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On stochastic quantization and metastable dynamics

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Doctoral dissertation

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Abstract

Stochastic quantization is a method for constructing probability measures as stationary measures of stochastic partial differential equations (SPDEs). Such measures arise in the study of quantum field theory. The method has classically used parabolic SPDEs, i.e., nonlinear stochastic heat equations. A more recent, active area of research is hyperbolic stochastic quantization, where nonlinear wave equations are used instead.

This dissertation is concerned with the interplay between statistics of the measures and dynamics of the quantization equations. This connection is studied for two simple quantum field theories, namely the ϕ^4 model and the sine-Gordon model. The ϕ^4 model is analyzed with hyperbolic methods.

The first article is about the construction of the ϕ^4 measure as a stationary measure of a hyperbolic SPDE on the two-dimensional plane. It extends an earlier hyperbolic result that required a periodic domain, and complements the existing parabolic theory on the plane.

The second article studies the hyperbolic dynamics when the field has two energetically optimal stable configurations. In this case the stochastic system is metastable, randomly jumping between the neighbourhoods of these energy minima. Expected frequency of these jumps is computed when the equation is posed on suitably small tori in two and three dimensions. This extends similar earlier results on two-dimensional parabolic dynamics.

The third article continues on the theme of metastability, now for a parabolic SPDE with a sine nonlinearity that gives infinitely many potential minima. An earlier work of Berglund and Gentz is adapted to compute expected transition times of the model on a one-dimensional torus.

Tiivistelmä

Stokastinen kvantisointi on keino rakentaa todennäköisyysmittoja stokastisten osittaisdifferentiaaliyhtälöiden tasapainomittoina. Sitä sovelletaan kvanttikenttäteorian tutkimuksessa. Menetelmä on perinteisesti pohjautunut parabolisiin yhtälöihin eli epälineaarisiin stokastisiin lämpöyhtälöihin. Tuoreempi, tällä hetkellä aktiivinen tutkimusala käyttää niiden sijasta hyperbolisia eli aaltoyhtälöitä.

Tässä väitöstyössä tarkastellaan tasapainomittojen tilastollisten ominaisuuksien ja kvantisointiyhtälöiden dynaamisen käytöksen välistä yhteyttä. Tarkastelu kohdistuu ϕ^4- ja sine-Gordon-malleihin, jotka ovat kaksi yksinkertaista kvanttikenttäteoriaa. Näistä ϕ^4-mallia tutkitaan hyperbolisin menetelmin.

Ensimmäinen artikkeli käsittelee ϕ^4-mitan konstruoimista tasapainomittana aaltoyhtälölle, joka on määritelty kaksiulotteisessa tasossa. Se laajentaa aiempaa tulosta, jossa aaltoyhtälö on määritelty toruksella, ja täydentää olemassaolevaa tason parabolista kvantisointiteoriaa.

Toisessa artikkelissa tutkitaan aaltoyhtälöä, jolla on kaksi pienimmän energian tasapainotilaa. Tällöin stokastinen systeemi on metastabiili eli se hyppii näiden energiaminimien välillä. Artikkelissa johdetaan odotusarvo näiden hyppyjen taajuudelle, kun yhtälö on määritelty riittävän pienellä toruksella kahdessa tai kolmessa ulottuvuudessa. Tämä rinnastuu aiempiin samankaltaisiin tuloksiin paraboliselle yhtälölle kahdessa ulottuvuudessa.

Kolmas artikkeli jatkaa metastabiiliuden parissa, mutta käsittelee parabolista yhtälöä, jossa sinifunktion määräämä epälineaarisuus johtaa äärettömän moneen energiaminimiin. Artikkelissa lasketaan odotusarvo hyppyjen väliselle ajalle, kun yhtälö on määritelty yksiulotteisella toruksella. Menetelmä perustuu Berglundin ja Gentzin aiempaan artikkeliin.

List of included articles

This dissertation consists of an introduction and three articles:

[**I**] N. Barashkov and P. Laarne. *Invariance of ϕ^4 measure under nonlinear wave and Schrödinger equations on the plane*. Submitted for publication; the included version is significantly revised from arXiv:2211.16111v4.

[**II**] N. Barashkov and P. Laarne. *Transition state theory for the hyperbolic ϕ^4 model*. Submitted for publication; the included version is a revised version of arXiv:2410.03495v1.

[**III**] P. Laarne. *Metastable transition times of the 1D dynamical sine-Gordon model*. Submitted for publication; preprint at arXiv:2509.13806v1.

The author contributions to the articles are as follows:

[**I**] The main idea was developed by NB, and PL contributed to the analysis. Both authors contributed equally to the writing.

