Applying extreme value theory and tail risk measures to reduce portfolio losses

Petri Tötterman

Department of Finance and Statistics
Hanken School of Economics
Helsingfors

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Author:
Petri Tötterman

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Abstract:

Risks and returns of portfolios consisting of stock, bond, and commodity indices from, respectively, the U.S. and European markets, are analysed. Different risk measures and statistical distributions are utilised in the construction of the risk models, in purpose to assess the usefulness of the different models for risk–averse investors.

Models under investigation include the well-known Gaussian mean–variance model, the minimum–variance model, and the distribution mean in combination with Value–at–Risk (VaR) and expected shortfall (ES), calculated both from historical returns and from estimated generalised Pareto distribution (GPD) parameters.

The traditional Gaussian mean–variance model works well as a first estimate, but expected losses can be reduced utilising other risk measures. ES in particular appears to work consistently in combination with different confidence levels and distribution models. Moreover, calculations concerning ES are not noticeably more complicated, compared with other risk measures.

According to expectations, the portfolio risk is minimised when the minimum–variance model is applied. The simple strategy of holding equal fractions of all assets in the portfolio leads to absent returns in the best case, and in the worst case to disaster. The recommended strategy is to make a first estimate with the mean–variance model, and then adjust to investor preferences utilising mean–ES models.

Daily asset returns imply persistence of returns with relation to different risk measures. The different models have some forecasting power due to this persistence. However, GPD models in particular require an extensive amount of observations in order to produce reliable estimates.

All models are capable of reproducing the stock market rally of the 1990s and the current recession in their respective portfolio asset allocations. The optimised portfolio contained only stocks at the end of the nineties, whereas in early 2009 only bonds would be selected. Currently, precious metals constitute a noticeable share of the portfolio.

Utilising risk measures, particularly the ES, the possible losses of the investment portfolio can be reduced, however not eliminated. Required calculations are not complicated, and computer–based tools are available. The method described can be recommended for investors wishing to reduce their risk taking.

Keywords

Value at Risk, expected shortfall, generalised Pareto distribution, asset allocation, portfolio optimisation

Anything that can go wrong will go wrong.

(Murphy's Law)

Murphy was an optimist.

(O'Toole's commentary on Murphy's Law)

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1. Introduction

1.1. Portfolio selection

Risk and return are the essential criteria when choosing assets for an investment portfolio. Risk averse investors seek portfolios having a lower risk overall than risk taking investors, but all investors strive for the maximal return at their defined risk level. While return is easy to measure – be it a percentage or a cash amount – the definition of risk is more complicated.

Traditionally, portfolio analysis has been based on the mean–variance framework, where risk is defined as the variance of the portfolio return. Assuming returns follow a normal, or Gaussian[1], distribution, an optimal portfolio can be found with quite simple mathematics. However, the variance is indifferent between fluctuations on the positive and the negative sides of the expected return. Moreover, asset returns are frequently reported not to follow the Gaussian distribution, especially during periods of unstable financial markets.

Particularly, risk averse investors are assumed to be anxious about their portfolio not to loose value. This class of investors contains e.g. households, pension funds and foundations, being concerned about retaining their current level of wealth. Consequently, they should be interested in a risk measure putting emphasis on the *negative* tail of the asset return distribution. Luckily, such measures are in common use in the insurance industry.

Risk measures are commonly used in calculating solvency requirements. Perhaps the most featured measure is Value–at–Risk, mentioned in the New Basel Capital Accord (Basel Committee 2001) as one permitted measure of the capital requirements of a bank. Asset sensitivities against changes in volatility or interest rate, the so called *greeks*, are well known in fixed income and derivatives trading. However, in a portfolio containing e.g. stocks, greeks are of limited use.

Calculating the value of a risk measure frequently involves assumptions on the under-

[1]"Gaussian" will be used in this paper to denote the *statistical distribution,* with the purpose of freeing the word "normal" into more general use.

lying return distribution. Although the Gaussian distribution is well known and easily used in calculations, reality has often shown to disagree. Extreme value distributions put more emphasis on the tail area of the distributions and therefore are well known by the insurance industry. Unfortunately, the required mathematics is more complicated than with the Gaussian distribution.

A common household should not need to be troubled about the actual distribution of their asset returns. After all, a very powerful hedge can be achieved by diversifying. By investing its wealth in equal shares of different assets, a household can earn good return with reduced risk, yet avoiding fund managing and trading costs. Institutional investors, with appointed portfolio managers, are in a different situation. They have the motivation and the resources to benefit from the latest findings in their investment strategies, thereby justifying research projects in this field.

In summary, some portfolio selection principles are widely known and practised. Investors base their decisions on the expected return and risk of the outcome, frequently using the Gaussian mean–variance ratio as a measure of goodness. Moreover, investors are more anxious for loosing than for gaining, thus emphasising the importance of modelling expected losses properly. Asset returns do not follow the Gaussian distribution in non–normal times. Instead, returns exhibit skewness and kurtosis, and their volatility is clustered[2]. The mean–variance model is particularly inappropriate when the portfolio contains derivative instruments. Misspecified models may lead to decisions which cause inappropriate capital requirements and unintentional investment risk.

However, many more issues are still waiting for discovery. Although the principles of coherence are established, it is not well known, which risk measures are good and useful, let alone which one is the best, in a portfolio framework. Better measures of portfolio downside risk may exist, but are yet unknown or untested. New data, reflecting the ever–changing nature of the financial markets, may disclose further strengths and weaknesses in different models.

In this study, some well–known but so far infrequently utilised methods are applied in portfolio selection. Particularly, the Gaussian mean–variance approach is supplemented with tail risk measures, based on an extreme value distribution model.

[2]Volatility clustering implies, that large price changes are frequently followed by additional large price changes – either in the same or in the opposite direction. Similarly, small price changes are followed by further small price changes.

1.2. Contribution

This study contributes to the empirical research on the application of different statistical distributions and risk measures in the portfolio asset allocation problem. Previous research has rarely been conducted in a portfolio framework and has usually covered only U. S. market data.

The main contribution of this study is to complement the pioneering work of Ho et al. (2008) with some extensions. First, the less allocated NASDAQ index is replaced with a commodity asset, potentially leading to better diversification, and hence broader allocation of the portfolio assets. Second, the 3–month U.S. Treasury Bill prices are added, hence introducing a risk–free asset in the calculations. Third, the observed time period is extended from November, 1983, to December, 2009. Fourth, tests out of sample are added in order to estimate the forecasting power of the different asset allocation strategies. Fifth, comparable analysis with European assets is conducted.

Describing the expected shortfall and an applicable extreme value distribution in an accessible way is likely to be appreciated by practitioners, and may well be considered an additional benefit of this study.

1.3. Purpose of study

The main purpose of this study is to numerically compare the performance of different statistical models in the asset allocation problem. Model classification is based on the performance of portfolios, selected by the models. Both in–sample and out–of–sample tests are included in the study.

1.4. Limitations

The study is empirical by its nature, and aiming to analyse the past course of financial market events. Only the Gaussian, the historical, and the generalised Pareto distribution are analysed. Additionally, the risk measures are limited to the variance, Value–at–Risk (VaR) and expected shortfall (ES). Due to limited computational resources, portfolios of only three assets can be tested.

Investigated models are intended for measuring undiversifiable market risk. Thus, the analysis is concentrated on model risk and undiversifiable market risk. Other types of financial risk include e. g. credit risk and system risk, but these are not discussed.

Uncountable features can be calculated from portfolio returns. In this work, focus is intentionally on the risk measures. Hence, no trading strategies will be developed.

Portfolios are formed not from basic assets as stocks and bonds, but from indices on basic assets. In this way, the analysed portfolios can always be considered well diversified, and only undiversifiable market risk is left for treatment. Moreover, price index data are easily obtained.

Although very interesting, derivative assets and hedge funds are not included in the investigation, mainly due to difficulties in obtaining data.

1.5. Structure

In the second chapter, the theoretic foundations are laid. Different distribution models, risk measures, and their corresponding optimal portfolio definitions are presented. Several pictures are added for clarification.

Previous research on the topic is presented in Chapter 3. Starting with a theoretical paper, Artzner et al. (1999) define coherence for risk measures. Value-at-Risk (VaR) and Expected Shortfall (ES) with different distributions are investigated empirically in the papers of Harmantzis et al. (2006) and Bali & Theodossiou (2008). The research of Ho et al. (2008) is otherwise similar, but conducted in a portfolio framework.

A layout on the research methods and data is presented in Chapter 4. Descriptive statistics on the research data, with comparisons to previous work, are included in this chapter. Moreover, the reasoning behind and the practical arrangements used in the empirical study are described.

Chapter 5 contains the results of the empirical study. First, a study similar to Ho et al. (2008) is made, and the results are compared to ensure model compatibility. Then, the observed time period is extended to reach from January, 1982, to November, 2009, a commodity asset and a risk–free asset are introduced, and tests are extended out of sample. Eventually, a comparing study using European indices is conducted. In all studies, portfolios are formed according to the different criteria, and their performance is measured and compared.

In Chapter 6, conclusions are drawn, and suggestions for further research are laid. Areas of interest might include connecting different utility models with risk measures; finding useful rules–of–thumb for confidence levels with differing sample sizes; and analysis of mutual funds, hedge funds, and derivative portfolios.

2. Mathematics behind portfolio models

2.1. Modelling in general

Almost invariably, portfolio models are based on return and risk. A commonly used measure for return is the periodical change in the asset's value, usually expressed as the logarithmic change of price P from time $t-1$ to time t: $r_t = ln\left(\frac{P_t}{P_{t-1}}\right)$.

For the portfolio risk, a statistical distribution describing the returns is assumed. Risk is then defined as the dispersion measure of the distribution. Assuming that the investor has positive marginal utility, decreasing absolute risk aversion, and consistent preferences (Scott & Horvath 1980), expected utility can be approximated from a Taylor expansion around the expected wealth (Bali 2007):

$$\mathbb{E}\left[U\left(W\right)\right] \approx U\left(\bar{W}\right) + \frac{1}{2}U^{(2)}\left(\bar{W}\right)\sigma^2 + \frac{1}{3!}U^{(3)}\left(\bar{W}\right)s^3 + \frac{1}{4!}U^{(4)}\left(\bar{W}\right)\kappa^4, \tag{2.1}$$

where σ^2 denotes variance, s^3 skewness, and κ^4 kurtosis. It can be seen from equation 2.1, that expected utility increases with higher values of mean and skewness, but decreases with higher values of variance and kurtosis. A good measure for risk should reflect this property.

Theoretically, equation 2.1 could be applied in portfolio optimisation, assuming that the investor´s utility function is known. However, this assumption is not likely to be fulfilled. Utility models are usually too simplistic to realistically express investor behaviour in all situations. Most models, suggested in textbooks, would overemphasise the higher order terms of equation 2.1. Hence, models become over–sensitive on noise in market data.

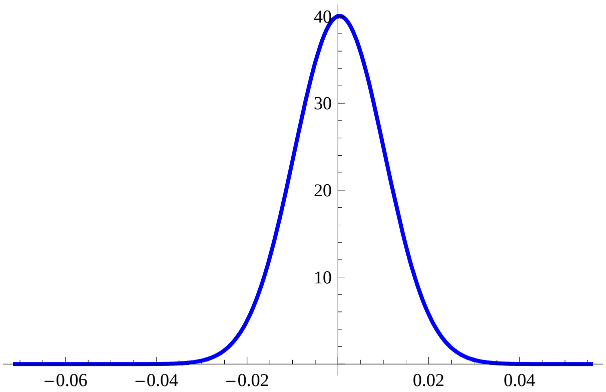

Figure 2.1.: The probability density function of a normal distribution. Here, log returns of the S&P 500 index between January 6, 1992 and October 5, 2007 are assumed to follow normal distribution.

2.2. Statistical distributions

2.2.1. Gaussian distribution

A Gaussian, or normal distribution in a variate x with mean μ and variance σ^2 is a statistic distribution with probability density function

$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)} \tag{2.2}$$

on the domain $x \in (-\infty, \infty)$. Graphically, the probability density function is shown in figure 2.1.

Gaussian distributions have many convenient properties, so random variates with unknown distributions are often assumed to be Gaussian, especially in physics and astronomy. Although this can be a dangerous assumption, it is often a good approximation due to the central limit theorem, stating that the mean of any set of variates with any distribution having a finite mean and variance tends to the Gaussian distribution .

As a Gaussian distribution is precisely defined by only two parameters, its mean μ and variance σ^2, it is easy to leverage for estimates and prognoses. However, many studies have shown that asset returns are not Gaussian distributed. Instead, they possess thick tails, i.e. extremely high and low values are more common than the Gaussian distribution would indicate.

Thick tails, peakedness, and skewness are mainly due to the documented (Mandelbrot 1963) volatility clustering, i. e. time varying volatility of asset returns. Large price

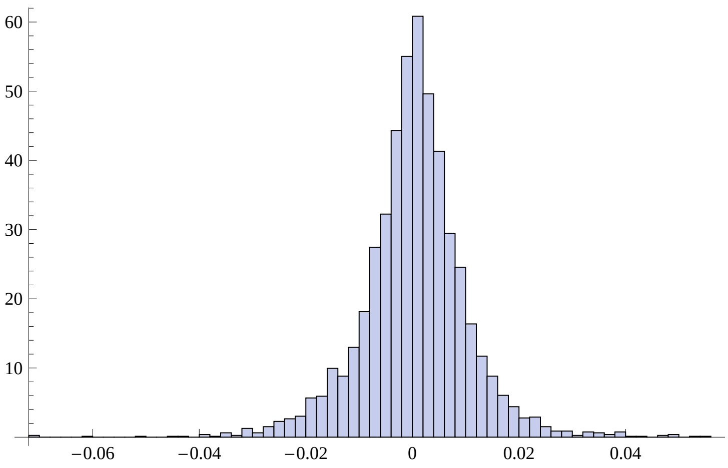

Figure 2.2.: The historical distribution of log returns of the S&P 500 index between January 6, 1992 and October 5, 2007.

changes are followed by additional large price changes, due to different interpretations of news among investors. Similarly, small price changes are followed by further small price changes.

Consequently, the assumption of Gaussian distributed returns works well in normal market situations, but underestimates risk at harsher times. A risk–averse investor would therefore prefer a model giving more emphasis to the extreme values, particularly losses.

2.2.2. Historical distribution

The historical, or empirical distribution is, obviously, a collection of historical asset returns. Countless databases of asset prices and returns exist, and at least for exchange traded assets there are no difficulties in finding adequate numbers. An example of a historical distribution is displayed in figure 2.2.

Historical distributions usually give a very accurate picture of returns in the past. However, past events can not predict possible future outcomes. Moreover, the lack of defining parameters renders the use of the historical distribution inconvenient.

Even worse, risk measure calculations require a large amount of data to be reliable. For instance, to calculate the expected shortfall with a confidence level of 99 %, 2 000 historical values would be needed to obtain a collection of 20 values forming the demanded 1 % of the sample. A small sample obviously leads to low accuracy of estimates.

2.2.3. Extreme value distributions

Low density regions of a distribution are known as the "tails" of the distribution. One reason why a model might fit poorly in the tails is that by definition, there are fewer data in the tails on which to base a choice of model, and so models are often chosen based on their ability to fit data near the centre of the model. Another reason might be that the distribution of real data is often more complicated than the usual parametric models.

In many applications, fitting the data in the tail is the main concern. Various alternative distributions have been suggested in order to describe the tails. Particularly, the insurance industry utilises extreme value distributions focusing on the tails, and flexible distributions expressing skewness and thick tails (Bali & Theodossiou 2008).

Extreme values can be modelled either by modelling the maxima of a collection of random variables, or by modelling the largest values over some high threshold. With the latter approach, supposing that $X_1 \dots X_n$ are n independent realisations of a random variable X with a distribution function $F(x)$, the distribution function of the excesses X_i over a certain high threshold U is given by

$$F_U(y) = \mathbb{P}\left[X - U \leq y \mid X > U\right] = \frac{F(U+y) - F(U)}{1 - F(U)}, y \geq 0 \tag{2.3}$$

Pickands (1975) theorem shows, that the *generalised Pareto distribution* (GPD) is the limiting distribution for $F_U(y)$, if U is a high threshold (Neftci 2000).

2.2.4. Generalised Pareto distribution

The GPD is a right-skewed distribution, parameterized with a shape parameter $\xi \neq 0$, and a scale parameter ϕ. It is often used in conjunction with a third threshold (or location) parameter u that shifts the lower limit away from zero. The shape parameter ξ is also called the tail index, and reflects the weight of the tail of the distribution. Moreover, the scale ϕ represents the volatility, whereas location u represents the average of the extremes (Bali 2003).

The probability density function in a variate x for the GPD with shape parameter $\xi \neq 0$, scale parameter ϕ, and threshold parameter u, is

$$P_{GPD}(x) = \left(\frac{1}{\phi}\right)\left[1 + \xi\left(\frac{x-u}{\phi}\right)\right]^{-\frac{1+\xi}{\xi}} \tag{2.4}$$

for $u < x$ when $\xi > 0$, or for $u < x < -\frac{\phi}{\xi}$ when $\xi < 0$.

The GPD nests the standard Pareto distribution when $\xi > 0$, the uniform distribution on $[-1, 0]$ when $\xi < 0$, and the standard exponential distribution when $\xi = 0$. Several recent research papers have utilised the GPD for modelling distribution tails, among them Harmantzis et al. (2006), Ho et al. (2008) and Bali & Theodossiou (2008).

One approach to distribution fitting involving the GPD is to use a non-parametric fit (the empirical cumulative distribution function, for example) in regions where there are many observations, and to fit the GPD to the tails of the data.

Estimating GPD parameters is generally not elementary, and several methods have been suggested, e.g. maximum-likelihood estimators, method-of-moments, and probability-weighted moments. Some computation algorithms require other parameters values to stay within defined limits — without any other reason — to allow the estimated parameter value to converge. This problem arises particularly with the maximum likelihood methods.

The α–quantile is frequently used as threshold value. Examples of parameter estimation methods can be found e.g. in Castillo & Hadli (1997) or Peng & Welsh (2001).

2.2.5. Other extreme value distributions

In a similar way, the *generalised extreme value distribution* (GEV) proposed by Jenkinson (1955) models extremes, i.e. maxima and minima, of independent and identically distributed (i. i. d.) random variables. GEV nests the three standard extreme value distributions – Frechet, Weibull, and Gumbel – by applying different values on ξ. However, estimates based on GEV tend to be less reliable, as real asset returns are usually not i. i. d. Additionally, as only extremes are used to model the GEV, information from the other tail values are lost (Bali 2007).

An even more general extreme value distribution is the *Box–Cox–GEV* (Box & Cox 1964) distribution, nesting the GEV and the GPD with an additional "Box–Cox GEV" parameter λ, having the probability density function

$$P_{BC}(x) = \left(\frac{1}{\phi}\right) \left[1 + \xi \left(\frac{x-u}{\phi}\right)\right]^{-\left(\frac{1+\xi}{\xi}\right)} \left\{ \exp\left[-\left(1 + \xi\left(\frac{x-u}{\phi}\right)\right)\right]^{-\frac{1}{\xi}} \right\}^{\lambda} \tag{2.5}$$

In other words, GEV (when $\lambda = 1$) and GPD (when $\lambda = 0$) are special cases of the Box–Cox–GEV distribution; the Frechet, Weibull, and Gumbel distributions are special cases of the GEV; and the standard Pareto, standard uniform, and standard exponential distributions are special cases of the GPD.

Obviously, the additional λ parameter increases the accuracy of the Box–Cox–GEV,

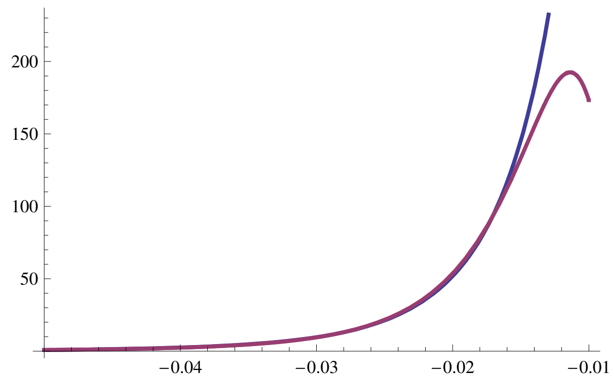

Figure 2.3.: Log returns from the S&P 500 index from January 4, 1950 to December 29, 2000, fitting values to GPD (blue line) with parameters $u = 0.01362$, $\phi = 0.00503$, $\xi = 0.1701$ and Box–Cox–GEV (red line) with parameters $u = 0.01485$, $\phi = 0.00440$, $\xi = 0.2083$, and $\lambda = 0.51$. Values are from Bali & Theodossiou (2008), table 2 on p. 421.

compared with GEV and GPD. In practice, however, parameter estimates become more tedious to obtain. Figure 2.3 illustrates the GPD and the Box–Cox–GEV.

2.2.6. Stable Paretian and other distributions

Mandelbrot (1963) suggested modelling asset returns with the stable Paretian distribution, which he claimed to produce good estimates for returns in wool production. However, comparable results have rarely been reported.

Countless possible statistical distributions exist, which put more emphasis on the distribution tails than the Gaussian model. For instance, the Student t is a well known and easily applied distribution, but in a portfolio framework, one essential parameter — the degree of freedom — has to be set arbitrarily.

Generally, most statistical distributions are symmetric, treating profits and losses similarly in a portfolio context. From a financial point of view, this is a dubious assumption. Hence, the qualities of extreme value distributions, treating positive and negative tails independently, appear even more attractive.

2.2.7. Autoregressive conditional heteroscedasticity

In order to cope with the clustered volatility of asset prices, the autoregressive conditional heteroscedasticity (ARCH) model by Engle (1982) is commonly used. Although efficient in time series analysis and estimation, ARCH is not a distribution model, directly applicable in the portfolio optimisation problem.

Instead, the asset return time series can first be estimated with e. g. an AR(1)–GARCH(1,1) model in order to eliminate the heteroscedasticity in the data. Residuals are then fed into an extreme value model. This kind of methodology has been used by e. g. McNeil & Frey (2000) and Bali (2007).

2.3. Risk measures

2.3.1. Variance and standard deviation

The variance of a random variable or distribution is the expected square deviation of that variable from its expected value or mean. If a random variable X has expected value (mean) $\mu = \mathbb{E}[X]$, then the variance $Var(X)$ of X is given by:

$$Var(X) = \mathbb{E}\left[(X-\mu)^2\right] \tag{2.6}$$

The unbiased estimator of the sample variance is defined as:

$$\sigma^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i-\mu)^2 \tag{2.7}$$

In the first mathematical model for portfolio selection (Markowitz 1952), the portfolio risk is quantified by the variance of its return. Markowitz showed, that given either an upper bound for the allowed risk or a lower bound for the acceptable return, the optimal portfolio can be found with quadratic programming.

Although the variance and standard deviation are easily calculated assuming Gaussian distribution, neither do distinguish between positive and negative deviations from the mean, which renders them less useful for a risk averse investor. Additionally, the variance does not exist if the expected value of the distribution is not defined, or the relevant integral diverges e.g. in the case of Pareto distribution. However, easy calculations make the variance of a normal distribution a good first–order approximation of portfolio risk.

2.3.2. Value at Risk

Value at Risk (VaR) is defined as the largest loss of an investment over a given time horizon at a certain confidence level. In other words, VaR can be interpreted as the largest potential loss in the best case — or the smallest potential loss in the worst case. An accessible introduction to VaR can be found in Jorion (2001).

VaR at the left quantile at level α (the α–quantile) of a random variable X is defined as

$$VaR_\alpha(X) = -q_\alpha(X) = -\inf\{x : P[X \leq x] \leq \alpha\} \tag{2.8}$$

Assuming normal distribution, if z_α at confidence level α is defined as $P[Z > z_\alpha] = \alpha$, then VaR at confidence level α is

$$VaR_\alpha = \mu - z_\alpha \sigma \tag{2.9}$$

Denoting the historical process of n observations $X_1, ..., X_n$ by $F_n(t) = \frac{1}{n}\sum_{i=1}^{n} I(X_i \leq t)$ where $I(X)$ is the indicator function and the distribution F is unknown, the α–quantile $F_n^{-1}(\alpha)$ can be estimated by

$$F_n^{-1}(\alpha) = X_{n(i)}, \alpha \in \left[\frac{i-1}{n}, \frac{i}{n}\right] \tag{2.10}$$

In other words, if the losses are ordered by size, historical VaR at quantile α is simply the loss at position i.

Assuming general Pareto distribution (GPD) with tail index ξ, scale ϕ, threshold u, and confidence level α, VaR is given (Bali 2003) as:

$$VaR_\alpha = u + \frac{\phi}{\xi}\left(\left(\alpha\frac{n}{N_u}\right)^{-\xi} - 1\right) \tag{2.11}$$

Here, n is the total number of observations in the sample, and N_u denotes the number of observations exceeding the threshold u, i. e. the extreme tail values.

