

Protoalgebraic and Equivalential Logics

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<p>In this master's thesis we study a lower part of the so called Leibniz hierarchy in abstract algebraic logic. Abstract algebraic logic concerns itself with taxonomic study of logics. The main classification of logics is the one into the Leibniz hierarchy according to the properties that the Leibniz operator has on the lattice of theories of a given logic. The Leibniz operator is a function that maps a theory of a logic to an indiscernability relation modulo the theory.</p> <p>We study here two of the main classes in the Leibniz hierarchy – protoalgebraic and equivalential logics – and some of their subclasses. We state and prove the most important characterizations for these classes. We also provide new characterizations for the class of finitely protoalgebraic logics – a class that has previously enjoyed only limited attention.</p> <p>We recall first some basic facts from universal algebra and lattice theory that are used in the remainder of the thesis. Then we present the abstract definition of logic we work with and give abstract semantics for any logic via logical matrices. We define logics determined by a class of matrices and show how to uniformly associate with a given logic a class of matrices so that the original logic and the logic determined by the class coincide – thus providing an abstract completeness theorem for any logic.</p> <p>Remainder of the thesis is dedicated to the study of protoalgebraic and equivalential logics. We provide three main families of characterizations for the various classes of logics. The first characterizations are completely syntactic via the existence of sets of formulae satisfying certain properties. The second family of characterizations is via the properties that the Leibniz operator has on the lattice of theories of a given logic. The third and final family of characterizations is via the closure properties of a canonical class of matrices – the class of reduced models – that we associate to any logic.</p>			
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Introduction

This master's thesis deals with a subfield of metalogic called abstract algebraic logic (AAL). AAL concerns itself with a taxonomic study of sentential logics. AAL serves as a general framework to study and compare different logics. As such, the aim of AAL is to bring order to the totality of all logics. Our general notion of logic is based on the Tarskian notion of consequence operator. [16]

In [17] Tarski made precise the connection between classical propositional calculus **CPC** and the class of Boolean algebras. Tarski showed that a formula is a theorem of **CPC** if and only if the formula is valid in the class of all Boolean algebras. He showed this by defining the binary relation of provable equivalence and showing that the quotient of the formula algebra modulo the relation is a Boolean algebra. This method, nowadays known as the Lindenbaum-Tarski method, was then used to show the connections between various other logics and classes of algebras, e.g. between the intuitionistic propositional calculus **IPC** and Heyting algebras.

In AAL, the focus shifts from the study of particular logics and their algebraic semantics to the process of algebraization itself. The shift can be likened to the shift in focus from the study of particular classes of algebras to universal algebra. AAL tries to generalize the Lindenbaum-Tarski method so that the method can be applied to a wider class of logics.

The main tool in AAL is the semantics of logical matrices. A matrix is a pair with an algebra of the same type as the language of the logic and a subset of the universe of the algebra called the set of designated elements. We associate to every logic a canonical class of matrices so that we can classify logics according to how well-behaved the class of matrices is and how strong the connection between the logic and the associated class is. Research in AAL has unearthed some deep theorems that connect a metalogical property of a logical system to an algebraic property of the associated class of matrices.

The main classification of logics in AAL is the one into what is called the Leibniz hierarchy. The name derives from Leibniz's law of identity of indiscernibles and a family of characterizations for the different levels of

the hierarchy via the properties of the Leibniz operator – a function that maps a theory of a logic to an indiscernability relation modulo the theory which is called here the Leibniz congruence determined by the theory. The Leibniz congruences play similar role in this general context as the relation of provable equivalence in the classical Lindenbaum-Tarski construction. The classification has proved to be very robust in a sense that the different levels of the hierarchy can be characterized by a multitude of means.

We study here a lower part of the Leibniz hierarchy – the classes of protoalgebraic and equivalential logics and some of their subclasses. We state and prove the central results about these classes. We also provide new characterizations for the class of finitely protoalgebraic logics that are very much in the spirit of existing characterizations for different levels of the hierarchy. The class of finitely protoalgebraic logics has received only little attention in the literature.

Equivalential logics were introduced by Prucnal and Wronski in a short note [15] and later studied extensively by Czelakowski [6, 7]. Equivalential logics have their Leibniz congruences definable by a possibly infinite set of formulas in two propositional variables that as a whole acts similarly to a biconditional connective in many familiar logics. Protoalgebraic logics were introduced by Blok and Pigozzi in [2]. Protoalgebraic logics also have their Leibniz congruences definable by a possibly infinite set of formulas, but now with using parameters.

The structure of the thesis In the first chapter we recall necessary basic facts from the fields of lattice theory and universal algebra. Here all the proofs are omitted. In the second chapter we introduce our definition of a logic and give abstract semantics via logical matrices for any logic. In the third and final chapter we study the protoalgebraic and equivalential logics. We'll give three distinct characterizations for both classes. The first one is fully syntactic. The second one is via the behavior of the so called Leibniz operator on the lattice of all theories of a given logic. The third and final characterization is via the closure properties of a canonical class of logical matrices that we assign to any logic.

On notation We try to minimize the use of brackets in the text. Hence, given a function $h: A \rightarrow B$ and $a \in A$, we write ha for the image of a under the function. Similarly for $X \subseteq A$ we write hX for the image of X under h , and for $Y \subseteq B$ we write $h^{-1}Y$ for the pre-image of Y under h . Also for binary relations R on A and S on B , we write hR for the set $\{\langle ha, hb \rangle : \langle a, b \rangle \in R\}$ and $h^{-1}S$ for the set $\{\langle a, b \rangle \in A \times A : \langle ha, hb \rangle \in S\}$. At certain points the

use of brackets is still necessary for unambiguous reading.

We use the notation $A \subseteq_{\omega} B$ to denote that A is a finite subset of B .

Chapter 1

Preliminaries

In this chapter we recall some basic facts from lattice theory and universal algebra that are used in what follows. For a more thorough introduction to the notions we refer the reader to [5].

1.1 Posets and Lattices

A partial order on a set A is a binary relation \leq satisfying the following conditions for any $a, b, c \in A$:

- $a \leq a$; (reflexivity)
- if $a \leq b$ and $b \leq a$, then $a = b$; (antisymmetry)
- if $a \leq b$ and $b \leq c$, then $a \leq c$. (transitivity)

We call a pair $\mathbf{A} = \langle A, \leq \rangle$, where A is a set and \leq is a partial order on A , a partially ordered set or a poset.

Given a poset $\mathbf{A} = \langle A, \leq \rangle$ and $X \subseteq A$, an element $a \in A$ is an upper-bound of X if $x \leq a$ for all $x \in X$. Similarly a is a lower-bound of X if $a \leq x$ for all $x \in X$. An element $a \in A$ is the meet of $X \subseteq A$ if a is a lower-bound of X and for all $b \in A$ it holds: if b is a lower-bound of X then $b \leq a$. Similarly an element a is the join of a set X if it is the least of all upper-bounds. The meet and the join of a set are unique if they exist. Given a set $X \subseteq A$ we denote the meet and join of X by $\bigwedge X$ and $\bigvee X$, respectively.

A poset $\mathbf{A} = \langle A, \leq \rangle$ is a lattice, if for each pair of elements $a, b \in A$ there exists both the meet and the join of $\{a, b\}$. It follows that in a lattice each non-empty finite subset has both the meet and the join. A lattice $\langle A, \leq \rangle$ is bounded if there are elements $0, 1 \in A$ such that $0 \leq a \leq 1$ for all $a \in A$.

It is well-known that we can give an equivalent algebraic characterization for lattices: we say that an algebra $\mathbf{A} = \langle A, \wedge, \vee \rangle$ with two binary operations is a lattice if for all $a, b, c \in A$ the following identities hold:

$$\begin{array}{ll} a \wedge a = a & a \vee a = a \\ a \wedge b = b \wedge a & a \vee b = b \vee a \\ a \wedge (b \wedge c) = (a \wedge b) \wedge c & a \vee (b \vee c) = (a \vee b) \vee c \\ a \wedge (a \vee b) = a & a \vee (a \wedge b) = a \end{array}$$

Given a lattice $\mathbf{A} = \langle A, \leq \rangle$ one can define operations \wedge and \vee on A as follows to obtain a lattice $\mathbf{A}^* = \langle A, \wedge, \vee \rangle$:

$$a \wedge b = \bigwedge \{a, b\} \text{ and } a \vee b = \bigvee \{a, b\}.$$

Conversely given a lattice $\mathbf{A} = \langle A, \wedge, \vee \rangle$ one can define a partial order \leq on A as follows to obtain a lattice $\mathbf{A}^+ = \langle A, \leq \rangle$:

$$a \leq b \text{ if and only if } a \wedge b = a.$$

Now it is easy to check that the two constructions are inverses of each other in a sense that $(\mathbf{A}^*)^+ = \mathbf{A}$ and $(\mathbf{A}^+)^* = \mathbf{A}$. We will freely take advantage of this twofold nature of lattices: sometimes it is natural to view them as ordered structures, sometimes as algebraic structures.

A lattice $\mathbf{A} = \langle A, \leq \rangle$ is complete if all subsets $X \subseteq A$ have both the meet and the join. Note that a complete lattice is always bounded. It is well-known that if in a poset $\mathbf{A} = \langle A, \leq \rangle$ the meet of every subset $X \subseteq A$ exists, then \mathbf{A} is a complete lattice. The same holds when all joins exist.

1.2 Basic Universal Algebra

An algebraic language is a set τ of function symbols. Each function symbol λ has an arity that is a natural number. We denote the arity of λ by $\tau(\lambda)$. Function symbols of arity 0 are also called constants. An algebra in language τ is a tuple $\mathbf{A} = \langle A; (\lambda^{\mathbf{A}})_{\lambda \in \tau} \rangle$, where A is a set and $\lambda^{\mathbf{A}}$ is a mapping from $A^{\tau(\lambda)}$ to A . Note that $A^0 = \{\emptyset\}$. In what follows, we consider always algebras in some fixed but arbitrary language τ . The only exception here are examples where we give the language explicitly. We denote algebras with boldface uppercase letters and their underlying sets or universes with corresponding plain uppercase letters.

Given an algebra \mathbf{A} a subset $B \subseteq A$ is closed if for all n -ary function symbols λ and all $a_1, \dots, a_n \in B$ we have that

$$\lambda^{\mathbf{A}}(a_1, \dots, a_n) \in B.$$

An algebra \mathbf{B} is a subalgebra of an algebra \mathbf{A} , when $B \subseteq A$ and for all n -ary function symbols λ and all $a_1, \dots, a_n \in B$ we have that

$$\lambda^{\mathbf{B}}(a_1, \dots, a_n) = \lambda^{\mathbf{A}}(a_1, \dots, a_n).$$

We write $\mathbf{B} \leq \mathbf{A}$ when \mathbf{B} is a subalgebra of \mathbf{A} . Note that the universe of a subalgebra of \mathbf{A} is a closed subset of \mathbf{A} and conversely each closed subset of \mathbf{A} is the universe of a unique subalgebra of \mathbf{A} .

A homomorphism from an algebra \mathbf{A} to an algebra \mathbf{B} is a mapping $h: A \rightarrow B$ such that for all n -ary function symbols $\lambda \in \tau$ and all $a_1, \dots, a_n \in A$ we have that

$$h\lambda^{\mathbf{A}}(a_1, \dots, a_n) = \lambda^{\mathbf{B}}(ha_1, \dots, ha_n).$$

We write either $h \in \text{hom}(\mathbf{A}, \mathbf{B})$ or simply $h: \mathbf{A} \rightarrow \mathbf{B}$ when h is a homomorphism from \mathbf{A} to \mathbf{B} . An injective homomorphism is an embedding and a surjective embedding is an isomorphism. When there is an isomorphism $h: \mathbf{A} \rightarrow \mathbf{B}$ we denote this by $h: \mathbf{A} \cong \mathbf{B}$ or simply $\mathbf{A} \cong \mathbf{B}$.

Given two algebras \mathbf{A} and \mathbf{B} , and a homomorphism $h: \mathbf{A} \rightarrow \mathbf{B}$, it is straightforward to show that both hA and $h^{-1}B$ are closed subsets of \mathbf{B} and \mathbf{A} , respectively. We denote corresponding subalgebras simply by $h\mathbf{A}$ and $h^{-1}\mathbf{B}$. Note also that an algebra \mathbf{B} is isomorphic to a subalgebra of \mathbf{A} if and only if there is an embedding $h: \mathbf{B} \rightarrow \mathbf{A}$.

A congruence of an algebra \mathbf{A} is an equivalence relation θ of A that respects the algebraic structure of \mathbf{A} in a sense that it satisfies the following for all n -ary function symbols λ and $a_1, \dots, a_n, b_1, \dots, b_n \in A$,

$$\text{if } \langle a_i, b_i \rangle \in \theta \text{ for all } 1 \leq i \leq n, \text{ then } \langle \lambda^{\mathbf{A}}(a_1, \dots, a_n), \lambda^{\mathbf{A}}(b_1, \dots, b_n) \rangle \in \theta.$$

We denote by $\text{Con}\mathbf{A}$ the set of all congruences of \mathbf{A} . The set $\text{Con}\mathbf{A}$ is closed under arbitrary intersections and hence forms a complete lattice under set-inclusion. We have a useful bottom-up characterization for the join of a family $\{\theta_i: i \in I\}$ of congruences of an algebra \mathbf{A} : $\langle a, b \rangle \in \bigvee_{i \in I} \theta_i$ if and only if there are $c_1, \dots, c_{n-1} \in A$ and $i_1, \dots, i_n \in I$ such that

$$a\theta_{i_1}c_1\theta_{i_2}c_2\theta_{i_3}\dots\theta_{i_{n-1}}c_{n-1}\theta_{i_n}b$$

When $\theta \in \text{Con}\mathbf{A}$, we may consider the quotient algebra \mathbf{A}/θ , where the underlying universe is the set of all equivalence classes of θ and the operations are defined as follows for all n -ary function symbols λ

$$\lambda^{\mathbf{A}/\theta}(a_1/\theta, \dots, a_n/\theta) = \lambda^{\mathbf{A}}(a_1, \dots, a_n)/\theta,$$

where a_i/θ denotes the equivalence class of a_i . It is a simple exercise to check that the operations are well-defined, i.e. they do not depend on the choice of representatives from the classes.

For each $\theta \in \text{Con}\mathbf{A}$, the canonical surjection $\pi_\theta: \mathbf{A} \rightarrow \mathbf{A}/\theta, a \mapsto a/\theta$ is a homomorphism. Conversely for any homomorphism $h: \mathbf{A} \rightarrow \mathbf{B}$, the kernel of h ,

$$\ker(h) = \{\langle a, b \rangle \in A \times A: ha = hb\}$$

is a congruence of \mathbf{A} . The First Isomorphism Theorem of Universal Algebra states that if $h: \mathbf{A} \rightarrow \mathbf{B}$ is a surjective homomorphism, then

$$\mathbf{A}/\ker(h) \cong \mathbf{B}.$$

1.3 Algebraic Constructions

Given a family $\{\mathbf{A}_i: i \in I\}$ of algebras, we define the direct product $\prod_I \mathbf{A}_i$ as follows: the underlying universe is the cartesian product $\prod_I A_i$, i.e. the set of all functions f from I to the disjoint union $\bigsqcup_I A_i$ such that $f(i) \in A_i$ for all $i \in I$. The operations are defined componentwise: for all n -ary function symbols λ and all $f_1, \dots, f_n \in \prod_I A_i$ we put

$$(\lambda \prod_I \mathbf{A}_i(f_1, \dots, f_n))i = \lambda^{\mathbf{A}_i}(f_1i, \dots, f_ni).$$

The projection function $\pi_j: \prod_I \mathbf{A}_i \rightarrow \mathbf{A}_j$ defined by $\pi_j f = fj$ is a surjective homomorphism for all $j \in I$.

Given a family $\{\mathbf{A}_i: i \in I\}$ of algebras, a subdirect product of the family is a subalgebra \mathbf{B} of the direct product $\prod_I \mathbf{A}_i$ such that for all $j \in I$ the restriction of the projection π_j to the set B is surjective. An algebra \mathbf{B} is representable as a subdirect product of a family of algebras if it is isomorphic to some subdirect product of the family. We call any embedding witnessing this a subdirect embedding.

A family \mathcal{F} of subsets of a set I is a filter when the following properties hold:

- $I \in \mathcal{F}$;
- if $X, Y \in \mathcal{F}$, then $X \cap Y \in \mathcal{F}$ for all $X, Y \subseteq I$;
- if $X \in \mathcal{F}$ and $X \subseteq Y$, then $Y \in \mathcal{F}$ for all $X, Y \subseteq I$.

A filter \mathcal{F} is an ultrafilter if moreover the following holds

- either $X \in \mathcal{F}$ or $I \setminus X \in \mathcal{F}$ for all $X \subseteq I$.

A family \mathcal{F} of subsets of a set I has the finite intersection property (f.i.p.) if $\bigcap \mathcal{G} \neq \emptyset$ for all finite subsets $\mathcal{G} \subseteq \mathcal{F}$. When a family $\mathcal{F} \subseteq \mathcal{P}(I)$ has the f.i.p.,

there is an ultrafilter \mathcal{U} of I such that $\mathcal{F} \subseteq \mathcal{U}$, i.e. \mathcal{F} can be extended to an ultrafilter of I .

Given a family $\{\mathbf{A}_i: i \in I\}$ of algebras and a filter \mathcal{F} of I define a binary relation $\sim_{\mathcal{F}}$ on $\prod_I \mathbf{A}_i$ as follows:

$$f \sim_{\mathcal{F}} g \text{ if and only if } \{i \in I: fi = gi\} \in \mathcal{F}.$$

It is straightforward to check that $\sim_{\mathcal{F}}$ is a congruence of the direct product. We call the quotient algebra $\prod_I \mathbf{A}_i / \sim_{\mathcal{F}}$ a filtered product of the family. We denote the filtered product also simply by $\prod_I \mathbf{A}_i / \mathcal{F}$. When \mathcal{F} is an ultrafilter, we call the filtered product an ultraproduct.

1.4 Term algebras

Let \mathbf{X} be any set, whose members we call variables. We define the set $T(\mathbf{X})$ of terms over \mathbf{X} as a set of strings over the alphabet $\tau \cup \mathbf{X}$ defined recursively as follows:

- the one-element string x is a term for all $x \in \mathbf{X}$;
- if λ is an n -ary function symbol and $\varphi_1, \dots, \varphi_n$ are terms, then the string $\lambda\varphi_1 \dots \varphi_n$ is a term.

We can define an algebra $\mathbf{T}(\mathbf{X})$ over $T(\mathbf{X})$ with operations defined as follows for all n -ary function symbols λ and terms $\varphi_1, \dots, \varphi_n$,

$$\lambda^{\mathbf{T}(\mathbf{X})}(\varphi_1, \dots, \varphi_n) = \lambda\varphi_1 \dots \varphi_n.$$

The algebra $\mathbf{T}(\mathbf{X})$ is the term algebra over \mathbf{X} .

Given an algebra \mathbf{A} , we can show that any function $h: \mathbf{X} \rightarrow A$ can be uniquely extended to a homomorphism $\hat{h}: \mathbf{T}(\mathbf{X}) \rightarrow \mathbf{A}$. Moreover, given a homomorphism $h: \mathbf{T}(\mathbf{X}) \rightarrow \mathbf{A}$ and a term φ , the value $h\varphi$ is uniquely determined by the values hx for all variables x that occur in the term φ . We use the notation $\varphi(x_1, \dots, x_n)$ to denote that the variables occurring in the formula φ are among x_1, \dots, x_n . When $\varphi(x_1, \dots, x_n)$ is a formula and $a_1, \dots, a_n \in A$ for some algebra \mathbf{A} , we use the notation $\varphi^{\mathbf{A}}(a_1, \dots, a_n)$ to denote the value of $\varphi(x_1, \dots, x_n)$ under a homomorphism that maps x_i to a_i for all $i \in \{1, \dots, n\}$.

An important property of term algebras is the following: let $f: \mathbf{A} \rightarrow \mathbf{B}$ be a surjective homomorphism and let $h: \mathbf{T}(\mathbf{X}) \rightarrow \mathbf{B}$ be a homomorphism. Then there is a homomorphism $h': \mathbf{T}(\mathbf{X}) \rightarrow \mathbf{A}$ such that $f \circ h' = h$. From this it follows that for any family $\{\mathbf{A}_i: i \in I\}$ of algebras and any filter \mathcal{F} of

I , a mapping $h: T(\mathbf{X}) \rightarrow \prod_I A_i/\mathcal{F}$ is a homomorphism if and only if there is a family of homomorphisms $\{h_i: \mathbf{T}(\mathbf{X}) \rightarrow \mathbf{A}_i: i \in I\}$ such that for all terms

$$h\varphi = \langle h_i\varphi: i \in I \rangle/\mathcal{F}.$$

We'll use this result regularly in what follows.

