On Dirichlet's (1829) paper on Fourier series

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Tiivistelmä: Tämän maisterintutkielman tavoitteena on tutustuttaa lukija Dirichletin kuuluisaan todistukseen Fourier-sarjojen suppenevuudesta. Työssäni olen pyrkinyt säilyttämään Dirichletin todistuksen hengen käymällä todistusta läpi Dirichletin itsensä kirjoittamalla tavalla. Sellaiset osuudet todistuksesta jotka Dirichlet sivuutti olen yrittänyt käydä läpi niinkuin Dirichlet olisi ne voinut käydä.

Työssä käsiteltävät Fourier-sarjat ovat trigonometrisiä sarjoja joiden avulla voidaan esittää tietyt ehdot täyttäviä funktiota. Todistuksen oleellinen osuus on siis osoittaa että suuri määrä funktio voidaan esittää trigonometristen sarjojen äärettömänä summana.

Tutkielma alkaa ensimmäisen kappaleen tiivistelmällä. Toisessa kappaleessa kerrotaan todistuksen tausta historiasta, sen merkittävyydestä sekä sen jälkeisistä tuloksista. Tämän jälkeen käydään läpi todistuksessa tarvittavia lauseita, määritelmiä ja laskuja kappaleissa kolme ja neljä. Kappaleessa viisi lasketaan integraali nollasta äärettömään funktiolle sin(x)/x, mitä käytetään seuraavassa kappaleessa. Kuudennessa ja viimeisessä kappaleessa käydään lopulta läpi Dirichletin todistus. Se on jaettu kahteen osaan. Ensimmäisessä osassa käydään läpi lauseita ja lemmoja, jotka sitten kootaan yhdeksi. Toisessa osassa käydään päätodistus läpi hyödyntämällä ensimmäistä osaa.

Hyvänä jatkotutkimuksen aiheena voisi olla Dirichletin todistuksen jälkeiset todistukset Fourier-sarjojen suppenemisesta. Dirichlet esitti omat riittävät ehtonsa Fourier-sarjojen suppenemiselle, mutta hänen jälkeensä on esitetty muitakin riittäviä ehtoja. Näiden kokoaminen ja vertailu voisivat olla varsin mielenkiintoinen tutkimuksen aihe.

Contents

Chapter 1

Introduction

This work goes through one of the most significant proofs of the 19th century. Peter Gustav Lejeune Dirichlet's proof on the convergence of Fourier series. Throughout the proof I have tried to only use concepts and ideas that were known at the time and which could have been used by Dirichlet when writing his proof, with the exception of Riemann integral which had not been defined at the time.

Dirichlet's proof solved the long debate of whether a vast number of functions be represented by Fourier series and opened up plenty of other questions to examine. Some of them have even led to the creation of new branches in mathematics and others are still out there waiting to be solved.

This was already significant in itself but the proof also introduced a new level of rigour at a time when rigour in analysis was sorely longed for. This gave a new standard for a rigorous proof that others could live up to. His work also shed some new light into the long controversy over the concept of a function by introducing a completely new type of function. Dirichlet presented a completely non-analytic function that was named after him as Dirichlet's function. Later he would give his own definition of a function which is still used today.

Chapter 2

History

2.1 Fourier series in the 18th century

During the 18th century, mathematicians worked with trigonometric series, especially in astronomical theory. The usefulness of such series in astronomy is apparent since they are periodic functions and astronomical phenomena are mainly periodic [12, p. 454].

Another source of interest for trigonometric series were partial differential equations which came up when mathematicians tried to solve the vibrating string problem. The problem was to find a function $u(x, t)$ of the space variable x and the time variable t describing the position of an elastic string of length L with fixed endpoints that was undergoing small vibrations [14, p. 85].

A trigonometric series is any series of the form

$$\frac{1}{2}a_0 + \sum_{k=1}^{\infty}[a_k \cos(kx) + b_k \sin(kx)],$$

with a_k and b_k constant. If such a series represents a function f, mainly

$$f(x) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty}[a_k \cos(kx) + b_k \sin(kx)], \tag{2.1}$$

then

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\,dx, \quad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx \tag{2.2}$$

for $k = 0, 1, 2, \ldots$. These are called the Fourier coefficients and the series is called the Fourier series of f. However, it should be noted that function f doesn't necessarily have the same values as the series. Indeed the series might diverge or converge to another function [4, p. 27-29].

In the 18th century, only some special cases of functions represented by trigonometric series were known. The following one was given by Leonhard Euler (1707-1783)

$$\frac{x}{2} = \sin x - \frac{1}{2}\sin(2x) + \frac{1}{3}\sin(3x) - \cdots ,$$

which is true for $-\pi < x < \pi$ [1, p. 4].

In 1757, while studying the perturbations of the sun, Alexis-Claude Clairaut (1713-1765) gave the Fourier coefficients in the case of cosine series expansion. He viewed the question of finding coefficients $A_0, A_1, A_2, \ldots$ in

$$f(x) = A_0 + 2\sum_{n=1}^{\infty} A_n \cos(nx).$$

and arrived at

$$A_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx,$$

which is the correct formula for A_n. Twenty years later, Euler derived the same formula [17, p. 528].

2.2 Fourier on Fourier series

In 1807 Jean Baptiste Joseph Fourier (1768-1830) submitted his paper *Théorie de la propagation de la chaleur* to the Paris Academy of Sciences, regarding heat conduction in various bodies [14, p. 89]. In the same paper Fourier also considered the problem of representing an "arbitrary function" in terms of a trigonometric series. He suggested that given an "arbitrary function" f on the closed bounded interval $[-L, L]$ there exists constants a_k and b_k such that

$$f(x) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty}\left[a_k \cos\left(\frac{k\pi x}{L}\right) + b_k \sin\left(\frac{k\pi x}{L}\right)\right],$$

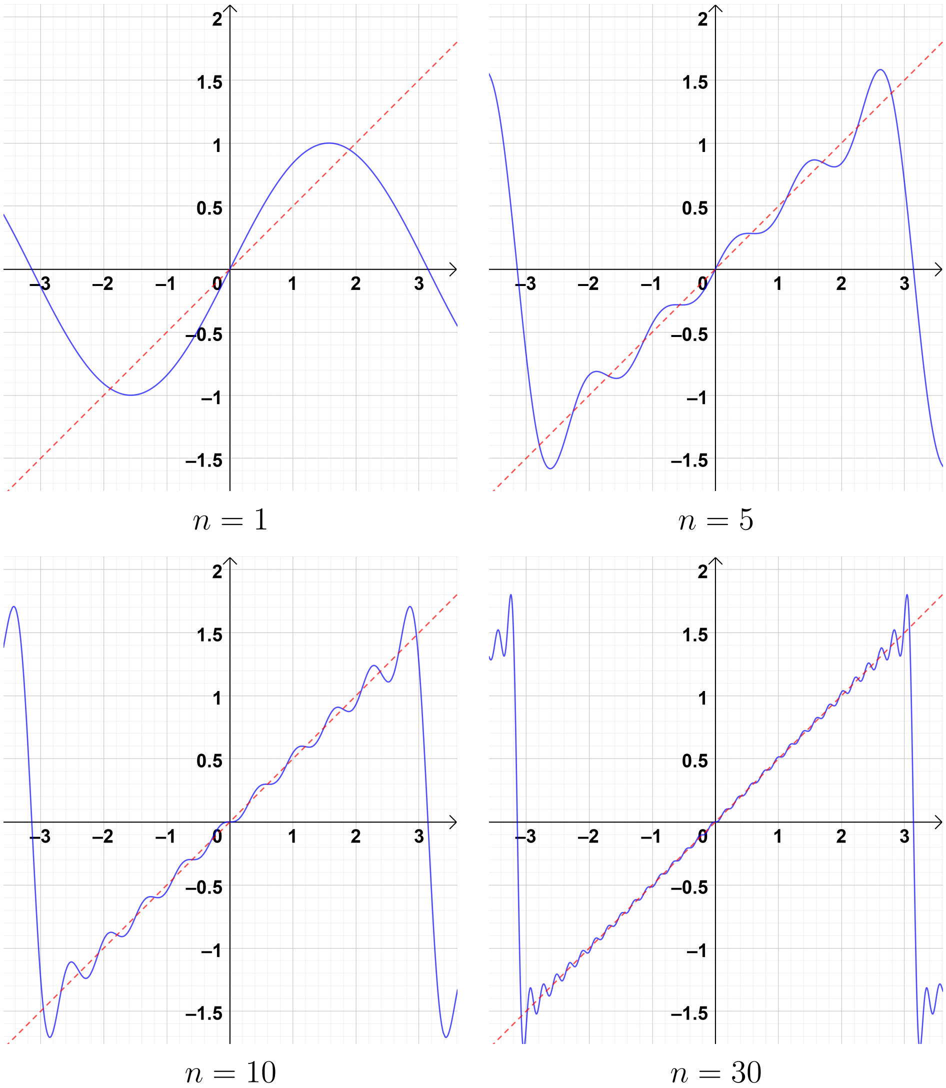

Figure 2.1: Graphs of $x/2$ (dashes) approximated by series $\sum_{k=1}^{n} \frac{1}{k} \sin(kx)(-1)^{(k+1)}$ (solid) with a varying number of terms.

Figure 2.2: Peter Gustav Lejeune Dirichlet

when $-L \leq x \leq L$ [14, p. 90]. The series becomes the same as (2.1) when $L = \pi$. He also gave the formula for Fourier coefficients (2.2)[1] (By this time Clairaut's and Euler's work had been forgotten.) [18, p. 4]. Fourier continued to work on heat conduction and published his most comprehensive treatment on the subject *Théorie analytique de la chaleur* (The analytical theory of heat) in 1822 [14, p. 91].