2.3.3. Expected shortfall

Expected shortfall (ES) is the expected value of distribution values to the left from the confidence threshold, i.e. the mean value of the losses that are greater than VaR, and is defined as

$$ES_\alpha(X) = -\mathbb{E}[X \mid X \leq q_\alpha(X)] = -\inf\{\mathbb{E}[X \mid A] : P[A] > \alpha\} \tag{2.12}$$

Assuming normal distribution, both ES and VaR are scalar multiples of the standard deviation, and consequently also of each other. ES becomes (Ho et al. 2008)

$$ES_\alpha(X) = \mathbb{E}[X \mid X > Z_\alpha\sigma] = \sigma\mathbb{E}\left[\frac{X}{\sigma} \mid \frac{X}{\sigma} > Z_\alpha\right] = \mu - \frac{e^{-Z_\alpha^2/2}}{\alpha\sqrt{2\pi}}\sigma \tag{2.13}$$

Historical ES can be estimated with (Harmantzis et al. 2006)

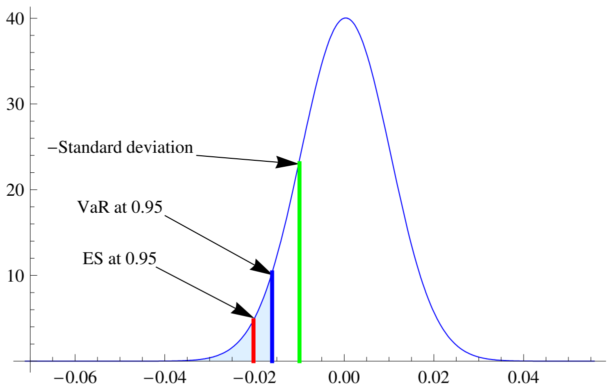

Figure 2.4.: The probability density function of a normal distribution, with markers for standard deviation (negative), VaR, and ES at 95 % confidence level. Here, log returns of the S&P 500 index between January 6, 1992 and October 5, 2007 are assumed to follow normal distribution. ES is the expected value, e.g. the mean, of values smaller than VaR, indicated by the shaded area.

$$ES_\alpha(X) = E\left[X \mid X > VaR_\alpha\right] = \frac{\sum_{i=n\alpha}^{n} X_{n(i)}}{n - n\alpha} \tag{2.14}$$

Here, X_n denotes the ordered sequence of n random values, $X_{n(i)}$ being the ith in the sequence..

Assuming general Pareto distribution (GPD) with tail index ξ, scale ϕ, threshold u, confidence level α, and VaR from (2.11), ES is

$$ES_\alpha(X) = VaR_\alpha + \frac{\phi + \xi(Var_\alpha - u)}{1-\xi} = \frac{Var_\alpha}{1-\xi} + \frac{\phi - \xi u}{1-\xi} = \frac{VaR_\alpha + \phi - \xi u}{1-\xi} \tag{2.15}$$

An illustration of the different risk measures under normal distribution can be found in figure 2.4.

2.4. Portfolio optimisation

The investment portfolio can be optimised for highest return at a given level of risk, alternatively for the lowest risk at a given level of return. Following Markowitz (1952), these optimisations can be formalised as following (vector entities denoted in bold):

$$
\begin{aligned}
\text{minimise} \quad & \mathbf{Var}\left[r_{\boldsymbol{\phi}}\right] \\
\text{subject to} \quad & \mathbb{E}\left[r_{\boldsymbol{\phi}}\right] \geq \alpha, \\
& \mathbf{1}^T \boldsymbol{\phi} = 1.
\end{aligned}
\tag{2.16}
$$

and

$$
\begin{aligned}
\text{maximise} \quad & \mathbb{E}\left[r_{\boldsymbol{\phi}}\right] \\
\text{subject to} \quad & \mathbf{Var}\left[r_{\boldsymbol{\phi}}\right] \leq \lambda, \\
& \mathbf{1}^T \boldsymbol{\phi} = 1.
\end{aligned}
\tag{2.17}
$$

respectively.

If neither an allowed risk level nor a required risk level can be quantified, an optimal combination of risk and return appears most suitable. Sharpe (1964) applied standard deviation, the square root of the variance, as risk measure, having the same unit as the return. Thus, calculating the Sharpe ratio of return and risk

$$
\frac{\mathbb{E}(r_{\boldsymbol{\phi}}) - r_f}{\sqrt{\mathbf{Var}(r_{\boldsymbol{\phi}})}}
$$

produces a meaningful combination of risk and return. As an optimisation rule, the Sharpe ratio can be applied:

$$
\begin{aligned}
\text{maximise} \quad & \frac{\mathbb{E}(r_{\boldsymbol{\phi}}) - r_f}{\sqrt{\mathbf{Var}\left[r_{\boldsymbol{\phi}}\right]}} \\
\text{subject to} \quad & \mathbf{1}^T \boldsymbol{\phi} = 1.
\end{aligned}
\tag{2.18}
$$

When VaR is used, the optimisation problem becomes

$$
\begin{aligned}
\text{maximise} \quad & \mathbb{E}\left[r_{\boldsymbol{\phi}}\right] \\
\text{subject to} \quad & P\left[r_{\boldsymbol{\phi}} \leq VaR_{\alpha}\right] \leq \alpha, \\
& \mathbf{1}^T \boldsymbol{\phi} = 1.
\end{aligned}
\tag{2.19}
$$

Consequently, the optimisation problem with ES can be expressed as:

$$
\begin{aligned}
\text{maximise} \quad & \mathbb{E}\left[r_{\boldsymbol{\phi}}\right] \\
\text{subject to} \quad & \mathbb{E}\left[r_{\boldsymbol{\phi}} \mid r_{\boldsymbol{\phi}} \leq VaR_{\alpha}\right] \leq ES_{\alpha}, \\
& \mathbf{1}^T \boldsymbol{\phi} = 1.
\end{aligned}
\tag{2.20}
$$

A performance index similar to the Sharpe ratio was developed by Campbell et al. (2001) to allocate financial assets by maximising expected return subject to VaR:

$$\begin{array}{ll} \text{maximise} & \frac{\mathbb{E}(r_{\boldsymbol{\phi}})-r_f}{VaR_\alpha} \\ \text{subject to} & \mathbf{1}^T\boldsymbol{\phi} = 1. \end{array} \tag{2.21}$$

Extending equation (2.21) to ES is straightforward:

$$\begin{array}{ll} \text{maximise} & \frac{\mathbb{E}(r_{\boldsymbol{\phi}})-r_f}{ES_\alpha} \\ \text{subject to} & \mathbf{1}^T\boldsymbol{\phi} = 1. \end{array} \tag{2.22}$$

If necessary, the additional condition prohibiting short sales:

$$\forall \phi_i \in \boldsymbol{\phi} : 0 \leq \phi_i \leq 1$$

can be added to all optimisation problems.

Generally, there is a strive to reduce the optimisation problems into e.g. linear programs for computational reasons.

2.5. Other portfolio strategies

2.5.1. Minimum variance

For a risk averse investor, one alternative portfolio selection principle is minimising the variance of the portfolio. In a Gaussian framework, the minimum–variance rule provides the combination of asset weights with the smallest variance, e. g. the smallest risk of return variations. Minimum variance is quite straightforward to calculate.

Following calculations are cited from Benninga (2008, pp. 291–301). Assuming return data for N assets over M periods, the return of asset i in period t is expressed as r_{it}, and the mean return of asset i is

$$\overline{r}_i = \frac{1}{M}\sum_{i=1}^{M} r_{it}$$

where $i = 1, ..., N$. Then, the *excess return*[1] matrix $\boldsymbol{A}$ can be defined as

[1] Return deviations from the sample mean.

$$\boldsymbol{A} = \begin{bmatrix} r_{11} - \overline{r}_1 & r_{21} - \overline{r}_2 & \cdots & r_{N1} - \overline{r}_N \\ r_{12} - \overline{r}_1 & r_{22} - \overline{r}_2 & \cdots & r_{N2} - \overline{r}_N \\ \vdots & \vdots & \ddots & \vdots \\ r_{1M} - \overline{r}_1 & r_{2M} - \overline{r}_2 & \cdots & r_{NM} - \overline{r}_N \end{bmatrix}$$

The matrix of asset return covariances, the sample *variance–covariance* matrix $\boldsymbol{S}$, can now be computed from

$$\boldsymbol{S} = [\boldsymbol{\sigma}_{ij}] = \frac{\boldsymbol{A}^T * \boldsymbol{A}}{M - 1}$$

Finally, the asset weight vector $\boldsymbol{MVP}$ of the minimum–variance portfolio can be computed from

$$\boldsymbol{MVP} = \frac{\mathbf{1} * \boldsymbol{S}^{-1}}{\mathbf{1} * \boldsymbol{S}^{-1} * \mathbf{1}^T} \tag{2.23}$$

where **1** is a column vector of 1's.

2.5.2. Hedging

In order to decrease portfolio risk, mainly three asset classes are of interest. Obviously, adding an asset with very low risk will reduce the risky part of the portfolio, hence decreasing total portfolio risk. Alternatively, an asset with opposite correlation to the risky asset will neutralise the total risk, provided that the opposite correlation is maintained in all situations. As a third alternative, derivatives on the risky asset can be used to hedge the portfolio risk. For instance, a portfolio containing S&P 500 index could be hedged buying put options on the index, thus establishing a low limit on the portfolio value.

2.6. Summary

The Gaussian mean–variance framework is easily understood, uncomplicated to compute, and works well in the average case. It supports concepts like diversification, and provides closed–form equations for different risk measures. However, its performance degrades as markets become unstable. In addition, derivatives are difficult to incorporate in the Gaussian framework.

Extreme value theory, on the other hand, focuses on rare events at the far ends, i. e. the interesting parts of the sample distribution. Extreme value theory is well known and often applied e. g. in meteorology, hydrology, agriculture, and insurance. Closed–form equations for different risk measures also exist.

VaR is well known in banking, and often used for estimating capital requirements. However, there are some complications with the properties of VaR if Gaussian distribution is abandoned. Normally, these can be mitigated with the application of ES. Although not as well known, ES is about as straight–forward to compute as VaR, and supported by contemporary packages.

3. Previous research

3.1. Artzner, Delbaen, Eber & Heath (1999)

In their paper, Artzner et al. define a set of four desirable properties for measures of risk, and call measures carrying these properties "coherent". Their definition of risk covers market risks as well as nonmarket risks, and market completeness is not assumed. They consider only one period of uncertainty for simplicity, but declare that an arbitrary time period can in general be considered.

Artzner et al. relate risk to the future net worth of a position, and consequently divide risks into the subsets of acceptable and unacceptable risks, i.e. positions having acceptable or unacceptable future net worth. Their definition of a measure of risk by means of a "reference instrument"[1] is a number $\rho(X)$, assigned by the measure ρ to the risk X, stating the minimum amount of cash *in excess of* the risky position X to be invested in the "reference instrument". The requirements on the risk measure ρ are stated as four axioms of translation invariance, subadditivity, positive homogeneity, and monotonicity.

Translation invariance

$$\{\rho : \mathcal{G} \to \mathbb{R} \mid \forall X \in \mathcal{G}, \alpha \in \mathbb{R} : \rho(X + \alpha \cdot r) = \rho(X) - \alpha\} \tag{3.1}$$

means that "adding the sure initial amount α to the initial position and investing it in the reference instrument r simply decreases the risk measure ρ by α".

Subadditivity

$$\{\rho : \mathcal{G} \to \mathbb{R} \mid \forall X_1, X_2 \in \mathcal{G} : \rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2)\} \tag{3.2}$$

can be stated as "a merger does not create extra risk", supporting the principle of diversification.

[1] Obviously, the "reference instrument" denominates the *risk–free* asset.

Positive homogeneity

$$\{\rho : \mathcal{G} \to \mathbb{R} \mid \forall \lambda \geq 0, X \in \mathcal{G} : \rho(\lambda X) = \lambda \rho(X)\} \tag{3.3}$$

implies that risk is linearly related to the size of the position.

Monotonicity

$$\{\rho : \mathcal{G} \to \mathbb{R} \mid \forall X, Y \in \mathcal{G} : X \leq Y \Leftrightarrow \rho(Y) \leq \rho(X)\} \tag{3.4}$$

states, that a position with lower future net worth has higher risk than a position with higher future net worth.

A risk measure satisfying these axioms is defined being *coherent*.

Moreover, Artzner et al. show, that unconditional VaR is coherent only if Gaussian distribution[2] is assumed. Otherwise, VaR does not comply with the requirement of subadditivity. Finally, Artzner et al. suggest a specific coherent risk measure called the *tail conditional expectation*, which is "the least expensive" in the set of risk measures that are "coherent and accepted by regulators". Expected shortfall and conditional VaR are other denominations for tail conditional expectation.

The results of Artzner et al. are valuable, although not very accessible to the risk–averse investor. Defining the set of coherent risk has lead to increased research interest in the field, opening the way for numerous empirical investigations.

3.2. Harmantzis, Miao & Chien (2006)

Artzner et al. were soon followed by several other papers comparing VaR and ES assuming different statistical distributions. Harmantzis et al. conduct a broad empirical investigation using both currencies and indices. They compare VaR and ES assuming Gaussian distribution, historical distribution, generalised Pareto, and stable Paretian distributions.

Using four currency exchange rates and six stock indices, Harmantzis et al. note the significant kurtosis of all assets, supporting the assumption that asset returns feature thick tails. Their empirical survey builds on back testing of one–day forecasts from rolling windows with a length of 126, 251, and 502 days, corresponding to a half, one, and two years. 95 % and 99 % confidence levels are used. Model estimates are compared

[2] Actually, the distribution must belong to the family of elliptic distributions, including for instance Gaussian and Student's t.

with actual losses, and a violation occurs if the actual loss is greater than the forecast. Model performance is measured by the number of violations.

Harmantzis et al. find, that the Stable and generalised Pareto models give the most accurate VaR estimates at 99 % confidence level, while the Gaussian and generalised Pareto are the most accurate at 95 % level. On the other hand, for both confidence levels, the historical distribution provides the most accurate ES estimates, with generalised Pareto being superior to the other models.

Like many other papers, Harmantzis et al. analyse different assets separately, with the purpose of comparing the performance of the *models*. However, few empirical studies utilise the most promising models.

3.3. Bali & Theodossiou (2008)

In their article, Bali & Theodossiou evaluate the performance of three extreme value distributions and four skewed fat–tail distributions in estimating unconditional and conditional VaR thresholds. Evaluated extreme value distributions are the generalised Pareto (GPD), the generalised extreme value (GEV), and the Box–Cox GEV. The article contributes by introducing the skewed fat–tail distributions i.e. the inversed hyperbolic sine (IHS), the exponential generalised beta of second kind (EGB2), the skewed generalised-t (SGT), and the skewed general error (SGED) distributions to VaR measures.

As data set, Bali & Theodossiou use daily logarithmic price changes of the S&P 500 composite index, the time period reaching from January, 1950 to December, 2000. Both kurtosis and negative skewness are reported with statistical significance, thus rejecting the assumption of normally distributed returns.

The extreme value distributions are analysed with a likelihood ratio test, applied on 5 % of the right and left tails of the empirical distribution. Both the GPD and the GEV are rejected in favour of the Box–Cox GEV.

The skewed fat–tail distributions are analysed using the complete data set and applying the Kolmogorov–Smirnov, Lilliefors, and likelihood ratio tests. As a result, the SGED, EGB2, and the normal distributions are rejected in favour of IHS and SGT.

Testing the performance of the different distributions is conducted by estimating the distribution parameters from a rolling 10–year sample, and evaluating against the returns during the subsequent year. In other words, the first estimation sample reaches from 1950 to 1959 and the first evaluation occurs for year 1960, yielding a total estimation sample of 41 years between 1960 and 2000.

Bali & Theodossiou find, that the SGT, IHS, and EGB2 distributions perform as well as the extreme value distributions in modelling the tail behaviour of the returns. The Box–Cox GEV performs better than the GPD and the GEV, however differences are really small. The result has potential implications in modelling loss processes, as finding the parameters of the skewed fat–tail distributions is simpler than with the extreme value distributions.

Another indication of the tests is, that actual VaR thresholds are time varying, which is not captured by unconditional VaR measures. An AR-GARCH model is hence suggested in order to eliminate heteroscedasticity from the data. Consequently, expected shortfall produces more accurate estimates than unconditional VaR. Interestingly, the skewed fat–tail distributions except the SGED perform better than some extreme value distributions in the estimation of conditional VaR thresholds.

3.4. Ho, Cadle & Theobald (2008)

The vast majority of the papers following Artzner et al. are mainly concerned with the statistical properties of the different measures. Very often, empirical tests cover only one asset, or single separate assets. Ho et al. are possibly the first to investigate risk measures in a *portfolio* framework.

Selecting the optimal portfolio in a risk measure framework constitutes a considerable computational effort, especially when the moving away from the normal distribution. Ho et al. simplify the task by using only three assets with precalculated portfolio weights, largely facilitating the calculations. Their data consisted of the S&P 500 index, the NASDAQ index, and a 10+ year US Government bond index, with daily log returns during a 15–year period.

They define four testable propositions, with the purpose of finding empirical evidence for or against these propositions.

Proposition 1. Strategies using variance, VaR, and ES produce exactly the same optimal portfolio, assuming portfolio returns follow a Gaussian distribution.

Proposition 2. Assuming the historical or extreme value distribution, different risk measures produce different optimal portfolios.

Proposition 3. Assuming the historical or extreme value distribution, the optimal portfolio contains a bigger allocation of safe assets than when Gaussian distribution is assumed.

Proposition 4. Using ES as risk measure reduces potential losses in optimal portfolios, compared with using VaR.

All propositions can be confirmed, thus supporting the mean–ES framework as a reasonable and consistent solution to the asset allocation problem.

Some minor issues with the study of Ho et al. concern their data. Precalculating the asset weights is equal to daily rebalancing of the portfolio, in order to maintain the asset weights. Clearly, this is impractical and hardly feasible. A more accurate picture of the portfolio evolution would be obtained by taking the precalculated asset weights on day 1, and recalculating the portfolio value for every trading day during the time period under consideration.

Another issue is the use of two strongly correlated assets – S&P 500 and NASDAQ – leading to sparse allocations of the NASDAQ index in the optimal portfolio. Surprisingly, they do not include a risk free asset, and hence their portfolios are not optimised subject to excess returns.

One possible reason for limiting the study to three assets might be the computational resources required. Particularly when investigating historical distributions, the most reliable algorithm includes iterating through all possible combinations of asset allocations. Consequently, the number of calculations required grows rapidly beyond feasible limits.

3.5. Previous thesis

At Hanken School of Economics, Lindström (2007) compares the performance of VaR and ES. In his Master´s thesis, Lindström conducts a back–testing experiment on 12 different asset classes, in the spirit of Harmantzis et al. (2006). As a result, ES is found to produce more accurate estimates for loss limit violations, when the estimation period is longer than 1 year. On shorter periods, VaR appears superior. A plausible explanation might be ES requiring a larger sample to provide reliable results. Only the empirical distribution model is analysed in the study, somewhat limiting the generality of the results.

3.6. Summary

Research in the field of risk measures has initially concentrated on the mathematical properties of the different measures and distributions. Usually, the risk properties of single assets are investigated. Only during the last few years, empirical research in a portfolio framework has started to arise.

It can be concluded from the previous research, that (unconditional) VaR is not the most suitable measure of the market risk of an investment. For instance, suppose two portfolios having equal initial investment and expected return, one being long in call options while the other being short in sell futures. Although these portfolios can be constructed to have equal VaR, their potential losses are quite different: the long call portfolio has very limited downside, while the short future portfolio may have nearly infinite loss in the worst case.

Gaussian models have proven inferior to more specialised models in calculating risk measures like VaR and ES. Evidence is frequently found in back–testing risk limit violations.

Extreme value distributions, the GPD in particular, have proven qualities, rendering them very useful in assessing downside risk. These qualities include focusing on the distribution tails; ability of asymmetrical estimation of the sample distribution; ready equations for calculating risk measures from the distribution parameter estimates; and available computation support.

An increasing amount of research has been published, where different risk measures and extreme value theory are applied in a portfolio framework, in connection with out–of–sample testing.

4. Research methods and data

4.1. Assumptions

Daily log returns are assumed to follow Gaussian distribution up to ± 1 standard deviations from the sample mean, and the tails are assumed to follow generalised Pareto distribution. These assumptions build partly on the fact, that return distributions are frequently reported to possess thick tails, skewness and kurtosis; and partly on the good results achieved with GPD, reported by Ho et al. (2008) and Bali & Theodossiou (2008). Moreover, Bali (2007) gives closed–form equations for VaR and ES using GPD parameters, further facilitating the task.

4.2. Methodology

In order to compare benefits and drawbacks of the different risk measures, they must be systematically calculated from a representative sample. Therefore, model correctness is first ensured by comparing the results with Ho et al. using a similar dataset. In practice, similar tests are run, applying the same statistical models and risk measures. For comparison, the rolling time frame tests are also repeated.

With the calibrated model, a new set of tests is run. After testing the complete dataset for different risk measures, a rolling time frame test is conducted. Using the results, an out–of–sample test of the forecasting power of the different models can be performed.

Essentially, the study concentrates on constructing optimal asset portfolios with a variety of different criteria. When portfolios are built for rolling investment periods, changes in asset allocations are observed. Goodness of selection criteria can be assessed e. g. by estimates for the rebalancing need, expected loss, and practical feasibility.

A portfolio selection rule, leading to frequent rebalancing, is considered instable, or non-robust. Frequent rebalancing leads to excess trading costs, quickly reducing the possible gain of the strategy. Therefore, the stability of the optimal portfolio is an important property of a good selection rule.

Results expected include first the confirmation of the four propositions of Ho et al. In subsequent tests, it is expected that VaR and ES provide smoother changes in allocations, i. e. less rebalancing needs. Additionally, basing the allocation rule on ES is expected to induce lower potential losses than VaR.

4.3. Datasets

In order to calibrate the models with the findings of Ho et al., a similar dataset is used. Dataset 1 contains daily prices for the S&P 500 index (^SPC), NASDAQ index (^IXIC), and 10-year US Treasury notes (US10Y), between January 6, 1992 and October 5, 2007. Ho et al. used a "US Government 10+ year bond price index (USG5PR)", which was not available[1] for this study. However, US10Y is assumed to provide a sufficiently accurate approximation. Statistic summaries[2] of the datasets used by Ho et al. and by this project are shown in Table 4.1.

Ho et al.:	**SPX**	**CCMP**	**USG5PR**	Dataset 1:	**GSPC**	**IXIC**	**US10Y**
R (%)	0.032	0.038	0.004	R (%)	0.033	0.039	0.006
σ (%)	0.980	1.511	0.540	σ (%)	0.993	1.536	0.576
R/σ (%)	3.259	2.489	0.805	R/σ (%)	3.322	2.530	0.960
Skewness	-0.127	-0.002	-0.312	Skewness	-0.101	-0.006	-0.373
Kurtosis	7.381	9.035	4.145	Kurtosis	7.044	8.749	5.209
Jarque-Bera	3 295	6 233	291	Jarque-Bera	2 721	5 483	902
Minimum (%)	-7.11	-10.17	-2.85	Minimum (%)	-7.11	-10.17	-0.41
Correlation	SPX	CCMP	USG5PR	**Correlation**	GSPC	IXIC	US10Y
SPX	1			GSPC	1		
CCMC	0.838	1		IXIC	0.836	1	
USG5PR	-0.398	-0.104	1	US10Y	-0.039	-0.109	1

Table 4.1.: Data of Ho et al. (2008) and Dataset 1 of the current project. Both datasets cover the S&P 500 (SPX/GSPC), NASDAQ (CCMP/IXIC), and the 10–year US government bond (USG5PR/US10Y) price indices between January 6, 1992 and October 5, 2007.

Apparently, the datasets are very similar, except the bond price index which could not be reproduced precisely. As US10Y has better return per standard deviation ratio, it is likely to gain a larger share in the portfolios than the USG5PR index used by Ho et al.

[1] For some reason, USG5PR seems to be unknown by Yahoo and Bloomberg.

[2] The Jarque-Bera test is a goodness-of-fit measure of departure from normality, based on the sample kurtosis and skewness.

Dataset 2:	**GSPC**	**XAU**	**TNC**	**3–m Bill**
R (% p. a.)	7.400	1.874	3.192	0.365
σ (%)	1.171	2.429	0.660	0.064
R/σ (%)	2.507	0.306	1.918	2.263
Skewness	-1.359	0.128	0.172	0.556
Kurtosis	33.572	8.816	7.923	24.065
Jarque-Bera	257 671	7 923	6 664	121 710
Maximum (%)	10.957	23.674	7.5	0.814
Minimum (%)	-22.900	-20.025	-3.9	-0.744
Correlation	GSPC	XAU	TNC	
GSPC	1			
XAU	0.173	1		
TNC	0.008	-0.088	1	

Table 4.2.: Dataset 2 of the current project. Dataset 2 covers the S&P 500 (GSPC), the 10–year US government bond (TNX), the gold and silver (XAU) industry, and the 3–month US Treasury Bill price indices between December 20, 1983 and December 14, 2009, a total of 6 560 observations.

Moreover, the negative skewness (except for the NASDAQ index) and kurtosis significantly greater than 3 contradict the assumption of Gaussian distribution.

Extending the observation period to reach from December, 1983, to December, 2009, produces Dataset 2, summarised in table 4.2. Replacing the NASDAQ index with a commodity index (the gold and silver industry price index ^XAU) having less covariance with the main stock and bond indices, is expected to produce portfolios with even better diversification and higher appeal to risk averse investors. Additionally, the 3–month US Treasury Bill is introduced, representing the risk–free asset.