Chapter 2

Logics and logical matrices

Adopting a more logical terminology over the algebraic one, we call the term algebra over some fixed countably infinite set X of variables the formula algebra and we denote it by \mathbf{Fm} . We call members of the algebra formulas. We denote the underlying set of all formulas by Fm . We call endomorphisms of the formula algebra substitutions. We use the notation $\varphi(x/\psi)$, to denote the value of φ under the substitution that maps x to ψ and all other variables to themselves.

We start by giving an abstract definition of a logic that encompasses a very wide family of (sentential) logics. After that we define the notion of a logical matrix and introduce logics defined by classes of matrices. Then we show how to uniformly associate a class of matrices to a given logic so that the given logic and the logic determined by the class coincide thus providing an abstract completeness theorem for any logic. We end this chapter by having a closer look on finitary logics, i.e. logics satisfying an abstract compactness property.

Definition 2.0.1. A logic is a pair $\mathcal{L} = \langle \mathbf{Fm}, \vdash_{\mathcal{L}} \rangle$, where \mathbf{Fm} is the formula algebra and $\vdash_{\mathcal{L}}$ is a subset of $\mathcal{P}(\text{Fm}) \times \text{Fm}$ satisfying the following three conditions:

- If $\varphi \in \Gamma$, then $\Gamma \vdash_{\mathcal{L}} \varphi$; (reflexivity)
- If $\Delta \vdash_{\mathcal{L}} \varphi$ and $\Gamma \vdash_{\mathcal{L}} \delta$ for all $\delta \in \Delta$, then $\Gamma \vdash_{\mathcal{L}} \varphi$; (cut)
- If $\Gamma \vdash_{\mathcal{L}} \varphi$, then $\sigma\Gamma \vdash_{\mathcal{L}} \sigma\varphi$ for all substitutions σ . (structurality)

A logic \mathcal{L} is finitary if moreover it satisfies the following condition:

- If $\Gamma \vdash_{\mathcal{L}} \varphi$, then there is some finite $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash_{\mathcal{L}} \varphi$.

The relation $\vdash_{\mathcal{L}}$ is called a structural consequence relation. It is easy to see that the first two conditions imply the monotonicity of $\vdash_{\mathcal{L}}$: if $\Delta \subseteq \Gamma$ and $\Delta \vdash_{\mathcal{L}} \varphi$, then $\Gamma \vdash_{\mathcal{L}} \varphi$. We use the notation $\Gamma \vdash_{\mathcal{L}} \Delta$, when $\Gamma \vdash_{\mathcal{L}} \delta$ for all $\delta \in \Delta$.

A subset T of Fm is called an \mathcal{L} -theory if it is closed under consequence, i.e. $\varphi \in T$ whenever $T \vdash_{\mathcal{L}} \varphi$. The set of \mathcal{L} -theories is denoted by $\text{Th}\mathcal{L}$. It is easy to see that $\text{Th}\mathcal{L}$ is closed under arbitrary intersections. Hence $\text{Th}\mathcal{L}$ forms a complete lattice under set-inclusion. Given any set $\Gamma \subseteq \text{Fm}$, there exists the least \mathcal{L} -theory that contains Γ . We'll denote this by $C_{\mathcal{L}}\Gamma$. It is easy to check that $C_{\mathcal{L}}\Gamma = \{\varphi \in \text{Fm} : \Gamma \vdash_{\mathcal{L}} \varphi\}$. We call a formula φ an \mathcal{L} -theorem, if $\vdash_{\mathcal{L}} \varphi$, i.e. if $\emptyset \vdash_{\mathcal{L}} \varphi$.

We can also order logics of the same similarity type by positing $\mathcal{L} \leq \mathcal{L}'$ if $\vdash_{\mathcal{L}} \subseteq \vdash_{\mathcal{L}'}$. When $\mathcal{L} \leq \mathcal{L}'$ we say that \mathcal{L}' is stronger than \mathcal{L} , or that \mathcal{L} is weaker than \mathcal{L}' . It is again easy to verify that the intersection of any family of structural consequence relations is a structural consequence relation. Hence the logics of a given similarity type form a complete lattice under the above ordering. The following lemma provides us a bottom-up characterization for the join of a family of logics. The reader is advised to compare this characterization to the bottom-up characterization for a join of a family of congruences.

Lemma 2.0.2. *Let $\{\mathcal{L}_i : i \in I\}$ be a family of logics, and let $\mathcal{L} = \bigvee_I \mathcal{L}_i$. Then $\Gamma \vdash_{\mathcal{L}} \varphi$ if and only if there are $i_0, \dots, i_n \in I$ and $\Gamma_1, \dots, \Gamma_n \subseteq \text{Fm}$ such that*

$$\Gamma \vdash_{\mathcal{L}_{i_0}} \Gamma_1 \vdash_{\mathcal{L}_{i_1}} \cdots \vdash_{\mathcal{L}_{i_{n-1}}} \Gamma_n \vdash_{\mathcal{L}_{i_n}} \varphi.$$

Proof. Define \mathcal{L}' such that $\Gamma \vdash_{\mathcal{L}'} \varphi$ if and only if there are $i_0, \dots, i_n \in I$ and $\Gamma_1, \dots, \Gamma_n \subseteq \text{Fm}$ such that

$$\Gamma \vdash_{\mathcal{L}_{i_0}} \Gamma_1 \vdash_{\mathcal{L}_{i_1}} \cdots \vdash_{\mathcal{L}_{i_{n-1}}} \Gamma_n \vdash_{\mathcal{L}_{i_n}} \varphi.$$

Now it is straightforward to check that \mathcal{L}' is indeed a logic. Moreover clearly $\mathcal{L}_i \leq \mathcal{L}'$ for all $i \in I$. Let then \mathcal{L}_0 be any logic such that $\mathcal{L}_i \leq \mathcal{L}_0$ for all $i \in I$. We want to show that $\mathcal{L}' \leq \mathcal{L}_0$. So suppose $\Gamma \vdash_{\mathcal{L}'} \varphi$. Then there are some $i_0, \dots, i_n \in I$ and $\Gamma_1, \dots, \Gamma_n \subseteq \text{Fm}$ such that

$$\Gamma \vdash_{\mathcal{L}_{i_0}} \Gamma_1 \vdash_{\mathcal{L}_{i_1}} \cdots \vdash_{\mathcal{L}_{i_{n-1}}} \Gamma_n \vdash_{\mathcal{L}_{i_n}} \varphi.$$

But since $\mathcal{L}_i \subseteq \mathcal{L}_0$ for all $i \in I$, we have that $\Gamma \vdash_{\mathcal{L}_0} \varphi$. Thus $\mathcal{L}' = \bigvee_I \mathcal{L}_i$. \square

2.1 Logical matrices

In this section we introduce the notion of logical matrix. A logical matrix is a pair with an algebra and a subset of the algebra called the designated set of the matrix. Intuitively the designated set corresponds to the values of the algebra that we deem true. We show how any class of matrices determines a logic called the semantical consequence associated with the class. We extend the algebraic notions of homomorphism and subalgebra and the various algebraic constructions introduced in the preliminaries to matrices and investigate how the associated consequence relations relate to these notions and constructions.

Definition 2.1.1. A (logical) matrix is a pair $\langle \mathbf{A}, F \rangle$ where \mathbf{A} is an algebra and $F \subseteq A$.

Definition 2.1.2. Let \mathbf{M} be a class of matrices. Define logic $\mathcal{L}_{\mathbf{M}} = \langle \mathbf{Fm}, \vdash_{\mathbf{M}} \rangle$ as follows:

$$\Gamma \vdash_{\mathbf{M}} \varphi \iff \text{for all } \langle \mathbf{A}, F \rangle \in \mathbf{M} \text{ and for all } h: \mathbf{Fm} \rightarrow \mathbf{A} \\ \text{if } h\Gamma \subseteq F, \text{ then } h\varphi \in F$$

It's easy to verify that $\mathcal{L}_{\mathbf{M}}$ is indeed a logic for any class \mathbf{M} of matrices. When \mathbf{M} is a singleton, we denote $\mathcal{L}_{\{\langle \mathbf{A}, F \rangle\}}$ simply by $\mathcal{L}_{\langle \mathbf{A}, F \rangle}$.

Definition 2.1.3. Let $\langle \mathbf{A}, F \rangle$ and $\langle \mathbf{B}, G \rangle$ be matrices. A mapping $h: A \rightarrow B$ is a matrix homomorphism, if h is a homomorphism from \mathbf{A} to \mathbf{B} and $hF \subseteq G$. The homomorphism h is strong, if $hF = G$, and h is strict if $h^{-1}G = F$.

An injective strict homomorphism is an embedding. A surjective embedding is an isomorphism.

We say that $\langle \mathbf{A}, F \rangle$ is embeddable to $\langle \mathbf{B}, G \rangle$ – denoted $\langle \mathbf{A}, F \rangle \hookrightarrow \langle \mathbf{B}, G \rangle$ – if there is an embedding $h: \langle \mathbf{A}, F \rangle \rightarrow \langle \mathbf{B}, G \rangle$. When $h: \langle \mathbf{A}, F \rangle \rightarrow \langle \mathbf{B}, G \rangle$ is a strict surjective homomorphism, we say that $\langle \mathbf{B}, G \rangle$ is a strict homomorphic image of $\langle \mathbf{A}, F \rangle$ and that $\langle \mathbf{A}, F \rangle$ is a strict homomorphic pre-image of $\langle \mathbf{B}, G \rangle$. We say $\langle \mathbf{A}, F \rangle$ and $\langle \mathbf{B}, G \rangle$ are isomorphic – denoted $\langle \mathbf{A}, F \rangle \cong \langle \mathbf{B}, G \rangle$ – if there is an isomorphism $h: \langle \mathbf{A}, F \rangle \rightarrow \langle \mathbf{B}, G \rangle$. Then h^{-1} is also an isomorphism.

The following lemma shows that all the strict homomorphic images and pre-images of a given matrix determine the same logic. In particular two isomorphic matrices determine the same logic.

Lemma 2.1.4. *Let $\langle \mathbf{A}, F \rangle, \langle \mathbf{B}, G \rangle$ be matrices and let $h: \langle \mathbf{A}, F \rangle \rightarrow \langle \mathbf{B}, G \rangle$ be a strict homomorphism. Then $\mathcal{L}_{\langle \mathbf{B}, G \rangle} \leq \mathcal{L}_{\langle \mathbf{A}, F \rangle}$. Moreover, if h is onto, then $\mathcal{L}_{\langle \mathbf{B}, G \rangle} = \mathcal{L}_{\langle \mathbf{A}, F \rangle}$.*

Proof. Suppose first that $\Gamma \vdash_{\langle \mathbf{B}, G \rangle} \varphi$ and let $f: \mathbf{Fm} \rightarrow \mathbf{A}$ be such that $f\Gamma \subseteq F$. Then $hf\Gamma \subseteq G$ and so $hf\varphi \in G$. Thus $f\varphi \in F$.

Suppose then that h is onto and that $\Gamma \vdash_{\langle \mathbf{A}, F \rangle} \varphi$ and let $g: \mathbf{Fm} \rightarrow \mathbf{B}$ be such that $g\Gamma \subseteq G$. Then there is $f: \mathbf{Fm} \rightarrow \mathbf{A}$ such that $h \circ f = g$. Now $f\Gamma \subseteq F$ and so $f\varphi \in F$. Thus $g\varphi \in G$. \square

We can generalize the notion of a strict homomorphism and the first part of the previous lemma as follows.

Definition 2.1.5. A family $\{h_i: \langle \mathbf{A}, F \rangle \rightarrow \langle \mathbf{A}_i, F_i \rangle: i \in I\}$ of homomorphisms is said to be jointly strict if $\bigcap_I h_i^{-1}F_i = F$.

Lemma 2.1.6. *Let $\{h_i: \langle \mathbf{A}, F \rangle \rightarrow \langle \mathbf{A}_i, F_i \rangle: i \in I\}$ be a family of jointly strict homomorphisms. Then $\bigwedge_I \mathcal{L}_{\langle \mathbf{A}_i, F_i \rangle} \leq \mathcal{L}_{\langle \mathbf{A}, F \rangle}$.*

Proof. Suppose that $\Gamma \vdash_{\langle \mathbf{A}_i, F_i \rangle} \varphi$ for all $i \in I$ and let $g: \mathbf{Fm} \rightarrow \mathbf{A}$ be such that $g\Gamma \subseteq F$. Then $h_i g\Gamma \subseteq F_i$ for all $i \in I$. Hence $h_i g\varphi \in F_i$ for all $i \in I$ and so $g\varphi \in F$. \square

Definition 2.1.7. Let $\langle \mathbf{A}, F \rangle$ and $\langle \mathbf{B}, G \rangle$ be matrices. We say that $\langle \mathbf{B}, G \rangle$ is a submatrix of $\langle \mathbf{A}, F \rangle$ if \mathbf{B} is a subalgebra of \mathbf{A} and $G = B \cap F$. We use the notation $\langle \mathbf{B}, G \rangle \subseteq \langle \mathbf{A}, F \rangle$, when $\langle \mathbf{B}, G \rangle$ is a submatrix of $\langle \mathbf{A}, F \rangle$.

Thus $\langle \mathbf{B}, G \rangle$ is a submatrix of $\langle \mathbf{A}, F \rangle$ iff \mathbf{B} is a subalgebra of \mathbf{A} and the inclusion map is an embedding of matrices. A matrix $\langle \mathbf{A}, F \rangle$ is isomorphic to some subalgebra of $\langle \mathbf{B}, G \rangle$ iff $\langle \mathbf{A}, F \rangle$ is embeddable to $\langle \mathbf{B}, G \rangle$. Given a class \mathbf{M} of matrices, we denote

$$\mathbf{S}(\mathbf{M}) = \{\langle \mathbf{A}, F \rangle: \langle \mathbf{A}, F \rangle \hookrightarrow \langle \mathbf{B}, G \rangle \text{ for some } \langle \mathbf{B}, G \rangle \in \mathbf{M}\}.$$

Lemma 2.1.8. *Let $\langle \mathbf{A}, F \rangle$ be a matrix and let $\langle \mathbf{B}, G \rangle \subseteq \langle \mathbf{A}, F \rangle$. Then $\mathcal{L}_{\langle \mathbf{A}, F \rangle} \leq \mathcal{L}_{\langle \mathbf{B}, G \rangle}$.*

Proof. Suppose $\Gamma \vdash_{\langle \mathbf{A}, F \rangle} \varphi$ and let $h: \mathbf{Fm} \rightarrow \mathbf{B}$ be such that $h\Gamma \subseteq G$. Then $h\Gamma \subseteq F$ and so $h\varphi \in F$. Hence $h\varphi \in G$. \square

Definition 2.1.9. Let $\{\langle \mathbf{A}_i, F_i \rangle: i \in I\}$ be a family of matrices. A direct product $\prod_I \langle \mathbf{A}_i, F_i \rangle$ of the family is the matrix $\langle \prod_I \mathbf{A}_i, \prod_I F_i \rangle$, where $\prod_I \mathbf{A}_i$ is the direct product of the algebras $\{\mathbf{A}_i: i \in I\}$ and $\prod_I F_i$ is the cartesian product of the sets $\{F_i: i \in I\}$.

For a class \mathbf{M} of matrices we denote

$$\mathbf{P}(\mathbf{M}) = \{\langle \mathbf{A}, F \rangle : \langle \mathbf{A}, F \rangle \cong \prod_I \langle \mathbf{A}_i, F_i \rangle \text{ for some } \{\langle \mathbf{A}_i, F_i \rangle : i \in I\} \subseteq \mathbf{M}\}.$$

Lemma 2.1.10. *Let $\{\langle \mathbf{A}_i, F_i \rangle : i \in I\}$ be a family of matrices and let $\langle \mathbf{B}, G \rangle = \prod_I \langle \mathbf{A}_i, F_i \rangle$. Then $\bigwedge_I \mathcal{L}_{\langle \mathbf{A}_i, F_i \rangle} \leq \mathcal{L}_{\langle \mathbf{B}, G \rangle}$.*

Proof. Suppose $\Gamma \vdash_{\langle \mathbf{A}_i, F_i \rangle} \varphi$ for all $i \in I$ and let $h: \mathbf{Fm} \rightarrow \prod_I \mathbf{A}_i$ be such that $h\Gamma \subseteq \prod_I F_i$. Then $(h\gamma)i \subseteq F_i$ for all $i \in I$ and all $\gamma \in \Gamma$ and so $(h\varphi)i \in F_i$ for all $i \in I$. Thus $h\varphi \in \prod_I F_i$. \square

Definition 2.1.11. Let $\{\langle \mathbf{A}_i, F_i \rangle : i \in I\}$ be a family of matrices. A subdirect product of the family is a submatrix $\langle \mathbf{B}, G \rangle$ of the direct product $\prod_I \langle \mathbf{A}_i, F_i \rangle$ such that the restriction of the projection π_i to B is surjective for each $i \in I$. We use the notation $\langle \mathbf{B}, G \rangle \subseteq_{\text{SD}} \prod_I \langle \mathbf{A}_i, F_i \rangle$ when $\langle \mathbf{B}, G \rangle$ is a subdirect product of the family $\{\langle \mathbf{A}_i, F_i \rangle : i \in I\}$.

We say that a matrix $\langle \mathbf{B}, G \rangle$ is representable as a subdirect product of a family $\{\langle \mathbf{A}_i, F_i \rangle : i \in I\}$ when $\langle \mathbf{B}, G \rangle$ is isomorphic to some subdirect product of the family. We use the notation

$$\langle \mathbf{B}, G \rangle \hookrightarrow_{\text{SD}} \prod_I \langle \mathbf{A}_i, F_i \rangle$$

when $\langle \mathbf{B}, G \rangle$ is representable as a subdirect product of a family $\{\langle \mathbf{A}_i, F_i \rangle : i \in I\}$. We call any isomorphism witnessing this a subdirect embedding.

Given any class \mathbf{M} of matrices we denote

$$\mathbf{P}_s(\mathbf{M}) = \{\langle \mathbf{A}, F \rangle : \langle \mathbf{A}, F \rangle \hookrightarrow_{\text{SD}} \prod_I \langle \mathbf{A}_i, F_i \rangle \text{ for some } \{\langle \mathbf{A}_i, F_i \rangle : i \in I\} \subseteq \mathbf{M}\}.$$

The following is an immediate corollary to Lemmas 2.1.8 and 2.1.10.

Corollary 2.1.12. *If $\langle \mathbf{B}, G \rangle$ is representable as a subdirect product of the family $\{\langle \mathbf{A}_i, F_i \rangle : i \in I\}$, then $\bigwedge_I \mathcal{L}_{\langle \mathbf{A}_i, F_i \rangle} \leq \mathcal{L}_{\langle \mathbf{B}, G \rangle}$.*

2.2 Compatible congruences and reduced matrices

Given any matrix $\langle \mathbf{A}, F \rangle$ and $\theta \in \text{Con}\mathbf{A}$, we can consider the quotient of the matrix by the congruence, i.e. the matrix $\langle \mathbf{A}/\theta, F/\theta \rangle$, where \mathbf{A}/θ is the quotient of the algebra \mathbf{A} by θ and $F/\theta = \{a/\theta : a \in F\}$. This makes most sense when the congruence respects also the designated set of the matrix in a sense that it does not identify any element inside the set with an element outside the set. We study here these kinds of congruences and the associated quotient matrices.

Definition 2.2.1. Let \mathbf{A} be an algebra, $F \subseteq A$ and $\theta \in \text{Con}\mathbf{A}$. We say θ is compatible with F if the following holds for all $a, b \in A$,

$$\text{if } a \in F \text{ and } a\theta b, \text{ then } b \in F.$$

Lemma 2.2.2. Let \mathbf{A} be an algebra, $F \subseteq A$ and $\theta \in \text{Con}\mathbf{A}$. Then the following are equivalent:

- (i) θ is compatible with F .
- (ii) $a \in F$ if and only if $a/\theta \in F/\theta$.

Proof. (i) \Rightarrow (ii): Suppose θ is compatible with F . Clearly if $a \in F$, then $a/\theta \in F/\theta$. Suppose then that $a/\theta \in F/\theta$. Then there is $b \in F$ such that $a\theta b$, and so, by compatibility, $a \in F$.

(ii) \Rightarrow (i): Let $a, b \in A$ and suppose $a \in F$ and $a\theta b$. Then $b/\theta = a/\theta \in F/\theta$ and so, by (ii), $b \in F$. \square

Compatible congruences and strict homomorphisms are in a closed connection with each other as the following lemma and its corollary show.

Lemma 2.2.3. Let $\langle \mathbf{A}, F \rangle$ and $\langle \mathbf{B}, G \rangle$ be matrices and let $h: \langle \mathbf{A}, F \rangle \rightarrow \langle \mathbf{B}, G \rangle$ be a homomorphism.