[**II**] Both authors contributed equally to the analysis. Most of the manuscript was written by PL.

[**III**] Independent work of PL.

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I have the honour to call Dr Nikolay Barashkov my *Doktorvater*. He introduced me to the world of stochastic quantization and the people around it, and patiently helped me through every question and issue I came up with. I equally hope that I managed to learn some of his technical arsenal – at least I have solved some problems by asking myself "what would Nikolay do here?".

Prof. Christian Webb was a voice of reason and reassurement in the thesis steering group, and I learned a lot as conference co-organizer and teaching assistant with him. Prof. Pekka Pankka had the same official role, but through a few twists and turns I somehow also ended up as an extra member of his group – let me quickly say "quasiconformal" here so that this thesis qualifies – and had some of the best times of my career.

I've had the pleasure to meet many lovely colleagues during my travels. They are too numerous to be listed here, but I would like to thank Dr Giacomo Di Gesù in Rome and Prof. Vesa Julin in Jyväskylä for hosting me on research visits.

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I would not have begun this journey without my family. Dr Mom is a role model in ways that I cannot really explain in words. Dad taught me the art of telling stories, although *this one* I tried to keep true. Reija is a source of endless energy and entropy. And the dogs revealed that happiness lies in simple joys (namely: food, belly scratches, and tennis balls).

I would not have finished this journey without the two ladies of my life. I have walked my entire academic path with my wife Milla, a fellow doctor-to-be, sharing the highs and lows of both research and life. And it is our daughter Silja whose arrival gave both a whole new meaning to everything, and a very non-negotiable deadline for this thesis.

I'm reaching up and reaching out
I'm reaching for the random or
Whatever will bewilder me

Tool: Lateralus (2001)

Stay in one state
Nothing is stable

The National: Grease in Your Hair (2023)

Contents

Chapter 1

QFT in nine pages

1.1 Motivation

The objects studied in this thesis originate in *quantum field theory*. QFT is a cornerstone of modern twentieth-century physics. Various QFTs describe elementary particles and their interactions, and are tightly connected to models and questions from statistical mechanics.

For the purposes of this dissertation, we do not need to concern ourselves with the physical interpretation of QFTs.[a] We take the two models studied in [**I**]–[**III**] as given, although we give some heuristic explanation of them below. As we are interested in purely mathematical objects, we can take the following simplification as our starting point:

Postulate 1.1. *A quantum field theory is a probability measure on some suitable space of generalized functions, required to satisfy certain properties.*

In physics, a QFT is an operator-valued probability distribution on a d-dimensional Minkowski spacetime. When the field is analytically continued to imaginary time, time becomes a regular coordinate in the Euclidean space $\mathbb{R}^d$. The "certain properties" are then the Osterwalder–Schrader axioms [107, 108]; see also [61, Section 6.1]. Many pages of paper have been devoted to the path from quantum mechanics to the highly simplified definition above; some classical and modern expositions include [61, 70, 114, 118].

The prime example of an Euclidean quantum field theory is the ϕ^4_d model, which is the subject of [**I**] and [**II**]. It is a probability measure over (generalized) functions on $\mathbb{R}^d$ or the torus $\mathbb{T}^d$; we denote either domain by Λ^d below.[b] The probability density of a given function ϕ is formally equal to

$$\frac{1}{\mathcal{Z}(\varepsilon)} \exp\left(-\frac{1}{\varepsilon}\int_{\Lambda^d}\left[\frac{1}{4}\phi(x)^4 \pm \frac{m^2}{2}\phi(x)^2 + \frac{1}{2}|\nabla\phi(x)|^2\right]\mathrm{d}x\right)\prod_{x\in\Lambda^d}\mathrm{d}\phi(x). \tag{1.1}$$

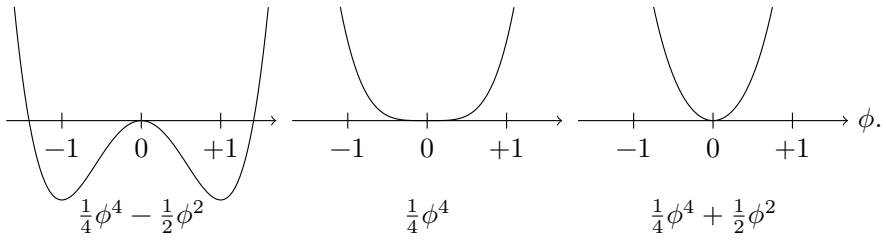
Here $\mathcal{Z}(\varepsilon)$ is a normalization constant, called the *partition function*, $\varepsilon > 0$ is called the temperature, and $m^2 \geq 0$ is a mass parameter. This expression is only formal since the infinite-dimensional product of Lebesgue measures is not

[a] As a further specialization, we only study *bosonic* QFTs, as opposed to *fermionic* QFTs.

