In addition, we show that the exponential moments of the Hölder coefficient $K^{1/\iota}$ are finite. This extends the results of [ASVY14] beyond Gaussianity. We note that the proof is fairly elementary, as it relies on the classical Paley–

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Zygmund inequality and a particular form of the Garsia–Rodemich–Rumsey (GRR) lemma introduced in Section 2.2.

We extend the the previous Hölder continuity result to fields $X = (X_t)_{t \in [0,1]^n}$ in a natural way, with the Euclidean norm $\|\cdot\|$ replacing the absolute value in (4.1.2); the proof also follows along analogous lines. However, our main focus in the case of fields is the rectangular increment $\square_y^n X(x)$ and the associated volume-like metric. In two dimensions, this is for $x = (x_1, x_2)$, $y = (y_1, y_2)$

$$\square_y^2 X(x) = X(x_1, x_2) - X(x_1, y_2) - X(y_1, x_2) + X(y_1, y_2).$$

Therefore in the multi-parameter case we assume that for all $p \geq 1$ we have

$$\mathbb{E} |\square_t^n X(s)|^p \leq C_0^p p^{p\iota} \left(\mathbb{E} |\square_t^n X(s)|^2 \right)^{p/2} \quad (4.1.3)$$

where $C_0 > 0$ is a generic fixed constant and $\iota \geq 0$ is a given parameter.

As in the case of processes, we show modulus of continuity- type results for fields, and characterise the joint Hölder continuity for fields satisfying (4.1.3). More specifically, we show that provided $X = (X_t)_{t \in [0,1]^n}$ satisfies (4.1.3), then for $\gamma = (\gamma_1, \dots, \gamma_n)$ we have

$$\forall \varepsilon = (\varepsilon_1, \dots, \varepsilon_n), \varepsilon_i > 0 : \quad |\square_t^n X(s)| \leq C(\omega) \prod_{j=1}^n |t_j - s_j|^{\gamma_j - \varepsilon_j},$$

if and only if

$$\forall \varepsilon = (\varepsilon_1, \dots, \varepsilon_n), \varepsilon_i > 0 : \quad \mathbb{E} |\square_t^n X(s)|^2 \leq C_\varepsilon \prod_{j=1}^n |t_j - s_j|^{2\gamma_j - \varepsilon_j}.$$

Similarly, as in the case of processes, we also show that the Hölder coefficient $K^{1/\iota}$ has finite exponential moments. The proof of the above result is essentially analogous to its one-dimensional counterpart, as it relies on an appropriate variant of the GRR lemma.

4.2 Article [B]

In this section we summarise our results concerning the existence and uniqueness of solutions to the one-dimensional boundary-value / initial value prob-

lem

$$\begin{cases} -\left(e^{\diamond(-X_\beta)} \times U'\right)' = f, \\ \text{boundary data.} \end{cases} \quad (4.2.1)$$

Above f is a deterministic, integrable function, $\times$ is either the usual scalar product on $\mathbb{R}$ or the Wick product $\diamond$ introduced in Section 3.2.3, and $X_\beta := \beta X$, where X is a log-correlated Gaussian field as in Definition 3.2.1, with $\beta < 1$ a scaling parameter. For boundary data we may take Dirichlet, Neumann, periodic, or Cauchy data.

As X_β is a generalised random function, the Wick exponential $e^{\diamond(-X_\beta)}$ cannot be defined pointwise; hence we understand the solutions of problem (4.2.1) through an approximation approach. Indeed, we convolve X_β with a smooth function to obtain a smooth random function $X_{\beta,\varepsilon}$, $\varepsilon > 0$ and consider the mollified, renormalised problem

$$\begin{cases} -C_\varepsilon \left(e^{\diamond(-X_{\beta,\varepsilon})} \times U'_\varepsilon\right)' = f, \\ \text{boundary data,} \end{cases} \quad (4.2.2)$$

and define the solution of (4.2.1) as the limit (in a suitable sense) of solutions U_ε to these approximated equations (4.2.2). The success of this procedure hinges on a reasonable choice of the normalisations C_ε (cf. Example 3.3.1); for example, in the case of $\times$ being the usual scalar product, we take $C_\varepsilon = \exp(\mathbb{E}[X_{\beta,\varepsilon}^2])$ to obtain well-defined solutions at the limit (for the problem with the Wick product, it suffices to take $C_\varepsilon \equiv 1$). We also show that the solution U of problem (4.2.1) admits the integral representation

$$U(t) = K + \int_0^t (\kappa + h(s)) \times e^{\diamond X_\beta(s)} ds, \quad t \in (0, T) \quad (4.2.3)$$

where the constants K and κ , arising from the boundary conditions on U_ε , are deterministic and random, respectively. Moreover, we denote $h(s) := -\int_0^s f(x) dx$. Above $e^{\diamond X_\beta(s)} ds$ is interpreted as the Gaussian multiplicative chaos measure.

Although the representation formulas in (4.2.3) turn out to be similar (and in some cases of boundary data, exactly the same), the techniques to establish the validity of these formulas are rather different. For the non-Wick prob-

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lem, we proceed by mollifying the field in a suitable manner, insert proper renormalisation into the equation, and then show the requisite convergence in probability in the space of continuous functions on $(0, T)$; this part of the argument essentially follows from the existence results of the GMC measure $e^{\diamond X_{\beta}(s)} ds$ itself. In the case of the problem with Wick product, again we start by from a smooth approximation of the log-correlated field. In this case however, we understand the solution as the limit of the S-transformed approximation solutions. Here the main obstacle is the inverse of a Wick product, as this can be a very rough object. Via the abstract machinery of Kondratiev distribution spaces we show that the solution is well-defined; the price to pay is that only in some particular cases of boundary data (e.g. Neumann) the solution is actually a $L^2(\Omega)$ function.

We note that irrespective of whether we consider (4.2.1) equipped with the Wick or the scalar products, the representation formula of the solution sees this difference essentially only in the product between the integrand and the GMC measure in (4.2.3). Motivated by these similarities, we consider formal projections to a subspace of $L^2(\Omega)$ of the form

$$U(t) = \int_{(0,T)} \varphi(t, a) e^{\diamond X_{\beta'}(s)} ds, \quad t \in (0, T) \quad (4.2.4)$$

where $\beta' < 1$ and $\varphi \in W_0^{-\frac{1-\beta'}{2}, 2}([0, T])$, the dual of the fractional Sobolev (Hilbert) space $W^{\frac{1-\beta'}{2}, 2}([0, T])$. In order to investigate this connection more thoroughly, we introduce the S-transform at a point $X_{\beta'}(s)$ directly, without mollification. As a result, we are able to show that these S-transforms characterise the $L^2(\Omega)$ subspace $L_{\beta'}^2(\Omega)$, which is by definition the subspace generated by the GMC measure $\exp(\diamond X_{\beta'}(s)) ds$.

4.3 Article [C]

In this section we summarise our results concerning the existence and uniqueness of solutions to the d -dimensional variant of the problem in the previous

section, equipped with periodic boundary data:

$$\begin{aligned} -\nabla \cdot (e^{\diamond(-X_\beta)} \diamond \nabla U) &= \nabla \cdot (e^{\diamond(-X_\beta)} \diamond \mathbf{F}), \quad \text{in } \mathbb{T}^d \\ y \mapsto U(y) &\text{ is } \mathbb{T}^d\text{-periodic,} \end{aligned} \quad (4.3.1)$$

Here $\mathbf{F} = (F_1, \dots, F_d)$ is a random element with real-valued random variables as its components, $X_\beta := \beta X$, where X is a log-correlated Gaussian field on $\mathbb{T}^d$, with $0 < \beta < \sqrt{d}$ being a scaling parameter, and $e^{\diamond X}$ denotes the Wick exponential of X . In Article [C] we show that this problem admits a solution given by

$$U(y) = \int_{\mathbb{T}^d} \varphi(y; z) e^{\diamond X_\beta(z)} dz, \quad y \in \mathbb{T}^d \quad (4.3.2)$$

where $\varphi \in W_0^{-(d-\beta^2)/2,2}(\mathbb{T}^d)$, and we interpret $e^{\diamond X_\beta(z)} dz$ as the GMC measure on $\mathbb{T}^d$. Moreover, if we consider $L_\beta^2(\Omega)$ as the subspace of $L^2(\Omega)$ generated by the GMC measure $e^{\diamond X_\beta(z)} dz$, we have that U given by (4.3.2) is the unique solution of (4.3.1) in $L_\beta^2(\Omega)$.

In the previous section, we understood solutions to the SDE via a smooth approximation approach. In this case, we proceed directly with the S-transform, defined in a suitable way on $L_\beta^2(\Omega)$ to accommodate the fact that no pointwise values of $X_\beta(z)$ are well-defined. This leads us to consider the deterministic periodic problem (here we denote by $u(\cdot, z)$ the S-transform of u at $X_\beta(z)$)

$$\begin{aligned} -\nabla \cdot (w(y; z) \nabla u(y; z)) &= \nabla \cdot (w(y; z) \mathbf{f}(y; z)), \quad \text{in } \mathbb{T}^d \\ y \mapsto u(y; z) &\text{ is } \mathbb{T}^d\text{-periodic,} \end{aligned} \quad (4.3.3)$$

where $w(y; z) := |y - z|^{\beta^2} \exp(-\beta^2 g(y - z))$, and $\mathbf{f} = (f_1, \dots, f_d)$. We see that the variational formulation of (4.3.3) resembles the Riesz potential operator; hence in order to show existence of the solution (and the representation formula), we show that a suitable integral operator $G_\beta: W^{-(d-\beta^2)/2,2}(\mathbb{T}^d) \rightarrow W^{(d-\beta^2)/2,2}(\mathbb{T}^d)$ is invertible. For this, we need as much regularity as possible (with respect to the z -variable) of the solution of the deterministic, S-transformed equation. This is achieved, as long as the derivatives of $\mathbf{f}$ (with respect to both its variables) are sufficiently integrable. Hence, the ideas of degenerate elliptic PDEs are the chief tools at our disposal, in particular bounds

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on the periodic Green function, alongside standard Sobolev / Besov space embeddings. Furthermore, since the deterministic equation (4.3.3) admits a unique solution, the injectivity of the operator G_β gives immediately the uniqueness for (4.3.2).

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Necessary and sufficient conditions for continuity of hypercontractive processes and fields[☆]

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ABSTRACT

Sample path properties of random processes are an interesting and extensively studied topic, especially in the case of Gaussian processes. In this article, we study the continuity properties of hypercontractive fields, providing natural extensions for some known Gaussian results beyond Gaussianity. Our results apply to both random processes and random fields alike.

1. Introduction

Sample path regularity of random processes and fields is an important and extensively studied topic in the literature, and can be used as a tool, for example, to study convergence of certain statistical estimators or to study convergence of numerical schemes for stochastic partial differential equations. Indeed, various application areas arise from the fact that sharp modulus of continuity estimates can be translated to supremum tail bounds, providing sharp tail behaviour of the supremum.

The problem is particularly well-studied in the case of Gaussian processes and fields. One of the earliest results in this direction is a sufficient condition due to Fernique (1964) who provided a sufficient condition for the sample path continuity involving the increment metric $d_X(s, t) = [\mathbb{E}(X_s - X_t)^2]^{1/2}$. Later on, Dudley (1967, 1973) established a necessary condition for the continuity of a Gaussian process X by using a metric entropy. While in general Dudley's condition is not sufficient, in the case of stationary Gaussian processes it turns out to be necessary and sufficient. Finally, general necessary and sufficient conditions for general Gaussian processes X were obtained by Talagrand (1987), in terms of metric entropies. Finally, while Talagrand's necessary and sufficient condition is rather complicated, a simple necessary and sufficient condition for Hölder continuity of Gaussian processes in terms of increment metric $d_X(s, t) = [\mathbb{E}(X_s - X_t)^2]^{1/2}$ was obtained in Azmoodeh et al. (2014), where the authors proved that the celebrated general Kolmogorov–Chentsov criterion for Hölder continuity is both necessary and sufficient condition for Gaussian processes.

While the topic is widely studied in the Gaussian case, the literature on modulus of continuity beyond Gaussianity is more limited. One general approach to obtain modulus of continuity for processes is to use Garsia–Rodemich–Rumsey lemma (Garsia et al., 1970). A multiparameter version was provided in Hu and Le (2013) where the authors obtained modulus of continuity estimates and joint Hölder continuity of solutions to certain stochastic partial differential equations driven by Gaussian noise. Finally, we mention a

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closely related article (Viens and Vizcarra, 2007), where the authors studied the case of so-called sub- n th Gaussian processes or Wiener processes, where n is an arbitrary integer. Such processes arise, roughly speaking, from processes of form $H_n(X_t)$, where X is a Gaussian process and H_n is the n th order Hermite polynomial. More precisely, in Viens and Vizcarra (2007) the authors provided sufficient conditions for continuity and, in the spirit of Talagrand, estimates for the tail of the supremum via metric entropy by using Malliavin calculus which is well-suited to the setting of a Gaussian Wiener space.

In this article we study sample path continuity for general hypercontractive processes and fields. That is, we assume that the higher-order moments of the increments satisfy

$$\mathbb{E}|X_t - X_s|^p \leq C(p) [\mathbb{E}|X_t - X_s|^2]^{p/2},$$

which allows us to deduce conditions in terms of the simple increment metric $\mathbb{E}|X_t - X_s|^2$. We show how the growth of the constant $C(p)$ translates into certain exponential moment bounds and tail estimates for the supremum. As a corollary, we extend the necessary and sufficient Kolmogorov–Chentsov criterion for the Hölder continuity beyond the Gaussian case studied in Azmoodeh et al. (2014). It is worth emphasising that, while our results can be used to cover and extend the known results on the Gaussian case (Azmoodeh et al., 2014) and in the case of sub- n th processes (Viens and Vizcarra, 2007), our results require only hypercontractivity and thus can be used in a more general framework where tools such as Malliavin calculus are not available. In our setting, our moment growth condition, Assumption 2.1, corresponds to the sub-Weibull distributions (see Vladimirova et al. (2020)) which generalise the widely studied and applied sub-Gaussian and sub-exponential distributions. Finally, our results extend naturally to random fields, and can be used to study joint Hölder continuity, akin to Hu and Le (2013), for instance.

The rest of the article is organised as follows. In Section 2 we introduce our notation and main results, while all the proofs and auxiliary lemmas are postponed to the supplementary material.

2. Necessary and sufficient conditions for continuity of hypercontractive processes and fields

We consider stochastic processes and fields, respectively, given by $X = (X_t)_{t \in K}$, where $K = [0, 1]$ or $K = [0, 1]^n$. We make use of the following hypercontractivity assumption:

Assumption 2.1. We suppose that for all $p \geq 1$ we have

$$\mathbb{E}|X_t - X_s|^p \leq C_0^p p^{\iota} [\mathbb{E}|X_t - X_s|^2]^{\frac{p}{2}}, \quad (2.1)$$

where $C_0 > 0$ is a generic fixed constant and $\iota \geq 0$ is a given parameter.

Remark 2.2. Condition (2.1) implies that the increment $X_t - X_s$ is a sub-Weibull random variable with parameter ι ; we recall that a random variable Z follows a sub-Weibull distribution with parameter $\iota \geq 0$ if there exists $c > 0$ such that $(\mathbb{E}[Z^p])^{1/p} \leq cp^\iota$ for all $p \geq 1$, see Vladimirova et al. (2020). This class is a generalisation of sub-Gaussian distributions ($\iota = 1/2$) and sub-exponential distributions ($\iota = 1$), and as such also contains random variables with bounded support. We note that a sub-exponentially distributed random variable is essentially a sub-Gaussian random variable squared. Both of these classes are studied extensively in the literature and have numerous applications; for details, we recommend the excellent textbook (Vershynin, 2018), and references therein. The sub-Weibull distribution has recently found widespread application within data science, for instance in dynamic factor models in econometrics (Barigozzi and Hallin, 2020), deep learning and stochastic optimisation (Wood and Dall’Anese, 2023), and in Bayesian neural networks (Vladimirova et al., 2020).

Remark 2.3. In general we could state hypercontractivity as

$$\mathbb{E}|X_t - X_s|^p \leq C(p) [\mathbb{E}|X_t - X_s|^2]^{\frac{p}{2}},$$

where $C(p)$ is a constant depending solely on p . We then could state our results by using a more general form of $C(p)$ with suitable growth in p , but this would result in an additional layer of notational complexity. For simplicity and to make the connection to sub-Weibull distributions more explicit, we restrict ourselves to the case $C(p) \leq C^p p^{\iota}$.

Remark 2.4. It is worth to note that our assumption implies the exponential moment assumption (up to an unimportant constant) of Viens and Vizcarra (2007), in the case $\iota = n$ is an integer, cf. proof of Theorem 2.9.

The following examples cover a large spectrum of situations that arise in mathematical modelling and should convince the reader that the class of processes satisfying Assumption 2.1 is large. The first three examples cover various processes arising naturally in Gaussian analysis, while Example 2.8 reveals that hypercontractivity can be achieved in many situations, whether or not one can apply Gaussian analysis and Malliavin calculus.

Example 2.5. If X is Gaussian, then it is well-known that

$$\mathbb{E}|X_t - X_s|^p = \frac{1}{\sqrt{\pi}} 2^{\frac{p}{2}} \Gamma\left(\frac{p+1}{2}\right) [\mathbb{E}|X_t - X_s|^2]^{\frac{p}{2}},$$

where Γ is the Euler Gamma function. By Stirling's approximation, we have

$$\Gamma(x+1) \sim \sqrt{2\pi x} e^{-x} x^x$$

for large x , and hence we may choose $\iota = \frac{1}{2}$ in Assumption 2.1.

Example 2.6. Let $\mathcal{H}$ be a separable, real Hilbert space, and $Z = \{Z(h) : h \in \mathcal{H}\}$ an isonormal Gaussian process on $\mathcal{H}$. We define the n th Hermite polynomial as $H_0(x) = 1$ and for $n \geq 1$ by

$$H_n(x) = (-1)^n e^{x^2/2} \frac{d}{dx} e^{-x^2/2}.$$

Denote by $\mathcal{H}_p$ the linear space generated by the class $\{H_p(Z(h)) : p \geq 0, h \in \mathcal{H}, \|h\|_{\mathcal{H}} = 1\}$. This linear space is called the p th Wiener chaos of Z . Then it is known (see, e.g. Nourdin and Peccati (2012)) that

$$[\mathbb{E}|F|^q]^{1/q} \leq [\mathbb{E}|F|^r]^{1/r} \leq \left(\frac{r-1}{q-1}\right)^{p/2} [\mathbb{E}|F|^q]^{1/q}.$$

Hence if X is a process living in a fixed Wiener chaos of order p (or in a finite linear combination of them with p as the highest chaos), we have (2.1) with $\iota = \frac{p}{2}$ and recover the processes studied in Viens and Vizcarra (2007). In particular, we recover the Gaussian case by setting $p = 1$.

Example 2.7. Solutions to several kinds of stochastic differential equations driven by Gaussian processes are expected to satisfy Assumption 2.1, while at the same these objects live in infinite amount of Wiener chaoses (that is, have non-finite chaos decompositions). For example, under certain technical conditions, solutions to certain stochastic differential equations driven by the fractional Brownian motion satisfies Assumption 2.1 with $\iota = \frac{1}{2}$, see Baudoin et al. (2016, Condition (ii) of Theorem 5.15, Proof of Theorem 5.16, and the references therein).

Example 2.8. Next, consider $Z_t = X_t Y_t$, where X and Y both satisfy Assumption 2.1 with parameters $C_{0,X}$ (resp. $C_{0,Y}$) and ι_X (resp. ι_Y), and for simplicity assume that both X and Y start at zero. Then it follows from a straightforward application of the Cauchy–Schwartz inequality that Z also satisfies Assumption 2.1 with $C_{0,Z} = 2^{\iota_X + \iota_Y} C_{0,X} C_{0,Y}$, and $\iota_Z = \iota_X + \iota_Y$. Moreover, a linear combination of processes satisfying Assumption 2.1 also satisfies the said assumption, see Vladimirova et al. (2020, Proposition 3.1). This covers, for instance, stochastic examples related to fractional splines, see Unser and Blu (2000); see also Monje et al. (2010), Sheng et al. (2011) for applications in signal processing.

We remark that these examples extend naturally to the case of random fields $X = (X_t)_{t \in [0,1]^n}$ by considering Gaussian fields (cf. Example 2.5), fields living in a fixed Wiener chaos (cf. Example 2.6), solutions to certain stochastic partial differential equation models as in Hu and Le (2013) (cf. Example 2.7), or spline smoothing in multiparameter setting (cf. Example 2.8).

We recall that if X is a stochastic process (resp. random field) on a probability space $(\Omega, \mathcal{F}, P)$ with an index set T , for any fixed random parameter $\omega \in \Omega$, the mapping $t \mapsto X_t(\omega) =: X_t$ defines the sample path. If $t \rightarrow t_0$ implies that $X_t \rightarrow X_{t_0}$ almost surely, we say that X is continuous at t_0 . As usual, when this property holds everywhere on the index set T , we simply say that X is a continuous stochastic process (resp. random field).

We begin with the following general result providing the modulus of continuity and certain moment estimates.

Theorem 2.9. Suppose that a continuous $X = (X_t)_{t \in [0,1]}$ satisfies Assumption 2.1 and suppose that $\mathbb{E}(X_t - X_s)^2 \leq \rho(|t - s|)^2$ for some non-decreasing, non-negative continuous function ρ , with $\rho(0) = 0$. Then

$$|X_t - X_s| \leq 8 \int_0^{|s-t|} \beta^{-t} \left(\log \left(\frac{4B}{u^2} \right) \right)^t d\rho(u) \quad (2.2)$$

for any $\beta > 0$, where

$$B = \int_0^1 \int_0^1 \exp \left(\beta \left(\frac{|X_t - X_s|}{\rho(|t - s|)} \right)^{\frac{1}{\iota}} \right) ds dt.$$

Moreover, for $\beta \in \left(0, \frac{e\iota}{C_0^{\frac{1}{\iota}}} \right)$, the random variable B has finite p -moments for all p satisfying

$$1 \leq p < \frac{e\iota}{\beta C_0^{\frac{1}{\iota}}}.$$

Remark 2.10. As in Viens and Vizcarra (2007), one could first study the existence of a continuous version in terms of metric entropies. That is, by considering the number N_ϵ of balls required to cover the interval $[0, T]$ with respect to the metric $d_X(s, t) = [\mathbb{E}(X_t - X_s)^2]^{1/2}$. Then one obtains continuity provided that

$$\int_0^\infty |\log N_\epsilon|' d\epsilon < \infty.$$

As we are interested in the modulus of continuity, we assume continuity a priori.

We obtain immediately the following corollaries.

Corollary 2.11. Suppose that $X = (X_t)_{t \in [0,1]}$ satisfies Assumption 2.1 and suppose that $\mathbb{E}(X_t - X_s)^2 \leq |t - s|^{2\alpha}$, for $\alpha \in (0, 1]$. Let $\beta \in \left(0, \frac{e\iota}{C_0^\frac{1}{\iota}}\right)$. Then there exists a random variable $C(\omega) = C(\beta, \omega)$ satisfying $\mathbb{E} \exp(\beta_0 C(\omega)^{1/\iota}) < \infty$ for any β_0 satisfying

$$\beta_0 < \frac{e\iota}{\left(8C_0 \cdot 3^{\max(\iota-1, 0)}\right)^{\frac{1}{\iota}}} \quad (2.3)$$

and a deterministic constant $C_d = C_d(\beta)$ such that

$$|X_t - X_s| \leq C(\omega)|t - s|^\alpha + C_d|t - s|^\alpha \left(\log \frac{1}{|t - s|}\right)^\iota. \quad (2.4)$$

In particular, we have

$$\limsup_{|t-s| \rightarrow 0} \frac{|X_t - X_s|}{|t - s|^\alpha \left(\log \frac{1}{|t-s|}\right)^\iota} \leq C_d. \quad (2.5)$$

Corollary 2.12. Suppose $X = (X_t)_{t \in [0,1]}$ satisfies Assumption 2.1 and assume that $\mathbb{E}(X_t - X_s)^2 \leq |t - s|^{2\alpha}$, for $\alpha \in (0, 1]$. Let $\beta \in \left(0, \frac{e\iota}{C_0^\frac{1}{\iota}}\right)$.

Then for any interval $I \subset [0, 1]$ and any $s \in I$ we have

$$P\left(\sup_{t \in I} |X_t - X_s| \geq u|I|^\alpha + C_d e^{-\alpha \iota^\iota}\right) \leq C(\beta_0) e^{-\beta_0 u^{1/\iota}},$$

where

$$C(\beta_0) = C(\beta_0, \beta) = \mathbb{E}\left(4B\beta_0\beta^{-1}\left(8 \cdot 3^{\max(\iota-1, 0)}\right)^{1/\iota}\right) < \infty$$

for any β_0 satisfying (2.3).

Remark 2.13. The above result is close in spirit to Viens and Vizcarra (2007, Theorem 3.1), and generalises naturally the well-known exponential decay of the supremum of Gaussian processes, in which case we would obtain

$$P(\sup_{t \in I} |X_t - X_s| \geq u|I|^\alpha + C_1) \leq C_2 e^{-C_3 u^2}$$

for constants C_1, C_2 , and C_3 .

As our final main theorem, we obtain the following characterisation of Hölder continuity: under Assumption 2.1, Kolmogorov continuity criterion is a necessary and sufficient condition for Hölder continuity. This extends the results of Azmoodeh et al. (2014) beyond Gaussian processes and covers, in particular, processes living in a finite sum of Wiener chaoses, cf. Example 2.6.

Theorem 2.14. Suppose that $X = (X_t)_{t \in [0,1]}$ satisfies Assumption 2.1. Then X is Hölder continuous of any order $\gamma < \alpha$, i.e. for any $\epsilon > 0$

$$|X_t - X_s| \leq C_\epsilon(\omega)|t - s|^{\alpha-\epsilon},$$

if and only if for any $\epsilon > 0$ we have

$$\mathbb{E}(X_t - X_s)^2 \leq C_\epsilon |t - s|^{2\alpha-\epsilon}. \quad (2.6)$$

Moreover, in this case the Hölder constant $C_\epsilon(\omega)$ of X satisfies

$$\mathbb{E} \exp\left(\beta C_\epsilon(\omega)^{\frac{1}{\iota}}\right) < \infty \quad (2.7)$$

for small enough $\beta > 0$ which depends only on C_0, α, ι , and ϵ .

Our results extend naturally to the case of random fields $X = (X_t)_{t \in [0,1]^n}$. We begin with the following result, which extends Theorem 2.14 to the case of fields in a natural way.

Proposition 2.15. Suppose that $X = (X_t)_{t \in [0,1]^n}$ satisfies Assumption 2.1. Then X is Hölder continuous of any order $\gamma < \alpha$, i.e. for any $\epsilon > 0$,

$$|X_t - X_s| \leq C_\epsilon(\omega)\|t - s\|^{\alpha-\epsilon},$$

if and only if for any $\epsilon > 0$ we have

$$\mathbb{E}(X_t - X_s)^2 \leq C_\epsilon \|t - s\|^{2\alpha-\epsilon}. \quad (2.8)$$

Moreover, in this case the Hölder constant of X satisfies

$$\mathbb{E} \exp \left(\beta C_\epsilon(\omega)^{\frac{1}{\iota}} \right) < \infty$$

for small enough $\beta > 0$ which depends only on C_0 , α , ι , and ϵ .

We next consider rectangular increments and joint continuity. We first introduce some notation, taken from Hu and Le (2013).

Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ be two elements in $\mathbb{R}^d$. For each integer $k = 1, 2, \dots, n$, we define

$$V_{k,y}x := (x_1, \dots, x_{k-1}, y_k, x_{k+1}, \dots, x_n).$$

Let f be a $\mathbb{R}^m$ -valued map on $\mathbb{R}^n$. We define the operator $V_{k,y}$ acting on f as follows:

$$V_{k,y}f(x) := f(V_{k,y}x).$$

It is simple to verify that $V_{k,y}V_{k,y}f(x) = V_{k,y}f(x)$ and that $V_{k,y}V_{l,y}f(x) = V_{l,y}V_{k,y}f(x)$ for any f .

Next, we define the joint (rectangular) increment of a function f on an n -dimensional rectangle,

$$\square_y^n f(x) = \prod_{k=1}^n (I - V_{k,y})f(x), \quad (2.9)$$

where I denotes the identity operator.

For a random field $X = (X_t)_{t \in [0,1]^n}$, let $d_X(t, s) := \sqrt{\mathbb{E}|\square_t^n X(s)|^2}$. We assume that the following condition, analogous to Assumption 2.1, is satisfied.

Assumption 2.16. We suppose that for all $p \geq 1$ we have

$$\mathbb{E}|\square_t^n X(s)|^p \leq C_0^p p^\iota d_X^\iota(t, s), \quad (2.10)$$

where $C_0 > 0$ is a generic fixed constant and $\iota \geq 0$ is a given parameter.

Theorem 2.17. Suppose that $X = (X_t)_{t \in [0,1]^n}$ is continuous and satisfies Assumption 2.16 and suppose that $d_X(t, s) \leq \prod_{j=1}^n \rho_j(|t_j - s_j|)$. Then

$$|\square_t^n X(s)| \leq 8^n \int_0^{|s_1 - t_1|} \dots \int_0^{|s_n - t_n|} \beta^{-\iota} \left(\log \left(\frac{4^n B}{u_1^2 \dots u_n^2} \right) \right)^\iota d\rho_1(u_1) \dots d\rho_n(u_n)$$

for any $\beta > 0$, where

$$B = \int_{[0,1]^n} \int_{[0,1]^n} \exp \left(\beta \left(\frac{|\square_t^n X(s)|}{\prod_{j=1}^n \rho_j(|t_j - s_j|)} \right)^{\frac{1}{\iota}} \right) ds dt.$$

Moreover, for $\beta \in \left(0, \frac{e\iota}{C_0^\iota} \right)$, the random variable B has finite p -moments for all p satisfying

$$1 \leq p < \frac{e\iota}{\beta C_0^{\frac{1}{\iota}}}.$$

The following corollary is analogous to Corollary 2.11, and extends some of the results in Hu and Le (2013) beyond Gaussianity.

Corollary 2.18. Suppose that $X = (X_t)_{t \in [0,1]^n}$ satisfies 2.16 and suppose that $d_X^2(t, s) \leq \prod_{j=1}^n |t_j - s_j|^{2\alpha_j}$. Let $\beta \in \left(0, \frac{e\iota}{C_0^\iota} \right)$. Then there exists a random variable $C(\omega) = C(\beta, \omega)$ satisfying $\mathbb{E} \exp(\beta_0 C^{1/\iota}(\omega)) < \infty$ for any β_0 satisfying

$$\beta_0 < \frac{e\iota}{(8^n C_0 \cdot 3^{\max(\iota-1, 0)})^{\frac{1}{\iota}}}, \quad (2.11)$$

and a deterministic constant $C_d = C_d(\beta)$, such that

$$|\square_t^n X(s)| \leq C(\omega) \prod_{j=1}^n |t_j - s_j|^{\alpha_j} + C_d \prod_{j=1}^n |t_j - s_j|^{\alpha_j} \left(\log \frac{1}{\prod_{j=1}^n |t_j - s_j|} \right)^\iota.$$

In particular, we have

$$\limsup_{\max_j |t_j - s_j| \rightarrow 0} \frac{|\square_t^n X(s)|}{\prod_{j=1}^n |t_j - s_j|^{\alpha_j} \left(\log \frac{1}{\prod_{j=1}^n |t_j - s_j|} \right)^\iota} \leq C_d.$$

Corollary 2.19. Suppose that $X = (X_t)_{t \in [0,1]^n}$ satisfies 2.16 and suppose that $d_X^2(t, s) \leq \prod_{j=1}^n |t_j - s_j|^{2\alpha_j}$. Let $\beta \in \left(0, \frac{e^t}{C_0^{\frac{1}{\alpha_j}}}\right)$. Then for any intervals $I_j \subset [0, 1]$ and any $s \in I = I_1 \times \cdots \times I_n$, we have

$$P\left(\sup_{t \in I} |\square_t^n X(s)| \geq u \prod_{j=1}^n |I_j|^{\alpha_j} + \tilde{C}\right) \leq C(\beta_0) e^{-\beta_0 u^{1/\alpha_j}},$$

where

$$C(\beta_0) = C(\beta_0, \beta) = \mathbb{E}\left[4B\beta_0\beta^{-1}(8.3^{\max(t-1,0)})^{1/\alpha_j}\right] < \infty$$

for any β_0 satisfying (2.11), and

$$\tilde{C} = C_d \max_{0 \leq x_j \leq 1} \prod_{j=1}^n |x_j|^{\alpha_j} \left(\log \frac{1}{\prod_{j=1}^n |x_j|}\right)^{\alpha_j}. \quad (2.12)$$

Similarly, Theorem 2.14 extends in a natural manner to the multiparameter case. Again, the proof is analogous to the proof of Theorem 2.14 and is thus left to the reader.