- (i) If h is strict, then $\ker(h)$ is compatible with F .
- (ii) If h is strong and $\ker(h)$ is compatible with F , then h is strict.

Proof. (i): Suppose h is strict. Suppose $a \in F$ and $\langle a, b \rangle \in \ker(h)$. Then $hb = ha \in G$, and so, by assumption, $b \in F$.

(ii): Suppose $\ker(h)$ is compatible with F and $hF = G$. Let $a \in A$ and suppose $ha \in G$. Now, by assumption, there is $b \in F$ such that $hb = ha$. Hence $\langle b, a \rangle \in \ker(h)$ and so, by compatibility, $a \in F$. \square

Corollary 2.2.4. Let $\langle \mathbf{A}, F \rangle$ be a matrix and let $\theta \in \text{Con}\mathbf{A}$. Then θ is compatible with F if and only if the canonical surjection π_θ is a strict homomorphism between $\langle \mathbf{A}, F \rangle$ and $\langle \mathbf{A}/\theta, F/\theta \rangle$.

The following theorem is a matrix analogue to the First Isomorphism Theorem in Universal Algebra.

Theorem 2.2.5. Let $h: \langle \mathbf{A}, F \rangle \rightarrow \langle \mathbf{B}, G \rangle$ be a strict surjective homomorphism. Then $\langle \mathbf{A}/\ker(h), F/\ker(h) \rangle \cong \langle \mathbf{B}, G \rangle$.

Proof. Define a mapping $\hat{h}: A/\ker(h) \rightarrow B$ as follows:

$$\hat{h}(a/\ker(h)) = ha.$$

Now it is straightforward to check that \hat{h} is indeed a matrix isomorphism. \square

Now we will turn our attention to the structure of the set congruences that are compatible with a fixed subset of an algebra. We'll show that there is a greatest congruence compatible with the subset and that the compatible congruences form a principal ideal of the lattice of all congruences. The greatest compatible congruence will have central role in what follows.

Lemma 2.2.6. *Let \mathbf{A} be an algebra, $F \subseteq A$ and $\{\theta_i: i \in I\}$ a family of congruences of \mathbf{A} compatible with F . Then $\bigvee_I \theta_i$ is compatible with F .*

Proof. Let $a, b \in A$ and suppose $a \in F$ and $\langle a, b \rangle \in \bigvee_I \theta_i$. Then there are $i_0, \dots, i_n \in I$ and $c_1, \dots, c_n \in A$ such that

$$a\theta_{i_0}c_1\theta_{i_1}\cdots\theta_{i_{n-1}}c_n\theta_{i_n}b.$$

Now, since each θ_i is compatible with F , we have that $b \in F$. Hence $\bigvee_I \theta_i$ is compatible with F . \square

Corollary 2.2.7. *Let \mathbf{A} be an algebra and let $F \subseteq A$. Then the congruences of \mathbf{A} compatible with F form a principal ideal of the lattice $\text{Con}\mathbf{A}$.*

Proof. It suffices to show that the compatible congruences form a down-set. So suppose $\theta, \theta' \in \text{Con}\mathbf{A}$, $\theta \subseteq \theta'$ and θ' is compatible with F . Now let $a \in F$ and suppose that $\langle a, b \rangle \in \theta$. Then $\langle a, b \rangle \in \theta'$ and so, by compatibility of θ' , we have that $b \in F$. Hence θ is compatible with F . \square

Definition 2.2.8. Given an algebra \mathbf{A} and $F \subseteq A$, the largest congruence of \mathbf{A} compatible with F is called the Leibniz congruence of \mathbf{A} determined by F , and it is denoted by $\Omega_{\mathbf{A}}F$.

As is customary, we drop the subscript when \mathbf{A} is the formula algebra and write $\Omega\Gamma$ instead of $\Omega_{\mathbf{Fm}\Gamma}$. Given an algebra \mathbf{A} we call the function that maps a subset F of A to $\Omega_{\mathbf{A}}F$ the Leibniz operator on \mathbf{A} . We denote the Leibniz operator on \mathbf{A} by $\Omega_{\mathbf{A}}$.

The names 'Leibniz congruence' and 'Leibniz operator' were introduced by Blok and Pigozzi in [3], but the idea of the Leibniz congruence dates back to [13].

Now we can restate the definition of compatible congruence as follows: $\theta \in \text{Con}\mathbf{A}$ is compatible with $F \subseteq A$, if $\theta \subseteq \Omega_{\mathbf{A}}F$. The following lemma gives an important characterization for the Leibniz congruences. It also explains the term Leibniz congruence – the congruence relation is an elementary version of Leibniz's law of identity of indiscernibles.

Lemma 2.2.9. *Let \mathbf{A} be an algebra and let $F \subseteq A$. Then*

$$\langle a, b \rangle \in \Omega_{\mathbf{A}}F \text{ if and only if for all } \varphi(x, \bar{z}) \in \text{Fm} \text{ and all } \bar{c} \in A, \\ \varphi^{\mathbf{A}}(a, \bar{c}) \in F \Leftrightarrow \varphi^{\mathbf{A}}(b, \bar{c}) \in F.$$

Proof. Suppose first that $\langle a, b \rangle \in \Omega_{\mathbf{A}}F$. Let $\varphi(x, \bar{z}) \in \text{Fm}$ and $\bar{c} \in A$. Then, since $\Omega_{\mathbf{A}}F$ is a congruence, $\langle \varphi^{\mathbf{A}}(a, \bar{c}), \varphi^{\mathbf{A}}(b, \bar{c}) \rangle \in \Omega_{\mathbf{A}}F$. Hence, by compatibility, $\varphi^{\mathbf{A}}(a, \bar{c}) \in F$ if and only if $\varphi^{\mathbf{A}}(b, \bar{c}) \in F$.

Define then a relation θ on A as follows: $a\theta b$ if for all $\varphi(x, \bar{z}) \in \text{Fm}$ and all $\bar{c} \in A$ we have

$$\varphi^{\mathbf{A}}(a, \bar{c}) \in F \Leftrightarrow \varphi^{\mathbf{A}}(b, \bar{c}) \in F.$$

Now it is easy to see that θ is congruence compatible with F . Hence $\theta \subseteq \Omega_{\mathbf{A}}F$. Hence, if for all $\varphi(x, \bar{z}) \in \text{Fm}$ and all $\bar{c} \in A$ we have

$$\varphi^{\mathbf{A}}(a, \bar{c}) \in F \Leftrightarrow \varphi^{\mathbf{A}}(b, \bar{c}) \in F,$$

then $\langle a, b \rangle \in \Omega_{\mathbf{A}}F$. □

We can state the above characterization more compactly when \mathbf{A} is the formula algebra.

Corollary 2.2.10. *Let $\Gamma \subseteq \text{Fm}$ and let $\alpha, \beta \in \text{Fm}$. Then*

$$\langle \alpha, \beta \rangle \in \Omega\Gamma \Leftrightarrow \text{for all } \varphi \in \text{Fm} \text{ and all } x \in \mathcal{X}, \\ \varphi(x/\alpha) \in \Gamma \text{ if and only if } \varphi(x/\beta) \in \Gamma$$

The following lemma shows that the Leibniz congruences behave nicely with respect to inverse images by homomorphisms.

Lemma 2.2.11. *Let \mathbf{A}, \mathbf{B} be algebras, $h: \mathbf{A} \rightarrow \mathbf{B}$ a homomorphism and $G \subseteq B$. Then the following hold.*

(i) $h^{-1}\Omega_{\mathbf{B}}G \subseteq \Omega_{\mathbf{A}}h^{-1}G$.

(ii) *If h is onto, then $h^{-1}\Omega_{\mathbf{B}}G = \Omega_{\mathbf{A}}h^{-1}G$.*

Proof. (i): It suffices to prove that $h^{-1}\Omega_{\mathbf{B}}G$ is compatible with $h^{-1}G$. So suppose $a \in h^{-1}G$ and $\langle a, b \rangle \in h^{-1}\Omega_{\mathbf{B}}G$. Then $ha \in G$ and $\langle ha, hb \rangle \in \Omega_{\mathbf{B}}G$. Hence $hb \in G$ and so $b \in h^{-1}G$.

(ii): Suppose h is onto and $\langle a, b \rangle \in \Omega_{\mathbf{A}}h^{-1}G$. Then for every $\varphi(x, \bar{z}) \in \text{Fm}$ and $\bar{c} \in A$, we have

$$\varphi^{\mathbf{A}}(a, \bar{c}) \in h^{-1}G \text{ if and only if } \varphi^{\mathbf{A}}(b, \bar{c}) \in h^{-1}G.$$

Hence for all $\varphi(x, \bar{z}) \in \text{Fm}$ and $\bar{c} \in A$, we have

$$\varphi^{\mathbf{B}}(ha, h\bar{c}) \in G \text{ if and only if } \varphi^{\mathbf{B}}(hb, h\bar{c}) \in G.$$

But, since h is onto, for all $\varphi(x, \bar{z})$ and $\bar{d} \in B$, we have

$$\varphi^{\mathbf{B}}(ha, \bar{d}) \in G \text{ if and only if } \varphi^{\mathbf{B}}(hb, \bar{d}) \in G.$$

Hence $\langle ha, hb \rangle \in \Omega_{\mathbf{B}}G$ and so $\langle a, b \rangle \in h^{-1}\Omega_{\mathbf{B}}G$. \square

Corollary 2.2.12. *Let $\langle \mathbf{A}, F \rangle, \langle \mathbf{B}, G \rangle$ be matrices and let $h: \langle \mathbf{A}, F \rangle \rightarrow \langle \mathbf{B}, G \rangle$ be a strict matrix homomorphism. Then the following hold*

(i) $h^{-1}\Omega_{\mathbf{B}}G \subseteq \Omega_{\mathbf{A}}F$;

(ii) *If h is surjective, then $h^{-1}\Omega_{\mathbf{B}}G = \Omega_{\mathbf{A}}F$.*

Lemma 2.2.9 shows that two elements of a matrix $\langle \mathbf{A}, F \rangle$ that are identified by the Leibniz congruence $\Omega_{\mathbf{A}}F$ cannot be distinguished by properties expressible in our language. We are interested in the matrices where no such redundancy occurs and we call such matrices reduced. The notion of a reduced matrix was introduced by Wójcicki in [18].

Definition 2.2.13. A matrix $\langle \mathbf{A}, F \rangle$ is called reduced if $\Omega_{\mathbf{A}}F = \text{id}_{\mathbf{A}}$. For any matrix $\langle \mathbf{A}, F \rangle$ the reduction of $\langle \mathbf{A}, F \rangle$ is the matrix $\langle \mathbf{A}/\Omega_{\mathbf{A}}F, F/\Omega_{\mathbf{A}}F \rangle$. We denote the reduction of $\langle \mathbf{A}, F \rangle$ also by $\langle \mathbf{A}, F \rangle^*$. When \mathbf{M} is a class of matrices, let $\mathbf{M}^* = \{\langle \mathbf{A}, F \rangle^* : \langle \mathbf{A}, F \rangle \in \mathbf{M}\}$.

By Lemma 2.1.4 and Corollary 2.2.4 any matrix and its reduction give rise to the same logic. Hence for any class \mathbf{M} of matrices we have that $\mathcal{L}_{\mathbf{M}} = \mathcal{L}_{\mathbf{M}^*}$. The following lemma collects some important properties of reductions of matrices and reduced matrices.

Lemma 2.2.14. (i) *The reduction of any matrix is reduced.*

(ii) *If $\langle \mathbf{A}, F \rangle$ is reduced, then $\langle \mathbf{A}, F \rangle \cong \langle \mathbf{A}, F \rangle^*$.*

(iii) *If $h: \langle \mathbf{A}, F \rangle \rightarrow \langle \mathbf{B}, G \rangle$ is strict and surjective homomorphism, then $\langle \mathbf{A}, F \rangle^* \cong \langle \mathbf{B}, G \rangle^*$.*

(iv) *If $h: \langle \mathbf{A}, F \rangle \rightarrow \langle \mathbf{B}, F \rangle$ is strict and surjective and $\langle \mathbf{A}, F \rangle$ is reduced, then $\langle \mathbf{B}, F \rangle$ is reduced and h is an isomorphism.*

(v) *If $h: \langle \mathbf{A}, F \rangle \rightarrow \langle \mathbf{B}, G \rangle$ is an isomorphism and $\langle \mathbf{B}, G \rangle$ is reduced, then $\langle \mathbf{A}, F \rangle$ is reduced.*

(vi) If $\theta \in \text{Con}\mathbf{A}$ is compatible with F and $\langle \mathbf{A}/\theta, F/\theta \rangle$ is reduced, then $\theta = \Omega_{\mathbf{A}}F$.

Proof. (i): Let $\langle \mathbf{A}, F \rangle$ be a matrix and consider $\langle \mathbf{A}', F' \rangle = \langle \mathbf{A}, F \rangle^*$. Now suppose that $\langle a/\Omega_{\mathbf{A}}F, b/\Omega_{\mathbf{A}}F \rangle \in \Omega_{\mathbf{A}'}F'$. Then for any $\varphi(x, \bar{z}) \in \text{Fm}$ and $\bar{c} \in A$ we have that

$$\varphi(a/\Omega_{\mathbf{A}}F, \bar{c}/\Omega_{\mathbf{A}}F) \in F' \text{ if and only if } \varphi(b/\Omega_{\mathbf{A}}F, \bar{c}/\Omega_{\mathbf{A}}F) \in F'.$$

Hence for any $\varphi(x, \bar{z})$ and any $\bar{c} \in A$ we have that

$$\varphi(a, \bar{c}) \in F \text{ if and only if } \varphi(b, \bar{c}) \in F.$$

Thus $\langle a, b \rangle \in \Omega_{\mathbf{A}}F$ and so $a/\Omega_{\mathbf{A}}F = b/\Omega_{\mathbf{A}}F$. Hence $\Omega_{\mathbf{A}'}F' = \text{id}_{\mathbf{A}'}$.

(ii): Follows from the fact that in this situation the canonical surjection $\pi_{\Omega_{\mathbf{A}}F}$ is injective.

(iii): Let $h: \langle \mathbf{A}, F \rangle \rightarrow \langle \mathbf{B}, G \rangle$ be strict and surjective homomorphism. Consider the strict and surjective homomorphism $\pi_{\Omega_{\mathbf{B}}G} \circ h$. Now, by Corollary 2.2.12,

$$\langle a, b \rangle \in \ker(\pi_{\Omega_{\mathbf{B}}G} \circ h) \Leftrightarrow \langle ha, hb \rangle \in \Omega_{\mathbf{B}}G \Leftrightarrow \langle a, b \rangle \in \Omega_{\mathbf{A}}F.$$

Hence, by Lemma 2.2.5, $\langle \mathbf{A}, F \rangle^* = \langle \mathbf{A}/\ker(\pi_{\Omega_{\mathbf{B}}G} \circ h), F/\ker(\pi_{\Omega_{\mathbf{B}}G} \circ h) \rangle \cong \langle \mathbf{B}, G \rangle^*$.

(iv): Let $h: \langle \mathbf{A}, F \rangle \rightarrow \langle \mathbf{B}, G \rangle$ be strict and surjective homomorphism and suppose that $\langle \mathbf{A}, F \rangle$ is reduced. Let $c, d \in B$ and suppose $\langle c, d \rangle \in \Omega_{\mathbf{B}}G$. Now, since h is surjective, there are $a, b \in A$ such that $ha = c$ and $hb = d$. Now, by Lemma 2.2.11, $\langle a, b \rangle \in h^{-1}\Omega_{\mathbf{B}}G = \Omega_{\mathbf{A}}F$. Hence $a = b$ and so $c = d$. Thus $\langle \mathbf{B}, G \rangle$ is reduced.

Moreover we need to show that h is injective. So let $a, b \in A$ and suppose $ha = hb$. Then $\langle ha, hb \rangle \in \Omega_{\mathbf{B}}G$ and so $\langle a, b \rangle \in h^{-1}\Omega_{\mathbf{B}}G = \Omega_{\mathbf{A}}F = \text{id}_{\mathbf{A}}$. Hence $a = b$.

(v): Let $h: \langle \mathbf{A}, F \rangle \rightarrow \langle \mathbf{B}, G \rangle$ be an isomorphism and suppose that $\langle \mathbf{B}, G \rangle$ is reduced. Suppose $\langle a, b \rangle \in \Omega_{\mathbf{A}}F = \Omega_{\mathbf{A}}h^{-1}G$. Then, by Corollary 2.2.12, $\langle ha, hb \rangle \in \Omega_{\mathbf{B}}G$, and so $ha = hb$. Now since h is injective we have that $a = b$ and so $\langle \mathbf{A}, F \rangle$ is reduced.

(vi): Let $\theta \in \text{Con}\mathbf{A}$ be compatible with $F \subseteq A$ and suppose that $\langle \mathbf{A}/\theta, F/\theta \rangle$ is reduced. By definition we have that $\theta \subseteq \Omega_{\mathbf{A}}F$. On the other hand the function

$$h: \langle \mathbf{A}/\theta, F/\theta \rangle \rightarrow \langle \mathbf{A}/\Omega_{\mathbf{A}}F, F/\Omega_{\mathbf{A}}F \rangle$$

defined by $h(a/\theta) = a/\Omega_{\mathbf{A}}F$ is clearly a strict and surjective homomorphism. Hence, by (iv), h is an isomorphism. Now if $\langle a, b \rangle \notin \theta$, then $a/\theta \neq b/\theta$ and so $a/\Omega_{\mathbf{A}}F \neq b/\Omega_{\mathbf{A}}F$, i.e. $\langle a, b \rangle \notin \Omega_{\mathbf{A}}F$. Thus $\theta = \Omega_{\mathbf{A}}F$. \square

2.3 Logical filters and matrix models

Let \mathcal{L} be a logic and let $\langle \mathbf{A}, F \rangle$ be a matrix. We say that $\langle \mathbf{A}, F \rangle$ is a model of \mathcal{L} if $\mathcal{L} \leq \mathcal{L}_{\langle \mathbf{A}, F \rangle}$. We denote the class of all models of \mathcal{L} by $\text{Mod}\mathcal{L}$. We use $\text{Mod}^*\mathcal{L}$ to denote the class of all reduced models of \mathcal{L} , i.e. matrices that are reduced and models of \mathcal{L} . The class $\text{Mod}^*\mathcal{L}$ will be our preferred semantics for all logics. Later we see how to classify logics by the closure properties this class has.

By Lemmas 2.1.8, 2.1.10 and 2.1.4 the class $\text{Mod}\mathcal{L}$ is closed under submatrices, direct products of matrices and strict homomorphic images and preimages. The class $\text{Mod}^*\mathcal{L}$ is closed under strict homomorphic images by lemma 2.2.14, but it does not need to be closed under subalgebras nor direct products.

If $\langle \mathbf{A}, F \rangle$ is a model of \mathcal{L} , we call F an \mathcal{L} -filter. Given an algebra \mathbf{A} we denote the set of all \mathcal{L} -filters on \mathbf{A} by $\text{Fi}_{\mathcal{L}}\mathbf{A}$. It is again easy to check that the intersection of any family of \mathcal{L} -filters on an algebra \mathbf{A} is itself an \mathcal{L} -filter. Hence the \mathcal{L} -filters of an algebra form a complete lattice under set-inclusion. The following lemma shows that the \mathcal{L} -filters of the formula algebra are exactly the \mathcal{L} -theories.

Lemma 2.3.1. *Let \mathcal{L} be a logic and let $T \subseteq \text{Fm}$. Then $T \in \text{Th}\mathcal{L}$ if and only if $T \in \text{Fi}_{\mathcal{L}}\mathbf{Fm}$.*

Proof. Suppose first that $T \in \text{Th}\mathcal{L}$. Suppose $\Gamma \vdash_{\mathcal{L}} \varphi$ and let $\sigma: \mathbf{Fm} \rightarrow \mathbf{Fm}$ such that $\sigma\Gamma \subseteq T$. By structurality we have that $\sigma\Gamma \vdash_{\mathcal{L}} \sigma\varphi$ and so, since $T \in \text{Th}\mathcal{L}$, we have that $\sigma\varphi \in T$.

Suppose then that $T \in \text{Fi}_{\mathcal{L}}\mathbf{Fm}$. Suppose $T \vdash_{\mathcal{L}} \varphi$. Now, since $T \subseteq T$ and the identity map is an endomorphism of \mathbf{Fm} , we have that $\varphi \in T$. Hence $T \in \text{Th}\mathcal{L}$. \square

We call a matrix of the form $\langle \mathbf{Fm}, T \rangle$, where $T \in \text{Th}\mathcal{L}$ a Lindenbaum model of \mathcal{L} . We denote the set of all Lindenbaum models of \mathcal{L} by $\text{LMod}\mathcal{L}$. We denote by $\text{LMod}^*\mathcal{L}$ the set of all isomorphic copies of reductions of the Lindenbaum models of \mathcal{L} .