2.3 Attempts to solve the convergence of Fourier series

In 1820 Siméon Denis Poisson (1781-1840) who was a rival to Fourier gave the first published treatment on the convergence of the Fourier series. His idea was to multiply the Fourier series $\sum a_n \cos(nx)$ (where the a_n's are given by the Fourier coefficients) by the terms of the geometric series $\sum p^n$ for $p \in (0, 1)$. The resulting series

$$\sum_{n=1}^{\infty} p^n a_n \cos(nx)$$

is convergent. He then put $p = 1$ in this expression and used questionable arguments to show that the result was $f(x)$. Problem was that this did not prove the original series to

[1]Fourier used the more general interval $[-L, L]$

be convergent [10, p. 179]. In 1822 Fourier gave his own proof in Théorie analytique de la chaleur, which too proved to be faulty [10, p. 179].

Based on Poisson's incorrect proof Augustin Louis Cauchy (1789-1857) gave his own proof in 1826 which too proved to be flawed, shown by the counterexample at the start of Peter Gustav Lejeune Dirichlet's (1805-1859) paper [5] [14, p. 94]. Cauchy's mistaken assumption was that if v_n approaches w_n as $n \to \infty$, and if $w_1 + w_2 + w_3 + \cdots$ converges then so must $v_1 + v_2 + v_3 + \cdots$. Dirichlet pointed out that if one defines

$$w_n = \frac{(-1)^n}{\sqrt{n}}, \quad v_n = \frac{(-1)^n}{\sqrt{n}} + \frac{1}{n},$$

then w_n approaches v_n and the series $\sum_{n=1}^{\infty} w_n$ converges, but the series $\sum_{n=1}^{\infty} v_n$ diverges. This collapsed Cauchy's argument [2, p. 218].

2.4 Dirichlet's proof

Dirichlet who had studied under Fourier [3, p. 156] gave his own proof *Sur la convergence des séries trigonométriques qui servent à représenter une fonction arbitraire entre des limites données* in 1829 which was a much refined version of the proof sketched by Fourier in his *Théorie analytique de la chaleur* [12, p. 966]. He started his paper by stating that the convergence of the Fourier series hasn't be proven before. Dirichlet goes on to say that he only knows one proof by Cauchy which is insufficient (see section 2.3). His proof then shows that bounded, piecewise monotonous and piecewise continuous functions have a converging Fourier series (see chapter 6 and Theorem 6.5.).

After the proof Dirichlet considers loosening the assumptions further. The function f can have an infinite number of discontinuities, if for every $a, b \in [-\pi, \pi]$ there exists such $r, s \in [a, b]$ that the function f is continuous in (r, s). He gives no proof to this claim. However, he argues for the restriction by pointing out that the terms of the series would otherwise lose all meaning. Dirichlet then gives an example to clarify. Let $c, d \in \mathbb{R}$ and $c \neq d$. Define function f as follows

$$f(x) = \begin{cases} c, & \text{if } x \in \mathbb{Q} \\ d, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases} \tag{2.3}$$

It's clear that function f is nowhere integrable when using Riemann integral. Hence it can't be substituted into the series because the terms which are definite integrals would lose all signification. Dirichlet then ends his paper by stating that to close this matter with the necessary clarity he will write about it in another paper.

Dirichlet never published the promised paper, but from a letter in 1853 to Carl Friedrich Gauss (1777-1855), it appears that he remained optimistic about the possible generalization of his conditions (as did Gauss). This was however in vain (see section 2.6) [10, p. 181].

2.5 Significance of Dirichlet's proof

Around the start of the 19th century, mathematicians began to be concerned about the looseness of concepts and proofs in vast branches of analysis. Rigorous analysis began with the works of Bernard Bolzano (1781-1848), Cauchy, Niels Henrik Abel (1802-1829) and Dirichlet [12, p. 947-948]. Much of the movement towards rigour consisted of the growing awareness that one can only use properties of functions that have been stated explicitly. Dirichlet was the first to live up to this ideal in his paper on Fourier series [5] [10, p. 158].

He was also the first to introduce a non-analytic function [10, p. 181] and to realise that not all functions can be integrated [2, p. 220]. He gave an example of a nowhere continuous function (2.3) that fills both of these conditions. A less general version of it is

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$$

which was named in his honour as Dirichlet's function [3, p. 158].

This came at a time when the concept of function was quite unclear. In 1837 in his german reprint[2] of his 1829 paper [5] Dirichlet gave his definition of a function [12, p. 950] that was widely accepted soon after and is still used today. It states that "y is a function of a variable x, defined on an interval $a < x < b$, if to every value of the variable x in this interval there corresponds a definite value of the variable y. Also, it is irrelevant in what way this correspondence is established" [15, p. 264].

When considering the $(n+1)$-st partial sum of Fourier series Dirichlet transformed it into (6.6) which is nowadays called the Dirichlet kernel [10, p. 180].

2.6 After Dirichlet's proof

Dirichlet continued to ponder on some questions raised by his work. How far can the assumptions on continuity be weakened? How many discontinuities may be admitted

[2] "Die Darstellung Ganz Willkürlicher Functionen Durch Sinus-und Cosinusreihen" (On the Representation of Completely Arbitrary Functions by Sine and Cosine Series)

without losing integrability? Is it possible to admit infinitely many discontinuities (even in the finite interval) [10, p. 263]?

In 1831 he became a professor in Berlin [10, p. 263]. There young Bernhard Riemann (1826-1866) studied under him and acquired an interest in the Fourier series. In 1854 Riemann took up the subject and tried to find the necessary and sufficient conditions that a function must satisfy so that at a point x in the interval $[-\pi, \pi]$ the Fourier series for $f(x)$ should converge to $f(x)$. He made some progress but was unable to find said conditions. This is still an open problem today. Riemann did manage to prove that if f is bounded and integrable in $[-\pi, \pi]$ then the Fourier coefficients (2.2) approach zero as n tends to infinity [12, p. 967-968].

For about fifty years after Dirichlet's work, it was commonly believed that the Fourier series of any function continuous in $[-\pi, \pi]$ converged to the function [12, p. 970]. This changed in 1873 when du Bois-Reymond gave an example of a continuous function whose Fourier series diverges at a point. This shattered the hope that Dirichlet had had about generalizing his conditions [10, p. 181].

In 1870 Georg Cantor (1845-1918) published a proof saying that if (2.1) converges to $f(x)$ for all x on $(0, 2\pi)$, then $f(x)$ cannot be represented by another trigonometric series converging to $f(x)$ for all x on $(0, 2\pi)$. In 1871 he improved the proof by showing that the representation remained unique if the requirements of convergence or convergence to $f(x)$ would be dropped for a finite set of exceptional points [13, p. 203]. This work eventually led him to create his acclaimed and controversial theory of sets [8, p. 204-205].

Many mathematicians took up the problem of finding sufficient conditions that the function f should have a Fourier series which converges to $f(x)$. One of them was Camille Jordan (1838-1922) who introduced Jordan's Criterion. It states that the Fourier series for the integrable function f converges to

$$\frac{1}{2}[f(x^-) + f(x^+)]$$

at every point for which there is a neighbourhood in which $f(x)$ is of bounded variation (see Definition 3.3 and compare to Theorem 6.5). He was also the one to introduce the concept of bounded variation which he did in the following way. Let $f(x)$ be bounded in $[a, b]$ and let $a = x_0, x_1, ..., x_{n-1}, x_n = b$ be a mode of division (partition) of this interval. Let $y_0, y_1, ..., y_{n-1}, y_n$ be the values of $f(x)$ at these points. Then for every partition

$$\sum_{r=0}^{n-1}(y_{r+1} - y_r) = f(b) - f(a)$$

we denote $t = \sum_{r=0}^{n-1} |y_{r+1} - y_r|$. Now for every mode of subdividing $[a, b]$ there is a t. If for every possible mode of division of $[a, b]$, the sums t have a least upper bound then f is defined to be of bounded variation in $[a, b]$ [12, p. 971].

In 1922, Andrey Kolmogorov (1903-1987) constructed the first example of an integrable function whose Fourier series diverges almost everywhere. He later improved the result from almost everywhere to everywhere [11, p. 303-304]. In 1966, Lennart Carleson (1928-) proved that the Fourier series of any square-integrable function $f : \mathbb{T} \to \mathbb{C}$ converges almost everywhere to f. This result is considered to be a cornerstone in the harmonic analysis of the 20th century [16, p. 1765].

Chapter 3

Theorems and Definitions

In this chapter we will introduce useful background information that will be used in the proof.

3.1 Theorems and Definitions

Theorem 3.1. *Let* $(x_n), (y_n)$ *and* (z_n) *be such sequences, that* $x_n \leq y_n \leq z_n$ *for all indices* n*. Let's assume that sequences* (x_n) *and* (z_n) *converge and* $\lim_{n\to\infty} x_n = a = \lim_{n\to\infty} z_n$*. Then sequence* (y_n) *converges and* $\lim_{n\to\infty} y_n = a$*.*

Proof. Intuitively Trivial. Theorem can be found in [9, Theorem 2.2.22. p. 49]. □

Theorem 3.2. *Let* $A \subset \mathbb{R}$*,* $f : A \to \mathbb{R}$ *and* $x_0 \in \mathbb{R}$*. Let's assume that there exists such* $r > 0$*, that* $(x_0 - r, x_0) \cup (x_0, x_0 + r) \subset A$*. Then the following are equivalent:*

(a) Function f *has at point* x_0 *limit* m*.*

(b) Let (y_n) *be a sequence, for which* $y_n \neq x_0$ *for all* $n \in \mathbb{N}_1$*. If sequence* (y_n) *converges towards value* x_0*, then sequence* $(f(y_n))$ *converges towards value* m*.*

Proof. Can be found in [9, Theorem 3.1.8. p. 61]. □

Definition 3.3. Let's assume $f : A \to \mathbb{R}$ and $x_0 \in \mathbb{R}$. Let's assume that there exists such $r > 0$, that $(x_0 - r, x_0) \subset A$. Then function f has at point x_0 left-hand limit m, if for every $\varepsilon > 0$ there exists such $\delta > 0$, that

$$|f(x) - m| < \varepsilon$$

for all $x \in A$, for which $x_0 - \delta < x < x_0$. Then we denote

$$\lim_{x \to x_0^-} f(x) = m \ \text{ or } \ f(x) \to m \,\text{when}\, x \to x_0^-.$$

Let's assume that there exists such $r > 0$, that $(x_0, x_0 + r) \subset A$. Then function f has at point x_0 right-hand limit m, if for every $\varepsilon > 0$ there exists such $\delta > 0$, that

$$|f(x) - m| < \varepsilon$$

for all $x \in A$, for which $x_0 < x < x_0 + \delta$. Then we denote

$$\lim_{x \to x_0^+} f(x) = m \ \text{ or } \ f(x) \to m \,\text{when}\, x \to x_0^+.$$

This definition can be found in [9, Definition 3.2.1. p. 63].