Theorem 2.20. Suppose that $X = (X_t)_{t \in [0,1]^n}$ satisfies Assumption 2.16. Then X is jointly Hölder continuous of any order $\gamma = (\gamma_1, \dots, \gamma_n)$ with $\gamma_j < \alpha_j$, i.e. for any $\epsilon = (\epsilon_1, \dots, \epsilon_n)$ with $\epsilon_i > 0$,

$$|\square_t^n X(s)| \leq C(\omega) \prod_{j=1}^n |t_j - s_j|^{\gamma_j - \epsilon_j},$$

if and only if for any $\epsilon = (\epsilon_1, \dots, \epsilon_n)$ with $\epsilon_i > 0$ we have

$$d_X^2(t, s) \leq C_\epsilon \prod_{j=1}^n |t_j - s_j|^{2\alpha_j - \epsilon_j}.$$

Moreover, in this case the Hölder constant of X satisfies

$$\mathbb{E} \exp\left(\beta C_\epsilon(\omega)^{\frac{1}{\alpha_j}}\right) < \infty$$

for small enough $\beta > 0$ which depends only on C_0 , α , t , and ϵ .

Data availability

No data was used for the research described in the article.

Appendix A. Supplementary data

Supplementary material related to this article can be found online at <https://doi.org/10.1016/j.spl.2024.110049>.

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[B]

1D stochastic pressure equation with log-correlated Gaussian coefficients

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1D STOCHASTIC PRESSURE EQUATION WITH LOG-CORRELATED GAUSSIAN COEFFICIENTS

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AND LAURI VIITASAARI

ABSTRACT. We study unique solvability for one dimensional stochastic pressure equation with diffusion coefficient given by the Wick exponential of log-correlated Gaussian fields. We prove well-posedness for Dirichlet, Neumann and periodic boundary data, and the initial value problem, covering the cases of both the Wick renormalization of the diffusion and of point-wise multiplication. We provide explicit representations for the solutions in both cases, characterized by the S -transform and the Gaussian multiplicative chaos measure.

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1. INTRODUCTION

We are studying existence and uniqueness of solutions U to the following second order linear stochastic boundary value problem on $[0, T] \subset \mathbb{R}$,

$$\begin{cases} -(e^{\diamond(-X_\beta(t))} \times U'(t))' &= f(t), & t \in (0, T), \\ \text{boundary data at} & \{0, T\}, \end{cases} \quad (1.1)$$

with deterministic $f \in L^1(0, T)$. The diffusion coefficient $e^{\diamond(-X_\beta)}$ is formally the Wick-exponential of a log-correlated scalar Gaussian random field X_β on $[0, T]$ with parameter $\beta \in (0, 1)$, and here the symbol $\times$ denotes either the usual point-wise product $\cdot$ in $\mathbb{R}$ or the Wick product $\diamond$. As now the diffusion coefficient $e^{\diamond(-X_\beta)}$ cannot be defined point-wise, we approximate X_β by its mollified version and define the solution to (1.1) as the limit (in a suitable sense) of the solutions of the approximate equations

$$\begin{cases} -C_\varepsilon(e^{\diamond(-X_{\beta,\varepsilon}(t))} \times U'_\varepsilon(t))' &= f(t), & t \in (0, T), \\ \text{boundary data at} & \{0, T\}, \end{cases}$$

where now C_ε is a suitably chosen normalization constant, depending on whether $\times$ corresponds to the usual product in which case $C_\varepsilon = e^{\mathbb{E}[X_{\beta,\varepsilon}^2]}$, or whether $\times$ corresponds to the Wick product in which case $C_\varepsilon \equiv 1$. We note that the normalization affects the boundary data, but is omitted here.

The problem is the one-dimensional stochastic pressure equation which can be used to model enhanced geothermal heating (the multidimensional case is tackled in a forthcoming paper [2]). The solutions to the stochastic pressure equation are steady state solutions to a creeping-water flow in porous media. The Wick exponential of X_β , i.e. $e^{\diamond(-X_\beta)}$ models the random permeability of the rock. For X_β replaced by white noise, the stochastic pressure equation has been studied with methods from white noise theory in [14–16, 25], see also [10, 31, 32] for the numerical analysis of these equations. In particular, the one dimensional problem with white noise has been discussed in [16, Subsection 4.6.3]. A common ingredient in showing existence and uniqueness to these types of equations is to employ some kind of integral transform (see, for instance [14, 27] and references therein).

Observing problem (1.1) closely, one finds that it can be written formally as the ordinary differential equation (ODE),

$$U'' + X'_\beta \times U' = -\frac{1}{e^{\diamond(-X_\beta)}} \times f,$$

which is generally ill-posed due to the lack of regularity for the realizations of X_β . Note that U is formally the stationary solution to the (1+1)-dimensional weighted random Kolmogorov equation with singular drift, compare with [9, 29],

$$-\frac{1}{e^{\diamond(-X_\beta)}} \times \partial_t u = \partial_x \partial_x u + \partial_x X_\beta \times \partial_x u + \frac{1}{e^{\diamond(-X_\beta)}} \times C f, \quad t > 0, \quad x \in (0, T).$$

The need of renormalization for stochastic partial differential equations of this type is already natural in one spatial dimension. See [4, 5, 12] for a systematic treatment of well-posedness and renormalization of singular stochastic PDEs.

For our stationary problem, we are treating both the point-wise product and the so-called Wick product, cf. [11, 20, 23]. See [16, Subsection 3.5] for a direct comparison in

the one-dimensional white noise case. From a certain asymptotic point of view, Wick multiplication corresponds to the Itô-stochastic integral and point-wise multiplication corresponds to the Stratonovich stochastic integral, see [18] for details. Wick products are the first order term in the so-called Wick-Malliavin expansion of the product of two random variables [30], so that in particular, Wick products preserve the expected value. We note that in the case of log-correlated fields, due to the lack of second moments, the white noise approach has to be transferred carefully using so-called Gaussian multiplicative chaos measures [21, 22].

As boundary data for (1.1), we are studying the cases of Dirichlet, Neumann and periodic boundary data, as well as the initial value problem (IVP). The different versions of boundary conditions and Wick vs. non-Wick for our problem are presented in the following Table 1. The corresponding explicit representations for the solutions are gathered in Table 2.

Dirichlet	Wick	Non-Wick
Equation	$-(e^{\diamond(-X_\beta)} \diamond U')' = f$	$-e^{\mathbb{E}[X_\beta^2]}(e^{\diamond(-X_\beta)} \cdot U')' = f$
Data	$U(0) = U_1,$ $U(T) = U_2$	$U(0) = U_1,$ $U(T) = U_2$
Neumann	Wick	Non-Wick
Equation	$-(e^{\diamond(-X_\beta)} \diamond U')' = f$	$-e^{\mathbb{E}[X_\beta^2]}(e^{\diamond(-X_\beta)} \cdot U')' = f$
Data	$e^{\diamond X_\beta(0)} \diamond U'(0) = U_1,$ $e^{\diamond X_\beta(T)} \diamond U'(T) = U_2$	$e^{\mathbb{E}[X_\beta^2]}e^{\diamond(-X_\beta(0))} \cdot U'(0) = U_1,$ $e^{\mathbb{E}[X_\beta^2]}e^{\diamond(-X_\beta(T))} \cdot U'(T) = U_2$
IVP	Wick	Non-Wick
Equation	$-(e^{\diamond(-X_\beta)} \diamond U')' = f$	$-e^{\mathbb{E}[X_\beta^2]}(e^{\diamond(-X_\beta)} \cdot U')' = f$
Data	$U(0) = U_1,$ $e^{\diamond X_\beta(0)} \diamond U'(0) = U_2$	$U(0) = U_1,$ $e^{\mathbb{E}[X_\beta^2]}e^{\diamond(-X_\beta(0))} \cdot U'(0) = U_2$
Periodic	Wick	Non-Wick
Equation	$-(e^{\diamond(-X_\beta)} \diamond U')' = f$	$-e^{\mathbb{E}[X_\beta^2]}(e^{\diamond(-X_\beta)} \cdot U')' = f$
Data	$U(0) = U(T),$ $e^{\diamond X_\beta(0)} \diamond U'(0) = e^{\diamond X_\beta(T)} \diamond U'(T)$	$U(0) = U(T),$ $e^{\diamond(-X_\beta(0))} \cdot U'(0) = e^{\diamond(-X_\beta(T))} \cdot U'(T)$

TABLE 1. Equations and corresponding boundary data to in Wick / non-Wick cases.

Note that in the solutions we have, with a slight abuse of notation, written $e^{\diamond X_\beta(s)}ds$ corresponding to the Gaussian multiplicative chaos measure $d\mu_\beta(s)$ (see Section 2.1 for details) which is not absolutely continuous. We obtain our solutions via an approximation argument. That is, we approximate the log-correlated field $X(s)$ and show that the solutions to the approximated equations converge, in a suitable sense. Actually, in Table 1 the normalization $e^{\mathbb{E}[X_\beta^2]}$ blows up in the limit, as the variance of the log-correlated field is unbounded, while it is not true along the approximating sequence. However, it turns out that this is the correct normalization to obtain a non-trivial solutions in the limit.

Solution	Dirichlet	Stochastic constant κ
Wick	$U(t) = U_1 + \int_0^t (\kappa + F(s)) \diamond e^{\diamond X_\beta(s)} ds$	$\left(U_2 - U_1 - \int_0^T F(s) e^{\diamond X_\beta(s)} ds \right) \diamond \left(\int_0^T e^{\diamond X_\beta(s)} ds \right)^{\diamond(-1)}$
Non-Wick	$U(t) = U_1 + \int_0^t (\kappa + F(s)) e^{\diamond X_\beta(s)} ds$	$\left(U_2 - U_1 - \int_0^T F(s) e^{\diamond X_\beta(s)} ds \right) \left(\int_0^T e^{\diamond X_\beta(s)} ds \right)^{-1}$
Solution	Neumann	Stochastic constant κ
Wick	$U(t) = \kappa + \int_0^t (U_1 + F(s)) e^{\diamond X(s)} ds$	Arbitrary, instead compatibility $U_2 - U_1 = - \int_0^T f(s) ds$
Non-Wick	$U(t) = \kappa + \int_0^t (U_1 + F(s)) e^{\diamond X(s)} ds$	Arbitrary, instead compatibility $U_2 - U_1 = - \int_0^T f(s) ds$
Solution	IVP	Stochastic constant κ
Wick	$U(t) = U_1 + \int_0^t (U_2 + F(s)) e^{\diamond X_\beta(s)} ds$	None
Non-Wick	$U(t) = U_1 + \int_0^t (U_2 + F(s)) e^{\diamond X_\beta(s)} ds$	None
Solution $U(t)$	Periodic	Stochastic constant κ
Wick	$C + \int_0^t (\kappa + F(s)) \diamond e^{\diamond X_\beta(s)} ds$	$- \int_0^T F(s) e^{\diamond X_\beta(s)} ds \left(\int_0^T e^{\diamond X_\beta(s)} ds \right)^{\diamond(-1)}$, f mean zero, C arbitrary
Non-Wick	$C + \int_0^t (\kappa + F(s)) e^{\diamond X_\beta(s)} ds$	$- \frac{\int_0^T F(s) e^{\diamond X_\beta(s)} ds}{\int_0^T e^{\diamond X_\beta(s)} ds}$, f mean zero, C arbitrary

TABLE 2. Solutions to Wick / non-Wick problems with different boundary data. Here we denote $F(t) = - \int_0^t f(s) ds$ and $e^{\diamond X_\beta(s)}$ is the Gaussian multiplicative chaos measure $\mu_\beta(ds)$.

Note also that rather interestingly, the Wick and path-wise equations admit solutions that are of the same form, with the only difference being, whether products are considered as ordinary product or Wick product. This can be viewed as a consequence of the fact that by using the ordinary product, we insert multiplicative deterministic renormalizations, while this can be omitted in the Wick case as Wick product is renormalizing in itself. Particularly this is visible if one considers special cases of initial value problem or the Neumann problem, in which the solution to the path-wise equation and to the Wick equation are equal, well-defined Hölder continuous random functions. On the other hand, in the Dirichlet case, the solution to the Wick equation can be viewed only through abstract white noise analysis involving Wick products, while the solution to the path-wise equation can be again viewed as a well-defined Hölder continuous random function.

Motivated by similarities in the representations, we also consider “projections” to the subspace of $L^2(\Omega)$ of the form, with suitably chosen $\beta' < 1$,

$$U(y) = \int_{(0,T)} \varphi(y, a) d\mu_{\beta'}(a), \quad y \in (0, T) \quad (1.2)$$

where $\varphi \in W_0^{-\frac{1-\beta'^2}{2}, 2}([0, T])$, the dual of the fractional Sobolev (Hilbert) space $W^{\frac{1-\beta'^2}{2}, 2}([0, T])$. This corresponds to the projections into a subspace that have a similar form to what is presented in Table 2, but having less moments if one chooses β' larger. In particular, for $\beta' = \beta < 1$ we recover the given solutions in the Neumann case or in the initial value problem. Our main contribution here is the definition of the so-called S -transform of random variables (1.2) at a point $X(z)$ of the log-correlated field (that do not admit

point-wise evaluation in the classical sense) directly, allowing us to employ Fourier analysis techniques to invert a potential operator between a suitable Sobolev space and its dual.

The rest of the article is organized as follows. In Section 2 we introduce some preliminaries. In particular, we recall basics of log-correlated fields and Gaussian multiplicative chaos measures. In Section 3 we prove, by means of approximation arguments, the existence of solutions to the renormalized path-wise equations. In Section 4 we use abstract Wick calculus to show existence of solutions to the Wick equations, again via approximation, and in Section 5 we study projections via S -transforms.

2. PRELIMINARIES

We begin by fixing some notation and by recalling some function spaces that will be used in the sequel. Throughout, we consider an open bounded interval $U = (0, T) \subset \mathbb{R}$. We denote by $C^k(U)$ (or by C^k in short) the space of functions $f: U \rightarrow \mathbb{R}$, such that f is k -times continuously differentiable on U . Similarly, we denote $C^\infty(U)$ the space of infinitely continuously differentiable functions, that is, $C^\infty(U) = \bigcap_{k=1}^\infty C^k(U)$. By C_0^k , and C_0^∞ we denote the corresponding spaces with the additional requirement that its elements have compact support, that is, the closure of

$$\{x \in U : f(x) \neq 0\}$$

is a compact subset of U .

For a measure space $(X, \mathcal{A}, \mu)$, we say that a function $f: X \rightarrow \mathbb{R}$ belongs to $L^p(X)$, $p \in [1, \infty)$, or L^p for short whenever confusion cannot arise, if f is Lebesgue measurable and the norm defined via the Lebesgue integral

$$\|f\|_{L^p(X)} = \left(\int_X |f(x)|^p d\mu(x) \right)^{1/p}$$

is finite. We will typically take $p = 2$, and shall concern ourselves with either the measure space being a probability space $(X, \mathcal{F}, \mathbb{P})$, or $(U, \mathcal{B}(U), dx)$ equipped with the Lebesgue measure dx and Borel σ -algebra $\mathcal{B}(U)$. The space $L^\infty(U)$ consists of Lebesgue measurable functions $f: U \rightarrow \mathbb{R}$ that are essentially bounded, that is, the quantity

$$\|u\|_{L^\infty(U)} = \operatorname{ess\,sup}_{x \in U} |f(x)| = \inf\{a \in \mathbb{R} : \mu(\{x \in U : f(x) > a\}) = 0\}$$

is finite.

Finally, we recall a sequence $Y_j, j = 1, 2, \dots$ of random variables taking values in a separable Banach space E converge in probability in E to Y if for all $\epsilon > 0$ we have

$$\lim_{j \rightarrow \infty} \mathbb{P}(\|Y_j - Y\|_E > \epsilon) = 0.$$

In particular, for $E = C(0, T)$ this means

$$\lim_{j \rightarrow \infty} \mathbb{P}(\sup_{0 \leq t \leq T} |Y_j(t) - Y(t)| > \epsilon) = 0$$

that is sometimes referred to as u.c.p. (uniform convergence in probability) in the probabilistic literature. Above and throughout, E -valued random variables are assumed to be Bochner measurable (see [7, Chapter 2] for details). In the sequel, we denote by

$L^p(\Omega; E)$, for $p \geq 1$, the space of Bochner measurable E -valued random variables such that $\mathbb{E}[\|Y\|_E^p] < \infty$.

2.1. Log-correlated Gaussian field. In this subsection we recall some basic facts on log-correlated Gaussian fields and Gaussian multiplicative chaos measures. For details, we refer to survey articles [8, 28] and references therein.

Definition 2.1 (Log-correlated Gaussian field). *A centered (distribution-valued) Gaussian process X on a bounded interval $U \subset \mathbb{R}$ is called a log-correlated field if it has covariance*

$$R(y, z) := R_X(y, z) := \mathbb{E}[X(y)X(z)] = \log \frac{1}{|z - y|} + g(z, y), \quad (2.1)$$

where we assume that $g \in C^\infty(U \times U) \cap L^\infty(U)$.

We interpret the covariance yielding the kernel of the covariance operator, as pointwise evaluations are not bounded:

$$\mathbb{E}[\langle X, \psi_1 \rangle \langle X, \psi_2 \rangle] = \int_{U \times U} R(y, z) \psi_1(y) \psi_2(z) dz dy$$

for any $\psi_1, \psi_2 \in C^\infty(U)$.

Remark 2.2. Note that the role of the function g in Definition 2.1 is to guarantee that (2.1) yields a well-defined covariance function. In the sequel we will assume that g depends only on the distance between the points z, y , that is, $g(y, z) = h(y - z)$ for some smooth function h . In particular this yields a scaling of the form $e^{R_X(y, z)} = e^{h(y-z)}|y - z|^{-1}$ for the exponential covariance.

In what follows, we will consider equations where the solution essentially inherits its regularity from the part of the covariance with the logarithmic singularity. Therefore, for the sake of simplicity we typically consider the case $g \equiv 0$.

Definition 2.3 (Wick exponential). *Let Z be a centered Gaussian random variable. We define the Wick exponential of Z by*

$$e^{\diamond Z} := e^{Z - \mathbb{E}[Z^2/2]}.$$

Remark 2.4. Wick exponential can also be expressed as

$$e^{\diamond Z} := e^{Z - \mathbb{E}[Z^2/2]} = \sum_{n=0}^{\infty} \frac{Z^{\diamond n}}{n!},$$

where $Z^{\diamond n}$ are Wick powers defined through Wick products, see Definition 4.2, and the convergence is in L^2 .

The family of Gaussian multiplicative chaos (=GMC) measures $d\mu_\beta$ on the real line, depending on a parameter $\beta \in (0, \sqrt{2})$, are random measures constructed from the Wick exponential of a log-correlated field X . The GMC measure on the real line is formally defined for all Borel sets $A \in \mathcal{B}(\mathbb{R})$ as the random measure μ_β given by

$$\mu_\beta(A) = \int_A e^{\beta X(y) - \frac{\beta^2}{2} \mathbb{E}[X(y)]} dy,$$

where X is a log-correlated Gaussian field as in Definition 2.1. The fact that such a measure exists follows by the next lemma. If θ is a smooth mollifier, we denote by X_ε the mollified field $X_\varepsilon = X * \theta_\varepsilon$, where $\theta_\varepsilon(y) = \varepsilon^{-1} \theta(\frac{y}{\varepsilon})$ for $\varepsilon > 0$. Then X_ε defines a centered and smooth Gaussian field. We have the following useful technical result:

Lemma 2.5 (Approximation lemma). *Let X be a log-correlated Gaussian field on $(0, T)$, and $\beta \in (0, 1)$ be a parameter. Let $0 < \varepsilon < \varepsilon'$, $x, y \in (0, T)$. Then the convolution approximation X_ε satisfies:*

- (i) X_ε has continuous realizations and $\mathbb{E}[X_\varepsilon(z)^2]$ is finite and independent of z .
- (ii) The sequence of random measures

$$d\mu_{\beta, \varepsilon}(y) := \mathbb{1}_{(0, T)}(y) \exp \left(\beta X_\varepsilon(y) - \frac{\beta^2}{2} \mathbb{E}[X_\varepsilon^2] \right) dy, \quad (2.2)$$

converges in probability (in the space of Radon measures equipped with the topology of weak convergence) towards a random measure $d\mu_\beta(y)$, supported on $[0, T]$, called the Gaussian multiplicative chaos measure.

- (iii) $\lim_{\varepsilon' \rightarrow 0} \mathbb{E}[X_\varepsilon(x)X_{\varepsilon'}(y)] = R(x, y)$, for $x \neq y$ fixed.

Before giving the proof, we remark that Definition 2.1 can naturally be extended to open sets $U \subset \mathbb{R}^d$. It can then be shown that Lemma 2.5 holds for GMC measures on $\mathbb{R}^d$ for $0 < \beta < \sqrt{d}$.

Proof of Lemma 2.5. The announced convergence in (ii) follows from standard results, see Theorem 2.3 in [28] and Theorem 1.1 in [22]. Moreover, (iii) follows from [22, Lemma 2.8]. For (i), it is well-known that, by definition via convolution, the field X_ε has continuous realizations. It remains to prove that the variance of $X_\varepsilon(z)$ is finite and does not depend on the point z . Now $X_\varepsilon(z) := (\theta_\varepsilon * X)(z) = \int_{\mathbb{R}} \theta_\varepsilon(z - y)X(y) dy$ which is centered. By the translational invariance of the Lebesgue measure we get

$$\begin{aligned} \mathbb{E}[X_\varepsilon(z)X_\varepsilon(z')] &= \mathbb{E} \left[\int_{\mathbb{R}} \theta_\varepsilon(z - y)X(y) dy \int_{\mathbb{R}} \theta_\varepsilon(z' - x)X(x) dx \right] \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \theta_\varepsilon(z - y)\theta_\varepsilon(z' - x)\mathbb{E}[X(y)X(x)] dy dx \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \theta_\varepsilon(z - y)\theta_\varepsilon(z' - x) \log \frac{1}{|x - y|} dy dx \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} \theta_\varepsilon(a)\theta_\varepsilon(b) \log \frac{1}{|z' - z - (b - a)|} da db \end{aligned}$$

from which we see that the variance $\mathbb{E}[X_\varepsilon(z)^2]$ is finite and in fact does not depend on z . Moreover, it is straightforward to see that the argument carries through if one adds $g(x, z) = h(x - z)$. This completes the proof. $\square$

For the reader's convenience, we record the fact that the GMC measure μ_β has positive p -moments up to order $\frac{2}{\beta^2}$ [28, Theorem 2.11]:

Theorem 2.6. *Let X be the log-correlated Gaussian field on $\mathbb{R}$, and $d\mu_\beta$ the associated Gaussian multiplicative chaos measure with parameter $0 < \beta < \sqrt{2}$. Then the measure μ_β admits finite p -moments for all $p \in (0, \frac{2}{\beta^2})$. In particular, $\mathbb{E}[\mu_\beta(A)] = |A|$ for any $A \in \mathcal{B}(\mathbb{R})$, where $|A|$ denotes the Lebesgue measure of A .*

In the present paper we shall restrict ourselves to the so-called L^2 -range, where $\beta < \sqrt{d} = 1$, and this will be our standing assumption from now on.

Note that by our definitions we have that

$$d\mu_{\beta,\varepsilon}(y) = e^{\diamond X_{\beta,\varepsilon}(y)} dy,$$

where $e^{\diamond X_{\beta,\varepsilon}}$ is the Wick exponential of the Gaussian field $X_{\beta,\varepsilon} = \beta X_\varepsilon$. Thus, it is natural to make use of the notation

$$e^{\diamond X_\beta} := d\mu_\beta.$$

Since X_β is centered (and hence X_β and $-X_\beta$ have same distribution), we obtain that similarly $e^{\diamond(-X_\beta)}$ appearing formally in our equation corresponds to a chaos measure equivalent in law to $d\mu_\beta$.

3. UNIQUE SOLVABILITY OF NON-WICK EQUATION

In this section we present our main results concerning the problem with the scalar product. That is, we prove the existence and uniqueness of the solution to the problem

$$\begin{cases} -e^{\mathbb{E}[X_\beta(t)^2]}(e^{\diamond(-X_\beta)} \cdot U')' = f & t \in (0, T) \\ \text{boundary data for } U \end{cases} \quad (3.1)$$

and the representation formula of the solution. Above X_β is the log-correlated Gaussian field with a parameter $\beta < 1$ and $f: [0, T] \rightarrow \mathbb{R}$ is a deterministic, integrable function. As X_β is a generalized random field with undefined variance, we proceed through approximation $X_{\beta,\varepsilon}$ and above and throughout (see, e.g. Theorem 3.3 below) the term $e^{\mathbb{E}[X_\beta(t)^2]}$ indicates a normalization $e^{\mathbb{E}[X_{\beta,\varepsilon}(t)^2]}$ in the approximating equation. More precisely, we will always interpret the solutions via the following definition.

Definition 3.1. Let X_ε be a convolution approximation of the log-correlated field X on $[0, T]$ so that $d\mu_{\beta,\varepsilon} \rightarrow d\mu_\beta$ in probability in the space of Radon measures as $\varepsilon \rightarrow 0$. Let U_ε be the corresponding solution of the problem

$$\begin{cases} -e^{\mathbb{E}[X_{\beta,\varepsilon}(t)^2]}(e^{\diamond(-X_{\beta,\varepsilon})} \cdot U'_\varepsilon)' = f & t \in (0, T) \\ \text{boundary data for } U_\varepsilon. \end{cases}$$

We say that the random process U is a solution to (3.1) if $\lim_{\varepsilon \rightarrow 0} U_\varepsilon \xrightarrow{P} U$ as $\varepsilon \rightarrow 0$ in the space $C(0, T)$.

Remark 3.2. Note that the solution U_ε is either unique or unique up to an additive constant, depending on the boundary data used.

We are now in place to state the solvability result for problem (3.1). Recall that below, all the boundary terms should be interpreted to hold for the approximating sequences and solutions. For example in the periodic case, condition $e^{\diamond(-X_\beta(0))}U'(0) = e^{\diamond(-X_\beta(T))}U'(T)$ is interpreted as $e^{\diamond(-X_{\beta,\varepsilon}(0))}U'_\varepsilon(0) = e^{\diamond(-X_{\beta,\varepsilon}(T))}U'_\varepsilon(T)$, for all $\varepsilon > 0$.

Theorem 3.3. Let $f: [0, T] \rightarrow \mathbb{R}$ be a deterministic, integrable function, and let $X_\beta = \beta X$, where X is a log-correlated Gaussian field on $[0, T]$, and $\beta < 1$. Consider the equation

$$-e^{\mathbb{E}[X_\beta(t)^2]}(e^{\diamond(-X_\beta(t))} \cdot U'(t))' = f(t), \quad t \in (0, T).$$

Denote $F(t) = -\int_0^t f(s)ds$, then coupled with:

- (i) Initial data: $U(0) = U_1$ and $e^{\mathbb{E}[X_\beta^2]}e^{\diamond(-X_\beta(0))} \cdot U'(0) = U_2$, where U_1 and U_2 are deterministic constants, the problem admits a unique solution $U \in L^2(\Omega; C(0, T))$ satisfying Definition 3.1. Moreover, this solution is given by

$$U(t) = U_1 + \int_0^t (U_2 + F(s)) e^{\diamond X_\beta(s)} ds, \quad t \in (0, T).$$

- (ii) Dirichlet data: $U(0) = U_1, U(T) = U_2$, where U_1 and U_2 are deterministic constants, the problem admits a unique solution $U \in L^2(\Omega; C(0, T))$ satisfying Definition 3.1. Moreover, this solution is given by

$$U(t) = U_1 + \int_0^t (\kappa + F(s)) e^{\diamond X_\beta(s)} ds, \quad t \in (0, T),$$

where the stochastic constant κ is given by

$$\kappa = \left(U_2 - U_1 - \int_0^T F(s) e^{\diamond X_\beta(s)} ds \right) \cdot \left(\int_0^T e^{\diamond X_\beta(s)} ds \right)^{-1},$$

with κ almost surely bounded.

- (iii) Neumann boundary conditions: $e^{\mathbb{E}[X_\beta^2]}e^{\diamond(-X_\beta(0))} \cdot U'(0) = U_1$ and $e^{\mathbb{E}[X_\beta^2]}e^{\diamond(-X_\beta(T))} \cdot U'(T) = U_2$, where U_1 and U_2 are deterministic constants, the problem admits a unique (up to additive stochastic constant) solution $U \in L^2(\Omega; C(0, T))$ satisfying Definition 3.1. Moreover, this solution is given by

$$U(t) = \int_0^t (U_1 + F(s)) e^{\diamond X_\beta(s)} ds, \quad t \in (0, T).$$

- (iv) Periodic boundary conditions: $U(0) = U(T), e^{\diamond(-X_\beta(0))} U'(0) = e^{\diamond(-X_\beta(T))} U'(T)$, assuming that $\int_0^T f(s) ds = 0$, the problem admits a unique (up to additive stochastic constant) solution $U \in L^2(\Omega; C(0, T))$ satisfying Definition 3.1. Moreover, this solution is given by

$$U(t) = \int_0^t (\kappa + F(s)) e^{\diamond X_\beta(s)} ds, \quad t \in (0, T),$$

and the stochastic constant κ is given by

$$\kappa = - \left(\int_0^T F(s) e^{\diamond X_\beta(s)} ds \right) \cdot \left(\int_0^T e^{\diamond X_\beta(s)} ds \right)^{-1},$$

with κ almost surely bounded.

Above $F(t) := -\int_0^t f(s) ds$, and recall that $e^{\diamond X_\beta(s)} ds$ denotes the Gaussian multiplicative chaos measure.

Remark 3.4. We remark that although we have restricted ourselves to the so-called L^2 -range of the Gaussian multiplicative chaos, i.e. $\beta^2 < 1$, the above Theorem 3.3 remains valid in the full range $\beta^2 < 2$. The key modification in the statement in this case is that the solution would belong only to $L^p(\Omega; C(0, T))$ for certain $p < 2$. This is a consequence of the fact that for $\beta \geq 1$, the Gaussian multiplicative chaos would have fewer moments, see Theorem 2.6 above.

Before moving on to the proof of Theorem 3.3, we make the following remarks on the sample path continuity of the solution, see also Remark 4.14 below.

Remark 3.5. Note that from the representations of solutions, we observe that the solutions are in fact almost surely Hölder continuous of certain order, depending on the parameter β . This follows immediately from the solution formulas and the fact that the chaos measure $d\mu_\beta$ have almost sure modulus of continuity:

$$\mu_\beta([s, t]) \leq C|t - s|^\eta,$$

where $\eta = \eta(\beta) > 0$ and C is an almost surely finite random constant. For this fact, see [3, Proposition 2.5] and the references therein.

Proof of Theorem 3.3. We prove the Theorem in the case of periodic boundary values, while the other cases can be proved with similar arguments. Let now $f \in L^1([0, T])$ be a mean-zero function and consider

$$-e^{\mathbb{E}[X_\beta(t)^2]}(e^{\diamond(-X_\beta(t))} \cdot U'(t))' = f(t), \quad t \in (0, T),$$

with periodic boundary conditions

$$\begin{aligned} U(0) &= U(T) \\ e^{\diamond(-X_\beta(0))} \cdot U'(0) &= e^{\diamond(-X_\beta(T))} \cdot U'(T). \end{aligned} \tag{3.2}$$

Step 1. Approximation of the field X with X_ε .

We use Lemma 2.5 to obtain a convolution approximation of the log-correlated field X_β , denoted by $X_{\beta, \varepsilon}$, with $\varepsilon > 0$, and we denote by U_ε the unknown function in the corresponding mollified differential equation.

Step 2. Solving the regularized equation. Now as the field $X_{\beta, \varepsilon}$ is well-defined and smooth, directly integrating (recall that $\mathbb{E}[X_{\beta, \varepsilon}(t)^2]$ is independent of t) the equation we have, for fixed $\omega \in \Omega$,

$$e^{\diamond(-X_{\beta, \varepsilon}(t))} U'_\varepsilon(t) - e^{\diamond(-X_{\beta, \varepsilon}(0))} U'_\varepsilon(0) = e^{-\mathbb{E}[X_{\beta, \varepsilon}^2]} F(t), \quad t \in (0, T)$$

where we denote $F(t) := -\int_0^t f(s) ds$. We note that we see here that (3.2) is satisfied precisely when f is mean-zero on $[0, T]$.