The following lemma shows that any inverse image of an \mathcal{L} -filter by a homomorphism will be itself an \mathcal{L} -filter and thus provides us a useful tool to construct new models for a logic.

Lemma 2.3.2. *Let \mathbf{A}, \mathbf{B} be algebras, $h: \mathbf{A} \rightarrow \mathbf{B}$ and $G \subseteq B$.*

- (i) *If $G \in \text{Fi}_{\mathcal{L}}\mathbf{B}$, then $h^{-1}G \in \text{Fi}_{\mathcal{L}}\mathbf{A}$.*
- (ii) *If h is onto and $h^{-1}G \in \text{Fi}_{\mathcal{L}}\mathbf{A}$, then $G \in \text{Fi}_{\mathcal{L}}\mathbf{B}$.*

Proof. (i): Suppose $\Gamma \vdash_{\mathcal{L}} \varphi$ and let $g: \mathbf{Fm} \rightarrow \mathbf{A}$ be such that $g\Gamma \subseteq h^{-1}G$. Then $hg\Gamma \subseteq G$ and so $hg\varphi \in G$. Hence $g\varphi \in h^{-1}G$.

(ii): Suppose $\Gamma \vdash_{\mathcal{L}} \varphi$ and let $g: \mathbf{Fm} \rightarrow \mathbf{B}$ be such that $g\Gamma \subseteq G$. Now there is $f: \mathbf{Fm} \rightarrow \mathbf{A}$ such that $h \circ f = g$. Hence $f\Gamma \subseteq h^{-1}G$ and so, by assumption, $f\varphi \in h^{-1}G$. Thus $g\varphi = hf\varphi \in G$. \square

We state one important immediate consequence of the above lemma explicitly below.

Corollary 2.3.3. *Let \mathcal{L} be a logic, let T be an \mathcal{L} -theory and let σ be a substitution. Then $\sigma^{-1}T$ is an \mathcal{L} -theory.*

Finally, we end this section by showing an abstract completeness theorem for all logics. We say that a logic \mathcal{L} is complete with respect to a class \mathbf{M} of matrices if $\mathcal{L} = \mathcal{L}_{\mathbf{M}}$. Then we also say that the class \mathbf{M} is adequate for \mathcal{L} . The following lemma shows that any logic has adequate semantics.

Lemma 2.3.4. *Let \mathcal{L} be a logic. Then \mathcal{L} is complete with respect to any class \mathbf{M} of matrices such that*

$$\text{LMod}\mathcal{L} \subseteq \mathbf{M} \subseteq \text{Mod}\mathcal{L}.$$

Proof. Clearly we have that $\mathcal{L} \leq \mathcal{L}_{\text{Mod}\mathcal{L}} \leq \mathcal{L}_{\mathbf{M}} \leq \mathcal{L}_{\text{LMod}\mathcal{L}}$. Suppose then that $\Gamma \not\vdash_{\mathcal{L}} \varphi$ and consider the Lindenbaum matrix $\langle \mathbf{Fm}, C_{\mathcal{L}}\Gamma \rangle$. Now $\Gamma \subseteq C_{\mathcal{L}}\Gamma$, but $\varphi \notin C_{\mathcal{L}}\Gamma$. Hence, the identity map witnesses that $\Gamma \not\vdash_{\text{LMod}\mathcal{L}} \varphi$. \square

Corollary 2.3.5. *Let \mathcal{L} be a logic. Then \mathcal{L} is complete with respect to any class \mathbf{M} of matrices such that*

$$\text{LMod}^*\mathcal{L} \subseteq \mathbf{M} \subseteq \text{Mod}^*\mathcal{L}.$$

2.4 Finitary logics

Recall that a logic \mathcal{L} is finitary if the following holds:

if $\Gamma \vdash_{\mathcal{L}} \varphi$, then there is a finite $\Delta \subseteq \Gamma$ such that $\Delta \vdash_{\mathcal{L}} \varphi$.

Given any logic \mathcal{L} we define a finitary companion \mathcal{L}^f of \mathcal{L} as follows:

$\Gamma \vdash_{\mathcal{L}^f} \varphi$ if there is a finite $\Delta \subseteq \Gamma$ such that $\Delta \vdash_{\mathcal{L}} \varphi$.

It is straightforward to check that \mathcal{L}^f is indeed a finitary logic. Note also that, if \mathcal{L} is finitary, then $\mathcal{L} = \mathcal{L}^f$.

Lemma 2.4.1. *Let \mathcal{L} be a logic. Then \mathcal{L}^f is the greatest finitary logic \mathcal{L}' such that $\mathcal{L}' \leq \mathcal{L}$.*

Proof. Clearly $\mathcal{L}^f \leq \mathcal{L}$. Now let \mathcal{L}' be a finitary logic such that $\mathcal{L}' \leq \mathcal{L}$ and suppose $\Gamma \vdash_{\mathcal{L}'} \varphi$. Then for some finite $\Delta \subseteq \Gamma$ we have that $\Delta \vdash_{\mathcal{L}'} \varphi$. Hence we have that $\Delta \vdash_{\mathcal{L}} \varphi$, and so $\Gamma \vdash_{\mathcal{L}^f} \varphi$. \square

Using the characterization given in Lemma 2.0.2, one easily observes the following.

Corollary 2.4.2. *Let $\{\mathcal{L}_i : i \in I\}$ be a family of finitary logics, then $\bigvee_I \mathcal{L}_i$ is finitary.*

Corollary 2.4.3. *The lattice of finitary logics is a sublattice of the lattice of all logics.*

Proof. We know that the join of any family of finitary logics is finitary. On the other hand it is also easy to see that the meet of a finite family of finitary logics is finitary. \square

By Corollary 2.4.2, finitary logics form a complete lattice. The following lemma provides a characterization for the meet of a family of finitary logics in this complete lattice. Let us denote the meet in the complete lattice of finitary logics by \bigwedge^f .

Lemma 2.4.4. *Let $\{\mathcal{L}_i : i \in I\}$ be a family of finitary logics. Then*

$$\bigwedge_{i \in I}^f \mathcal{L}_i = (\bigwedge_{i \in I} \mathcal{L}_i)^f.$$

Proof. First of all, we clearly have that

$$(\bigwedge_{i \in I} \mathcal{L}_i)^f \leq \bigwedge_{i \in I}^f \mathcal{L}_i.$$

Let then \mathcal{L}' be a finitary logic such that $\mathcal{L}' \leq \mathcal{L}_i$ for all $i \in I$. Then $\mathcal{L}' \leq \bigwedge_I \mathcal{L}_i$, and so, by Lemma 2.4.1, we have that $\mathcal{L}' \leq (\bigwedge_I \mathcal{L}_i)^f$. \square

We'll end this section by proving our first bridge theorem. Bridge theorems are theorems that relate a property of a logic to a property of a class of its models. We will show that a logic \mathcal{L} is finitary only if there is a class of models closed under ultraproducts adequate for \mathcal{L} . Given a family $\{\langle \mathbf{A}_i, F_i \rangle : i \in I\}$ of matrices, and a filter \mathcal{F} on the set I , we define the filtered product $\prod_I \langle \mathbf{A}_i, F_i \rangle / \mathcal{F}$ of the family to be the quotient matrix

$$\langle \prod_I \mathbf{A}_i / \mathcal{F}, \prod_I F_i / \mathcal{F} \rangle.$$

Given a class \mathbf{M} of matrices, $\mathbf{P}_f\mathbf{M}$ and $\mathbf{P}_u\mathbf{M}$ denote the classes of matrices isomorphic to filtered products and ultraproducts of members of \mathbf{M} , respectively.

Lemma 2.4.5. *Let \mathcal{L} be a logic. If $\mathbf{M} \subseteq \text{Mod}\mathcal{L}$, then $\mathbf{P}_f\mathbf{M} \subseteq \text{Mod}\mathcal{L}^f$. Moreover, if \mathcal{L} is complete with respect to \mathbf{M} , then \mathcal{L}^f is complete with respect to $\mathbf{P}_u\mathbf{M}$.*

Proof. Let $\{\langle \mathbf{A}_i, F_i \rangle : i \in I\} \subseteq \mathbf{M}$, let \mathcal{F} be a filter on I and consider $\langle \mathbf{A}, F \rangle = \prod_I \langle \mathbf{A}_i, F_i \rangle / \mathcal{F}$. Suppose $\Gamma \vdash_{\mathcal{L}^f} \varphi$ and let $h: \mathbf{Fm} \rightarrow \mathbf{A}$ be such that $h\Gamma \subseteq F$. Let $\{h_i: \mathbf{Fm} \rightarrow \mathbf{A}_i : i \in I\}$ be a family of homomorphism such that $h\varphi = \langle h_i\varphi : i \in I \rangle / \mathcal{F}$. Now, by assumption there is some finite $\Delta \subseteq \Gamma$ such that $\Delta \vdash_{\mathcal{L}} \varphi$. Now $h\Delta \subseteq F$ and so $\{i \in I : h_i\delta \in F_i\} \in \mathcal{F}$ for each $\delta \in \Delta$. Thus, since Δ is finite, $\{i \in I : h_i\Delta \subseteq F_i\} \in \mathcal{F}$ and so, by assumption, $\{i \in I : h_i\varphi \in F_i\} \in \mathcal{F}$. Thus $h\varphi \in F$.

Suppose then that \mathcal{L} is complete with respect to \mathbf{M} and suppose that $\Gamma \not\vdash_{\mathcal{L}^f} \varphi$. Then for all finite $\Delta \subseteq \Gamma$, we have that $\Delta \not\vdash_{\mathcal{L}} \varphi$. Now, since \mathcal{L} is complete with respect to \mathbf{M} , for any $\Delta \subseteq_{\omega} \Gamma$ there is some $\langle \mathbf{A}_\Delta, F_\Delta \rangle \in \mathbf{M}$ and $h_\Delta: \mathbf{Fm} \rightarrow \mathbf{A}_\Delta$ such that $h_\Delta\Delta \subseteq F_\Delta$, but $h_\Delta\varphi \notin F_\Delta$. Now for each $\Delta \subseteq_{\omega} \Gamma$, consider the set $\Delta^* = \{\Delta' \subseteq_{\omega} \Gamma : \Delta \subseteq \Delta'\}$. Now the family $\{\Delta^* : \Delta \subseteq_{\omega} \Gamma\}$ has the finite intersection property and so there is an ultrafilter \mathcal{U} that extends the family. Now consider the ultraproduct

$$\langle \mathbf{A}, F \rangle = \prod_{\Delta \subseteq_{\omega} \Gamma} \langle \mathbf{A}_\Delta, F_\Delta \rangle / \mathcal{U}$$

and let $h: \mathbf{Fm} \rightarrow \mathbf{A}$ be the homomorphism defined by

$$h\psi = \langle h_\Delta\psi : \Delta \subseteq_{\omega} \Gamma \rangle / \mathcal{U} \text{ for all } \psi \in \mathbf{Fm}.$$

Now $h\Gamma \subseteq F$, but $h\varphi \notin F$. Hence $\Gamma \not\vdash_{\mathbf{P}_u\mathbf{M}} \varphi$. \square

Corollary 2.4.6. *Let \mathcal{L} be a logic. Then the following conditions are equivalent:*

- (i) \mathcal{L} is finitary.
- (ii) $\text{Mod}\mathcal{L}$ is closed under filtered products.
- (iii) $\text{Mod}\mathcal{L}$ is closed under ultraproducts.

Proof. (i) \Rightarrow (ii): This follows from the first part of the previous lemma.

(ii) \Rightarrow (iii): This is completely trivial.

(iii) \Rightarrow (i): Suppose $\text{Mod}\mathcal{L}$ is closed under ultraproducts. Then $\mathbf{P}_u\text{Mod}\mathcal{L} \subseteq \text{Mod}\mathcal{L} \subseteq \text{Mod}\mathcal{L}^f$ and so by the second part of the previous lemma \mathcal{L}^f is complete with respect to $\text{Mod}\mathcal{L}$. Hence $\mathcal{L}^f = \mathcal{L}$ and so \mathcal{L} is finitary. \square

Corollary 2.4.7. *Let \mathcal{L} be a logic. Then the following are equivalent:*

- (i) \mathcal{L} is finitary.*
- (ii) \mathcal{L} is complete with respect to a class of matrices that is closed under filtered products.*
- (iii) \mathcal{L} is complete with respect to a class of matrices that is closed under ultraproducts.*

We say that a logic is strongly finite if it is complete with respect to a finite set of finite matrices. As a corollary to above theorem, we obtain that any strongly finite logic is finitary, since any ultraproduct of a finite family of finite matrices is isomorphic to some member of the family.

Chapter 3

Protoalgebraic and Equivalential Logics

In this chapter we investigate the classes of protoalgebraic and equivalential logics and their various subclasses. In the first section we give two distinct syntactic characterizations for protoalgebraic logics and then define equivalential logics as a certain special case of protoalgebraic logics. In the second section we prove that for both protoalgebraic logics and equivalential logics the Leibniz congruences of models of the logics are definable by a set of formulas – with parameters in the case of protoalgebraic logics and without in the case of equivalential logics.

In the third section we provide characterizations for the classes via the behavior of Leibniz operator on the lattice of \mathcal{L} -theories. In the fourth and final section we show how logics can be classified by the closure properties their classes of reduced models have.

We introduce protoalgebraic logics by presenting the original definition given by Blok and Pigozzi. [2]

Definition 3.0.1. A logic \mathcal{L} is protoalgebraic if for every \mathcal{L} -theory T and $\alpha, \beta \in \text{Fm}$ we have

$$\text{if } \langle \alpha, \beta \rangle \in \Omega T, \text{ then } T, \alpha \dashv\vdash_{\mathcal{L}} T, \beta$$

The above definition says that a logic is protoalgebraic if two formulas are interderivable relative to a theory, whenever they are indiscernible relative to the theory.

3.1 Syntactic characterizations

In this section we give two distinct syntactic characterizations for protoalgebraic logics. These syntactic characterizations were first proven by Blok and Pigozzi in [4]. In the end of the section we will define finitely protoalgebraic, equivalential and finitely equivalential logics as protoalgebraic logics satisfying slightly stronger syntactic conditions.

Many logics possess some form of implication connective, either primitive or definable. We show that having an implication – defined by a possibly infinite set of formulas – satisfying two natural and rather weak conditions adequately characterizes protoalgebraic logics.

Definition 3.1.1. A set $\Delta(x, y)$ of formulas in two variables x, y is a proto-implication for a logic \mathcal{L} , if the following two conditions hold

- (i) $\vdash_{\mathcal{L}} \Delta(x, x)$; (reflexivity)
- (ii) $x, \Delta(x, y) \vdash_{\mathcal{L}} y$. (Modus Ponens)

The other syntactic characterization we provide for protoalgebraic logics is a bit more unnatural, but we will see that it has great theoretical importance. Given a set $\Delta(x, y, \bar{z})$ of formulas with main variables x and y and parameters \bar{z} , and formulas φ and ψ , we denote by $\Delta(\langle \varphi, \psi \rangle)$ the set

$$\{\delta(\varphi, \psi, \bar{\chi}) : \delta(x, y, \bar{z}) \in \Delta(x, y, \bar{z}), \bar{\chi} \in \text{Fm}\}.$$

If the set of parameters is empty we drop the angle brackets and simply write $\Delta(\varphi, \psi)$ for the set

$$\{\delta(\varphi, \psi) : \delta(x, y) \in \Delta(x, y)\}.$$

It is good to notice that for any set $\Delta(x, y, \bar{z})$ the set $\Delta(\langle x, y \rangle)$ is closed under any substitution for which $x \mapsto x$ and $y \mapsto y$. Hence, if $\Delta(\langle x, y \rangle) \vdash_{\mathcal{L}} \varphi(x, y, \bar{z})$ where \bar{z} is disjoint from x and y , then $\Delta(\langle x, y \rangle) \vdash_{\mathcal{L}} \varphi(x, y, \bar{\chi})$ for all $\bar{\chi} \in \text{Fm}$.

Definition 3.1.2. We say that a set $\Delta(x, y, \bar{z})$ is a parameterized equivalence for a logic \mathcal{L} if the following three conditions hold:

- (i) $\vdash_{\mathcal{L}} \Delta(\langle x, x \rangle)$; (reflexivity)
- (ii) $x, \Delta(\langle x, y \rangle) \vdash_{\mathcal{L}} y$; (Modus Ponens)
- (iii) $\Delta(\langle x_1, y_1 \rangle), \dots, \Delta(\langle x_n, y_n \rangle) \vdash_{\mathcal{L}} \Delta(\langle \lambda x_1 \dots x_n, \lambda y_1 \dots y_n \rangle)$ for any n -ary function symbol λ . (simple replacement)

We say that a set $\Delta(x, y)$ with no parameters is an equivalence for a logic \mathcal{L} if it satisfies the conditions above. First we want to show that these conditions suffice to capture properties that we would expect from an equivalence.

Lemma 3.1.3. *Let \mathcal{L} be a logic and let $\Delta(x, y, \bar{z})$ be a parameterized equivalence for \mathcal{L} . Then the following hold:*

$$(i) \Delta(\langle x, y \rangle) \vdash_{\mathcal{L}} \Delta(\langle y, x \rangle); \quad (\text{symmetry})$$

$$(ii) \Delta(\langle x, y \rangle), \Delta(\langle y, z \rangle) \vdash_{\mathcal{L}} \Delta(\langle x, z \rangle); \quad (\text{transitivity})$$

$$(iii) \Delta(\langle x, y \rangle) \vdash_{\mathcal{L}} \Delta(\langle \varphi(z/x), \varphi(z/y) \rangle) \text{ for all formulas } \varphi \text{ and variables } z. \quad (\text{replacement})$$

Proof. We will first prove the replacement property. The proof is by induction on the structure of the formula φ . If φ is a variable, the claim is obvious. Suppose then that $\varphi = \lambda\varphi_1 \dots \varphi_n$ for some n -ary function symbol and formulas $\varphi_1, \dots, \varphi_n$. Now, by induction assumption,

$$\Delta(\langle x, y \rangle) \vdash_{\mathcal{L}} \Delta(\langle \varphi_i(z/x), \varphi_i(z/y) \rangle) \text{ for all } i \in \{1, \dots, n\}.$$

On the other hand, by simple replacement,

$$\bigcup_{i=1}^n \Delta(\langle \varphi_i(z/x), \varphi_i(z/y) \rangle) \vdash_{\mathcal{L}} \Delta(\langle \lambda\varphi_1 \dots \varphi_n(z/x), \lambda\varphi_1 \dots \varphi_n(z/y) \rangle).$$

Hence, by cut,

$$\Delta(\langle x, y \rangle) \vdash_{\mathcal{L}} \Delta(\langle \varphi(z/x), \varphi(z/y) \rangle).$$

Next we prove that the symmetry property holds. We want to show that $\Delta(\langle x, y \rangle) \vdash_{\mathcal{L}} \varphi(y, x, \bar{z})$ for all $\varphi(x, y, \bar{z}) \in \Delta(x, y, \bar{z})$. The claim follows from this. Now let v be a variable disjoint from x, y and \bar{z} and consider the formula $\varphi(v, x, \bar{z})$. Now, by replacement,

$$\Delta(\langle x, y \rangle) \vdash_{\mathcal{L}} \Delta(\langle \varphi(x, x, \bar{z}), \varphi(y, x, \bar{z}) \rangle).$$

On the other hand, by reflexivity, $\vdash_{\mathcal{L}} \varphi(x, x, \bar{z})$ and so, by Modus Ponens and cut, we have that

$$\Delta(\langle x, y \rangle) \vdash_{\mathcal{L}} \varphi(y, x, \bar{z}).$$

Finally we prove transitivity. Let $\varphi(x, y, \bar{z}) \in \Delta(x, y, \bar{z})$, let \bar{v} be a sequence of variables disjoint from x, y and \bar{z} . Now, by replacement,

$$\Delta(\langle y, z \rangle) \vdash_{\mathcal{L}} \Delta(\langle \varphi(x, y, \bar{v}), \varphi(x, z, \bar{v}) \rangle).$$

Now $\varphi(x, y, \bar{v}) \in \Delta(\langle x, y \rangle)$. Hence, by monotonicity, Modus Ponens and cut,

$$\Delta(\langle x, y \rangle), \Delta(\langle y, z \rangle) \vdash_{\mathcal{L}} \varphi(x, z, \bar{v})$$

and then the claim follows again by structurality. \square

Finally, before moving forward, we show that we could have just as well exchanged the simple replacement property in our definition by the (full) replacement.