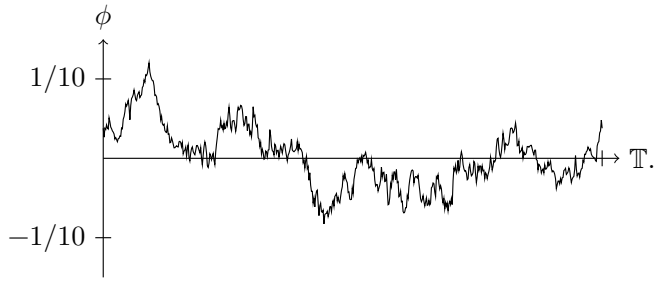
[b] It is also possible to use cylinders $\mathbb{R}^m \times \mathbb{T}^{d-m}$ or general manifolds, but we only consider $\mathbb{T}^d$ and $\mathbb{R}^d$ in this thesis.

well-defined, and in $d \geq 2$ there are divergencies that must be *renormalized*. The goal of Sections 1.2 and 1.3 is to give a proper meaning to (1.1).

Unless otherwise stated, we assume the field ϕ to take real values. Depending on the sign of the quadratic mass term, the potential given by the first two terms takes various forms:



This potential is applied on every point of Λ^d, and the $|\nabla\phi|$ term penalizes rapid changes of ϕ. A sample from a positive-mass ϕ_1^4 model on $\mathbb{T}$ could look like



The ϕ_1^4 measure is illustrative and easy to analyze but physically not very meaningful, as it corresponds to a $0 + 1$-dimensional spacetime. The higher-dimensional models are connected to interesting physical phenomena. We shall make a brief detour into statistical mechanics in order to illustrate this, even though the discussion is not relevant for the rest of the thesis.

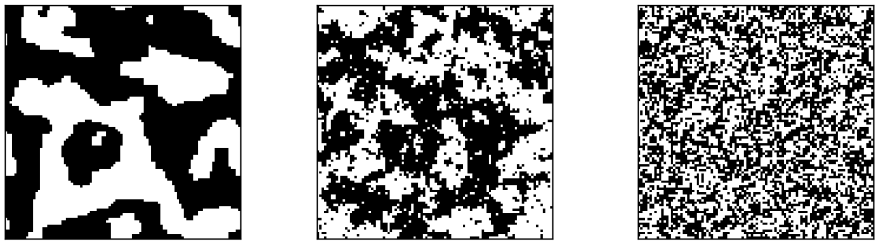
The *Ising model* is a simplified model of a ferromagnet. Atoms are modelled by points on a lattice, each pointing either "up" or "down" signified by ± 1. These *spins* are random, emulating thermal motion, but each atom also tries to align with its nearest neighbours. A region is magnetized when most atoms are aligned to the same direction.

The probability density of this model is

$$\frac{1}{\mathcal{Z}(\varepsilon)} \exp\left(\frac{1}{\varepsilon} \sum_{x \sim y} \sigma(x)\sigma(y)\right), \tag{1.2}$$

where $\sigma(x)$ gives the spin of a lattice point x, $\sim$ is the nearest neighbour relation, and the density is with respect to the Bernoulli distribution on each atom.

The following images show three simulations of the Ising model on a periodic 100×100 lattice, with increasing values of ε:



In the left figure ε is small, meaning that individual spins are heavily penalized for differing from their neighbours. The lattice is separated into clear *phases*, not unlike those of liquid and solid in a melting block of ice.

In the right figure ε is large, allowing for so much single-site randomness that there is no discernible pattern at all. It turns out that there is a precise value for ε where the Ising model changes between these two behaviours – this is an example of an order-disorder transition [106, 110]. Such a sharp transition actually happens in magnets heated to the so-called *Curie temperature*.[c] The middle figure approximates this phase transition.

Even though the Ising model is discrete and the ϕ^4 model is continuous, the latter can be approximated by various modified versions the former [64, 71, 92, 115]. Properties of the ϕ^4 model, like triviality when $d \geq 4$ and the existence of phase transitions, have been proved via Ising techniques [2, 62].