We also introduce a new abbreviation that will be used later on.

$$f(y^+) = \lim_{x \to y^+} f(x)$$
$$f(y^-) = \lim_{x \to y^-} f(x)$$

Definition 3.4. Let's assume $f : A \to \mathbb{R}$ and $x_0 \in A$ such that $(x_0 - r, x_0 + r) \subset A$ for some $r > 0$. Function f is continuous at point x_0, if

$$\lim_{x \to x_0} f(x) = f(x_0).$$

In other words function f is continuous at point x_0, if for every $\varepsilon > 0$ there exists such $\delta > 0$, that

$$|f(x) - f(x_0)| < \varepsilon$$

when $|x - x_0| < \delta$.

This definition can be found in [9, Definition 4.1.1. p. 73].

Definition 3.5. Let's assume $f : A \to \mathbb{R}$ and $x_0 \in A$ such that $(x_0 - r, x_0] \subset A$ for some $r > 0$. Function f is left continuous at point x_0, if

$$\lim_{x \to x_0^-} f(x) = f(x_0).$$

Let's assume then, that $[x_0, x_0 + r) \subset A$ for some $r > 0$. Function f is right continuous at point x_0, if

$$\lim_{x \to x_0^+} f(x) = f(x_0).$$

This definition can be found in [9, Definition 4.1.4. p. 75].

Definition 3.6. Let's assume $f : (a, b) \to \mathbb{R}$. Function f is piecewise continuous, if there exists such $n \in \mathbb{N}$ and points $x_1, ..., x_n \in (a, b)$, that f is continuous in the set $(a, b) \setminus x_1, ..., x_n$ and that limits $\lim_{x \to x_i^+} f(x)$ and $\lim_{x \to x_i^-} f(x)$ exist (and are finite) for all $i = 1, ..., n$.

This definition can be found in [9, Definition 4.1.17. p. 79].

Theorem 3.7. *L'Hôpital's rule:*
Let $f, g : (a, b) \to \mathbb{R}$ be differentiable functions. Let's assume that $\lim_{x \to a^+} f(x) = 0 = \lim_{x \to a^+} g(x)$, that $g(x) \neq 0$ and $g'(x) \neq 0$ for all $x \in (a, b)$. If the limit

$$\lim_{x \to a^+} \frac{f'(x)}{g'(x)}$$

exists (finite or infinite), then

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{f'(x)}{g'(x)}.$$

Proof. Can be found in [9, Theorem 6.5.2. p. 137]. □

During Dirichlet's time the only known rigorously defined integral was Cauchy integral. However it is a special case of the more known Riemann integral and hence we shall use Riemann integral instead in our proof.

Definition 3.8. A Cauchy integral is a definite integral of a continuous function of one real variable. Let $f(x)$ be a continuous function on an interval $[a, b]$ and let $a = x_0 < \cdots < x_{i-1} < x_i < \cdots < x_n = b, \Delta x_i = x_i - x_{i-1}, i = 1, ..., n$. The limit

$$\lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_{i-1}) \Delta x_i,$$

is called the definite integral in Cauchy's sense of $f(x)$ over $[a, b]$ and is denoted by

$$\int_a^b f(x)\, dx.$$

The Cauchy integral is a particular case of the Riemann integral [6].

Definition 3.9. Consider a function f which is given on an interval $[a, b]$. Let $a = x_0 < x_1 < \cdots < x_n = b$ is a partition (subdivision) of the interval $[a, b]$ and $\Delta x_i = x_i - x_{i-1}$, where $i = 1, ..., n$. The sum

$$\sigma = f(\xi_1)\Delta x_1 + \cdots + f(\xi_i)\Delta x_i + \cdots + f(\xi_n)\Delta x_n, \tag{3.10}$$

where $x_{i-1} \leq \xi_i \leq x_i$, is called the Riemann sum corresponding to the given partition of $[a, b]$ by the points x_i and to the sample of points ξ_i. The number I is called the limit of the Riemann sums (3.10) as $\max_i \Delta x_i \to 0$ if for any $\varepsilon > 0$ a $\delta > 0$ can be found such that $\max_i \Delta x_i < \delta$ implies the inequality $|\sigma - I| < \varepsilon$. If the Riemann sums have a finite limit I as $\max_i \Delta x_i \to 0$, then the function f is called Riemann integrable over $[a, b]$, where $a < b$. The limit is known as the definite Riemann integral of f over $[a, b]$, and is written as

$$\int_a^b f(x)\, dx. \tag{3.11}$$

When $a = b$ then, by definition,

$$\int_a^a f(x)\, dx = 0,$$

and when $a > b$ the integral (3.11) is defined using the equation

$$\int_b^a f(x)\, dx = - \int_a^b f(x)\, dx.$$

A necessary and sufficient condition for the Riemann integrability of f over $[a, b]$ is the boundedness of f on this interval and the zero value of the Lebesgue measure of the set of all points of discontinuity of f contained in $[a, b]$ [7].

Theorem 3.12. *continuous function defined in the closed interval $[a, b]$ is integrable in the interval $[a, b]$.*

Proof. Omitted. Can be found in [9, Theorem 7.2.3. p. 155]. □

Corollary 3.13. *Piecewise continuous function defined in the closed interval $[a, b]$ is integrable in the interval $[a, b]$.*

Proof. Omitted. Can be found in [9, Corollary 7.2.5. p. 156]. □

Theorem 3.14. *generalized integral mean value theorem:*

Let functions f and h be be an integrable in the interval $[a, b]$, $h(x) \geq 0$ for all $x \in [a, b]$, and

$$\int_a^b h(x)\,dx > 0.$$

Then

$$\inf_{x \in [a,b]} f(x) \leq \frac{\int_a^b f(x)h(x)\,dx}{\int_a^b h(x)\,dx} \leq \sup_{x \in [a,b]} f(x)$$

and if f is continuous, then there exists such $z \in (a, b)$ that

$$\frac{\int_a^b f(x)h(x)\,dx}{\int_a^b h(x)\,dx} = f(z).$$

Proof. Can be found in [9, Theorem 8.3.3. p. 187]. □

Definition 3.15. Uniform convergence:

Let's assume that (f_n) is a sequence of functions in set $A \subset \mathbb{R}$. Then (f_n) converges uniformly towards function $f : A \to \mathbb{R}$ in set A, if for every $\varepsilon > 0$ there exists an $n_\varepsilon \in \mathbb{N}_1$, such that

$$\sup_{x \in A} |f_n(x) - f(x)| < \varepsilon,$$

when $n > n_\varepsilon$. Then we say that

$$f_n \to f \text{ uniformly on the set A.}$$

This definition can be found in [9, Theorem 11.1.2. p. 237].

Theorem 3.16. *Limit of Integration:*
Assume that the functions $f_n : [a, b] \to \mathbb{R}$ are integrable, $n \in \mathbb{N}$ and $f : [a, b] \to \mathbb{R}$. If f_n converges uniformly to f on $[a, b]$, then the limit function f is integrable and

$$\lim_{n\to\infty} \int_a^b f_n(x)\, dx = \int_a^b f(x)\, dt.$$

Proof. Can be found in [9, Theorem 11.2.1. p. 242]. □

Definition 3.17. Let's assume that (x_n) is a sequence. We define a new sequence (s_n) by setting $s_1 = x_1$, $s_2 = x_1 + x_2$ and in general

$$s_n = \sum_{k=1}^{n} x_k = x_1 + x_2 + x_3 + \cdots + x_n, \quad n \in \mathbb{N}_1.$$

The pair $((x_n), (s_n))$ is called a series. The number x_n is the n:th term of the series and number s_n is the n:th partial sum of the series. The series is denoted by

$$\sum_{k=1}^{n} x_k \quad \text{or} \quad x_1 + x_2 + x_3 + \cdots .$$

This definition can be found in [9, Theorem 13.1.1. p. 269].

Definition 3.18. Series $\sum_{k=1}^{\infty} x_k$ converges, if the sequence of it's partial sums (s_n) converges, meaning that there exists an $s \in \mathbb{R}$, such that $\lim_{n\to\infty} s_n = s$. Then s is the sum of the series and we denote

$$s = \sum_{k=1}^{\infty} x_k.$$

If the series doesn't converge, then it diverges.

This definition can be found in [9, Theorem 13.1.3. p. 270].

Theorem 3.19. *(Leibniz's test) Let's assume that*

(a) numbers x_k are non-negative for all $k \in \mathbb{N}_1$

(b) $x_k \geq x_{k+1}$ *for all* $k \in \mathbb{N}_1$

(c) $\lim_{k\to\infty} x_k = 0$.