Lemma 3.1.4. *Let \mathcal{L} be a logic and let $\Delta(x, y, \bar{z})$ be a set of formulas that satisfies reflexivity, Modus Ponens and replacement with respect to \mathcal{L} . Then $\Delta(x, y, \bar{z})$ is a parameterized equivalence.*

Proof. We need to show that $\Delta(x, y, \bar{z})$ satisfies simple replacement. We've already seen that replacement together with Modus Ponens imply transitivity. Now let λ be an n -ary function symbol and let $x_1, \dots, x_n, y_1, \dots, y_n$ be variables. Now, by replacement,

$$\Delta(\langle x_1, y_1 \rangle) \vdash_{\mathcal{L}} \Delta(\langle \lambda x_1 \dots x_n, \lambda y_1 x_2 \dots x_n \rangle),$$

$$\Delta(\langle x_i, y_i \rangle) \vdash_{\mathcal{L}} \Delta(\langle \lambda y_1 \dots y_{i-1} x_i \dots x_n, \lambda y_1 \dots y_i x_{i+1} \dots x_n \rangle)$$

for all $1 < i < n$ and

$$\Delta(\langle x_n, y_n \rangle) \vdash_{\mathcal{L}} \Delta(\langle \lambda y_1 \dots y_{n-1} x_n, \lambda y_1 \dots y_n \rangle).$$

Hence, by transitivity and cut,

$$\Delta(\langle x_1, y_1 \rangle), \dots, \Delta(\langle x_n, y_n \rangle) \vdash_{\mathcal{L}} \Delta(\langle \lambda x_1 \dots x_n, \lambda y_1 \dots y_n \rangle).$$

□

Next we show that all parameterized equivalences of a logic are deductively equivalent.

Lemma 3.1.5. *Let \mathcal{L} be a logic and let $\Delta(x, y, \bar{z})$ and $\Delta'(x, y, \bar{v})$ be parameterized equivalences of \mathcal{L} . Then*

$$\Delta(\langle x, y \rangle) \dashv\vdash_{\mathcal{L}} \Delta'(\langle x, y \rangle).$$

Proof. Let $\varphi(x, y, \bar{\chi}) \in \Delta'(\langle x, y \rangle)$. Now, by replacement, we have that

$$\Delta(\langle x, y \rangle) \vdash_{\mathcal{L}} \Delta(\langle \varphi(x, x, \bar{\chi}(y/x)), \varphi(x, y, \bar{\chi}) \rangle).$$

On the other hand

$$\varphi(x, x, \bar{\chi}(y/x)), \Delta(\langle \varphi(x, x, \bar{\chi}(y/x)), \varphi(x, y, \bar{\chi}) \rangle) \vdash_{\mathcal{L}} \varphi(x, y, \bar{\chi}).$$

But, by reflexivity, $\vdash_{\mathcal{L}} \varphi(x, x, \bar{\chi}(y/x))$, and so, by cut,

$$\Delta(\langle x, y \rangle) \vdash_{\mathcal{L}} \varphi(x, y, \bar{\chi}).$$

Hence $\Delta(\langle x, y \rangle) \vdash_{\mathcal{L}} \Delta'(\langle x, y \rangle)$. The other direction is proved similarly. □

Lemma 3.1.6. *Let \mathcal{L} be a logic and let $\Delta(x, y, \bar{z}), \Delta'(x, y, \bar{v}) \subseteq \text{Fm}$. Suppose*

$$\Delta(\langle x, y \rangle) \dashv\vdash_{\mathcal{L}} \Delta'(\langle x, y \rangle).$$

Then $\Delta(x, y, \bar{z})$ is a parameterized equivalence for \mathcal{L} if and only if $\Delta'(x, y, \bar{v})$ is.

Proof. Suppose that $\Delta(x, y, \bar{z})$ is a parameterized equivalence for \mathcal{L} . By assumption, $\Delta(\langle x, x \rangle) \vdash_{\mathcal{L}} \Delta'(x, x, \bar{v})$. Hence $\vdash_{\mathcal{L}} \Delta'(x, x, \bar{v})$ and so, by structurality, we have that $\vdash_{\mathcal{L}} \Delta'(\langle x, x \rangle)$. On the other hand we have that

$$x, \Delta'(\langle x, y \rangle) \vdash_{\mathcal{L}} x, \Delta(\langle x, y \rangle) \vdash_{\mathcal{L}} y$$

and so, by cut, we have that $x, \Delta'(\langle x, y \rangle) \vdash_{\mathcal{L}} y$.

To check that replacement holds let $\varphi \in \text{Fm}$ and let \bar{w} be a sequence of variables disjoint from $\{x, y\} \cup \text{var}(\varphi)$. Now, by assumption

$$\Delta'(\langle x, y \rangle) \vdash_{\mathcal{L}} \Delta(\langle x, y \rangle) \vdash_{\mathcal{L}} \Delta(\langle \varphi(z/x), \varphi(z/y) \rangle) \vdash_{\mathcal{L}} \Delta'(\varphi(z/x), \varphi(z/y), \bar{w}).$$

Hence, by structurality and cut, $\Delta'(\langle x, y \rangle) \vdash_{\mathcal{L}} \Delta'(\langle \varphi(z/x), \varphi(z/y) \rangle)$.

Hence $\Delta'(x, y, \bar{v})$ is a parameterized equivalence for \mathcal{L} . The other implication is proved completely similarly. \square

For any logic \mathcal{L} , we introduce a canonical set of formulas $\Sigma_{\mathcal{L}}$ as follows

$$\Sigma_{\mathcal{L}} = \{\varphi \in \text{Fm} : \vdash_{\mathcal{L}} \varphi(y/x)\}.$$

For any logic $\Sigma_{\mathcal{L}}$ is an \mathcal{L} -theory, since $\Sigma_{\mathcal{L}} = \sigma^{-1}C_{\mathcal{L}}\emptyset$ for a substitution σ that maps y to x and all other variables to themselves. The following lemma collects two important properties of the set $\Sigma_{\mathcal{L}}$.

Lemma 3.1.7. *Let \mathcal{L} be a logic.*

- (i) *If σ is a substitution that satisfies $(\sigma x)(y/x) = (\sigma y)(y/x)$, then $\sigma\Sigma_{\mathcal{L}} \subseteq \Sigma_{\mathcal{L}}$;*
- (ii) *$\langle x, y \rangle \in \Omega\Sigma_{\mathcal{L}}$.*

Proof. (i) Let σ be a substitution satisfying $(\sigma x)(y/x) = (\sigma y)(y/x)$. Suppose $\varphi \in \Sigma_{\mathcal{L}}$. Then $\vdash_{\mathcal{L}} \varphi(y/x)$. Now, by structurality, we have also that $\vdash_{\mathcal{L}} (\sigma\varphi(y/x))(y/x)$. On the other hand, it is easy to show by induction that $(\sigma\varphi(y/x))(y/x) = (\sigma\varphi)(y/x)$. Hence $\vdash_{\mathcal{L}} (\sigma\varphi)(y/x)$ and so $\sigma\varphi \in \Sigma_{\mathcal{L}}$.

- (ii) Let $\varphi \in \text{Fm}$ and let $z \in \text{var}(\varphi)$. Now clearly $\varphi(z/x)(y/x) = \varphi(z/y)(y/x)$ and so, we have that $\varphi(z/x) \in \Sigma_{\mathcal{L}}$ if and only if $\varphi(z/y) \in \Sigma_{\mathcal{L}}$. Hence $\langle x, y \rangle \in \Omega\Sigma_{\mathcal{L}}$. □

For any logic \mathcal{L} the set $\Sigma_{\mathcal{L}}$ is almost a parameterized equivalence as the following lemma shows. We'll see that a logic is protoalgebraic only in the case that $\Sigma_{\mathcal{L}}$ is indeed a parameterized equivalence.

Lemma 3.1.8. *Let \mathcal{L} be a logic. Then $\Sigma_{\mathcal{L}}$ satisfies reflexivity and replacement with respect to \mathcal{L} .*

Proof. The fact that $\Sigma_{\mathcal{L}}$ satisfies reflexivity is immediate from the definition. Now let $\varphi(x, y, \bar{z}) \in \Sigma_{\mathcal{L}}$, let $\psi \in \text{Fm}$ and let $\bar{\chi} \in \text{Fm}$. We want to show that $\varphi(\psi(v/x), \psi(v/y), \bar{\chi}) \in \Sigma_{\mathcal{L}}$. This suffices to show replacement. Consider any substitution σ such that $x \mapsto \psi(v/x)$, $y \mapsto \psi(v/y)$ and $\bar{z} \mapsto \bar{\chi}$. Now it is straightforward to check that $(\sigma x)(y/x) = (\sigma y)(y/x)$. Hence, by the Lemma 3.1.7 above, we have that

$$\varphi(\psi(v/x), \psi(v/y), \bar{\chi}) = \sigma\varphi(x, y, \bar{z}) \in \sigma\Sigma_{\mathcal{L}} \subseteq \Sigma_{\mathcal{L}}.$$

□

Now we can show that the two syntactic properties defined above are equivalent and both adequately characterize protoalgebraic logics.

Theorem 3.1.9. *Let \mathcal{L} be a logic. The following conditions are equivalent:*

- (i) \mathcal{L} is protoalgebraic;
- (ii) There is a proto-implication for \mathcal{L} ;
- (iii) There is a parameterized equivalence for \mathcal{L} .

Proof. (i) \Rightarrow (iii): Suppose \mathcal{L} is protoalgebraic and consider $\Sigma_{\mathcal{L}}$. By the Lemma 3.1.8, $\Sigma_{\mathcal{L}}$ satisfies reflexivity and replacement. Thus we want to show that $\Sigma_{\mathcal{L}}$ satisfies Modus Ponens. By Lemma 3.1.7, we have that $\langle x, y \rangle \in \Omega\Sigma_{\mathcal{L}}$. Hence, by protoalgebraizability, we have that $x, \Sigma_{\mathcal{L}} \vdash_{\mathcal{L}} y$.

(iii) \Rightarrow (ii): Let $\Delta(x, y, \bar{z})$ be a parameterized equivalence for \mathcal{L} and consider the substitution σ that maps x to x and all other variables to y . Now it is clear that $\sigma\Delta(\langle x, y \rangle)$ is an implication set for \mathcal{L} .

(ii) \Rightarrow (i): Let $\Delta(x, y)$ be a proto-implication for \mathcal{L} . Let $T \in \text{Th}\mathcal{L}$ and suppose that $\langle \alpha, \beta \rangle \in \Omega T$. Then we have that $\Delta(\alpha, \alpha) \subseteq T$ if and only if $\Delta(\alpha, \beta) \subseteq T$. But $\vdash_{\mathcal{L}} \Delta(\alpha, \alpha)$ and so $\Delta(\alpha, \alpha) \subseteq T$. Hence $\Delta(\alpha, \beta) \subseteq T$. On the other hand $\alpha, \Delta(\alpha, \beta) \vdash_{\mathcal{L}} \beta$ and so, by monotonicity, we have that $\alpha, T \vdash_{\mathcal{L}} \beta$. Similarly one proves that $\beta, T \vdash_{\mathcal{L}} \alpha$. Hence \mathcal{L} is protoalgebraic. □

The following corollary is immediate from proof of the previous theorem.

Corollary 3.1.10. *A logic \mathcal{L} is protoalgebraic if and only if $\Sigma_{\mathcal{L}}$ is a parameterized equivalence for \mathcal{L} .*

Example 3.1.11. It follows from the previous theorem that the only protoalgebraic logic without theorems is the almost inconsistent logic, i.e. the logic whose only theories are \emptyset and Fm . The empty set serves as a protoimplication for this logic.

The following lemma provides us a useful criterion to determine whether a set of formulas is a parameterized equivalence for a given protoalgebraic logic.

Lemma 3.1.12. *Let \mathcal{L} be a protoalgebraic logic and let $\Delta(x, y, \bar{z}) \subseteq \text{Fm}$. $\Delta(x, y, \bar{z})$ is a parameterized equivalence for \mathcal{L} if and only if $\Delta(x, y, \bar{z}) \subseteq \Sigma_{\mathcal{L}}$ and $\langle x, y \rangle \in \Omega C \Delta(\langle x, y \rangle)$.*

Proof. Suppose first that $\Delta(x, y, \bar{z})$ is a parameterized equivalence for \mathcal{L} . Then, by Lemma 3.1.5,

$$\Delta(\langle x, y \rangle) \dashv\vdash_{\mathcal{L}} \Sigma_{\mathcal{L}}.$$

Hence $\Delta(x, y, \bar{z}) \subseteq \Sigma_{\mathcal{L}}$. Also, because $\langle x, y \rangle \in \Omega \Sigma_{\mathcal{L}}$ and $C_{\mathcal{L}} \Delta(\langle x, y \rangle) = \Sigma_{\mathcal{L}}$, we have that $\langle x, y \rangle \in \Omega C_{\mathcal{L}} \Delta(\langle x, y \rangle)$.

Suppose then that $\Delta(x, y, \bar{z}) \subseteq \Sigma_{\mathcal{L}}$ and $\langle x, y \rangle \in \Omega C_{\mathcal{L}} \Delta(\langle x, y \rangle)$. Then $\Delta(\langle x, y \rangle) \subseteq \Sigma_{\mathcal{L}}$, and so $\Sigma_{\mathcal{L}} \vdash_{\mathcal{L}} \Delta(\langle x, y \rangle)$. On the other we have that $\varphi(y/x) \in C_{\mathcal{L}} \Delta(\langle x, y \rangle)$ if and only if $\varphi \in C_{\mathcal{L}} \Delta(\langle x, y \rangle)$ for all $\varphi \in \Sigma_{\mathcal{L}}$. But $\vdash_{\mathcal{L}} \varphi(y/x)$ for all $\varphi \in \Sigma_{\mathcal{L}}$ and so $\Sigma_{\mathcal{L}} \subseteq C_{\mathcal{L}} \Delta(\langle x, y \rangle)$, that is $\Delta(\langle x, y \rangle) \vdash_{\mathcal{L}} \Sigma_{\mathcal{L}}$. Hence, by Lemma 3.1.6, $\Delta(x, y, \bar{z})$ is a parameterized equivalence for \mathcal{L} . \square

Now we define the other classes of logics we will consider as special cases of protoalgebraic logics.

Definition 3.1.13. Let \mathcal{L} be a protoalgebraic logic.

- (i) \mathcal{L} is equivalential if there is a parameter-free equivalence $\Delta(x, y)$ for \mathcal{L} .
- (ii) \mathcal{L} is finitely protoalgebraic if there is a finite parametrised equivalence $\Delta(x, y, \bar{z})$ for \mathcal{L} .
- (iii) \mathcal{L} is finitely equivalential if there is a finite parameter-free equivalence $\Delta(x, y)$ for \mathcal{L} .
- (iv) \mathcal{L} is unitarily protoalgebraic if there is a formula $\varphi(x, y, \bar{z})$ such that $\{\varphi(x, y, \bar{z})\}$ is a parametrised equivalence for \mathcal{L} .

- (v) \mathcal{L} is unitarily equivalential if there is a formula $\varphi(x, y)$ such that $\{\varphi(x, y)\}$ is a parameter-free equivalence for \mathcal{L} .

When logic \mathcal{L} is unitarily protoalgebraic (equivalential) and the singleton $\{\varphi(x, y, \bar{z})\}$ witnesses this, we call the formula $\varphi(x, y, \bar{z})$ an equivalence formula for \mathcal{L} . Note that even if $\Delta(x, y, \bar{z})$ is finite, the set $\Delta(\langle x, y \rangle)$ is infinite in case the set of parameters is non-empty.

Note that if a logic \mathcal{L} belongs to any of the six classes of logics defined above, then any logic \mathcal{L}' stronger than \mathcal{L} belongs to the same classes.

Remark 3.1.14. We have seen two distinct syntactic characterizations for protoalgebraic logics. Thus we had two options in defining finitely protoalgebraic logics. We hope that the reasons for opting to choose to define finitely protoalgebraic logics as logics having finite parameterized equivalence instead of finite proto-implication come apparent in the next section.

Example 3.3.9 shows that having a finite proto-implication does not imply that a logic has also a finite parameterized equivalence. On the other hand in Example 3.1.15 below we have a logic that is unitarily protoalgebraic but that does not have a finite proto-implication.

It would be natural to think that any logic that is both finitely protoalgebraic and equivalential is also finitely equivalential. Unfortunately this is not the case as the following counterexample shows. In the tradition of giving biblical names to the counterexamples [11], I call the following logic The Doubting Thomas Logic. The third argument in the ternary function symbol here plays the role of Doubting Thomas.

Example 3.1.15 (The Doubting Thomas Logic). Consider the language of a single a ternary function symbol λ . Let

$$\Delta(x, y) = \{\lambda(x, y, \delta) : \delta \in \text{Fm}(x, y)\},$$

where $\text{Fm}(x, y)$ is the set of all formulas obtained from variables x and y . Let **DT** be the least logic satisfying the following conditions:

- $\vdash_{\mathcal{L}} \lambda(x, x, z)$
- $x, \Delta(x, y) \vdash_{\mathcal{L}} y$
- $\Delta(x_1, y_1), \Delta(x_2, y_2), \Delta(x_3, y_3) \vdash_{\mathcal{L}} \lambda(\lambda(x_1, x_2, x_3), \lambda(y_1, y_2, y_3), z)$

Now **DT** is clearly equivalential with $\Delta(x, y)$ as a set of equivalence formulas. It is also unitarily protoalgebraic with $\lambda(x, y, z)$ as an parameterized equivalence formula.

On the other hand **DT** does not have a finite proto-implication, since clearly for any finite set of formulas Γ , we have that $C_{\mathbf{DT}}\Gamma = \Gamma \cup C_{\mathbf{DT}}\emptyset$. Thus **DT** does not have a finite parameter-free equivalence. Hence **DT** is not finitely equivalential.

The following little lemma characterizes the logics that have a finite proto-implication.

Lemma 3.1.16. *A logic \mathcal{L} has a finite proto-implication if and only if \mathcal{L}^f is protoalgebraic.*

Proof. If \mathcal{L} has a finite proto-implication $\Delta(x, y)$, then $\Delta(x, y)$ is a proto-implication for \mathcal{L}^f also, and so \mathcal{L}^f is protoalgebraic.

On the other hand if \mathcal{L}^f is protoalgebraic it has a proto-implication $\Delta(x, y)$. But then by definition of \mathcal{L}^f , there is a finite $\Delta'(x, y) \subseteq \Delta(x, y)$ such that $x, \Delta'(x, y) \vdash_{\mathcal{L}} y$. Hence \mathcal{L} has a finite proto-implication. \square

We end this section by giving a characterization for equivalential logics analogous to the one for protoalgebraic logics in Corollary 3.1.10.

Lemma 3.1.17. *A logic \mathcal{L} is equivalential if and only if $\Sigma_{\mathcal{L}} \cap \text{Fm}(x, y)$ is an equivalence for \mathcal{L} .*

Proof. Suppose first that \mathcal{L} is equivalential. Then there is $\Delta(x, y) \subseteq \Sigma_{\mathcal{L}}$ that is an equivalence for \mathcal{L} . Now, by Lemma 3.1.5, we have that $\Delta(x, y) \vdash_{\mathcal{L}} \Sigma_{\mathcal{L}}$. Hence, by monotonicity, $\Sigma_{\mathcal{L}} \cap \text{Fm}(x, y) \vdash \Sigma_{\mathcal{L}}$. On the other hand, clearly $\Sigma_{\mathcal{L}} \vdash_{\mathcal{L}} \Sigma_{\mathcal{L}} \cap \text{Fm}(x, y)$. Now, since $\Sigma_{\mathcal{L}}$ is a parameterized equivalence for \mathcal{L} , by Lemma 3.1.6, $\Sigma_{\mathcal{L}} \cap \text{Fm}(x, y)$ is an equivalence for \mathcal{L} .

On the other hand if $\Sigma_{\mathcal{L}} \cap \text{Fm}(x, y)$ is an equivalence for \mathcal{L} , then \mathcal{L} is equivalential. \square

3.2 Definability of the Leibniz congruences

Analogously to the definitions given above, given a set $\Delta(x, y, \bar{z})$ of formulas, an algebra \mathbf{A} and $a, b \in A$, we define

$$\Delta^{\mathbf{A}}(\langle a, b \rangle) = \{\varphi^{\mathbf{A}}(a, b, \bar{c}) : \varphi(x, y, \bar{z}) \in \Delta(x, y, \bar{z}), \bar{c} \in A\}.$$

Given an \mathcal{L} -model $\langle \mathbf{A}, F \rangle$ and a set $\Delta(x, y, \bar{z})$, we define a relation $\Delta_{\mathbf{A}}F$ on A as follows:

$$\langle a, b \rangle \in \Delta_{\mathbf{A}}F \Leftrightarrow \Delta^{\mathbf{A}}(\langle a, b \rangle) \subseteq F.$$

We call $\Delta_{\mathbf{A}}F$ the analytical relation on $\langle \mathbf{A}, F \rangle$ determined by $\Delta(x, y, \bar{z})$. The relation is not necessarily even an equivalence, but if it is a congruence of \mathbf{A} compatible with F , then it coincides with the Leibniz congruence.