In [**III**] we discuss the *sine-Gordon* model.[d] Its probability density is

$$\frac{1}{\mathcal{Z}(\beta)} \exp\biggl(- \int_{\Lambda^d} \biggl[\frac{1}{\beta} \cos(\beta\phi(x)) + \frac{m^2}{2}\phi(x)^2 + \frac{1}{2}|\nabla\phi(x)|^2 \biggr] \mathrm{d}x \biggr) \prod_{x \in \Lambda^d} \mathrm{d}\phi(x). \quad (1.3)$$

This equation is mostly studied when $d = 1$ or 2; again it must be renormalized when $d = 2$. In two dimensions the range of $\beta > 0$ is restricted, as we discuss at the end of Section 1.3.

The physical background of this model is not as easy to explain. The two-dimensional model arises from *Coulomb gas*, a model of electrically charged particles. The parameter β denotes inverse temperature: at low β, the gas mostly consists of individual particles, whereas at higher β the particles begin combining into dipoles and multipoles. See e.g. [39, 55, 58, 77].

The one-dimensional model is linked to a nonlinear wave equation that has infinitely many conserved quantities and exhibits solitons. It is connected to equations in various branches of mathematics. [46, 112]

The model is mathematically interesting in that the nonlinearity is not polynomial but still rather elementary. This makes its renormalization and analysis suitably challenging.

[c] The reader is encouraged to search for a video demonstration of this fact.

[d] The name is a play on the Klein–Gordon equation underlying these models.

1.2 Gaussian measures

QFT measures can be constructed rigorously as limits of approximation schemes. Let us first illustrate the idea with a Gaussian measure, which is defined by the part common to (1.1) and (1.3).

On $\mathbb{T}^d$ functions can be approximated via lattice restriction or truncated Fourier series. Here we use the latter approach. Since

$$\frac{1}{\varepsilon}\int_{\mathbb{T}^d} \frac{m^2}{2}\phi(x)^2 + \frac{1}{2}|\nabla\phi(x)|^2 \,\mathrm{d}x = \frac{1}{\varepsilon}\sum_{n\in\mathbb{Z}^d} \frac{m^2+n^2}{2}|\hat{\phi}(n)|^2, \tag{1.4}$$

we can write the Gaussian as the $N\to\infty$ limit of probability measures

$$\propto \exp\left(-\frac{1}{\varepsilon}\sum_{n\in\mathbb{Z}^d} \frac{m^2+n^2}{2}|\hat{\phi}(n)|^2\right) \prod_{\substack{n\in\mathbb{Z}^d\\ |n|\le N}} \mathrm{d}\hat{\phi}(n). \tag{1.5}$$

Even though the product does not converge on its own, these weighted measures do have a well-defined limit.

Either the trigonometric or complex exponential basis can be used in the construction. The same limit would be attained by lattice regularization. This is called the *ultraviolet limit* as we pass to smaller length scales when $N\to\infty$. The limiting object is the *Gaussian free field* (GFF) on $\mathbb{T}^d$, of covariance $\varepsilon(m^2-\Delta)^{-1}$.

Remark 1.2. The covariance is well-defined only when $m^2 > 0$, or if the wavenumbers $|n|$ are also restricted from below. The ϕ^4 measure can be defined also when the mass term is negative, since we can add some mass to the Gaussian measure by subtracting it from the fourth-order term (which stays bounded from below).

The extension of the measure to functions over $\mathbb{R}^d$ is called the *infrared limit*. One approach is to construct the field on increasingly large tori $L\mathbb{T}^d$ and pass $L\to\infty$. We discuss this case more in Section 2.4.

The resulting Gaussian measures are supported on infinite-dimensional spaces of (generalized) functions. There are many equivalent ways to view the measure and its support, but for us it is natural to use Besov spaces [8, Chapter 2]:

Definition 1.3. Let $(\Delta_k)_{k\ge -1}$ be dyadic Fourier multipliers as in [8, Section 2.2]. For $p, r \in [1,\infty]$ and $s\in\mathbb{R}$ we define the *non-homogeneous Besov space* on $\mathbb{R}^d$ as the completion of $C_c^\infty(\mathbb{R}^d)$ with respect to the norm

$$\|u\|_{B^s_{p,r}(\mathbb{R}^d)} := \left[\sum_{k\ge -1} 2^{krs}\|\Delta_k u\|^r_{L^p(\mathbb{R}^d)}\right]^{1/r}, \tag{1.6}$$

with the usual modification if $r=\infty$. We shorten $H^s := B^s_{2,2}$ and $\mathcal{C}^s := B^s_{\infty,\infty}$.

The idea is that Δ_k picks frequencies in an annulus of scale 2^k and Δ_{-1} captures all frequencies near the origin.[e] It is also possible to take the L^p norm

[e] The *homogeneous* Besov space is defined with dyadic annuli all the way down. Such a space does not see the zero mode of functions, and some embedding results are different.

with respect to a weighted Lebesgue measure [66, 93]. A Besov space over $\mathbb{T}^d$ is defined as the completion of $C^\infty(\mathbb{T}^d)$ when the L^p norm is restricted to $\mathbb{T}^d$.