For European observations, the iTraxx *DAX index* (GDAXI), iTraxx *German Government bond index* (XBCT), and iTraxx *Deutsche Bank liquid commodity index* (XDBC) are used. iTraxx started in 2007, and prices are thus available only since, necessitating a shorter comparison period than originally intended. The 3–month Euribor rate is included to represent the risk–free asset. Dataset 3 in table 4.3 contains the European indices.

Due to the sampling period coinciding with the global financial turmoil, descriptive statistics on Dataset 3 display extreme properties, particularly the R/σ ratio of the government bond index. Consequently, it is expected that the government bond will dominate all optimal portfolios.

Comparing Datasets 2 and 3, ^XAU and XDBC are not fully compatible, as the latter

Dataset 3:	**GDAXI**	**XBCT**	**XDBC**	3–m Euribor
R (% p.a.)	-14.225	6.155	-2.913	1.789
σ (% p.a.)	31.440	2.288	26.615	0.347
R/σ (%)	-2.85	16.949	-0.689	32.432
Skewness	0.315	-0.366	-0.217	0.661
Kurtosis	8.259	3.962	5.334	9.077
Jarque-Bera	695	36	140	957
Maximum (%)	10.800	0.533	7.914	0.123
Minimum (%)	-7.435	-0.511	-7.072	-0.106
Correlation	GDAXI	XBCT	XDBC	
GDAXI	1			
XBCT	-0.401	1		
XDBC	0.442	-0.237	1	

Table 4.3.: Dataset 3 of the current project. Dataset 3 includes the iTraxx DAX (GDAXI), the German government bond (XBCT), the Deutsche Bank liquid commodity (XDBC), and the 3–month Euribor indices between October 10, 2007 and January 22, 2010.

contains 12 other commodities in addition to gold prices. However, as diversifying assets for the stock and bond indices, they defend their place.

4.4. Practical arrangements

Data for US observations, i.e. Datasets 1 and 2, is obtained from Yahoo Finance (`http://finance.yahoo.com`). US Treasury Bill and 10–year bond prices are from Federal Reserve Statistical Release (`www.federalreserve.gov/releases`). European data is obtained from Deutsche Bank x-Trackers (`http://www.dbxtrackers.de`) ETF prices for the indices mentioned. Euribor rates are from the Euribor site (`www.euribor.org`).

As risk–free assets, 3–month Treasury Bill respective Euribor rates are used. Money–market rates r_t for period t are reported as annualised, and the corresponding zero–coupon bond price is $100 - r_t$. Log returns are hence calculated from the obtained annualised rates with formula $R_t = \ln(\frac{100-r_t}{100-r_{t-1}})$.

Bond index values are given as yield y_t for period t, calculated directly from the yield curve. Price indices are hence calculated straightforwardly with formula $R_t = \frac{100}{(1+\frac{y_t}{100})^{10}}$ assuming a maturity of exactly 10 years.

To reproduce the results of Ho et al., portfolios are built from Dataset 1. Mean–variance, Value-at-Risk, and expected shortfall are used as selection criteria for optimal asset weights. According to Ho et al., the Gaussian distribution, the historical distribution, and an extreme value distribution are investigated. Risk measures of the historical distribution are computed from quintiles of the dataset. For the extreme value distribution, the generalised Pareto distribution is applied, according to Bali (2003). Risk measures are computed from the parameter estimates.

Dataset 2 extends the time frame of Ho et al. and is supposed to bring additional information concerning investor behaviour and asset allocation during a global recession. Moreover, out–of–sample tests are conducted, and a risk–free asset is used in portfolio performance comparisons.

A similar analysis, however for a much more limited time frame, is made using Dataset 3. For comparisons between Europe and U. S., datasets 2 and 3 are used.

All portfolio calculations, statistical analysis, and back–testing are computed with *R* software[3]. Program codes can be found in Appendix A at the end of this report.

Usually, optimal portfolios can not be found with maximum or minimum finding functions, as either iteration limits stop execution prematurely. Alternatively, only a local maximum is found. The optimal portfolio is therefore calculated through iterations in a loop, covering asset weights in steps of 1 percentage point, as demonstrated by Bensalah (2002). It is hence assumed sufficient to report asset weights with accuracy of 1 % of the total portfolio value.

Naturally, the mean–variance and minimum–variance portfolios are based on Gaussian distribution parameters, and are hence optimised using simple matrix calculus.

[3] *R* is an open–source implementation of the *S* programming language, and very similar to the *S+* package, newer versions included. Good things in life are free!

5. Empirical results

5.1. Comparisons with Ho et al.

5.1.1. Full sample period

For calibrating and validating the model, an attempt is made to reproduce the results of Ho et al. (2008). Using the full sample period of dataset 1, optimal portfolio allocations are calculated using mean–variance, mean–VaR, and mean–ES ratios as selection rules, also applying the Gaussian distribution, the historical returns, and the generalised Pareto distribution, respectively. Results are reported in tables 5.1 to 5.3, together with the results of Ho et al. for comparison.

Following Ho et al., *daily* returns and standard deviations are reported. Consequently, values in tables 5.1 to 5.4 and A.1 to A.4[1] in appendix A are quite small, and it is likely difficult to perceive any significant differences. VaR and ES values are computed from their respective statistical distribution parameters, and values in different tables are hence not directly comparable.

It can be seen from table 5.1, that assuming Gaussian distribution necessarily leads to identical portfolio allocations, independently of which risk measure is in use. This result is consistent with earlier research, with equations (2.9) and (2.13), and confirms proposition 1 of Ho et al.

An investor maximising the portfolio return per unit of risk should select a portfolio containing 64 % of the S&P 500 index, and 36 % 10–year bonds. The share of bonds is 3 percentage points (about 9 %) larger than the share reported by Ho et al., due to the slightly higher R/σ ratio of the bond in Dataset 1.

Proceeding with the historical distribution, with results displayed in table 5.2, further differences to Ho et al. can be detected. VaR values of Dataset 1 are clearly lower, whereas ES values are slightly higher than in Ho et al. Consequently, the share of bonds is considerably higher in the mean–VaR optimal portfolios, and lower in the mean–ES optimal portfolios, than reported by Ho et al.

[1] In tables 5.4 and A.1 – A.4, results from the rolling time periods are reported.

	MV		VaR	99 %	99 %	ES	95 %	99 %
Asset	*Ho*	Dataset 1	*Ho*	Dataset 1		*Ho*	Dataset 1	
S&P 500	*67 %*	64 %	*67 %*	64 %	64 %	*67 %*	64 %	64 %
NASDAQ	*0 %*	0 %	*0 %*	0 %	0 %	*0 %*	0 %	0 %
US Bond	*33 %*	36 %	*33 %*	36 %	36 %	*33 %*	36 %	36 %
Return								
R (%)	*0.023*	0.023	*0.023*	0.023	0.023	*0.023*	0.023	0.023
s.d. (%)	*0.673*	0.661	*0.673*	0.661	0.661	*0.673*	0.661	0.661
R/s.d. (%)	*3.393*	3.496	*3.393*	3.496	3.496	*3.393*	3.496	3.496
Skew	*-0.118*	-0.099	*-0.118*	-0.099	-0.099	*-0.118*	-0.099	-0.099
Kurt	*5.936*	5.583	*5.936*	5.583	5.583	*5.936*	5.583	5.583
VaR								
95 %	*1.085*	1.064	*1.085*	1.064	1.064	*1.085*	1.064	1.064
99 %	*1.543*	1.515	*1.543*	1.515	1.515	*1.543*	1.515	1.515
ES								
95 %	*1.366*	1.364	*1.366*	1.364	1.364	*1.366*	1.364	1.364
99 %	*1.771*	1.762	*1.771*	1.762	1.762	*1.771*	1.762	1.762

Table 5.1.: Comparison of the portfolio assuming normal distribution. Results of Ho et al., Table 2a are *emphasised*, and results of the current research are in normal font. Data from period between January 6, 1992 and October 5, 2007.

Asset allocations in table 5.2 are different from the Gaussian mean–variance optimal portfolio in table 5.1. Moreover, they exhibit consistently lower standard deviations than the mean–variance optimal portfolio. Propositions 2 and 3 of Ho et al. can hence be confirmed for the historical distribution part. Contradicting the expectations, mean–VaR produces equal risk measure values as mean–ES at 95 % confidence level, but lower at 99 % confidence level.

Even smaller differences can be observed from the GPD model in Table 5.3. Portfolio allocations differ less than 7 percentage points, and both VaR and ES measures are very close to Ho et al. Curiously, mean–VaR still produces higher risk measure values than mean–ES at 95 % confidence level, but lower at 99 % confidence level.

Optimal portfolio asset allocations in table 5.3 are consistently different and exhibit lower risk values than the Gaussian mean–variance optimal portfolio in table 5.1. Propositions 2 and 3 of Ho et al. can hence be confirmed.

Generally, lower risk measure values are obtained using mean–ES than mean–VaR, except for the 99 % confidence level. Of course, some issue with Dataset 1 is conceivable. However, unless the data issue can be resolved, Proposition 4 can not be confirmed.

	VaR	**95 %**	**VaR**	**99 %**	**ES**	**95 %**	**ES**	**99 %**
Asset	*Ho*	Dataset 1	*Ho*	Dataset 1	*Ho*	Dataset 1	*Ho*	Dataset 1
S&P 500	*74 %*	63 %	*54 %*	48 %	*60 %*	63 %	*55 %*	*60 %*
NASDAQ	*0 %*	0 %	*1 %*	4 %	*0 %*	0 %	*0 %*	*0 %*
US Bond	*26 %*	37 %	*45 %*	48 %	*40 %*	37 %	*45 %*	*40 %*
Return								
R (%)	*0.025*	0.023	*0.020*	0.020	*0.021*	0.023	*0.020*	0.022
s.d. (%)	*0.733*	0.653	*0.584*	0.586	*0.618*	0.653	*0.582*	0.631
R/s.d. (%)	*3.382*	3.496	*3.348*	3.425	*3.381*	3.496	*3.351*	3.490
Skew	*-0.121*	-0.100	*-0.119*	-0.130	*-0.118*	-0.100	*-0.120*	-0.106
Kurt	*6.330*	5.535	*5.238*	5.105	*5.519*	5.535	*5.227*	5.393
VaR								
95 %	*1.192*	1.060	*0.973*	0.957	*1.023*	1.060	*0.963*	1.039
99 %	*1.912*	1.663	*1.480*	1.419	*1.599*	1.663	*1.479*	1.595
ES								
95 %	*1.674*	1.459	*1.309*	1.302	*1.389*	1.459	*1.304*	1.406
99 %	*2.494*	2.101	*1.883*	1.895	*2.019*	2.101	*1.876*	2.021

Table 5.2.: Comparison of the portfolio allocations from the historical distribution. Results of Ho et al., Table 2b are *emphasised*, and results of the current research are in normal font. Data from period between January 6, 1992 and October 5, 2007.

5.1.2. Rolling window analysis

For further examination of the different portfolio selection methods, 11 rolling windows in steps of 20 days are constructed. For each window, the calibration time is 3776 days, and the full test covers 3776 + (11 - 1) * 20 = 3976 daily observations. Due to slight differences in obtaining data, the windows do not match the analysis of Ho et al. exactly, but sufficiently well to provide similar results.

A comparison between Ho et al. and Dataset 1 under the Gaussian mean–variance framework is displayed in Table 5.4. According to Ho et al., the largest allocation of S&P 500 should occur in the 8^{th} period, while the largest bond allocation occurs in the 5^{th} period.

Evidently, optimal allocations from Dataset 1 follow a similar pattern as reported by Ho et al. in table 5.4, further confirming that the results are comparable. As already shown, the mean–VaR and mean–ES allocation rules produce equal portfolio allocations as the Gaussian mean–variance rule, and are hence excluded from further comparisons.

Continuing with the historical distribution, the mean–VaR method at 95 %, 97 %, 99 %, and 99.5 % confidence levels is compared with previous results in table A.1. Generally,

	VaR	**95 %**	**VaR**	**99 %**	**ES**	**95 %**	**ES**	**99 %**
Asset	*Ho*	Dataset 1	*Ho*	Dataset 1	*Ho*	Dataset 1	*Ho*	Dataset 1
S&P 500	*74 %*	63 %	*51 %*	56 %	*62 %*	62 %	*56 %*	57 %
NASDAQ	*0 %*	0 %	*5 %*	1 %	*0 %*	0 %	*0 %*	1 %
US Bond	*26 %*	37 %	*44 %*	43 %	*38 %*	38 %	*44 %*	42 %
Return								
R (%)	*0.025*	0.023	*0.020*	0.021	*0.021*	0.023	*0.020*	0.022
s.d. (%)	*0.733*	0.653	*0.602*	0.611	*0.633*	0.646	*0.589*	0.619
R/s.d. (%)	*3.382*	3.496	*3.334*	3.475	*3.388*	3.495	*3.359*	3.480
Skew	*-0.121*	-0.100	*-0.116*	-0.114	*-0.118*	-0.102	*-0.119*	-0.110
Kurt	*6.330*	5.535	*5.354*	5.267	*5.639*	5.487	*5.285*	5.308
VaR								
95 %	*1.192*	1.059	*1.035*	1.014	*1.053*	1.050	*0.999*	1.025
99 %	*1.912*	1.688	*1.513*	1.554	*1.631*	1.665	*1.497*	1.575
ES								
95 %	*1.674*	1.457	*1.351*	1.361	*1.425*	1.439	*1.322*	1.377
99 %	*2.494*	2.142	*1.953*	1.980	*2.098*	2.112	*1.915*	2.004

Table 5.3.: Comparison of the portfolio allocations from the generalised Pareto distribution. Results of Ho et al., Table 2c are *emphasised*, and results of the current research are in normal font. Data from period between January 6, 1992 and October 5, 2007.

the highest and lowest S&P 500 index allocations occur simultaneously in both samples.

Some issue with the 99 % confidence level VaR computations is evident from table A.1, both with Dataset 1 and with Ho et al. Although markets were subject to considerable — mostly downward — motion, hardly any rebalancing occurs. At 95 % confidence level, the allocation pattern follows the mean–variance method, at slightly more conservative asset weights. Again, results from Dataset 1 are in agreement with Ho et al. As a detail, the NASDAQ index becomes allocated at 99.5 % confidence level in the study of Ho et al.. In Dataset 1, NASDAQ is allocated at 99 % confidence level, but not at 99.5 %.

Compared with the mean–VaR framework, the mean–ES framework provides more stability and less rebalancing need, as evident from table A.2. Particularly, the standard deviations of the mean–ES allocations are clearly lower than those of the other methods. However, all approaches provide similar results in, that the highest equity index allocations occur around period 8, and the lowest around period 4.

To further support the compatibility with the results of Ho et al., the generalised Pareto distribution is estimated at 97 % threshold.

Optimal allocations with the mean–VaR rule at 95 % level in table A.3 feature quite pe-

MV	*Ho*			Dataset 1		
Period	*SP500*	*NDAQ*	*Bond*	SP500	NDAQ	Bond
1	*65*	*0*	*35*	63	0	37
2	*63*	*0*	*37*	61	0	39
3	*61*	*0*	*39*	60	0	40
4	*60*	*0*	*40*	57	0	43
5	*59*	*0*	*41*	58	0	42
6	*63*	*0*	*37*	59	0	41
7	*68*	*0*	*32*	66	0	34
8	*69*	*0*	*31*	71	0	29
9	*68*	*0*	*32*	68	0	32
10	*65*	*0*	*35*	64	0	36
11	*66*	*0*	*34*	66	0	34
mean	*64*	*0*	*36*	63	0	37
s.d.	*3.38*	*0*	*3.38*	4.25	0	4.25

Table 5.4.: Asset allocation (%) in rolling windows applying mean and variance from the Gaussian distribution. Results of Ho et al., Table 3a are *emphasised*, and results of the current research are in normal font. Data from period between January 6, 1992 and October 5, 2007.

culiar conduct. Allocation movements appear over–amplified, producing excessive rebalancing needs. On average, applying a higher confidence level leads to more conservative allocations than with lower confidence levels, as expected.

Finally, the mean–ES portfolios from the GPD are presented in table A.4. As expected, the allocations are relatively stable, compared with other allocation rules. Higher confidence levels leads to marginally different allocations, notably by diversifying with the NASDAQ index. However, the 95 % level leads to least rebalancing need in this case.

According to Tables 4.1 and 5.1, the results in the current project are not very far from the results of Ho et al., indicating that the models used are compatible. Use of different bond indices seems the most plausible reason for deviations in the results. Potentially, the bond index could be applied as a risk–free asset, enabling portfolio optimisation with respect to excess return.

5.2. US market data

5.2.1. Full sample period

Testing Dataset 2, US market data between 1983 and 2009, constitutes the essential part of this study. Using the full sample period of Dataset 2, optimal portfolio allocations are calculated using mean–variance, mean–VaR, and mean–ES ratios as selection rules, applying the Gaussian distribution, the historical returns, and the generalised Pareto distribution, respectively. To compare different approaches, a minimum–variance and a constant–fraction portfolio are added.

Results are reported in tables 5.5 to A.14. Note that *annual* returns and standard deviations are reported. The Sharpe ratio is computed using the 3–month Treasury bill as risk–free asset. Again, VaR and ES values are computed from their respective statistical distribution parameters, and values in different tables are not directly comparable.

Gaussian	**Mean variance**	**Min variance**	**Even 1/3**
Asset			
S&P 500	44 %	21 %	33 %
Gold & Silver	0 %	5 %	33 %
US Bond	56 %	74 %	33 %
Return			
R (% p. a.)	5.043	3.994	4.155
s.d. (%)	0.637	0.562	0.968
Sharpe	0.031	0.028	0.017
Max (%)	6.486	5.540	9.602
Min (%)	-9.628	-4.769	-11.148
VaR			
95 %	1.027	0.908	1.575
99 %	1.461	1.291	2.234
ES			
95 %	1.313	1.159	1.996
99 %	1.696	1.498	2.579

Table 5.5.: Comparison of asset allocations over the complete time period assuming Gaussian distribution. Data from period between December 20, 1983 and December 14, 2009.

It can be seen from Table 5.5, that the mean–variance approach provides superior return, compared with the minimum–variance and the constant–fraction portfolios. As expected, minimum–variance leads to the lowest risk measures. Note that the mean–variance approach leads to a nearly twice as big Sharpe ratio than staying with the simple constant–fraction portfolio.

An investor maximising the portfolio return per unit of risk should select a portfolio containing 44 % of the S&P 500 index, and 56 % 10–year bonds, and can expect 5 % annual return. In comparison, the minimum–variance investor gains 4 % annual return, with an only slightly lower Sharpe ratio, the largest losses being only half of the mean–variance portfolio losses.

Historical	**VaR**				**ES**			
Conf. level	**95 %**	**97 %**	**99 %**	**99.5 %**	**95 %**	**97 %**	**99 %**	**99.5 %**
Asset								
S&P 500	42 %	47 %	47 %	47 %	45 %	45 %	37 %	35 %
Gold & Silver	0 %	2 %	1 %	8 %	0 %	0 %	0 %	0 %
US Bond	58 %	51 %	52 %	45 %	55 %	55 %	63 %	65 %
Return								
R (% p. a.)	4.959	5.143	5.157	5.064	5.086	5.086	4.749	4.665
s.d. (%)	0.626	0.654	0.654	0.677	0.642	0.642	0.603	0.596
Sharpe	3.144	3.119	3.129	2.967	3.141	3.141	3.125	3.107
Max (%)	6.532	5.866	6.142	5.490	6.463	6.463	6.647	6.693
Min (%)	-9.154	-10.582	-10.460	-11.310	-9.865	-9.865	-7.969	-7.495
VaR								
95 %	0.940	1.000	0.996	1.015	0.973	0.973	0.912	0.911
99 %	1.601	1.610	1.596	1.733	1.597	1.597	1.564	1.565
ES								
95 %	1.396	1.456	1.455	1.521	1.430	1.430	1.352	1.339
99 %	2.279	2.442	2.429	2.609	2.361	2.361	2.162	2.124

Table 5.6.: Comparison of asset allocations over the complete time period assuming historical distribution. Data from period between December 20, 1983 and December 14, 2009.

A comparison of portfolios using historical data are presented in table 5.6. Except for VaR at 95 % confidence levels, risk measures obtained from historical data are clearly higher than values estimated from the Gaussian distribution. Moreover, the mean–ES approach provides consistently lower risk measures with higher confidence levels, whereas mean–VaR does not. Compared with mean–variance, mean–ES leads to further increased Sharpe ratio.

The GPD distribution in Table 5.7 features similar properties as the historical distribution. Applying the mean–ES approach provides consistently lower risk measures with higher confidence levels, which can also be seen from decreasing maximum losses. On the contrary, mean–VaR does not appear consistent. In fact, mean–VaR works quite

GPD	VaR				ES			
Conf. level	**95 %**	**97 %**	**99 %**	**99.5 %**	**95 %**	**97 %**	**99 %**	**99.5 %**
Asset								
S&P 500	42 %	47 %	47 %	47 %	45 %	44 %	40 %	37 %
Gold & Silver	0 %	2 %	2 %	0 %	0 %	0 %	0 %	0 %
US Bond	58 %	51 %	51 %	53 %	55 %	56 %	60 %	63 %
Return								
R (% p. a.)	4.959	5.143	5.143	5.170	5.086	5.043	4.875	4.749
s.d. (%)	0.626	0.654	0.654	0.655	0.642	0.637	0.616	0.603
Sharpe	3.144	3.119	3.119	3.134	3.141	3.144	3.141	3.125
Max (%)	6.532	5.866	5.866	6.417	6.463	6.486	6.578	6.647
Min (%)	-9.154	-10.582	-10.582	-10.339	-9.865	-9.628	-8.680	-7.969
VaR								
95 %	0.940	1.000	1.000	0.999	0.973	0.965	0.929	0.912
99 %	1.626	1.641	1.641	1.657	1.643	1.632	1.606	1.586
ES								
95 %	1.392	1.461	1.461	1.457	1.428	1.416	1.374	1.350
99 %	2.255	2.470	2.470	2.418	2.343	2.315	2.211	2.154

Table 5.7.: Comparison of asset allocations applying risk measures from the generalised Pareto distribution. Data from period between December 20, 1983, and December 14, 2009.

unexpectedly with the GPD approach, in that higher confidence levels lead to higher risk (and return) levels.

5.2.2. Rolling window analysis

For further examination of the different portfolio selection methods, 56 rolling windows in steps of 63 days are constructed. For each window, the calibration time is 3030 days, and the full test covers 3030 + (57 - 1) * 63 = 6558 daily observations. There are 21 trading days each month, and 63 days consequently form a quartile.

Moreover, to assess the forecasting power of the different approaches, allocations for a given quartile are multiplied with returns during the next quartile. A statistic summary over the time series obtained is presented at the bottom of tables 5.12 – 5.21.

In table A.5, asset allocations according to the Gaussian mean–variance approach are presented. The dotcom boom during years 1997–2000 can easily be identified from the suggested 100 % allocation of the stock index. On the contrary, the global financial crisis is evident from the 5 % stock index allocation in the first quartile of 2009. During 2009, the XAU index takes a growing share of the portfolio, indicating precious metals providing a "safe haven" (Hillier et al. 2006; Jaffe 1989) for the period. A graph over portfolio allocations with this approach can be find in Figure 5.1.

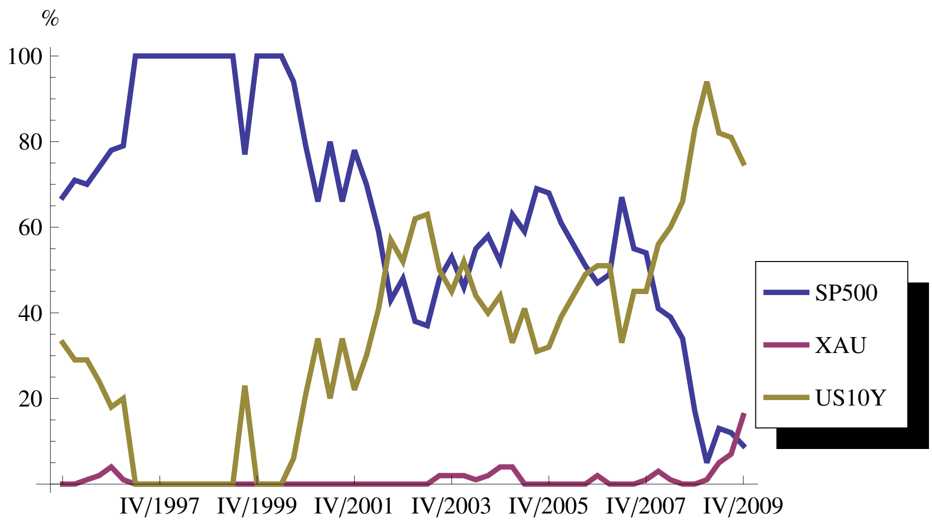

Figure 5.1.: Portfolio allocations with the Gaussian mean–variance approach.

Continuing with the historical distribution, applying mean–VaR at 95 % confidence level (95 % tail) leads to very similar allocations as the mean–variance approach, as indicated in table A.6. However, the pattern of precious metals as a "safe haven" can be detected also during the years 1996–1997, 2003, and 2004–2005. Generally, the 1990s appear as a period of veritable growth, whereas the mid–2000's implies a return to a longer time average.