Then the alternating series $\sum_{k=1}^{\infty}(-1)^{k-1}x_k$ *and* $\sum_{k=1}^{\infty}(-1)^{k}x_k$ *converge.*

Proof. Can be found in [9, Theorem 14.1.3. p. 290]. □

Theorem 3.20. *Let's assume that the conditions of Theorem 3.19 apply. Then for alternating series* $\sum_{k=1}^{\infty}(-1)^{k-1}x_k$ *and* $\sum_{k=1}^{\infty}(-1)^{k}x_k$

$$|s - s_n| \leq x_{n+1} \quad \textit{for all} \ \ n \in \mathbb{N}_1.$$

Proof. Can be found in [9, Theorem 14.1.5. p. 291]. □

Chapter 4

Auxiliary results

This chapter introduces some lemmas and calculations that are needed in the coming chapters.

4.1 Some Lemmas

Lemma 4.1. *Let $f : A \to \mathbb{R}$ and $x_0 \in \mathbb{R}$. Let's assume that there exists such $r > 0$, that $(x_0, x_0 + r) \subset A$. Then the following are equivalent:*

(a) The function f has at point x_0 right hand limit m.

(b) Let (y_n) be a sequence, for which $y_n > x_0$ for all $n \in \mathbb{N}_1$. If the sequence (y_n) converges towards value x_0, then sequence $(f(y_n))$ converges towards value m.

Proof. Follows from Theorem 3.2 and Definition 3.3. □

Lemma 4.2. *The following inequalities hold.*

(a) $\sin x \leq x$ *when* $x \geq 0$.

(b) $\frac{x}{2} \leq \sin x$ *when* $0 \leq x \leq 1$.

(c) $|\sin x - x| \leq x^3$ *when* $x \geq 0$.

(d) $|\frac{\sin x}{x}| \leq 1$.

Proof. (a) Inequality $\sin x \leq x$ is true when $x - \sin x \geq 0$. Let's denote $f(x) = x - \sin x$. Now $f'(x) = 1 - \cos x \geq 1 - 1 = 0$. The function f is therefore increasing. Now $x - \sin x = f(x) \geq f(0) = 0 - \sin 0 = 0$.

(b) Inequality $\frac{x}{2} \leq \sin x$ is true when $\sin x - \frac{x}{2} \geq 0$. Let's denote $f(x) = \sin x - \frac{x}{2}$. Then $f'(x) = \cos x - \frac{1}{2} \geq \cos 1 - \frac{1}{2} > 0$. The function f is therefore increasing. Now $\sin x - \frac{x}{2} = f(x) \geq f(0) = \sin 0 - \frac{0}{2} = 0$.

(c) Using (a) we have that inequality $|\sin x - x| \leq x^3$ is true when $x^3 - x + \sin x \geq 0$. Let's denote $f(x) = x^3 - x + \sin x$. Then $f'(x) = 3x^2 - 1 + \cos x$ and $f''(x) = 6x - \sin x \geq 5x \geq 0$. Since $f''(x) \geq 0$ we know that f' is increasing. Since $f'(0) = 3 \cdot 0^2 - 1 + \cos 0 = 0 - 1 + 1 = 0$ and f' is increasing we know that $f'(x) \geq 0$ and f is increasing. Since $f(0) = 0^3 - 0 + \sin 0 = 0$ and f is increasing then $x^3 - x + \sin x = f(x) \geq 0$.

(d) $\left|\frac{\sin x}{x}\right| \leq \left|\frac{x}{x}\right| = 1$.

□

4.2 Auxiliary results

Theorem 4.3. *Let $h \in (0, \frac{\pi}{2}]$ and let $f : [0, h] \to \mathbb{R}$ be continuous. Assume that f is positive and decreasing. Assume further that $k \in \mathbb{N}$ and r is the greatest integer for which $\frac{r\pi}{k} < h$. Then the series*

$$\sum_{v=1}^{r} \int_{(v-1)\pi/k}^{v\pi/k} \frac{\sin(kx)}{\sin x} f(x)\, dx$$

is alternating and each term is smaller then the one before in absolute value.[1]

Proof. We know that since $0 < x < h \leq \frac{\pi}{2}$ then $\sin x$ is always positive and $f(x)$ is positive by definition. For this reason, only $\sin kx$ can change the sign of the integral. The integration interval of the first integral is $[0, \frac{\pi}{k}]$. Hence $kx \in [0, \pi]$ and $\sin kx \geq 0$. The first integral is therefore positive. The second integration interval is $[\frac{\pi}{k}, \frac{2\pi}{k}]$. Hence $kx \in [\pi, 2\pi]$ and $\sin kx \leq 0$. The second integral is therefore negative. Let's further prove that these new integrals are alternately positive and negative.

Let $v \in \mathbb{N}$ and $v + 1 \leq r$. Now

$$\int_{(v-1)\pi/k}^{v\pi/k} \frac{\sin(kx)}{\sin x} f(x)\, dx, \qquad \int_{v\pi/k}^{(v+1)\pi/k} \frac{\sin(kx)}{\sin x} f(x)\, dx$$

are two consecutive integrals. If we replace x by $x + \frac{\pi}{k}$ in the second integral, it becomes

[1] The Lemma follows Dirichlet's original proof.

$$\int_{(v-1)\pi/k}^{v\pi/k} \frac{\sin(kx+\pi)}{\sin(x+\pi/k)} f\left(x+\frac{\pi}{k}\right)\, dx.$$

Which using a known theorem $\sin(x+\pi) = -\sin(x)$ is the same as

$$-\int_{(v-1)\pi/k}^{v\pi/k} \frac{\sin(kx)}{\sin(x+\pi/k)} f\left(x+\frac{\pi}{k}\right)\, dx.$$

Note that $0 \le \frac{(v-1)\pi}{k} \le x < x+\frac{\pi}{k} \le \frac{v\pi}{k}+\frac{\pi}{k} = \frac{(v+1)\pi}{k} \le \frac{r\pi}{k} < h \le \frac{\pi}{2}$ and especially $x, x+\frac{\pi}{k} \in [0, \frac{\pi}{2}]$. Therefore both integrals have the same sign over $[\frac{(v-1)\pi}{k}, \frac{v\pi}{k}]$. Since the second integral has a minus sign in front of it, they in fact have a different sign.

Each integral is also smaller then the one before in absolute value. This can be proved in the following way. By definition f is a decreasing function and therefore

$$f(x) \ge f\left(x+\frac{\pi}{k}\right).$$

On the other hand since $x, x+\frac{\pi}{k} \in [0, \frac{\pi}{2}]$

$$\sin x < \sin\left(x+\frac{\pi}{k}\right).$$

From these two inequality's it follows that

$$\frac{f(x)}{\sin x} > \frac{f\left(x+\frac{\pi}{k}\right)}{\sin\left(x+\frac{\pi}{k}\right)}.$$

This holds for all values of x between the limits $(v-1)\frac{\pi}{k}$ and $v\frac{\pi}{k}$. Which shows that each integral is smaller then the one before in absolute value.

□

Lemma 4.4. *As $m \to \infty$, over the natural numbers, then*

$$\lim_{m\to\infty} \int_{m\pi}^{(m+1)\pi} \frac{\sin x}{x}\, dx = 0.$$

Proof.

$$\left|\int_{m\pi}^{(m+1)\pi} \frac{\sin x}{x}\,dx - 0\right| \le \int_{m\pi}^{(m+1)\pi} \left|\frac{\sin x}{x}\right| dx \le \int_{m\pi}^{(m+1)\pi} \frac{1}{x}\,dx$$
$$\le \int_{m\pi}^{(m+1)\pi} \frac{1}{m\pi}\,dx = \Big/_{m\pi}^{(m+1)\pi} \frac{x}{m\pi} = \frac{(m+1)\pi - m\pi}{m\pi}$$
$$= \frac{1}{m} \to 0 \quad \text{when} \quad m \to \infty.$$

□

Lemma 4.5. *: Let's assume that $v \in \mathbb{N}_1$. As $k \to \infty$, over the natural numbers, then*

$$\lim_{k\to\infty} \int_{(v-1)\pi/k}^{v\pi/k} \frac{\sin(kx)}{\sin x}\,dx = \int_{(v-1)\pi}^{v\pi} \frac{\sin y}{y}\,dy.$$

Proof. It is clear that $x \in [0, \infty)$. We begin by examining the integral

$$\int_{(v-1)\pi/k}^{v\pi/k} \frac{\sin(kx)}{\sin x}\,dx.$$

Let's replace x by y/k, where y is the new variable. The integral then changes into[2]

$$\int_{(v-1)\pi}^{v\pi} \frac{\sin y}{k \sin\left(\frac{y}{k}\right)}\,dy.$$

Now we only need to prove:

$$\lim_{k\to\infty} \int_{(v-1)\pi}^{v\pi} \frac{\sin y}{k \sin\left(\frac{y}{k}\right)}\,dy = \int_{(v-1)\pi}^{v\pi} \frac{\sin y}{y}\,dy.$$

Using Theorem 3.16 we only need to prove that $\frac{\sin y}{k\sin\left(\frac{y}{k}\right)}$ converges uniformly to $\frac{\sin y}{y}$. Let's use the definition of uniform convergence 3.15 to do this. We start by examining the following expression using Lemma 4.2 and the knowledge that $\frac{y}{k} \ge 0$

[2]Dirichlet did the substitution given here in his paper but left the examination of the limit for the reader.