Besov spaces support useful embedding and multiplication properties, as listed in [**I**, Section 2] and [**II**, Section 2.2]. Different authors use slightly varying definitions of Besov spaces, but they are generally equivalent, at least outside the endpoints $p, r = \infty$.

It then follows that the Gaussian free field on $\mathbb{T}^d$ is a measure supported on $\mathcal{C}^{1-d/2-}$. The assumption $m^2 > 0$ is required for the zero Fourier mode to have a proper distribution. The case $m^2 = 0$ can be handled by fixing the zero mode to a constant.

Theorem 1.4. *Let ϕ be sampled from the Gaussian free field on $\mathbb{T}^d$ with $m^2 > 0$. Then $\mathbb{E}\,\|\phi\|_{\mathcal{C}^s} < \infty$ for any $s < 1 - d/2$.*

Idea of proof. Let us write $s = 1 - d/2 - \kappa$ for some $\kappa > 0$. By Besov embedding theorems, we have

$$\|\phi\|_{\mathcal{C}^{1-d/2-\kappa}} \lesssim \|\phi\|_{B^{1-d/2-\kappa/2}_{p,p}} \tag{1.7}$$

for some large p (dependent on κ). Then hypercontractivity [114, Theorem I.22] and translation invariance of Gaussian measures give us that

$$\mathbb{E}\,|\Delta_k\phi(x)|^p \le (p-1)^p \left[\mathbb{E}\,|\Delta_k\phi(0)|^2\right]^{p/2}. \tag{1.8}$$

Then we can estimate

$$\begin{aligned}
\mathbb{E}\,|\Delta_k\phi(0)|^2 &= \sum_{\ell,n\in\mathbb{Z}^d} \Delta_k(\ell)\Delta_k(n)\,\mathbb{E}[\hat\phi(\ell)\hat\phi(n)] \\
&\le \sum_{\ell\in\operatorname{supp}\Delta_k} \frac{1}{m^2+\ell^2} \\
&\lesssim \int_{2^k}^{2^{k+1}} x^{d-1}x^{-2}\,\mathrm{d}x,
\end{aligned} \tag{1.9}$$

which is of order $2^{k(d-2)}$. This leaves us to directly calculate

$$\begin{aligned}
\mathbb{E}\,\|\phi\|^p_{B^{1-d/2-\kappa/2}_{p,p}} &= \sum_{k\ge -1} 2^{kp(1-d/2-\kappa/2)}\,\mathbb{E}\,\|\Delta_k\phi\|^p_{L^p} \\
&\lesssim \sum_{k\ge -1} 2^{kp(1-d/2-\kappa/2)}2^{-kp(1-d/2)} \\
&\lesssim \sum_{k\ge -1} 2^{-kp\kappa/2}.
\end{aligned} \tag{1.10}$$

This is a convergent geometric sum. □

The measures (1.1) and (1.3) then have the formal density e^{-V} with respect to a Gaussian, where

$$V(\phi) = \frac{1}{4\varepsilon}\int_{\Lambda^d} \phi(x)^4\,\mathrm{d}x \quad \text{or} \quad V(\phi) = \frac{1}{\beta}\int_{\Lambda^d} \cos(\beta\phi(x))\,\mathrm{d}x \tag{1.11}$$

respectively. The approximate measures really have Gaussian densities, but the ultraviolet and infrared limit measures might not be absolutely continuous with respect to the free field (see e.g. [11]).

The normalization constant $\mathcal{Z}$ can be seen as an expectation of e^{-V} with respect to the Gaussian. Perhaps the easiest way to compute such expectations is the variational *Boué–Dupuis formula*:

Theorem 1.5. *In all of the following $\mathbb{E}$ is expectation with respect to a regularized Gaussian $\phi_T = \rho_T * \phi$, where the smooth cutoff ρ_T approaches a Dirac delta as $T \to \infty$. Assume that the nonlinearity V is* tame*, meaning $\mathbb{E}\,|V|^p + \mathbb{E}\,e^{-qV} < \infty$ for some Hölder conjugates $1/p + 1/q = 1$. Then*

$$-\log \mathbb{E}\, e^{-V(\phi_T)} = \inf_{v \in \mathbb{H}_a} \mathbb{E}\left[V(\phi_T + I_T[v]) + \frac{1}{2}\int_0^T \|v_t\|_{L^2}^2 \,\mathrm{d}t \right].$$

Here $\mathbb{H}_a$ is the space of progressively measurable processes that are almost surely in $L^2(\mathbb{R}_+ \times \Lambda^d)$, and I_T is a regularizing operator related to the cutoff.