A simple computation from Definition 2.3 yields for any convolution approximation $X_{\beta, \varepsilon}$ of X_β that it holds

$$e^{\diamond(-X_{\beta, \varepsilon}(t))} = \frac{1}{e^{\mathbb{E}[X_{\beta, \varepsilon}(t)^2]} e^{\diamond X_{\beta, \varepsilon}(t)}}, \quad t \in (0, T).$$

Using the above identity, after rearranging gives

$$U'_\varepsilon(t) = \left(\frac{1}{e^{\diamond X_{\beta, \varepsilon}(0)}} U'_\varepsilon(0) + F(t) \right) e^{\diamond X_{\beta, \varepsilon}(t)}, \quad t \in (0, T).$$

Integrating this expression up to $t \in (0, T)$ and using the boundary condition $U_\varepsilon(0) = U_\varepsilon(T)$ we have

$$U_\varepsilon(t) = U_\varepsilon(0) + \int_0^t (\kappa_\varepsilon + F(s)) e^{\diamond X_{\beta, \varepsilon}(s)} ds,$$

where the random constant κ_ε is defined to be

$$\kappa_\varepsilon := \frac{1}{e^{\diamond X_{\beta,\varepsilon}(0)}} U'_\varepsilon(0) = - \frac{\int_0^T F(s) e^{\diamond X_{\beta,\varepsilon} ds}}{\int_0^T e^{\diamond X_{\beta,\varepsilon} ds}}. \quad (3.3)$$

Step 3. Convergence of the approximate solution. In Step 2 we obtained the solution formula for the approximated equation as

$$U_\varepsilon(t) = U_\varepsilon(0) + \int_0^t \left(F(s) + \kappa_\varepsilon \right) e^{\diamond X_{\beta,\varepsilon}(s)} ds, \quad t \in (0, T) \quad (3.4)$$

with $\kappa_\varepsilon = - \frac{\int_0^T F(s) e^{\diamond X_{\beta,\varepsilon} ds}}{\int_0^T e^{\diamond X_{\beta,\varepsilon} ds}}$, for any $\varepsilon > 0$. Here

$$|\kappa_\varepsilon| \leq \sup_{s \in [0, T]} |F(s)| \leq \int_0^T |f(s)| ds < \infty.$$

In order to complete the proof, we need the following result.

Lemma 3.6. *Assume that μ_n, μ ($n \geq 1$) are random positive measures on $[0, 1]$ such that $\mu_n \rightarrow \mu$ in probability (with respect to the weak star convergence of measures on $[0, 1]$). Assume that μ is almost surely non-atomic. Let $h \in C([0, 1])$ and write $f(x) := \int_0^x h(y) d\mu(y)$, and $f_n(x) := \int_0^x h(y) d\mu_n(y)$. Then $f_n \rightarrow f$ in probability in the space $C([0, 1])$.*

Proof. Denote $A := \|h\|_\infty$. Fix an integer $N \geq 2$ and choose a continuous partition of unity of tent-functions $\{\psi_k : k = 0, \dots, 2N\}$ by setting $\psi_k(x) = \max(0, 1 - 2N|x - k(2N)^{-1}|)$, $x \in [0, 1]$. Let $\varepsilon = \varepsilon(\omega) := \sup_{|x-y| \leq 1/N} \mu([x, y])$. Set also

$$A_{k,N} = \left\{ \omega : \left| \int_0^1 \psi_k(y) d\mu_n(y) - \int_0^1 \psi_k(y) d\mu(y) \right| \leq N^{-2} \right\}$$

and

$$B_{k,N} = \left\{ \omega : \left| \int_0^1 h(y) \psi_k(y) d\mu_n(y) - \int_0^1 h(y) \psi_k(y) d\mu(y) \right| \leq N^{-2} \right\}.$$

By the assumption we may choose n_0 so large that for $n \geq n_0$ it holds, for any $k \in \{0, \dots, 2N\}$ and $n \geq n_0$, that

$$\mathbb{P}(A_{k,N} \cap B_{k,N}) \geq 1 - \frac{1}{N(2N+1)}. \quad (3.5)$$

Assume that $n \geq n_0$ and that the events $A_{k,N}, B_{k,N}$ happen for all $k \in \{0, \dots, 2N\}$, which occurs at least with probability $1 - 1/N$, and let $x, y \in [0, 1]$ be arbitrary such that $|x - y| \leq 1/N$. Since the measures μ_n, μ are positive we may majorize $\chi_{[x,y]}$ by using at most three of the functions ψ_k , it follows that

$$\mu_n([x, y]) \leq 3(\varepsilon + 1/N^2).$$

More generally, for any $x, y \in [0, 1]$ we may express $\chi_{[x,y]}$ as a sum of at most $2N + 1$ of the functions ψ_k and, in addition, two halves of a single tent function ψ_k . Then clearly

$$\begin{aligned} |f_n(x) - f(x)| &\leq (2N + 1)N^{-2} + 2A(\varepsilon + 3(\varepsilon + 1/N^2)) < 8A\varepsilon + 9 \max(1, A)/N \\ &< 9(A + 1)(\varepsilon + 1/N). \end{aligned}$$

Let next $\delta > 0$ be given. We fix N so large that $\mathbb{P}(9(A+1)\varepsilon < \delta/2) \geq 1 - \delta/2$ and $9(A+1)/N < \delta/2$. Then it follows for $n \geq n_0$ that

$$\mathbb{P}(\|f - f_n\|_{C([0,1])} < \delta) \geq 1 - \delta/2 - 1/N > 1 - \delta.$$

This completes the proof. $\square$

We continue the proof of the Theorem. By the weak convergence in probability of $d\mu_{\beta,\varepsilon}$ to $d\mu_\beta$, and the fact that μ_β is non-atomic almost surely, $\kappa_\varepsilon \rightarrow \kappa$ in probability. Since h is continuous, the convergence in probability in the space $C([0, T])$ now follows from Lemma 3.6. Put together, the above argument shows that we have obtained the unique solution of the problem. Finally, the fact that $U \in L^2(\Omega; C(0, T))$ in cases (i) and (iii) follows directly from the representation formula together with the fact that h is bounded and

$$\sup_{0 \leq t \leq T} \mu_\beta([0, t]) = \mu_\beta([0, T]) \in L^2(\Omega),$$

since $\beta < 1$. Similarly, $U \in L^2(\Omega; C(0, T))$ in case (iv), since then $|\kappa| \leq \sup_{0 \leq s \leq T} |F(s)|$ almost surely. To conclude, in case (ii) we have

$$\frac{U_2 - U_1}{\mu_\beta([0, T])} \int_0^t |F(s)| e^{\diamond X_\beta(s)} ds \leq (U_2 - U_1) \sup_{0 \leq s \leq T} |F(s)|$$

from which similar arguments as above yields $U \in L^2(\Omega; C(0, T))$. This completes the whole proof. $\square$

4. UNIQUE SOLVABILITY OF WICK EQUATION

In this section we consider the problem

$$\begin{cases} -(e^{\diamond(-X_\beta(t))} \diamond U'(t))' &= f(t), & t \in (0, T) \\ \text{boundary data for } U. \end{cases} \quad (4.1)$$

We state and prove an existence and uniqueness result for integrable, deterministic functions f . Above the symbol $\diamond$ denotes the Wick product, see Definition 4.2 below, and $e^{\diamond X_\beta}$ denotes the Wick exponential of X_β , as in Definition 2.3. The boundary data can be taken to be initial values, Dirichlet, Neumann or periodic. Again, as point-wise evaluations do not make sense for the log-correlated field X_β , we define the solution to the problem in (4.1) via approximations of the field using Lemma 2.5. Then, we convert this to a family of deterministic equations by taking an S -transform (see the definition below). These deterministic equations have unique solutions, from this we get a solution to (4.1) by using the invertibility of the S -transform.

We first recall some elementary facts about Gaussian L^2 -spaces, and then define the S -transform, one of the main tools of the remainder of the paper.

We recall that a Gaussian linear space $\mathcal{H}$ is a vector space of real-valued random variables, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that each random variable follows the Gaussian distribution with mean zero. Each $h \in \mathcal{H}$ has finite variance; therefore $\mathcal{H} \subset L^2(\Omega, \mathcal{F}, \mathbb{P})$. If $\mathcal{H}$ is a closed subspace of $L^2(\Omega, \mathcal{F}, \mathbb{P})$, we say that $\mathcal{H}$ is a Gaussian Hilbert space. It is well known that if $\mathcal{G} \subset L^2(\Omega, \mathcal{F}, \mathbb{P})$ is a Gaussian linear space, then the closure of $\mathcal{G}$ is a Gaussian Hilbert space.

Definition 4.1 (Gaussian Hilbert space generated by Gaussian families). *Let $\mathcal{H}$ be a Gaussian Hilbert space. For any collection $\{\xi_\alpha\}$ of centered jointly normal random variables in $\mathcal{H}$, the closed linear span of $\{\xi_\alpha\}$ in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ is called the Gaussian Hilbert space generated by $\{\xi_\alpha\}$.*

We also recall the basic Wiener-Itô chaos decomposition of L^2 : For $n \geq 0$, let $\mathcal{P}_n(\mathcal{H})$ be the closure in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ of the linear space

$$\mathcal{P}_n(\mathcal{H}) = \{p(\xi_1, \dots, \xi_m) : p \text{ is a polynomial of degree } \leq n, \xi_1, \dots, \xi_m \in \mathcal{H}, m < \infty\},$$

and set $K^n := \overline{\mathcal{P}_n(\mathcal{H})} \cap \overline{\mathcal{P}_{n-1}(\mathcal{H})}^\perp$, with $K^0 = \overline{\mathcal{P}_0(\mathcal{H})}$. The space K^n is called the n^{th} (homogeneous) chaos. It is clear from the construction that $K^n \subset L^p(\Omega, \mathcal{F}, \mathbb{P})$ for any $p \geq 1$. We also remark that, by construction the first chaos K_1 is composed of Gaussian random variables. More importantly, we have (see, for instance [20, Theorem 2.6]) the orthogonal decomposition

$$\bigoplus_{n=0}^{\infty} K^n = L^2(\Omega, \mathcal{F}_{\mathcal{H}}, \mathbb{P}) =: L^2_{\mathcal{H}}(\Omega),$$

where $\mathcal{F}_{\mathcal{H}}$ is the σ -algebra generated by the Gaussian random variables in $\mathcal{H}$. Actually, the space $L^2_{\mathcal{H}}(\Omega)$ is spanned by the Wick exponential of Gaussian random variables $h \in \mathcal{H}$, see [20, Corollary 3.40] and Definition 2.3. The space $L^p_{\mathcal{H}}(\Omega)$ for $p \geq 1$ is defined as $\mathcal{F}_{\mathcal{H}}$ -measurable random variables that have finite p moment. Also, note that $e^{\diamond h} \in L^p_{\mathcal{H}}(\Omega)$ for all $p \geq 1$. Furthermore, one may prove that $K^n \subset L^p_{\mathcal{H}}(\Omega)$ for all $p, n \geq 1$.

Definition 4.2 (Wick product). *Let $Z \in K^m, Y \in K^n$. We define the Wick product of Z and Y by*

$$Z \diamond Y = \pi_{m+n}(ZY),$$

where π_n denotes the orthogonal projection of $L^2_{\mathcal{H}}(\Omega)$ onto K^n . The operator $\diamond$ can by bilinearity be extended to a binary operator on the space of all elements of $L^2_{\mathcal{H}}(\Omega)$ having finite chaos decomposition.

We define the Wick powers of $Z^{\diamond n}, n = 0, 1, 2, \dots$ inductively as

$$\begin{cases} Z^{\diamond 0} &= 1, \\ Z^{\diamond n} &= Z \diamond Z^{\diamond(n-1)}, \quad n = 1, 2, \dots \end{cases}$$

Definition 4.3 (S -transform). *Let $\mathcal{H}$ be a Gaussian Hilbert space, and let $Z \in L^2_{\mathcal{H}}(\Omega)$. We call the mapping $S: \mathcal{H} \rightarrow \mathbb{R}$ given by*

$$(SZ)(h) = \mathbb{E} \left[Z e^{\diamond h} \right], \quad h \in \mathcal{H}, \quad (4.2)$$

the S -transform of Z evaluated at h . Actually, it suffices to assume $Z \in L^p_{\mathcal{H}}(\Omega)$ for some $p > 1$.

Remark 4.4. We remark that by the density of $e^{\diamond h}$ in $L^2_{\mathcal{H}}(\Omega)$, it is a routine task to show that the S -transform is a linear, injective map on $L^2_{\mathcal{H}}(\Omega)$: that is, if $(SZ_1)(h) = (SZ_2)(h)$ for all $h \in \mathcal{H}$, then $Z_1 = Z_2$ almost surely. In fact, it is sufficient to have $(SZ_1)(h) = (SZ_2)(h)$ for all $h \in \mathcal{A}$, where $\mathcal{A}$ is a dense subset of $\mathcal{H}$. For details, see [20, Theorem 16.11, Corollary 3.40].

The Wick product acts like an ordinary product under S -transform.

Lemma 4.5. *Let $\mathcal{H}$ be a Gaussian Hilbert space, and $Y \in K^m, Z \in K^n$. Then we have*

$$S(Y \diamond Z)(h) = S(Y)(h) \cdot S(Z)(h), \quad \text{for all } h \in \mathcal{H}.$$

The content of Lemma 4.5 can be used as a defining property of a general Wick product.

Definition 4.6 (General Wick-product). *Let $Y, Z \in \bigcup_{q \geq 1} L_{\mathcal{H}}^q(\Omega)$, and $p > 1$. We say that the Wick product of Z and Y exists in $L_{\mathcal{H}}^p(\Omega)$ if there exists an element $V \in L_{\mathcal{H}}^p(\Omega)$ such that*

$$(SY)(h) \cdot (SZ)(h) = (SV)(h),$$

for all $h \in \mathcal{H}$, and we denote V by $Y \diamond Z$.

It is in general not true that for $Y, Z \in L_{\mathcal{H}}^p(\Omega)$, the Wick product $Y \diamond Z$ is still an element of $L_{\mathcal{H}}^p(\Omega)$ or even $L_{\mathcal{H}}^1(\Omega)$ (or that it even exists), see e.g. [11] or [20, Example 16.33]. To overcome this, one can enlarge the space and consider random variables $Y, Z \in (\mathcal{S})_{-1}$, where $(\mathcal{S})_{-1}$ is a stochastic distribution space (Hida space), see Appendix A. The definition of the Wick product extends naturally to this case, with a benefit that then $Y \diamond Z$ is a well-defined element of $(\mathcal{S})_{-1}$ as well. Similarly, one can define the S -transform of the elements $Y, Z \in (\mathcal{S})_{-1}$, and we have the expected identity¹

$$(S(Y \diamond Z))(h) = (SY)(h) \cdot (SZ)(h)$$

for suitable test objects $h \in \mathcal{H}$. We note however, that in this case the S -transform can be defined only for Gaussian objects h that are small enough in suitable associated norms. More precisely, the choice of the associated norm depends on the subspace $(\mathcal{S})_{-1,-q} \subset (\mathcal{S})_{-1}$ one wishes to characterize by S -transform, where we have

$$(\mathcal{S})_{-1} = \bigcup_q (\mathcal{S})_{-1,-q}.$$

For $(\mathcal{S})_{-1,-q}$, the test points h are Gaussian random variables such that $e^{\diamond h} \in (\mathcal{S})_{1,q}$ (see Definition A.15). In particular, this holds true if $\mathbb{E}[h^2]$ is small enough, where the bound depends only on q . In practice, this can be achieved by a multiplication with a small enough constant λ and testing on Gaussian points λh . Regardless, the S -transform (now defined on a restricted class of Gaussian random variables) characterizes random variables in $(\mathcal{S})_{-1,-q}$ uniquely. As a particular example, we note that whenever Y and Z are elements of spaces $L_{\mathcal{H}}^p(\Omega) \subset (\mathcal{S})_{-1}$, the generalized Wick product and S -transform equals to the ones given in (4.2) and Definition 4.6. In this case, the S -transform can be defined for any test point $h \in \mathcal{H}$ and the S -transform characterizes random variables uniquely. In the sequel, for the sake of notational simplicity we always use $h \in \mathcal{H}$ as a test point when taking S -transforms. The reader should keep in mind however that one should consider $\lambda h \in \mathcal{H}$ for small enough λ if necessary, despite the fact that we have suppressed λ from the notation. For an interested reader, we have also collected basic definitions and results on generalized stochastic distribution spaces and Wick products into Appendix A. For a more throughout treatment of the topic in the white noise setting, see [11], [19], or [23] and references therein. In what follows, we naturally assume that

¹Actually, one could equivalently define the S -transform first and then define the Wick product through this identity, see e.g. [23].

our log-correlated field is constructed from the white noise, allowing to use white noise analysis techniques.

For the reader's convenience we record some properties of Wick calculus in the next lemma. That is, the Wick product is commutative, associative and distributive, and it reduces to the usual scalar product when one of the factors is deterministic. On top of that, we collect some elementary properties of the Wick exponential to the same lemma. The proofs of the identities below (in the case of $L^2_{\mathcal{H}}(\Omega)$ random variables) can be found for instance in [20].

Lemma 4.7. *Let $X, Y, Z \in (\mathcal{S})_{-1}$. Then the Wick product is commutative, associative, and distributive, i.e.*

$$\begin{aligned} X \diamond Y &= Y \diamond X, \\ X \diamond (Y \diamond Z) &= (X \diamond Y) \diamond Z, \\ X \diamond (Y + Z) &= X \diamond Y + X \diamond Z. \end{aligned} \tag{4.3}$$

Moreover, for any deterministic C we have

$$X \diamond C = X \cdot C. \tag{4.4}$$

Finally, for Gaussian random variables X and Y the Wick exponential satisfies

$$e^{\diamond(X+Y)} = e^{\diamond X} \diamond e^{\diamond Y} \tag{4.5}$$

$$e^{\diamond(X+Y)} = e^{-\mathbb{E}[XY]} e^{\diamond X} e^{\diamond Y}, \tag{4.6}$$

Remark 4.8. In particular, taking $Y = -X$ in (4.6) immediately results in

$$e^{\diamond(-X)} = \frac{1}{e^{\mathbb{E}[X^2]} e^{\diamond X}},$$

upon recalling that by definition $e^{\diamond 0} = 1$. The above identity justifies the inclusion of the $e^{\diamond(-X)}$ -term in problem (4.1) above, in addition to the fact that the log-correlated Gaussian field X is symmetric: thus $e^{\diamond(-X)}$ and $e^{\diamond X}$ obey the same law.

Random variables $X \in (\mathcal{S})_{-1}$ with non-zero (generalized) expectation also admit a multiplicative Wick inverse $X^{\diamond(-1)} \in (\mathcal{S})_{-1}$ defined through identity

$$X \diamond X^{\diamond(-1)} = 1. \tag{4.7}$$

Remark 4.9. In terms of S -transforms, we obtain

$$(SX)(h) \cdot (SX^{\diamond(-1)})(h) = 1 \iff (SX^{\diamond(-1)})(h) = ((SX)(h))^{-1}, \tag{4.8}$$

for the Wick inverse $X^{\diamond(-1)}$. In particular, we have

$$\begin{aligned} \frac{(SX)(h)}{(SY)(h)} &= (SX)(h) \cdot ((SY)(h))^{-1} = (SX)(h) \cdot (SY^{\diamond(-1)})(h) \\ &= (S(X \diamond Y^{\diamond(-1)}))(h). \end{aligned}$$

We will return to this point in the proof of Theorem 4.12.

Remark 4.10. The Wick inverse in (4.7) can be a very poorly behaving object in general. Indeed, even in the simple case $X(t) = 1 + \alpha Z$ with $Z \sim N(0, 1)$ and $\alpha \in \mathbb{R}$, one can show by using the Wiener-Itô chaos decomposition that, formally, we have

$$X(t)^{\diamond(-1)} = 1 + \sum_{k=1}^{\infty} (-\alpha)^k H_k(Z),$$

where H_k denotes the normalized Hermite polynomial. Note that the above series converges in $L^2_{\mathcal{H}}(\Omega)$ only for $|\alpha| < 1$. This example already illustrates that even for Gaussian random variables X the inverse $X^{\diamond(-1)}$ in general is not a well-defined random variable (in some $L^p_{\mathcal{H}}(\Omega)$ space), but rather only a generalized function in the white noise space $(\mathcal{S})_{-1}$. For the computation of the Wick inverse in the white noise case, we refer to [17].

Definition 4.11. Let X_ε be a convolution approximation of the log-correlated field X on $[0, T]$ so that $d\mu_{\beta, \varepsilon} \rightarrow d\mu_\beta$ in probability in the space of Radon measures (see Lemma 2.5) as $\varepsilon \rightarrow 0$. Let U_ε be the corresponding solution of the problem

$$\begin{cases} -(e^{\diamond(-X_\varepsilon(t))} \diamond U'_\varepsilon(t))' &= f(t), & t \in (0, T) \\ \text{boundary data for } U_\varepsilon. \end{cases}$$

In the above, boundary data can be any of the cases (i)–(iv) in Theorem 4.12. We say that U with $U(t) \in (\mathcal{S})_{-1}$ for all $t \in (0, T)$ is a solution of (4.1), if we have

$$\lim_{\varepsilon \rightarrow 0} (SU_\varepsilon(t))(h) = (SU(t))(h)$$

for all $t \in (0, T)$ and for all² Hida test points h .

We remark that although the convergence specified in Definition 4.11 is even weaker than the weak $L^2(\Omega)$ convergence, with some choices of boundary data we can replace the convergence in the S -transform side with the usual weak L^2 -convergence. In particular, this is the case with Neumann boundary data or in the initial value problem.

Theorem 4.12. Let $f: [0, T] \rightarrow \mathbb{R}$ be a deterministic, integrable function, and let $X_\beta = \beta X$ where X is the log-correlated Gaussian field on $(0, T)$, with $0 < \beta < 1$. Consider the equation

$$-(e^{\diamond(-X_\beta(t))} \diamond U'(t))' = f(t), \quad t \in (0, T).$$

Denote $F(t) = -\int_0^t f(s)ds$, then coupled with:

- (i) Initial data: $U(0) = U_1$ and $e^{\diamond(-X_\beta(0))} \diamond U'(0) = U_2$, where U_1 and U_2 are deterministic constants, the problem admits a unique solution U on $(0, T)$ in the sense of Definition 4.11. The solution is given by

$$U(t) = U_1 + \int_0^t (U_2 + F(s)) \diamond e^{\diamond X_\beta(s)} ds, \quad t \in (0, T)$$

²More precisely, there exists some q large enough so that $U_\varepsilon \in (\mathcal{S})_{-1, -q}$, and the test points h are chosen such that $e^{\diamond h} \in (\mathcal{S})_{1, q}$.

- (ii) Dirichlet data: $U(0) = U_1$, and $U(T) = U_2$, where U_1 and U_2 are deterministic constants, the problem admits a unique solution U on $(0, T)$ in the sense of Definition 4.11. The solution is given by

$$U(t) = U_1 + \int_0^t (\kappa + F(s)) \diamond e^{\diamond X_\beta(s)} ds, \quad t \in (0, T)$$

where the stochastic constant κ is

$$\kappa := \left(U_2 - U_1 - \int_0^T F(s) e^{\diamond X_\beta(s)} ds \right) \diamond \left(\int_0^T e^{\diamond X_\beta(s)} ds \right)^{\diamond(-1)}.$$

- (iii) Neumann boundary conditions: $e^{\diamond(-X_\beta(0))} \diamond U'(0) = U_1$ and $e^{\diamond(-X_\beta(T))} \diamond U'(T) = U_2$, where U_1 and U_2 are deterministic constants, the problem admits a unique (up to an additive stochastic constant) solution U on $(0, T)$ in the sense of Definition 4.11. The solution is given by

$$U(t) = \int_0^t (U_1 + F(s)) \diamond e^{\diamond X_\beta(s)} ds, \quad t \in (0, T).$$

- (iv) Periodic boundary conditions: $U(0) = U(T)$, and $e^{\diamond(-X_\beta(0))} \diamond U'(0) = e^{\diamond(-X_\beta(T))} \diamond U'(T)$, assuming that $\int_0^T f(s) ds = 0$, the problem admits a unique (up to an additive stochastic constant) solution U on $(0, T)$ in the sense of Definition 4.11. The solution is given by

$$U(t) = \int_0^t (\kappa + F(s)) \diamond e^{\diamond X_\beta(s)} ds, \quad t \in (0, T), \quad (4.9)$$

and the stochastic constant κ is

$$\kappa := - \left(\int_0^T F(s) e^{\diamond X_\beta(s)} ds \right) \diamond \left(\int_0^T e^{\diamond X_\beta(s)} ds \right)^{\diamond(-1)}. \quad (4.10)$$

Recall that $e^{\diamond X_\beta(s)} ds$ denotes the Gaussian multiplicative chaos measure.

Remark 4.13. We remark that cases (i) and (iii) above the solutions actually belong to $L^2(\Omega; C(0, T))$ and coincides with the solution to the renormalized non-Wick equation, see Theorem 3.3. This follows from Theorem 3.3, since in these cases

$$\int_0^t (U_1 + F(s)) \diamond e^{\diamond X_\beta(s)} ds = \int_0^t (U_1 + F(s)) e^{\diamond X_\beta(s)} ds$$

by the fact that $U_1 + F(s)$ is deterministic.

Remark 4.14. While the solution to the “non-Wick” problem

$$e^{\mathbb{E}[X_\beta(t)^2]} (e^{\diamond(-X_\beta(t))} U')' = f(t),$$

with given boundary data is a well-defined Hölder continuous process, see Remark 3.5, this is not the case for the Wick equation in general. Indeed, the reason being the (Wick) multiplication with a Wick inverse in the constant κ , cf. Remark 4.9. The exceptions are the cases (i) and (iii), where the Wick product reduces to the ordinary scalar product.

Remark 4.15. We remark that for $D \subset \mathbb{R}^n$ the pressure equation

$$\begin{cases} -\nabla(K(x) \diamond \nabla U(x)) = f(x), & x \in D \\ U(x) = 0, & x \in \partial D, \end{cases}$$

where K is a re-normalized exponential of Gaussian white noise, has been studied in the literature. Common to some approaches in the literature and ours is the idea to transform the problem into a deterministic problem for which PDE techniques can be applied. For example, in [14] the authors obtain a representation formula for the solution as a $(\mathcal{S})_{-1}$ -valued random variable by using the Hermite transform, which is closely related to S -transform employed in the present paper. The key difference here is that for the white noise, upon Hermite (or S -) transformation the left-hand-side of the equation defines a uniformly elliptic operator and, thus is amenable to standard analytic methods. For a related result, see also [27]. Finally, we note that in [24] (see also [16]), the authors solve a one-dimensional variant of the above problem,

$$-(e^{\diamond X(t)} \diamond U')' = f(t), \quad t \in (0, T),$$

directly by Wick calculus identities in Lemma 4.7. Indeed, integrating the equation, applying (4.3), (4.7) (since $\mathbb{E}[e^{\diamond X}] = 1$), (4.6) and (4.4) we have

$$U'(t) = -\left(e^{\diamond X(t)}\right)^{\diamond(-1)} \int_0^t f(s) ds + C,$$

where C is an arbitrary random constant. Upon integrating once more, the above yields a formula for the solution in the distribution space $(\mathcal{S})_{-1}$.

Before proceeding with the proof of Theorem 4.12 we need the following technical lemma to handle the stochastic constant κ .

Lemma 4.16. *Let $X_\varepsilon \rightarrow X$ in $L^2_{\mathcal{H}}(\Omega)$, where $\mathbb{E}[X] \neq 0$, be such that*

$$\sup_{\varepsilon > 0} \mathbb{E}[X_\varepsilon^{-2}] < \infty.$$

Then there exists q that is independent of ε such that $X_\varepsilon^{\diamond(-1)} \rightarrow X^{\diamond(-1)}$ in $(\mathcal{S})_{-1,-q}$.

Proof. By [17, Proposition 3.1] we have $X_\varepsilon^{\diamond(-1)} \rightarrow X^{\diamond(-1)}$ in $(\mathcal{S})_{-1,-q}$ for some q if $(SX_\varepsilon^{\diamond(-1)})(h) \rightarrow (SX^{\diamond(-1)})(h)$ and

$$(SX_\varepsilon^{\diamond(-1)})(h) \leq C e^{K|h|_{2,p}^2}$$

for constants C, K and $p > 0$ that does not depend on ε ³. The first assertion is clear from (4.8) together with the fact $X_\varepsilon \rightarrow X$ in $L^2_{\mathcal{H}}(\Omega)$. For the second assertion, note that by Cauchy-Schwarz inequality we obtain since $\mathbb{E}[e^{\diamond h}] = 1$,

$$\mathbb{E}[X_\varepsilon e^{\diamond h}] \geq \frac{\mathbb{E}[e^{\diamond h}]}{\mathbb{E}\left[\frac{e^{\diamond h}}{X_\varepsilon}\right]} \geq \frac{\mathbb{E}[e^{\diamond h}]}{\mathbb{E}[(e^{\diamond h})^2] \mathbb{E}[X_\varepsilon^{-2}]} \geq C e^{-\mathbb{E}[h^2]},$$

³For the precise definition and properties of the norm $|h|_{2,p}$, see e.g. [26].

where the last inequality follows from the assumption and the fact that $\mathbb{E}[(e^{\diamond h})^2] = e^{\mathbb{E}[h^2]}$, see [20, Corollary 3.37]. Hence, again by (4.8), we obtain

$$(SX_{\varepsilon}^{\diamond(-1)})(h) \leq C e^{\mathbb{E}[h^2]}$$

which completes the proof since $\mathbb{E}[h^2] \leq |h|_{2,p}^2$ for any $p > 0$, see [26, pp. 214]. $\square$

Proof of Theorem 4.12. We prove this for only in the case of periodic boundary data, as again the other cases can be proved with similar arguments. Moreover, we set $U(T) = U(0) = 0$ (recall that the solution is unique up to additive constant). In this case the approximated equation is given by

$$-\left(e^{\diamond(-X_{\beta,\varepsilon}(t))} \diamond U'_{\varepsilon}(t)\right)' = f(t), \quad t \in (0, T) \quad (4.11)$$

with boundary conditions

$$\begin{aligned} U_{\varepsilon}(0) &= U_{\varepsilon}(T) \\ e^{\diamond(-X_{\beta,\varepsilon}(0))} \diamond U'_{\varepsilon}(0) &= e^{\diamond(-X_{\beta,\varepsilon}(T))} \diamond U'_{\varepsilon}(T). \end{aligned}$$

Integrating once and Wick multiplying with the inverse $(e^{\diamond(-X_{\beta,\varepsilon}(t))})^{\diamond(-1)} = e^{\diamond X_{\beta,\varepsilon}(t)}$ yields

$$U'_{\varepsilon}(t) = \left(-\int_0^t f(s) ds + \kappa_{\varepsilon}\right) \diamond e^{\diamond X_{\beta,\varepsilon}(t)}$$

for some stochastic constant κ_{ε} . Integrating once more yields

$$U_{\varepsilon}(t) = \int_0^t (\kappa_{\varepsilon} + F(s)) \diamond e^{\diamond X_{\beta,\varepsilon}(s)} ds, \quad (4.12)$$

where the boundary conditions implies that κ_{ε} is given by

$$\kappa_{\varepsilon} = -\left(\int_0^T F(s) e^{\diamond X_{\beta,\varepsilon}(s)} ds\right) \diamond \left(\int_0^T e^{\diamond X_{\beta,\varepsilon}(s)} ds\right)^{\diamond(-1)}.$$

Taking the S -transform now gives

$$(SU_{\varepsilon}(t))(h) = (S\kappa_{\varepsilon})(h) \left(S\left[\int_0^t e^{\diamond X_{\beta,\varepsilon}(s)} ds\right]\right)(h) + \left(S\left[\int_0^t F(s) e^{\diamond X_{\beta,\varepsilon}(s)} ds\right]\right)(h).$$

Note that here we have, for all $t \in (0, T)$,

$$\int_0^t e^{\diamond X_{\beta,\varepsilon}(s)} ds \rightarrow \int_0^t e^{\diamond X_{\beta}(s)} ds$$

and

$$\int_0^t F(s) e^{\diamond X_{\beta,\varepsilon}(s)} ds \rightarrow \int_0^t F(s) e^{\diamond X_{\beta}(s)} ds$$

in $L^2_{\mathcal{H}}(\Omega)$. Since the random variable $\int_0^T e^{\diamond X_{\beta,\varepsilon}(s)} ds$ has all negative moments finite, uniformly in ε^4 . By Lemma 4.16, it follows that there exists q large enough, independent

⁴This can be easily deduced from the Kahane inequality, see e.g. [28, Theorem 2.1], or by adding smoothing to the proof in [1, Appendix B].

of ε , such that

$$\kappa_\varepsilon = - \left(\int_0^T F(s) e^{\diamond X_{\beta,\varepsilon}(s)} ds \right) \diamond \left(\int_0^T e^{\diamond X_{\beta,\varepsilon}(s)} ds \right)^{\diamond(-1)} \in (\mathcal{S})_{-1,-q}.$$

This further implies that the solution $U_\varepsilon(t) \in (\mathcal{S})_{-1,-q}$ for large enough q that does not depend on ε , and in particular this determines the class of test points h given by the condition $e^{\diamond h} \in (\mathcal{S})_{1,q}$. Now passing to the limit and using the above convergences in $L^2_{\mathcal{H}}(\Omega)$ (that implies convergence of the S -transforms on any test point h) yields, for any h such that $e^{\diamond h} \in (\mathcal{S})_{1,q}$ where now q is independent of ε ,

$$\lim_{\varepsilon \rightarrow 0} (SU_\varepsilon(t))(h) = (S\kappa)(h) \left(S \int_0^t e^{\diamond X_\beta(s)} ds \right)(h) + \left(S \int_0^t F(s) e^{\diamond X_\beta(s)} ds \right)(h),$$

where

$$\kappa = - \left(\int_0^T F(s) e^{\diamond X_\beta(s)} ds \right) \diamond \left(\int_0^T e^{\diamond X_\beta(s)} ds \right)^{\diamond(-1)} \in (\mathcal{S})_{-1,-q}.$$

This provides us the claimed $U(t)$ as the solution. Finally, the uniqueness follows from the invertibility of the S -transform. This completes the proof. $\square$

5. S -TRANSFORM AT X_β AND PROJECTIONS INTO A SUBSPACE

By Theorem 3.3 and Theorem 4.12, the solutions to problems (3.1) and (4.1) respectively are very similar, and essentially generated by random variables of the form

$$\int_0^T \varphi(y, s) e^{\diamond X_\beta(s)} ds$$

for some suitable $\varphi(y, s)$. Motivated by this, we consider projections into a subspace $L^2_{\beta'}(\Omega) \subset L^2(\Omega)$ generated by random variables of form

$$\int_0^T \varphi(y, s) e^{\diamond X_{\beta'}(s)} ds$$

with appropriately chosen $\beta' < 1$. Choosing $\beta' \neq \beta$, corresponds to projecting the solution into a subspace with less / higher order moments, cf. Theorem 2.6. To this end, we introduce the S -transform at “a point” $X_{\beta'}(s)$ directly without mollification, and show that these S -transforms characterize our subspace $L^2_{\beta'}(\Omega)$, defined as the subspace generated by the GMC measure $d\mu_{\beta'}$.