Lemma 3.2.1. *Let $\langle \mathbf{A}, F \rangle$ be a matrix and let $\Delta(x, y, \bar{z}) \subseteq \text{Fm}$.*

(i) *If $\Delta_{\mathbf{A}}F$ is reflexive, then $\Omega_{\mathbf{A}}F \subseteq \Delta_{\mathbf{A}}F$.*

(ii) *If $\Delta_{\mathbf{A}}F$ is a congruence of \mathbf{A} compatible with F , then $\Omega_{\mathbf{A}}F = \Delta_{\mathbf{A}}F$.*

Proof. (i) Suppose $\Delta_{\mathbf{A}}F$ is reflexive and suppose $\langle a, b \rangle \in \Omega_{\mathbf{A}}F$. Let $\bar{c} \in A$. Then for each $\varphi(x, y, \bar{z}) \in \Delta(x, y, \bar{z})$ we have that

$$\varphi^{\mathbf{A}}(a, a, \bar{c}) \in F \text{ if and only if } \varphi^{\mathbf{A}}(a, b, \bar{c}) \in F.$$

Now, by assumption, $\varphi^{\mathbf{A}}(a, a, \bar{c}) \in F$ for all $\varphi(x, y, \bar{z}) \in \Delta(x, y, \bar{z})$. Hence $\varphi^{\mathbf{A}}(a, b, \bar{c}) \in F$ for all $\varphi(x, y, \bar{z}) \in \Delta(x, y, \bar{z})$, that is $\Delta^{\mathbf{A}}(a, b, \bar{c}) \subseteq F$. Hence $\Delta^{\mathbf{A}}(\langle a, b \rangle) \subseteq F$ and so $\langle a, b \rangle \in \Delta_{\mathbf{A}}F$.

(ii) This is an immediate consequence of (i). \square

We say that $\Delta(x, y, \bar{z})$ defines the Leibniz congruence of a matrix $\langle \mathbf{A}, F \rangle$, when $\Delta_{\mathbf{A}}F = \Omega_{\mathbf{A}}F$. We show that any parameterized equivalence for a logic defines the Leibniz congruences of the models of the logic and that any set that defines the Leibniz congruences of all models of a logic is a parameterized equivalence for the logic.

Theorem 3.2.2. *Let \mathcal{L} be a logic and let $\Delta(x, y, \bar{z}) \subseteq \text{Fm}$. Then the following are equivalent:*

(i) *$\Delta(x, y, \bar{z})$ is a parametrised equivalence for \mathcal{L} .*

(ii) *$\Delta_{\mathbf{A}}F = \Omega_{\mathbf{A}}F$ for every $\langle \mathbf{A}, F \rangle \in \text{Mod}\mathcal{L}$.*

(iii) *$\Delta_{\mathbf{A}}F = \text{id}_{\mathbf{A}}$ for every $\langle \mathbf{A}, F \rangle \in \text{Mod}^*\mathcal{L}$.*

Proof. (i) \Rightarrow (ii): Suppose $\Delta(x, y, \bar{z})$ is a parametrised equivalence for \mathcal{L} and let $\langle \mathbf{A}, F \rangle \in \text{Mod}\mathcal{L}$. Now, by reflexivity condition, we have that $\Delta_{\mathbf{A}}F$ is reflexive. Hence, by previous lemma, $\Omega_{\mathbf{A}}F \subseteq \Delta_{\mathbf{A}}F$.

To prove the other inclusion, we first prove the following claim.

Claim 1. *$\Delta_{\mathbf{A}}F$ is symmetric.*

Proof of the claim. Suppose $\langle a, b \rangle \in \Delta_{\mathbf{A}}F$. Let $\varphi(x, y, \bar{z}) \in \Delta(x, y, \bar{z})$ and let $\bar{c} \in A$. Then, by replacement, we have that $\Delta^{\mathbf{A}}(\langle \varphi(a, a, \bar{c}), \varphi(b, a, \bar{c}) \rangle) \subseteq F$. On the other hand, by reflexivity, $\varphi(a, a, \bar{c}) \in F$. Hence, by Modus Ponens, $\varphi(b, a, \bar{c}) \in F$. Hence $\Delta^{\mathbf{A}}(\langle b, a \rangle) \subseteq F$ and so $\langle b, a \rangle \in \Delta_{\mathbf{A}}F$. \square

Suppose then that $\langle a, b \rangle \in \Delta_{\mathbf{A}}F$. Let $\varphi(x, \bar{z}) \in \text{Fm}$ and $\bar{c} \in A$. Then, by Claim, replacement and Modus Ponens, we have that

$$\varphi^{\mathbf{A}}(a, \bar{c}) \in F \text{ if and only if } \varphi^{\mathbf{A}}(b, \bar{c}) \in F.$$

Hence $\langle a, b \rangle \in \Omega_{\mathbf{A}}F$.

(ii) \Rightarrow (i): We prove that $\Delta(x, y, \bar{z})$ is a parameterized equivalence for \mathcal{L} using the fact that \mathcal{L} is complete with respect to $\text{Mod}\mathcal{L}$.

- Now since $\Delta^{\mathbf{A}}(\langle a, a \rangle) \subseteq F$ for all $\langle \mathbf{A}, F \rangle \in \text{Mod}\mathcal{L}$ and $a \in A$, we have that $\vdash_{\mathcal{L}} \Delta(\langle x, x \rangle)$.
- Let $\langle \mathbf{A}, F \rangle \in \text{Mod}\mathcal{L}$ and let $h: \mathbf{Fm} \rightarrow \mathbf{A}$ be such that

$$h[\{x\} \cup \Delta(\langle x, y \rangle)] \subseteq F.$$

Consider the submatrix $\langle \mathbf{A}_0, F_0 \rangle = \langle h\mathbf{Fm}, h\mathbf{Fm} \cap F \rangle$ of $\langle \mathbf{A}, F \rangle$. Now, by assumption, $\Delta^{\mathbf{A}}(hx, hy, \bar{c}) \subseteq F_0$ for every $\bar{c} \in h\mathbf{Fm}$. Hence $\langle hx, hy \rangle \in \Delta_{\mathbf{A}_0}F_0 = \Omega_{\mathbf{A}_0}F_0$. Now, since $hx \in F_0$, we have that $hy \in F_0$. Hence $hy \in F$.

- Let $\langle \mathbf{A}, F \rangle \in \text{Mod}\mathcal{L}$ and let $h: \mathbf{Fm} \rightarrow \mathbf{A}$ be such that $h\Delta(\langle x, y \rangle) \subseteq F$. Now consider again $\langle \mathbf{A}_0, F_0 \rangle = \langle h\mathbf{Fm}, h\mathbf{Fm} \cap F \rangle$. Now again $\langle hx, hy \rangle \in \Delta_{\mathbf{A}_0}F_0$. Now, since $\Delta_{\mathbf{A}_0}F_0$ is a congruence,

$$\langle \varphi(hx, \bar{c}), \varphi(hy, \bar{c}) \rangle \in \Delta_{\mathbf{A}_0}F_0$$

for any $\varphi(x, \bar{z}) \in \text{Fm}$ and any $\bar{c} \in A_0$. Thus

$$h\Delta(\langle \varphi(x, \bar{z}), \varphi(y, \bar{z}) \rangle) \subseteq F_0 \subseteq F.$$

(ii) \Rightarrow (iii): This is trivial.

(iii) \Rightarrow (ii): Suppose $\Omega_{\mathbf{A}}F = \text{id}_{\mathbf{A}}$ for every $\langle \mathbf{A}, F \rangle \in \text{Mod}^*\mathcal{L}$. Let $\langle \mathbf{A}, F \rangle \in \text{Mod}\mathcal{L}$. Now for $a, b \in A$ we have that

$$\begin{aligned} \langle a, b \rangle \in \Omega_{\mathbf{A}}F &\Leftrightarrow a/\Omega_{\mathbf{A}}F = b/\Omega_{\mathbf{A}}F \\ &\Leftrightarrow \Delta^{\mathbf{A}/\Omega_{\mathbf{A}}F}(\langle a/\Omega_{\mathbf{A}}F, b/\Omega_{\mathbf{A}}F \rangle) \subseteq F/\Omega_{\mathbf{A}}F \\ &\Leftrightarrow \Delta^{\mathbf{A}}(\langle a, b \rangle) \subseteq F \\ &\Leftrightarrow \langle a, b \rangle \in \Delta_{\mathbf{A}}F \end{aligned}$$

□

Corollary 3.2.3. *A logic \mathcal{L} is protoalgebraic (equivalential) if and only if there exists a (parameter-free) set $\Delta(x, y, \bar{z})$ of formulas that defines the Leibniz congruences of its models.*

The following two corollaries are important in applications.

Corollary 3.2.4. *Let \mathcal{L} be a protoalgebraic logic. Then for any $T \in \text{Th}\mathcal{L}$ and for any surjective substitution σ we have that*

$$\text{if } \langle \alpha, \beta \rangle \in \Omega T, \text{ then } \langle \sigma\alpha, \sigma\beta \rangle \in \Omega C_{\mathcal{L}}\sigma T.$$

Proof. Let $T \in \text{Th}\mathcal{L}$, let σ be a surjective substitution and suppose that $\langle \alpha, \beta \rangle \in \Omega T$. Let $\Delta(x, y, \bar{z})$ be a parameterized equivalence for \mathcal{L} . Now $\Delta(\langle \alpha, \beta \rangle) \subseteq T$. Hence, by surjectivity, $\Delta(\langle \sigma\alpha, \sigma\beta \rangle) \subseteq \sigma T$. Thus $\langle \sigma\alpha, \sigma\beta \rangle \in \Omega C_{\mathcal{L}}\sigma T$. \square

Corollary 3.2.5. *Let \mathcal{L} be a protoalgebraic logic. Then for any finite set of formulas X and any substitution σ we have that*

$$\text{if } \langle \alpha, \beta \rangle \in \Omega C_{\mathcal{L}}X, \text{ then } \langle \sigma\alpha, \sigma\beta \rangle \in \Omega C_{\mathcal{L}}\sigma X.$$

Proof. Let X be a finite set of formulas, let σ be any substitution and suppose that $\langle \alpha, \beta \rangle \in \Omega C_{\mathcal{L}}X$. Let $\Delta(x, y, \bar{z})$ be a parameterized equivalence for \mathcal{L} . Now $\Delta(\langle \alpha, \beta \rangle) \subseteq C_{\mathcal{L}}X$. Now let σ' be a surjective substitution that agrees with σ on $\text{var}(\{\alpha, \beta\} \cup X)$. Now, by previous corollary, $\langle \sigma'\alpha, \sigma'\beta \rangle \in \Omega C_{\mathcal{L}}\sigma' C_{\mathcal{L}}X$. Then, by structurality, $\langle \sigma'\alpha, \sigma'\beta \rangle \in \Omega C_{\mathcal{L}}\sigma' X$. But then $\langle \sigma\alpha, \sigma\beta \rangle \in \Omega C_{\mathcal{L}}\sigma X$. \square

For equivalential logics we can prove a stronger version of corollary 3.2.4 that serves as a characterization for equivalential logics inside the class of protoalgebraic logics.

Theorem 3.2.6. *Let \mathcal{L} be a protoalgebraic logic. Then \mathcal{L} is equivalential if and only if for all \mathcal{L} -theories T and all substitutions σ we have that*

$$\text{if } \langle \alpha, \beta \rangle \in \Omega T, \text{ then } \langle \sigma\alpha, \sigma\beta \rangle \in \Omega C_{\mathcal{L}}\sigma T.$$

Proof. Suppose first that \mathcal{L} is equivalential and let $\Delta(x, y)$ be an equivalence for \mathcal{L} . Let T be an \mathcal{L} -theory and let σ be any substitution. Suppose that $\langle \alpha, \beta \rangle \in \Omega T$. Then $\Delta(\alpha, \beta) \subseteq T$ and so $\Delta(\sigma\alpha, \sigma\beta) \subseteq \sigma T$. Hence $\langle \sigma\alpha, \sigma\beta \rangle \in \Omega C_{\mathcal{L}}\sigma T$.

Suppose then that the property above holds for all \mathcal{L} -theories and all substitutions σ . Now we know that $\langle x, y \rangle \in \Omega \Sigma_{\mathcal{L}}$. Let σ be the substitution that maps x to x and all other variables to y . Now, by assumption, $\langle x, y \rangle \in \Omega C_{\mathcal{L}}\sigma \Sigma_{\mathcal{L}}$. Thus, by Lemma 3.1.12, $\sigma \Sigma_{\mathcal{L}}$ is an equivalence for \mathcal{L} . \square

3.3 Characterizations via the Leibniz operator

In this section, we show how to characterise the protoalgebraic and equivalential logics by the properties that the Leibniz operator has on the lattice of theories. In all but one case these characterizations lift up to the lattice of \mathcal{L} -filters of an arbitrary algebra. These characterizations are often very useful in practice when trying to determine the place of a particular logic inside the Leibniz Hierarchy.

The operator approach in metalogical studies was initiated by Blok and Pigozzi in [2]. There they showed that a logic \mathcal{L} is protoalgebraic if and only if the Leibniz operator is monotone on the lattice of \mathcal{L} -theories. In [4], they showed how to lift the result to arbitrary algebras. The characterizations for equivalential and finitely equivalential logics are due to Herrmann [12] and Blok and Pigozzi [4], respectively.

We'll first prove the characterization for protoalgebraic logics. Then we give the characterizations for equivalential, finitely equivalential and unitarily equivalential logics. Finally we prove the characterizations for finitely protoalgebraic and unitarily protoalgebraic logics. This is because the last two results generalize the characterizations for finitely equivalential and unitarily equivalential logics. Hence it is natural to present the characterizations in this order.

We say that $\Omega_{\mathbf{A}}$ is monotone on $\text{Fi}_{\mathcal{L}}\mathbf{A}$, when $F \subseteq G$ only if $\Omega_{\mathbf{A}}F \subseteq \Omega_{\mathbf{A}}G$ for all $F, G \in \text{Fi}_{\mathcal{L}}\mathbf{A}$. We say that $\Omega_{\mathbf{A}}$ is meet-continuous on $\text{Fi}_{\mathcal{L}}\mathbf{A}$, when for every family $\{F_i : i \in I\}$ of \mathcal{L} -filters of \mathbf{A} , we have that

$$\bigcap_I \Omega_{\mathbf{A}}F_i = \Omega_{\mathbf{A}} \bigcap_I F_i.$$

These notions extend naturally to any complete sublattice of $\text{Fi}_{\mathcal{L}}\mathbf{A}$.

Theorem 3.3.1. *Let \mathcal{L} be a logic. Then the following are equivalent:*

- (i) \mathcal{L} is protoalgebraic;
- (ii) Ω is monotone on $\text{Th}\mathcal{L}$;
- (iii) Ω is meet-continuous on $\text{Th}\mathcal{L}$.
- (iv) $\Omega_{\mathbf{A}}$ is monotone on $\text{Fi}_{\mathcal{L}}\mathbf{A}$ for every algebra \mathbf{A} .
- (v) $\Omega_{\mathbf{A}}$ is meet-continuous on $\text{Fi}_{\mathcal{L}}\mathbf{A}$ for every algebra \mathbf{A} .

Proof. (i) \Rightarrow (iv): Suppose \mathcal{L} is protoalgebraic, let \mathbf{A} be an algebra and let $F, G \in \text{Fi}_{\mathcal{L}}\mathbf{A}$ be such that $F \subseteq G$. Now, by Corollary 3.2.3, there is a set $\Delta(x, y, \bar{z}) \subseteq \text{Fm}$ that defines the Leibniz congruences on $\text{Mod}\mathcal{L}$. Now, since clearly $\Delta_{\mathbf{A}}F \subseteq \Delta_{\mathbf{A}}G$, we have that $\Omega_{\mathbf{A}}F \subseteq \Omega_{\mathbf{A}}G$.

(iv) \Rightarrow (v): Let \mathbf{A} be an algebra and let $F_i \in \text{Fi}_{\mathcal{L}}\mathbf{A}, i \in I$. Now $\bigcap_I \Omega_{\mathbf{A}}F_i \subseteq \Omega_{\mathbf{A}} \bigcap_I F_i$, since $\bigcap_I \Omega_{\mathbf{A}}F_i$ is compatible with $\bigcap_I F_i$. On the other hand, by monotonicity, $\Omega_{\mathbf{A}} \bigcap_I F_i \subseteq \Omega_{\mathbf{A}}F_i$ for all $i \in I$, and so, $\Omega_{\mathbf{A}} \bigcap_I F_i \subseteq \bigcap_I \Omega_{\mathbf{A}}F_i$.

(v) \Rightarrow (iii): This is trivial.

(iii) \Rightarrow (ii): Suppose Ω is meet-continuous on $\text{Th}\mathcal{L}$ and let $T_1, T_2 \in \text{Th}\mathcal{L}$ be such that $T_1 \subseteq T_2$. Then $T_1 \cap T_2 = T_1$ and so, by meet-continuity, $\Omega T_1 = \Omega(T_1 \cap T_2) = \Omega T_1 \cap \Omega T_2$. Hence $\Omega T_1 \subseteq \Omega T_2$.

(ii) \Rightarrow (i): It suffices to show that $\Sigma_{\mathcal{L}}$ satisfies Modus Ponens. We know that $\langle x, y \rangle \in \Omega\Sigma_{\mathcal{L}}$. Hence, by monotonicity of Ω , we have that $\langle x, y \rangle \in \Omega C_{\mathcal{L}}(\Sigma_{\mathcal{L}} \cup \{x\})$. Now, by compatibility, $x, \Sigma_{\mathcal{L}} \vdash_{\mathcal{L}} y$. \square

We say that $\Omega_{\mathbf{A}}$ commutes with inverse images by homomorphisms when for any $h: \mathbf{A} \rightarrow \mathbf{B}$ and any $G \in \text{Fi}_{\mathcal{L}}\mathbf{B}$ we have that $\Omega_{\mathbf{A}}h^{-1}G = h^{-1}\Omega_{\mathbf{B}}G$. Similarly we say that Ω commutes with inverse images by substitutions when for any substitution σ and any \mathcal{L} -theory T we have that $\Omega\sigma^{-1}T = \sigma^{-1}\Omega T$.

Theorem 3.3.2. *Let \mathcal{L} be a logic. Then the following are equivalent:*

- (i) \mathcal{L} is equivalential;
- (ii) Ω is monotone on $\text{Th}\mathcal{L}$ and commutes with inverse images by substitutions.
- (iii) $\Omega_{\mathbf{A}}$ is monotone on $\text{Fi}_{\mathcal{L}}\mathbf{A}$ and commutes with inverse images by homomorphisms for any algebra \mathbf{A} .

Proof. (i) \Rightarrow (iii): Suppose \mathcal{L} is equivalential. Then \mathcal{L} is, in particular, protoalgebraic and so $\Omega_{\mathbf{A}}$ is monotone on $\text{Fi}_{\mathcal{L}}\mathbf{A}$ for any algebra \mathbf{A} .

Let then $h: \mathbf{A} \rightarrow \mathbf{B}$ and $G \in \text{Fi}_{\mathcal{L}}\mathbf{B}$. Let $\Delta(x, y)$ be an equivalence for \mathcal{L} . Now

$$\begin{aligned} \langle a, b \rangle \in h^{-1}\Omega_{\mathbf{B}}G &\Leftrightarrow \langle ha, hb \rangle \in \Omega_{\mathbf{B}}G \\ &\Leftrightarrow \Delta(ha, hb) \subseteq G \\ &\Leftrightarrow h\Delta(a, b) \subseteq G \\ &\Leftrightarrow \Delta(a, b) \subseteq h^{-1}G \\ &\Leftrightarrow \langle a, b \rangle \in \Omega_{\mathbf{A}}h^{-1}G \end{aligned}$$

Here the last equivalence follows, since $h^{-1}G \in \text{Fi}_{\mathcal{L}}\mathbf{A}$.

(iii) \Rightarrow (ii): This is trivial.

(ii) \Rightarrow (i): Now, since Ω is monotone on $\text{Th}\mathcal{L}$, we have, by Theorem 3.3.1, that \mathcal{L} is protoalgebraic. Now, by Theorem 3.2.6, it suffices to show that for all \mathcal{L} -theories and all substitutions σ it holds that

$$\text{if } \langle \alpha, \beta \rangle \in \Omega T \text{ then } \langle \sigma\alpha, \sigma\beta \rangle \in \Omega C_{\mathcal{L}}\sigma T.$$

So let T be an \mathcal{L} -theory and let σ be a substitution. Now $T \subseteq \sigma^{-1}C_{\mathcal{L}}\sigma T$ and so, by monotonicity and the assumption,

$$\Omega T \subseteq \Omega\sigma^{-1}C_{\mathcal{L}}\sigma T = \sigma^{-1}\Omega C_{\mathcal{L}}\sigma T,$$

and so $\sigma\Omega T \subseteq \Omega C_{\mathcal{L}}\sigma T$. □

We say that a family $\{F_i : i \in I\}$ of \mathcal{L} -filters of some algebra is directed if for any $i, j \in I$ there is $k \in I$ such that $F_i \cup F_j \subseteq F_k$. This is equivalent to saying that for any finite $J \subseteq I$ there is $k \in I$ such that $\bigcup_{i \in J} F_i \subseteq F_k$. We say that $\Omega_{\mathbf{A}}$ is continuous on $\text{Fi}_{\mathcal{L}}\mathbf{A}$ if for any directed family $\{F_i : i \in I\} \subseteq \text{Fi}_{\mathcal{L}}\mathbf{A}$ such that $\bigcup_I F_i \in \text{Fi}_{\mathcal{L}}\mathbf{A}$, we have that

$$\Omega_{\mathbf{A}} \bigcup_I F_i = \bigcup_I \Omega_{\mathbf{A}} F_i.$$

This notion again extends naturally to any complete sublattice of $\text{Fi}_{\mathcal{L}}\mathbf{A}$.