The precise statement can be found in [10, Section 2]. This formula originates with the work of Boué and Dupuis [20] for bounded functions of d-dimensional Brownian motion. It was extended to abstract Wiener spaces by Zhang [128], and the tameness condition was introduced by Üstünel [123]. The original result has been applied in numerous large-deviations results for stochastic differential equations. Barashkov and Gubinelli [10] introduced it to QFT computations, in which it has become quite popular.[f]

1.3 Renormalization

Equation (1.1) cannot be used as is to define the ϕ_d^4 measure when $d \geq 2$: The Gaussian free field is almost surely in $\mathcal{C}^{1-d/2-}$ but not in $\mathcal{C}^{1-d/2}$. This means that the field is not a function but a distribution. Distributions cannot generally be evaluated pointwise or multiplied together. This is a problem for defining the fourth-order potential.

There is however a way to remove infinite divergencies in a way that also makes physical sense [126]. To motivate this procedure, we first begin with a basic result on Gaussian random variables.

Theorem 1.6 (Isserlis' theorem)**.** *Let $X_1, \dots, X_n$ be real-valued, jointly Gaussian and centred random variables. Then*

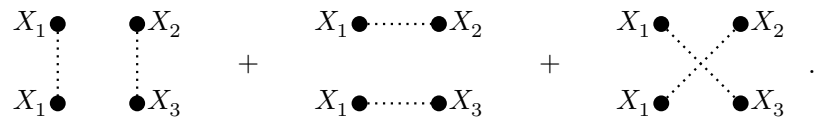
$$\mathbb{E}[X_1 \cdots X_n] = \sum_{\mathcal{P}} \prod_{(i,j)\in\mathcal{P}} \mathbb{E}[X_i X_j],$$

where $\mathcal{P}$ is summed over all possible pairings of $\{1, \dots, n\}$.

Proof. See e.g. [14, Theorem 4.2.2]. □

[f] As of mid-December 2025, zbMATH lists 49 references to [10].

This result can be interpreted in a graphical way where dots represent Gaussian variables and lines between them indicate pairing. Let us illustrate this with a concrete example. We can represent $\mathbb{E}[X_1^2 X_2 X_3]$ as



It is worth noting that the two last terms are the same – there is usually a combinatorial element to using Isserlis' theorem. It also follows that a product of odd number (with multiplicity) of Gaussians always has zero expectation.

Given a Gaussian variable X, its k'th *Wick power* $:X^k:$ is defined through the Hermite polynomial $H_k(X, \mathbb{E}\,X^2)$; the first few are

$$:X^0: = 1, \quad :X^1: = X, \quad :X^2: = X^2 - \mathbb{E}\,X^2, \tag{1.12}$$
$$:X^3: = X^3 - 3X\,\mathbb{E}\,X^2, \quad :X^4: = X^4 - 6X^2\,\mathbb{E}\,X^2 + 3\,\mathbb{E}\,X^2.$$

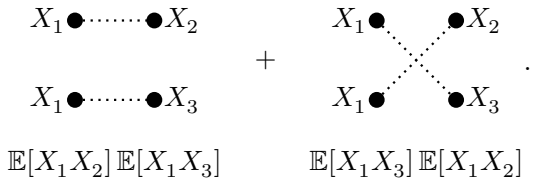
There is then the important relation:

Theorem 1.7. *Let X and Y be real-valued, centred Gaussian random variables. Then for all $n, m = 0, 1, \ldots$ we have*

$$\mathbb{E}[:X^n::Y^m:] = \begin{cases} n!\,\mathbb{E}[XY]^n, & \text{if } n = m, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. See e.g. [14, Lemma 4.2.9]. □

Wick ordering reduces the set of possible pairings by prohibiting self-pairings. Let us modify the previous example and consider $\mathbb{E}[:X_1^2:X_2X_3]$,



Here the pairing $\mathbb{E}[X_1^2]\,\mathbb{E}[X_2X_3]$ is cancelled by the Wick ordering.

Corollary 1.8. $\mathbb{E}:X^n: = 0$ *for all* $n = 1, 2, \ldots$.