For the sake of generality and the purposes of a forthcoming paper studying the d -dimensional case, we introduce our definitions concerning log-correlated fields and GMC measures on a smooth bounded domain $D \subset \mathbb{R}^d$ from which we naturally recover our one-dimensional case by choosing $D = (0, T)$. In the general d -dimensional case, we consider values $\beta < \sqrt{d}$ corresponding to the L^2 -range of the GMC measure. In the sequel, we always assume D to be a smooth and bounded domain. We also need to define spaces $W^{s,2}(D)$ and its topological dual $W_0^{-s,2}(D)$:

Definition 5.1. *Let $s \in \mathbb{R}$, then $W^{s,2}(\mathbb{R}^d)$ is defined as*

$$W^{s,2}(\mathbb{R}^d) = \{f \in S'(\mathbb{R}^d) : \mathcal{F}f \in L^1_{loc}(\mathbb{R}^d) \text{ s.t. } \|f\|_{W^{s,2}(\mathbb{R}^d)}^2 = \int_{\mathbb{R}^d} (1+|\xi|^2)^s |\mathcal{F}f(\xi)|^2 d\xi < \infty\},$$

where $\mathcal{F}f$ denotes the Fourier transform and S' denotes the space of Schwartz distributions.

Let $D \subset \mathbb{R}^d$ be smooth and bounded, then the space $W^{s,2}(D)$ is defined as restrictions $f|_D$ of functions $f \in W^{s,2}(\mathbb{R}^d)$ into D , equipped with the norm

$$\|f\|_{W^{s,2}(D)} = \inf\{\|g\|_{W^{s,2}(\mathbb{R}^d)}, g|_D = f\} \quad (5.1)$$

The topological dual of $W^{s,2}(D)$ (in the standard distributional duality sense) is simply $W_0^{-s,2}(D)$, i.e. the closure of $C_0^\infty(D)$ in $W^{-s,2}(D)$. We consider the following subspace of $L^2(\Omega)$.

Definition 5.2 ($L_\beta^2(\Omega)$ space generated by GMC $d\mu_\beta$). *Let $\beta \in (0, \sqrt{d})$. The space $L_\beta^2(\Omega) := L^2(\Omega, \mathbb{P}, \mathcal{F}_{d\mu_\beta})$ generated by the GMC measure $d\mu_\beta$ is the linear space spanned by random variables of the form*

$$\int_D \varphi(z) d\mu_\beta(z), \quad \varphi \in C^\infty(D). \quad (5.2)$$

Furthermore, L_β^2 is closed in $L^2(\Omega)$.

In order to characterize the random variables $Z \in L_\beta^2(\Omega)$ and define the S -transform at $X_\beta(s)$, we introduce the operator acting on $\phi : D \rightarrow \mathbb{R}$

$$\begin{aligned} G\phi(z) &= G_\alpha\phi(z) := \int_D \exp(\alpha \mathbb{E}[X(z)X(y)]) \phi(y) dy \\ &= \int_D |z - y|^{-\alpha} e^{\alpha g(z,y)} \phi(y) dy, \end{aligned} \quad (5.3)$$

where $\alpha \in (0, d)$ and X is a log-correlated Gaussian field as in Definition 2.1. Note that if $g \equiv 0$ in (2.1), then the operator G reduces to the Riesz potential on D . We will later on apply the operator G_α with the choices $\alpha \in \{\beta^2, \beta'^2, \beta\beta'\}$.

For the invertibility of the S -transform, we need to assume that the log-correlated field has the following additional property.

Definition 5.3. *We say that our log-correlated field $X_\beta := \beta X$ is β -non-degenerate on a bounded domain $D \subset \mathbb{R}^d$ if it satisfies the following property: there is a constant $c_0 > 0$ such that for each $\psi \in C_0^\infty(D)$ it holds that*

$$\int_D \psi(z) G_{\beta^2} \psi(z) dz \geq c_0 \|\psi\|_{W_0^{-s_\beta, 2}(D)}^2, \quad (5.4)$$

where $s_\beta = \frac{d-\beta^2}{2}$.

Our next result verifies that the class of such fields is not empty.

Lemma 5.4. *There are log-correlated fields X on D such that the fields $X_\beta := \beta X$ are β -non-degenerate on D for all $\beta \in (0, \sqrt{d})$.*

Proof. Let $\psi_0 \in C_0^\infty(\mathbb{R}^d)$ be a positive cut-off function equal to 1 on $B(0, 4R)$, where $R := \text{diam}(D)$. Since the Fourier-transform of $\log(1/|\cdot|)$ is equal (up to a multiplicative constant) to $|\xi|^{-d}$ for large ξ , we infer that the Fourier-transform of $\psi_0 \log(1/|\cdot|)$ is a smooth function that is positive outside a suitable neighborhood of the origin. We then

choose a radially symmetric and positive $r \in C_0^\infty(\mathbb{R}^d)$ such that $\mathcal{F}(\psi_0 \log(1/|\cdot|)) + r \geq 0$ everywhere. Then

$$\psi_0(z - y) \log(1/|z - y|) + h(z - y), \quad \text{where } h := \mathcal{F}^{-1}r,$$

is a covariance on $\mathbb{R}^d$, and we denote by Y the Gaussian distribution valued field with this covariance. We claim that the field $X := Y|_D$ is the field we are looking for.

For that end, by using the assumption on the support of ψ_0 we note that the operator $G := G_{\beta^2}$ for the field X has the kernel $H(z - y)$ ($z, y \in D$), where

$$H(z) := e^{\beta^2 r(z)} |z|^{-\beta^2}.$$

We extend H to all of $\mathbb{R}^d$ by this formula and note that

$$a := \mathcal{F}\left(e^{\beta^2 r(z)} - 1\right) = \sum_{n=1}^{\infty} \frac{\beta^{2n} r^{*n}}{n!}$$

where $r^{*n} = r * r * \dots * r$ is the n -fold convolution. Thus, $a \geq 0$ and a is a Schwarz function as it is the Fourier transform of a test function. Thus,

$$\mathcal{F}H = |\cdot|^{-2s_\beta} + a * |\cdot|^{-2s_\beta}$$

that decays as $|\cdot|^{-2s_\beta}$. The claim follows, since if $\psi \in C_0^\infty(D)$, we obtain by the previous estimate and Plancherel identity that for some constant c independent of ψ that

$$\int_D \psi(z)(G\psi)(z) dz = \int_{\mathbb{R}^d} \mathcal{F}H(\xi) |\mathcal{F}\psi(\xi)|^2 d\xi \geq c \|\psi\|_{W_0^{-s_\beta, 2}(\mathbb{R}^d)}^2,$$

where the last inequality follows as $\mathcal{F}H(\xi) \geq c|\xi|^{-2s_\beta}$. □

Lemma 5.5. *Let $\alpha \in (0, d)$. Assume that the domain D is smooth and bounded and the field X is β -non-degenerate. Then the mapping G_α given in (5.3) extends to a bijective map*

$$G : W_0^{-\frac{d-\alpha}{2}, 2}(D) \rightarrow W^{\frac{d-\alpha}{2}, 2}(D).$$

Proof. Let $s_\alpha = (d - \alpha)/2$. We first verify that G is bounded between the stated spaces. For that end, let G_0 denote the operator with the kernel $H_0(z - y)$, where

$$H_0(z) := |z|^{2s_\alpha - d} \psi_0(z),$$

with ψ_0 as in the previous lemma. Now

$$\mathcal{F}H_0 = \mathcal{F}\psi_0 * |\cdot|^{-2s_\alpha} \lesssim (1 + |\cdot|^2)^{-s_\alpha}.$$

Since G_0 is a convolution operator, we immediately obtain that $G_0 : W_0^{-s_\alpha, 2}(D) \rightarrow W^{s_\alpha, 2}(D)$. In order to consider the operator G , denote for $\lambda \in \mathbb{R}^d$ by M_λ the multiplication operator $f \mapsto e^{i\lambda \cdot x} f$. The norm of M_λ on both of the spaces $W_0^{-s_\alpha, 2}(D)$ and $W^{s_\alpha, 2}(D)$ is bounded by $c(1 + |\lambda|)^a$ with some fixed a , since on the Fourier side M_λ corresponds to translation by λ . The action of G in our situation is given by the kernel

$k(z, y)H_0(z - y)$, where $k(z, y) := \psi_0(z)\psi_0(y)e^{\alpha g(z, y)}$. The boundedness of G is then seen by writing

$$G = G_0 + \int_{\mathbb{R}^d \times \mathbb{R}^d} \mathcal{F}k(\lambda_1, \lambda_2) M_{\lambda_1} G_0 M_{\lambda_2} d\lambda_1 d\lambda_2$$

and use the fact that $\mathcal{F}k$ is a Schwarz test function.

Now the rest of the statement follows easily just by the non-degeneracy of $X_{\sqrt{\alpha}}$ and the fact that because D is smooth, the spaces $W_0^{-s_{\alpha}, 2}(D)$ and $W^{s_{\alpha}, 2}(D)$ are dual spaces of each others with respect to the L^2 -pairing. Namely, the non-degeneracy condition implies that $G : W_0^{-s_{\alpha}, 2}(D) \rightarrow W^{s_{\alpha}, 2}(D)$ is lower bounded. Hence, it is injective and its image, call it M , is a closed subspace of $W^{s_{\alpha}, 2}(D)$. If M would not be all of $W^{s_{\alpha}, 2}(D)$, we could find a non-zero dual element $\psi \in W_0^{-s_{\alpha}, 2}(D)$ which is orthogonal to all elements in M . Especially ψ should annihilate $G\psi$, but this contradicts the non-degeneracy assumption. $\square$

Remark 5.6. Note that by the above arguments we actually see that $G_{\delta} : W_0^{\alpha+\delta-d, 2}(D) \rightarrow W^{\alpha, 2}(D)$ for any $\delta \in (0, d)$ and $\alpha > 0$ such that $\alpha + \delta < d$. Indeed, when G_{δ} is the convolution operator with kernel H_{δ} , we have $\mathcal{F}(G_{\delta}u) = \mathcal{F}H_{\delta}\mathcal{F}u$, where

$$\mathcal{F}H_{\delta} \lesssim (1 + |\cdot|^2)^{-\frac{d-\delta}{2}}.$$

Thus we have

$$\begin{aligned} \int_{\mathbb{R}^d} (1 + |\xi|^2)^{\alpha} |\mathcal{F}(G_{\delta}u)(\xi)|^2 d\xi &= \int_{\mathbb{R}^d} (1 + |\xi|^2)^{\alpha} |\mathcal{F}H_{\delta}(\xi)\mathcal{F}u(\xi)|^2 d\xi \\ &\lesssim \int_{\mathbb{R}^d} (1 + |\xi|^2)^{\alpha+\delta-d} |\mathcal{F}u(\xi)|^2 d\xi \end{aligned}$$

which is finite for any $u \in W_0^{\alpha+\delta-d, 2}(\mathbb{R}^d)$ by Definition 5.1. Taking into account (5.1), this implies that G_{δ} is a bounded operator from $W_0^{\alpha+\delta-1, 2}(D)$ into $W^{\alpha, 2}(D)$. Finally, the case of general operator G follows by the arguments of the proof of Lemma 5.5.

We obtain immediately the following result characterizing random variables in $L_{\beta}^2(\Omega)$.

Corollary 5.7. *Let $D \subset \mathbb{R}^d$ be a smooth and bounded domain and let $\beta \in (0, \sqrt{d})$. Suppose X is a β -non-degenerate and let $Z \in L_{\beta}^2(\Omega)$. Then Z has the form*

$$Z = \int_D \varphi(z) d\mu_{\beta}(z),$$

for some $\varphi \in W_0^{-s_{\beta}, 2}(D)$, where $s_{\beta} = \frac{d-\beta^2}{2}$.

Proof. For $Z = \int_D \varphi(z) d\mu_{\beta}(z)$ with $\varphi \in C^{\infty}(D)$ we have

$$\begin{aligned} \mathbb{E}[Z^2] &= \int_D \int_D \varphi(z)\varphi(y) \exp(\beta^2 \mathbb{E}[X(y)X(z)]) dy dz \\ &= \int_D \varphi(z) G_{\beta^2} \varphi(z) dz \approx \|\varphi\|_{W_0^{-s_{\beta}, 2}([0, T])}^2 \end{aligned}$$

by Definition 5.3 and the continuity of G_{β^2} . The claim now follows by taking the closure in $L^2(\Omega)$. $\square$

We are now ready to define the S -transform of square integrable random variables at a point $X_\beta(z)$. For this, observe that for $V \in L_\beta^2(\Omega)$ of form $V = \int_D \psi(z) d\mu_\beta(z)$, where $\psi \in W_0^{-s_\beta, 2}(D)$ with $s_\beta = \frac{d-\beta^2}{2}$, formal computations yield

$$ZV = \int_D \psi(z) Z e^{\diamond X_\beta(z)} dz$$

for any $Z \in L^2(\Omega)$ from which taking expectation gives, formally,

$$\mathbb{E}[ZV] = \int_D \psi(z) (SZ)(X_\beta(z)) dz.$$

More precisely, the mapping $\psi \mapsto \mathbb{E}[ZV]$ is a continuous functional on $W_0^{-s_\beta, 2}(D)$. By duality, there exists $\iota \in W^{s_\beta, 2}(D)$ such that

$$\mathbb{E}[ZV] = \int_D \psi(z) \iota(z) dz.$$

This leads to the following definition.

Definition 5.8 (S -transform in $L_\beta^2(\Omega)$). *Let $Z \in L^2(\Omega)$ and set $s_\beta = \frac{d-\beta^2}{2}$. For $z \in D$ we identify $(SZ)(X_\beta(z))$ as the (unique) element of $W^{s_\beta, 2}(D)$ defined through duality*

$$\mathbb{E}[ZV] = \int_D \psi(z) (SZ)(X_\beta(z)) dz$$

for all $V = \int_D \psi(z) d\mu_\beta(z) \in L_\beta^2(\Omega)$.

Note that the definition makes sense for almost every z which is sufficient for our purposes. The following result allows us to characterize the subspace $L_\beta^2(\Omega)$ (or projections into it) via the S -transform at points $X_\beta(z)$.

Lemma 5.9. *Let $Z \in L^2(\Omega)$ with a decomposition $Z = Z_\beta + Z'_\beta$, where $Z_\beta = \int_D \varphi(s) d\mu_\beta(s) \in L_\beta^2(\Omega)$ is the orthogonal projection into the subspace $L_\beta^2(\Omega) \subset L^2(\Omega)$. Then we have*

$$(SZ_\beta)(X_\beta(z)) = (G_{\beta^2} \varphi)(z)$$

and

$$(SZ'_\beta)(X_\beta(z)) = 0.$$

Proof. The claim $(SZ'_\beta)(X_\beta(z)) = 0$ follows immediately from Definition 5.8 by observing that then

$$\int_D \psi(z) (SZ'_\beta)(X_\beta(z)) dz = 0 \quad \forall \psi \in W_0^{-s_\beta, 2}(D).$$

For Z_β , obtain that

$$\mathbb{E}[Z_\beta V] = \mathbb{E} \left[\int_D \int_D \psi(z) \varphi(y) d\mu_\beta(z) d\mu_\beta(y) \right] = \int_D \psi(z) (G_{\beta^2} \varphi)(z) dz,$$

where now, by Lemma 5.5, $(G_{\beta^2} \varphi)(z) \in W^{s_\beta, 2}(D)$. This completes the proof. $\square$

This leads to the following characterization result of random variables in $L^2_\beta(\Omega)$ via the S -transform:

Lemma 5.10. *The S -transform at points $X_\beta(z)$ determine random variables $Z \in L^2_\beta(\Omega)$ uniquely.*

Proof. Suppose that for random variables $Z_1 = \int_D \varphi_1(y) d\mu_\beta(y)$ and $Z_2 = \int_D \varphi_2(y) d\mu_\beta(y)$ we have $(SZ_1)(X_\beta(z)) = (SZ_2)(X_\beta(z))$ for almost every $z \in D$. Then $G_{\beta^2} \varphi_1(z) = G_{\beta^2} \varphi_2(z)$ for almost every $z \in D$ from which $\varphi_1 = \varphi_2$ follows by Lemma 5.5. $\square$

We now leave the general d -dimensional setting and focus on the case $d = 1$ with $D = (0, T)$. Hence, in the sequel, we have $s_\beta = \frac{1-\beta^2}{2}$. By using the S -transform, we are able to give explicit expressions for the projections into $L^2_{\beta'}(\Omega)$ of random variables in $L^2_\beta(\Omega)$, for $0 < \beta, \beta' < 1$.

Proposition 5.11. *Let $\beta, \beta' \in (0, 1)$ and let*

$$Z = \int_D \varphi(z) d\mu_\beta(z) \in L^2_\beta(\Omega)$$

for some $\varphi \in W_0^{-\frac{1-\beta^2}{2}, 2}(0, T)$. Then the projection of Z into the subspace $L^2_{\beta'}(\Omega)$ is given by

$$Z_{\beta'} = \int_0^T \varphi_{\beta'}(z) d\mu_{\beta'}(z),$$

where $\varphi_{\beta'} \in W_0^{-\frac{1-(\beta')^2}{2}, 2}(0, T)$ solves $G_{\beta\beta'} \varphi = G_{(\beta')^2} \varphi_{\beta'}$.

Proof. Let $Z = \int_0^T \varphi(z) d\mu_\beta(z) \in L^2_\beta(\Omega)$ and $V = \int_0^T \psi(z) d\mu_{\beta'}(z) \in L^2_{\beta'}(\Omega)$ be a test random variable. It is enough to check that $\mathbb{E}[Z_{\beta'} V] = \mathbb{E}[ZV]$ for each $\psi \in W_0^{-\frac{1-(\beta')^2}{2}, 2}(0, T)$. Here we have

$$\mathbb{E}[Z_{\beta'} V] = \int_0^T \psi(z) G_{(\beta')^2} \varphi_{\beta'}(z) dz.$$

Similarly, Definition 5.8 yields

$$\mathbb{E}[ZV] = \int_0^T \psi(z) (SZ)(X_{\beta'}(z)) dz$$

for $(SZ)(X_{\beta'}(z)) \in W^{\frac{1-(\beta')^2}{2}, 2}(0, T)$ while formal computations as above gives

$$\mathbb{E}[ZV] = \int_0^T \psi(z) G_{\beta\beta'} \varphi(z) dz, \tag{5.5}$$

where now $\psi \in W_0^{-\frac{1-(\beta')^2}{2}, 2}(0, T)$ and $\varphi \in W_0^{-\frac{1-\beta^2}{2}, 2}(0, T)$. We now claim that

$$G_{\beta\beta'} \varphi(z) \in W^{\frac{1-(\beta')^2}{2}, 2}(0, T). \tag{5.6}$$

Using claim (5.6), we see that (5.5) follows, since from Definition 5.8, the uniqueness of the S -transform gives $(SZ)(X_{\beta'}(z)) = G_{\beta\beta'} \varphi(z)$. Now Lemma 5.5 gives that $G_{(\beta')^2}$ is invertible and hence $\mathbb{E}[Z_{\beta'} V] = \mathbb{E}[ZV]$ holds if and only if $G_{\beta\beta'} \varphi = G_{(\beta')^2} \varphi_{\beta'}$.

For the claim (5.6) note that, by Remark 5.6, we have $G_{\beta\beta'} : W_0^{\alpha+\beta\beta'-1,2}(0,T) \rightarrow W^{\alpha,2}(0,T)$ for any $\alpha \in (0, 1 - \beta\beta')$. In particular, this holds for $\alpha = \frac{1}{2} - \frac{(\beta')^2}{2}$ yielding $G_{\beta\beta'} : W_0^{-\frac{1}{2}-\frac{(\beta')^2}{2}+\beta\beta',2}(0,T) \rightarrow W^{\frac{1}{2}-\frac{(\beta')^2}{2},2}(0,T)$. Now (5.6) follows by noting that $W_0^{-\frac{1-\beta^2}{2}}(0,T) \subset W_0^{-\frac{1}{2}-\frac{(\beta')^2}{2}+\beta\beta',2}(0,T)$ since

$$-\frac{1}{2} - \frac{(\beta')^2}{2} + \beta\beta' \leq -\frac{1-\beta^2}{2}$$

is trivially true. Hence, (5.6) is proved. $\square$

We now address projections of solutions to (4.1) into $L_{\beta'}^2(\Omega)$, for some given $\beta' \geq \beta$. As the solution given by Theorem 4.12 is a proper element of $L^2(\Omega)$ only in the case of initial value problem or with Neumann boundary condition, we define the (generalized) projection via the corresponding S -transformed deterministic equation, in the spirit of Definition 4.11. To this end, recall that by taking the S -transform in (4.1) on a test point h gives us, after interchanging the order of expectation and differentiation, a deterministic equation

$$\left(e^{-\mathbb{E}[hX_\beta(\cdot)]} u'_h(\cdot) \right)' = f(\cdot),$$

where $u_h(\cdot) = SU(\cdot)(h)$ is the S -transform of the stochastic solution U tested at h . Similarly, the terms $e^{\diamond(-X_\beta(0))} \diamond U'(0)$ and $e^{\diamond(-X_\beta(T))} \diamond U'(T)$ in the boundary conditions are translated into $e^{-\mathbb{E}[hX_\beta(0)]} u'_h(0)$ and $e^{-\mathbb{E}[hX_\beta(T)]} u'_h(T)$. Hence, assuming that $U(y) \in L^2(\Omega)$ and if we are interested in the projection into $L_{\beta'}^2(\Omega)$, we would formally choose $h = X_{\beta'}(z)$. This motivates the following definition.

Definition 5.12. *For any solution to (4.1), its projection into $L_{\beta'}^2(\Omega)$ for $\beta' \in (0, 1)$ is a random function $U(y)$ such that $U(y) \in L_{\beta'}^2(\Omega)$ for all $y \in (0, T)$ and $\tilde{u}_z(y) = (SU(y))(X_{\beta'}(z)) \in W^{s_{\beta'},2}(0, T)$ solves the deterministic equation*

$$\left(e^{-\beta\beta'\mathbb{E}[X(z)X(\cdot)]} \tilde{u}'_z(\cdot) \right)' = f(\cdot) \quad (5.7)$$

with the corresponding boundary terms given by

(i) Initial data:

$$\tilde{u}_z(0) = U_1 \quad \text{and} \quad e^{-\beta\beta'\mathbb{E}[X(z)X(0)]} \tilde{u}'_z(0) = U_2,$$

(ii) Dirichlet data:

$$\tilde{u}_z(0) = U_1 \quad \text{and} \quad \tilde{u}_z(T) = U_2,$$

(iii) Neumann boundary conditions:

$$e^{-\beta\beta'\mathbb{E}[X(z)X(0)]} \tilde{u}'_z(0) = U_1 \quad \text{and} \quad e^{-\beta\beta'\mathbb{E}[X(z)X(T)]} \tilde{u}'_z(T) = U_2,$$

(iv) Periodic boundary conditions:

$$\tilde{u}_z(0) = \tilde{u}_z(T) \quad \text{and} \quad e^{-\beta\beta'\mathbb{E}[X(z)X(0)]} \tilde{u}'_z(0) = e^{-\beta\beta'\mathbb{E}[X(z)X(T)]} \tilde{u}'_z(T).$$

Remark 5.13. It turns out that this definition makes sense. Indeed, in the case of the initial value problem or with Neumann boundary conditions, the solution to (4.1) is actually an element of $L^2_\beta(\Omega)$ with suitable kernel φ , and hence we can apply Proposition 5.11 to compute the projection into $L^2_{\beta'}(\Omega)$ that corresponds to the one obtained through Definition 5.12, cf. theorem below. In particular for $\beta' = \beta$, the S -transformed equation (5.7) evaluated at $X_\beta(z)$ yields precisely the solutions given in Theorem 4.12.

Theorem 5.14. *Let $\beta < 1$ and $\beta' \in [\beta, 1)$. Then for any solution u to (4.1) and any $y \in (0, T)$, the projection of U into $L^2_{\beta'}(\Omega)$ (in the sense of Definition 5.12) is given by*

$$U(y) = \int_0^T \varphi(y, a) d\mu_{\beta'}(a)$$

with

$$\varphi(y, a) = G_{\beta/2}^{-1}[\tilde{u}_z(y)](a),$$

and $\tilde{u}_z(y)$ is the solution to the deterministic problem (5.7).

The proof makes use of the following elementary lemma. For the reader's convenience, we present a short proof.

Lemma 5.15. *Let $s \in (0, 1)$ and let $T < \infty$ be fixed. Let $f, g \in W^{s,2}(0, T)$.*

- *If $f, g \in L^\infty(0, T)$, then $fg \in W^{s,2}(0, T)$.*
- *If $f, 1/g \in L^\infty(0, T)$, then $f/g \in L^\infty(0, T) \cap W^{s,2}(0, T)$.*

Proof. Recall that the norm of $W^{s,2}(0, T)$ can be given as

$$\|f\|_{W^{s,2}(0,T)} = \|f\|_{L^2(0,T)} + \left(\int_0^T \int_0^T \frac{|f(x) - f(y)|^2}{|x - y|^{1+2s}} dx dy \right)^{1/2}.$$

Since $[0, T]$ is compact, we have $L^\infty(0, T) \subset L^2(0, T)$ and hence it suffices to consider the seminorm part. By using triangle and Jensen's inequality

$$|f(x)g(x) - f(y)g(y)|^2 \leq 2[|f(x)|^2|g(x) - g(y)|^2 + |g(y)|^2|f(x) - f(y)|^2]$$

gives us the first claim, since $f, g \in L^\infty(0, T) \cap W^{s,2}(0, T)$. Similarly, for the second claim, we write

$$\left| \frac{f(x)}{g(x)} - \frac{f(y)}{g(y)} \right| \leq \frac{|f(x) - f(y)|}{|g(x)|} + \frac{|f(y)|}{|g(x)g(y)|} |g(x) - g(y)|$$

from which the claim follows since $f, 1/g \in L^\infty(0, T)$ and $f, g \in W^{s,2}(0, T)$. $\square$

Proof of Theorem 5.14. It is a routine exercise to check that the unique solution to equation (5.7) is given by

$$\tilde{u}_z(t) = \tilde{u}_z(0) + \int_0^t \left[e^{\beta\beta'\mathbb{E}[X(z)X(y)]} e^{-\beta\beta'\mathbb{E}[X(z)X(0)]} \tilde{u}'_z(0) + e^{\beta\beta'\mathbb{E}[X(z)X(y)]} F(y) \right] dy,$$

where $u(0)$ and $\tilde{u}'_z(0)$ are determined by the boundary/initial conditions. Hence, in order to prove the result it suffices to show that this solution is an element of $W^{s_{\beta'},2}(0, T)$

w.r.t. the test point z . Indeed, then it follows that the projection of $U(t)$ into $L^2_{\beta'}(\Omega)$ has a representation

$$U_{\beta'}(t) = \int_0^T \left[G_{\beta'^2}^{-1} \tilde{u}_\cdot(t) \right] (z) d\mu_{\beta'}(z).$$

Observe now that $\tilde{u}_z(t)$ can be written as

$$\tilde{u}_z(t) = \tilde{u}_z(0) + e^{-\beta\beta'\mathbb{E}[X(z)X(0)]} \tilde{u}'_z(0) (G_{\beta\beta'} I_{[0,t]})(z) + (G_{\beta\beta'} [FI_{[0,t]}])(z). \quad (5.8)$$

Here we have $FI_{[0,t]} \in W^{-\delta,2}(0,T)$ for any $\delta > 0$, and thus, thanks to Lemma 5.5, we have $G_{\beta\beta'} [FI_{[0,t]}] \in W^{\frac{1-\beta\beta'}{2},2}(0,T)$. Moreover, [6, Proposition 2.1] implies that $W^{\frac{1-\beta\beta'}{2},2}(0,T) \subset W^{s_{\beta'},2}(0,T)$ since $1 - \beta\beta' \geq 1 - \beta'^2$. Thus,

$$G_{\beta\beta'} [FI_{[0,t]}] \in W^{s_{\beta'},2}(0,T)$$

and hence, by linearity, it suffices to consider the terms

$$\tilde{u}_z(0) + e^{-\beta\beta'\mathbb{E}[X(z)X(0)]} \tilde{u}'_z(0) (G_{\beta\beta'} I_{[0,t]})(z)$$

arising from boundary conditions. For the initial value problem we have $\tilde{u}_z(0) = U_1$ and $e^{-\beta\beta'\mathbb{E}[X(z)X(0)]} \tilde{u}'_z(0) = U_2$ from which the claim follows directly by the above argument (by replacing F with a constant function 1). Similarly, with Neumann boundary conditions we may assume (by shifting with an arbitrary constant in the solution if necessary) that $\tilde{u}_z(0) = 0$. Thus, this case follows directly by the above argument as well. With Dirichlet data we see that (5.8) at time T becomes

$$U_2 = \tilde{u}_z(T) = U_1 + e^{-\beta\beta'\mathbb{E}[X(z)X(0)]} \tilde{u}'_z(0) (G_{\beta\beta'} I_{[0,T]})(z) + (G_{\beta\beta'} [FI_{[0,T]}])(z)$$

from which we get

$$e^{-\beta\beta'\mathbb{E}[X(z)X(0)]} \tilde{u}'_z(0) = \frac{U_2 - U_1 - (G_{\beta\beta'} [FI_{[0,T]}])(z)}{(G_{\beta\beta'} I_{[0,T]})(z)}.$$

Inserting the above into (5.8), we are left to prove that

$$(G_{\beta\beta'} I_{[0,t]})(z) \cdot \frac{U_2 - U_1 - (G_{\beta\beta'} [FI_{[0,T]}])(z)}{(G_{\beta\beta'} I_{[0,T]})(z)} \in W^{s_{\beta'},2}(0,T).$$

Note that here

$$\begin{aligned} (G_{\beta\beta'} I_{[0,t]})(z) &= \int_0^T |z - y|^{-\beta\beta'} e^{\beta\beta'g(z,y)} I_{[0,t]}(y) dy \\ &= \int_0^t |z - y|^{-\beta\beta'} e^{\beta\beta'g(z,y)} dy. \end{aligned}$$

Since g is a bounded function and $\beta\beta' < 1$, it follows that, for any $t \in (0, T)$, $G_{\beta\beta'} I_{[0,t]}(z)$ is $1 - \beta\beta'$ Hölder continuous. Thus, for any $\beta' \geq \beta$ we have $G_{\beta\beta'} I_{[0,t]} \in L^\infty([0, T]) \cap W^{s_{\beta'},2}(0, T)$. Indeed, for $\beta' > \beta$ this follows from $C^{1-\beta\beta'}(0, T) \subset W^{s_{\beta'},2}(0, T)$ since

$$1 - \beta\beta' \geq 1 - (\beta')^2$$

while for $\beta' = \beta$ this follows directly from the fact $I_{[0,t]} \in W_0^{-s_\beta, 2}(0, T)$ and Lemma 5.5. Moreover, for any $t \in (0, T)$ the function $G_{\beta\beta'} I_{[0,t]}$ is bounded from below, and hence

$$\frac{1}{G_{\beta\beta'} I_{[0,t]}} \in L^\infty([0, T]) \cap W^{s_{\beta'}, 2}(0, T) \quad \text{for any } t \in (0, T).$$

Arguing similarly we also see that $U_2 - U_1 - (G_{\beta\beta'} F I_{[0,T]})(z) \in L^\infty([0, T]) \cap W^{s_{\beta'}, 2}(0, T)$ and hence the claim follows from Lemma 5.15. Treating the periodic case with similar arguments completes the whole proof. $\square$

Remark 5.16. As pointed out in Remark 5.13, for the initial value problem or in the case of Neumann boundary conditions, choosing $\beta' = \beta$ in Theorem 5.14 yields precisely the solutions provided in Theorem 4.12. On the contrary, with Dirichlet boundary conditions we obtain that “the projection” of the Wick product

$$\kappa \diamond \int_0^t e^{\diamond X_\beta(s)} ds$$

into $L_\beta^2(\Omega)$ leads to the term (on the S -transformed side)

$$(G_{\beta^2} I_{[0,t]})(z) \cdot \frac{U_2 - U_1 - (G_{\beta^2} F I_{[0,T]})(z)}{(G_{\beta^2} I_{[0,T]})(z)}$$

which is precisely what one would expect when taking the S -transform at $X_\beta(z)$ on $\kappa \diamond \int_0^t e^{\diamond X_\beta(s)} ds$ and using identities $S(X \diamond Y) = (SX)(SY)$ and $(SX^{\diamond(-1)}) = 1/(SX)$. This computation is of course only formal as S -transform at points $X_\beta(z)$ is not defined in the Hida distribution space $(\mathcal{S})_{-1}$.