$$\left|\frac{\sin y}{k\sin\left(\frac{y}{k}\right)}-\frac{\sin y}{y}\right| = \left|\frac{\sin y}{y}\right|\left|\frac{y-k\sin\left(\frac{y}{k}\right)}{k\sin\left(\frac{y}{k}\right)}\right| \leq \left|\frac{y-k\sin\left(\frac{y}{k}\right)}{k\sin\left(\frac{y}{k}\right)}\right|$$
$$= \left|\frac{\frac{y}{k}-\sin\left(\frac{y}{k}\right)}{\sin\left(\frac{y}{k}\right)}\right| \leq \frac{\left(\frac{y}{k}\right)^3}{|\sin\left(\frac{y}{k}\right)|}.$$

Lemma 4.2 also tells us that for all $0 \leq x \leq 1$ we have $\frac{x}{2} \leq \sin x \leq x$. We know that $(v-1)\pi \leq y \leq v\pi$. Let us choose $k_\varepsilon \in \mathbb{N}$ so that $k_\varepsilon \geq v\pi$. Now when $k > k_\varepsilon$ then $\frac{y}{k} \leq \frac{v\pi}{k} < \frac{v\pi}{v\pi} = 1$. Hence $0 \leq \frac{y}{k} \leq 1$ and we get $\sin\left(\frac{y}{k}\right) \geq \frac{y}{2k}$ but also that $|\sin\left(\frac{y}{k}\right)| = \sin\left(\frac{y}{k}\right)$. Now

$$\frac{\left(\frac{y}{k}\right)^3}{|\sin\left(\frac{y}{k}\right)|} = \frac{\left(\frac{y}{k}\right)^3}{\sin\left(\frac{y}{k}\right)} \leq \frac{\left(\frac{y}{k}\right)^3}{\left(\frac{y}{2k}\right)} = 2\left(\frac{y}{k}\right)^2.$$

Let us further assume that $k_\varepsilon \geq v\pi\sqrt{\frac{2}{\varepsilon}}$. Now when $k > k_\varepsilon$ we get

$$2\left(\frac{y}{k}\right)^2 \leq 2\left(\frac{v\pi}{k}\right)^2 < 2\left(\frac{v\pi}{v\pi\sqrt{\frac{2}{\varepsilon}}}\right)^2 = 2\left(\sqrt{\frac{\varepsilon}{2}}\right)^2 = \varepsilon.$$

We have now proved that the convergence is uniform since using definition 3.15 for every $\varepsilon > 0$ we can choose $k_\varepsilon = \max[v\pi, v\pi\sqrt{\frac{2}{\varepsilon}}]$ and then

$$\sup_{x\in[0,\infty)}\left|\frac{\sin y}{k\sin\left(\frac{y}{k}\right)}-\frac{\sin y}{y}\right| < \varepsilon,$$

when $k > k_\varepsilon$.

□

Lemma 4.6. *Let's assume that for $v, k \in \mathbb{N}$, $z_{v_k} \in [(v-1)\frac{\pi}{k}, v\frac{\pi}{k}]$ and $h \in (0, \frac{\pi}{2}]$. Assume that $f : [0,h] \to \mathbb{R}$ and f is continuous in $[0,h]$. Then for any fixed $v \geq 1$:*

$$\lim_{k\to\infty} f(z_{v_k}) = f(0)$$

Proof. Clearly $\lim_{k\to\infty}(v-1)\frac{\pi}{k} = 0$ and $\lim_{k\to\infty} v\frac{\pi}{k} = 0$. Using Theorem 3.1 we get that $\lim_{k\to\infty} z_{v_k} = 0$. Since $\lim_{k\to\infty} z_{v_k} = 0$ we can use Lemma 4.1 and get that $\lim_{k\to\infty} f(z_{v_k}) = f(0)$.

□

Lemma 4.7. *Let's assume that $k \in \mathbb{N}$. We have*

$$\cos(kx)\int_{-\pi}^{\pi} f(y)\cos(ky)\,dy + \sin(kx)\int_{-\pi}^{\pi} f(y)\sin(ky)\,dy = \int_{-\pi}^{\pi} f(y)\cos[k(y-x)]\,dy.$$

Proof. Rearranging terms and using Lemma 5.1

$$\begin{aligned}
&\cos(kx)\int_{-\pi}^{\pi} f(y)\cos(ky)\,dy + \sin(kx)\int_{-\pi}^{\pi} f(y)\sin(ky)\,dy \\
&= \int_{-\pi}^{\pi} \cos(kx)f(y)\cos(ky)\,dy + \int_{-\pi}^{\pi} \sin(kx)f(y)\sin(ky)\,dy \\
&= \int_{-\pi}^{\pi} \cos(kx)f(y)\cos(ky)\,dy + \sin(kx)f(y)\sin ky\,dy \\
&= \int_{-\pi}^{\pi} f(y)[\cos(kx)\cos(ky)\,dy + \sin(kx)\sin(ky)]\,dy \\
&= \int_{-\pi}^{\pi} f(y)\cos[k(y-x)]\,dy.
\end{aligned}$$

□

Chapter 5

Computation of the integral $\int_0^\infty \frac{\sin x}{x}\,dx$

5.1 Computations

This chapter will go through an elementary proof of Lemma $\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$ and some Lemmas for the convenience of the proof. Dirichlet used the Lemma but omitted the proof in his own paper.

Lemma 5.1. *For* $x \in \mathbb{R}$*.*

(a) $\sin x = \sin(\pi - x) = \sin(x + 2n\pi)$

(b) $\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$

(c) $\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

Proof. Omitted. □

Lemma 5.2. *For* $x \in \mathbb{R}$*.*

$$\frac{\sin(2n+1)x}{\sin x} = 1 + 2\sum_{k=1}^{n} \cos(2kx)$$

Proof. Lemma 5.1 gives us the following identities

$$\begin{cases} \sin(a+b) = \sin a \cos b + \cos a \sin b \\ \sin(a-b) = \sin a \cos b - \cos a \sin b. \end{cases}$$

Subtracting the second one from the first gives us

$$\sin(a+b) - \sin(a-b) = 2\cos a \sin b.$$

Set $a = 2kx$ and $b = x$ and obtain

$$\sin(2k+1)x - \sin(2k-1)x = 2\cos(2kx)\sin x.$$

When summing these from $k = 1$ to $k = n$ most of the terms on the left hand side will cancel out and we get

$$\sin(2n+1)x - \sin x = 2\sin x \sum_{k=1}^{n} \cos(2kx).$$

Then $\sin x$ can be added to both sides

$$\sin(2n+1)x = \sin x(1 + 2\sum_{k=1}^{n} \cos(2kx)).$$

Dividing by $\sin x$ we get

$$\frac{\sin(2n+1)x}{\sin x} = 1 + 2\sum_{k=1}^{n} \cos(2kx).$$

□

Lemma 5.3.

$$\lim_{n\to\infty} \int_0^{n\pi/(2n+1)} \frac{\sin(2n+1)x}{\sin x}\, dx = \frac{\pi}{2}$$

Proof. Using Lemma 5.2 we have that

$$\int_0^{\pi/2} \frac{\sin(2n+1)x}{\sin x}\, dx = \int_0^{\pi/2} \left(1 + 2\sum_{k=1}^{n} \cos(2kx)\right) dx = \frac{\pi}{2}.$$

Let's examine the difference with the alleged limit. Notice that $\frac{n\pi}{2n+1} \geq \frac{\pi}{3}$. It's easy to see that when $\frac{\pi}{3} \leq x \leq \frac{\pi}{2}$ then $\sin x \geq \frac{1}{2}$. Now

$$\begin{aligned}
&\left| \int_0^{n\pi/(2n+1)} \frac{\sin(2n+1)x}{\sin x}\, dx - \frac{\pi}{2} \right| \\
=&\left| \int_0^{\pi/2} \frac{\sin(2n+1)x}{\sin x}\, dx - \frac{\pi}{2} - \int_{n\pi/(2n+1)}^{\pi/2} \frac{\sin(2n+1)x}{\sin x}\, dx \right| \\
=&\left| \int_{n\pi/(2n+1)}^{\pi/2} \frac{\sin(2n+1)x}{\sin x}\, dx \right| \leq \int_{n\pi/(2n+1)}^{\pi/2} \left| \frac{\sin(2n+1)x}{\sin x} \right| dx \\
\leq& \int_{n\pi/(2n+1)}^{\pi/2} \frac{1}{\frac{1}{2}}\, dx \leq 2\left(\frac{\pi}{2} - \frac{n\pi}{2n+1}\right) = 2\left(\frac{2\pi n + \pi - 2\pi n}{2(2n+1)}\right) \\
=&\frac{\pi}{2n+1} \to 0 \quad \text{when} \quad n \to \infty.
\end{aligned}$$

□

Lemma 5.4.

$$\int_0^\infty \frac{\sin y}{y}\, dy = \lim_{M\to\infty} \int_0^M \frac{\sin y}{y}\, dy = \frac{\pi}{2}$$

Proof. Let's denote

$$I_n = \int_0^{n\pi/(2n+1)} \frac{\sin(2n+1)x}{\sin x}\, dx = \sum_{k=1}^{n} \int_{(k-1)\pi/(2n+1)}^{k\pi/(2n+1)} \frac{\sin(2n+1)x}{\sin x}\, dx.$$

From Lemma 4.3 we know that I_n is a decreasing sequence of alternating terms. With the help of Lemma 5.3 this converges to $\frac{\pi}{2}$. Now for subsequent partial sums

$$\int_0^{2m\pi/(2n+1)} \frac{\sin(2n+1)x}{\sin x}\, dx < I_n < \int_0^{(2m+1)\pi/(2n+1)} \frac{\sin(2n+1)x}{\sin x}\, dx$$

for any such m that $2m+1 \leq n$. With the help of Lemma 4.5 when taking the limit $\lim_{n\to\infty}$ we get

$$\int_0^{2m\pi} \frac{\sin y}{y}\, dy \leq \frac{\pi}{2} \leq \int_0^{(2m+1)\pi} \frac{\sin y}{y}\, dy.$$