Mean–VaR at 99 % confidence level, presented in table A.7, also provides similar results. However, the safe haven of precious metals in 1996—1997 is even more evident. It can also be noted that mean–VaR at both 95 % and 99 % confidence level leads to slightly higher risk — and return — than the mean–variance approach, against the expectations.

Surprisingly, applying mean–ES at 95 % confidence level, as featured in table A.8, provides lower risk with most measures, but also higher return than the previous approaches. Obviously, the mean–ES approach has some very favourable properties in a longer run.

As indicated by the previous approaches, precious metals form a safe haven in 2003—2005 and, most distinctly, in 2009. However, the 1996—1997 safe haven, suggested by mean–variance and mean–VaR approaches, has disappeared with the mean–ES approach.

Moreover, the mean–ES approach at 99 % confidence level, in table A.9, leads to the most conservative allocations, and consequently the lowest risk measures and overall return. The minimum daily return, in other words the largest expected loss, is evidently smaller than with other approaches.

Returning to the GPD model, mean–VaR at 95 % confidence level (95 % threshold) leads to almost identical allocations as the historical distribution model, as indicated in table A.10. A 100 % allocation of bonds in the 1st quarter of 2009 can be interpreted as the low–mark of the depression.

The GPD mean–VaR approach at 99 % confidence level (95 % threshold) provides improved performance compared with 95 % level, as evident from table A.11. Consequently, it seems clear, that mean–VaR approaches do not produce consistent patterns, concerning the confidence level.

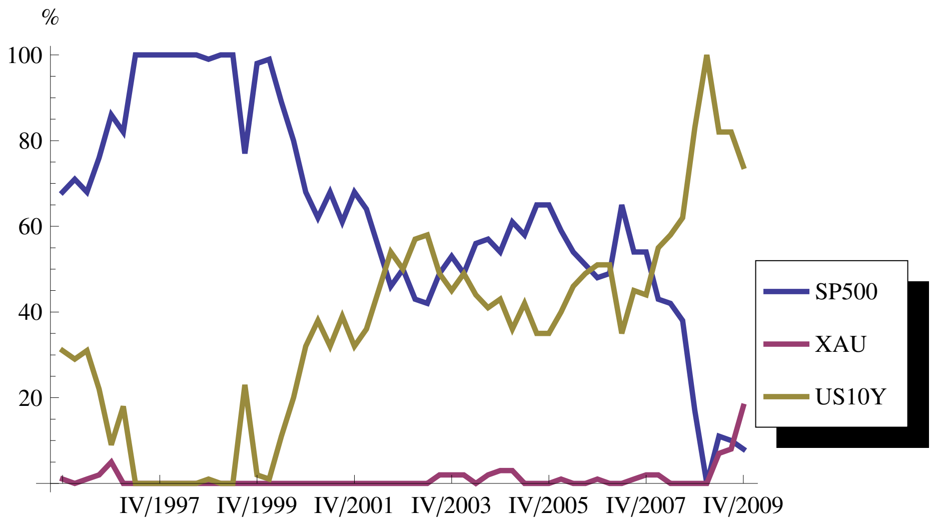

Figure 5.2.: Portfolio allocations with the GPD mean–ES approach, 95 % threshold.

Superior performance is evident with the GPD mean–ES approach at 95 % confidence level (95 % threshold), featured in table A.12. Although risk measures are slightly higher than with the historical distribution model, expected return is also higher. Again, the safe haven of precious metals can reliably be detected only in 1996 and 2009. A graph over portfolio allocations with this approach can be find in Figure 5.2.

Eventually, the GPD mean–ES approach at 99 % confidence level (95 % threshold) in table A.13 provides more conservative allocations, however with higher risk than the historical distribution model. On the contrary to the mean–VaR approach, mean–ES allocations are consistent with regard to the chosen confidence level.

As comparison, rolling time frame estimates from the minimum–variance approach are

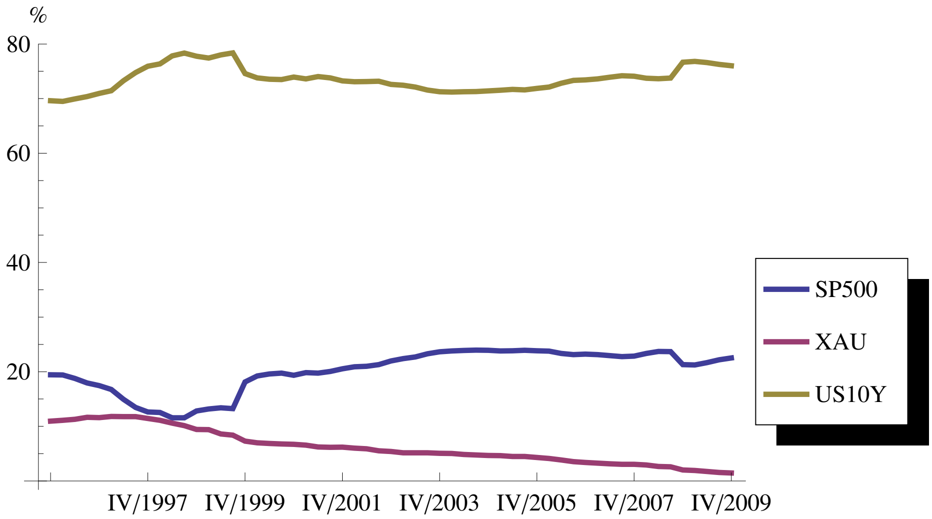

Figure 5.3.: Portfolio allocations with the minimum–variance approach.

presented in table A.14. Evidently, minimum–variance produces allocations with lower risk than the other approaches. Asset allocations are very stable, compared with the other approaches. However, the expected return is consequently much lower. Precious metals occupy a considerable, but constantly decreasing fraction of the portfolio, which makes a noticeable difference to the other approaches. A graph over portfolio allocations with this approach can be find in Figure 5.3.

Approach	**Average allocations (%)**			**Average performance**			
	S&P 500	XAU	Bond	Return	SD	Max	Min
Mean–variance	64.1	0.1	34.8	2.947	0.852	4.989	-7.113
VaR 95% hist.	65.2	2.1	32.6	3.214	0.852	4.920	-7.371
VaR 99 % hist.	65.5	2.2	32.3	3.026	0.857	5.231	-7.014
ES 95 % hist.	62.5	1.1	36.4	3.316	0.835	5.100	-7.113
ES 99 % hist.	56.9	0.7	42.4	2.004	0.753	4.948	-5.853
VaR 95 % GPD	65.1	2.1	32.8	3.084	0.853	5.100	-7.451
VaR 99 % GPD	61.0	1.4	37.6	3.286	0.816	5.166	-6.875
ES 95 % GPD	63.0	1.1	35.9	3.443	0.839	5.100	-7.113
ES 99 % GPD	57.4	0.7	41.9	2.628	0.754	5.011	-5.927
Minimum–variance	20.1	6.2	73.6	1.496	0.514	4.518	-3.165
Constant–fraction	33.3	33.3	33.3	4.155	0.968	9.602	-11.148

Table 5.8.: Summary of the rolling time frame tests for comparing allocation approach performance.

A summary of the results from the rolling time frame forecasting tests is featured in table 5.8. It may first appear surprising, that the constant–fraction portfolio earns the highest expected return. However, the other approaches tested are aimed rather at minimising expected losses than maximising the return. Moreover, all returns are relative to the portfolio risk, which is considerably high in the constant–fraction case.

Gaussian	**Mean variance**	**Min variance**	**Even 1/3**
Asset			
DAX	1 %	2.856 %	33.333 %
Commodities	1 %	1.067 %	33.333 %
German Bond	98 %	96.077 %	33.333 %
Return			
R (% p. a.)	5.861	5.476	-3.661
s.d. (% p.a.)	2.103	2.016	16.180
Sharpe	17.556	17.108	-0.014
Max (%)	0.525	0.507	5.007
Min (%)	-0.480	-0.518	-4.863
VaR			
95 %	0.195	0.187	1.691
99 %	0.285	0.274	2.386
ES			
95 %	0.273	0.262	2.102
99 %	0.353	0.338	2.717

Table 5.9.: Comparison of the portfolio allocations over the full sample period assuming Gaussian distribution. Data from period between October 10, 2007 and January 22, 2010.

5.3. European market data

5.3.1. Full sample period

With the calibrated model, another test set was run using the data in Dataset 3, European asset returns proxied by Deutsche Bank ETF prices between October 10, 2007 and January 22, 2010. Using the full sample period of Dataset 3, optimal portfolio allocations are calculated using mean–variance, mean–VaR, and mean–ES ratios as selection rules, and applying the Gaussian distribution, the historical returns, and the generalised Pareto distribution, respectively. To compare different approaches, a minimum–variance and a constant–fraction portfolio are added.

Results are reported in tables 5.9 to 5.11. Note that *annual* returns and standard deviations are reported. The Sharpe ratio is computed using the 3–month Euribor rate as risk–free asset. Again, VaR and ES values are computed from their respective statistical distribution parameters, and values in different tables are not directly comparable.

Descriptive statistics over the Gaussian mean–variance, minimum–variance, and constant–fraction portfolio are presented in table 5.9. Evidently, during an unusually low stock

Historical	VaR				ES			
Conf. level	**95 %**	**97 %**	**99 %**	**99.5 %**	**95 %**	**97 %**	**99 %**	**99.5 %**
Asset								
DAX	0 %	3 %	1 %	1 %	0 %	0 %	0 %	0 %
Commodities	0 %	3 %	3 %	0 %	0 %	1 %	2 %	1 %
German Bond	100 %	94 %	96 %	99 %	100 %	99 %	98 %	99 %
Return								
R (% p. a.)	6.156	5.272	5.679	5.952	6.156	6.065	5.974	6.065
s.d. (%)	2.288	2.092	2.097	2.158	2.288	2.217	2.178	2.217
Sharpe	0.169	0.159	0.171	0.174	0.169	0.172	0.173	0.172
Max (%)	0.533	0.509	0.529	0.523	0.533	0.535	0.537	0.535
Min (%)	-0.511	-0.660	-0.519	-0.510	-0.511	-0.482	-0.469	-0.482
VaR								
95 %	0.205	0.182	0.203	0.204	0.205	0.202	0.209	0.202
99 %	0.427	0.354	0.333	0.378	0.427	0.394	0.386	0.394
ES								
95 %	0.315	0.283	0.294	0.301	0.315	0.310	0.307	0.310
99 %	0.467	0.461	0.440	0.433	0.467	0.445	0.433	0.445

Table 5.10.: Comparison of the portfolio allocations over the complete time period assuming historical distribution. Data from period between October 10, 2007 and January 22, 2010.

market, the constant–fraction approach leads to disastrous portfolio performance, compared with the other approaches. Again, the lowest risk measure values are obtained by the minimum–variance approach, as expected. Differences from the mean–variance approach are small.

Optimal allocations from the historical samples applying both VaR and ES are found in table 5.10. Again, mean–ES features better consistency than mean–VaR. The apparent inconsistency with mean–ES at 99 % confidence level may be due to rounding errors.

Applying the GPD, as featured in table 5.11, both mean–VaR and mean–ES show consistently decreasing risk measure values with increased confidence levels, except for the 95 % VaR.

During such short period of exceptionally low stock market, no apparent differences between the different approaches can be detected. With all approaches, except the constant–fraction, allocation is close to 100 % bonds and a negligible share of the stock indices.

GPD	VaR				ES			
Conf. level	**95 %**	**97 %**	**99 %**	**99.5 %**	**95 %**	**97 %**	**99 %**	**99.5 %**
Asset								
DAX	0 %	0 %	0 %	0 %	0 %	0 %	0 %	0 %
Commodities	0 %	0 %	0 %	3 %	0 %	0 %	3 %	3 %
German Bond	100 %	100 %	100 %	97 %	100 %	100 %	97 %	97 %
Return								
R (% p. a.)	6.155	6.155	6.155	5.896	6.155	6.155	5.896	5.896
s.d. (%)	2.288	2.288	2.288	2.171	2.288	2.288	2.171	2.171
Sharpe	0.169	0.169	0.169	0.171	0.169	0.169	0.171	0.171
Max (%)	0.532	0.532	0.532	0.538	0.532	0.532	0.538	0.538
Min (%)	-0.511	-0.511	-0.511	-0.473	-0.511	-0.511	-0.473	-0.473
VaR								
95 %	0.207	0.207	0.207	0.218	0.207	0.207	0.218	0.218
99 %	0.392	0.392	0.392	0.371	0.392	0.392	0.371	0.371
ES								
95 %	0.319	0.319	0.319	0.310	0.319	0.319	0.310	0.310
99 %	0.446	0.446	0.446	0.419	0.446	0.446	0.419	0.419

Table 5.11.: Comparison of the portfolio allocations applying risk measures from the generalised Pareto distribution. Data from period between October 10, 2007 and January 22, 2010.

5.3.2. Rolling window analysis

For further examination of the different portfolio selection methods, 12 rolling windows in steps of 21 days are constructed. For each window, the calibration time is 357 days, and the full test covers 357 + (12 - 1) * 21 = 588 daily observations. There are 21 trading days each month on average.

Moreover, to assess the forecasting power of the different approaches, allocations for a given month are multiplied with returns during the next month. A statistic summary over the time series obtained is presented at the bottom of tables A.15 — A.17.

Asset allocations according to the Gaussian mean–variance and minimum–variance approaches are presented in table A.15. There is some weak evidence of stock market recovery in the mean–variance forecasts. On the contrary, the minimum–variance approach is still decreasing its share of the stock index. Commodities are allocated only by the minimum–variance approach, and only by a marginal share. Expected return is considerably higher with the minimum–variance approach, against expectations.

Continuing with the historical sample, allocations with the mean–VaR approach are very similar to the Gaussian results, as evident from table A.16. Risk levels land in between the mean–variance and the minimum–variance results. The 99.5 % confidence level does not decrease risk measures, but reduces rebalancing need. Differences are however small.

Results applying the Mean–VaR approach are almost identical to the mean–variance approach. Curiously, using the 99.5 % confidence level leads to increased expected return, in combination with reduced risk levels and rebalancing need. A more conservative approach obviously performs better during periods of troubled stock markets. Again, the differences are very small.

Intentionally, the rolling window analysis should also have covered the GPD model estimates. However, the testing procedure could not produce a fit of the GPD from the 357–day datasets. No results can hence be reported. Apparently, the dataset is too small for the estimation algorithm to allow parameter values to converge.

Approach	Average allocations (%)			Average performance			
	DAX	XDBC	XBCT	Return	SD	Max	Min
Mean–variance	0.1	0	99.9	5.816	1.873	0.289	-0.438
VaR 95% hist.	0.9	0	99.1	5.866	1.866	0.288	-0.438
VaR 99.5 % hist.	0.7	0	99.3	5.873	1.863	0.288	-0.439
ES 95 % hist.	0.2	0	99.8	5.765	1.887	0.288	-0.438
ES 99.5 % hist.	0.6	0	99.4	5.816	1.873	0.288	-0.438
Minimum–variance	2.4	1.2	96.4	6.755	1.667	0.317	-0.423
Constant–fraction	33.3	33.3	33.3	-3.661	16.180	5.007	-4.863

Table 5.12.: Summary of the rolling time frame tests for comparing allocation approach performance.

A summary of the results from the rolling time frame forecasting tests is featured in table 5.12. It may first appear surprising, that the minimum–variance portfolio earns the highest expected return, in addition to reduced risk. However, the period under investigation is very discouraging against stock investments, and hence the least risky approach is likely to produce the best results. The other approaches tested are intended to reduce portfolio risk relative to the mean–variance approach, but are not expected to challenge the minimum–variance approach in this respect. Poor forecasting power among the other approaches offers another plausible explanation.

6. Conclusions

6.1. Findings from methods used

Abandoning the traditional Gaussian mean–variance approach is related with several promising features. Applying specialised models, known issues with asset returns can better be taken into consideration. Intentionally, the idea of applying downside risk measures in asset allocation is likely to appeal a risk–averse investor.

Extreme value distributions (particularly the GPD) in combination with VaR and ES are not overwhelmingly tedious to use. Existing software packages[1] enable relatively rapid model building and calculation, at least when the portfolio consists of a limited number of assets.[2]

Comparing the mean–variance approach, the minimum variance approach, a constant–fraction approach, and mean–VaR and mean–ES from the historical sample and a GPD model, detectable deviations in results can be found. Generally, differences between mean–variance, mean–VaR, and mean–ES approaches are small. The mean–ES approach features steady consistence with regard to the different portfolio risk measures, hence making it the favoured allocation method.

The four propositions of Ho et al. (2008) can be confirmed, both using a similar dataset and extending the observation period. As a result, use of ES in portfolio allocation and risk calculations is particularly recommended. However, differences with regard to other allocation approaches are mostly small.

The sample sizes of Dataset 1 (3 776 observations) and Dataset 2 (3 030 observations) are sufficient for the GPD rolling window analysis. Using a 97 % threshold for Dataset 1, tails still contain 112 observations, whereas Dataset 2 is analysed using a 95 % threshold, hence providing tails consisting of 151 observations. On the contrary, Dataset 3 only contained 357 observations for each rolling window, leading to only 18 observations for tail estimation. The testing procedure is unable to produce parameter estimates from

[1] The approach described requires the use of computer software for calculations. Fortunately, free packages exist, proven to provide reliable results.

[2] As a consequence, VaR and ES calculations from an extreme value distribution are quite unlikely to appear as university exam questions. As homework assignments, they defend their place.

such small sample.

Most likely, this is caused by the computing algorithm for maximum likelihood estimation. Several authors (e.g. Castillo & Hadli 1997; Peng & Welsh 2001) report peculiarities with different estimation algorithms. On the other hand, Ciliberti et al. (2009) show that insufficient data leads to an increasing estimation error in portfolio optimisation, and the existence of a distinct threshold where the error diverges. The situation could have been avoided by the application of simulation methods, which, however, is out of the scope of this work.

Alternatives to maximal likelihood estimation exist, but comparing between different estimation methods is beyond the scope for this study. At least the probability weighted moments method is supported in *R*, and implementing other methods is feasible. A good overview on different methods can be find in de Zea Bermudez & Kotz (2010*a*) and de Zea Bermudez & Kotz (2010*b*)

Apparently, there exists a smallest useful sample size in order to conduct analysis based on the GPD. In literature, tails of around 100 observations are suggested (McNeil & Frey 2000). A tail of 100 observations at a 95 % threshold requires a sample of 2 000 observations. Using daily returns, the observation period would extend to nearby eight years. Moreover, this conclusion would suggest that the results obtained from Dataset 3 are unreliable, due to insufficient data. The results of Ciliberti et al. support this suggestion.

Selecting an appropriate tail threshold level, and VaR and ES confidence levels, are still open issues. Evidently, there is a relation between the available sample size, the chosen levels, and the reliability of the results obtained. Some rules of thumb would have been appreciated in the analysis.

Perhaps the best overall approach is to execute several tests, utilising different approaches. The traditional Gaussian mean–variance approach serves as a good standard, and the mean–ES results from both the historical sample and the GPD estimates provide possible alternatives. Hence, diversification benefits can be enjoyed also in decision making.

6.2. Findings from data

Considering Dataset 1, all approaches do assign the highest and lowest allocation of the S&P 500 index, and consequently also the 10–year US bond allocation, at the same time period. Asset weights deviate slightly, but usually stay inside 5 percentage points.

From Dataset 2, it is evident that all approaches, except the minimum–variance, honour the dot–com boom of the late 1990s. In the mid–2000's, asset allocations tend to their average for the whole observation period. The global financial crisis becomes apparent in portfolios only towards the end of 2008. Most approaches estimate, that the low–water mark in stock markets has passed during early 2009, and that the U. S. stock market is slowly recovering.

Replacing the NASDAQ index with a precious metal industry index in Dataset 2 did not change optimal asset allocations much. However, evidently there are times when precious metals serve as a safe haven for investors. Although these periods reportedly (Hillier et al. 2006) exist only for short times — less than one month — it is possible to achieve diversification benefits during a longer time. Especially now, in the aftermath of the global banking crisis, precious metals provide an alternative to be considered.

According to tables 5.5, 5.6 and 5.7, returns from Dataset 2 appear considerably higher for full–period portfolios than for rolling window portfolios in table 5.8. It seems evident, that quarterly estimates do not feature sufficient return forecasting power to base an investment strategy on. However, the largest expected losses are observably reduced by rebalancing, thus justifying the strategy in defensive purpose.

From the apparently robust portfolio allocations detected in the rolling window analysis of Dataset 2, some forecasting power with regard to potential loss reduction is evident among the different approaches. Moreover, it is reasonable to suggest that the ratios of return to risk in Dataset 2 are persistent for long periods, hence enabling the forecasting capabilities of the test approaches.

Due to the troubled estimation algorithm, all comparing tests could not be executed with Dataset 3. It is yet evident, that every approach, except the constant–fraction, provides close to 100 % allocation of bonds in the optimal portfolio. No signs of stock market recovery can be perceived so far. The short observation period precludes closer comparisons between European and U. S. investment circumstances further.

Both VaR and ES are frequently utilised in recent research, primarily in assessing bank capital requirements, but also in portfolio optimisation. For instance, Annaert et al. (2009) and Caillault & Guégan (2009) independently report ES producing consistently less risky portfolio allocations than VaR. Adam et al. (2008) construct portfolios of hedge funds using different risk measures, finding only small differences between the resulting portfolios, hence supporting the robustness assumption of optimal portfolio allocations.

Maximum losses frequently appear higher during full sample periods than in rolling windows, indicating that rebalancing with regard to some risk measure is beneficial. Utilising VaR, expected losses are reduced, compared with the Gaussian variance, whereas ES leads to the lowest potential losses. Hence, utilising ES and the GPD, potential portfolio losses can be reduced with nearly intact returns, and this study has therewith served its purpose.

6.3. Suggestions for further research

Several further research ideas came into thought during the conduct of this study. The following topics appear promising:

Theoretical foundation. The connection between investor utility and tail risk measures needs to be established. Particularly, utility functions, potentially optimised utilising tail risk measures from extreme value distributions, are to be defined. Similarly, the possible use of positive tail extremes should be justified by theoretical reasoning.

Practical considerations. Although the combination of expected shortfall and generalised Pareto distribution appears feasible, several questions concerning practical testing arrangements remain. Some open questions are, how the confidence level should be chosen; how the threshold level, and hence the tail sample size, should be chosen; which estimation method should be utilised; and, whether there is a relation between the optimal confidence level and the size, frequency, or any other property of the sample.

Extreme returns. Apparently, tail risk measures from the negative part of the return distribution give appropriate information to reduce potential portfolio losses. Similarly, analysing the positive tail should provide information about extraordinarily high profits. Hence, in future portfolio analysis, both tails should be investigated. Additionally, some sort of AR–GARCH preprocessing should be incorporated, on order to filter out volatility clustering in asset returns.

Mutual funds. As portfolios of a larger set of assets, the risk and return properties of mutual funds can be analysed applying tail risk measures and the GPD. Numerous research projects concerning mutual fund performance are already accomplished, and both initial data and comparable results are likely available. Analysis covering hedge funds would be of particular interest.

Portfolios of derivatives. Being difficult to analyse in the traditional mean–variance framework, portfolios containing derivative assets could also be analysed applying tail risk measures and the GPD. Structured products, recently introduced to a broader class of investors, would be of particular interest presently.

6.4. Summary

In this study, portfolios of stock, bond, and commodity indices from the U. S. and European markets are analysed, both in a traditional mean–variance framework, and utilising different tail risk measures from the historical sample and the generalised Pareto distribution estimators. The traditional mean–variance framework apparently works well as a first estimate, but expected losses are further reduced, when tail risk measures are applied. Particularly, the expected shortfall can be recommended as a good risk measure, due to consistent behaviour with different distribution models and confidence levels, in combination with relatively straight–forward computations.

As expected, the lowest levels of risk are obtained when portfolios are optimised for minimum variance. A simplistic equal–fraction approach, however, is in the best case suboptimal, and in the worst case disastrous. For best results, the Gaussian mean–variance approach should be used in combination with the mean–ES approach.

Daily asset return series exhibit persistent return–to–risk ratios, hence enabling portfolio risk reduction when tail risk related approaches are utilised. However, sufficiently large samples of observations are required, in order to obtain reliable estimators.

Extraordinary returns from the stock market during the 1990s are indicated from optimal asset allocations with each approach. Moreover, the global financial crisis is clearly evident from high bond allocations in the late 2000's. Commodities, precious metals in particular, provide a safe haven at infrequent periods. One such period is currently present.

Utilising tail risk measures, especially the ES, in combination with the historical sample and a GPD estimate is feasible with contemporary software tools. Indeed, sufficiently capable open–source packages are available. This approach is consistent with regard to confidence levels and distribution models, and can hence be recommended for a broad class of risk averse investors.

7. Svensk sammanfattning

7.1. Bakgrund

Investerare bygger sina placeringsbeslut på investeringens förväntade avkastning i förhållande till risken. Sedan introduktionen av den moderna portföljteorin (Markowitz 1952) har Gaussfördelningens[1] medelvärde använts som avkastningens väntevärde, och standardavvikelsen som riskmått. Uppställningen är matematiskt enkel och lättfattlig, och portföljoptimering kan utföras effektivt genom att tillämpa linjär programmering.