APPENDIX A. HIDA-KONDRATIEV DISTRIBUTION SPACES

In this section we briefly introduce the stochastic distribution spaces $(\mathcal{S})_\rho$ which morally are stochastic counterparts of the Schwartz space of rapidly decreasing functions $\mathcal{S}$ in real analysis, with $\mathcal{S}(\mathbb{R}^d) \subset L^2(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d)$, where $\mathcal{S}'$ is the dual of $\mathcal{S}$. We will list here only elementary definitions and properties without proofs; for a more thorough treatment, see for instance [16] or [23] and references therein.

A.1. Definitions and basic properties. We start by recalling that via the Bochner-Minlos theorem we obtain a probability measure μ on $\mathcal{B}(\mathcal{S}'(\mathbb{R}^d))$, called the one-dimensional white noise probability measure, such that

$$\mathbb{E}_\mu[e^{i\langle \cdot, \varphi \rangle}] = e^{-\frac{1}{2}\|\varphi\|^2}, \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^d).$$

Then, the resulting $L^2(\mu)$ -space admits a basis composed of essentially (suitably normalized) Hermite polynomials H_α , for $\alpha \in \mathbb{N}^{\mathbb{N}}$.

Definition A.1 (White noise). *The 1-dimensional, d -parameter (smoothed) white noise is the map*

$$w: \mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d) \rightarrow \mathbb{R}$$

defined by

$$w(\varphi) = \langle w, \varphi \rangle, \quad w \in \mathcal{S}'(\mathbb{R}^d), \varphi \in \mathcal{S}(\mathbb{R}^d).$$

The multidimensional case is treated in the usual way by tensorization: indeed, if m is a natural number, we define

$$\mathcal{S} := \prod_{i=1}^m \mathcal{S}(\mathbb{R}^d), \quad \mathcal{S}' := \prod_{i=1}^m \mathcal{S}'(\mathbb{R}^d), \quad \mathcal{B} := \prod_{i=1}^m \mathcal{B}(\mathcal{S}'(\mathbb{R}^d))$$

and equip $\mathcal{S}'$ with the product measure

$$\mu_m = \mu_1 \times \mu_1 \times \cdots \times \mu_1,$$

where μ_1 is the 1-dimensional white noise probability measure.

We then have

$$\int_{\mathcal{S}'} e^{i\langle \omega, \phi \rangle} d\mu_m(\omega) = e^{-\frac{1}{2}\|\phi\|^2}, \quad \text{for all } \phi \in \mathcal{S}.$$

Here $\langle \omega, \phi \rangle = \langle \omega_1, \phi_1 \rangle + \cdots + \langle \omega_m, \phi_m \rangle$, where $\langle \omega_k, \phi_k \rangle$ is the action of $\omega_k \in \mathcal{S}'(\mathbb{R}^d)$ on $\phi_k \in \mathcal{S}(\mathbb{R}^d)$ for each $k = 1, \dots, m$. Furthermore, denoting $\mathcal{K} = \oplus_{k=1}^m L^2(\mathbb{R}^d)$ the orthogonal direct sum of identical copies of $L^2(\mathbb{R}^d)$, we have

$$\|\phi\| = \|\phi\|_{\mathcal{K}} = \left(\sum_{k=1}^m \|\phi\|_{L^2(\mathbb{R}^d)}^2 \right)^{1/2} = \left(\sum_{k=1}^m \int_{\mathbb{R}^d} \phi_k^2(x) dx \right)^{1/2}.$$

The triplet $(\mathcal{S}', \mathcal{B}, \mu_m)$ is called the *d-parameter multidimensional white noise probability space*, and the parameter m is referred to as the *white noise dimension*.

Definition A.2 (M-dimensional white noise). *The m-dimensional (smoothed) white noise vector*

$$\mathbf{w}: \mathcal{S} \times \mathcal{S}' \rightarrow \mathbb{R}^m$$

is defined by

$$\mathbf{w}(\phi) = \mathbf{w}(\phi, \omega) = (\langle \omega_1, \phi_1 \rangle, \dots, \langle \omega_m, \phi_m \rangle),$$

for $\omega = (\omega_1, \dots, \omega_m) \in \mathcal{S}'$ and $\phi = (\phi_1, \dots, \phi_m) \in \mathcal{S}$.

We will next define a basis for $L^2(\mu_m)$. We start by defining the Hermite functions $\xi_n(x)$ by setting

$$\xi_n(x) = \pi^{-\frac{1}{4}} ((n-1)!)^{-1/2} e^{-\frac{1}{2}x^2} h_{n-1}(\sqrt{2}x), \quad n = 1, 2, \dots$$

where $h_n(x)$ are the classical orthogonal Hermite polynomials defined by

$$h_n(x) = (-1)^n e^{\frac{1}{2}x^2} \frac{d^n}{dx^n} \left(e^{-\frac{1}{2}x^2} \right), \quad n = 0, 1, 2, \dots \quad (\text{A.1})$$

We remark that since the Hermite functions ξ_n are essentially rescaled versions of Hermite polynomials, we have $\xi_n \in \mathcal{S}(\mathbb{R})$ for all $n \in \mathbb{N}$, and moreover, $\{\xi_n\}$ is an orthonormal basis for $L^2(\mathbb{R})$.

Now, let $\delta = (\delta_1, \dots, \delta_d) \in \mathbb{N}^d$ be a d-dimensional multi-index, and let $\delta^{(j)} = (\delta_1^{(j)}, \dots, \delta_d^{(j)})$ be the *j*th multi-index number in some fixed ordering of all d-dimensional multi-indices δ . We assume that this ordering occurs in increasing order, that is,

$$i < j \implies \delta_1^{(i)} + \delta_2^{(i)} + \dots + \delta_d^{(i)} \leq \delta_1^{(j)} + \dots + \delta_d^{(j)}.$$

Now define

$$\eta_j = \xi_{\delta^{(j)}} = \xi_{\delta_1^{(j)}} \otimes \cdots \otimes \xi_{\delta_d^{(j)}}, \quad j = 1, 2, \dots$$

Note that the collection $\{\eta_j\}$ constitutes a basis for $L^2(\mathbb{R}^d)$.

We will next construct an orthogonal basis for $L^2(\mu_m)$, and for this we will need to consider multi-indices of arbitrary length. We will denote by $\mathcal{J}$ the space of all sequences $\alpha = (\alpha_1, \alpha_2, \dots)$ with all coordinates $\alpha_j \in \mathbb{N}$ such that $\alpha_j \neq 0$ for only finitely many indices j .

Definition A.3 ($m=1$). *Let $\alpha = (\alpha_1, \alpha_2, \dots) \in \mathcal{J}$. Then we define*

$$H_\alpha(\omega) = H_\alpha^{(1)}(\omega) := \prod_{i=1}^{\infty} h_{\alpha_i}(\langle \omega, \eta_i \rangle), \quad \omega \in \mathcal{S}'(\mathbb{R}^d) \quad (\text{A.2})$$

where h_{α_i} is the Hermite polynomial of degree α_i given by (A.1).

We then define the multidimensional variant of (A.2):

Definition A.4 ($m > 1$). *Let $\Gamma = (\gamma^{(1)}, \dots, \gamma^{(m)}) \in \mathcal{J}^m = \mathcal{J} \times \cdots \times \mathcal{J}$, and $\omega = (\omega_1, \dots, \omega_m) \in \mathcal{S}$. Define*

$$\mathbf{H}_\Gamma(\omega) = \mathbf{H}_\Gamma^{(m)}(\omega) := \prod_{i=1}^m H_{\gamma^{(i)}}^{(1)}(\omega_i),$$

where each $H_{\gamma^{(i)}}^{(1)}(\omega_i)$ is as in (A.2).

These indeed constitute a basis:

Theorem A.5. *For any $m \geq 1$ the family $\{H_\alpha\}_{\alpha \in \mathcal{J}} = \{\mathbf{H}_\Gamma^{(m)}\}_{\Gamma \in \mathcal{J}^m}$ constitutes an orthonormal basis for $L^2(\mu_m)$. Moreover, if $\alpha = (\alpha_1, \alpha_2, \dots) \in \mathcal{J}$, we have*

$$\|H_\alpha\|_{L^2(\mu)}^2 = \alpha! = \alpha_1! \alpha_2! \cdots$$

Next, we fix parameter $d \geq 1$, the white noise dimension $m \geq 1$ and the state space dimension $N \geq 1$. Let

$$\mathbf{L}^2(\mu_m) = \bigoplus_{k=1}^N L^2(\mu_m).$$

Applying Theorem A.5 to each component of $\mathbf{L}^2(\mu_m)$ we obtain the Wiener-Itô chaos expansion:

Theorem A.6. *Every $f \in \mathbf{L}^2(\mu_m)$ has a unique representation*

$$f(\omega) = \sum_{\alpha \in \mathcal{J}} c_\alpha H_\alpha(\omega)$$

where $c_\alpha \in \mathbb{R}^N$ for all α . In addition, we have the isometry

$$\|f\|_{\mathbf{L}^2(\mu_m)}^2 = \sum_{\alpha \in \mathcal{J}} \alpha! |c_\alpha|^2,$$

where $|c_\alpha|^2 = c_\alpha \cdot c_\alpha$.

We fix some notation used in the sequel: let $\alpha = (\alpha_1, \dots, \alpha_j, \dots) \in \mathcal{J}$, and $\beta = (\beta_1, \dots, \beta_j, \dots) \in (\mathbb{R}^N)_c$ (each $\beta_j \in \mathbb{R}$, with only finitely many non-zero elements), we will use the notation

$$\alpha^\beta = \alpha_1^{\beta_1} \alpha_2^{\beta_2} \dots \alpha_j^{\beta_j} \dots,$$

where $\alpha_j^0 = 1$. In addition, if $\gamma = (\gamma_1, \dots, \gamma_j, \dots) \in (\mathbb{R}^N)_c$, we write

$$(2\mathbb{N})^\gamma := \prod_j (2j)^{\gamma_j}.$$

To motivate the stochastic distribution spaces we will consider next, we remark that by Theorem A.6 we know that if a random element f has the formal expansion

$$f(\omega) = \sum_{\alpha \in \mathcal{J}} c_\alpha H_\alpha(\omega),$$

then $\sum_{\alpha \in \mathcal{J}} \alpha! |c_\alpha|^2 < \infty$ is enough to guarantee that f belongs to $\mathbf{L}^2(\mu_m)$. To explore the classes of random elements outside this framework, one introduces different summability criteria for the coefficients c_α . Hence, we have:

Definition A.7 (Stochastic test function spaces). *For $\rho \in [-1, 1]$ and $r \in \mathbb{R}$, let $(\mathcal{S})_{\rho, r}^N$ consist of those $F = \sum_{\alpha \in \mathcal{J}} a_\alpha H_\alpha \in \mathbf{L}^2(\mu_m)$ such that*

$$\|F\|_{\rho, r}^2 := \sum_{\alpha \in \mathcal{J}} a_\alpha^2 (\alpha!)^{1+\rho} (2\mathbb{N})^{r\alpha} < \infty.$$

Here $c_\alpha^2 = \sum_{k=1}^N (c_\alpha^{(k)})^2$, for $c_\alpha \in \mathbb{R}^N$.

Definition A.8 (Stochastic distribution spaces). *Let N be a natural number. For $0 \leq \rho \leq 1, q \in \mathbb{R}$, let $(\mathcal{S})_{-\rho, -q}^N$ consist of all formal expansions*

$$F = \sum_{\alpha} b_\alpha H_\alpha, \quad b_\alpha \in \mathbb{R}$$

such that

$$\|F\|_{-\rho, -q}^2 := \sum_{\alpha} b_\alpha^2 (\alpha!)^{1-\rho} (2\mathbb{N})^{-q\alpha} < \infty.$$

We regard $(\mathcal{S})_{-\rho, -r}^N$ as the dual of $(\mathcal{S})_{\rho, r}^N$ with respect to the action

$$\langle F, f \rangle = \sum_{\alpha} (b_\alpha, c_\alpha) \alpha!, \quad (\text{A.3})$$

whenever $F \in (\mathcal{S})_{-\rho, -r}^N$ and $f \in (\mathcal{S})_{\rho, r}^N$. Above (a, b) denotes the usual inner product on $\mathbb{R}^N$. This action is well-defined, since

$$\begin{aligned} \sum_{\alpha} |(b_\alpha, c_\alpha)| \alpha! &= \sum_{\alpha} |b_\alpha, c_\alpha| (\alpha!)^{\frac{1-\rho}{2}} (\alpha!)^{\frac{1+\rho}{2}} (2\mathbb{N})^{-\frac{q\alpha}{2}} (2\mathbb{N})^{\frac{q\alpha}{2}} \\ &\leq \sum_{\alpha} \left(\sum_{\alpha} b_\alpha^2 (\alpha!)^{\frac{1-\rho}{2}} (2\mathbb{N})^{-q\alpha} \right)^{1/2} \left(c_\alpha^2 (\alpha!)^{\frac{1+\rho}{2}} (2\mathbb{N})^{q\alpha} \right)^{1/2} < +\infty. \end{aligned}$$

For $\rho \in [0, 1]$ the space $(\mathcal{S})_{\rho}^N$ is defined as the projective limit (intersection) of the spaces $(\mathcal{S})_{\rho, r}^N$ with respect to $r \in \mathbb{R}$, while similarly $(\mathcal{S})_{-\rho}^N$ is defined to be the inductive

limit (union) of the spaces $(\mathcal{S})_{-\rho, -r}^N$. We remark that for $\rho = 0, N = 1$ the spaces $(\mathcal{S})_0^N$ and $(\mathcal{S})_{-0}^N$ are typically denoted by $(\mathcal{S})$ and $(\mathcal{S})^*$, and referred to as the Hida test function and distribution spaces, respectively. For details on these spaces in particular, see the monograph [13].

In general for $0 \leq \rho \leq 1$ we have the inclusions

$$(\mathcal{S})_1^N \subset (\mathcal{S})_\rho^N \subset (\mathcal{S})_0^N \subset L^2(\mu) \subset (\mathcal{S})_{-0}^N \subset (\mathcal{S})_{-\rho}^N \subset (\mathcal{S})_{-1}^N,$$

where each embedding is continuous and dense. In particular, when $F, G \in L^2(\mu)$, then (A.3) reduces simply to

$$\langle F, G \rangle = \mathbb{E}[FG].$$

Let $\{\eta_j\}$ be the orthonormal basis of $L^2(\mathbb{R})$. We remark that the 1-dimensional white noise process, defined via the formal expansion

$$W(x) = \sum_{j=1}^{\infty} \eta_j(x) H_{e(j)}, \quad x \in \mathbb{R},$$

belongs to the space $(\mathcal{S})_{-0}$.

A.2. Wick product.

Definition A.9 (Wick product). *For F and G , two elements of $(\mathcal{S})_{-1}^N$, that is,*

$$F = \sum_{\alpha} a_{\alpha} H_{\alpha}(\omega), \quad G = \sum_{\beta} b_{\beta} H_{\beta}(\omega),$$

with $a_{\alpha}, b_{\beta} \in \mathbb{R}^N$, the Wick product $F \diamond G$ is defined by

$$F \diamond G = \sum_{\alpha, \beta} (a_{\alpha}, b_{\beta}) H_{\alpha+\beta}(\omega).$$

Above (a_{α}, b_{β}) denotes the usual inner product on $\mathbb{R}^N$.

One key advantage of the spaces $(\mathcal{S})_1^N$ and $(\mathcal{S})_{-1}^N$ is that they are closed with respect to the Wick product: if $F, G \in (\mathcal{S})_1^N$, then $F \diamond G \in (\mathcal{S})_1^N$; similarly if $F, G \in (\mathcal{S})_{-1}^N$, then $F \diamond G \in (\mathcal{S})_{-1}^N$. We remark that it is well-known that for example $L^p(\mu)$ spaces are not closed with respect to the Wick product (see [11]). In addition, in the stochastic distribution space $(\mathcal{S})_{-1}^N$ the Wick product is commutative, distributive and associative.

For any natural number k , we define the Wick powers $X^{\diamond k}$ of $X \in (\mathcal{S})_{-1}^N$ inductively by setting $X^{\diamond 0} = 1$ and then

$$X^{\diamond k} = X \diamond X^{\diamond(k-1)}, \quad k > 1.$$

Definition A.10 (generalized expectation). *Let $X = \sum_{\alpha \in \mathcal{J}} c_{\alpha} H_{\alpha}(\omega) \in (\mathcal{S})_{-1}^N$. The vector $c_0 \in \mathbb{R}^N$ is called the generalized expectation of X , denoted by $\mathbb{E}[X]$.*

We note that when $X \in L^p(\mu)$ for some $p > 1$, the generalized expectation coincides with the usual expectation of X .

Definition A.11 (Wick exponential). *Let $X \in (\mathcal{S})_{-1}$. The Wick exponential of X is defined by*

$$e^{\diamond X} = \sum_{n=0}^{\infty} \frac{1}{n!} X^{\diamond n}.$$

Definition A.12 (Wick inverse). *Let $X \in (\mathcal{S})_{-1}^N$ with $\mathbb{E}[X] \neq 0$. We define the Wick inverse of X , denoted by $X^{\diamond(-1)}$, as the element of $(\mathcal{S})_{-1}^N$ having the property that*

$$X \diamond X^{\diamond(-1)} = 1.$$

A.3. $(\mathcal{S})_{\rho,r}^N$ spaces and the S -transform. Recall that for $\rho \in [0, 1]$ the space $(\mathcal{S})_{\rho}^N$ is the projective limit (intersection) of the spaces $(\mathcal{S})_{\rho,r}^N$ as in Definition A.7 with respect to $r \in \mathbb{R}$, and similarly $(\mathcal{S})_{-\rho}^N$ is the inductive limit (union) of the spaces $(\mathcal{S})_{-\rho,-r}^N$ as in Definition A.8. In particular, if $F \in (\mathcal{S})_{-1}$, then there exists $q < \infty$ such that $F \in (\mathcal{S})_{-1,-q}$.

Lemma A.13.

- (a) *Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $q < \infty$. Then there exists $\varepsilon > 0$ such that for all $\lambda \in \mathbb{R}$ with $|\lambda| < \varepsilon$ we have*

$$e^{\diamond \lambda w(\varphi)} \in (\mathcal{S})_{1,q}.$$

- (b) *For all $\rho < 1$ we have*

$$e^{\diamond \lambda w(\varphi)} \in (\mathcal{S})_{\rho}$$

for all $\lambda \in \mathbb{R}$.

Definition A.14 (Hida test points). *Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $\lambda \in \mathbb{R}$. For given λ , we call the random variables $\lambda w(\varphi)$ the Hida test points.*

The name in Definition A.14 stems from the fact that if $|\lambda|$ is small enough, then $e^{\diamond \lambda w(\varphi)} \in ((\mathcal{S})_{-1,-q})^* = (\mathcal{S})_{1,q}$ and they are dense in $(\mathcal{S})_{1,q}$. Consequently, we can define the S -transform that determines random variables $F \in (\mathcal{S})_{-1,-q}$ uniquely.

Definition A.15 (S -transform).

- (i) *Let $F \in (\mathcal{S})_{-1}$ and let $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Then the S -transform of F at $\lambda w(\varphi)$, is defined for all real numbers λ with $|\lambda|$ small enough, by*

$$(SF)(\lambda w(\varphi)) = \langle F, e^{\diamond \lambda w(\varphi)} \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the action of $F \in (\mathcal{S})_{-1,-q}$ on $e^{\diamond \lambda w(\varphi)} \in ((\mathcal{S})_{-1,-q})^* = (\mathcal{S})_{1,q}$.

- (ii) *Let $F \in (\mathcal{S})_{-\rho}$ for some $\rho < 1$ and let $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Then the S -transform of F at $\lambda w(\varphi)$ is defined by*

$$(SF)(\lambda w(\varphi)) = \langle F, e^{\diamond \lambda w(\varphi)} \rangle$$

for all $\lambda \in \mathbb{R}$.

Theorem A.16 (Analytic extension). *Suppose $F = \sum_{\alpha \in \mathcal{J}} a_\alpha H_\alpha \in (\mathcal{S})_{-1}$ and let $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Then if $\lambda \in \mathbb{R}$ with $|\lambda|$ small enough, we have*

$$(SF)(\lambda w(\varphi)) = \sum_{\alpha \in \mathcal{J}} \lambda^{|\alpha|} a_\alpha(\varphi, \eta)^\alpha,$$

where $(\varphi, \eta)^\alpha = (\varphi, \eta_1)^{\alpha_1} (\varphi, \eta_2)^{\alpha_2} \dots$

Theorem A.17. *Suppose $F, G \in (\mathcal{S})_{-1}$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Then if $|\lambda|$ is small enough, we have*

$$S(F \diamond G)(\lambda w(\varphi)) = (SF)(\lambda w(\varphi)) \cdot (SG)(\lambda w(\varphi)).$$

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Renormalized stochastic pressure equation with log-correlated Gaussian coefficients

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RENORMALIZED STOCHASTIC PRESSURE EQUATION WITH LOG-CORRELATED GAUSSIAN COEFFICIENTS

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AND LAURI VIITASAARI

ABSTRACT. We study periodic solutions to the following divergence-form stochastic partial differential equation with Wick-renormalized gradient on the d -dimensional flat torus $\mathbb{T}^d$,

$$-\nabla \cdot \left(e^{\diamond(-\beta X)} \diamond \nabla U \right) = \nabla \cdot (e^{\diamond(-\beta X)} \diamond \mathbf{F}),$$

where X is the log-correlated Gaussian field, $\mathbf{F}$ is a random vector and $\diamond$ denotes the Wick product. The problem is a variant of the stochastic pressure equation, in which U is modeling the pressure of a creeping water-flow in crustal rock that occurs in enhanced geothermal heating. In the original model, the Wick exponential term $e^{\diamond(-\beta X)}$ is modeling the random permeability of the rock. The porosity field is given by a log-correlated Gaussian random field βX , where $\beta < \sqrt{d}$. We use elliptic regularity theory in order to define a notion of a solution to this (a priori very ill-posed) problem, via modifying the S -transform from Gaussian white noise analysis, and then establish the existence and uniqueness of solutions. Moreover, we show that the solution to the problem can be expressed in terms of the Gaussian multiplicative chaos measure.

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1. Introduction

The stochastic pressure equation on a domain $V \subset \mathbb{R}^d$ is given by the following stochastic partial differential equation (SPDE)

$$\begin{cases} -\nabla \cdot (e^\xi \times \nabla U) = \nabla \cdot (e^\xi \times \mathbf{f}), \\ \text{boundary conditions on } \partial V, \end{cases}$$

where ξ is a realization of a Gaussian random field on V . The product “ $\times$ ” is usually given by some algebraic renormalization, where many authors have considered the Wick-product “ $\diamond$ ”, and set $e^\xi = e^{\diamond W}$, the Wick exponential of white noise W , see [16] and the references therein.

We are studying the question of existence and uniqueness of solutions to the following divergence-form SPDE

$$-\nabla \cdot \left(e^{\diamond(-\beta X)} \diamond \nabla U \right) = \nabla \cdot \left(e^{\diamond(-\beta X)} \diamond \mathbf{F} \right), \quad (1.1)$$

on the d -dimensional flat torus $\mathbb{T}^d$ with dimension $d \geq 2$. Here $0 < \beta < \sqrt{d}$ is a parameter, and X is realization of a log-correlated Gaussian random field on $\mathbb{T}^d$, and $\mathbf{F}$ is a suitable random vector. This continues our work [1] in which we study the one-dimensional case.

The log-correlated Gaussian field has a covariance function with a specific logarithmic-type singularity on the diagonal:

$$\mathbb{E}[X(y)X(z)] = \log \frac{1}{|y - z|_{\mathbb{T}^d}} + g(y, z), \quad (1.2)$$

where g is typically assumed to be smooth, and $|\cdot|_{\mathbb{T}^d}$ denotes the distance in the torus. It is clear that X cannot be a function (but is close to being one, see [20]), and in fact X can be obtained as a special case of the so-called fractional Gaussian fields, see for instance [31]. The inclusion of the log-correlated field in the problem is motivated by geophysical modeling, which we discuss in detail in Subsection 1.1 below.

In general, we do not expect weak solutions to (1.1) to have sufficient regularity to compensate for the lack of regularity of the random factor, namely the log-correlated

Gaussian field. The known fact that there is no canonical way to define the product of two distributions, is tackled by “renormalizing” the equation. This process can be viewed as a local reparametrization leading to an implicit dependence of the solution to the specific choice of smooth approximations. In recent years, this method has been applied systematically in the context of so-called singular SPDEs in [4, 5, 11, 12, 24, 33].

In (1.1) we work, albeit indirectly, with the Wick product, which most resembles the Itô stochastic integral, see [17] for details. In fact, as already observed by [16, pages 141–191] in the white noise case, the choice of Wick products leads to the property that the stochastic pressure equation becomes unbiased given deterministic data. In the one-dimensional case of the Itô integral, this is a consequence of its local martingale property. The Wick product is known in particle physics from Wick’s theorem [39], which can be reformulated in terms of Gaussian integration by parts [19]. This is useful in Liouville quantum field theory to derive the conformal Ward identities [25, 35]. An earlier appearance is made in the probabilistic Isserlis’ theorem on higher-order moments of the multivariate Gaussian normal distribution [18].

Wick renormalization leads naturally to the so-called S -transform, which mimics a variant of the Laplace transform with random test-elements and constitutes a characterization of random elements with second moments, see [19]. For the given range of parameters, we define (a variant of) the S -transform at test points $X_\beta(z)$ in the $L^2_\beta(\Omega)$ space of random variables of form

$$\int_{\mathbb{T}^d} \varphi(z) d\mu_\beta(z), \quad \varphi \in C^\infty(\mathbb{T}^d),$$

where $d\mu_\beta(z)$ denotes the Gaussian multiplicative chaos measure (see (2.3) below) with parameter β . Applying the S -transform to equation (1.1) results in a family of weighted, divergence-form, deterministic PDEs, parameterized by $z \in \mathbb{T}^d$:

$$\begin{cases} -\nabla \cdot (w(\cdot; z) \nabla u(\cdot; z)) = \nabla \cdot (w(\cdot; z) \mathbf{f}(\cdot; z)), & \text{in } \mathbb{T}^d, \\ y \mapsto u(y; z) \text{ is } \mathbb{T}^d\text{-periodic for all } z \in \mathbb{T}^d \end{cases} \quad (1.3)$$

for the weight $w(y; z) := |y - z|^{\beta^2} e^{-\beta^2 g(y, z)}$, which can be tackled by well-known methods of degenerate elliptic PDEs. We show that these deterministic equations admit unique solutions $u(\cdot; z)$, which have suitable regularity in the z variable, and this in turn allows the inversion of the S -transform, provided that the S -transform of $\mathbf{F}$ has sufficient regularity. This leads to our main result on the existence, uniqueness and representation of solutions to (1.1), which is outlined below:

Theorem 1.1. *Let X_β be a log-correlated Gaussian field with scaling parameter $0 < \beta < \sqrt{d}$, and let $\mathbf{F}$ be a random vector with sufficient regularity on its S -transform. Then the problem (1.1) admits a solution (in the sense of Definition*

3.2). In particular, the solution admits a representation given by

$$U(y) = \int_{\mathbb{T}^d} \varphi(y; z) d\mu_\beta(z), \quad y \in \mathbb{T}^d, \quad (1.4)$$

where φ is deterministic, and where $d\mu_\beta$ denotes the Gaussian multiplicative chaos measure with parameter β .

For the precise statement and the proof of the main result, see Theorem 3.3 in Section 3 below.

1.1. Relation to the physical model. Motivated by the St1 Deep Heat project in Espoo [23], we are interested in modeling the high pressure creeping water-flow in crustal rock that occurs in enhanced geothermal heating (EGH). The EGH system consists of two 7-km-deep boreholes into the bedrock of the Nordic countries. Water is pushed into one hole and creeps through the bedrock into the second hole, heated by the geothermal heat in the process, such that the hot water can be extracted for energy. Understanding the fluid flow is essential for understanding the heat extraction.

The modeling of fluid flow in rock has been widely studied, mainly due to the oil industry, for instance in the stimulation of oil wells. For porous rock types the homogenized problem describes the flow fairly well [9, 29, 30]. It has long been known that the creeping water-flow in porous media can be well described by Darcy's law, given as

$$\frac{\kappa(x)}{\nu} \nabla P = v(x).$$

Here $v(x)$ describes the flow velocity, P is the pressure, κ is the permeability and ν the (constant) viscosity of water. From preservation of “mass/energy” we get that the pressure P of the creeping flow satisfies the following parabolic equation

$$\frac{\partial P}{\partial t} - \nabla \cdot (\nu^{-1} \kappa(x) \nabla P(x, t)) = f.$$

As such, in our case we are interested in the steady state this system, i.e. where $\partial P / \partial t = 0$, specifically

$$-\nabla \cdot (\nu^{-1} \kappa(x) \nabla P(x, \cdot)) = f. \quad (1.5)$$

In engineering applications, we cannot measure the porosity/permeability of the rock other than locally around the borehole. Hence the uncertainty about the surrounding rock is modeled as a random field. It has been observed from the borehole measurements that the spatial correlation of the crustal crack distribution follows a power-law scaling that is called $1/k$ scaling. This was actually first noted by Hewett et al. [15] for the variances of borehole logs of oil and gas reservoir formations. To-doeschuk et al. [37] further noted that the sequence of seismic reflection coefficients computed from a well log had a power law spatial frequency spectrum. Leary [26, 28],

on the other hand, found the power law scaling spectra for deep borehole spatial fluctuation sequences in sonic velocity and resistivity values closely related to fracture content in crustal granite. Recently, the power-law scaling has also been observed in wellbore porosity logs, see Leary et al. [27] and the references therein. While the borehole measurements only contains information in one direction (depth), we can view them as a 1d projection of the 3d field modeling the rock porosity.

The permeability is observed to be log-normally distributed and is related to the exponential of porosity (see [27] and the references therein). Thus, a natural model for permeability would be $\kappa = e^{\alpha\gamma + \beta\phi}$ for some given parameters $\alpha, \beta, \gamma > 0$, where γ is mean porosity and ϕ is normally distributed. In order to have the correct $1/k$ scaling properties suggested by empirical evidence, a natural choice (in dimension $d = 3$) is to consider $\phi \sim \text{FGF}_{3/2}(\mathbb{R}^3)$ with FGF standing for fractional Gaussian field that corresponds to the log-correlated Gaussian field. However, this is not really well defined (we need to consider a suitably renormalized exponential). Furthermore, $\alpha\gamma + \beta\phi$ cannot, strictly speaking, be interpreted as porosity which can only take values between 0 and 1. However, since the permeability can be arbitrarily close to zero (solid rock), this seems to be a plausible model for permeability. In our approach we consider arbitrary dimension $d \geq 2$ (the case $d = 1$ already being treated in [1]), and we need a bound $\beta < \sqrt{d}$. Notice carefully, that we assume above that ϕ has zero mean, so in practice the choice of β scales the size of fluctuations of porosity, not the absolute size. For example, in [27], $\alpha\gamma$ is observed to be of size 3 to 4, and in our analysis this is simply a multiplicative constant for the equation and it plays no significant role.