Proof. Set $m = 0$ in Theorem 1.7. □

It is now possible to extend the Wick ordering to the Gaussian free field on $\mathbb{T}^2$. If the sum in (1.4) is truncated to finitely many terms, then $\phi(x)$ is a well-defined Gaussian variable for any $x \in \mathbb{T}^d$. These variables can be Wick-ordered, and they converge to a limit as the truncation is removed. The next result is an analogue of Theorem 1.4:

Theorem 1.9. *Let $(\phi_T)_{T\geq 0}$ be sampled from the T-truncated massive GFF on $\mathbb{T}^2$. Fix $\kappa > 0$ and $n = 0, 1, \ldots$. Then $:\phi_T^n:$ converges to $:\phi^n: \in \mathcal{C}^{-\kappa}(\mathbb{T}^2)$ in $L^p(\mathbb{P})$ sense for any $1 \leq p < \infty$.*

Proof. [40, Lemma 3.2]. □

The truncation can be done in many ways; article [**II**] uses an approach based on a continuous cutoff. The diagrams can be extended to incorporate information about both the cutoff and Fourier wavenumbers. These diagrams are explained in detail in [95]; let us only outline the basic idea via two examples.

Let now a dot represent a standard Gaussian random variable indexed by time $t \in \mathbb{R}$. The *Brownian motion* is the time integral of such a variable; integration is represented by a solid line. Any Fourier mode of the Gaussian free field can then be written as[g]

$$\hat{\phi}(n) = \frac{1}{\sqrt{m^2+n^2}} \int_0^\infty \sigma_t(n)\, \mathrm{d}B_n(t) = \quad , \tag{1.13}$$

where $\sigma_t(n)$ is a smooth cutoff function chosen so that $\mathbb{E}\,|\hat{\phi}(n)|^2 = (m^2+n^2)^{-1}$, and B_n is a Brownian motion.

Multiplication is indicated by joining trees. Wavenumbers of the joined trees must sum to the target wavenumber, and the integrals simplify by the rules for independent Brownian motions. For example, any Fourier mode of the second Wick power $\widehat{:\phi^2:}(n)$ can be written as

$$= \sum_{k+\ell=n} \frac{1}{\sqrt{(m^2+k^2)(m^2+\ell^2)}} \int_0^\infty \mathrm{d}B_k(t) \int_0^t \mathrm{d}B_\ell(s)\sigma_t(k)\sigma_s(\ell). \tag{1.14}$$

These diagrams can then be used to analyze more complicated interactions; e.g.

would correspond to $\mathbb{E}(:\phi^2:)^2$, with self-contractions within a tree again removed by the Wick ordering.

Equipped with these tools, we can now define the ϕ_2^4 measure as the limit of

$$\frac{1}{\mathcal{Z}(\varepsilon,T)} \exp\left(-\frac{1}{\varepsilon}\int_{\mathbb{T}^2} \frac{1}{4}{:}\phi_T(x)^4{:}\,\mathrm{d}x\right) \mathrm{dGFF}^{(T)}_{\varepsilon(m^2-\Delta)^{-1}}(\phi_T), \tag{1.15}$$

when the truncation $T \to \infty$. The only difference to (1.1) is that the fourth-order term is now Wick-ordered.

The three-dimensional Gaussian free field is harder to analyze since its samples have regularity $\phi \in \mathcal{C}^{-1/2-}(\mathbb{T}^3)$. It can be shown that $:\phi^2: \in \mathcal{C}^{-1-}(\mathbb{T}^3)$, but

[g] This diagram is colloquially known as the "lollipop".

$:\phi^3:$ makes sense only after some averaging [95]. It follows that the ϕ^4_3 measure contains further divergent terms. It can be defined as

$$\frac{1}{\mathcal{Z}(\varepsilon,T)}\exp\biggl(-\frac{1}{\varepsilon}\int_{\mathbb{T}^3}\Bigl[\frac{1}{4}{:}\phi_T(x)^4{:}+\gamma_T{:}\phi_T(x)^2{:}+\delta_T\Bigr]\,\mathrm{d}x\biggr)\mathrm{dGFF}^{(T)}_{\varepsilon(m^2-\Delta)^{-1}}(\phi_T), \tag{1.16}$$

where γ_T and δ_T are certain divergent constants (see e.g. [**II**, Theorem 5.2]). Strictly speaking δ_T is not necessary as it will be cancelled by $\mathcal{Z}(\varepsilon,T)$, but it simplifies notation to collect all divergent stochastic terms in one place.

In four and more dimensions the model can no longer be renormalized: if a renormalization scheme leads to a well-defined limit measure, the limit is necessarily a Gaussian measure. This was known for $d\geq 5$ already in 1980s [1, 56], and $d=4$ was quite recently shown by Aizenman and Duminil-Copin [2].