Metoden har dessvärre sina svagheter. Gaussfördelningen är en statistisk modell som bygger på den centrala gränsvärdesteorin, och blir mera exakt ju närmare medelvärdet man ligger. Med andra ord, modellen fungerar bäst för vanliga utfall av slumpmässiga händelser. Däremot, vid osäkra tider, då en bra modell skulle vara mest värdefull, fungerar Gaussmodellen dåligt.

Sedan Mandelbrot (1963) har otaliga undersökningar visat, att tillgångars avkastningar inte följer Gaussfördelningen. Investerarnas varierande tolkningar av marknadsnyheter leder till volatilitetsklustrar, dvs. stora prisändringar uppåt eller neråt följs av flera stora prisändringar mot samma eller motsatta håll. På samma sätt följs små prisändringar av flera små prisändringar. Volatilitetsklustrarna har som följd att prisändringarna inte är Gaussfördelade, utan visar tecken på skevhet, toppighet och tjocka ändor. Med andra ord, både de mest väntade och de mest oväntade utfallen är mycket vanligare än vad Gaussfördelningen förutsäger, medan utfallen "mittemellan" är ovanligare.

Gaussfördelningens varians används ofta som riskmått. Variansen och standardavvikelsen (kvadratroten av variansen) är symmetriska och gör ingen skillnad mellan positiva och negativa avvikelser i avkastningarna. En riskavers investerare däremot är mycket angelägen om att undvika förluster (negativ avkastning), medan vinsterna gärna får vara hur stora som helst. Detta sammanhang kan inte förklaras bra med variansmodellen.

Det skulle alltså vara fördelaktigt med en modell, som beaktar både avkastningarnas fördelning med vassa toppar och tjocka svansar, och investerarens motvilja till förluster.

[1] Benämningen *Gaussfördelning* används istf *normalfördelning* för att spara uttrycket *normal* som benämning för vanliga utfall.

Value–at–Risk (VaR) beskriver den största sannolika förlusten med en given konfidensnivå, och används allmänt t. ex. vid uträkning av bankers kapitalkrav enligt Baselfördraget (Basel Committee 2001). En omfattande beskrivning av VaR finns t.ex. i Jorion (2001). Vidare finns olika förbättringar av VaR, av vilka Expected Shortfall (ES), den förväntade förlusten vid en given konfidensnivå, är den förmånligaste ur uträkningssynpunkt.

Otaliga statistiska modeller har föreslagits, börjande från Mandelbrot (1963) som ansåg att Stable–fördelningen kunde bäst avbilda avkastningar från textilindustrin. Extremvärdesteori, som beskriver ytterliga utfall, har teoretisk sett väldigt fina egenskaper för modellering av fördelningar med tjocka ändor. En kombination av riskmått och extremvärdesteori kunde alltså leda till en bättre modell för uträkning av portföljrisk än den traditionella medeltal–variansmodellen.

Trots sina svagheter har den traditionella medeltal–variansmodellen behållit sin popularitet. Orsaken är dess lättbegriplighet och enkla uträkningar, som tillämpas genomgående vid ekonomutbildning runtom i världen. Dessutom ger modellen ett första estimat, som ofta är tillräckligt för investeringsbeslut. En mera avancerad modell kan knappast ersätta medeltal–variansmodellen helt, men däremot komplettera den genom att öka noggrannheten i beräkningarna.

7.2. Motivering och syfte

Att avkastningar inte följer Gaussfördelningen p.g.a. volatilitetsklustrering är välkänt. Man känner också bra till, att medeltal–variansmodellen fungerar bäst vid normala tider, samt modellens svårigheter med uträkningar då portföljen innehåller derivat. Felaktigt specifierade modeller kan leda till otillräckliga krav på säkerhetskapital, investeringar med oavsiktligt höga risker med mera. För rådande osäkra tider skulle det behövas en riskmodell med bättre anknytning till verkligheten.

Det finns otaliga användbara statistiska modeller och riskmått med lovande teoretiska egenskaper. Hittills har det dock varit ovanligt att undersöka modellers egenskaper vid utvärdering av portföljer med flera tillgångar. Bland de första undersökningarna med portföljer jämför Ho et al. (2008) portföljsammansättningar baserade på olika riskmått i början av den rådande finanskrisen. Detta arbete utvecklar resonemanget hos Ho et al. på flera sätt.

NASDAQ–indexet, som mycket sällan ingår i portföljerna i Ho et al.:s modeller, ersätts med en ädelmetallsindex i syfte att uppnå en ännu bredare diversifiering i portföljerna, då ädelmetaller antas ha en låg korrelation med industrins konjunkturer. Som den riskfria tillgången används priserna på 3–månaders US Treasury Bill, så att Sharpe–index kan uträknas för portföljerna. Tidsperioden för insamlad data förlängs till att omfatta tiden från november 1983 till december 2009. Testerna utökas med prover utanför samplet i syfte att undersöka modellernas förmåga att välja portfölj för en kommande tidsperiod, dvs. att förutse utvecklingen på finansmarknaden. Avslutningsvis utförs jämförande undersökning med uppgifter från den europeiska marknaden för att upptäcka möjliga skillnader mellan utvecklingen i USA och Europa.

Arbetets huvudsakliga syfte är att kvantitativt jämföra olika statistiska modellers prestationsförmåga vid val av portföljtillgångar. Prestationen beskrivs av portföljens förväntade avkastning i relation till förväntad osäkerhet och möjlig förlust.

7.3. Tidigare forskning

Extremvärdesfördelningar har redan länge använts inom meteorologi och limnologi, och har visat sig användbara inom finansiell ekonomi. Efter att VaR fick ett "genombrott" då det accepterades som riskmått i Basel–fördraget, framfördes mycket kritik mot VaR p.g.a. dess svagheter.

Artzner et al. (1999) definierar fyra önskvärda egenskaper för riskmått, och kallar riskmått med dessa egenskaper *koherenta.* De ställer positionens risk i förhållande till positionens framtida nettovärde, och skiljer därmed mellan acceptabla och oacceptabla risker som positioner med acceptabel eller oacceptabel framtida värde. Ett referensinstrument, dvs. en riskfri tillgång, används som hjälpmedel för att definiera positionens risk. Kraven på riskmått uppställs som följande fyra axiomer.

1. *Invarians i omvandling* betyder, att genom att tillägga ett bestämt belopp α till utgångspositionen och investera det i referensinstrumentet, minskas riskmåttet med α.
2. *Subadditivitet* innebär, att en kombination av flera tillgångar inte ökar risken, utan positionens risk är högst detsamma som de enskilda tillgångarnas.
3. *Positiv enhetlighet* anger, att positionens risk är linjärt beroende av dess storlek.
4. *Monitonitet* betyder, att positioner med lägre framtida värde har större risk än positioner med högre framtida värde.

Ett riskmått som uppfyller dessa axiomer anses koherent. Artzner et al. visar, att obetingad VaR är koherent endast då man antar att avkastningarna följer en elliptisk fördelning, t.ex. Gauss eller Student–t, eftersom subadditivitet inte gäller i annat fall. Slutligen föreslår de användandet av en betingad VaR, som är räknemässigt det förmånligaste koherenta riskmåttet. Vanliga benämningar för betingad VaR är bl.a. *tail conditional expectation* och *expected shortfall*.

Harmantzis et al. (2006) jämför VaR och ES uträknade med Gaussfördelning, historiska data, generaliserad Pareto- och Stable Pareto -fördelningarna. Som datamaterial använder de fyra valutakurser och sex börsindex. De jämför förluster estimerade ur modellerna med de verkliga förlusterna. Om den verkliga förlusten överstiger estimatet, har det skett en överträdelse, och prestationerna mäts genom antalet överträdelser.

Enligt Harmantzis et al. ger Paretofördelningarna de bästa resultaten för VaR, medan bästa ES–estimaten fås ur historiska data. Generaliserad Paretofördelning är genomgående bättre än de andra fördelningsmodellerna. På samma sätt som i annan tidigare forskning undersöks tillgångarna skilt för sig, och syftet är att utreda de olika modellernas prestationsförmåga.

Bali & Theodossiou (2008) genomför en ännu mera utförlig utredning med sju olika statistiska fördelningar på Standard & Poor –indexet från år 1950 till 2000. Av extremvärdesfördelningarna undersöks GEV, GPD samt Box–Cox GEV. Dessutom ingår fyra skevhetsfördelningar.

Resultaten visar enligt Bali & Theodossiou att skevhetsfördelningarna avbildar tjocka ändor alldeles likvärdigt med extremvärdesfördelningarna. Då skevhetsfördelningarnas parametrar är lättare att hitta än extremvärdesfördelningarnas, underlättas modelleringen av förlustprocesser.

Vidare tyder resultaten på, att verkliga VaR–gränsvärden varierar med tiden, vilket inte kan upptäckas med obetingad VaR. I detta fall fungerar ES mycket bättre än VaR. Bali & Theodossiou föreslår därför, att datamaterialet filtreras först med en AR–GARCH (1, 1) –modell för att eliminera heteroskedasticitet i materialet.

Ho et al. (2008) avviker från tidigare undersökningar genom att analysera en portfölj av flera tillgångar med hjälp av VaR och ES samt en extremvärdesfördelning. På grund av räknekapaciteten som behövs, använder Ho et al. endast tre olika tillgångar. Då två av dessa är amerikanska aktieindex med hög inbördes korrelation, blir resultatet i praktiken portföljer av ett aktieindex (S&P 500) och ett obligationsindex (US 10–åriga).

För att klargöra skillnader mellan de olika modellerna, ställer Ho et al. fyra testbara påståenden.

1. Portföljstrategier med varians, VaR och ES som riskmått leder till exakt samma portföljer, om man antar att avkastningarna följer Gaussfördelningen.
2. Olika riskmått leder till olika portföljer, om man utgår från historiska data eller en extremvärdesfördelning.
3. Om man utgår från historiska data eller extremvärdesfördelning, ingår det mera tillgångar av låg risknivå i portföljen, jämfört med Gaussfördelning.
4. Då ES används som riskmått, blir den optimala portföljens eventuella förluster mindre än då VaR används.

Alla påståenden kan bestyrkas, och medeltal–ES -modellen kan därför anses ge konsistenta och tillförlitliga resultat för val av tillgångar i en investeringsportfölj.

Sammanfattningsvis kan det konstateras, att forskningen i ämnet riktades först på de olika modellernas matematiska egenskaper, i syfte att hitta fördelar med att använda ett visst riskmått eller fördelning. Genomgående har man analyserat enskilda tillgångar, och portföljer har kommit i bilden först under de senaste åren.

Från den tidigare forskningen står det klart, att VaR inte är det allra lämpligaste riskmåttet. Det är lätt att inse, att det går att bilda två portföljer med samma VaR, av vilka den ena består av köpoptioner och den andra av säljfuturer. Trots lika VaR är portföljernas potentiella förlust väldigt olika.

Modeller, som utgår från Gaussfördelningen, har visat sig underlägsna mot mera specialiserade modeller vid uträkning av riskmått som VaR och ES. Extremvärdesfördelningar har speciellt lovande egenskaper för modellering av sällsynta händelser.

Under det senaste året har flera undersökningar med ES och derivatportföljer publicerats. Av någon anledning förekommer de i journaler (Quantitative Finance, Journal of Portfolio Management, Journal of Banking and Finance) vars senaste årgång inte finns tillgänglig i Hankens bibliotek, och återges därför inte i detta arbete.

7.4. Uppställning

I den empiriska undersökningen bildas optimala portföljer av tre tillgångar genom att använda standardavvikelse, VaR samt ES som riskmått, samt Gaussfördelning, historiska data och generaliserad Paretofördelning som statistisk modell. Dessutom uträknas optimala portföljer med Gaussfördelningens minsta varians, och en portfölj med jämna (1/3) andelar av varje tillgång. Urvalets medeltal används som väntevärde för avkastningen, och korta positioner tillåts inte. Portföljen med det högsta jämförelsetalet, dvs. avkastning i förhållande till riskmått, anses bäst.

Först upprepas testen av Ho et al.:s fyra påståenden. Som datamaterial används dataserie 1, som innehåller dagliga priser för S&P 500 –indexet, NASDAQ –indexet och Förenta Staternas 10–åriga statsobligation från tiden mellan 6 januari 1992 och 5 oktober 2007. Aktieindexerna erhålls direkt från Yahoo Finance, medan obligationspriserna är uträknade ur centralbankens (Federal Reserve) noteringar. Ur priserna uträknas sedan dagliga logaritmiska avkastningar. Resultaten jämförs med Ho et al. för att försäkra att modellerna i detta arbete är riktigt uppgjorda.

I följande steg utvidgas datamaterialet i dataserie 2 att omfatta tiden mellan 20 december 1983 och 14 december 2009. Då Ho et al. redan konstaterat, att NASDAQ–aktieindexet knappt alls blir allokerat i portföljerna, ersätts NASDAQ–indexet med en index för guld– och silverindustri. Ädelmetaller antas ha mindre korrelation med industrins konjunkturer, och förväntas därför erbjuda diversifieringsfördelar, som denna undersökning kan upptäcka. Dessutom införs 3–månaders Treasury Bill som riskfri tillgång, så att Sharpe–kvoten av de optimala portföljerna ur de olika modellerna kan räknas ut. Som jämförelsetal används förhållandet mellan portföljens överloppsavkastning, dvs. differensen mellan portföljens avkastning och den riskfria tillgångens avkastning, och riskmått.

Slutligen genomförs samma undersökning med dataserie 3, som består av motsvarande europeiska index. I detta arbete används iTraxx indexfonder för tyska aktieindex, tyska statsobligationer och tyska likvida förnödenheter. Eftersom iTraxx grundades först 2007, täcker Dataset 3 endast tidsperioden från 10 oktober 2007 till 22 januari 2010. Som riskfri tillgång används 3–månaders Euriborränta. Som jämförelsetal används förhållandet mellan portföljens överloppsavkastning och riskmått.

Undersökningen går ut på att först ta fram deskriptiv statistik på datamaterialet. Samtidigt uträknas riskmått för hela urvalet. Uträkningarna görs med programmet *R*, en variant av *S–Plus* med öppet källkod. Formlerna för uträkning av riskmåtten finns angivna av bl. a. Neftci (2000), Bali (2003) och Marimoutou et al. (2009).

Sedan indelas tidsperioden i "rullande fönster" för att undersöka hur allokeringen av tillgångar i optimala portföljer ändrar under kortare tidsperioder. Slutligen testas modellernas prognosförmåga genom att välja portfölj för en tidsperiod på basen av parametrarna från föregående period. Optimeringsekvationer i sluten form finns endast för Gaussfördelningens standardavvikelse och minimi-varians. De övriga modellerna är oelegant uträknade med "rå styrka", dvs. avkastningar och riskmått räknas ut skilt för varje möjlig portföljsammansättning med 1 % noggrannhet, och den bästa portföljen väljs ut.

Programkoden för arbetet finns som bilaga i appendix B. På grund av den korta inlärningstiden uppfyller programkoden inte några höga kvalitetskrav. Förhoppningsvis hjälper den till att förstå hur detta arbete genomfördes i praktiken, och kan fungera som underlag till kommande arbeten.

7.5. Resultat

Undersökning av dataserie 1 visar, att Ho et al.:s fyra påståenden gäller. Då avkastningarna antas följa Gaussfördelningen, blir alla riskmåtten produkter av standardavvikelsen och en koefficient, och den optimala portföljen blir följaktligen alltid densamma. Om man däremot utgår från historiska data eller GPD, blir det skillnader i sammansättningen av den optimala portföljen. Modellerna med VaR och ES väljer mera ränteinstrument i portföljen än modellen med standardavvikelse, och eventuella förluster blir oftast mindre då ES används.

Ur tabellerna 5.1 – 5.4 kan man utläsa, att dataserie 1 är mycket likt Ho et al.:s datamaterial. Då resultaten också stämmer med Ho et al.:s upptäckter, kan de i detta arbete använda portföljmodellerna anses vara riktiga.

Analysen av dataserie 2 börjar med en jämförelse mellan medeltal–variansportföljen, minimivariansportföljen och en portfölj med lika stora andelar. Medeltal–variansportföljen visar sig ge den största avkastningen och det högsta Sharpemåttet, medan den lika delade portföljen har största standardavvikelsen och lägsta Sharpemåttet. Minimivariansportföljen ger cirka 20 % lägre avkastning än medeltal–variansportföljen, men den största förlusten hos medeltal–variansportföljen är över dubbelt större än hos minimiviariansportföljen. En jämförelse av riskmåtten visar, att skillnaderna i ES mellan portföljerna är större än skillnaderna i VaR, som i sin tur är större än skillnaderna i standardavvikelse.

Eftersom det anses konstaterat, att Gaussfördelningen leder till samma portföljer oavsett riskmått, undersöks VaR- och ES -modellerna endast med historiska data och GPD. Det visar sig, att medeltal–ES -modellen ger portföljsammansättningar, vars risk minskar konsekvent, då konfidensnivån stiger. VaR -modellerna uppför sig mindre regelbundet i jämförelse.

Resultaten av testen med "rullande föster" visar, att alla modeller kan rätt bra återge både högkonjunkturen under slutet av 1990–talet, nedgången efter IT–bubblans punktering, uppgången i mitten av 2000–talet och den rådande finanskrisen. Skillnaderna hos de optimala portföljsammansättningarna med olika modeller är dock inte stora. Ädelmetallsindexet blir väldigt sällan invalt i portföljerna, och endast minimivariansportföljerna innehåller märkbara andelar av ädelmetallsindexet.

På samma sätt som tidigare, producerar ES–modellerna portföljer, vars risk följer konsekvent urvalets konfidensnivå. Den högsta avkastningen uppnås genom medeltal–ES -modellen med GPD–fördelning. Den lägsta risken erhålls med minimivariansmodellen. Grovt taget är sambandet mellan dessa modeller sådant, att minimivariansen ger halva avkastningen och halva risken jämfört med medeltal–ES. Investeraren måste alltså definiera sina preferenser med omsorg för att kunna välja mellan dessa modeller.

Europeiska data i dataserie 3 kommer från en kort tidsperiod under en världsomfattande lågkonjunktur. Skillnaderna mellan de olika portföljmodellerna blir därför knappt märkbara, och alla modeller väljer nästan uteslutande obligationer i portföljen. Följaktligen visar sig strategin med lika andelar närmast katastrofal på kort sikt.

Undersökningen med "rullande fönster" leder inte till nya upptäckter om marknadens beteende. Däremot saknas resultat med GPD–fördelningen, då testuppställningen inte kan räkna ut värden för riskmåtten. Orsaken torde bero på att maximum likelihood -algoritmen som används i R inte kan framställa tillförlitliga parameterestimat med alltför små urval.

7.6. Avslutning

Avsteg från den traditionella Gaussfördelade medeltal–variansmodellen är förknippat med många fördelar. Teoretiskt sett kan kända svårigheter med avkastningars fördelningar undvikas, och användningen av specialiserade riskmått bör tilltala investerare med ovilja mot risktagning. Extremvärdesfördelningar, i synnerhet GPD, i kombination med riskmått som VaR och ES är inte speciellt krävande att använda, och datoriserade hjälpmedel med inbyggt stöd finns att tillgå.

En jämförelse mellan medeltal–variansmodellen, minimivariansmodellen och lika-andelsmodellen med medeltal–VaR och medeltal–ES från historiska data och extremvärdesfördelningar ger som resultat, att skillnaderna mellan de optimala portföljerna inte är stora, med undantag för lika-andelsmodellen. Medeltal–ES -modellen fungerar konsekvent med olika fördelningar och konfidensnivåer, och leder till de minsta eventuella förlusterna i förhållande till avkastning. Då Ho et al.:s fyra påståenden också kan bestyrkas, är medeltal–ES följaktligen den mest favoriserade av de testade modellerna.

Analysen av dataserie 3 är ofullständig pga. alltför litet urval. Detta slags problem kan övervinnas vid kommande undersökningar t.ex. genom simulering.

Estimering av sannolikhetsfördelningarnas parametrar sker i detta arbete genom Maximum Likelihood -metoden. Det är rätt enkelt att utnyttja alternativa estimeringsmetoder, t. ex. Method of Moments, Probability Weighted Moments m. fl., och följaktligen kunde valet av konfidensnivå och estimeringsmetod i förhållande till urvalsstorleken undersökas mera ingående. En täckande översikt av metoderna finns i de Zea Bermudez & Kotz (2010*a*) och de Zea Bermudez & Kotz (2010*b*).

De undersökta modellerna producerar stabila portföljallokeringar ur den synpunkten, att det sällan behövs stora ombalanseringar mellan perioderna. Vid en jämförelse av portföljinnehav för hela observationsperioden med ombalansering kvartalsvis, visar det sig, att de största förlusterna reduceras märkbart vid ombalansering. Därmed kan modellerna anses äga en viss prognostiseringsförmåga.

Genom att ersätta ett aktieindex med ett ädelmetallsindex går det att upptäcka tidsperioder, då ädelmetaller erbjuder en "skyddshamn" för placeringar. Ädelmetallsindexet upptas i portföljerna i betydande grad endast i slutet av undersökningsperioden. Ädelmetaller erbjuder alltså en "skyddshamn" för närvarande. Då aktieindexets andel är på stigande och obligationsindexet på avtagande, kan det tolkas, att modellerna påstår att den värsta recessionen i USA är förbi.

Som möjliga framtida undersökningsobjekt kan följande områden föreslås:

1. Utvärdering av olika estimeringsmetoder för extremvärdesfördelningarnas parametrar, samt tumregler för val av konfidensnivå och urvalsstorlek.

2. Jämförelser mellan olika placeringsfonder och fondförvaltare, utgående från modellerna i detta arbete.

3. Inverkan av fördelningens positiva ända på portföljens förväntade avkastning.

Som en konklusion av arbetet står, att den traditionella Gaussfördelade medeltal–variansmodellen fungerar bra i vanliga fall, och är mycket användbar på grund av sin enkelhet. Investerare, som vill ytterligare minska på risken för förluster, rekommenderas använda en medeltal–ES -modell från en extremvärdesfördelning. Detta gäller i all synnerhet, om portföljen innehåller derivat. ES fungerar konsekvent med valet av fördelningsmodell och konfidensnivå, och investeraren kan prova sig fram till önskad risknivå. Hjälpmedel för beräkningarna finns fritt tillgängliga.

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Appendices

A. Collected tables

A.1. Comparisons with Ho et al.

	VaR	**95 %**	**VaR**	**99 %**
Period	*Ho*	Dataset 1	*Ho*	Dataset 1
1	*74 / 0 / 26*	65 / 0 / 35	*54 / 0 / 46*	48 / 4 / 48
2	*71 / 0 / 29*	65 / 0 / 35	*54 / 0 / 46*	50 / 3 / 47
3	*66 / 0 / 34*	65 / 0 / 35	*54 / 0 / 46*	50 / 3 / 47
4	*71 / 0 / 29*	64 / 0 / 36	*54 / 0 / 46*	48 / 4 / 48
5	*66 / 0 / 34*	65 / 0 / 35	*54 / 0 / 46*	48 / 4 / 48
6	*66 / 0 / 34*	65 / 0 / 35	*54 / 0 / 46*	48 / 4 / 48
7	*81 / 0 / 19*	70 / 0 / 30	*67 / 0 / 33*	48 / 4 / 48
8	*81 / 0 / 19*	82 / 0 / 18	*67 / 0 / 33*	48 / 4 / 48
9	*72 / 0 / 28*	65 / 0 / 35	*55 / 0 / 45*	48 / 4 / 48
10	*72 / 0 / 28*	64 / 0 / 36	*55 / 0 / 45*	48 / 4 / 48
11	*72 / 0 / 28*	64 / 0 / 36	*55 / 0 / 45*	48 / 4 / 48
mean	*72 / 0 / 28*	67 / 0 / 33	*57 / 0 / 43*	48 / 4 / 48
s.d.	*5.25 / 0 / 5.25*	5.09 / 0 / 5.09	*5.14 / 0 / 5.14*	0.77 / 0.39 / 0.39

	VaR	**97 %**	**VaR**	**99.5 %**
Period	*Ho*	Dataset 1	*Ho*	Dataset 1
1	*64 / 0 / 36*	59 / 0 / 41	*58 / 4 / 38*	59 / 0 / 41
2	*64 / 0 / 36*	59 / 0 / 41	*59 / 2 / 39*	59 / 0 / 41
3	*64 / 0 / 36*	59 / 0 / 41	*49 / 3 / 48*	59 / 0 / 41
4	*64 / 0 / 36*	59 / 0 / 41	*49 / 3 / 48*	58 / 0 / 42
5	*61 / 0 / 39*	44 / 4 / 52	*49 / 3 / 48*	59 / 0 / 41
6	*61 / 0 / 39*	59 / 0 / 41	*58 / 4 / 38*	59 / 0 / 41
7	*64 / 0 / 36*	62 / 0 / 38	*58 / 4 / 38*	59 / 0 / 41
8	*64 / 0 / 36*	62 / 0 / 38	*58 / 4 / 38*	59 / 0 / 41
9	*64 / 0 / 36*	62 / 0 / 38	*58 / 4 / 38*	59 / 0 / 41
10	*65 / 0 / 35*	62 / 0 / 38	*58 / 4 / 38*	59 / 0 / 41
11	*65 / 0 / 35*	62 / 0 / 38	*58 / 4 / 38*	59 / 0 / 41
mean	*64 / 0 / 36*	59 / 0 / 41	*56 / 4 / 41*	59 / 0 / 41
s.d.	*1.36 / 0 / 1.36*	4.97 / 1.15 / 3.88	*4.27 / 0.69 / 4.62*	0.29 / 0 / 0.29

Table A.1.: Asset allocation (%) in rolling windows applying Value–at–Risk from the historical distribution. Results of Ho et al., Table 3a are *emphasised*, and results of the current research are in normal font. Data from period between January 6, 1992 and October 5, 2007.