On a related literature, let us comment that a discussion about mathematical modeling of the permeability with the exponential of a Gaussian field would not be complete without mentioning the works of Holden et al., see [16] and the references therein. They use the exponential of smoothed white noise as the permeability. In comparison to our approach, the unfortunate side-effect of using smoothed white noise is that there is a finite correlation length which violates the correlation length of cm to km scale from [27].

1.2. Notation. Throughout the article we use the following notation. Let $f \in L^1(\mathbb{T}^d)$. Then the Fourier coefficients of f are given by, for $n \in \mathbb{Z}^d$,

$$\hat{f}(n) = \int_{\mathbb{T}^d} f(x) e^{-ix \cdot n} dx.$$

Definition 1.2. Let $s \in \mathbb{R}$, then $W^{s,2}(\mathbb{T}^d)$ is defined as

$$W^{s,2}(\mathbb{T}^d) = \{f \in S'(\mathbb{T}^d) \text{ s.t. } \|f\|_{W^{s,2}(\mathbb{T}^d)}^2 = \sum_{n \in \mathbb{Z}^d} \langle n \rangle^{2s} |\hat{f}(n)|^2 < \infty\},$$

where $S'(\mathbb{T}^d)$ is the space of tempered distributions on $\mathbb{T}^d$, and $\hat{f}(n)$ denotes the Fourier coefficients of f , and $\langle n \rangle^2 = 1 + |n|^2 = 1 + \sum_{k=1}^d n_k^2$.

We also need to define a suitable distance on the torus $\mathbb{T}^d$. For this purpose, we define $\varphi : \mathbb{R}^d \mapsto \mathbb{R}$ as a 2π -periodic function in each variable such that $\varphi(u) = |u|$ when $|u| < 1/3$ and $\varphi(u)$ is positive and C^∞ -smooth in the set $\mathbb{R}^d \setminus (2\pi\mathbb{Z})^d$. Then φ yields function $\tilde{\varphi}$ on $\mathbb{T}^d$ satisfying

$$\tilde{\varphi}(x + (2\pi\mathbb{Z})^d) = \varphi(x), \quad x \in \mathbb{R}^d.$$

Then we may define the distance between two points $x, y \in \mathbb{T}^d$ by

$$d_{\mathbb{T}^d}(x, y) := |x - y|_{\mathbb{T}^d} := \tilde{\varphi}(x - y). \quad (1.6)$$

This distance function behaves exactly as Euclidean distance for small distances and it is smooth outside the diagonal.

In the sequel, we will simply write $\|f\|_{s,2}$ instead of $\|f\|_{W^{s,2}(\mathbb{T}^d)}$ whenever confusion cannot arise. We also note that the topological dual of $W^{s,2}(\mathbb{T}^d)$ (in the standard distributional duality sense) is simply $W^{-s,2}(\mathbb{T}^d)$.

For a weight w and a domain $\Omega \subset \mathbb{T}^d$, we also use the notation $W^{m,p}(\Omega; w)$, where $1 \leq p < \infty$, for the weighted Sobolev space defined as the completion of $C^\infty(\Omega)$ under the norm

$$\|u\|_{W^{m,p}(\Omega; w)} := \|u\|_{L^p(\Omega; w)} + \sum_{1 \leq |\alpha| \leq m} \|\partial_x^\alpha u\|_{L^p(\Omega; w)}, \quad (1.7)$$

where α is a multi-index and where

$$\|u\|_{L^p(\Omega; w)} := \left(\int_{\Omega} |u(x)|^p w(x) dx \right)^{\frac{1}{p}}.$$

We will often use the shorthand notation $d\mu_z = w(x)dx$, and we will, as customary, denote $H^1(\Omega; w) := W^{1,2}(\Omega; w)$.

1.3. The plan. The rest of the article is organized as follows. In Section 2 we discuss basic facts on log-correlated fields and Gaussian multiplicative measures, cf. Section 2.1, and we define the S -transform on log-correlated fields and study its basic properties, cf. Section 2.2. In Section 3 we motivate and describe precisely what we mean by a solution to the problem (1.1). We also formulate and prove our main result, by using regularity estimates for the deterministic equation (1.3). Section 4 is devoted to the required regularity estimates, and we end the paper with a short appendix on basic facts about Muckenhoupt weights needed in Section 4.

2. Log-correlated fields and the S -transform

2.1. Basic facts on log-correlated fields and Gaussian multiplicative chaos measures. In this section we recall some basic facts on log-correlated Gaussian fields and Gaussian multiplicative chaos (GMC) measures in the d -torus. For details on GMC measures in general, we refer to survey articles [6], [34] and references therein. For log-correlated fields in the d -torus, see also [32].

Definition 2.1 (Log-correlated Gaussian field). A distribution-valued centered Gaussian field X in a d -torus $\mathbb{T}^d$ is called a *log-correlated Gaussian field* if it has covariance

$$R(y, z) := R_X(y, z) := \mathbb{E}[X(y)X(z)] = \log \frac{1}{|y - z|_{\mathbb{T}^d}} + g(y, z), \quad (2.1)$$

where we assume that $g \in C^\infty(\mathbb{T}^d \times \mathbb{T}^d)$ and $|\cdot|_{\mathbb{T}^d}$ denotes the distance in the torus.

Log-correlated fields on $\mathbb{T}^d$ exist, and a natural way to realize such a field is through the series

$$X(y) = \sum_{n \in \mathbb{Z}^d} \frac{A_n}{\langle n \rangle^{d/2}} e^{in \cdot y}, \quad (2.2)$$

where A_n are mutually independent complex standard Gaussian random variables, and convergence is in the sense of distributions. Recall that above we denote $\langle n \rangle = (1 + n^2)^{1/2}$. Such a field has covariance (see e.g. [32])

$$R(y - z) = \text{Cov}(X(y), X(z)) = -\log |y - z|_{\mathbb{T}^d} + g(y - z),$$

where $g \in C^\infty(\mathbb{T}^d)$. Throughout, we consider the exact field given by (2.2).

Remark 2.2. Since the covariance is unbounded, the point-wise evaluations $X_\beta(z)$ are not well-defined, and the precise interpretation for the covariance is that it yields the kernel of the covariance operator:

$$\mathbb{E}[\langle X, \psi_1 \rangle \langle X, \psi_2 \rangle] = \int_{\mathbb{T}^d \times \mathbb{T}^d} R(y, z) \psi_1(y) \psi_2(z) dz dy$$

for any $\psi_1, \psi_2 \in C^\infty(\mathbb{T}^d)$.

In the sequel, we will also consider a scaling parameter $\beta > 0$, and a log-correlated Gaussian field with scaling parameter β , βX , written simply as X_β . The GMC measure $d\mu_\beta$, depending on a parameter $0 < \beta < \sqrt{2d}$, is a random measure constructed from the log-correlated field X formally¹ as the measure

$$d\mu_\beta(y) = e^{\diamond X_\beta(y)} dy, \quad (2.3)$$

where

$$e^{\diamond X_\beta(y)} := \exp \left(X_\beta(y) - \frac{\beta^2}{2} \mathbb{E}[X(y)^2] \right) \quad (2.4)$$

denotes the Wick-exponential. Note that again here the representation is only formal, as we have $\mathbb{E}[X(y)^2] = \infty$ and the point-wise evaluations $X_\beta(y)$ do not exist.

¹To construct the measure precisely, one can approximate the field, e.g. through mollification and show that the corresponding sequence of random measures converge in probability in the space of Radon measures.

In the present paper we shall restrict ourselves to the so-called L^2 -range, where $\beta < \sqrt{d}$, and this also will be our standing assumption from now on. In this case, by using the notation $\mu_\beta(\psi) := \int_{\mathbb{T}^d} \psi(z) d\mu_\beta(z)$ we obtain for any $\psi \in L^\infty(\mathbb{T}^d)$ that

$$\begin{aligned} \mathbb{E}(\mu_\beta(\psi))^2 &:= \mathbb{E} \left[\int_{\mathbb{T}^d} \psi(z) e^{\diamond X_\beta(z)} dz \right]^2 = \mathbb{E} \left[\int_{\mathbb{T}^d} \psi(z) e^{\beta X(z) - \frac{\beta^2}{2} \mathbb{E}[X(z)^2]} dz \right]^2 \\ &\leq \|\psi\|_\infty^2 \int_{\mathbb{T}^d \times \mathbb{T}^d} \mathbb{E} \left[e^{\diamond \beta X(z)} e^{\diamond \beta X(y)} \right] dz dy \\ &= \|\psi\|_\infty^2 \int_{\mathbb{T}^d \times \mathbb{T}^d} e^{\beta^2 \mathbb{E}[X(z)X(y)]} dz dy = \|\psi\|_\infty^2 \int_{\mathbb{T}^d \times \mathbb{T}^d} e^{\beta^2 R(z,y)} dz dy \\ &= \|\psi\|_\infty^2 \int_{\mathbb{T}^d \times \mathbb{T}^d} |z - y|_{\mathbb{T}^d}^{-\beta^2} e^{\beta^2 g(z-y)} dz dy \end{aligned} \quad (2.5)$$

which is finite as $\beta < \sqrt{d}$. Note that we have used the fact that, for any centered Gaussian random variables H_1 and H_2 , we have

$$\mathbb{E} \left[e^{\diamond H_1} e^{\diamond H_2} \right] = e^{\mathbb{E}(H_1 H_2)},$$

and we performed the computation only formally, but as mentioned before, it can be validated by first replacing X by the approximated field X_ε and letting then $\varepsilon \rightarrow 0^+$ (this is essentially as in [2, Theorem 1.1.]).

2.2. S -transforms and log-correlated fields. In this section we briefly introduce subspaces $L_\beta^2(\Omega)$ that are characterized by (extensions of) S -transforms at field points $X_\beta(z)$. We follow the ideas introduced in [1] and extend them to the torus $\mathbb{T}^d$ case. The subspace $L_\beta^2(\Omega)$ is given by the following definition.

Definition 2.3 ($L_\beta^2(\Omega)$ space generated by GMC $d\mu_\beta$). Let $\beta \in (0, \sqrt{d})$. The space $L_\beta^2(\Omega) := L^2(\Omega, \mathbb{P}, \mathcal{F}_{d\mu_\beta})$ generated by the GMC measure $d\mu_\beta$ is the closed linear space spanned by random variables of the form

$$\int_{\mathbb{T}^d} \varphi(z) d\mu_\beta(z), \quad \varphi \in C^\infty(\mathbb{T}^d). \quad (2.6)$$

In order to characterize the random variables $Z \in L_\beta^2(\Omega)$ and define the S -transform at $X_\beta(s)$, we introduce the operator acting on sufficiently regular functions $\phi : \mathbb{T}^d \rightarrow \mathbb{R}$

$$\begin{aligned} G\phi(z) &= G_\alpha \phi(z) := \int_{\mathbb{T}^d} \exp(\alpha \mathbb{E}[X(z)X(y)]) \phi(y) dy \\ &= \int_{\mathbb{T}^d} |z - y|_{\mathbb{T}^d}^{-\alpha} e^{\alpha g(z-y)} \phi(y) dy, \end{aligned} \quad (2.7)$$

where $\alpha \in (0, d)$ and X is a log-correlated Gaussian field as in Definition 2.1. The motivation for the operator stems from the fact that for random variables $Z_1 =$

$\int_{\mathbb{T}^d} \varphi_1(z) d\mu_\beta(z)$ and $Z_2 = \int_{\mathbb{T}^d} \varphi_2(z) d\mu_\beta(z)$, we can (again formally) compute the covariance and get

$$\begin{aligned} \mathbb{E}[Z_1 Z_2] &= \mathbb{E} \left[\int_{\mathbb{T}^d} \varphi_1(z) d\mu_\beta(z) \int_{\mathbb{T}^d} \varphi_2(z) d\mu_\beta(z) \right] \\ &= \mathbb{E} \left[\int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \varphi_1(z) \varphi_2(y) e^{\diamond X_\beta(z)} e^{\diamond X_\beta(y)} dy dz \right] \\ &= \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \varphi_1(z) \varphi_2(y) e^{\beta^2 R(z,y)} dy dz \\ &= \int_{\mathbb{T}^d} \varphi_1(z) G_{\beta^2} \varphi_2(z) dz \\ &= \langle \varphi_1, G_{\beta^2} \varphi_2 \rangle_{L^2(\mathbb{T}^d)}. \end{aligned}$$

As in [1], our characterization is based on the bijectivity properties of the operator G_α that we study next. We begin with the following simple lemma that eventually provides the required decay of the Fourier coefficients.

Lemma 2.4. *For $\alpha \in (0, d)$ and a symmetric function $\phi \in C_0^\infty(\mathbb{R}^d)$ with $\phi(0) > 0$, let $h : \mathbb{R}^d \rightarrow \mathbb{R}$ be given by $h(x) = \phi(x)|x|^{-\alpha}$. Define the Fourier transform of h by*

$$\hat{h}(\xi) = \int_{\mathbb{R}^d} h(x) e^{-i\xi \cdot x} dx.$$

Then there exists constants c , C , and K such that, for all $|\xi| \geq K$, we have

$$c|\xi|^{\alpha-d} \leq \hat{h}(\xi) \leq C|\xi|^{\alpha-d}.$$

Proof. Throughout the proof, we denote by K' any generic constant that may change from line to line. We first observe that $\hat{h}$ is a continuous function. Hence it suffices to prove that there exists $C' > 0$ such that

$$\lim_{|\xi| \rightarrow \infty} |\xi|^{d-\alpha} \hat{h}(\xi) = C'.$$

By convolution theorem we have $\hat{h}(\xi) = K' \left(\hat{\phi} * |\cdot|^{\alpha-d} \right) (\xi)$. Set $R = \sqrt{|\xi|} > 1$. We split the integral over the whole space in the following way:

$$\begin{aligned} |\xi|^{d-\alpha} \left(\hat{\phi} * |\cdot|^{\alpha-d} \right) (\xi) &= \int_{\mathbb{R}^d} \hat{\phi}(y) |\xi|^{d-\alpha} |\xi - y|^{\alpha-d} dy \\ &= \int_{B_R(0)} \hat{\phi}(y) |\xi|^{d-\alpha} |\xi - y|^{\alpha-d} dy \\ &\quad + \int_{B_R^c(0) \cap \{|y-\xi| > |\xi|/2\}} \hat{\phi}(y) |\xi|^{d-\alpha} |\xi - y|^{\alpha-d} dy \\ &\quad + \int_{B_R^c(0) \cap \{|y-\xi| < |\xi|/2\}} \hat{\phi}(y) |\xi|^{d-\alpha} |\xi - y|^{\alpha-d} dy. \end{aligned}$$

As ϕ is a smooth function, it holds that $|\widehat{\phi}(y)| \lesssim |y|^{-k}$ for any natural number k . We apply this to the second and third terms above. For the second term we obtain, since $|y - \xi| > |\xi|/2$,

$$\begin{aligned} & \left| \int_{B_R^c(0) \cap \{|y-\xi| > |\xi|/2\}} \widehat{\phi}(y) |\xi|^{d-\alpha} |\xi - y|^{\alpha-d} dy \right| \\ & \lesssim \frac{1}{2^{\alpha-d}} \int_{|y| > R} |y|^{-k} dy \\ & \lesssim R^{-k+1} \end{aligned}$$

that converges to zero for $k > 1$ and since $R = \sqrt{|\xi|} \rightarrow \infty$. Similarly for the third term, since now $|\widehat{\phi}(y)| \lesssim |y|^{-k} \lesssim |\xi|^{-k}$,

$$\begin{aligned} & \left| \int_{B_R^c(0) \cap \{|y-\xi| < |\xi|/2\}} \widehat{\phi}(y) |\xi|^{d-\alpha} |\xi - y|^{\alpha-d} dy \right| \\ & \lesssim |\xi|^{-k+d-\alpha} \int_{B_R^c(0) \cap \{|y-\xi| < |\xi|/2\}} |\xi - y|^{\alpha-d} dy \\ & \lesssim |\xi|^{-k+d} \end{aligned}$$

that converges to zero for $k > d$. For the remaining term, we first note that on $B_R(0)$ we have

$$|\xi|^{d-\alpha} |\xi - y|^{\alpha-d} \leq K'$$

for some constant $K' > 0$, since $|y - \xi| \approx |\xi|$ (as $|y| \leq R = \sqrt{|\xi|}$). Hence we apply the Dominated Converge Theorem (as $\widehat{\phi}$ is integrable) to get

$$\lim_{|\xi| \rightarrow \infty} \int_{B_R(0)} \widehat{\phi}(y) |\xi|^{d-\alpha} |\xi - y|^{\alpha-d} dy = \int_{\mathbb{R}^d} \widehat{\phi}(y) dy = (2\pi)^d \phi(0) > 0,$$

which establishes the result. $\square$

Lemma 2.5. *There exists constants c and C such that the Fourier coefficients of the kernel $H_\alpha = e^{ag(x)} |x|_{\mathbb{T}^d}^{-\alpha}$ with $h \in C^\infty(\mathbb{T}^d)$ satisfy*

$$c\langle n \rangle^{\alpha-d} \leq \widehat{H}_\alpha(n) \leq C\langle n \rangle^{\alpha-d} \quad (2.8)$$

for all $n \in \mathbb{Z}^d$.

Proof. Recall that our field is given by

$$X(y) = \sum_n \frac{A_n}{\langle n \rangle^{d/2}} e^{in \cdot y} \quad (2.9)$$

which has the covariance (see e.g. [32])

$$R(y - z) = \text{Cov}(X(y), X(z)) = -\log |y - z|_{\mathbb{T}^d} + g(y - z),$$

where $g \in C^\infty(\mathbb{T}^d)$. In other words, we are computing the Fourier coefficients of $H_\alpha(x) = e^{\alpha R(x)}$. Note also that it suffices to prove that the coefficients are strictly positive and that (2.8) holds for large enough $|n| \geq N_0$. We begin by showing that the coefficients are strictly positive. Note that it follows from (2.9) that R has Fourier coefficients

$$\hat{R}(n) = \langle n \rangle^{-d}.$$

Now

$$e^{\alpha R(x)} = \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} R^k(x)$$

and hence

$$\widehat{H}_\alpha(n) = \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} \widehat{R}^k(n).$$

Now by the convolution theorem and the fact $\hat{R}(n) > 0$ for each $n \in \mathbb{Z}^d$, we obtain $\widehat{H}_\alpha(n) > 0$. It remains to prove that lower and upper bounds in (2.8) hold for large enough $|n|$. To this end, by identifying H_α with a periodic function on $\mathbb{R}^d$, we note that decay of the Fourier coefficients is determined by the singularity $|x|_{\mathbb{T}^d}^{-\alpha}$. Namely, write

$$|x|_{\mathbb{T}^d}^{-\alpha} e^{\alpha g(x)} = \rho_1(x) |x|^{-\alpha} + \rho_2(x),$$

where $\rho_1, \rho_2 \in C^\infty(\mathbb{T}^d)$ and ρ_1 satisfies $\rho_1(0) > 0$ and $\text{supp}(\rho_1) \subset B(0, \frac{1}{3})$. The Fourier coefficients of ρ_2 decay rapidly. In turn, by interpreting the first term as a function in $\mathbb{R}^d$ it is enough to consider its Fourier transform evaluated at points $\xi = n \in \mathbb{Z}^d$. Now the result follows from Lemma 2.4. This completes the proof. $\square$

Lemma 2.6. *Let $s \in \mathbb{R}$. Then the mapping G_α given in (2.7) extends to a bijective map*

$$G_\alpha : W^{s+\alpha-d,2}(\mathbb{T}^d) \rightarrow W^{s,2}(\mathbb{T}^d).$$

Proof. Let $s \in \mathbb{R}$. Since $G_\alpha u = H_\alpha * u$ is a convolution operator with kernel H_α , we have, thanks to convolution theorem and Lemma 2.5, that

$$\begin{aligned} \|G_\alpha u\|_{s,2}^2 &= \sum_{n \in \mathbb{Z}^d} [\widehat{G_\alpha u}(n)]^2 \langle n \rangle^{2s} \\ &= \sum_{n \in \mathbb{Z}^d} [\widehat{H}_\alpha(n)]^2 [\widehat{u}(n)]^2 \langle n \rangle^{2s} \\ &\approx \sum_{n \in \mathbb{Z}^d} [\widehat{u}(n)]^2 \langle n \rangle^{2s+2\alpha-2d} \\ &= \|u\|_{s+\alpha-d,2}^2. \end{aligned}$$

This shows that $G_\alpha : W^{s+\alpha-d,2}(\mathbb{T}^d) \rightarrow W^{s,2}(\mathbb{T}^d)$ is bounded and lower bounded. Since the image is obviously dense, it is actually an isomorphism. This completes the proof. $\square$

Now that the mapping properties of the operator G_α are established, we can characterize random variables in the subspace $L^2_\beta(\Omega)$ and define the S -transforms, similar to [1]. While the proofs are analogous to the ones presented in [1], for the reader's convenience we present the key steps.

Corollary 2.7. *Let $\beta \in (0, \sqrt{d})$ and suppose $Z \in L^2_\beta(\Omega)$. Then Z has the form*

$$Z = \int_{\mathbb{T}^d} \varphi(z) d\mu_\beta(z), \quad (2.10)$$

for some $\varphi \in W^{-s_\beta, 2}(\mathbb{T}^d)$, where $s_\beta = \frac{d-\beta^2}{2}$.

Proof. For $Z = \int_{\mathbb{T}^d} \varphi(z) d\mu_\beta(z)$ with $\varphi \in C^\infty(\mathbb{T}^d)$ we have

$$\begin{aligned} \mathbb{E}[Z^2] &= \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \varphi(z) \varphi(y) \exp(\beta^2 \mathbb{E}[X(y)X(z)]) dy dz \\ &= \int_{\mathbb{T}^d} \varphi(z) G_{\beta^2} \varphi(z) dz \approx \|\varphi\|_{W^{-s_\beta, 2}(\mathbb{T}^d)}^2 \end{aligned}$$

by Lemma 2.6. The claim now follows by taking the closure in $L^2(\Omega)$. $\square$

Remark 2.8. The integral in (2.10) is understood as the limit in $L^2(\Omega)$ when $\varphi \in W^{-s_\beta, 2}(\mathbb{T}^d)$ is approximated with a sequence $\varphi_n \in C^\infty(\mathbb{T}^d)$, and our computation above shows that this limit is independent of the sequence φ_n .

Let us next recall that the concept of S -transform of $Z \in L^2(\Omega)$ evaluated at a centered Gaussian random variable h is given by

$$(SZ)(h) = \mathbb{E} \left[Z e^{\diamond h} \right], \quad (2.11)$$

where

$$e^{\diamond h} := e^{h - \mathbb{E}[h^2]/2}$$

is again the Wick exponential. It is well-known that S -transform characterizes the random variables in the $L^2(\Omega)$ space of random variables measurable with respect to the sigma-algebra generated by the Gaussian random variables. Indeed, this follows from the fact that such space is spanned by the Wick exponentials $e^{\diamond h}$, see e.g. [19]. Moreover, the S -transform can be generalized to random elements in the so-called Hida space, in which one can also define the so-called Wick product through the identity

$$(S(Z \diamond Y))(h) = (SZ)(h) \cdot (SY)(h), \quad (2.12)$$

see [22] for details.

For our purposes, we follow the ideas of our previous article [1] and define the concept of S -transform also at evaluation points $X_\beta(z)$, characterized by projections onto the subspace $L^2_\beta(\Omega)$. After that, we can define the Wick product and our solution through identity (2.12). Indeed, an element $V \in L^2_\beta(\Omega)$ can be expressed as

$V = \int_{\mathbb{T}^d} \psi(z) d\mu_\beta(z)$, where $\psi \in W^{-s_\beta, 2}(\mathbb{T}^d)$ with $s_\beta = \frac{d-\beta^2}{2}$, a formal computation gives, for any $Z \in L^2(\Omega)$, that

$$\mathbb{E}[ZV] = \mathbb{E} \left[\int_{\mathbb{T}^d} \psi(z) Z e^{\diamond X_\beta(z)} dz \right] = \int_{\mathbb{T}^d} \psi(z) (SZ)(X_\beta(z)) dz.$$

Note also that the mapping $\psi \mapsto \mathbb{E}[ZV]$ is a continuous functional on $W^{-s_\beta, 2}(\mathbb{T}^d)$. Hence, by duality, there exists $\iota \in W^{s_\beta, 2}(\mathbb{T}^d)$ such that

$$\mathbb{E}[ZV] = \int_{\mathbb{T}^d} \psi(z) \iota(z) dz.$$

This leads to the following definition.

Definition 2.9 (*S-transform in $L^2_\beta(\Omega)$*). Let $Z \in L^2(\Omega)$ and set $s_\beta = \frac{d-\beta^2}{2}$. For $z \in \mathbb{T}^d$ we identify $(SZ)(X_\beta(z))$ as the (unique) element of $W^{s_\beta, 2}(\mathbb{T}^d)$ defined through duality

$$\mathbb{E}[ZV] = \int_D \psi(z) (SZ)(X_\beta(z)) dz$$

for all $V = \int_{\mathbb{T}^d} \psi(z) d\mu_\beta(z) \in L^2_\beta(\Omega)$.

Remark 2.10. Note that we have defined $(SZ)(X_\beta(z))$ as a function in $W^{s_\beta, 2}(\mathbb{T}^d)$, defined for almost all $z \in \mathbb{T}^d$.

Our next result shows how *S*-transform is linked to the projections into $L^2_\beta(\Omega)$ (cf. [1, Lemma 5.9]).

Lemma 2.11. *Let $Z \in L^2(\Omega)$ with a decomposition $Z = Z_\beta + Z'_\beta$, where $Z_\beta = \int_{\mathbb{T}^d} \varphi(s) d\mu_\beta(s) \in L^2_\beta(\Omega)$ is the orthogonal projection onto the subspace $L^2_\beta(\Omega) \subset L^2(\Omega)$. Then we have*

$$(SZ_\beta)(X_\beta(z)) = (G_{\beta^2} \varphi)(z)$$

and

$$(SZ'_\beta)(X_\beta(z)) = 0.$$

Proof. The claim $(SZ'_\beta)(X_\beta(z)) = 0$ follows immediately from Definition 2.9 by observing that then

$$\int_{\mathbb{T}^d} \psi(z) (SZ'_\beta)(X_\beta(z)) dz = 0 \quad \forall \psi \in W^{-s_\beta, 2}(\mathbb{T}^d).$$

For Z_β , obtain that

$$\mathbb{E}[Z_\beta V] = \mathbb{E} \left[\int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \psi(z) \varphi(y) d\mu_\beta(z) d\mu_\beta(y) \right] = \int_{\mathbb{T}^d} \psi(z) (G_{\beta^2} \varphi)(z) dz,$$

where now, by Lemma 2.6, $(G_{\beta^2} \varphi)(z) \in W^{s_\beta, 2}(\mathbb{T}^d)$. This completes the proof. $\square$

It follows from the bijectivity of G_α that the *S*-transform characterizes all random variables in $L^2_\beta(\Omega)$ uniquely. For the proof, see also [1, Proof of Lemma 5.10].

Lemma 2.12. *The S -transform at points $X_\beta(z)$ determine random variables $Z \in L^2_\beta(\Omega)$ uniquely.*

We end this section by the following result allowing us to compute projections of random variables in $L^2_\beta(\mathbb{T}^d)$ into $L^2_{\beta'}(\mathbb{T}^d)$. For an analogous result in the univariate setting, see [1, Proposition 5.11].

Proposition 2.13. *Let $\beta, \beta' \in (0, \sqrt{d})$ and let*

$$Z = \int_{\mathbb{T}^d} \varphi(z) d\mu_\beta(z) \in L^2_\beta(\Omega)$$

for some $\varphi \in W^{-\frac{d-\beta^2}{2}, 2}(\mathbb{T}^d)$. Denote the projection of Z into the subspace $L^2_{\beta'}(\Omega)$ by

$$Z_{\beta'} = \int_{\mathbb{T}^d} \varphi_{\beta'}(z) d\mu_{\beta'}(z).$$

The function $\varphi_{\beta'} \in W^{-\frac{d-(\beta')^2}{2}, 2}(\mathbb{T}^d)$ solves $G_{\beta\beta'}\varphi = G_{(\beta')^2}\varphi_{\beta'}$.

Proof. Let $Z = \int_{\mathbb{T}^d} \varphi(z) d\mu_\beta(z) \in L^2_\beta(\Omega)$ and $V = \int_{\mathbb{T}^d} \psi(z) d\mu_{\beta'}(z) \in L^2_{\beta'}(\Omega)$ be a test random variable. It is enough to check that $\mathbb{E}[Z_{\beta'}V] = \mathbb{E}[ZV]$ for each $\psi \in W^{-\frac{d-(\beta')^2}{2}, 2}(\mathbb{T}^d)$. Here we have

$$\mathbb{E}[Z_{\beta'}V] = \int_{\mathbb{T}^d} \psi(z) G_{(\beta')^2} \varphi_{\beta'}(z) dz.$$

Similarly, Definition 2.9 yields

$$\mathbb{E}[ZV] = \int_{\mathbb{T}^d} \psi(z) (SZ)(X_{\beta'}(z)) dz$$

for $(SZ)(X_{\beta'}(z)) \in W^{\frac{d-(\beta')^2}{2}, 2}(\mathbb{T}^d)$ while formal computations as above gives, assuming (2.14) is true,

$$\mathbb{E}[ZV] = \int_{\mathbb{T}^d} \psi(z) G_{\beta\beta'} \varphi(z) dz, \quad (2.13)$$

where $\psi \in W^{-\frac{d-(\beta')^2}{2}, 2}(\mathbb{T}^d)$ and $\varphi \in W^{-\frac{d-\beta^2}{2}, 2}(\mathbb{T}^d)$. We claim that indeed

$$G_{\beta\beta'} \varphi(z) \in W^{\frac{d-(\beta')^2}{2}, 2}(\mathbb{T}^d). \quad (2.14)$$

Using (2.14), we see that (2.13) follows, since from Definition 2.9, the uniqueness of the S -transform gives $(SZ)(X_{\beta'}(z)) = G_{\beta\beta'} \varphi(z)$. Now Lemma 2.6 gives that $G_{(\beta')^2} : W^{-\frac{d-(\beta')^2}{2}, 2}(\mathbb{T}^d) \rightarrow W^{\frac{d-(\beta')^2}{2}, 2}(\mathbb{T}^d)$ is invertible and hence $\mathbb{E}[Z_{\beta'}V] = \mathbb{E}[ZV]$ holds if and only if $G_{\beta\beta'} \varphi = G_{(\beta')^2} \varphi_{\beta'}$. For the claim (2.14) note that, by Lemma 2.6, we have $G_{\beta\beta'} : W^{s+\beta\beta'-d, 2}(\mathbb{T}^d) \rightarrow W^{s, 2}(\mathbb{T}^d)$ for any $s \in \mathbb{R}$. In particular, this holds

for $s = \frac{d}{2} - \frac{(\beta')^2}{2}$ yielding $G_{\beta\beta'} : W^{-\frac{d}{2} - \frac{(\beta')^2}{2} + \beta\beta', 2}(\mathbb{T}^d) \rightarrow W^{\frac{d}{2} - \frac{(\beta')^2}{2}, 2}(\mathbb{T}^d)$. Now (2.14) follows by noting that $W^{-\frac{d}{2} - \frac{(\beta')^2}{2}}(\mathbb{T}^d) \subset W^{-\frac{d}{2} - \frac{(\beta')^2}{2} + \beta\beta', 2}(\mathbb{T}^d)$ since

$$-\frac{d}{2} - \frac{(\beta')^2}{2} + \beta\beta' \leq -\frac{d - \beta^2}{2}$$

for arbitrary $\beta, \beta' \in \mathbb{R}$. Hence, (2.14) is proved. $\square$

3. Main result

In this section, we first present a heuristic computation which motivates the definition of the solution of our problem. Indeed, consider formally the stochastic PDE

$$\begin{aligned} -\nabla \cdot (e^{\diamond(-X_\beta)} \diamond \nabla U) &= \nabla \cdot (e^{\diamond(-X_\beta)} \diamond \mathbf{F}), \quad \text{in } \mathbb{T}^d \\ y \mapsto U(y) &\text{ is } \mathbb{T}^d\text{-periodic,} \end{aligned} \quad (3.1)$$

where X_β is the log-correlated Gaussian field with scaling parameter $\beta < \sqrt{d}$, and $\mathbf{F} = (F_1, \dots, F_d)$ is a random element with real-valued random variables as its components.