Theorem 3.3.3. *Let \mathcal{L} be a logic. Then the following are equivalent:*

- (i) \mathcal{L} is finitely equivalential.
- (ii) Ω is continuous on $\text{Th}\mathcal{L}$.
- (iii) $\Omega_{\mathbf{A}}$ is continuous on $\text{Fi}_{\mathcal{L}}\mathbf{A}$, for all algebras \mathbf{A} .

Proof. (i) \Rightarrow (iii): Suppose \mathcal{L} is finitely equivalential and let $\Delta(x, y)$ be a finite equivalence for \mathcal{L} . Let \mathbf{A} be an algebra and let $\langle F_i : i \in I \rangle$ be a directed family of \mathcal{L} -filters of \mathbf{A} such that $\bigcup_I F_i \in \text{Fi}_{\mathcal{L}}\mathbf{A}$. Since \mathcal{L} is in particular protoalgebraic, $\Omega_{\mathbf{A}}$ is monotone on $\text{Fi}_{\mathcal{L}}\mathbf{A}$. By monotonicity, we have $\bigcup_I \Omega_{\mathbf{A}} F_i \subseteq \Omega_{\mathbf{A}} \bigcup_I F_i$. Suppose then that $\langle a, b \rangle \in \Omega_{\mathbf{A}} \bigcup_I F_i$. Then $\Delta^{\mathbf{A}}(a, b) \subseteq \bigcup_I F_i$. Now, since $\Delta(x, y)$ is finite, there is $i \in I$ such that $\Delta^{\mathbf{A}}(a, b) \subseteq F_i$. Hence $\langle a, b \rangle \in \Omega_{\mathbf{A}} F_i$ and so $\langle a, b \rangle \in \bigcup_I \Omega_{\mathbf{A}} F_i$.

(iii) \Rightarrow (ii): This is trivial.

(ii) \Rightarrow (i): First of all, it is easy to see that continuity of Ω implies the monotonicity of Ω . Hence \mathcal{L} is protoalgebraic. Now consider the directed family

$$\{C_{\mathcal{L}}\Delta(x, y, \bar{z}) : \Delta(x, y, \bar{z}) \subseteq_{\omega} \Sigma_{\mathcal{L}}\}.$$

Now, since $\langle x, y \rangle \in \Omega\Sigma_{\mathcal{L}}$, by continuity we have that there is finite $\Delta(x, y, \bar{z}) \subseteq \Sigma_{\mathcal{L}}$ such that $\langle x, y \rangle \in \Omega C_{\mathcal{L}}\Delta(x, y, \bar{z})$. Now let σ be a substitution that maps y to y and all other variables to x . Then, by Corollary 3.2.5, we have that $\langle x, y \rangle \in \Omega C_{\mathcal{L}}\sigma\Delta(x, y, \bar{z})$. But then $\sigma\Delta(x, y, \bar{z})$ is a finite equivalence for \mathcal{L} by Lemma 3.1.12. \square

Let \mathbf{A} be an algebra. We say that $\Omega_{\mathbf{A}}$ commutes with closed unions of $\text{Fi}_{\mathcal{L}}\mathbf{A}$ when for any family $\{F_i : i \in I\} \subseteq \text{Fi}_{\mathcal{L}}\mathbf{A}$ such that $\bigcup_I F_i \in \text{Fi}_{\mathcal{L}}\mathbf{A}$, we have that

$$\bigcup_I \Omega_{\mathbf{A}} F_i = \Omega_{\mathbf{A}} \bigcup_I F_i.$$

This notion again extends naturally to any complete sublattice of $\text{Fi}_{\mathcal{L}}\mathbf{A}$.

Theorem 3.3.4. *Let \mathcal{L} be a logic. Then the following are equivalent:*

- (i) \mathcal{L} is unitarily equivalential.
- (ii) Ω commutes with closed unions of $\text{Th}\mathcal{L}$.
- (iii) $\Omega_{\mathbf{A}}$ commutes with closed unions of $\text{Fi}_{\mathcal{L}}\mathbf{A}$ for all algebras \mathbf{A} .

Proof. (i) \Rightarrow (iii): Suppose \mathcal{L} is unitarily equivalent and let $\varphi(x, y)$ be an equivalence formula for \mathcal{L} . Let \mathbf{A} be an algebra and let $\{F_i : i \in I\}$ be a family of \mathcal{L} -filters of \mathbf{A} such that $\bigcup_I F_i$ is an \mathcal{L} -filter. Again, by monotonicity of $\Omega_{\mathbf{A}}$, we have that $\bigcup_I \Omega_{\mathbf{A}} F_i \subseteq \Omega_{\mathbf{A}} \bigcup_I F_i$. Suppose then that $\langle a, b \rangle \in \Omega_{\mathbf{A}} \bigcup_I F_i$. Then $\varphi(a, b) \in \bigcup_I F_i$ and so $\varphi(a, b) \in F_i$ for some $i \in I$. Hence $\langle a, b \rangle \in \Omega_{\mathbf{A}} F_i$ for some $i \in I$ and so $\langle a, b \rangle \in \bigcup_I \Omega_{\mathbf{A}} F_i$.

(iii) \Rightarrow (ii): This is again obvious.

(ii) \Rightarrow (i): Suppose Ω commutes with closed unions of $\text{Th}\mathcal{L}$. Again this clearly implies monotonicity of Ω . Now consider the family

$$\{C_{\mathcal{L}}\varphi(x, y, \bar{z}) : \varphi(x, y, \bar{z}) \in \Sigma_{\mathcal{L}}\}.$$

Now, since $\langle x, y \rangle \in \Omega\Sigma_{\mathcal{L}}$, by assumption, $\langle x, y \rangle \in \Omega C_{\mathcal{L}}\varphi(x, y, \bar{z})$ for some $\varphi(x, y, \bar{z}) \in \Sigma_{\mathcal{L}}$. Let σ be a substitution that maps y to y and all other variables to x . Then, by Corollary 3.2.5, $\langle x, y \rangle \in \Omega C_{\mathcal{L}}\sigma\varphi(x, y, \bar{z})$. Hence $\sigma\varphi(x, y, \bar{z})$ is an equivalence formula for \mathcal{L} by Lemma 3.1.12. \square

In order to give a characterization for finitely protoalgebraic logics we need to introduce one more notion. Given a logic \mathcal{L} and a set of variables X we say that an \mathcal{L} -theory T is X -invariant if $\sigma T \subseteq T$ for all substitutions σ such that $\sigma x = x$ for all $x \in X$. We say that T is invariant if it is \emptyset -invariant. We denote the set of X -invariant \mathcal{L} -theories by $\text{Th}_{\text{inv}}^X \mathcal{L}$ and the

set of invariant \mathcal{L} -theories by $\text{Th}_{\text{inv}}\mathcal{L}$. It is easy to verify that $\text{Th}_{\text{inv}}^X\mathcal{L}$ is a complete sublattice of $\text{Th}\mathcal{L}$ for all sets of variables X .

First note that to show that a logic is protoalgebraic it is enough to consider only $\{x, y\}$ -invariant theories, since we have the following extension of Theorem 3.3.1.

Theorem 3.3.5. *Let \mathcal{L} be a logic. Then the following are equivalent.*

- (i) \mathcal{L} is protoalgebraic.
- (ii) Ω is monotone on $\text{Th}_{\text{inv}}^{xy}\mathcal{L}$.
- (iii) Ω is meet-continuous on $\text{Th}_{\text{inv}}^{xy}\mathcal{L}$.

Proof. (i) \Rightarrow (iii): This is a special case of 3.3.1.

(iii) \Rightarrow (ii): This is clear.

(ii) \Rightarrow (i): The same argument as in the implication from (ii) to (i) in 3.3.1 works also now. The theories considered there are $\{x, y\}$ -invariant. \square

Now we can give the characterization for finitely protoalgebraic logics.

Theorem 3.3.6. *Let \mathcal{L} be a logic. Then the following are equivalent:*

- (i) \mathcal{L} is finitely protoalgebraic.
- (ii) Ω is continuous on $\text{Th}_{\text{inv}}^{xy}\mathcal{L}$.

Proof. (i) \Rightarrow (ii): Suppose \mathcal{L} is finitely protoalgebraic and let $\Delta(x, y, \bar{z})$ be a finite parameterized equivalence for \mathcal{L} . Let then $\{T_i : i \in I\} \subseteq \text{Th}_{\text{inv}}^{xy}\mathcal{L}$. Now, by monotonicity of Ω , we have that $\bigcup_I \Omega T_i \subseteq \Omega \bigcup_I T_i$.

Suppose then that $\langle \alpha, \beta \rangle \in \Omega \bigcup_I T_i$. Then $\Delta(\langle \alpha, \beta \rangle) \subseteq \bigcup_I T_i$, and so for a sequence \bar{v} of variables disjoint from $\{x, y\} \cup \text{var}(\{\alpha, \beta\})$, we have $\Delta(\alpha, \beta, \bar{v}) \subseteq \bigcup_I T_i$. Now, since $\Delta(x, y, \bar{z})$ is finite, there is some $i \in I$ such that $\Delta(\alpha, \beta, \bar{v}) \subseteq T_i$. Now, since T_i is closed under substitutions that map x to x and y to y , $\Delta(\langle \alpha, \beta \rangle) \subseteq T_i$, and so $\langle \alpha, \beta \rangle \in \Omega T_i$. Thus $\langle \alpha, \beta \rangle \in \bigcup_I \Omega T_i$.

(ii) \Rightarrow (i): Suppose Ω continuous on $\text{Th}_{\text{inv}}^{xy}\mathcal{L}$. Then, by Theorem 3.3.5, \mathcal{L} is protoalgebraic. Now consider the family

$$\{C_{\mathcal{L}}\Delta(\langle x, y \rangle) : \Delta(x, y, \bar{z}) \subseteq_{\omega} \Sigma_{\mathcal{L}}\} \subset \text{Th}_{\text{inv}}^{xy}\mathcal{L}.$$

The family is clearly directed and the union of the family is $\Sigma_{\mathcal{L}}$. Now $\langle x, y \rangle \in \Omega \Sigma_{\mathcal{L}}$ and so, by continuity, there is some $\Delta(x, y, \bar{z}) \subseteq_{\omega} \Sigma_{\mathcal{L}}$, such that $\langle x, y \rangle \in \Omega C_{\mathcal{L}}\Delta(\langle x, y \rangle)$. Now, by Lemma 3.1.12, $\Delta(x, y, \bar{z})$ is a parameterized equivalence for \mathcal{L} . Hence \mathcal{L} is finitely protoalgebraic. \square

Notice that the characterization given by the previous theorem is local in a sense that it does not lift to arbitrary algebras unlike the other characterizations we have encountered. The proof uses the freeness of \mathbf{Fm} in an essential way.

Theorem 3.3.7. *Let \mathcal{L} be a logic. Then the following are equivalent:*

- (i) \mathcal{L} is unitarily protoalgebraic.
- (ii) Ω commutes with closed unions of $\text{Th}_{\text{inv}}^{xy} \mathcal{L}$.

Proof. The proof is similar to what we have encountered already and is thus omitted. \square

As an application of the previous results we can prove that a finitary logic is finitely equivalential if and only if it is both equivalential and finitely protoalgebraic.

Lemma 3.3.8. *Let \mathcal{L} be a finitary logic that is both finitely protoalgebraic and equivalential. Then \mathcal{L} is finitely equivalential.*

Proof. Let $\Delta(x, y)$ be an equivalence for \mathcal{L} and consider the family of finite subsets of $\Delta(x, y)$. Now since \mathcal{L} is finitary, we have that

$$C_{\mathcal{L}}\Delta(x, y) = \bigcup_{\Delta'(x, y) \subseteq_{\omega} \Delta(x, y)} C_{\mathcal{L}}\Delta'(x, y).$$

Now, since the family $\{C_{\mathcal{L}}\Delta'(x, y) : \Delta'(x, y) \subseteq_{\omega} \Delta(x, y)\}$ is clearly a directed family of $\{x, y\}$ -invariant theories, we have, by 3.3.6, that

$$\Omega C_{\mathcal{L}}\Delta(x, y) = \bigcup_{\Delta'(x, y) \subseteq_{\omega} \Delta(x, y)} \Omega C_{\mathcal{L}}\Delta'(x, y).$$

Now, since $\langle x, y \rangle \in \Omega C_{\mathcal{L}}\Delta(x, y)$, we have that $\langle x, y \rangle \in \Omega C_{\mathcal{L}}\Delta'(x, y)$ for some finite $\Delta'(x, y) \subseteq \Delta(x, y)$. Hence, by 3.1.12, $\Delta'(x, y)$ is a finite equivalence for \mathcal{L} . \square

We end this section by giving an example of a logic that is equivalential, but not finitely protoalgebraic and hence not finitely equivalential. We use Theorem 3.3.6 above to prove the negative result. The example is the local modal consequence relation associated with the class of all modal Kripke models.

Example 3.3.9. We'll first recall the basics of Kripke semantics for modal logics. A Kripke model is a triple $M = \langle W, R, V \rangle$, where W is a set of *worlds*, R is a binary relation on W and V is a mapping from the set of propositional variables to subsets of W . Given a Kripke model $M = \langle W, R, V \rangle$ we define when a formula φ in the language $\langle \wedge, \neg, \Box \rangle$ is true in a world w in the model – denoted $M, w \Vdash \varphi$ – recursively as follows:

- $M, w \Vdash x$ if and only if $w \in Vx$;
- $M, w \Vdash \neg\varphi$ if and only if $M, w \not\Vdash \varphi$;
- $M, w \Vdash \varphi \wedge \psi$ if and only if $M, w \Vdash \varphi$ and $M, w \Vdash \psi$;
- $M, w \Vdash \Box\varphi$ if and only if for all $w' \in W$, wRw' implies $M, w' \Vdash \varphi$.

Other Boolean connectives are definable from \wedge and \neg as usual, and so we will use them freely.

Now we are ready to define the logic lK . We define that $\Gamma \vdash_{lK} \varphi$ if and only if for all Kripke models $M = \langle W, R, V \rangle$ and all $w \in W$ we have,

$$\text{if } M, w \Vdash \gamma \text{ for all } \gamma \in \Gamma, \text{ then } M, w \Vdash \varphi.$$

One can easily check that the relation defined is indeed a structural consequence relation.

We show that the logic lK is finitary by a simple ultraproduct construction. Let Γ be a set of formulas and let φ be a single formula. Suppose that for all finite subset $\Gamma' \subseteq \Gamma$ it holds $\Gamma' \not\vdash_{lK} \varphi$. We show that $\Gamma \not\vdash_{lK} \varphi$. Let I be the set of finite subsets of Γ and for all $i \in I$ let $M_i = \langle W_i, R_i, V_i \rangle$ be model and let $w_i \in W_i$ be such that $M_i, w_i \Vdash i$, but $M_i, w_i \not\vdash \varphi$. Now for each $i \in I$ let $i^* = \{j \in I : i \subseteq j\}$. Now the family $\{i^* : i \in I\}$ has the finite intersection property and so there is an ultrafilter \mathcal{U} on I extending the family. Now define a Kripke model $M = \langle W, R, V \rangle$ as follows:

- $W = \prod_I W_i / \mathcal{U}$;
- fRg if and only if $\{i \in I : (fi)R_i(gi)\} \in \mathcal{U}$;
- $f \in Vx$ if and only if $\{i \in I : fi \in V_i x\} \in \mathcal{U}$.

Now it is straightforward to check that for all $f \in W$ and all formulas ψ it holds $M, f \Vdash \psi$ if and only if $\{i \in I : M_i, fi \Vdash \psi\} \in \mathcal{U}$. Now let $w_0 = \langle w_i : i \in I \rangle / \mathcal{U}$. Now, by definition, $M, w_0 \Vdash \Gamma$, but $M, w_0 \not\vdash \varphi$. Hence $\Gamma \not\vdash_{lK} \varphi$.

We define that $\Box^0\varphi = \varphi$ and $\Box^{n+1}\varphi = \Box\Box^n\varphi$. First we claim that $\Delta(x, y) = \{\Box^n(x \leftrightarrow y) : n \in \omega\}$ is an equivalence for lK . Clearly we have

that $\vdash_{lK} \Delta(x, x)$, since $x \leftrightarrow x$ holds in every world. Let then $M = \langle W, R, V \rangle$ be a Kripke model and let $w \in W$ and suppose that

$$M, w \Vdash \Delta(x, y) \text{ and } M, w \Vdash x.$$

Then, in particular $M, w \Vdash x \leftrightarrow y$ and so clearly $M, w \Vdash y$.

To prove that $\Delta(x, y)$ satisfies simple replacement with respect to lK we need to consider the different connectives.

(\neg): Suppose $M, w \Vdash \Delta(x, y)$. Let $n \in \omega$ and let $w' \in M$ be such that $wR^n w'$. We want to show that $M, w' \Vdash \neg x \leftrightarrow \neg y$. Now, by assumption, $M, w' \Vdash x \leftrightarrow y$ and so $M, w' \Vdash \neg x \leftrightarrow \neg y$. Thus $M, w \Vdash \Box^n(\neg x \leftrightarrow \neg y)$ and so $M, w \Vdash \Delta(\neg x, \neg y)$.

(\wedge): Suppose $M, w \Vdash \Delta(x_1, y_1)$ and $M, w \Vdash \Delta(x_2, y_2)$. Let $n \in \omega$ and let $w' \in M$ be such that $wR^n w'$. We want to show that $M, w' \Vdash x_1 \wedge x_2 \leftrightarrow y_1 \wedge y_2$. Now, by assumption, $M, w' \Vdash x_1 \leftrightarrow y_1$ and $M, w' \Vdash x_2 \leftrightarrow y_2$, and thus $M, w' \Vdash x_1 \wedge x_2 \leftrightarrow y_1 \wedge y_2$. Hence $M, w \Vdash \Box^n(x_1 \wedge x_2 \leftrightarrow y_1 \wedge y_2)$ and so $M, w \Vdash \Delta(x_1 \wedge x_2, y_1 \wedge y_2)$.

(\Box): Suppose $M, w \Vdash \Delta(x, y)$. Let $n \in \omega$ and let $w' \in M$ be such that $wR^n w'$. We want to show that $M, w' \Vdash \Box x \leftrightarrow \Box y$. Suppose that $M, w' \Vdash \Box x$, but suppose towards a contradiction that $M, w' \not\Vdash \Box y$. Then there is $w'' \in M$ such that $w'R w''$ and $M, w'' \not\Vdash y$. Then, since $M, w' \Vdash \Box x$ we have that $M, w'' \Vdash x$. But $wR^{n+1} w''$ and so, by assumption $M, w'' \Vdash x \leftrightarrow y$. Hence $M, w'' \Vdash y$. A contradiction. Thus $M, w' \Vdash \Box y$. The other direction is proved similarly. Thus $M, w' \Vdash \Box x \leftrightarrow \Box y$, and so $M, w \Vdash \Box^n(\Box x \leftrightarrow \Box y)$. Hence $M, w \Vdash \Delta(\Box x, \Box y)$.

Next we will show that no finite subset of $\Delta(x, y)$ is an equivalence for lK and then, using Theorem 3.3.6, we show that this implies that lK is not finitely protoalgebraic.

Let $\Delta'(x, y)$ be a finite subset of $\Delta(x, y)$. Let n be the largest number such that $\Box^n(x \leftrightarrow y) \in \Delta'(x, y)$. Now define a Kripke model $M = \langle W, R, V \rangle$ as follows: Let $W = \{0, 1, \dots, n, n+1\}$ and let $k_1 R k_2$ if and only if $k_2 = k_1 + 1$. Let V be a valuation such that $Vx = \emptyset$ and $Vy = \{n+1\}$. Now clearly $M, 0 \Vdash \Delta'(x, y)$, but $M, 0 \not\Vdash \Delta'(\Box x, \Box y)$. Hence $\Delta'(x, y)$ is not an equivalence for lK .