	ES	**95 %**	**ES**	**99 %**
Period	*Ho*	Dataset 1	*Ho*	Dataset 1
1	*60 / 0 / 40*	62 / 0 / 38	*55 / 0 / 45*	60 / 0 / 40
2	*59 / 0 / 41*	58 / 0 / 42	*54 / 0 / 46*	60 / 0 / 40
3	*58 / 0 / 42*	57 / 0 / 43	*53 / 0 / 47*	59 / 0 / 41
4	*57 / 0 / 43*	55 / 0 / 45	*53 / 0 / 47*	58 / 0 / 42
5	*57 / 0 / 43*	56 / 0 / 44	*53 / 0 / 47*	59 / 0 / 41
6	*59 / 0 / 41*	57 / 0 / 43	*54 / 0 / 46*	59 / 0 / 41
7	*62 / 0 / 38*	64 / 0 / 36	*56 / 0 / 44*	61 / 0 / 39
8	*63 / 0 / 37*	66 / 0 / 34	*56 / 0 / 44*	62 / 0 / 38
9	*61 / 0 / 59*	65 / 0 / 35	*56 / 1 / 43*	62 / 0 / 38
10	*60 / 0 / 40*	62 / 0 / 38	*56 / 0 / 44*	61 / 0 / 39
11	*60 / 0 / 40*	63 / 0 / 37	*56 / 0 / 44*	61 / 0 / 39
mean	*60 / 0 / 40*	60 / 0 / 40	*55 / 0 / 45*	60 / 0 / 40
s.d.	*1.91 / 0 / 1.91*	3.76 / 0 / 3.76	*1.35 / 0.30 / 1.47*	1.27 / 0 / 1.27

	ES	**97 %**	**ES**	**99.5 %**
Period	*Ho*	Dataset 1	*Ho*	Dataset 1
1	*57 / 0 / 43*	58 / 0 / 42	*53 / 0 / 47*	58 / 0 / 42
2	*56 / 0 / 44*	58 / 0 / 42	*52 / 0 / 48*	57 / 0 / 43
3	*55 / 0 / 45*	57 / 0 / 43	*51 / 0 / 49*	57 / 0 / 43
4	*54 / 0 / 46*	55 / 0 / 45	*51 / 0 / 49*	57 / 0 / 43
5	*54 / 0 / 46*	56 / 0 / 44	*51 / 0 / 49*	56 / 1 / 43
6	*56 / 0 / 44*	56 / 0 / 44	*53 / 0 / 47*	57 / 0 / 43
7	*58 / 0 / 42*	60 / 0 / 40	*55 / 0 / 45*	59 / 0 / 41
8	*58 / 0 / 42*	62 / 0 / 38	*55 / 0 / 45*	64 / 3 / 33
9	*58 / 0 / 42*	60 / 0 / 40	*55 / 0 / 45*	57 / 5 / 38
10	*57 / 0 / 43*	58 / 0 / 42	*53 / 0 / 47*	59 / 0 / 41
11	*57 / 0 / 43*	60 / 0 / 40	*55 / 0 / 45*	59 / 0 / 41
mean	*56 / 0 / 44*	58 / 0 / 42	*53 / 0 / 47*	58 / 1 / 41
s.d.	*1.50 / 0 / 1.50*	2.04 / 0 / 2.04	*1.70 / 0 / 1.70*	2.09 / 1.59 / 2.93

Table A.2.: Asset allocation (%) in rolling windows applying expected shortfall from the empirical distribution. Results of Ho et al., Table 3a are *emphasised*, and results of the current research are in normal font. Data from period between January 6, 1992 and October 5, 2007.

	VaR	**95 %**	**VaR**	**99 %**
Period	*Ho*	Dataset 1	*Ho*	Dataset 1
1	*70 / 0 / 30*	77 / 0 / 23	*52 / 4 / 44*	53 / 0 / 47
2	*70 / 0 / 30*	77 / 0 / 23	*54 / 1 / 45*	53 / 0 / 47
3	*70 / 0 / 30*	43 / 2 / 55	*56 / 0 / 44*	53 / 0 / 47
4	*69 / 0 / 31*	43 / 2 / 55	*56 / 0 / 44*	53 / 0 / 47
5	*66 / 0 / 34*	43 / 2 / 55	*53 / 3 / 44*	53 / 0 / 47
6	*70 / 0 / 30*	43 / 2 / 55	*53 / 3 / 44*	53 / 0 / 47
7	*74 / 0 / 26*	98 / 0 / 2	*52 / 4 / 44*	53 / 0 / 47
8	*77 / 0 / 23*	98 / 0 / 2	*53 / 4 / 43*	53 / 0 / 47
9	*67 / 0 / 33*	99 / 0 / 1	*52 / 4 / 44*	53 / 2 / 45
10	*67 / 0 / 33*	43 / 2 / 55	*52 / 4 / 44*	53 / 0 / 47
11	*67 / 0 / 33*	75 / 0 / 25	*52 / 4 / 44*	53 / 0 / 47
mean	*70 / 0 / 30*	67 / 1 / 32	*53 / 3 / 44*	53 / 0 / 47
s.d.	*3.29 / 0 / 3.29*	24.68 / 1.04 / 23.70	*1.54 / 1.66 / 0.45*	0 / 0.60 / 0.60

	VaR	**97 %**	**VaR**	**99.5 %**
Period	*Ho*	Dataset 1	*Ho*	Dataset 1
1	*67 / 0 / 33*	59 / 0 / 41	*54 / 1 / 45*	53 / 0 / 47
2	*66 / 0 / 34*	59 / 0 / 41	*54 / 1 / 45*	53 / 0 / 47
3	*66 / 0 / 34*	59 / 0 / 41	*56 / 0 / 44*	53 / 0 / 47
4	*66 / 0 / 34*	59 / 0 / 41	*56 / 0 / 44*	53 / 0 / 47
5	*52 / 2 / 46*	44 / 4 / 52	*50 / 2 / 48*	53 / 0 / 47
6	*66 / 0 / 34*	59 / 0 / 41	*56 / 0 / 44*	53 / 0 / 47
7	*66 / 0 / 34*	62 / 0 / 38	*52 / 2 / 44*	53 / 0 / 47
8	*66 / 0 / 34*	62 / 0 / 38	*53 / 4 / 43*	53 / 0 / 47
9	*67 / 0 / 33*	62 / 0 / 38	*52 / 4 / 44*	53 / 2 / 45
10	*67 / 0 / 33*	59 / 0 / 41	*52 / 4 / 44*	53 / 0 / 47
11	*67 / 0 / 33*	62 / 0 / 38	*52 / 4 / 44*	53 / 2 / 45
mean	*65 / 0 / 35*	59 / 0 / 41	*53 / 2 / 44*	53 / 0 / 47
s.d.	*4.37 / 0.60 / 3.77*	4.87 / 1.15 / 3.78	*2.01 / 1.73 / 1.29*	0 / 0.77 / 0.77

Table A.3.: Asset allocation (%) in rolling windows applying Value–at–Risk from the generalised Pareto distribution with 97 % threshold. Results of Ho et al., Table 3b are *emphasised*, and results of the current research are in normal font. Data from period between January 6, 1992 and October 5, 2007.

	ES	95 %	ES	99 %
Period	*Ho*	Dataset 1	*Ho*	Dataset 1
1	*59 / 0 / 41*	59 / 0 / 41	*55 / 0 / 45*	60 / 0 / 40
2	*59 / 0 / 41*	59 / 0 / 41	*55 / 0 / 45*	58 / 0 / 42
3	*58 / 0 / 42*	59 / 0 / 41	*52 / 0 / 48*	58 / 0 / 42
4	*54 / 0 / 46*	59 / 0 / 41	*52 / 0 / 48*	57 / 0 / 43
5	*52 / 2 / 46*	59 / 0 / 41	*52 / 0 / 48*	57 / 0 / 43
6	*58 / 0 / 42*	59 / 0 / 41	*52 / 0 / 48*	57 / 0 / 43
7	*66 / 0 / 34*	62 / 0 / 38	*56 / 0 / 44*	61 / 0 / 39
8	*66 / 0 / 34*	62 / 0 / 38	*56 / 0 / 44*	62 / 1 / 37
9	*60 / 0 / 40*	62 / 0 / 38	*55 / 1 / 44*	59 / 2 / 39
10	*60 / 0 / 40*	59 / 0 / 41	*56 / 0 / 44*	58 / 2 / 40
11	*60 / 0 / 40*	62 / 0 / 38	*56 / 0 / 44*	59 / 2 / 39
mean	*59 / 0 / 41*	60 / 0 / 40	*54 / 0 / 46*	59 / 1 / 40
s.d.	*4.20 / 0.60 / 3.88*	1.45 / 0 / 1.45	*1.85 / 0.30 / 1.91*	1.60 / 0.88 / 1.97

	ES	97 %	ES	99.5 %
Period	*Ho*	Dataset 1	*Ho*	Dataset 1
1	*57 / 0 / 43*	59 / 0 / 41	*53 / 0 / 47*	58 / 2 / 40
2	*56 / 0 / 44*	58 / 0 / 42	*51 / 0 / 49*	59 / 0 / 41
3	*55 / 0 / 45*	58 / 0 / 42	*50 / 0 / 50*	59 / 0 / 41
4	*54 / 0 / 46*	55 / 0 / 45	*50 / 0 / 50*	59 / 0 / 41
5	*55 / 0 / 45*	57 / 0 / 43	*50 / 0 / 50*	56 / 3 / 41
6	*56 / 0 / 44*	58 / 0 / 42	*51 / 0 / 49*	58 / 2 / 40
7	*59 / 0 / 41*	59 / 0 / 41	*55 / 0 / 45*	59 / 3 / 38
8	*60 / 0 / 40*	62 / 0 / 38	*55 / 0 / 45*	62 / 2 / 36
9	*58 / 2 / 40*	60 / 0 / 40	*55 / 0 / 45*	60 / 3 / 37
10	*55 / 1 / 44*	59 / 0 / 41	*54 / 0 / 46*	58 / 3 / 39
11	*55 / 1 / 44*	60 / 0 / 40	*55 / 0 / 45*	61 / 2 / 37
mean	*56 / 0 / 43*	59 / 0 / 41	*53 / 0 / 47*	59 / 2 / 39
s.d.	*1.91 / 0.67 / 2.05*	1.72 / 0 / 1.72	*2.25 / 0 / 2.25*	1.54 / 1.19 / 1.80

Table A.4.: Asset allocation (%) in rolling windows applying expected shortfall from the generalised Pareto distribution with 97 % threshold. Results of Ho et al., Table 3b are *emphasised*, and results of the current research are in normal font. Data from period between January 6, 1992 and October 5, 2007.

A.2. US market data

Mean–Variance. Gaussian distribution.							
	SP500	XAU	US10Y		SP500	XAU	US10Y
IV/1995	67	0	33				
I/1996	71	0	29	I/2003	38	0	62
II/1996	70	1	29	II/2003	37	0	63
III/1996	74	2	24	III/2003	48	2	50
IV/1996	78	4	18	IV/2003	53	2	45
I/1997	79	1	20	I/2004	46	2	52
II/1997	100	0	0	II/2004	55	1	44
III/1997	100	0	0	III/2004	58	2	40
IV/1997	100	0	0	IV/2004	52	4	44
I/1998	100	0	0	I/2005	63	4	33
II/1998	100	0	0	II/2005	59	0	41
III/1998	100	0	0	III/2005	69	0	31
IV/1998	100	0	0	IV/2005	68	0	32
I/1999	100	0	0	I/2006	61	0	39
II/1999	100	0	0	II/2006	56	0	44
III/1999	77	0	23	III/2006	51	0	49
IV/1999	100	0	0	IV/2006	47	2	51
I/2000	100	0	0	I/2007	49	0	51
II/2000	100	0	0	II/2007	67	0	33
III/2000	94	0	6	III/2007	55	0	45
IV/2000	79	0	21	IV/2007	54	1	45
I/2001	66	0	34	I/2008	41	3	56
II/2001	80	0	20	II/2008	39	1	60
III/2001	66	0	34	III/2008	34	0	66
IV/2001	78	0	22	IV/2008	17	0	83
I/2002	70	0	30	I/2009	5	1	94
II/2002	59	0	41	II/2009	13	5	82
III/2002	43	0	57	III/2009	12	7	81
IV/2002	48	0	52	IV/2009	9	16	75
ASSETS				Max	4.989	% p.a.	
Average	64.1 %	0.1 %	34.8 %	Min	-7.113	% p.a.	
St. dev.	25.9	0.25	24.8	VaR 95%	1.390		
APPROACH				VaR 99%	1.970		
Return	2.947	% p.a.		ES 95%	1.757		
SD	0.852			ES 99%	2.271		

Table A.5.: Asset allocation (%) forecast performance in rolling windows for the Gaussian mean–variance approach. Data from period between December 20, 1983, and December 14, 2009.

Mean–VaR, 95% confidence level. Historical distribution.							
	SP500	XAU	US10Y		SP500	XAU	US10Y
IV/1995	88	1	11				
I/1996	95	4	1	I/2003	36	0	64
II/1996	87	6	7	II/2003	36	1	63
III/1996	95	5	0	III/2003	42	4	54
IV/1996	95	5	0	IV/2003	60	5	35
I/1997	94	5	1	I/2004	41	2	57
II/1997	93	7	0	II/2004	67	0	33
III/1997	92	7	1	III/2004	63	5	32
IV/1997	96	4	0	IV/2004	43	4	53
I/1998	97	3	0	I/2005	63	5	32
II/1998	96	4	0	II/2005	67	0	33
III/1998	100	0	0	III/2005	67	0	33
IV/1998	99	0	1	IV/2005	67	0	33
I/1999	100	0	0	I/2006	67	0	33
II/1999	100	0	0	II/2006	50	1	49
III/1999	100	0	0	III/2006	40	0	60
IV/1999	98	0	2	IV/2006	40	0	60
I/2000	98	0	2	I/2007	40	0	60
II/2000	96	0	4	II/2007	68	0	32
III/2000	94	0	6	III/2007	49	0	51
IV/2000	78	0	22	IV/2007	55	1	44
I/2001	62	0	38	I/2008	40	0	60
II/2001	81	0	19	II/2008	39	0	61
III/2001	70	0	30	III/2008	37	0	63
IV/2001	70	0	30	IV/2008	22	0	78
I/2002	70	0	30	I/2009	7	1	77
II/2002	49	0	51	II/2009	13	10	77
III/2002	43	0	57	III/2009	13	10	77
IV/2002	40	0	60	IV/2009	10	21	69
ASSETS				Max	4.920	% p.a.	
Average	65.2 %	2.1 %	32.6 %	Min	-7.371	% p.a.	
St. dev.	27.3	3.7	26.8	VaR 95%	1.357		
APPROACH				VaR 99%	2.221		
Return	3.214	% p.a.		ES 95%	1.972		
SD	0.852			ES 99%	3.091		

Table A.6.: Asset allocation (%) forecast performance in rolling windows for the mean–VaR approach using the historical distribution at 95 % confidence level. Data from period between December 20, 1983, and December 14, 2009.

Mean–VaR 99%. Historical distribution.							
	SP500	XAU	US10Y		SP500	XAU	US10Y
IV/1995	59	9	32				
I/1996	93	7	0	I/2003	39	0	61
II/1996	93	7	0	II/2003	47	0	53
III/1996	86	9	5	III/2003	59	0	41
IV/1996	86	10	4	IV/2003	59	1	40
I/1997	86	9	5	I/2004	53	2	45
II/1997	94	4	2	II/2004	59	0	41
III/1997	94	3	3	III/2004	59	1	40
IV/1997	97	2	1	IV/2004	59	1	40
I/1998	98	2	0	I/2005	59	1	40
II/1998	98	2	0	II/2005	59	0	41
III/1998	98	2	0	III/2005	67	0	33
IV/1998	99	1	0	IV/2005	67	0	33
I/1999	99	1	0	I/2006	59	0	41
II/1999	83	0	17	II/2006	57	0	43
III/1999	72	0	28	III/2006	57	0	43
IV/1999	100	0	0	IV/2006	57	0	43
I/2000	96	3	1	I/2007	57	0	43
II/2000	96	3	1	II/2007	66	0	34
III/2000	96	3	1	III/2007	59	0	41
IV/2000	67	0	33	IV/2007	58	0	42
I/2001	62	0	38	I/2008	56	0	44
II/2001	66	0	34	II/2008	53	0	47
III/2001	66	0	34	III/2008	43	0	57
IV/2001	66	0	34	IV/2008	16	1	83
I/2002	65	0	35	I/2009	0	4	96
II/2002	62	0	38	II/2009	7	7	86
III/2002	48	0	52	III/2009	6	9	85
IV/2002	59	0	41	IV/2009	13	20	67
ASSETS				Max	5.231	% p.a.	
Average	65.5 %	2.2 %	32.3 %	Min	-7.014	% p.a.	
St. dev.	24.7	3.8	24.5	VaR 95%	1.341		
APPROACH				VaR 99%	2.251		
Return	3.026	% p.a.		ES 95%	1.974		
SD	0.857			ES 99%	3.118		

Table A.7.: Asset allocation (%) forecast performance in rolling windows for the mean–VaR approach using the historical distribution at 95 % confidence level. Data from period between December 20, 1983, and December 14, 2009.

Mean–ES, 95 % confidence level. Historical distribution.							
	SP500	XAU	US10Y		SP500	XAU	US10Y
IV/1995	66	0	34				
I/1996	71	0	29	I/2003	43	0	57
II/1996	69	1	30	II/2003	43	0	57
III/1996	72	2	26	III/2003	50	2	48
IV/1996	80	4	16	IV/2003	54	2	44
I/1997	81	0	19	I/2004	48	2	50
II/1997	100	0	0	II/2004	56	0	44
III/1997	100	0	0	III/2004	56	2	42
IV/1997	100	0	0	IV/2004	54	4	43
I/1998	100	0	0	I/2005	61	3	36
II/1998	100	0	0	II/2005	58	0	42
III/1998	100	0	0	III/2005	65	0	35
IV/1998	100	0	0	IV/2005	65	0	35
I/1999	100	0	0	I/2006	59	1	40
II/1999	94	0	6	II/2006	54	0	46
III/1999	70	0	30	III/2006	51	0	49
IV/1999	97	0	3	IV/2006	48	1	51
I/2000	100	0	0	I/2007	49	0	51
II/2000	89	0	11	II/2007	65	0	35
III/2000	79	0	21	III/2007	54	1	45
IV/2000	67	0	33	IV/2007	54	2	44
I/2001	60	0	40	I/2008	43	2	55
II/2001	68	0	32	II/2008	42	0	58
III/2001	61	0	39	III/2008	38	0	62
IV/2001	67	0	33	IV/2008	17	0	83
I/2002	64	0	36	I/2009	0	0	100
II/2002	56	0	44	II/2009	11	7	82
III/2002	46	0	54	III/2009	10	8	82
IV/2002	49	0	51	IV/2009	8	18	74
ASSETS				Max	5.100	% p.a.	
Average	62.5 %	1.1 %	36.4 %	Min	-7.113	% p.a.	
St. dev.	24.9	2.8	23.7	VaR 95%	1.297		
APPROACH				VaR 99%	2.159		
Return	3.316	% p.a.		ES 95%	1.917		
SD	0.835			ES 99%	3.058		

Table A.8.: Asset allocation (%) forecast performance in rolling windows for the mean–ES approach using the historical distribution at 99 % confidence level. Data from period between December 20, 1983, and December 14, 2009.

Mean–ES, 99 %confidence level. Historical distribution.							
	SP500	XAU	US10Y		SP500	XAU	US10Y
IV/1995	46	0	54				
I/1996	49	0	51	I/2003	43	0	57
II/1996	54	0	46	II/2003	43	0	57
III/1996	55	0	45	III/2003	54	0	46
IV/1996	60	0	40	IV/2003	58	0	42
I/1997	57	0	43	I/2004	54	0	46
II/1997	75	0	25	II/2004	60	0	40
III/1997	75	0	25	III/2004	61	0	39
IV/1997	75	0	25	IV/2004	58	1	41
I/1998	85	0	15	I/2005	62	0	38
II/1998	84	0	16	II/2005	61	0	39
III/1998	74	0	26	III/2005	63	0	37
IV/1998	74	0	26	IV/2005	62	0	38
I/1999	77	0	23	I/2006	61	0	39
II/1999	75	0	25	II/2006	59	0	41
III/1999	56	0	44	III/2006	56	0	44
IV/1999	82	0	18	IV/2006	54	1	45
I/2000	79	0	21	I/2007	55	0	45
II/2000	73	0	27	II/2007	62	0	38
III/2000	69	0	31	III/2007	60	0	40
IV/2000	66	0	34	IV/2007	60	0	40
I/2001	61	0	39	I/2008	47	3	50
II/2001	66	0	34	II/2008	45	0	55
III/2001	62	0	38	III/2008	40	0	60
IV/2001	67	0	33	IV/2008	17	0	83
I/2002	66	0	34	I/2009	5	0	95
II/2002	62	0	38	II/2009	11	7	82
III/2002	45	0	55	III/2009	11	8	81
IV/2002	53	0	47	IV/2009	0	18	82
ASSETS				Max	4.948	% p.a.	
Average	56.9 %	0.7 %	42.4 %	Min	-5.853	% p.a.	
St. dev.	18.3	2.7	16.7	VaR 95%	1.187		
APPROACH				VaR 99%	1.903		
Return	2.004	% p.a.		ES 95%	1.705		
SD	0.753			ES 99%	2.630		

Table A.9.: Asset allocation (%) forecast performance in rolling windows for the mean–ES approach using the historical distribution at 99 % confidence level. Data from period between December 20, 1983, and December 14, 2009.

Mean–VaR, 95% confidence level. Generalised Pareto distribution, 95% tail.							
	SP500	XAU	US10Y		SP500	XAU	US10Y
IV/1995	90	1	9				
I/1996	95	4	1	I/2003	36	0	64
II/1996	87	6	7	II/2003	36	1	63
III/1996	95	5	0	III/2003	42	4	54
IV/1996	95	5	0	IV/2003	60	5	35
I/1997	94	5	1	I/2004	39	2	59
II/1997	93	7	0	II/2004	67	0	33
III/1997	93	7	0	III/2004	63	5	32
IV/1997	96	4	0	IV/2004	43	4	53
I/1998	97	3	0	I/2005	63	5	32
II/1998	96	4	0	II/2005	67	0	33
III/1998	100	0	0	III/2005	67	0	33
IV/1998	99	0	1	IV/2005	67	0	33
I/1999	100	0	0	I/2006	67	0	33
II/1999	100	0	0	II/2006	50	1	49
III/1999	100	0	0	III/2006	40	0	60
IV/1999	98	0	2	IV/2006	40	0	60
I/2000	98	0	2	I/2007	40	0	60
II/2000	96	0	4	II/2007	68	0	32
III/2000	94	0	6	III/2007	49	0	51
IV/2000	77	0	23	IV/2007	55	1	44
I/2001	63	0	37	I/2008	40	0	60
II/2001	81	0	19	II/2008	39	0	61
III/2001	70	0	30	III/2008	37	0	63
IV/2001	70	0	30	IV/2008	23	0	77
I/2002	69	0	31	I/2009	0	0	100
II/2002	47	0	53	II/2009	14	9	77
III/2002	43	0	57	III/2009	13	1	77
IV/2002	40	0	60	IV/2009	10	21	69
ASSETS				Max	5.100	% p.a.	
Average	65.1 %	2.1 %	32.8 %	Min	-7.451	% p.a.	
St. dev.	27.6	3.6	27.2	VaR 95%	1.356		
APPROACH				VaR 99%	2.295		
Return	3.084	% p.a.		ES 95%	1.971		
SD	0.853			ES 99%	3.128		

Table A.10.: Asset allocation (%) forecast performance in rolling windows for the mean–VaR approach using the GPD at 95 % confidence level. Data from period between December 20, 1983, and December 14, 2009.