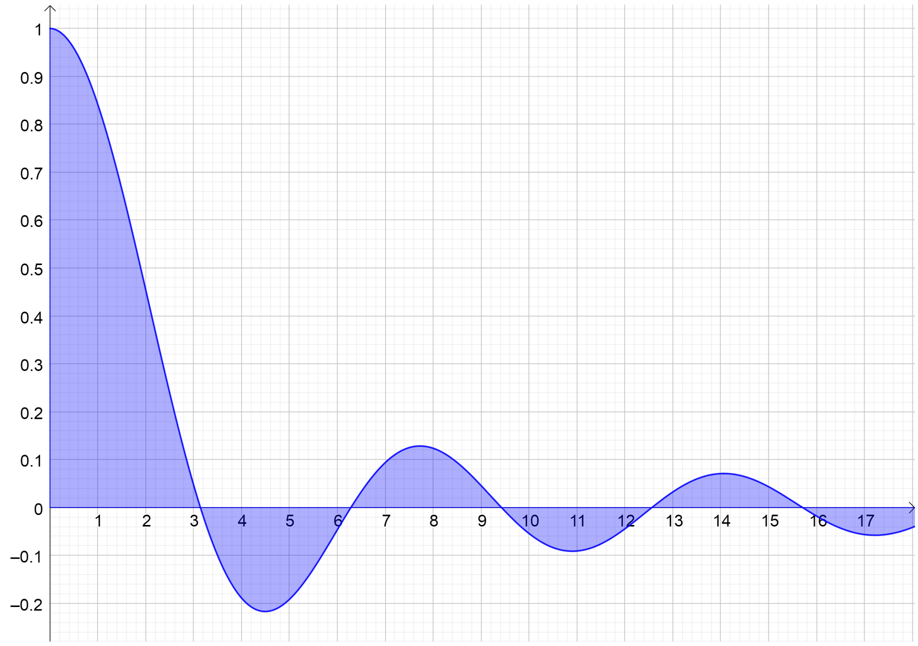

Figure 5.1: The graph of the function $\frac{\sin x}{x}$.

Let's examine the difference of the integrals

$$\begin{aligned}\int_0^{(2m+1)\pi} \frac{\sin y}{y}\,dy - \int_0^{2m\pi} \frac{\sin y}{y}\,dy = \int_{2m\pi}^{(2m+1)\pi} \frac{\sin y}{y}\,dy &\le \int_{2m\pi}^{(2m+1)\pi} \frac{1}{y}\,dy \\ &\le \frac{1}{2m\pi}[(2m+1)\pi - 2m\pi] \\ &= \frac{\pi}{2m\pi} = \frac{1}{2m} \to 0 \quad \text{when} \quad m \to \infty.\end{aligned}$$

Therefore

$$\lim_{m\to\infty} \int_0^{(2m+1)\pi} \frac{\sin y}{y}\,dy = \lim_{m\to\infty} \int_0^{2m\pi} \frac{\sin y}{y}\,dy = \frac{\pi}{2}.$$

Combining these we have

$$\lim_{m\to\infty} \int_0^{m\pi} \frac{\sin y}{y}\,dy = \frac{\pi}{2}.$$

Let's assume that $M > 0$ and choose such m, that $m\pi < M \leq (m+1)\pi$. Now

$$\begin{aligned}\int_0^M \frac{\sin y}{y}\,dy - \int_0^{m\pi} \frac{\sin y}{y}\,dy &= \int_{m\pi}^M \frac{\sin y}{y}\,dy \leq \int_{m\pi}^M \frac{1}{y}\,dy \\ &\leq \frac{1}{m\pi}[M - m\pi] \leq \frac{1}{m\pi}[(m+1)\pi - m\pi] \\ &= \frac{\pi}{m\pi} = \frac{1}{m} \to 0 \quad \text{when} \quad M \to \infty\end{aligned}$$

Therefore

$$\int_0^\infty \frac{\sin y}{y}\,dy = \lim_{M\to\infty} \int_0^M \frac{\sin y}{y}\,dy = \frac{\pi}{2}.$$

□

Chapter 6

Dirichlet's proof

6.1 Chapter introduction

In this chapter we follow Dirichlet's original proof from 1829. For convenience reasons this has been divided into 2 parts. First we go through some Theorems and Lemmas and stack them all as Theorem 6.4. In the second part we will use it to prove the main result of Dirichlet's paper, which is our Theorem 6.5.

6.2 Part 1

Theorem 6.1. *Let $h \in (0, \frac{\pi}{2}]$ and let $f : [0,h] \to \mathbb{R}$ be continuous. Assume that the f is positive and decreasing* [1]*. Then the following holds (when $k \to \infty$ over the natural numbers)*

$$\lim_{k\to\infty} \int_0^h \frac{\sin(kx)}{\sin x} f(x)\,dx = \frac{\pi}{2} f(0).$$

Proof. We begin by dividing the integral

$$\int_0^h \frac{\sin(kx)}{\sin x} f(x)\,dx = \int_0^{\pi/k} \frac{\sin(kx)}{\sin x} f(x)\,dx + \int_{\pi/k}^{2\pi/k} \frac{\sin(kx)}{\sin x} f(x)\,dx + ...$$
$$+ \int_{(r-1)\pi/k}^{r\pi/k} \frac{\sin(kx)}{\sin x} f(x)\,dx + \int_{r\pi/k}^{h} \frac{\sin(kx)}{\sin x} f(x)\,dx,$$

[1]Here Dirichlet used the assumption that the function was strictly decreasing and later noted that strictly part wasn't needed at any point.

where r is the greatest integer for which $\frac{r\pi}{k} < h$. Therefore we have $r+1$ new integrals. From Lemma 4.3 and using analogous deduction for the penultimate and last integral we know that the series is alternating, first term being positive and each term is smaller then the one before in absolute value.

We can now simplify the v:th integral, which is:

$$\int_{(v-1)\pi/k}^{v\pi/k} \frac{\sin(kx)}{\sin x} f(x)\, dx.$$

Let us denote $h(x) = \frac{\sin kx}{\sin x}$. By definition $f : [(v-1)\frac{\pi}{k}, v\frac{\pi}{k}] \to \mathbb{R}$ is a continuous function. $h : [(v-1)\frac{\pi}{k}, v\frac{\pi}{k}] \to \mathbb{R}$ is also continuous and hence it's integrable and also always non negative. Now we can use Theorem 3.14 and find a $z_{v_k} \in [(v-1)\frac{\pi}{k}, v\frac{\pi}{k}]$ for which

$$\int_{(v-1)\pi/k}^{v\pi/k} \frac{\sin(kx)}{\sin x} f(x)\, dx = f(z_{v_k}) \int_{(v-1)\pi/k}^{v\pi/k} \frac{\sin(kx)}{\sin x}\, dx.$$

Since function f is strictly decreasing then for all $x \in [(v-1)\frac{\pi}{k}, v\frac{\pi}{k}]$ it holds that $f(x) \in [f(v\frac{\pi}{k}), f((v-1)\frac{\pi}{k})]$ and therefore $f(z_{v_k}) \in [f(v\frac{\pi}{k}), f((v-1)\frac{\pi}{k})]$.

Using Lemma 4.5 we get that

$$\lim_{k\to\infty} \int_{(v-1)\pi/k}^{v\pi/k} \frac{\sin(kx)}{\sin x}\, dx = \int_{(v-1)\pi}^{v\pi} \frac{\sin y}{y}\, dy.$$

We know from Lemma 5.4 that integral $\int_0^\infty \frac{\sin x}{x}\, dx$ converges to $\frac{\pi}{2}$. This integral can be divided into an infinite sum of integrals, the first integral taken from $x=0$ to $x=\pi$, the second from $x=\pi$ to $x=2\pi$, and so on. Let's denote this infinite sequence by

$$S = \int_0^\pi \frac{\sin y}{y}\, dy - \int_\pi^{2\pi} \frac{\sin y}{y}\, dy + \int_{2\pi}^{3\pi} \frac{\sin y}{y}\, dy - \text{etc.}$$

Theorem 5.4 now tells us that S is convergent and has sum of $\frac{\pi}{2}$.

Let's now return to the limit

$$\lim_{k\to\infty} \int_0^h \frac{\sin(kx)}{\sin x} f(x)\, dx$$

and determine it's limit. Choose an even integer m and suppose that m remains fixed while k increases. The number $r+1$ (the number of integrals), which increases continually with k will eventually surpass the fixed number m, however large it is chosen.