Now fix $y \in \mathbb{T}^d$, and take the S -transform of (3.1) on Gaussian Z , which does not depend on y . Hence the equation becomes, using Fubini's Theorem and $S(X \diamond Y)(Z) = (SX)(Z) \cdot (SY)(Z)$,

$$-\nabla \cdot (Se^{\diamond(-X_\beta(y))}(Z) \cdot (S\nabla U(y))(Z)) = \nabla \cdot ((Se^{\diamond(-X_\beta(y))}(Z) \cdot (S\mathbf{F}(y))(Z)) \quad (3.2)$$

Now, recall that since

$$e^{\diamond(X+Y)} = e^{-\mathbb{E}XY} e^{\diamond X} e^{\diamond Y},$$

and therefore $(Se^{\diamond(-X_\beta(y))})(Z) = e^{-\mathbb{E}[X_\beta(y)Z]}$, so that (3.2) becomes

$$-\nabla \cdot (e^{-\mathbb{E}[X_\beta(y)Z]}(S\nabla U(y))(Z)) = \nabla \cdot (e^{-\mathbb{E}[X_\beta(y)Z]}(S\mathbf{F}(y))(Z)).$$

Now, choosing $Z = X_\beta(z)$, for fixed $z \in \mathbb{T}^d$, this becomes

$$-\nabla \cdot (e^{-\beta^2 R(y;z)} \nabla u(y;z)) = \nabla \cdot (e^{-\beta^2 R(y;z)} \mathbf{f}(y;z)), \quad (3.3)$$

where we have denoted $R(y;z) := \mathbb{E}[X(y)X(z)]$, and $u(y;z) = (SU(y))(X_\beta(z)) = \mathbb{E}[U(y)e^{\diamond X_\beta(z)}]$, which is well-defined by Definition 2.9, and $\mathbf{f}(y;z) = (S\mathbf{F}(y))(X_\beta(z))$.

Hence, we see that the problem in (3.1) has a deterministic counterpart on the S -transform side, parameterized by z , given by

$$\begin{aligned} -\nabla \cdot (w(y;z) \nabla u(y;z)) &= \nabla \cdot (w(y;z) \mathbf{f}(y;z)), \quad \text{in } \mathbb{T}^d \\ y \mapsto u(y;z) &\text{ is } \mathbb{T}^d\text{-periodic,} \end{aligned} \quad (3.4)$$

where

$$w_z(y) := w(y;z) = |y - z|^{\beta^2} e^{-\beta^2 g(y-z)}$$

with g smooth and bounded.

Definition 3.1. We say that (3.4) is solvable if, for a fixed $z \in \mathbb{T}^d$, there exists a unique (up to constant) function $u(\cdot; z) \in W^{1,2}(\mathbb{T}^d; w_z)$, where $W^{1,2}(\mathbb{T}^d; w_z)$ denotes the weighted Sobolev space (see (1.7)), that satisfies (3.4).

Based on the previous computation, we consider solutions of (3.1) in the following way:

Definition 3.2. Let X_β be a log-correlated Gaussian field with scaling parameter $0 < \beta < \sqrt{d}$. Assume that $\mathbf{f}(y; z) = (f_1(y; z), \dots, f_d(y; z))$ satisfies, for every $z \in \mathbb{T}^d$ and $j = 1, \dots, d$, $f_j(\cdot; z) \in L^2(\mathbb{T}^d)$. We say that (3.1) is solvable in the space $L_\beta^2(\Omega)$, whenever the solution of the S -transformed, deterministic PDE, for all $z \in \mathbb{T}^d$,

$$\begin{aligned} -\nabla_y \cdot (w(y; z) \nabla_y u(y; z)) &= \nabla_y \cdot (w(y; z) \mathbf{f}(y; z)), \quad \text{in } \mathbb{T}^d \\ y \mapsto u(y; z) &\text{ is } \mathbb{T}^d\text{-periodic,} \end{aligned} \quad (3.5)$$

satisfies, for any fixed $y \in \mathbb{T}^d$, $u(y; \cdot) \in W^{s_\beta, 2}(\mathbb{T}^d)$.

We note the asymmetric role of the variables y and z above in Definition 3.2; while the equation, parameterized by z , contains derivatives only in the y -variable, in order to guarantee the existence of the inverse $G_{\beta^2}^{-1}u$, one needs regularity from u in the z -variable, see Lemma 2.6.

We note that unpacking the definitions of the space $L_\beta^2(\Omega)$, the S -transform, and the operator G in Definitions 2.3 and 2.9 and equation (2.7) respectively, Definition 3.2 yields us a representation of the solution as

$$U(y) = \int_{\mathbb{T}^d} G_{\beta^2}^{-1} u(y; z) d\mu_\beta(z) = \int_{\mathbb{T}^d} \varphi(y; z) d\mu_\beta(z), \quad x \in \mathbb{T}^d$$

where $\varphi \in W^{-s_\beta, 2}(\mathbb{T}^d)$, with $s_\beta = \frac{d-\beta^2}{2}$, and $d\mu_\beta$ is a Gaussian multiplicative chaos measure with parameter $\beta < \sqrt{d}$. Moreover, Lemma 2.12 guarantees uniqueness in this representation class – although we remark that we have uniqueness only in the subspace $L_\beta^2(\Omega)$, not in the entirety of the ambient space $L^2(\Omega)$.

Next, we state our main result concerning the solvability of the periodic problem in (3.1):

Theorem 3.3. Let X_β be a log-correlated Gaussian field with scaling parameter $0 < \beta < \sqrt{d}$, and let $\mathbf{F} \in (L^2(\Omega))^d$ with $\mathbf{f}(y; z) := (S\mathbf{F}(y))(X_\beta(z))$ satisfying

$$\sum_{0 \leq |\alpha_1 + \alpha_2| \leq \lfloor \frac{d+1}{2} \rfloor} \|\partial_y^{\alpha_1} \partial_z^{\alpha_2} \mathbf{f}\|_{L_z^p(\mathbb{T}^d; L_y^q(\mathbb{T}^d; w_z))} < \infty,$$

for some $q > 2d - \varepsilon$ and $p > d/\gamma$, where $\gamma = \gamma(\beta, d, q) \in (0, 1)$ and $\varepsilon = \varepsilon(\beta, d) > 0$.

Then the problem (3.1) admits a solution in the sense of Definition 3.2. In particular, there exists a function $z \mapsto \varphi(y; z) = (G_{\beta^2}^{-1} u(y; \cdot))(z) \in W^{-s_\beta, 2}(\mathbb{T}^d)$ with

$s_\beta = \frac{d-\beta^2}{2}$ such that

$$U(y) = \int_{\mathbb{T}^d} \varphi(y; z) d\mu_\beta(z), \quad y \in \mathbb{T}^d.$$

Above $d\mu_\beta$ is a Gaussian multiplicative chaos measure with parameter β .

Remark 3.4. Before giving the proof of Theorem 3.3, we make some remarks on the right-hand side $\mathbf{f}$:

- In the simple case when $\mathbf{F}$ is deterministic (hence $\mathbf{f}$ is independent of the z -variable) and has sufficient regularity in the y -variable, the theorem guarantees the existence of a solution.
- More generally, consider now some non-deterministic $\mathbf{F}$, which belongs to $\left(L^2_\beta(\Omega)\right)^d$ (or its orthogonal projection), that is, $\mathbf{F}$ is of the form

$$\mathbf{F}(y) = \int_{\mathbb{T}^d} \varphi(y, z) d\mu_\beta(z),$$

for some function $\varphi(y, \cdot) \in (W^{\eta, 2}(\mathbb{T}^d))^d$. Hence $(S\mathbf{F})(X_\beta(z)) = G_{\beta^2}(\varphi(y, \cdot))(z)$, which has mapping properties to $(W^{\gamma, 2}(\mathbb{T}^d))^d$ given by Lemma 2.6. Hence provided sufficient regularity in both y - and z -variables for $\varphi(y, z)$, the theorem guarantees the existence of a solution.

Proof of Theorem 3.3. We start by noting that it is a general fact of weighted degenerate elliptic partial differential equations that (3.5) admits a solution u in the weighted Sobolev space $W^{1, 2}(\mathbb{T}^d; w_z)$, where $w(y; z) = |y - z|^{\beta^2} e^{-\beta^2 g(y-z)}$ [8, Theorem 2.2.]. In order for u to satisfy Definition 3.2, we need sufficient regularity for u in the z -variable. Indeed, we show that $z \mapsto u(y; z) \in W^{s_\beta, 2}(\mathbb{T}^d)$ uniformly in y , where $s_\beta = \frac{d-\beta^2}{2}$, and then invert via the operator G in Lemma 2.6.

By Theorem 4.1 below we have that $z \mapsto u(y; z) \in W^{d/2, 2}(\mathbb{T}^d)$ uniformly in y . Recall that the space $W^{d/2, 2}(\mathbb{T}^d)$ embeds into $W^{s_\beta, 2}(\mathbb{T}^d)$ continuously (see, for example [36, Theorem 1, p.82]). By Lemma 2.6 the operator G_{β^2} is a bijection between $W^{-s_\beta, 2}(\mathbb{T}^d)$ and $W^{s_\beta, 2}(\mathbb{T}^d)$, and therefore there exists a unique element φ of $W^{-s_\beta, 2}(\mathbb{T}^d)$ such that $(G_{\beta^2}^{-1}u(y; \cdot))(z) = \varphi(y; z)$ for all $y \in \mathbb{T}^d$; and hence

$$U(y) = \int_{\mathbb{T}^d} \varphi(y; z) d\mu_\beta(z), \quad y \in \mathbb{T}^d$$

is a solution of (3.1), where $d\mu_\beta$ is the GMC measure with parameter $\beta < \sqrt{d}$ on $\mathbb{T}^d$. Finally, uniqueness of the solution in the space $L^2_\beta(\Omega)$ is immediate from the bijectivity of G_{β^2} . This completes the proof. $\square$

4. Regularity of solutions of the deterministic PDE

In this section, we prove the required regularity result for the deterministic equation, Theorem 4.1, used in the proof of our main result. Unless stated otherwise,

throughout we let $w_z(\cdot) := w(\cdot; z)$ denote a periodic function in $\mathbb{T}^d$ that can be written as $w(x; z) = w(x - z)$ for $x, z \in \mathbb{T}^d$, satisfying $w(x) \sim |x|^{\beta^2}$ for $x \in \mathbb{T}^d$, for some $0 < \beta^2 < d$. In particular, this is the case for the weight $w_z(y) = |y - z|^{\beta^2} e^{-\beta^2 g(y-z)}$. Here and in the sequel, we identify $\mathbb{T}^d$ with $[-1, 1]^d$ with periodic boundary conditions. As our distance we use the distance introduced in Section 1.2, that behaves as Euclidean distance on small scales. As in what follows we are essentially interested only in local behavior around the singular pole, for notational simplicity we work only with the Euclidean distance.

We first remark that the weight $\hat{w} = |x|^{\beta^2}$ is a Muckenhoupt- A_2 (see [8] and Appendix A). Specifically the estimate

$$\sup_B \frac{1}{|B|} \int_B \hat{w} dx \left(\frac{1}{|B|} \int_B \hat{w}^{-1} dx \right) \leq \frac{1}{d^2 - \beta^4}$$

holds true, where the supremum is taken over all Euclidean balls. Although we will not use this form of the estimate, it is worth noting that $\hat{w}$ is a Muckenhoupt- A_2 exactly when $\beta^2 < d$. Let $A_2(w)$ denote the A_2 constant (see [8]). Then from the above estimate we obtain $A_2(w) \leq \frac{C(w)}{d^2 - \beta^4}$.

Let next $z \in \mathbb{T}^d$ be fixed and recall that the weighted Sobolev space $W^{m,p}(\Omega; w_z)$ for $1 \leq p < \infty$ is given by the completion of $C^\infty(\Omega)$ under

$$\|u\|_{W^{m,p}(\Omega; w_z)} := \|u\|_{L^p(\Omega; w_z)} + \sum_{1 \leq |\alpha| \leq m} \|\partial_x^\alpha u\|_{L^p(\Omega; w_z)}.$$

In the sequel, we use notation $d\mu_z = w(x; z)dx$ and if there is no danger of confusion, we will suppress the z and write simply w or $d\mu$ instead.

For our proofs, we also define the Euclidean balls with center at x as $B_r(x) := \{y \in \mathbb{R}^d : |y - x| < r\}$, and $B_r := B_r(0)$ for short. Finally, for an A_2 weight w we define the mass of a set $E \subset \mathbb{R}^d$ as $w(E) = \int_E w(x)dx$. The weighted normalized norms for balls $B_r(x) \subset \mathbb{T}^d$ as

$$\|u\|_{\underline{W}^{m,p}(B_r(x); w_z)} := \sum_{0 \leq |\alpha| \leq m} \left(\int_{B_r(x)} |\partial_x^\alpha u|^p d\mu_z \right)^{\frac{1}{p}},$$

where the average in the integral is with respect to the measure $d\mu_z$. The norm $\underline{L}^p$ is defined similarly.

Our main result of this section is the following regularity result.

Theorem 4.1. *Let $u(\cdot; z) \in H^1(\mathbb{T}^d; w)$ be a weak solution to the equation*

$$\begin{cases} -\nabla \cdot (w(\cdot; z) \nabla u(\cdot; z)) = \nabla \cdot (w(\cdot; z) \mathbf{f}(\cdot; z)), & \text{in } \mathbb{T}^d, \\ \int_{\mathbb{T}^d} u(x; z) dx = 0, \end{cases}$$

for every $z \in \mathbb{T}^d$. Then, there exists $\varepsilon = \varepsilon(\beta, d) > 0$ such that if

$$\sum_{0 \leq |\alpha_1 + \alpha_2| \leq \lfloor \frac{d+1}{2} \rfloor} \|\partial_x^{\alpha_1} \partial_z^{\alpha_2} \mathbf{f}\|_{L_z^p(\mathbb{T}^d; L_x^q(\mathbb{T}^d; w_z))} < \infty,$$

for $q > 2d - \varepsilon$ and $p > d/\gamma$ where $\gamma = \gamma(\beta, d, q) \in (0, 1)$. Then

$$u(x; \cdot) \in H^{d/2}(\mathbb{T}^d), \quad (4.1)$$

uniformly in x .

We will prove this theorem by using regularity theory for weighted equations, see [7, 8], which we record in the next subsection.

Note that from $w \in L^\infty(\mathbb{T}^d)$, it follows that $W^{d/2, q}(\mathbb{T}^d; dx) \subset W^{d/2, q}(\mathbb{T}^d; w)$ by basic estimates as in [14, Lemma 1.13].

4.1. Basic regularity and Green function bounds.

Lemma 4.2. *Let w be an A_2 weight and let $u \in H^1(B_1; w)$ be a weak solution in B_1 of*

$$-\nabla \cdot (w \nabla u) = \nabla \cdot (w \mathbf{f}).$$

There exists $\varepsilon = \varepsilon(A_2(w)) > 0$ such that if $f \in L^q(B_1, \mathbb{R}^d; w)$ for $q > 2d - \varepsilon$, then u is Hölder continuous. Specifically, for $\varrho < 1$ there exists a constant $C = C(A_2(w), d) > 1$ and an exponent $\alpha = \alpha(A_2(w), d) \in (0, 1)$ such that

$$\sup_{B_\varrho} |u| + \frac{\text{osc}_{B_\varrho} u}{\varrho^\alpha} \leq C \left(\|u\|_{\underline{L}^2(B_1; w)} + \|\mathbf{f}\|_{\underline{L}^q(B_1; w)} \right). \quad (4.2)$$

Furthermore, it holds

$$\int_{B_\varrho} |\nabla u|^2 d\mu \lesssim \varrho^{2\alpha-2} \left(\|\mathbf{f}\|_{\underline{L}^q(B_1; w)}^2 + \int_{B_1} |\nabla u|^2 d\mu \right), \quad (4.3)$$

where $d\mu = w dx$.

Proof. In the following we assume without loss of generality that $\int_{B_1} u d\mu = 0$, and we note that estimate (4.2) is just [8, Theorem 2.3.15].

To prove (4.3), note that (4.2) and the weighted Poincaré inequality, see [8], gives that for $|x| < 1/2$ we have

$$\begin{aligned} |u(x) - u(0)|^2 &\lesssim |x|^{2\alpha} \left(\|u\|_{\underline{L}^2(B_1; w)}^2 + \|\mathbf{f}\|_{\underline{L}^q(B_1; w)}^2 \right) \\ &\lesssim |x|^{2\alpha} \left(\|\nabla u\|_{\underline{L}^2(B_1; w)}^2 + \|\mathbf{f}\|_{\underline{L}^q(B_1; w)}^2 \right). \end{aligned} \quad (4.4)$$

Now take a cutoff function $\chi_{B_\varrho} \leq \phi \leq \chi_{B_{2\varrho}}$ with $|\nabla \phi| \lesssim \frac{1}{\varrho}$, and test the equation with $\psi = (u(x) - u(0))\phi^2$. We get

$$\begin{aligned} 2 \int \nabla u \cdot \nabla \phi \cdot \phi(u(x) - u(0)) d\mu + \int |\nabla u|^2 \phi^2 d\mu \\ = 2 \int \mathbf{f} \cdot \nabla \phi \cdot \phi(u(x) - u(0)) d\mu + \int \mathbf{f} \cdot \nabla u \phi^2 d\mu. \end{aligned}$$

Using Young's inequality for products, we get the classical Caccioppoli estimate

$$\int |\nabla u|^2 \phi^2 d\mu \lesssim \int |\mathbf{f}|^2 \phi^2 d\mu + \int |\nabla \phi|^2 (u(x) - u(0))^2 d\mu.$$

Now

$$\int |u(x) - u(0)|^2 |\nabla \phi|^2 d\mu \lesssim \frac{\sup_{x \in B_{2\varrho}} |u(x) - u(0)|^2}{\varrho^2} w(B_{2\varrho}),$$

and, using Hölder's inequality,

$$\int_{B_\varrho} |\mathbf{f}|^2 d\mu \leq w(B_\varrho)^{\frac{q-2}{q}} \left(\int_{B_\varrho} |\mathbf{f}|^q d\mu \right)^{2/q}.$$

Consequently,

$$\int_{B_\varrho} |\nabla u|^2 d\mu \lesssim w(B_{2\varrho})^{\frac{q-2}{q}-1} \left(\int_{B_{2\varrho}} |\mathbf{f}|^q d\mu \right)^{2/q} + \frac{\sup_{x \in B_{2\varrho}} |u(x) - u(0)|^2}{\varrho^2}.$$

Finally, by (4.4) there exists another $\gamma = \gamma(A_2(w), d) \in (0, 1)$ such that

$$\int_{B_\varrho} |\nabla u|^2 d\mu \lesssim \varrho^{2\gamma-2} \left(\|\mathbf{f}\|_{L^q(B_1; w)}^2 + \int_{B_1} |\nabla u|^2 d\mu \right).$$

This completes the proof. $\square$

If we already know a bound on the $\|\mathbf{f}\|_{\underline{L}^2(B_\varrho; w)}$ norm in terms of powers of ϱ , we can bypass the L^q norm, and obtain the same regularity as in Lemma 4.2:

Lemma 4.3. *Let w be an A_2 weight and let $u \in H^1(B_1; w)$ be a weak solution to*

$$\nabla \cdot (w \nabla u) = \nabla \cdot (w \mathbf{f}) \quad (4.5)$$

in B_1 . For all $\rho < 1$, let $f \in L^2(B_1, \mathbb{R}^d; w)$ satisfy the growth estimate

$$\int_{B_\varrho} |\mathbf{f}|^2 d\mu \lesssim M \varrho^{2\gamma-2} \int_{B_1} |\mathbf{f}|^2 d\mu \quad (4.6)$$

for some exponent $\gamma = \gamma(A_2(w), d) \in (0, 1)$ and $d\mu = w dx$. Then there exists an exponent $\hat{\gamma} = \hat{\gamma}(A_2(w), d, \gamma) \in (0, 1)$ and a constant $C = C(A_2(w), d, \gamma) \geq 1$ such that

$$\int_{B_\varrho} |\nabla u|^2 d\mu \leq C \varrho^{2\hat{\gamma}-2} \left(\int_{B_1} |\nabla u|^2 d\mu + M \int_{B_1} |\mathbf{f}|^2 d\mu \right).$$

Proof. Let $\varrho < R \leq 1$ and construct the comparison solution v that solves

$$\begin{cases} \nabla \cdot (w \nabla v) = 0, & \text{in } B_R. \\ v = u, & \partial B_R. \end{cases}$$

Denote $\hat{u} = u - v$. Then $\hat{u}$ solves (4.5) with $\hat{u} \in H_0^1(B_R; w)$; thus we can test (4.5) with $\hat{u}$ itself, which gives after applying Young's inequality together with the Poincaré inequality, that

$$\int_{B_R} |\nabla \hat{u}|^2 d\mu \lesssim \int_{B_R} |\mathbf{f}|^2 d\mu.$$

Also, by direct computation and applying Lemma 4.2 we get

$$\begin{aligned} \int_{B_\varrho} |\nabla u|^2 d\mu &\lesssim \int_{B_\varrho} |\nabla v|^2 d\mu + \int_{B_\varrho} |\nabla \hat{u}|^2 d\mu \\ &\lesssim \frac{w(B_\varrho)}{w(B_R)} \left(\frac{\varrho}{R}\right)^{2\alpha-2} \int_{B_R} |\nabla v|^2 d\mu + \int_{B_R} |\nabla \hat{u}|^2 d\mu \\ &\lesssim \frac{w(B_\varrho)}{w(B_R)} \left(\frac{\varrho}{R}\right)^{2\alpha-2} \int_{B_R} |\nabla u|^2 d\mu + \int_{B_R} |\mathbf{f}|^2 d\mu. \end{aligned}$$

Now by (4.6) we have by a standard iteration inequality, see for instance [13, Lemma 3.4], that

$$\int_{B_\varrho} |\nabla u|^2 d\mu \lesssim \left(\frac{\varrho}{R}\right)^{2\hat{\alpha}-2} \int_{B_R} |\nabla u|^2 d\mu + \varrho^{2\hat{\alpha}-2} M \int_{B_R} |\mathbf{f}|^2 d\mu,$$

which completes the proof. $\square$

We finally end with a regularity result for the polynomially bounded weight w_z away from the pole z . We will later use this to derive bounds for the Green function for polynomially weighted equations.

Lemma 4.4. *Let $z^* \in \mathbb{R}^d$, $x^* \in \mathbb{R}^d \setminus \{z\}$, and let*

$$r = \frac{\min(|x^* - z^*|, 1)}{16}. \quad (4.7)$$

For $w_{z^*}(x) = |x - z^*|^{\beta^2}$, let $u \in H^1(B_{8r}(x^*); w_{z^*})$ be a solution of

$$\nabla \cdot (w_{z^*} \nabla u) = 0 \quad \text{in } B_{8r}(x^*).$$

Then there exist a constant $C(\beta, d) < \infty$ such that

$$\sup_{B_r(x^*)} |\nabla u(x)| \leq Cr^{-1} \inf_a \int_{B_{8r}(x^*)} |u - a|^2 d\mu_{z^*}. \quad (4.8)$$

Proof. Since for any $y \in B_{8r}(x^*)$ we have $|y - z^*| \geq 8r$, it follows that, for any $i = 1, \dots, d$ and for $\hat{w} = \frac{w_{z^*}}{\inf_{B_{8r}(x^*)} w_{z^*}}$, we have

$$\sup_{B_{8r}(x^*)} \hat{w} \leq C(\beta), \quad \text{and}, \quad \sup_{B_{8r}(x^*)} |\partial_{x_i} \hat{w}| \leq \frac{C(\beta)}{r}.$$

As such, $v = u - a$ solves the equation $\nabla \cdot (\hat{w} \nabla v) = 0$, where the coefficient $\hat{w}$ is Lipschitz continuous with Lipschitz constant C/r and $\hat{w} \geq 1$. Now the regularity theory for such equations, see display (3.17) in [13], implies that

$$\sup_{x \in B_r(x^*)} |\nabla v| \lesssim \frac{1}{r} \int_{B_{8r}(x^*)} v^2 d\mu_{z^*},$$

proving the claim. $\square$

Lemma 4.5. *There exists a periodic Green function $(x, y, z) \mapsto G(x, y; z)$ solving, for every $z, y \in \mathbb{T}^d$, the equation*

$$\begin{cases} -\nabla \cdot (w(\cdot; z) \nabla G(\cdot, y; z)) = \delta(\cdot - y) - \frac{1}{\int_{\mathbb{T}^d} dz} & \text{in } \mathbb{T}^d, \\ \int_{\mathbb{T}^d} G(x, y; z) dx = 0. \end{cases} \quad (4.9)$$

Moreover, the Green function is symmetric in the first two variables, that is, for every $x, y, z \in \mathbb{T}^d$,

$$G(x, y; z) = G(y, x; z). \quad (4.10)$$

Furthermore, there exists a constant $C(\beta, d) < \infty$ such that, for every $x, y, z \in \mathbb{T}^d$, it holds for $i = 0, 1$ that

$$(|x - y| \wedge |y - z|)^i |\nabla_y^i G(x, y; z)| \lesssim \begin{cases} (|x - z| \vee |y - z|)^{-\beta^2} |x - y|^{2-d}, & d > 2, \\ (|x - z| \vee |y - z|)^{-\beta^2} \log \frac{C}{|x - y|}, & d = 2. \end{cases} \quad (4.11)$$

Proof. It is clear from basic existence theory that for periodic f , the following problem has a unique periodic solution (with $\int_{\mathbb{T}^d} u dy = 0$)

$$-\nabla \cdot (w(\cdot; z) \nabla u) = f - \int_{\mathbb{T}^d} f dy.$$

Now, we can represent the map $f \rightarrow u$ via an integral kernel (see [7, 10]), i.e.

$$u(x; z) = \int_{\mathbb{T}^d} G(x, y; z) f(y) dy.$$

It can be shown using standard methods (see [7, 10]) that such G is unique, periodic, and satisfies (4.9).

For the rest of the proof, denote G_{per} the periodic Green function and let G_{Σ} be the Green function for the unit ball $B_1 = \Sigma$. Let $y \in B_{1/2}$ and denote $u_z = G_{\text{per}}(\cdot, y; z) - G_{\Sigma}(\cdot, y; z)$, then

$$\begin{cases} -\nabla \cdot (w(\cdot; z) \nabla u_z) = -1, & \text{in } \Sigma, \\ u_z = G_{\text{per}}(\cdot, y; z), & \text{on } \partial \Sigma. \end{cases}$$

By the maximum principle (see [8, Theorem 2.2.3]) we know that

$$|u_z| \leq C + \sup_{\partial \Sigma} G_{\text{per}}(\cdot, y; z).$$

Next, by compactness we know that the right-hand side is bounded by a constant $C(d, \beta)$, i.e. $|u_z| \leq C$ in Σ . Hence, it suffices to bound G_{Σ} from above.

By [7, Theorem 3.3] we have that

$$G_{\Sigma}(x, y; z) \leq C \int_{|x-y|}^1 \frac{r}{w(B_r(x); z)} dr$$

By symmetry, we may assume that $|y - x| \leq |z - x|$. First, we have the lower bound

$$w(B_r(x); z) \geq \frac{1}{C} \int_{B_r(x)} |y - z|^{\beta^2} dy \geq C(r \vee |x - z|)^{\beta^2} r^d$$

from which we deduce that

$$\begin{aligned} & \int_{|x-y|}^1 \frac{r}{w(B_r(x); z)} dr \\ & \leq C \int_{|x-y|}^1 r^{2-d} (r \vee |x-z|)^{-\beta^2} \frac{dr}{r} \\ & \leq C |x-z|^{-\beta^2} \int_{|x-y|}^{|x-y| \vee |x-z|} r^{2-d} \frac{dr}{r} + C \int_{|x-y| \vee |x-z|}^1 r^{2-d-\beta^2} \frac{dr}{r}. \end{aligned}$$

The first integral admits the estimate

$$\int_{|x-y|}^{|x-y| \vee |x-z|} r^{2-d} \frac{dr}{r} \leq C \begin{cases} |x-y|^{2-d} \chi_{\{|x-z| > |x-y|\}} & \text{if } d > 2, \\ \log \left(\frac{|x-y| \vee |x-z|}{|x-y|} \right) & \text{if } d = 2 \end{cases}$$

and the second (since $d \geq 2$ and $\beta^2 > 0$)

$$\int_{|x-y| \vee |x-z|}^1 r^{2-d-\beta^2} \frac{dr}{r} \leq C(|x-y| \vee |x-z|)^{2-d-\beta^2}.$$

By $|y - z| \leq |x - z|$ we see that $|x - y| \leq 2|x - z|$, and hence for $d > 2$ we have

$$\begin{aligned} \int_{|x-y|}^1 \frac{r}{w(B_r(x); z)} dr & \lesssim |x-z|^{-\beta^2} |x-y|^{2-d} + |x-z|^{2-d-\beta^2} \\ & \lesssim |x-z|^{-\beta^2} |x-y|^{2-d}. \end{aligned}$$

In the case $d = 2$ we get (since $|x - z| < C$)

$$\begin{aligned} \int_{|x-y|}^1 \frac{r}{w(B_r(x); z)} dr &\lesssim |x-z|^{-\beta^2} \log \frac{|x-z|}{|x-y|} + |x-z|^{-\beta^2} \\ &\lesssim |x-z|^{-\beta^2} \log \frac{|x-z|}{|x-y|} \lesssim |x-z|^{-\beta^2} \log \frac{C}{|x-y|}. \end{aligned}$$

This proves (4.11) in the case $i = 0$, as $x, y, z \in \mathbb{T}^d$ and we know that the right-hand side is bounded from below. Thus, we can dispense with the additive constant in passing from G_Σ to G_{per} by simply choosing C large enough.

To prove the gradient estimate we first consider the fact that if we are away from the diagonal, i.e. $x \neq y$ then $v(\cdot) = G_\Sigma(\cdot, y; z)$ solves

$$\nabla \cdot (w_z \nabla v) = 0.$$

As such Lemma 4.4 implies that whenever $|x - y| \wedge |y - z| > 16\varrho$ we get

$$|\nabla_y G_\Sigma(x, y; z)| \leq \frac{1}{\varrho} \left(\frac{1}{w(B_\varrho(y); z)} \int_{B_\varrho(y)} G_\Sigma(x, y'; z)^2 d\mu_z(y') \right)^{1/2}.$$

Assembling the estimates, as $\varrho < 1$, we obtain

$$|\nabla_y G_\Sigma(x, y; z)| \lesssim \frac{1}{\varrho} G_\Sigma(x, y; z),$$

and we complete the proof by arguing as for G_{per} above and then applying (4.11) for $i = 0$. $\square$

4.2. Proof of Theorem 4.1. Fix $x^*, z^* \in \mathbb{T}^d$ such that $x^* \neq z^*$, and let $r = \min(|x^* - z^*|, 1)/16^2$. In the sequel, we use w_{z^*} and u_{z^*} to denote functions $w(\cdot; z^*)$ and $u(\cdot; z^*)$ whenever confusion cannot arise.

Formally, one could differentiate u with respect to z leading to a new equation and then try to derive L^∞ bounds for the solution. However, then the right-hand side would contain $\partial_{z_i} w$ which is not in $L^2(w)$. To overcome this, we first shift the pole, and then derive a corrector expansion and L^∞ bounds for the correctors. The proof is divided into four steps. In the step 1, we shift the pole, while the corrector expansion is derived in step 2. Correctors in the expansion are then estimated in step 3, and in step 4 we conclude the proof.

Step 1. Shifting the pole: We introduce the following shift transformation (i.e. change of coordinates): Let $\Phi_{z^*}(x) = x + \eta(x - z^*)h$, where $\chi_{B_r} \leq \eta \leq \chi_{B_{2r}}$ with $\|\nabla^k \eta\|_{L^\infty(\mathbb{R}^d)} \leq C_k r^{-k}$. Note that if $|h| \leq cr$ for small constant c , then Φ_{z^*} is actually a diffeomorphism.

²Note that as we work on $\mathbb{T}^d$, the distance $|x^* - z^*|$ is always bounded and hence the minimum with 1 is not required. For the reader's convenience, we have taken the minimum with 1 to emphasize that we are interested in the small scales proportional to $|x^* - z^*|$.

Next we observe that since $w(x; z) = w(x - z)$, we also have $w(x + h; z + h) = w(x; z)$, which will be useful later. Ultimately, we will establish an expansion of the form

$$u(\Phi_{z^*}(\cdot); z^*) - u(\Phi_{z^*}(\cdot); z^* + h) = \sum_{1 \leq |\alpha| \leq d/2} v^{(\alpha)}(x^*; z^*) h^\alpha + o(|h|^{d/2}),$$

for $x \in (B_{4r}(z^*))^c$ and $v^{(\alpha)}(x^*; z^*) = \partial_z^\alpha u(x^*; z^*)$. Then we establish L^∞ bounds for the correctors $v^{(\alpha)}$ and this will give us the desired regularity for u in terms of z .