Mean–VaR, 99 % confidence level. Generalised Pareto distribution, 95% tail.							
	SP500	XAU	US10Y		SP500	XAU	US10Y
IV/1995	62	3	35				
I/1996	62	3	35	I/2003	49	0	51
II/1996	65	2	33	II/2003	47	1	52
III/1996	80	2	18	III/2003	46	3	51
IV/1996	78	4	18	IV/2003	55	2	43
I/1997	79	2	19	I/2004	48	1	51
II/1997	99	1	0	II/2004	58	0	42
III/1997	88	2	10	III/2004	57	1	42
IV/1997	97	1	2	IV/2004	55	2	43
I/1998	91	2	7	I/2005	57	1	42
II/1998	97	1	2	II/2005	58	0	42
III/1998	96	1	3	III/2005	59	0	41
IV/1998	95	1	4	IV/2005	59	0	41
I/1999	95	0	5	I/2006	59	0	41
II/1999	93	0	7	II/2006	59	0	41
III/1999	69	0	31	III/2006	59	0	41
IV/1999	97	0	3	IV/2006	48	3	49
I/2000	100	0	0	I/2007	48	3	49
II/2000	69	0	31	II/2007	61	1	38
III/2000	67	0	33	III/2007	59	0	41
IV/2000	69	0	31	IV/2007	58	1	41
I/2001	56	0	44	I/2008	45	1	54
II/2001	62	0	38	II/2008	45	0	55
III/2001	62	0	38	III/2008	38	0	62
IV/2001	65	0	35	IV/2008	18	0	82
I/2002	59	0	41	I/2009	0	2	98
II/2002	59	0	41	II/2009	11	6	83
III/2002	47	1	52	III/2009	5	8	87
IV/2002	49	0	51	IV/2009	11	19	70
ASSETS				Max	5.166	% p.a.	
Average	61.0 %	1.4 %	37.6 %	Min	-6.875	% p.a.	
St. dev.	23.2	2.8	22.1	VaR 95%	1.251		
APPROACH				VaR 99%	2.201		
Return	3.286	% p.a.		ES 95%	1.868		
SD	0.816			ES 99%	3.001		

Table A.11.: Asset allocation (%) forecast performance in rolling windows for the mean–VaR approach using the GPD at 99 % confidence level. Data from period between December 20, 1983, and December 14, 2009.

Mean–ES, 95% confidence level. Generalised Pareto distribution, 95% threshold.							
	SP500	XAU	US10Y		SP500	XAU	US10Y
IV/1995	68	1	31				
I/1996	71	0	29	I/2003	43	0	57
II/1996	68	1	31	II/2003	42	0	58
III/1996	76	2	22	III/2003	49	2	49
IV/1996	86	5	9	IV/2003	53	2	45
I/1997	82	0	18	I/2004	49	2	49
II/1997	100	0	0	II/2004	56	0	44
III/1997	100	0	0	III/2004	57	2	41
IV/1997	100	0	0	IV/2004	54	3	43
I/1998	100	0	0	I/2005	61	3	36
II/1998	100	0	0	II/2005	58	0	42
III/1998	100	0	0	III/2005	65	0	35
IV/1998	99	0	1	IV/2005	65	0	35
I/1999	100	0	0	I/2006	59	1	40
II/1999	100	0	0	II/2006	54	0	46
III/1999	77	0	23	III/2006	51	0	49
IV/1999	98	0	2	IV/2006	48	1	51
I/2000	99	0	1	I/2007	49	0	51
II/2000	89	0	11	II/2007	65	0	35
III/2000	80	0	20	III/2007	54	1	45
IV/2000	68	0	32	IV/2007	54	2	44
I/2001	62	0	38	I/2008	43	2	55
II/2001	68	0	32	II/2008	42	0	58
III/2001	61	0	39	III/2008	38	0	62
IV/2001	68	0	32	IV/2008	17	0	83
I/2002	64	0	36	I/2009	0	0	100
II/2002	55	0	45	II/2009	11	7	82
III/2002	46	0	54	III/2009	10	8	82
IV/2002	50	0	50	IV/2009	8	18	74
ASSETS				Max	5.100	% p.a.	
Average	63.0 %	1.1 %	35.9 %	Min	-7.113	% p.a.	
St. dev.	25.2	2.8	24.0	VaR 95%	1.306		
APPROACH				VaR 99%	2.243		
Return	3.443	% p.a.		ES 95%	1.921		
SD	0.839			ES 99%	3.080		

Table A.12.: Asset allocation (%) forecast performance in rolling windows for the mean–ES approach using the GPD at 95 % confidence level. Data from period between December 20, 1983, and December 14, 2009.

Mean–ES, 99 % confidence level. Generalised Pareto distribution, 95% threshold.							
	SP500	XAU	US10Y		SP500	XAU	US10Y
IV/1995	51	0	49				
I/1996	54	0	46	I/2003	47	0	53
II/1996	56	0	44	II/2003	47	0	53
III/1996	68	1	31	III/2003	52	0	48
IV/1996	65	3	32	IV/2003	55	0	45
I/1997	69	1	30	I/2004	52	0	48
II/1997	79	0	21	II/2004	55	0	45
III/1997	80	0	20	III/2004	57	0	43
IV/1997	84	0	16	IV/2004	55	1	44
I/1998	87	0	13	I/2005	61	0	39
II/1998	85	0	15	II/2005	57	0	43
III/1998	78	0	22	III/2005	61	0	39
IV/1998	77	0	23	IV/2005	61	0	39
I/1999	83	0	17	I/2006	60	0	40
II/1999	79	0	21	II/2006	57	0	43
III/1999	54	0	46	III/2006	54	0	46
IV/1999	80	0	20	IV/2006	52	1	47
I/2000	80	0	20	I/2007	52	0	48
II/2000	72	0	28	II/2007	60	0	40
III/2000	69	0	31	III/2007	57	0	43
IV/2000	63	0	37	IV/2007	57	1	42
I/2001	59	0	41	I/2008	48	2	50
II/2001	66	0	34	II/2008	46	0	54
III/2001	58	0	42	III/2008	36	0	64
IV/2001	69	0	31	IV/2008	15	0	85
I/2002	62	0	38	I/2009	4	1	95
II/2002	62	0	38	II/2009	10	7	83
III/2002	48	0	52	III/2009	9	9	82
IV/2002	52	0	48	IV/2009	4	16	80
ASSETS				Max	5.011	% p.a.	
Average	57.4 %	0.7 %	41.9 %	Min	-5.927	% p.a.	
St. dev.	19.0	2.6	17.6	VaR 95%	1.189		
APPROACH				VaR 99%	1.972		
Return	2.628	% p.a.		ES 95%	1.696		
SD	0.754			ES 99%	2.622		

Table A.13.: Asset allocation (%) forecast performance in rolling windows for the mean–ES approach using the GPD at 99 % confidence level. Data from period between December 20, 1983, and December 14, 2009.

Minimum–variance approach. Gaussian distribution.							
	SP500	XAU	US10Y		SP500	XAU	US10Y
IV/1995	19.43962	10.94660	69.61379				
I/1996	19.39438	11.10239	69.50323	I/2003	22.39877	5.163253	72.43798
II/1996	18.76859	11.27987	69.95154	II/2003	22.71311	5.16066	72.12623
III/1996	17.95520	11.66802	70.37678	III/2003	23.27460	5.151911	71.5735
IV/1996	17.45444	11.58794	70.95762	IV/2003	23.65554	5.075712	71.26875
I/1997	16.73936	11.80878	71.45186	I/2004	23.79055	5.020327	71.18912
II/1997	14.94130	11.77470	73.284	II/2004	23.90068	4.841602	71.25772
III/1997	13.47039	11.77001	74.7596	III/2004	23.96751	4.757628	71.27487
IV/1997	12.64497	11.43709	75.91794	IV/2004	23.91389	4.663387	71.42272
I/1998	12.52642	11.11556	76.35802	I/2005	23.81393	4.649439	71.53663
II/1998	11.57442	10.59146	77.83411	II/2005	23.82706	4.489222	71.68372
III/1998	11.55634	10.11555	78.32812	III/2005	23.91018	4.485721	71.6041
IV/1998	12.81786	9.430164	77.75197	IV/2005	23.83164	4.299196	71.86916
I/1999	13.18023	9.399157	77.42061	I/2006	23.75738	4.121274	72.12135
II/1999	13.38935	8.595305	78.01535	II/2006	23.35022	3.84855	72.80123
III/1999	13.22795	8.400855	78.3712	III/2006	23.14173	3.525168	73.3331
IV/1999	18.13367	7.283312	74.58302	IV/2006	23.20935	3.371911	73.41874
I/2000	19.23428	6.987067	73.77865	I/2007	23.12336	3.264279	73.61236
II/2000	19.58128	6.861364	73.55736	II/2007	22.94221	3.131614	73.92617
III/2000	19.73112	6.785854	73.48303	III/2007	22.76533	3.041318	74.19335
IV/2000	19.34294	6.705554	73.95151	IV/2007	22.85146	3.038414	74.11013
I/2001	19.82270	6.572179	73.60512	I/2008	23.33532	2.939904	73.72477
II/2001	19.74149	6.234315	74.0242	II/2008	23.69652	2.662538	73.64094
III/2001	20.03328	6.169252	73.79747	III/2008	23.66702	2.584958	73.74802
IV/2001	20.54842	6.199564	73.25202	IV/2008	21.30814	2.018645	76.67321
I/2002	20.89852	6.023709	73.07777	I/2009	21.24467	1.928824	76.8265
II/2002	21.00722	5.885657	73.10713	II/2009	21.65846	1.745782	76.59576
III/2002	21.28816	5.536162	73.17568	III/2009	22.17887	1.550489	76.27064
IV/2002	21.98223	5.408614	72.60916	IV/2009	22.53185	1.456973	76.01118
ASSETS				Max	4.518	% p.a.	
Average	20.1 %	6.2 %	73.6 %	Min	-3.165	% p.a.	
St. dev.	3.8	3.2	2.3	VaR 95%	0.840		
APPROACH				VaR 99%	1.190		
Return	1.496	% p.a.		ES 95%	1.061		
SD	0.514			ES 99%	1.371		

Table A.14.: Asset allocation (%) forecast performance in rolling windows for the Gaussian minimum–variance approach. Data from period between December 20, 1983, and December 14, 2009.

A.3. European market data

Mean and Minimum Variance, Gaussian distribution.						
	Mean–variance			Minimum–variance		
	DAX	COMM	BOND	DAX	COMM	BOND
Feb 2009	0	0	100	3	1	96
Mar 2009	0	0	100	3	1	96
Apr 2009	0	0	100	3	1	96
May 2009	0	0	100	3	1	96
Jun 2009	0	0	100	3	1	96
Jul 2009	0	0	100	2	1	97
Aug 2009	0	0	100	2	1	97
Sep 2009	0	0	100	2	1	97
Oct 2009	1	0	99	2	1	97
Nov 2009	2	0	98	2	1	97
Dec 2009	2	0	98	2	1	97
Jan 2010	2	0	98	2	1	97
ASSETS						
Average	0.1 %	0 %	99.9 %	2.4 %	1.2 %	96.4 %
St. dev.	0.0	0	0.0	0.3	0.1	0.2
APPROACH						
Return	5.816	% p.a.		6.755	% p.a.	
SD	1.873	% p.a.		1.667	% p.a.	
Max	0.289	% p.a.		0.317	% p.a.	
Min	-0.438	% p.a.		-0.423	% p.a.	
VaR 95%	0.171			0.146		
VaR 99%	0.251			0.218		
ES 95%	0.243			0.217		
ES 99%	0.314			0.280		

Table A.15.: Asset allocation (%) forecast performance in rolling windows for the Gaussian mean–variance and minimum–variance approaches. Data from period between October 10, 2007 and January 22, 2010.

Mean–VaR. Historical distribution.						
	VaR 95 %			VaR 99.5 %		
	DAX	COMM	BOND	DAX	COMM	BOND
Feb 2009	0	0	100	0	0	100
Mar 2009	0	0	100	0	0	100
Apr 2009	0	0	100	0	0	100
May 2009	0	0	100	0	0	100
Jun 2009	0	0	100	0	0	100
Jul 2009	0	0	100	0	0	100
Aug 2009	1	0	99	1	0	99
Sep 2009	0	0	100	1	0	99
Oct 2009	1	0	99	1	0	99
Nov 2009	4	0	96	2	0	98
Dec 2009	2	0	98	2	0	98
Jan 2010	3	0	97	2	0	98
ASSETS						
Average	0.9 %	0 %	99.1 %	0.7 %	0 %	99.3 %
St. dev.	1.3	0	1.3	0.8	0	0.8
APPROACH						
Return	5.866	% p.a.		5.873	% p.a.	
SD	1.866	% p.a.		1.863	% p.a.	
Max	0.288	% p.a.		0.288	% p.a.	
Min	-0.438	% p.a.		-0.439	% p.a.	
VaR 95%	0.159			0.161		
VaR 99%	0.332			0.332		
ES 95%	0.268			0.268		
ES 99%	0.419			0.419		

Table A.16.: Asset allocation (%) forecast performance in rolling windows for the Gaussian mean–variance approach. Data from period between October 10, 2007 and January 22, 2010.

Mean–ES. Historical distribution.						
	ES 95 %			ES 99.5 %		
	DAX	COMM	BOND	DAX	COMM	BOND
Feb 2009	0	0	100	0	0	100
Mar 2009	0	0	100	0	0	100
Apr 2009	0	0	100	0	0	100
May 2009	0	0	100	0	0	100
Jun 2009	0	0	100	0	0	100
Jul 2009	0	0	100	0	0	100
Aug 2009	0	0	100	0	0	100
Sep 2009	0	0	100	0	0	100
Oct 2009	0	0	100	1	0	99
Nov 2009	1	0	99	2	0	98
Dec 2009	1	0	99	2	0	98
Jan 2010	1	0	99	2	0	98
ASSETS						
Average	0.2 %	0 %	99.8 %	0.6 %	0 %	99.4 %
St. dev.	0.4	0	0.4	0.9	0	0.9
APPROACH						
Return	5.765	% p.a.		5.816	% p.a.	
SD	1.887	% p.a.		1.873	% p.a.	
Max	0.288	% p.a.		0.288	% p.a.	
Min	-0.438	% p.a.		-0.438	% p.a.	
VaR 95%	0.175			0.167		
VaR 99%	0.332			0.332		
ES 95%	0.270			0.269		
ES 99%	0.419			0.419		

Table A.17.: Asset allocation (%) forecast performance in rolling windows for the empirical mean–ES approach. Data from period between October 10, 2007 and January 22, 2010.