This done we divide the integrals into two groups. First group will contain the first m integrals, the second group will contain the rest. The sum of the first group is

$$f(z_{1_k})\int_0^{\pi/k}\frac{\sin(kx)}{\sin x}\,dx-f(z_{2_k})\int_{\pi/k}^{2\pi/k}\frac{\sin(kx)}{\sin x}\,dx+\cdots-f(z_{m_k})\int_{(m-1)\pi/k}^{m\pi/k}\frac{\sin(kx)}{\sin x}\,dx.$$

The sum of the second group whose number of terms increases continually with k has for it's first terms

$$f(z_{(m+1)_k})\int_{m\pi/k}^{(m+1)\pi/k}\frac{\sin(kx)}{\sin x}\,dx-f(z_{(m+2)_k})\int_{(m+1)\pi/k}^{(m+2)\pi/k}\frac{\sin(kx)}{\sin x}\,dx+\cdots$$

Let's consider these two groups separately. Sum of first group is easy to determine. Using Lemma 4.6 we get that quantities $f(z_{1_k}), f(z_{2_k}), ..., f(z_{m_k})$ all converge $f(0)$ as, m remaining fixed, k increases without limit. On the other hand quantities

$$\int_0^{\pi/k}\frac{\sin(kx)}{\sin x}\,dx-\int_{\pi/k}^{2\pi/k}\frac{\sin(kx)}{\sin x}\,dx+\cdots-\int_{(m-1)\pi/k}^{m\pi/k}\frac{\sin(kx)}{\sin x}\,dx$$

converge in the same circumstances to the respective limits

$$\int_0^{\pi}\frac{\sin x}{x}\,dx-\int_{\pi}^{2\pi}\frac{\sin x}{x}\,dx+\cdots-\int_{(m-1)\pi}^{m\pi}\frac{\sin x}{x}\,dx.$$

Therefore the sum converges to the limit

$$\left(\int_0^{\pi}\frac{\sin x}{x}\,dx-\int_{\pi}^{2\pi}\frac{\sin x}{x}\,dx+\cdots-\int_{(m-1)\pi}^{m\pi}\frac{\sin x}{x}\,dx\right)f(0)=S_mf(0).$$

Since m could have been chosen to be as large as we want, we obtain

$$\lim_{m\to\infty}S_mf(0)=Sf(0)=\frac{\pi}{2}f(0).$$

Let's then consider the sum of the second group. The conditions of Theorem 3.20 apply and hence the sum of these terms is positive like it's first term

$$f(z_{(m+1)_k}) \int_{m\pi/k}^{(m+1)\pi/k} \frac{\sin(kx)}{\sin x}\, dx$$

and the sum has a value smaller then this term. The first term converges to the limit

$$f(0) \int_{m\pi}^{(m+1)\pi} \frac{\sin x}{x}\, dx.$$

Since m could have been chosen to be as large as we want, we next examine

$$\lim_{m\to\infty} f(0) \int_{m\pi}^{(m+1)\pi} \frac{\sin x}{x}\, dx.$$

Using Lemma 4.4 we see that this converges to

$$\lim_{m\to\infty} f(0) \int_{m\pi}^{(m+1)\pi} \frac{\sin x}{x}\, dx = f(0) \cdot 0 = 0.$$

Combining the sums of both groups, we get that

$$\lim_{k\to\infty} \int_0^h \frac{\sin(kx)}{\sin x} f(x)\, dx = \frac{\pi}{2} f(0).$$

□

Dirichlet then loosens the assumptions for function f from Theorem 6.1.

Lemma 6.2. *Let $h \in (0, \frac{\pi}{2}]$ and let $f : [0,h] \to \mathbb{R}$ be continuous. Assume one of the options.*

(a) The function f is constant. [2]

(b) The function f is decreasing.

[2]Here Dirichlet noted that the assumption of f being strictly decreasing was never needed in the previous theorem.

(c) The function f is increasing.

Then the following holds (when $k \to \infty$, over the natural numbers)

$$\lim_{k\to\infty} \int_0^h \frac{\sin(kx)}{\sin x} f(x)\, dx = \frac{\pi}{2} f(0).$$

Proof. (a) Let's first assume that $f(x) = 1$. Using Theorem 6.1 we have that

$$\lim_{k\to\infty} \int_0^h \frac{\sin(kx)}{\sin x}\, dx = \frac{\pi}{2}.$$

Let's then assume that $f(x) = c$, where $c \in \mathbb{R}$. Then

$$\begin{aligned}\lim_{k\to\infty} \int_0^h f(x) \frac{\sin(kx)}{\sin x}\, dx &= \lim_{k\to\infty} \int_0^h c \frac{\sin(kx)}{\sin x}\, dx = \lim_{k\to\infty} c \int_0^h \frac{\sin(kx)}{\sin x}\, dx \\ &= c\frac{\pi}{2} = f(x)\frac{\pi}{2}.\end{aligned}$$

(b) It is possible that $f(x) \leq 0$ for some $x \in [0, h]$. However we can always find a $c \in \mathbb{R}$ such that $c + f(x) > 0$ for all $x \in [0, h]$. Using Theorem 6.1 we get

$$\begin{aligned}\lim_{k\to\infty} \int_0^h \frac{\sin(kx)}{\sin x} f(x)\, dx &= \lim_{k\to\infty} \left[\int_0^h f(x) \frac{\sin(kx)}{\sin x}\, dx + \int_0^h c \frac{\sin(kx)}{\sin x}\, dx - \int_0^h c \frac{\sin(kx)}{\sin x}\, dx \right] \\ &= \lim_{k\to\infty} \left[\int_0^h (f(x) + c) \frac{\sin(kx)}{\sin x}\, dx - \int_0^h c \frac{\sin(kx)}{\sin x}\, dx \right] \\ &= \lim_{k\to\infty} \int_0^h (f(x) + c) \frac{\sin(kx)}{\sin x}\, dx - \lim_{k\to\infty} \int_0^h c \frac{\sin(kx)}{\sin x}\, dx \\ &= (f(0) + c)\frac{\pi}{2} - c\frac{\pi}{2} = \frac{\pi}{2} f(0).\end{aligned}$$

(c) If function $f : [0, h] \to \mathbb{R}$ is increasing, then $-f$ is a decreasing function. Now using Theorem 6.1 we get that

$$\lim_{k\to\infty} \int_0^h \frac{\sin(kx)}{\sin x} (-f(x))\, dx = -\frac{\pi}{2} f(0)$$

and consequently

$$\lim_{k\to\infty}\int_0^h \frac{\sin(kx)}{\sin x} f(x)\,dx = \frac{\pi}{2} f(0).$$

□

Dirichlet then compiles conditions from Lemma 6.2 to Lemma 6.3 and also considers the case when the integral doesn't start from 0.

Lemma 6.3. *Let $h \in (0, \frac{\pi}{2}]$ and let $f : [s, h] \to \mathbb{R}$ be continuous and either decreasing or increasing. Then the following statements hold (when $k \to \infty$, over the natural numbers)*

(a) if $s = 0$ then $\lim_{k\to\infty}\int_s^h \frac{\sin(kx)}{\sin x} f(x)\,dx = \frac{\pi}{2} f(0)$.

(b) if $s \in (0, h)$ then $\lim_{k\to\infty}\int_s^h \frac{\sin(kx)}{\sin x} f(x)\,dx = 0$.

Proof. (a) This follows from Lemma 6.2.

(b) We know that when $f : [0, h] \to \mathbb{R}$ is a continuous function that is either decreasing or increasing then

$$\lim_{k\to\infty}\int_0^h \frac{\sin(kx)}{\sin x} f(x)\,dx = \frac{\pi}{2} f(0).$$

Now if $s \in [0, h)$, then analogously

$$\lim_{k\to\infty}\int_0^s \frac{\sin(kx)}{\sin x} f(x)\,dx = \frac{\pi}{2} f(0).$$

Combining this information and we get

$$\begin{aligned}\lim_{k\to\infty}\int_s^h \frac{\sin(kx)}{\sin x} f(x)\,dx &= \lim_{k\to\infty}\left[\int_0^h \frac{\sin(kx)}{\sin x} f(x)\,dx - \int_0^s \frac{\sin(kx)}{\sin x} f(x)\,dx\right] \\ &= \frac{\pi}{2} f(0) - \frac{\pi}{2} f(0) = 0.\end{aligned}$$

□

Dirichlet then goes on to consider what would happen if the assumption of continuity was to be dropped at endpoints s and f. We would then allow $f(s^+) \neq f(s)$ and $f(h^-) \neq f(h)$. He then noted the statement of the following Theorem to be true but gave no proof.

Theorem 6.4. *Let $h \in (0, \frac{\pi}{2}]$, $s \in [0, h)$ and let $f : [s, h] \to \mathbb{R}$ be continuous and either decreasing or increasing in open interval (s, h) . Then the following statements hold (when $k \to \infty$, over the natural numbers)*

(a) if $s = 0$ then $\lim_{k\to\infty} \int_s^h \frac{\sin(kx)}{\sin x} f(x)\, dx = \frac{\pi}{2} f(0^+)$.

(b) if $s \in (0, h)$ then $\lim_{k\to\infty} \int_s^h \frac{\sin(kx)}{\sin x} f(x)\, dx = 0$.

Proof. Let's define a new function F as follows:

$$F(x) = \begin{cases} f(x), & \text{if } s < x < h \\ f(x^+), & \text{if } x = s \\ f(x^-), & \text{if } x = h \end{cases}$$

F is then a continuous function and can differ from f in only two points. The claim follows from 6.3.

□

6.3 Part 2

Until now Dirichlet's proof has focused on proving Theorem 6.4. With that proven we can now focus on Theorem 6.5 which is the main result of his paper.

Theorem 6.5. *Let's assume the following: Function $f : [-\pi, \pi] \to \mathbb{R}$ is bounded, piecewise continuous and piecewise monotonous*[3].

Then the Fourier series of f converges as follows:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\, dy + \frac{1}{\pi}\sum_{k=1}^{\infty}[\cos(kx)\int_{-\pi}^{\pi} f(y)\cos(ky)\, dy + \sin(kx)\int_{-\pi}^{\pi} f(y)\sin(ky)\, dy]$$
$$= \begin{cases} \frac{1}{2}[f(x^-) + f(x^+)], & \text{if } -\pi < x < \pi \\ \frac{1}{2}[f(\pi^-) + f(-\pi^+)], & \text{if } x = \pm\pi \end{cases}$$

[3]Dirichlet used the phrasing that the function is bounded, has a finite number of discontinuities and a finite number of local maxima and minima.