Suppose then towards a contradiction that lK is nonetheless finitely protoalgebraic – perhaps the finite parameterized equivalence is lurking behind some other corner. Now since lK is finitary we have that

$$C_{lK} \Delta(x, y) = \bigcup_{\Delta' \subseteq_{\omega} \Delta} C_{lK} \Delta'(x, y).$$

Now the family $\{C_{lK}\Delta'(x, y) : \Delta'(x, y) \subseteq_{\omega} \Delta(x, y)\}$ is clearly a directed family of $\{x, y\}$ -invariant lK -theories. Hence by Theorem 3.3.6, we have that

$$\Omega C_{lK}\Delta(x, y) = \bigcup_{\Delta' \subseteq_{\omega} \Delta} \Omega C_{lK}\Delta'(x, y)$$

Now, since $\langle x, y \rangle \in \Omega C_{lK}\Delta(x, y)$, we have that $\langle x, y \rangle \in \Omega C_{lK}\Delta'(x, y)$ for some $\Delta'(x, y) \subseteq_{\omega} \Delta(x, y)$. Hence, by Lemma 3.1.12, $\Delta'(x, y)$ is an equivalence for lK against what we have just proven.

Note also that $x \rightarrow y$ is a proto-implication for lK . Hence lK is an example of a logic that has a finite proto-implication but no finite parameterized equivalence.

3.4 Characterizations via the closure properties of reduced models

In this final section of the chapter, we show how to characterize the different classes of logics by the closure properties of the class of reduced models of a logic. The first two characterizations – for protoalgebraic and equivalential logics – were obtained independently by Czelakowski [8] and Blok and Pigozzi [4].

Theorem 3.4.1. *Let \mathcal{L} be a logic. \mathcal{L} is protoalgebraic if and only if $\text{Mod}^*\mathcal{L}$ is closed under subdirect products.*

Proof. Suppose first that \mathcal{L} is protoalgebraic. Let $\{\langle \mathbf{A}_i, F_i \rangle : i \in I\}$ be a family of reduced models of \mathcal{L} and let $\langle \mathbf{B}, G \rangle \subseteq_{\text{SD}} \prod_I \langle \mathbf{A}_i, F_i \rangle$. Now $\langle \mathbf{B}, G \rangle \in \text{Mod}\mathcal{L}$.

Let $\Delta(x, y, \bar{z})$ be a parameterized equivalence for \mathcal{L} . Suppose $\langle f, g \rangle \in \Omega_{\mathbf{B}}G$. Then $\Delta^{\mathbf{B}}(\langle f, g \rangle) \subseteq G$ and so, since $\pi_i \upharpoonright B$ is surjective for all $i \in I$, we have that $\Delta^{\mathbf{A}_i}(\langle fi, gi \rangle) \subseteq F_i$ for all $i \in I$. Hence $\langle fi, gi \rangle \in \Omega_{\mathbf{A}_i}F_i$ for all $i \in I$, and so, because $\langle \mathbf{A}_i, F_i \rangle \in \text{Mod}^*\mathcal{L}$ for all $i \in I$, we have that $fi = gi$ for all $i \in I$. Thus $f = g$ and so $\langle \mathbf{B}, G \rangle \in \text{Mod}^*\mathcal{L}$.

Suppose then that $\text{Mod}^*\mathcal{L}$ is closed under subdirect products. We will show that $\Omega_{\mathbf{A}}$ is monotone on $\text{Fi}_{\mathcal{L}}\mathbf{A}$ for every algebra \mathbf{A} .

Let \mathbf{A} be an algebra and let $F, G \in \text{Fi}_{\mathcal{L}}\mathbf{A}$ be such that $F \subseteq G$. Now let $\theta = \Omega_{\mathbf{A}}F \cap \Omega_{\mathbf{A}}G$. Then θ is compatible with F . Consider now the mapping

$$h : \langle \mathbf{A}/\theta, F/\theta \rangle \rightarrow \langle \mathbf{A}/\Omega_{\mathbf{A}}F, F/\Omega_{\mathbf{A}}F \rangle \times \langle \mathbf{A}/\Omega_{\mathbf{A}}G, G/\Omega_{\mathbf{A}}G \rangle$$

defined by $h(a/\theta) = \langle a/\Omega_{\mathbf{A}}F, a/\Omega_{\mathbf{A}}G \rangle$. Now it is straightforward to check that h is a subdirect embedding. Hence, by assumption, $\langle \mathbf{A}/\theta, F/\theta \rangle$ is reduced. Thus, by Lemma 2.2.14, $\theta = \Omega_{\mathbf{A}}F$ and so $\Omega_{\mathbf{A}}F \subseteq \Omega_{\mathbf{A}}G$. \square

In order to give the characterization for the equivalential logics we first prove a lemma that provides sufficient and necessary conditions for a class of reduced models to be closed under submatrices.

Lemma 3.4.2. *Let \mathcal{L} be a logic. Then the following are equivalent.*

- (i) $\text{Mod}^*\mathcal{L}$ is closed under submatrices.
- (ii) For all $\langle \mathbf{A}, F \rangle \in \text{Mod}\mathcal{L}$ and all $\langle \mathbf{B}, G \rangle \subseteq \langle \mathbf{A}, F \rangle$,

$$\Omega_{\mathbf{B}}G = \Omega_{\mathbf{A}}F \cap B \times B.$$

Proof. (i) \Rightarrow (ii): Suppose $\text{Mod}^*\mathcal{L}$ is closed under submatrices. Let $\langle \mathbf{B}, G \rangle \subseteq \langle \mathbf{A}, F \rangle \in \text{Mod}\mathcal{L}$. Now consider $\theta = \Omega_{\mathbf{A}}F \cap B \times B$. It is easy to see that θ is compatible with G . Now consider the mapping $h: \langle \mathbf{B}/\theta, G/\theta \rangle \rightarrow \langle \mathbf{A}/\Omega_{\mathbf{A}}F, F/\Omega_{\mathbf{A}}F \rangle$ defined by $h(a/\theta) = a/\Omega_{\mathbf{A}}F$. It is straightforward to check that h is an embedding. Hence, by assumption, $\langle \mathbf{B}/\theta, G/\theta \rangle \in \text{Mod}^*\mathcal{L}$ and so, by Lemma 2.2.14, $\theta = \Omega_{\mathbf{B}}G$.

(ii) \Rightarrow (i): Let $\langle \mathbf{A}, F \rangle \in \text{Mod}^*\mathcal{L}$ and let $\langle \mathbf{B}, G \rangle \subseteq \langle \mathbf{A}, F \rangle$. Then

$$\Omega_{\mathbf{B}}G = \Omega_{\mathbf{A}}F \cap B \times B = \text{id}_{\mathbf{A}} \cap B \times B = \text{id}_{\mathbf{B}}$$

and so $\langle \mathbf{B}, G \rangle \in \text{Mod}^*\mathcal{L}$. □

Theorem 3.4.3. *Let \mathcal{L} be a logic. \mathcal{L} is equivalential if and only if $\text{Mod}^*\mathcal{L}$ is closed under submatrices and direct products.*

Proof. Suppose first that \mathcal{L} is equivalential. Then, by Theorem 3.4.1, $\text{Mod}^*\mathcal{L}$ is closed under subdirect products and so, in particular, under direct products.

Let now $\Delta(x, y)$ be an equivalence for \mathcal{L} and let $\langle \mathbf{A}, F \rangle \in \text{Mod}^*\mathcal{L}$. Let $\langle \mathbf{B}, G \rangle$ be a submatrix of $\langle \mathbf{A}, F \rangle$. We want to show that $\Omega_{\mathbf{B}}G = \text{id}_{\mathbf{B}}$. Suppose $\langle a, b \rangle \in \Omega_{\mathbf{B}}G$. Then $\Delta^{\mathbf{B}}(a, b) \subseteq G$ and so $\Delta^{\mathbf{A}}(a, b) \subseteq F$. Hence $\langle a, b \rangle \in \Omega_{\mathbf{A}}F$. Thus, by assumption, $a = b$.

Suppose then that Mod^* is closed under submatrices and direct products. Then, in particular, $\text{Mod}^*\mathcal{L}$ is closed under subdirect products, and so \mathcal{L} is protoalgebraic. We will show that $\Delta(x, y) = \Sigma_{\mathcal{L}} \cap \text{Fm}(x, y)$ is an equivalence for \mathcal{L} . Let $T = C_{\mathcal{L}}\Delta(x, y)$, and consider the Lindenbaum matrices $\langle \mathbf{Fm}, \Sigma_{\mathcal{L}} \rangle$ and $\langle \mathbf{Fm}, T \rangle$. Now $\langle \mathbf{Fm}(x, y), \Delta(x, y) \rangle$ is a submatrix of both of them. We know that $\langle x, y \rangle \in \Omega\Sigma_{\mathcal{L}}$ and so by previous lemma, $\langle x, y \rangle \in \Omega_{\mathbf{Fm}(x, y)}\Delta(x, y)$. Hence $\langle x, y \rangle \in \Omega T = \Omega C_{\mathcal{L}}\Delta(x, y)$. Hence, by 3.1.12, $\Delta(x, y)$ is an equivalence for \mathcal{L} . □

In order to give a characterization for finitely protoalgebraic logics, we need to assume some further properties from the logic, namely that the finitary companion of the logic is protoalgebraic. Previously Czelakowski [8] has proven that a finitary protoalgebraic logic \mathcal{L} is finitely protoalgebraic if and only if $\text{Mod}^*\mathcal{L}$ is closed under ultraproducts. We obtain Czelakowski's result as an immediate corollary.

Theorem 3.4.4. *Let \mathcal{L} be a logic such that \mathcal{L}^f is protoalgebraic. Then the following are equivalent:*

- (i) \mathcal{L} is finitely protoalgebraic.
- (ii) $\mathbf{P}_f\text{Mod}^*\mathcal{L} \subseteq \text{Mod}^*\mathcal{L}^f$.
- (iii) $\mathbf{P}_u\text{Mod}^*\mathcal{L} \subseteq \text{Mod}^*\mathcal{L}^f$.

Proof. Note first that, by assumption, both \mathcal{L} and \mathcal{L}^f are protoalgebraic. Also, because $C_{\mathcal{L}}\emptyset = C_{\mathcal{L}^f}\emptyset$, we have that $\Sigma_{\mathcal{L}} = \Sigma_{\mathcal{L}^f}$.

(i) \Rightarrow (ii): Suppose \mathcal{L} is finitely protoalgebraic. We assume for simplicity that there is only one parameter in the finite parameterized equivalence $\Delta(x, y, z)$ for \mathcal{L} . The general case does not offer any further difficulties. Let $\{\langle \mathbf{A}_i, F_i \rangle : i \in I\} \subseteq \text{Mod}^*\mathcal{L}$, let \mathcal{F} be a filter on I and consider the filtered product

$$\langle \mathbf{A}, F \rangle = \prod_I \langle \mathbf{A}_i, F_i \rangle / \mathcal{F}.$$

We know that $\langle \mathbf{A}, F \rangle \in \text{Mod}\mathcal{L}^f$. Let us prove that $\langle \mathbf{A}, F \rangle$ is reduced. Let $f, g \in \prod_I A_i$ be such that $f/\mathcal{F} \neq g/\mathcal{F}$. Hence $I_0 = \{i \in I : f_i = g_i\} \notin \mathcal{F}$. Then, since $\langle \mathbf{A}_i, F_i \rangle$ is reduced for every $i \in I$,

$$I \setminus I_0 = \{i \in I : \exists \delta(x, y, z) \in \Delta(x, y, z) \text{ and } \exists c_i \in A_i \text{ s.t. } \delta^{\mathbf{A}_i}(f_i, g_i, c_i) \notin F_i\}.$$

Now let $h : I \rightarrow \bigcup_I A_i$ be such that $hi \in A_i$ and

$$I \setminus I_0 = \{i \in I : \exists \delta(x, y, z) \in \Delta(x, y, z) \text{ s.t. } \delta^{\mathbf{A}_i}(f_i, g_i, hi) \notin F_i\}.$$

Then

$$I_0 = \{i \in I : \Delta^{\mathbf{A}_i}(f_i, g_i, hi) \subseteq F_i\}.$$

Now, since $\Delta(x, y, z)$ is finite, $\Delta^{\mathbf{A}}(f/\mathcal{F}, g/\mathcal{F}, h/\mathcal{F}) \not\subseteq F$. Thus

$$\Sigma_{\mathcal{L}^f}^{\mathbf{A}}(\langle f/\mathcal{F}, g/\mathcal{F} \rangle) = \Sigma_{\mathcal{L}}^{\mathbf{A}}(\langle f/\mathcal{F}, g/\mathcal{F} \rangle) \not\subseteq F,$$

and so $\langle f/\mathcal{F}, g/\mathcal{F} \rangle \notin \Omega_{\mathbf{A}}F$.

(ii) \Rightarrow (iii): This is obvious.

(iii) \Rightarrow (i): By Theorem 3.2.2 it suffices to prove the following claim.

Claim 2. *There exists a finite $\Delta(x, y, \bar{z}) \subseteq \Sigma_{\mathcal{L}}$ such that for all $\langle \mathbf{A}, F \rangle \in \mathbf{Mod}^* \mathcal{L}$ it holds that if $\Delta(\langle a, b \rangle) \subseteq F$, then $a = b$.*

Proof of the claim. Suppose towards a contradiction that the claim does not hold. Then for any finite $\Delta(x, y, \bar{z}) \subseteq \Sigma_{\mathcal{L}}$ there is $\langle \mathbf{A}_{\Delta}, F_{\Delta} \rangle \in \mathbf{Mod}^* \mathcal{L}$ and $a_{\Delta}, b_{\Delta} \in A_{\Delta}$ such that $\Delta(\langle a_{\Delta}, b_{\Delta} \rangle) \subseteq F_{\Delta}$ but $a_{\Delta} \neq b_{\Delta}$.

Now for each $\Delta \subseteq_{\omega} \Sigma_{\mathcal{L}}$, let $\Delta^* = \{\Delta' \subseteq_{\omega} \Sigma_{\mathcal{L}} : \Delta \subseteq \Delta'\}$. Now the family $\{\Delta^* : \Delta \subseteq_{\omega} \Sigma_{\mathcal{L}}\}$ has the finite intersection property and so there exists an ultrafilter \mathcal{U} that contains the family.

Now consider the ultraproduct

$$\langle \mathbf{A}, F \rangle = \prod_{\Delta \subseteq_{\omega} \Sigma_{\mathcal{L}}} \langle \mathbf{A}_{\Delta}, F_{\Delta} \rangle / \mathcal{U}$$

and let $\alpha = \langle a_{\Delta} : \Delta \subseteq_{\omega} \Sigma_{\mathcal{L}} \rangle / \mathcal{U}$ and $\beta = \langle b_{\Delta} : \Delta \subseteq_{\omega} \Sigma_{\mathcal{L}} \rangle / \mathcal{U}$. Now

$$\Sigma_{\mathcal{L}^f}(\langle \alpha, \beta \rangle) \subseteq F,$$

and so $\langle \alpha, \beta \rangle \in \Omega_{\mathbf{A}} F$, since \mathcal{L}^f is protoalgebraic. But, on the other hand, $\alpha \neq \beta$ and so $\langle \mathbf{A}, F \rangle$ is not reduced, which is against the assumption. \square

\square

Corollary 3.4.5. *Let \mathcal{L} be protoalgebraic logic. Then the following are equivalent.*

- (i) \mathcal{L} is finitary and finitely protoalgebraic.
- (ii) $\mathbf{Mod}^* \mathcal{L}$ is closed under filtered products.
- (iii) $\mathbf{Mod}^* \mathcal{L}$ is closed under ultraproducts.

When it comes to finitely equivalential logics, the picture becomes a lot simpler. Notice that a logic \mathcal{L} is finitely equivalential if and only if \mathcal{L}^f is finitely equivalential. Compare this to the Example 3.1.15 where we have a finitely protoalgebraic logic whose finitary companion is not even protoalgebraic. We have now also an analogue of Theorem 3.4.4 for equivalential logics.

Theorem 3.4.6. *Let \mathcal{L} be a logic such that \mathcal{L}^f is equivalential. Then the following are equivalent:*

- (i) \mathcal{L} is finitely equivalential.
- (ii) $\mathbf{P}_f \mathbf{Mod}^* \mathcal{L} \subseteq \mathbf{Mod}^* \mathcal{L}^f$.

(iii) $\mathbf{P}_u \text{Mod}^* \mathcal{L} \subseteq \text{Mod}^* \mathcal{L}^f$.

Proof. The implication from (i) to (ii) follows from 3.4.4. The implication from (ii) to (iii) is trivial. Finally the implication from (iii) to (i) is proved similarly to the corresponding one in 3.4.4 noting that $\Sigma_{\mathcal{L}} \cap \text{Fm}(x, y)$ is an equivalence for both \mathcal{L} and \mathcal{L}^f . \square

Corollary 3.4.7. *Let \mathcal{L} be an equivalential logic. Then the following are equivalent.*

- (i) \mathcal{L} is finitary and finitely equivalential.
- (ii) $\text{Mod}^* \mathcal{L}$ is closed under filtered products.
- (iii) $\text{Mod}^* \mathcal{L}$ is closed under ultraproducts.

We'll end this section with another example. We'll present a class of logics that are all protoalgebraic but not equivalential. We use Theorem 3.4.3 to prove the negative result.

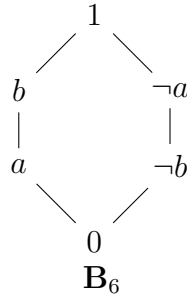
Example 3.4.8. An ortholattice is an algebra $\langle A, \wedge, \vee, \neg, 0, 1 \rangle$ such that

- $\langle A, \wedge, \vee, 0, 1 \rangle$ is a bounded lattice;
- $\neg\neg a = a$ for all $a \in A$;
- $\neg a \wedge a = 0$ for all $a \in A$;
- $\neg(a \wedge b) = \neg a \vee \neg b$ for all $a, b \in A$.

An ortholattice is orthomodular if it satisfies moreover the following

$$\text{if } a \leq b \text{ then } b = a \vee (\neg a \wedge b).$$

We denote the class of all ortholattices by **OL** and the class of all orthomodular lattices, i.e. orthomodular ortholattices, by **OML**. One can show that an ortholattice **A** is orthomodular if and only if the Benzene ring **B**₆ is not (isomorphic to) a subalgebra of **A** (see e.g. [1]).



Consider now the matrix $\langle \mathbf{B}_6, \{1\} \rangle$. It is easy to see that the principal congruence determined by $\langle a, b \rangle$ is compatible with $\{1\}$. Hence $\langle a, b \rangle \in \Omega_{\mathbf{B}_6}\{1\}$ and so $\langle \mathbf{B}_6, \{1\} \rangle$ is not reduced.

Now given a class \mathbf{K} of ortholattices we define the orthologic $\mathcal{L}_{\mathbf{K}}$ as the logic determined by the class $\{\langle \mathbf{A}, \{1\} \rangle : \mathbf{A} \in \mathbf{K}\}$. Orthologic \mathcal{L} is orthomodular if $\mathcal{L}_{\mathbf{OML}} \leq \mathcal{L}$. It's easy to see that the formula $\neg x \vee y$ serves as a proto-implication for all orthologics. Hence any orthologic is protoalgebraic. We show using Theorem 3.4.3 that any non-orthomodular orthologic is not equivalential by showing that the class of reduced models is not closed under submatrices.

Now let \mathbf{K} be a class of ortholattices where not all are orthomodular. Consider the class $\mathbf{M} = \{\langle \mathbf{A}/\Omega_{\mathbf{A}}\{1\}, \{1\}/\Omega_{\mathbf{A}}\{1\} \rangle : \mathbf{A} \in \mathbf{K}\}$. Now $\mathcal{L}_{\mathbf{K}} = \mathcal{L}_{\mathbf{M}}$. It's easy to see that any homomorphic image of an ortholattice is an ortholattice and so $\mathbf{A}/\Omega_{\mathbf{A}}\{1\}$ is an ortholattice for all $\mathbf{A} \in \mathbf{K}$. On the other hand $\mathbf{A}/\Omega_{\mathbf{A}}\{1\}$ is not orthomodular for some $\mathbf{A} \in \mathbf{K}$, since $\mathcal{L}_{\mathbf{K}}$ is not orthomodular. Hence $\langle \mathbf{B}_6, \{1\} \rangle \leq \langle \mathbf{A}/\Omega_{\mathbf{A}}\{1\}, \{1\}/\Omega_{\mathbf{A}}\{1\} \rangle$ for some $\mathbf{A} \in \mathbf{K}$. But as noted above $\langle \mathbf{B}_6, \{1\} \rangle$ is not reduced and so $\mathbf{Mod}^* \mathcal{L}_{\mathbf{K}}$ is not closed under submatrices. Hence, by Theorem 3.4.3, $\mathcal{L}_{\mathbf{K}}$ is not equivalential.

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