B. *R* program codes

Admittedly, the coding quality does not reach any acceptable level, the most obvious reason being the relatively short period of acquaintance with the *R* programming language. With better coding conventions, it is quite straightforward to construct a class library for tail risk based portfolio analysis. Apart from these reservations, the supplied code may help the reader to understand the approach used in this study. Moreover, possible errors in the analysis may be tracked from the code.

```
options(STERM='iESS', editor='emacsclient')
library(fBasics)
library(fExtremes)

#
# Initialisations
db <- read.csv("db.csv", sep=",", header=TRUE)
DAX <- 100*db[[2]]; BOND <- 100*db[[3]]; COMM <- 100*db[[4]]; EU3M <-
    100*db[[5]]
# DAX <- 100*DAX
# BOND <- 100*BOND

#
ARRAYSIZE <- length(EU3M)
r <- array(c(DAX, COMM, BOND), dim=c(ARRAYSIZE, 3))
ret <- t(r)
w <- c(0.67, 0.0, 0.33)
wpf = as.timeSeries(crossprod(ret, w))
wfit = gpdFit(-wpf, u = quantile(-wpf, 0.95), type = "mle", information =
     "observed")

#
# Own functions
#

myJBTest <- function(x) {
xJB <- jarqueberaTest(x);
xJB@test[[1]][[1]]
}
```

```
StatSummary3 <- function(asset1, asset2, asset3) {
cat("R (%)      :", mean(asset1), mean(asset2), mean(asset3), "\n")
cat("SD (%)     :", stdev(asset1), stdev(asset2), stdev(asset3), "\n")
cat("R/SD (%)   :", mean(asset1)/stdev(asset1), mean(asset2)/stdev(asset2)
    , mean(asset3)/stdev(asset3),"\n")
cat("Skewness   :", skewness(asset1), skewness(asset2), skewness(asset3),
    "\n")
cat("Kurtosis   :", kurtosis(asset1)+3, kurtosis(asset2)+3, kurtosis(
    asset3)+3, "\n")
cat("JarqueBera:", myJBTest(asset1), myJBTest(asset2), myJBTest(asset3),
    "\n")
cat("Minimum    :", min(asset1), min(asset2), min(asset3), "\n")
cat("\n")
cat("Asset 1    :", cor(asset1, asset1), "\n")
cat("Asset 2    :", cor(asset1, asset2), cor(asset2,asset2), "\n")
cat("Asset 3    :", cor(asset1, asset3), cor(asset2, asset3), cor(asset3,
    asset3), "\n")
}

SampleSummary <- function(asset1) {
cat("R (%) p.a.:", 252 * mean(asset1), "\n")
cat("SD (%)     :", stdev(asset1), "\n")
cat("R/SD (%)   :", mean(asset1)/stdev(asset1), "\n")
cat("Skewness   :", skewness(asset1), "\n")
cat("Kurtosis   :", kurtosis(asset1) + 3, "\n")
cat("JarqueBera:", myJBTest(asset1), "\n")
cat("Maximum    :", max(asset1), "\n")
cat("Minimum    :", min(asset1), "\n")
cat("\n")
cat("VaR 95%    :", VaR(asset1, 0.05, "sample", "lower")[[1]], "\n")
cat("VaR 97%    :", VaR(asset1, 0.03, "sample", "lower")[[1]], "\n")
cat("VaR 99%    :", VaR(asset1, 0.01, "sample", "lower")[[1]], "\n")
cat("VaR 99,5% :", VaR(asset1, 0.005, "sample", "lower")[[1]], "\n")
}

VaRnorm <- function(dist, p) {
qnorm(p, mean(dist), stdev(dist))
}

ESnorm <- function(dist, p) {
-exp(-(qnorm(p, 0, 1)^2)/2)/(p*sqrt(2*pi))*stdev(dist)
}
```

```
MVnorm.sharpe <- function(dist, riskfree) {
mean(dist-riskfree)/stdev(dist)
}

VaRnorm.sharpe <- function(dist, riskfree, p) {
mean(dist-riskfree)/qnorm(p, mean(dist), stdev(dist))
}

ESnorm.sharpe <- function(dist, riskfree, p) {
mean(dist-riskfree) / (exp(-(qnorm(p, 0, 1)^2)/2)/(p*sqrt(2*pi))*stdev(
    dist))
}

VaRhist.sharpe <- function(dist, riskfree, p) {
-mean(dist - riskfree)/VaR(dist, p, "sample", "lower")[[1]]
}

EShist.sharpe <- function(dist, riskfree, p) {
-mean(dist - riskfree)/CVaR(dist, p, "sample", "lower")[[1]]
}

VaRgpd <- function(dist, p) {
pf <- as.timeSeries(dist)
fit <- gpdFit(-pf, u = quantile(-pf, 0.95), type = "mle", information = "
    observed")
tailRisk(fit, p)[[2]]
}

ESgpd <- function(dist, p) {
pf <- as.timeSeries(dist)
fit <- gpdFit(-pf, u = quantile(-pf, 0.95), type = "mle", information = "
    observed")
tailRisk(fit, p)[[3]]
}

VaRgpd.sharpe <- function(dist, riskfree, p) {
mean(dist - riskfree)/VaRgpd(dist, p)
}

ESgpd.sharpe <- function(dist, riskfree, p) {
mean(dist - riskfree)/ESgpd(dist, p)
}

GaussianSummary <- function(asset1) {
```

```
  cat("Gaussian Summary:\n")
  cat("R/SD (%)   :", mean(asset1)/stdev(asset1), "\n")
  cat("R (% p.a.):", 252 * mean(asset1), "\n")
  cat("SD (%)     :", sqrt(252) * stdev(asset1), "\n")
  cat("Skewness   :", skewness(asset1), "\n")
  cat("Kurtosis   :", kurtosis(asset1) + 3, "\n")
  cat("JarqueBera:", myJBTest(asset1), "\n")
  cat("Maximum    :", max(asset1), "\n")
  cat("Minimum    :", min(asset1), "\n")
  cat("\n")
cat("VaR 95%   :", VaRnorm(asset1, 0.05), "\n")
cat("VaR 97%   :", VaRnorm(asset1, 0.03), "\n")
cat("VaR 99%   :", VaRnorm(asset1, 0.01), "\n")
cat("VaR 99,5% :", VaRnorm(asset1, 0.005), "\n")
cat("\n")
cat("ES 95%    :", ESnorm(asset1, 0.05), "\n")
cat("ES 97%    :", ESnorm(asset1, 0.03), "\n")
cat("ES 99%    :", ESnorm(asset1, 0.01), "\n")
cat("ES 99,5%  :", ESnorm(asset1, 0.005), "\n")
}

EmpiricSummary <- function(asset1) {
    cat("Empiric Summary:\n")
cat("R/SD (%)  :", mean(asset1)/stdev(asset1), "\n")
cat("R (%)     :", mean(asset1), "\n")
cat("SD (%)    :", stdev(asset1), "\n")
cat("Skewness  :", skewness(asset1), "\n")
cat("Kurtosis  :", kurtosis(asset1) + 3, "\n")
cat("JarqueBera:", myJBTest(asset1), "\n")
cat("Maximum   :", max(asset1), "\n")
cat("Minimum   :", min(asset1), "\n")
cat("\n")
cat("VaR 95%   :", -VaR(asset1, 0.05, "sample", "lower")[[1]], "\n")
cat("VaR 97%   :", -VaR(asset1, 0.03, "sample", "lower")[[1]], "\n")
cat("VaR 99%   :", -VaR(asset1, 0.01, "sample", "lower")[[1]], "\n")
cat("VaR 99,5% :", -VaR(asset1, 0.005, "sample", "lower")[[1]], "\n")
cat("\n")
cat("ES 95%    :", -CVaR(asset1, 0.05, "sample", "lower")[[1]], "\n")
cat("ES 97%    :", -CVaR(asset1, 0.03, "sample", "lower")[[1]], "\n")
cat("ES 99%    :", -CVaR(asset1, 0.01, "sample", "lower")[[1]], "\n")
cat("ES 99,5%  :", -CVaR(asset1, 0.005, "sample", "lower")[[1]], "\n")
}

GpdSummary <- function(asset1) {
```

```
    cat("GPD Summary:\n")
cat("R/SD (%)   :", mean(asset1)/stdev(asset1), "\n")
cat("R (%)      :", mean(asset1), "\n")
cat("SD (%)     :", stdev(asset1), "\n")
cat("Skewness   :", skewness(asset1), "\n")
cat("Kurtosis   :", kurtosis(asset1) + 3, "\n")
cat("JarqueBera:", myJBTest(asset1), "\n")
cat("Maximum    :", max(asset1), "\n")
cat("Minimum    :", min(asset1), "\n")
cat("\n")
cat("VaR 95%    :", VaRgpd(asset1, 0.95), "\n")
cat("VaR 97%    :", VaRgpd(asset1, 0.97), "\n")
cat("VaR 99%    :", VaRgpd(asset1, 0.99), "\n")
cat("VaR 99,5% :", VaRgpd(asset1, 0.995), "\n")
cat("\n")
cat("ES 95%     :", ESgpd(asset1, 0.95), "\n")
cat("ES 97%     :", ESgpd(asset1, 0.97), "\n")
cat("ES 99%     :", ESgpd(asset1, 0.99), "\n")
cat("ES 99,5%   :", ESgpd(asset1, 0.995), "\n")
}

#
# Now Finding optimal portfolio with different criteria
#
GetMVnorm <- function(sample, riskfree, accuracy) {
result <- c(-1,0,0,0)
for (i in 0:accuracy) {
        for (j in 0:(accuracy-i)) {
                weight <- c(i/100, j/100, (accuracy-i-j)/100)
                portf <- crossprod(sample, weight)
                t <- c(MVnorm.sharpe(portf,riskfree), weight)
                if (t[[1]] > result[[1]]) result <- t
                }
        }
        cat("Optimal MV / Gaussian portfolio:  ", result, "\n")
        return(c(result[[2]], result[[3]], result[[4]]))
}

FindMVnorm <- function(accuracy) {
        Amv <- GetMVnorm(ret, EU3M, accuracy)
        Fmv <- crossprod(ret, Amv)
        cat("*** Best MV-Gaussian allocation ***\n")
        cat("Weights = ", Amv, "\n")
        #cat("SDs    = ", colSds(Amv), "\n")
```

```
        cat("=========================\n")
        cat("Mean p.a. = ", mean(Fmv) * 252, "\n")
        cat("SD        = ", stdev(Fmv) * sqrt(252), "\n")
        cat("R / SD    = ", mean(Fmv) / stdev(Fmv), "\n")
        cat("Skewness  = ", skewness(Fmv), "\n")
        cat("Kurtosis  = ", kurtosis(Fmv) + 3, "\n")
        cat("Max       = ", max(Fmv), "\n")
        cat("Min       = ", min(Fmv), "\n")
        cat("VaR (95%) = ", -VaRnorm(Fmv, 1-0.95), "\n")
        cat("VaR (99%) = ", -VaRnorm(Fmv, 1-0.99), "\n")
        cat("ES (95%)  = ", -ESnorm(Fmv, 1-0.95), "\n")
        cat("ES (99%)  = ", -ESnorm(Fmv, 1-0.99), "\n")
}

GetMinVar <- function(sample) {
        vcv <- crossprod(sample - colMeans(sample), sample - colMeans(
           sample)/dim(sample)[1])
        gmv <-(inv(vcv) %*% t(t(c(1, 1, 1)))) / sum(inv(vcv) %*% t(t(c(1,
            1, 1))))
                cat("Minimum variance portfolio:  ", 100 * gmv, "\n")
        return(gmv)
        }

FindMinVar <- function() {
        Amv <- GetMinVar(t(ret))
        Fmv <- t(ret) %*% Amv
        cat("*** Best Minimum Variance allocation ***\n")
        cat("Weights = ", Amv, "\n")
        #cat("SDs   = ", colSds(Amv), "\n")
        cat("=========================\n")
        cat("Mean p.a. = ", mean(Fmv) * 252, "\n")
        cat("SD        = ", stdev(Fmv) * sqrt(252), "\n")
        cat("R / SD    = ", mean(Fmv) / stdev(Fmv), "\n")
        cat("Skewness  = ", skewness(Fmv), "\n")
        cat("Kurtosis  = ", kurtosis(Fmv) + 3, "\n")
        cat("Max       = ", max(Fmv), "\n")
        cat("Min       = ", min(Fmv), "\n")
        cat("VaR (95%) = ", -VaRnorm(Fmv, 1-0.95), "\n")
        cat("VaR (99%) = ", -VaRnorm(Fmv, 1-0.99), "\n")
        cat("ES (95%)  = ", -ESnorm(Fmv, 1-0.95), "\n")
        cat("ES (99%)  = ", -ESnorm(Fmv, 1-0.99), "\n")
        }
```

```
GetVaRnorm <- function(sample, riskfree, accuracy, p) {
        result <- c(0,0,0,0)
        for (i in 0:accuracy) {
                for (j in 0:(accuracy-i)) {
                        weight <- c(i/100, j/100, (accuracy-i-j)/100)
                        portf <- crossprod(sample, weight)
                        t <- c(VaRnorm.sharpe(portf, riskfree, p), weight
                            )
                        if (t[[1]] > result[[1]]) result <- t
                }
        }
cat("Optimal VaR / Gaussian portfolio: ", result, ", p =", p, "\n")
return(c(result[[2]], result[[3]], result[[4]]))
}

FindVaRnorm <- function(accuracy, p) {
        Amv <- GetVaRnorm(ret, EU3M, accuracy, p)
        Fmv <- crossprod(ret, Amv)
        cat("*** Best VaR-Gaussian allocation ***\n")
        cat("Weights = ", Amv, "\n")
        #cat("SDs    = ", colSds(Amv), "\n")
        cat("========================\n")
        cat("Mean p.a. = ", mean(Fmv) * 252, "\n")
        cat("SD        = ", stdev(Fmv) * sqrt(252), "\n")
        cat("R / SD    = ", mean(Fmv) / stdev(Fmv), "\n")
        cat("Skewness  = ", skewness(Fmv), "\n")
        cat("Kurtosis  = ", kurtosis(Fmv) + 3, "\n")
        cat("Max       = ", max(Fmv), "\n")
        cat("Min       = ", min(Fmv), "\n")
        cat("VaR (95%) = ", -VaRnorm(Fmv, 1-0.95), "\n")
        cat("VaR (99%) = ", -VaRnorm(Fmv, 1-0.99), "\n")
        cat("ES (95%)  = ", -ESnorm(Fmv, 1-0.95), "\n")
        cat("ES (99%)  = ", -ESnorm(Fmv, 1-0.99), "\n")
}

GetESnorm <- function(sample, riskfree, accuracy, p) {
        result <- c(0,0,0,0)
        for (i in 0:accuracy) {
                for (j in 0:(accuracy-i)) {
                        weight <- c(i/100, j/100, (accuracy-i-j)/100)
                        portf <- crossprod(sample, weight)
                        t <- c(ESnorm.sharpe(portf, riskfree, p), weight)
                        if (t[[1]] > result[[1]]) result <- t
                }
```

```
        }
cat("Optimal ES / Gaussian portfolio:  ", result, ", p =", p,  "\n")
return(c(result[[2]], result[[3]], result[[4]]))
}

FindESnorm <- function(accuracy, p) {
        Amv <- GetESnorm(ret, EU3M, accuracy, p)
        Fmv <- crossprod(ret, Amv)
        cat("*** Best ES-Gaussian allocation ***\n")
        cat("Weights = ", Amv, "\n")
        #cat("SDs   = ", colSds(Amv), "\n")
        cat("========================\n")
        cat("Mean p.a. = ", mean(Fmv) * 252, "\n")
        cat("SD        = ", stdev(Fmv) * sqrt(252), "\n")
        cat("R / SD    = ", mean(Fmv) / stdev(Fmv), "\n")
        cat("Skewness  = ", skewness(Fmv), "\n")
        cat("Kurtosis  = ", kurtosis(Fmv) + 3, "\n")
        cat("Max       = ", max(Fmv), "\n")
        cat("Min       = ", min(Fmv), "\n")
        cat("VaR (95%) = ", -VaRnorm(Fmv, 1-0.95), "\n")
        cat("VaR (99%) = ", -VaRnorm(Fmv, 1-0.99), "\n")
        cat("ES (95%)  = ", -ESnorm(Fmv, 1-0.95), "\n")
        cat("ES (99%)  = ", -ESnorm(Fmv, 1-0.99), "\n")
}


GetVaRhist <- function(sample, riskfree, accuracy, p) {
        result <- c(-1,0,0,0)
        for (i in 0:accuracy) {
                for (j in 0:(accuracy - i)) {
                        weight <- c(i/100, j/100, (accuracy - i - j)/100)
                        portf <- crossprod(sample, weight)
                        t <- c(VaRhist.sharpe(portf, riskfree, (1 - p)),
                            weight)
                        if (t[[1]] > result[[1]]) result <- t
                }
        }
        cat("Optimal VaR / Empiric portfolio: ", result, ", p =", p, "\n
            ")
        return(c(result[[2]], result[[3]], result[[4]]))
}


FindVaRhist <- function(accuracy, p) {
```

```
        Amv <- GetVaRhist(ret, EU3M, accuracy, p)
        Fmv <- crossprod(ret, Amv)
        cat("*** Best VaR-historic allocation ***\n")
        cat("Weights = ", Amv, "\n")
        #cat("SDs   = ", colSds(Amv), "\n")
        cat("========================\n")
        cat("Mean p.a. = ", mean(Fmv) * 252, "\n")
        cat("SD        = ", stdev(Fmv) * sqrt(252), "\n")
        cat("R / SD    = ", mean(Fmv) / stdev(Fmv), "\n")
        cat("Skewness  = ", skewness(Fmv), "\n")
        cat("Kurtosis  = ", kurtosis(Fmv) + 3, "\n")
        cat("Max       = ", max(Fmv), "\n")
        cat("Min       = ", min(Fmv), "\n")
        cat("VaR (95%) = ", -VaR(Fmv, 1-0.95), "\n")
        cat("VaR (99%) = ", -VaR(Fmv, 1-0.99), "\n")
        cat("ES (95%)  = ", -CVaR(Fmv, 1-0.95), "\n")
        cat("ES (99%)  = ", -CVaR(Fmv, 1-0.99), "\n")
}


GetEShist <- function(sample, riskfree, accuracy, p) {
        result <- c(-1,0,0,0)
        for (i in 0:accuracy) {
                for (j in 0:(accuracy - i)) {
                        weight <- c(i/100, j/100, (accuracy - i - j)/100)
                        portf <- crossprod(sample, weight)
                        t <- c(EShist.sharpe(portf, riskfree, (1-p)),
                            weight)
                        if (t[[1]] > result[[1]]) result <- t
                }
        }
        cat("Optimal ES / Empiric portfolio:  ", result, ", p =", p, "\n
            ")
        return(c(result[[2]], result[[3]], result[[4]]))
}


FindEShist <- function(accuracy, p) {
        Amv <- GetEShist(ret, EU3M, accuracy, p)
        Fmv <- crossprod(ret, Amv)
        cat("*** Best ES-historic allocation ***\n")
        cat("Weights = ", Amv, "\n")
        #cat("SDs   = ", colSds(Amv), "\n")
        cat("========================\n")
```

```
        cat("Mean p.a. = ", mean(Fmv) * 252, "\n")
        cat("SD        = ", stdev(Fmv) * sqrt(252), "\n")
        cat("R / SD    = ", mean(Fmv) / stdev(Fmv), "\n")
        cat("Skewness  = ", skewness(Fmv), "\n")
        cat("Kurtosis  = ", kurtosis(Fmv) + 3, "\n")
        cat("Max       = ", max(Fmv), "\n")
        cat("Min       = ", min(Fmv), "\n")
        cat("VaR (95%) = ", -VaR(Fmv, 1-0.95), "\n")
        cat("VaR (99%) = ", -VaR(Fmv, 1-0.99), "\n")
        cat("ES (95%)  = ", -CVaR(Fmv, 1-0.95), "\n")
        cat("ES (99%)  = ", -CVaR(Fmv, 1-0.99), "\n")
}



GetVaRgpd <- function(sample, riskfree, accuracy, p) {
        result <- c(-1,0,0,0)
        for (i in 0:accuracy) {
                for (j in 0:(accuracy - i)) {
                        weight <- c(i/accuracy, j/accuracy, (accuracy - i
                             - j)/accuracy)
                        portf <- crossprod(sample, weight)
                        t <- c(VaRgpd.sharpe(portf, riskfree, p), weight)
                        if (t[[1]] > result[[1]]) result <- t
                }
        }
        cat("Optimal VaR / Extreme portfolio: ", result, ", p =", p, "\n
            ")
        return(c(result[[2]], result[[3]], result[[4]]))
}

FindVaRgpd <- function(accuracy, p) {
        Amv <- GetVaRgpd(ret, EU3M, accuracy, p)
        Fmv <- crossprod(ret, Amv)
        cat("*** Best VaR-GPD allocation ***\n")
        cat("Weights = ", Amv, "\n")
        #cat("SDs   = ", colSds(Amv), "\n")
        cat("========================\n")
        cat("Mean p.a. = ", mean(Fmv) * 252, "\n")
        cat("SD        = ", stdev(Fmv) * sqrt(252), "\n")
        cat("R / SD    = ", mean(Fmv) / stdev(Fmv), "\n")
        cat("Skewness  = ", skewness(Fmv), "\n")
        cat("Kurtosis  = ", kurtosis(Fmv) + 3, "\n")
        cat("Max       = ", max(Fmv), "\n")
```

```
        cat("Min       = ", min(Fmv), "\n")
        cat("VaR (95%) = ", VaRgpd(Fmv, 0.95), "\n")
        cat("VaR (99%) = ", VaRgpd(Fmv, 0.99), "\n")
        cat("ES (95%)  = ", ESgpd(Fmv, 0.95), "\n")
        cat("ES (99%)  = ", ESgpd(Fmv, 0.99), "\n")
}


GetESgpd <- function(sample, riskfree, accuracy, p) {
        result <- c(-1,0,0,0)
        for (i in 0:accuracy) {
                for (j in 0:(accuracy - i)) {
                        weight <- c(i/accuracy, j/accuracy, (accuracy - i
                             - j)/accuracy)
                        portf <- crossprod(sample, weight)
                        t <- c(ESgpd.sharpe(portf, riskfree, p), weight)
                        if (t[[1]] > result[[1]]) result <- t
                }
        }
        cat("Optimal ES / Extreme portfolio: ", result, ", p =", p, "\n")
        return(c(result[[2]], result[[3]], result[[4]]))
}

FindESgpd <- function(accuracy, p) {
        Amv <- GetESgpd(ret, EU3M, accuracy, p)
        Fmv <- crossprod(ret, Amv)
        cat("*** Best ES-GPD allocation ***\n")
        cat("Weights = ", Amv, "\n")
        #cat("SDs   = ", colSds(Amv), "\n")
        cat("========================\n")
        cat("Mean p.a. = ", mean(Fmv) * 252, "\n")
        cat("SD        = ", stdev(Fmv) * sqrt(252), "\n")
        cat("R / SD    = ", mean(Fmv) / stdev(Fmv), "\n")
        cat("Skewness  = ", skewness(Fmv), "\n")
        cat("Kurtosis  = ", kurtosis(Fmv) + 3, "\n")
        cat("Max       = ", max(Fmv), "\n")
        cat("Min       = ", min(Fmv), "\n")
        cat("VaR (95%) = ", VaRgpd(Fmv, 0.95), "\n")
        cat("VaR (99%) = ", VaRgpd(Fmv, 0.99), "\n")
        cat("ES (95%)  = ", ESgpd(Fmv, 0.95), "\n")
        cat("ES (99%)  = ", ESgpd(Fmv, 0.99), "\n")
}

#
```

```
# and, ladies and gentlemen, here come the finders
#

# we have a sample of 588
# analyze 12 periods of 21 days
# 588 - (21 * 12-1) = 348

INSAMPLE <- 348
# should be PERLEN
QUARTALE <- 21
PERIODS  <- 11
ASSETS   <- 3
NOBS <- (PERIODS * QUARTALE)

FindBestMVnorm <- function() {
        Wmv <- numeric()
        for (i in 0 : PERIODS) {
                from <- i * QUARTALE + 1
                to <- from + INSAMPLE - 1
                X <- DAX[from : to]
                Y <- COMM[from : to]
                Z <- BOND[from : to]
                RF <- EU3M[from : to]
                r <- array(c(X, Y, Z), dim = c(INSAMPLE, ASSETS))
                rt <- t(r)
                wPeriod <- GetMVnorm(rt, RF, 100)
                Wmv <- c(Wmv, rep(wPeriod, times = QUARTALE))
        }
        dim(Wmv) <- c(ASSETS, NOBS + QUARTALE)

        INFROM <- INSAMPLE + 1
        INTO <- INSAMPLE + NOBS
#       cat("Dim(Wmv) = ", dim(Wmv), "\n")
        Rmv <- array(c(DAX[INFROM:INTO], COMM[INFROM:INTO], BOND[INFROM:
            INTO]), dim = c(NOBS, ASSETS))
        Smv <- numeric()
#       cat ("Dim(Rmv) = ", dim(Rmv), "\n")

        for ( i in 1:(NOBS) )
                Smv <- c( Smv, sum( Rmv[i, ] * t(Wmv)[i, ]) )
#       cat("Dim(Smv) = ", dim(Smv), "\n")

        cat("*** Best MV-Gaussian allocation ***\n")
        cat("Means = ", rowMeans(Wmv), "\n")
```

```
        cat("SDs   = ", rowSds(Wmv), "\n")
        cat("=======================\n")
        cat("Mean p.a. = ", 252 * mean(Smv), "\n")
        cat("SD        = ", stdev(Smv) * sqrt(252), "\n")
        cat("R / SD    = ", mean(Smv) / stdev(Smv), "\n")
        cat("Max       = ", max(Smv), "\n")
        cat("Min       = ", min(Smv), "\n")
        cat("VaR (95%) = ", -VaRnorm(Smv, 1-0.95), "\n")
        cat("VaR (99%) = ", -VaRnorm(Smv, 1-0.99), "\n")
        cat("ES (95%)  = ", -ESnorm(Smv, 1-0.95), "\n")
        cat("ES (99%)  = ", -ESnorm(Smv, 1-0.99), "\n")
}


FindBestVaRhist <- function(p) {
        Wmv <- numeric()
        for (i in 0 : PERIODS) {
                from <- i * QUARTALE + 1
                to <- from + INSAMPLE - 1
                X <- DAX[from : to]
                Y <- COMM[from : to]
                Z <- BOND[from : to]
                RF <- EU3M[from : to]
                r <- array(c(X, Y, Z), dim = c(INSAMPLE, ASSETS))
                rt <- t(r)
                wPeriod <- GetVaRhist(rt, RF, 100, p)
                Wmv <- c(Wmv, rep(wPeriod, times = QUARTALE))
        }
        dim(Wmv) <- c(ASSETS, NOBS + QUARTALE)

        INFROM <- INSAMPLE + 1
        INTO <- INSAMPLE + NOBS
        Rmv <- array(c(DAX[INFROM:INTO], COMM[INFROM:INTO], BOND[INFROM:
            INTO]), dim = c(NOBS, ASSETS))
        Smv <- numeric()

        for ( i in 1:NOBS )
                Smv <- c( Smv, sum( Rmv[i, ] * t(Wmv)[i, ]) )

        cat("*** Best VaR-Historic allocation ***\n")
        cat("Means = ", rowMeans(Wmv), "\n")
        cat("SDs   = ", rowSds(Wmv), "\n")
        cat("=======================\n")
        cat("Mean p.a. = ", 252 * mean(Smv), "\n")
```

```
        cat("SD         = ", stdev(Smv) * sqrt(252), "\n")
        cat("R / SD     = ", mean(Smv) / stdev(Smv), "\n")
        cat("Max        = ", max(Smv), "\n")
        cat("Min        = ", min(Smv), "\n")
        cat("VaR (95%) = ", -VaR(Smv, 1-0.95), "\n")
        cat("VaR (99%) = ", -VaR(Smv, 1-0.99), "\n")
        cat("ES (95%)  = ", -CVaR(Smv, 1-0.95), "\n")
        cat("ES (99%)  = ", -CVaR(Smv, 1-0.99), "\n")
}

FindBestEShist <- function(p) {
        Wmv <- numeric()
        for (i in 0 : PERIODS) {
                from <- i * QUARTALE + 1
                to <- from + INSAMPLE - 1
                X <- DAX[from : to]
                Y <- COMM[from : to]
                Z <- BOND[from : to]
                RF <- EU3M[from : to]
                r <- array(c(X, Y, Z), dim = c(INSAMPLE, ASSETS))
                rt <- t(r)
                wPeriod <- GetEShist(rt, RF, 100, p)
                Wmv <- c(Wmv, rep(wPeriod, times = QUARTALE))
        }
        dim(Wmv) <- c(ASSETS, NOBS + QUARTALE)

        INFROM <- INSAMPLE + 1
        INTO <- INSAMPLE + NOBS
        Rmv <- array(c(DAX[INFROM:INTO], COMM[INFROM:INTO], BOND[INFROM:
            INTO]), dim = c(NOBS, ASSETS))
        Smv <- numeric()

        for ( i in 1:NOBS )
                Smv <- c( Smv, sum( Rmv[i, ] * t(Wmv)[i, ]) )

        cat("*** Best ES-Historic allocation ***\n")
        cat("Means = ", rowMeans(Wmv), "\n")
        cat("SDs   = ", rowSds(Wmv), "\n")
        cat("=======================\n")
        cat("Mean p.a. = ", 252 * mean(Smv), "\n")
        cat("SD         = ", stdev(Smv) * sqrt(252), "\n")
        cat("R / SD     = ", mean(Smv) / stdev(Smv), "\n")
        cat("Max        = ", max(Smv), "\n")
        cat("Min        = ", min(Smv), "\n")
```

```
        cat("VaR (95%) = ", -VaR(Smv, 1-0.95), "\n")
        cat("VaR (99%) = ", -VaR(Smv, 1-0.99), "\n")
        cat("ES (95%)  = ", -CVaR(Smv, 1-0.95), "\n")
        cat("ES (99%)  = ", -CVaR(Smv, 1-0.99), "\n")
}

FindBestVaRgpd <- function(p) {
        Wmv <- numeric()
        for (i in 0 : PERIODS) {
                from <- i * QUARTALE + 1
                to <- from + INSAMPLE - 1
                X <- DAX[from : to]
                Y <- COMM[from : to]
                Z <- BOND[from : to]
                RF <- EU3M[from : to]
                r <- array(c(X, Y, Z), dim = c(INSAMPLE, ASSETS))
                rt <- t(r)
                wPeriod <- GetVaRgpd(rt, RF, 2, p)
                Wmv <- c(Wmv, rep(wPeriod, times = QUARTALE))
        }
        dim(Wmv) <- c(ASSETS, NOBS + QUARTALE)
        cat("Dim(Wmv) = ", dim(Wmv), "\n")
        INFROM <- INSAMPLE + 1
        INTO <- INSAMPLE + NOBS
        Rmv <- array(c(DAX[INFROM:INTO], COMM[INFROM:INTO], BOND[INFROM:
            INTO]), dim = c(NOBS, ASSETS))
        Smv <- numeric()

        for ( i in 1:NOBS )
                Smv <- c( Smv, sum( Rmv[i, ] * t(Wmv)[i, ]) )

        cat("*** Best VaR-GPD allocation ***\n")
        cat("Means = ", rowMeans(Wmv), "\n")
        cat("SDs   = ", rowSds(Wmv), "\n")
        cat("========================\n")
        cat("Mean p.a. = ", 252 * mean(Smv), "\n")
        cat("SD        = ", stdev(Smv) * sqrt(252), "\n")
        cat("R / SD    = ", mean(Smv) / stdev(Smv), "\n")
        cat("Max       = ", max(Smv), "\n")
        cat("Min       = ", min(Smv), "\n")
        cat("VaR (95%) = ", VaRgpd(Smv, 0.95), "\n")
        cat("VaR (99%) = ", VaRgpd(Smv, 0.99), "\n")
        cat("ES (95%)  = ", ESgpd(Smv, 0.95), "\n")
        cat("ES (99%)  = ", ESgpd(Smv, 0.99), "\n")
```

```
}

FindBestESgpd <- function(p) {
        Wmv <- numeric()
        for (i in 0 : PERIODS) {
                from <- i * QUARTALE + 1
                to <- from + INSAMPLE - 1
                X <- DAX[from : to]
                Y <- COMM[from : to]
                Z <- BOND[from : to]
                RF <- EU3M[from : to]
                r <- array(c(X, Y, Z), dim = c(INSAMPLE, ASSETS))
                rt <- t(r)
                wPeriod <- GetESgpd(rt, RF, 10, p)
                Wmv <- c(Wmv, rep(wPeriod, times = QUARTALE))
        }
        dim(Wmv) <- c(ASSETS, NOBS + QUARTALE)

        INFROM <- INSAMPLE + 1
        INTO <- INSAMPLE + NOBS
        Rmv <- array(c(DAX[INFROM:INTO], COMM[INFROM:INTO], BOND[INFROM:
            INTO]), dim = c(NOBS, ASSETS))
        Smv <- numeric()

        for ( i in 1:NOBS )
                Smv <- c( Smv, sum( Rmv[i, ] * t(Wmv)[i, ]) )

        cat("*** Best ES-GPD allocation ***\n")
        cat("Means = ", rowMeans(Wmv), "\n")
        cat("SDs   = ", rowSds(Wmv), "\n")
        cat("=======================\n")
        cat("Mean p.a. = ", 252 * mean(Smv), "\n")
        cat("SD        = ", stdev(Smv) * sqrt(252), "\n")
        cat("R / SD    = ", mean(Smv) / stdev(Smv), "\n")
        cat("Max       = ", max(Smv), "\n")
        cat("Min       = ", min(Smv), "\n")
        cat("VaR (95%) = ", VaRgpd(Smv, 0.95), "\n")
        cat("VaR (99%) = ", VaRgpd(Smv, 0.99), "\n")
        cat("ES (95%)  = ", ESgpd(Smv, 0.95), "\n")
        cat("ES (99%)  = ", ESgpd(Smv, 0.99), "\n")
}

FindBestMinVar <- function() {
        Wmv <- numeric()
```

```
        for (i in 0 : PERIODS) {
                from <- i * QUARTALE + 1
                to <- from + INSAMPLE - 1
                X <- DAX[from : to]
                Y <- COMM[from : to]
                Z <- BOND[from : to]
#               RF <- EU3M[from : to]
                r <- array(c(X, Y, Z), dim = c(INSAMPLE, ASSETS))
#               rt <- t(r)
                wPeriod <- GetMinVar(r)
                Wmv <- c(Wmv, rep(wPeriod, times = QUARTALE))
        }
        dim(Wmv) <- c(ASSETS, NOBS + QUARTALE)

        INFROM <- INSAMPLE + 1
        INTO <- INSAMPLE + NOBS
        Rmv <- array(c(DAX[INFROM:INTO], COMM[INFROM:INTO], BOND[INFROM:
            INTO]), dim = c(NOBS, ASSETS))
        Smv <- numeric()

        for ( i in 1:NOBS )
                Smv <- c( Smv, sum( Rmv[i, ] * t(Wmv)[i, ]) )

        cat("*** Minimum Variance allocation ***\n")
        cat("Means = ", rowMeans(Wmv), "\n")
        cat("SDs   = ", rowSds(Wmv), "\n")
        cat("========================\n")
        cat("Mean p.a. = ", 252 * mean(Smv), "\n")
        cat("SD        = ", stdev(Smv) * sqrt(252), "\n")
        cat("R / SD    = ", mean(Smv) / stdev(Smv), "\n")
        cat("Max       = ", max(Smv), "\n")
        cat("Min       = ", min(Smv), "\n")
        cat("VaR (95%) = ", -VaRnorm(Smv, 1-0.95), "\n")
        cat("VaR (99%) = ", -VaRnorm(Smv, 1-0.99), "\n")
        cat("ES (95%)  = ", -ESnorm(Smv, 1-0.95), "\n")
        cat("ES (99%)  = ", -ESnorm(Smv, 1-0.99), "\n")
        }
# #### that's all folks!
```

List of Symbols

AR	Autoregression
ARCH	Autoregressive conditional heteroscedasticity
ES	Expected shortfall, also known as conditional VaR
ETF	Exchange traded fund
GARCH	General autoregressive conditional heteroscedasticity
GEV	Generalised extreme value distribution
GPD	Generalised Pareto distribution
VaR	Value at Risk

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