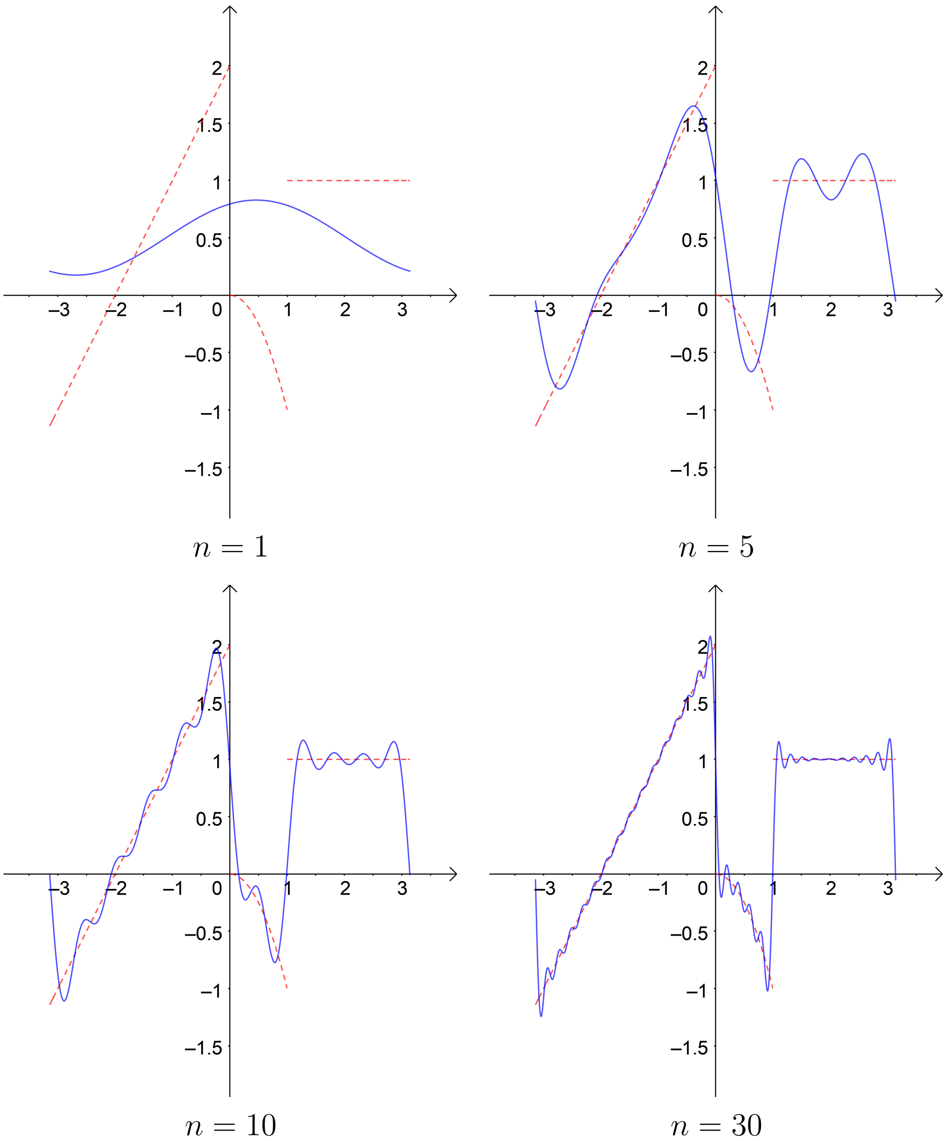

Figure 6.1: Graphs of

$$f(x) = \begin{cases} x+2, & \text{if } -\pi \leq x < 0 \\ -x^2, & \text{if } 0 \leq x < 1 \\ 1, & \text{if } 1 \leq x \leq \pi \end{cases}$$

(dashes) approximated by $n+1$ first terms of it's Fourier series (solid).

Proof. Let's assume that $x \in [-\pi, \pi]$ and $f : [-\pi, \pi] \to \mathbb{R}$. The Fourier series of f is now of the form

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\,dy + \frac{1}{\pi}\sum_{k=1}^{\infty}[\cos(kx)\int_{-\pi}^{\pi} f(y)\cos(ky)\,dy + \sin(kx)\int_{-\pi}^{\pi} f(y)\sin(ky)\,dy].$$

Let's now consider the $n+1$ first terms of this series[4]. They are

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\,dy + \frac{1}{\pi}\sum_{k=1}^{n}[\cos(kx)\int_{-\pi}^{\pi} f(y)\cos(ky)\,dy + \sin(kx)\int_{-\pi}^{\pi} f(y)\sin(ky)\,dy].$$

Using Lemma 4.7 we get that this equals

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\left(\frac{1}{2} + \sum_{k=1}^{n}\cos[k(y-x)]\right)\,dy$$

which with Lemma 5.2 becomes the same as

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\frac{\sin[(n+\frac{1}{2})(y-x)]}{2\sin\left[\frac{1}{2}(y-x)\right]}\,dy. \tag{6.6}$$

Let's omit the $\frac{1}{\pi}$ for now. The integral

$$\int_{-\pi}^{\pi} f(y)\frac{\sin[(n+\frac{1}{2})(y-x)]}{2\sin\left[\frac{1}{2}(y-x)\right]}\,dy$$

can be divided into 2 parts:

$$\int_{-\pi}^{x} f(y)\frac{\sin[(n+\frac{1}{2})(y-x)]}{2\sin\left[\frac{1}{2}(y-x)\right]}\,dy \quad \text{and} \quad \int_{x}^{\pi} f(y)\frac{\sin[(n+\frac{1}{2})(y-x)]}{2\sin\left[\frac{1}{2}(y-x)\right]}\,dy.$$

Let's create a new variable z such that $y = x - 2z$ in the first integral and $y = x + 2z$ in the second integral. The integrals now become

[4] Dirichlet used a different notation and therefore considered $2n+1$ first terms.

$$\int_0^{\frac{\pi+x}{2}} f(x-2z)\frac{\sin[(2n+1)z]}{\sin z}\,dz \quad \text{and} \quad \int_0^{\frac{\pi-x}{2}} f(x+2z)\frac{\sin[(2n+1)z]}{\sin z}\,dz.$$

Let's examine the second integral first

$$\int_0^{\frac{\pi-x}{2}} f(x+2z)\frac{\sin[(2n+1)z]}{\sin z}\,dz.$$

Note that $x+2z \in [-\pi,\pi]$.

1. If $x=\pi$, then $\frac{\pi-x}{2}=0$ and the integral just becomes 0.
2. If $0 \le x < \pi$, then $0 < \frac{\pi-x}{2} \le \frac{\pi}{2}$. It should be noted that function f can have several breaks in continuity and turning points with values $x+2z \in [-\pi,\pi]$

   Let's denote these points by $l, l', l'', ..., l^{(v)}$. The integral can now be composed to the following

$$\int_0^{l} f(x+2z)\frac{\sin[(2n+1)z]}{\sin z}\,dz + \int_l^{l'} f(x+2z)\frac{\sin[(2n+1)z]}{\sin z}\,dz + \cdots + \int_{l^{(v)}}^{\frac{\pi-x}{2}} f(x+2z)\frac{\sin[(2n+1)z]}{\sin z}\,dz$$

   Now using Theorem 6.4 these all converge to 0 when n increases except for the first one which converges to $\frac{\pi}{2}f(x^+)$

3. If $-\pi \le x < 0$

   (a) if $0 \le z < \frac{\pi}{2}$ then we have already handled it in part 2. This converges to $\frac{\pi}{2}f(x^+)$

   (b) if $\frac{\pi}{2} \le z \le \frac{\pi-x}{2}$, then let's replace z by $\pi-u$, u being a new variable. Now the integral becomes

$$\int_{\frac{\pi+x}{2}}^{\frac{\pi}{2}} f(x+2\pi-2u)\frac{\sin[(2n+1)(\pi-u)]}{\sin(\pi-u)}\,dz.$$

   Which using Lemma 5.1 becomes the same as

$$\int_{\frac{\pi+x}{2}}^{\frac{\pi}{2}} f(x+2\pi-2u)\frac{\sin[(2n+1)u]}{\sin u}\,dz.$$

This can be analogously decomposed to several others like the preceding one.

i. if $x \neq -\pi$, then all these integrals converge to 0

ii. if $x = -\pi$, then $\frac{\pi+x}{2} = 0$ and integral converges to $\frac{\pi}{2}f(\pi^-)$[5]

Summarizing all this we get that the second integral has the following values:

(a) if $x = \pi$, the value is 0

(b) if $x = -\pi$, the value is $\frac{\pi}{2}[f(\pi^-) + f(-\pi^+)]$

(c) if $-\pi < x < \pi$, the value is $\frac{\pi}{2}f(x^+)$

The first integral is entirely analogous to the second. Applying similar considerations we get the summary of integrals values:

(a) if $x = \pi$, the value is $\frac{\pi}{2}[f(\pi^-) + f(-\pi^+)]$

(b) if $x = -\pi$, the value is 0

(c) if $-\pi < x < \pi$, the value is $\frac{\pi}{2}f(x^-)$

Summing these we now we have that

$$\int_{-\pi}^{\pi} f(y)\frac{\sin[(n+\frac{1}{2})(y-x)]}{2\sin\left[\frac{1}{2}(y-x)\right]}\,dy = \begin{cases} \frac{\pi}{2}[f(x^-)+f(x^+)], & \text{if } -\pi < x < \pi \\ \frac{\pi}{2}[f(\pi^-)+f(-\pi^+)], & \text{if } x = \pm\pi \end{cases}$$

and hence

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(y)\frac{\sin[(n+\frac{1}{2})(y-x)]}{2\sin\left[\frac{1}{2}(y-x)\right]}\,dy = \begin{cases} \frac{1}{2}[f(x^-)+f(x^+)], & \text{if } -\pi < x < \pi \\ \frac{1}{2}[f(\pi^-)+f(-\pi^+)], & \text{if } x = \pm\pi \end{cases}$$

We have now proven that if a bounded function f, is piecewise continuous and piecewise monotonous between $[-\pi, \pi]$. Then the Fourier series of f is convergent and converges as stated.

□

[5]Lemma 6.4 works analogously when variable given to function has a minus sign.

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