By the definition of $u(x; z)$, we have for any $\phi \in C^\infty(\mathbb{T}^d)$ that

$$\int_{\mathbb{T}^d} w(x; z^* + h) \mathbf{f}(x; z^* + h) \nabla \phi(x) dx = \int_{\mathbb{T}^d} w(x; z^* + h) \nabla u(x; z^* + h) \cdot \nabla \phi(x) dx.$$

To proceed, denote first $\hat{w} := w(\Phi_{z^*}(\cdot); z^* + h)$, $\hat{u} := u(\Phi_{z^*}(\cdot); z^* + h)$ and $\hat{\mathbf{f}} := \mathbf{f}(\Phi_{z^*}(\cdot); z^* + h)$. Considering the change of variables (for h small enough) $x = \Phi_{z^*}(y)$, then $dx = |\det \nabla \Phi_{z^*}(y)| dy$ and if we denote $\hat{\phi}(y) = \phi(\Phi_{z^*}(y))$, then substituting $u(x; z^* + h) = \hat{u}(\Phi_{z^*}^{-1}(x))$ we have that

$$\begin{aligned} \int_{\mathbb{T}^d} w(\Phi_{z^*}(y); z^* + h) [(\nabla \Phi_{z^*})^{-T} \nabla_y \hat{u}(y; z^* + h)] \cdot [(\nabla \Phi_{z^*})^{-T} \nabla_y \hat{\phi}(y)] |\det(\nabla \Phi_{z^*}(\cdot))| dy \\ = \int_{\mathbb{T}^d} \hat{w} (\nabla \Phi_{z^*})^{-1} (\nabla \Phi_{z^*})^{-T} |\det(\nabla \Phi_{z^*}(\cdot))| \nabla \hat{u} \cdot \nabla \hat{\phi} dy. \end{aligned}$$

Thus, we see that $\hat{u}$ solves the equation

$$-\nabla \cdot (\hat{w} (\nabla \Phi_{z^*})^{-1} (\nabla \Phi_{z^*})^{-T} |\det(\nabla \Phi_{z^*}(\cdot))| \nabla \hat{u}) = \nabla \cdot (|\det(\nabla \Phi_{z^*}(\cdot))| \hat{w} (\nabla \Phi_{z^*})^{-1} \hat{\mathbf{f}}),$$

and we get

$$-\nabla \cdot \hat{w} \nabla \hat{u} = \nabla \cdot \mathbf{a} \nabla \hat{u} + \nabla \cdot (|\det(\nabla \Phi_{z^*}(\cdot))| \hat{w} (\nabla \Phi_{z^*})^{-1} \hat{\mathbf{f}}),$$

where

$$\mathbf{a} = \hat{w}(\cdot) \left((\nabla \Phi_{z^*}(\cdot))^{-1} (\nabla \Phi_{z^*}(\cdot))^{-T} |\det(\nabla \Phi_{z^*}(\cdot))| - \mathbf{I}_d \right).$$

Note that $\mathbf{a} = 0$ in $B_r(z^*)$. Now this equation together with the equation for u ,

$$\begin{aligned} -\nabla \cdot (w_{z^*} \nabla (\hat{u} - u_{z^*})) &= -\nabla \cdot (w_{z^*} \mathbf{f}_{z^*}) - \nabla \cdot (w_{z^*} \nabla \hat{u}) \\ &= \nabla \cdot ((\hat{w} - w_{z^*}) \nabla \hat{u}) + \nabla \cdot \mathbf{a} \nabla \hat{u} + \nabla \cdot (w_{z^*} \tilde{\mathbf{f}}) \\ &= \nabla \cdot (w_{z^*} \mathbf{b} \nabla (\hat{u} - u_{z^*})) + \nabla \cdot (w_{z^*} \mathbf{b} \nabla u_{z^*}) + \nabla \cdot (w_{z^*} \tilde{\mathbf{f}}), \end{aligned} \tag{4.12}$$

where the matrix $\mathbf{b}$ is defined as

$$\mathbf{b} = \left(\frac{\hat{w}}{w_{z^*}} - 1 \right) \mathbf{I}_d + \frac{\hat{w}}{w_{z^*}} \left((\nabla \Phi_{z^*}(\cdot))^{-1} (\nabla \Phi_{z^*}(\cdot))^{-T} |\det(\nabla \Phi_{z^*}(\cdot))| - \mathbf{I}_d \right),$$

and

$$w_{z^*} \tilde{\mathbf{f}} = |\det(\nabla \Phi_{z^*}(\cdot))| \hat{w} (\nabla \Phi_{z^*})^{-1} \hat{\mathbf{f}} - w_{z^*} \mathbf{f}_{z^*}.$$

Here $\mathbf{b}$ is as smooth as the correlation function. Indeed,

$$\frac{\hat{w}(x)}{w(x; z^*)} = \frac{w(x - z^* - (1 - \eta)h)}{w(x - z^*)},$$

so that in $B_r(z^*)$ we have that $\hat{w}/w_{z^*} = 1$, and hence $\mathbf{b} = 0$ in $B_r(z^*)$. Next observe that by our assumptions on w we have

$$\frac{\hat{w}}{w(x; z^*)} \lesssim \left(\frac{r + |h|}{r} \right)^{\beta^2}$$

which immediately gives the bounds

$$\left| \frac{\hat{w}}{w(x; z^*)} - 1 \right| \lesssim (1 + |h|/r)^{\beta^2} - 1 = \beta^2(|h|/r) + o(|h|/r)$$

and

$$\left| \left((\nabla \Phi_{z^*}(\cdot))^{-1} (\nabla \Phi_{z^*}(\cdot))^{-T} | \det(\nabla \Phi_{z^*}(\cdot)) | - \mathbf{I}_d \right) \right| \leq c|h|/r.$$

Hence, we have

$$|\mathbf{b}| \leq c|h|/r. \quad (4.13)$$

Thus, $\mathbf{b}/|h|$ is bounded, and we can choose $|h| < r/c$ such that $|\mathbf{b}| < 1/2$.

Step 2. Corrector expansion: In this step we derive the expansions.

First, note that for any function $\tilde{u}$, denoting $v = \hat{u} - u_{z^*} - \tilde{u}$ and using (4.12), we obtain

$$\begin{aligned} -\nabla \cdot (w_{z^*} \nabla v) &= \nabla \cdot (w_{z^*} \mathbf{b} \nabla v) + \nabla \cdot (w_{z^*} \mathbf{b} \nabla u_{z^*}) + \nabla \cdot (w_{z^*} (1 + \mathbf{b}) \nabla \tilde{u}) \\ &\quad + \nabla \cdot (w_{z^*} \tilde{\mathbf{f}}). \end{aligned}$$

After rearrangement, this gives

$$-\nabla \cdot (w_{z^*} (1 + \mathbf{b}) \nabla v) = \nabla \cdot (w_{z^*} \mathbf{b} \nabla u_{z^*}) + \nabla \cdot (w_{z^*} (1 + \mathbf{b}) \nabla \tilde{u}) + \nabla \cdot (w_{z^*} \tilde{\mathbf{f}}). \quad (4.14)$$

Now let us define the expansions (in the variable h , and for any $m \in \mathbb{N}$) for the involved terms:

$$\begin{aligned} v_m &= \hat{u} - u_{z^*} - \sum_{1 \leq |\alpha| \leq m} v^{(\alpha)} h^\alpha, \quad \mathbf{b} = \sum_{1 \leq |\alpha| \leq m} \mathcal{E}^{(\alpha)} h^\alpha + \chi_{(B_r(z^*))^c} o(|h|^m) \\ \tilde{\mathbf{f}} &= \sum_{1 \leq |\alpha| \leq m} \mathcal{F}^{(\alpha)} h^\alpha + E_{\mathbf{f}, m}, \end{aligned} \quad (4.15)$$

where small- o notation $o(|h|^m)$ means that the remainder term may depend on r . The error term $E_{\mathbf{f}, m}$ satisfies for q given in the statement of Theorem 4.1,

$$\|E_{\mathbf{f}, m}\|_{L^q(\mathbb{T}^d; w_{z^*})} = o(|h|^m),$$

which again may depend on r . Furthermore, using the calculations below (4.12) and some elementary computations we obtain bounds for the coefficients in the expansions, namely

$$\begin{cases} |\mathcal{E}^{(\alpha)}| & \lesssim \frac{1}{|x-z^*|^{|\alpha|}} \chi_{B_r^c}(z^*) \\ \|\mathcal{F}^{(\alpha)}\|_{L^q(\mathbb{T}^d; w_{z^*})} & \lesssim r^{-|\alpha|} \sum_{0 \leq |\alpha_1 + \alpha_2| \leq |\alpha|} \|(\partial_x^{\alpha_1} \partial_z^{\alpha_2} \mathbf{f})(\cdot; z^*)\|_{L^q(\mathbb{T}^d; w_{z^*})}, \end{cases} \quad (4.16)$$

and the implied constants are independent of h and r .

As the expansion of v_m above is formal, in the following we will define the functions $v^{(\alpha)}$ so that we will have $v_m = o(|h|^m)$ in L^∞ . Here we have, for the sake of simplicity in the notation, omitted the division by $\alpha!$ (Taylor expansions can be recovered simply by multiplying with $\alpha!$). Note also that by (4.13), we do not have a zero order term in the expansion of $\mathbf{b}$ or $\tilde{\mathbf{f}}$.

Now, by using the definition of v_m , we get from (4.14) that

$$\begin{aligned} -\nabla \cdot (w_{z^*}(1 + \mathbf{b})\nabla v_m) &= \nabla \cdot (w_{z^*} \mathbf{b} \nabla u_{z^*}) + \sum_{|\alpha| \leq m} \nabla \cdot (w_{z^*}(1 + \mathbf{b})\nabla v^{(\alpha)}) h^\alpha \\ &\quad + \nabla \cdot (w_{z^*} \tilde{\mathbf{f}}). \end{aligned} \quad (4.17)$$

Now we will define the functions $v^{(\alpha)}$ so that we have $v_m = o(|h|^m)$. To do this, we first expand the right-hand side of (4.17) using the expansion in (4.15), leading to

$$\begin{aligned} -\nabla \cdot (w_{z^*}(1 + \mathbf{b})\nabla v_m) &= \nabla \cdot (w_{z^*} \sum_{1 \leq |\alpha| \leq m} \mathcal{E}^{(\alpha)} h^\alpha \nabla u_{z^*}) \\ &\quad + \sum_{|\alpha| \leq m} \nabla \cdot \left(\sum_{1 \leq |\hat{\alpha}| \leq m} w_{z^*} (\mathcal{E}^{(\hat{\alpha})} h^{\hat{\alpha}}) \nabla v^{(\alpha)} \right) h^\alpha + \sum_{|\alpha| \leq m} \nabla \cdot (w_{z^*} \nabla v^{(\alpha)}) h^\alpha \\ &\quad + \sum_{1 \leq |\alpha| \leq m} \nabla \cdot (w_{z^*} \mathcal{F}^{(\alpha)}) h^\alpha + o(|h|^m). \end{aligned} \quad (4.18)$$

Matching the powers of h^α in the right-hand side of the above we get that the functions $v^{(\alpha)}$ should solve

$$\begin{aligned} -\nabla \cdot (w_{z^*} \nabla v^{(\alpha)}) &= \nabla \cdot (w_{z^*} \mathcal{E}^{(\alpha)} \nabla u_{z^*}) + \sum_{\substack{1 \leq |\alpha_2|, |\alpha_1| \leq m, \\ \alpha_1 + \alpha_2 = \alpha}} \nabla \cdot (w_{z^*} \mathcal{E}^{(\alpha_2)} \nabla v^{(\alpha_1)}) \\ &\quad + \nabla \cdot (w_{z^*} \mathcal{F}^{(\alpha)}) \end{aligned} \quad (4.19)$$

for $1 \leq |\alpha| \leq m$. We can now define the functions $v^{(\alpha)}$ inductively by solving (4.19) in $\mathbb{T}^d$ (with periodic boundary conditions) for $1 \leq |\alpha| \leq m$, where we note that at each step the right-hand side has the correct regularity to ensure solvability. This follows from (4.16), i.e. the expansion coefficients $\mathcal{E}^{(\alpha)}$ vanish in $B_r(z^*)$, and

$\mathcal{F}^{(\alpha)} \in L^q(\mathbb{T}^d; w_{z^*})$. Now, inserting our definition of $v^{(\alpha)}$ into (4.18) we get

$$\begin{aligned} -\nabla \cdot (w_{z^*}(1 + \mathbf{b})\nabla v_m) &= \sum_{m+1 \leq |\alpha|} h^\alpha \nabla \cdot (w_{z^*} \mathcal{E}^{(\alpha)} \nabla u_{z^*}) \\ &\quad + \sum_{\substack{1 \leq |\alpha| \leq m, 1 \leq |\hat{\alpha}| \leq m, \\ m+1 \leq |\hat{\alpha} + \alpha|}} \nabla \cdot (w_{z^*} \mathcal{E}^{(\hat{\alpha})} \nabla v^{(\alpha)}) h^{\hat{\alpha} + \alpha} \\ &\quad + \sum_{m+1 \leq |\alpha|} h^\alpha \nabla \cdot (w_{z^*} \mathcal{F}^{(\alpha)}). \end{aligned}$$

In order to show $v_m = o(|h|^m)$, it suffices to use Lemma 4.2 and note that the quantities $\mathcal{E}^{(\alpha)} \nabla u_{z^*}$, $\mathcal{E}^{(\hat{\alpha})} \nabla v^{(\alpha)}$, and $\mathcal{F}^{(\alpha)}$ appearing on the right-hand side belong to $L^{2d-\varepsilon}$, which follows from construction and the estimate (4.16) together with the assumption of the theorem.

Step 3. Local boundedness of the correctors: In this step we prove an L^∞ estimate for $v^{(\alpha)}$. Recall that $v^{(\alpha)}$ for $1 \leq |\alpha| \leq m$ solves (4.19). Here, as the right-hand side of (4.19) is the divergence of a periodic function, we may represent $v^{(\alpha)}$ using the Green function. After integration by parts, we obtain

$$\begin{aligned} v^{(\alpha)}(x^*) &= - \int_{\mathbb{T}^d} \nabla_y G(x^*, y; z^*) \cdot (w_{z^*} \mathcal{E}^{(\alpha)} \nabla u_{z^*}) dy \\ &\quad - \sum_{\substack{1 \leq |\alpha_2|, |\alpha_1| \leq m, \\ \alpha_1 + \alpha_2 = \alpha}} \int_{\mathbb{T}^d} \nabla_y G(x^*, y; z^*) \cdot (w_{z^*} \mathcal{E}^{(\alpha_2)} \nabla v^{(\alpha_1)}) dy \\ &\quad + \int_{\mathbb{T}^d} \nabla_y G(x^*, y; z^*) w_{z^*}(y) \mathcal{F}^{(\alpha)}(y) dy \\ &= I_1 + I_2 + I_3. \end{aligned}$$

We estimate the terms one by one. For this, let us consider the rings

$A_j = (B_{2^j r}(x^*) \setminus B_{2^{j-1} r}(x^*)) \setminus B_r(z^*)$ for $j = 0, \dots, j_0$, where $A_0 = B_r(x^*) \setminus B_r(z^*) = B_r(x^*)$, and $2^{j_0} r = C$, and note that $\mathcal{E}^{(\alpha)} = 0$ on the ball $B_r(z^*)$. Using Lemma 4.5 and (4.16) we have for $d > 2$ that

$$\begin{aligned} I_1 &= \sum_{j=0}^{j_0} \int_{A_j} \nabla_y G(x^*, y; z^*) \cdot (w_{z^*} \mathcal{E}^{(\alpha)} \nabla u_{z^*}) dy \\ &\lesssim \sum_{j=0}^{j_0} \int_{A_j} \frac{(|x^* - z^*| \vee |y - z^*|)^{-\beta^2} |x^* - y|^{2-d} |y - z^*|^{\beta^2} |\mathcal{E}^{(\alpha)} \nabla u_{z^*}|}{(|x^* - y| \wedge |y - z^*|)} dy \\ &\lesssim r^{-|\alpha|} \int_{A_0} |x^* - y|^{1-d} |\nabla u_{z^*}| dy + \sum_{j=1}^{j_0} (2^j r)^{1-|\alpha|} \int_{A_j} |\nabla u_{z^*}| d\mu_{z^*}, \end{aligned} \quad (4.20)$$

where we have also taken advantage of the fact that on the sets A_j , $j > 0$ we are at a distance of the order $2^j r$ away from the pole z^* , and hence $\frac{\sup w_{z^*}}{\inf w_{z^*}} \lesssim C$.

Let us begin by bounding the first term on the right, and for simplicity let us assume that $x^* = 0$. Alongside with Lemma 4.2, we obtain

$$\begin{aligned}
\int_{A_0} |y|^{1-d} |\nabla u_{z^*}| dy &= \sum_k \int_{B_{2^{-k}r} \setminus B_{2^{-k+1}r}} |y|^{1-d} |\nabla u_{z^*}| dy \\
&\lesssim \sum_k \frac{|2^{-k}r|}{|B_{2^{-k}r} \setminus B_{2^{-k+1}r}|} \int_{B_{2^{-k}r} \setminus B_{2^{-k+1}r}} |\nabla u_{z^*}| dy \\
&\lesssim \sum_k |2^{-k}r| \left(\int_{B_{2^{-k}r} \setminus B_{2^{-k+1}r}} |\nabla u_{z^*}|^2 d\mu_{z^*} \right)^{1/2} \\
&\lesssim \sum_k |2^{-k}r| (2^{-k}r)^{\gamma-1} \|\mathbf{f}(\cdot; z^*)\|_{L^q(\mathbb{T}^d; w_{z^*})} \\
&\lesssim r^\gamma \|\mathbf{f}(\cdot; z^*)\|_{L^q(\mathbb{T}^d; w_{z^*})}.
\end{aligned} \tag{4.21}$$

Above, we use the fact that w_{z^*} is essentially constant on the annulus $B_{2^{-k}r} \setminus B_{2^{-k+1}r}$. Repeating a similar argument for the second term in (4.20) we get

$$\sum_{j=1} (2^j r)^{1-|\alpha|} \int_{A_j} |\nabla u_{z^*}| w_{z^*} dy \lesssim r^{\gamma-|\alpha|} \|\mathbf{f}(\cdot; z^*)\|_{L^q(\mathbb{T}^d; w_{z^*})}. \tag{4.22}$$

Assembling (4.20)–(4.22) finally gives

$$I_1 \lesssim r^{\gamma-|\alpha|} \|\mathbf{f}(\cdot; z^*)\|_{L^q(\mathbb{T}^d; w_{z^*})}. \tag{4.23}$$

Repeating a similar argument for I_2 we are led to estimate, for $j = j_0, j_0 - 1, \dots$, terms

$$\int_{A_j} |\nabla v^{(\alpha)}| d\mu_{z^*}.$$

Since $v^{(\alpha)}$ solves (4.19) we get by testing with $v^{(\alpha)}$, Hölder's inequality, (4.16), and an induction argument that

$$\begin{aligned}
\int_{\mathbb{T}^d} |\nabla v^{(\alpha)}|^2 d\mu_{z^*} &\lesssim \int_{\mathbb{T}^d} |\mathcal{E}^{(\alpha)}|^2 |\nabla u_{z^*}|^2 d\mu_{z^*} + \sum_{\substack{1 \leq |\alpha_2|, |\alpha_1| \leq m, \\ \alpha_1 + \alpha_2 = \alpha}} \int_{\mathbb{T}^d} |\mathcal{E}^{(\alpha_2)}|^2 |\nabla v^{(\alpha_1)}|^2 d\mu_{z^*} \\
&\quad + \int_{\mathbb{T}^d} |\mathcal{F}^{(\alpha)}|^2 d\mu_{z^*} \\
&\lesssim r^{-2|\alpha|} \sum_{0 \leq |\alpha_1 + \alpha_2| \leq |\alpha|} \|(\partial_x^{\alpha_1} \partial_z^{\alpha_2} \mathbf{f})(\cdot; z^*)\|_{L^q(\mathbb{T}^d; w_{z^*})}^2.
\end{aligned} \tag{4.24}$$

Applying Lemmas 4.2 and 4.3 and (4.16) and (4.24), and again using an induction argument we get

$$\int_{B_\varrho} |\nabla v^{(\alpha)}|^2 d\mu_{z^*} \lesssim \varrho^{2\gamma-2} r^{-2|\alpha|} \sum_{0 \leq |\alpha_1 + \alpha_2| \leq |\alpha|} \|(\partial_x^{\alpha_1} \partial_z^{\alpha_2} \mathbf{f})(\cdot; z^*)\|_{L^q(\mathbb{T}^d; w_{z^*})}^2. \tag{4.25}$$

Using (4.25) we can bound I_2 similarly as the term I_1 and obtain

$$I_2 \lesssim C(|\alpha|)r^{\gamma-|\alpha|} \left(\|\nabla u\|_{L^2(\mathbb{T}^d, w)} + \sum_{0 \leq |\alpha_1 + \alpha_2| \leq |\alpha|} \|(\partial_x^{\alpha_1} \partial_z^{\alpha_2} \mathbf{f})(\cdot; z^*)\|_{L^q(\mathbb{T}^d; w_{z^*})}^2 \right).$$

Finally, for I_3 it suffices to consider the integral over $A = B_{64r}(x^*) \setminus B_r(z^*)$. Outside $B_{64r}(x^*)$, $\Phi_{z^*}(x) = x$ and no prefactors of the form $r^{-|\alpha|}$ appears in the integrals, furthermore there is no singularity of the Greens function in this region. We would also like to point out that inside $B_r(z^*)$ we have $\Phi_{z^*}(x) = x + h$, and $\hat{w}/w = 1$ (see Step 2) and again no $r^{-|\alpha|}$ prefactor appears. In order to estimate the integral over A , we note that for $q > 2d - \varepsilon$, the conjugate $p = \frac{q}{q-1}$ satisfies $p < \frac{d}{d-1}$, provided $\varepsilon < d$, or equivalently, $(1-d)p + d > 0$. Hence Hölder's inequality, (4.16) for $\mathcal{F}^{(\alpha)}$, and the pointwise bounds for $\nabla_y G$ in Lemma 4.5 as in the case I_1 gives

$$\begin{aligned} & \int_A \nabla_y G(x^*, y; z^*) w_{z^*}(y) \mathcal{F}^{(\alpha)}(y) dy \\ & \leq \|\mathcal{F}^{(\alpha)}\|_{L^q(\mathbb{T}^d; w_{z^*})} \left[\int_A |\nabla_y G(x^*, y; z^*)|^p w_{z^*}(y) dy \right]^{1/p} \\ & \lesssim r^{-|\alpha|} \sum_{0 \leq |\alpha_1 + \alpha_2| \leq |\alpha|} \|(\partial_x^{\alpha_1} \partial_z^{\alpha_2} \mathbf{f})(\cdot; z^*)\|_{L^q(\mathbb{T}^d; w_{z^*})} \left[\int_A |\nabla_y G(x^*, y; z^*)|^p w_{z^*}(y) dy \right]^{1/p}, \end{aligned}$$

where

$$\left[\int_A |\nabla_y G(x^*, y; z^*)|^p w_{z^*}(y) dy \right]^{1/p} \lesssim \left[\int_{B_r(x^*)} |y - x^*|^{(1-d)p} dy \right]^{1/p} \lesssim r^{1-d+d/p}$$

that leads to

$$I_3 \lesssim r^{\gamma-|\alpha|} \sum_{0 \leq |\alpha_1 + \alpha_2| \leq |\alpha|} \|(\partial_x^{\alpha_1} \partial_z^{\alpha_2} \mathbf{f})(\cdot; z^*)\|_{L^q(\mathbb{T}^d; w_{z^*})}.$$

Putting all this together, we finally arrive at the estimate

$$|v^{(\alpha)}(x^*)| \lesssim C(|\alpha|)r^{\gamma-|\alpha|} \sum_{0 \leq |\alpha_1 + \alpha_2| \leq |\alpha|} \|(\partial_x^{\alpha_1} \partial_z^{\alpha_2} \mathbf{f})(\cdot; z^*)\|_{L^q(\mathbb{T}^d; w_{z^*})}.$$

This gives the required estimate in the case $d > 2$. For the case $d = 2$, by using the arguments above we get the same estimates as in (4.20)–(4.22) and (4.25) except with an extra $\log(1/r)$ factor. As such, we recover the same type of bound, namely

$$|v^{(\alpha)}(x^*)| \lesssim C(|\alpha|)r^{\hat{\gamma}-|\alpha|} \sum_{0 \leq |\alpha_1 + \alpha_2| \leq |\alpha|} \|(\partial_x^{\alpha_1} \partial_z^{\alpha_2} \mathbf{f})(\cdot; z^*)\|_{L^q(\mathbb{T}^d; w_{z^*})},$$

where now $\hat{\gamma} < \gamma$.

Step 4. Concluding the proof:

Putting together all the steps, we have proved that for $z \in \mathbb{T}^d \setminus B_{4r}(x)$, $r = |x - z|/16$ and $|h| < r/2$, it holds

$$u(x; z) = u(x; z + h) + \sum_{1 \leq |\alpha| \leq m} v_r^{(\alpha)}(x; z) h^\alpha + o(|h|^m).$$

This implies

$$\begin{aligned} |\partial_z^\alpha u(x; z)| &= |v_r^{(\alpha)}(x; z)| \alpha! \\ &\lesssim \sum_{0 \leq |\alpha_1 + \alpha_2| \leq |\alpha|} \|(\partial_x^{\alpha_1} \partial_z^{\alpha_2} \mathbf{f})(\cdot; z)\|_{L^q(\mathbb{T}^d; w_z)} |x - z|^{\gamma - |\alpha|}. \end{aligned} \quad (4.26)$$

To complete the proof, note that if d is even we get immediately from (4.26) that

$$\|u(x; \cdot)\|_{H^{d/2}(\mathbb{T}^d)} \lesssim \sum_{0 \leq |\alpha_1 + \alpha_2| \leq d/2} \left(\int_{\mathbb{T}^d} \|\partial_x^{\alpha_1} \partial_z^{\alpha_2} \mathbf{f}(\cdot; z)\|_{L^q(\mathbb{T}^d; w_z)}^p dz \right)^{1/p}$$

for $p > \frac{d}{\gamma}$, and the implied constant does not depend on x .

In the case that d is odd, we can use the Brezis-Mironescu [3] interpolation theorem to estimate

$$\|u(x, \cdot)\|_{H^{\frac{d}{2}}} \leq C \|u(x, \cdot)\|_{W^{\frac{d-1}{2}, \frac{2d}{d-1}}}^{\frac{1}{2}} \|u(x, \cdot)\|_{W^{\frac{d+1}{2}, \frac{2d}{d+1}}}^{\frac{1}{2}}. \quad (4.27)$$

To bound the terms on the right hand side of the above, we integrate (4.26) w.r.t z to get that

$$\|u(x, \cdot)\|_{W^{\frac{d+1}{2}, \frac{2d}{d+1}}(\mathbb{T}^d)}^{\frac{2d}{d+1}} \lesssim \sum_{0 \leq |\alpha_1 + \alpha_2| \leq \frac{d+1}{2}} \int_{\mathbb{T}^d} \|\partial_x^{\alpha_1} \partial_z^{\alpha_2} \mathbf{f}(\cdot; z)\|_{L^q(\mathbb{T}^d; w_z)}^{\frac{2d}{d+1}} |x - z|^{(\gamma - \frac{d+1}{2}) \frac{2d}{d+1}} dz.$$

Now using Hölder's inequality we get that for $p_+ > \frac{d+1}{2\gamma}$ it holds

$$\|u(x, \cdot)\|_{W^{\frac{d+1}{2}, \frac{2d}{d+1}}(\mathbb{T}^d)}^{p_+ \frac{2d}{d+1}} \lesssim \sum_{0 \leq |\alpha_1 + \alpha_2| \leq \frac{d+1}{2}} \int_{\mathbb{T}^d} \|\partial_x^{\alpha_1} \partial_z^{\alpha_2} \mathbf{f}(\cdot; z)\|_{L^q(\mathbb{T}^d; w_z)}^{p_+ \frac{2d}{d+1}} dz,$$

where the implied constant is independent of x . This bounds the first term on the right hand side of (4.27). Similar calculations give that for $p_- > \frac{d-1}{2\gamma}$ it holds

$$\|u(x, \cdot)\|_{W^{\frac{d-1}{2}, \frac{2d}{d-1}}(\mathbb{T}^d)}^{p_- \frac{2d}{d-1}} \lesssim \sum_{0 \leq |\alpha_1 + \alpha_2| \leq \frac{d-1}{2}} \int_{\mathbb{T}^d} \|\partial_x^{\alpha_1} \partial_z^{\alpha_2} \mathbf{f}(\cdot; z)\|_{L^q(\mathbb{T}^d; w_z)}^{p_- \frac{2d}{d-1}} dz,$$

where again the implied constant is independent of x . This bounds the second term on the right hand side of (4.27). Finally, if

$$\sum_{0 \leq |\alpha_1 + \alpha_2| \leq \frac{d+1}{2}} \int_{\mathbb{T}^d} \|\partial_x^{\alpha_1} \partial_z^{\alpha_2} \mathbf{f}(\cdot; z)\|_{L^q(\mathbb{T}^d; w_z)}^p dz < \infty,$$

for some $p > \frac{d}{\gamma}$, then the right hand side of (4.27) is finite. This concludes the proof of Theorem 4.1.

Appendix A. Muckenhoupt weights

Definition A.1. If $0 < w < \infty$ a.e., we say that $w \in A_1$ (i.e., belongs to the *Muckenhoupt class* A_1) if there exists a constant $c(1, w) \geq 1$ such that

$$\int_B w \, dx \leq c(1, w) \operatorname{ess\,inf}_B w$$

for every ball $B \in \mathbb{R}^d$.

Furthermore, we say that $w \in A_p$, $1 < p < \infty$ if there exists a constant $c(p, w) > 0$ such that

$$\int_B w \, dx \leq c(p, w) \left(\int_B w^{1/(1-p)} \, dx \right)^{1-p}$$

for every ball $B \in \mathbb{R}^d$.

We have that $A_1 \subset A_p$. If $w \in A_p$ and $h \in \mathbb{R}^d$, $a \in \mathbb{R} \setminus \{0\}$, then $w(\cdot + h) \in A_p$ and $w(a \cdot) \in A_p$.

Example A.2. For $p > 1$, We have that $|\cdot|^\delta \in A_p$ if and only if $-d < \delta < d(p-1)$. Also, we have that $|\cdot|^\delta \in A_1$ if and only if $-d < \delta \leq 0$. We refer to [38, Chapter IX, Proposition 3.2, Corollary 4.4] for the well-known proofs.

Lemma A.3. Let $U \subset \mathbb{R}^d$ be an open bounded domain. Let $\beta \in (0, \sqrt{d})$ and let

$$R(x; z) = e^{g(x,z)} |x-z|^{-\beta^2 \phi(x)\phi(z)}, \quad x, z \in \mathbb{R}^d,$$

where $g \in C_b(\mathbb{R}^d \times \mathbb{R}^d)$, and where ϕ is a compactly supported smooth function such that $\chi_{U'} \leq \phi \leq \chi_U$ with $U' := \{x \in U : \operatorname{dist}(x, \partial U) > r_0\}$.

Then $R(\cdot; z) \in A_1$ for any $z \in \mathbb{R}^d$.

Proof. Expand R as follows,

$$R(x; z) = \exp(g(x, z) - \beta^2(\phi(x) - \phi(z))\phi(z) \log |x-z| - \beta^2 \phi^2(z) \log |x-z|).$$

Clearly, by translation invariance and Example A.2, $x \mapsto \exp(-\beta^2 \phi^2(z) \log |x-z|)$ is an element of A_1 for any z . By the Coifman-Rochberg characterization of A_1 -weights [38, Chapter IX, Theorem 3.4], it suffices to prove that

$$x \mapsto \exp(g(x, z) - \beta^2(\phi(x) - \phi(z))\phi(z) \log |x-z|) \tag{A.1}$$

is bounded above and below away from zero. By the mean value theorem, there exists $\theta = \theta_{x,z} \in (0, 1)$, such that

$$\begin{aligned} & |g(x, z) - \beta^2(\phi(x) - \phi(z))\phi(z) \log |x-z|| \\ & \leq \|g\|_\infty + \beta^2 \|\phi\|_\infty |\nabla \phi(\theta x + (1-\theta)z)| |x-z| \log |x-z| \\ & \leq \|g\|_\infty + \begin{cases} \beta^2 \operatorname{Lip}(\phi) |x-z| \log |x-z|, & \text{if } \theta x + (1-\theta)z \in U, \\ 0, & \text{if } \theta x + (1-\theta)z \in \mathbb{R}^d \setminus U. \end{cases} \end{aligned}$$

However, $|x-z| \log |x-z| \chi_U(\theta x + (1-\theta)z)$ is bounded as is U . Hence the factor (A.1) is bounded from above and below away from zero. $\square$

Corollary A.4. *Let $\eta \in C_0^\infty(\mathbb{R}^d)$, $\eta \geq 0$, $\int_{\mathbb{R}^d} \eta(x) dx = 1$. Then the weight*

$$x \mapsto (R(\cdot; z) * \eta)(x) := \int_{\mathbb{R}^d} R(x-y; z) \eta(y) dy$$

belongs to the class A_1 for any z .

Proof. By Lemma A.3, we may apply [21, Lemma 2.1]. $\square$

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