Although we only discuss the 1D sine-Gordon model in [**III**], let us briefly explain the renormalization of the 2D model. The Wick ordering can be extended to convergent power series. Following [114, Eq. (I.18a)], we have for a Gaussian field ϕ the identity

$${:}\exp(\alpha\phi){:}=\exp\biggl(\alpha\phi-\frac{\alpha^2\,\mathbb{E}\,\phi^2}{2}\biggr). \tag{1.17}$$

This implies that a sine or cosine term can be Wick-ordered as

$${:}\sin(\beta\phi){:}=\exp\biggl(\frac{\beta^2\,\mathbb{E}\,\phi^2}{2}\biggr)\sin(\beta\phi). \tag{1.18}$$

Wick ordering suffices to make the sine-Gordon measure

$$\frac{1}{\mathcal{Z}(\beta)}\exp\biggl(-\int_{\mathbb{T}^2}\frac{1}{\beta}{:}\cos(\beta\phi_T(x)){:}\,\mathrm{d}x\biggr)\mathrm{dGFF}^{(T)}_{(m^2-\Delta)^{-1}}(\phi_T) \tag{1.19}$$

well-defined in the range $0<\beta^2<4\pi$. Similarly to the ϕ^4_3 model, a further divergent term appears at $\beta^2=4\pi$ and must be renormalized. At each threshold $\beta^2=8\pi n/(n+1)$ a new divergent term appears: these thresholds are 4π, $16\pi/3$, 6π, and so on. At $\beta^2=8\pi$ and beyond the model is expected to be no longer renormalizable. See e.g. the introductions of [38, 77].

Chapter 2

Stochastic quantization

2.1 Invariant measures

Parisi and Wu [109] introduced in the 1980s a simple-sounding idea: if the QFT measure is a stationary measure of some stochastic system, then we can analyze samples from the system to understand properties of the measure. This process is called stochastic quantization.

If a measure is of form

$$\frac{1}{\mathcal{Z}} \exp\left(-\frac{1}{\varepsilon}\int_{\mathbb{T}^d} V(\phi)\,\mathrm{d}x\right) \mathrm{dGFF}_{\varepsilon(m^2-\Delta)^{-1}}(\phi), \tag{2.1}$$

then it is at least formally invariant under the stochastic partial differential equation (SPDE)

$$\partial_t u(x,t) + (m^2-\Delta)u(x,t) = -V'(u(x,t)) + \sqrt{2\varepsilon}\xi, \tag{2.2}$$

where $x \in \mathbb{T}^d$ and ξ is a spacetime white noise measure (Gaussian with identity covariance, independent for any two points in time). This is because (2.1) is the *Gibbs measure* of the energy of *Langevin dynamics* corresponding to (2.2).

Definition 2.1. Let Φ_t denote the (random) flow of an SPDE. A measure μ is invariant under the SPDE if $\mu(A) = \mu(\Phi_t^{-1}A)$ for all measurable sets A and $t > 0$.

Remark 2.2. The parameter t is often called 'time'. It is however an auxiliary variable and *not* related to the spacetime Λ^d of a QFT. When we study dynamics of SPDEs in Chapter 3, we study *properties of the construction* and not the evolution of the quantum field itself.

Some basic questions are then:

- Is there a unique invariant measure?
- Is the SPDE well-defined locally (t in short interval) or globally (all $t \geq 0$)?
- Can the full SPDE be approximated by finite-dimensional systems?

There are some generic results for existence of invariant measures, like the Krylov–Bogoliubov theorem [41, Theorem 11.7]. It can be used to show the invariance of (2.2) under general conditions on V; see e.g. [14, Section 3.5.2]. Uniqueness is implied e.g. by the strong Feller property [41, Section 11.2.2].

For our purposes it is useful to work with finite-dimensional approximate measures, since they are also compatible with the constructions of the previous chapter. Then it is necessary to show that invariance is preserved in the limit (such as in [**I**, Section 4.4]).

Usually the first step in stochastic quantization is to show existence of solutions and invariance on a short time interval $t \in [0, \tau]$. We discuss this in the next section. The next step is to extend these properties to all $t \geq 0$. This is often done with an argument of Bourgain [22].

The basic idea is simple: If initial data for (2.2) is sampled from the invariant measure, then the law of the solutions at time τ is again the invariant measure. Hence we can restart the flow from this distribution, and get solutions up to time 2τ, and so on.

In practice τ is not found uniformly for all initial data; it might depend e.g. on the norm of initial data. We then need to restrict to a "nice" set of data.

Fix $T > 0$, and assume that the initial data is such that a solution stays in the nice set up to time τ with probability $1 - p$. At $t = \tau, 2\tau, ...$ we intersect the solution set with this set of nice data. This lets us piece together T/τ intervals such that the solution always stays in the nice set up to T. By a union bound, the final intersection has probability greater than $1 - (T/\tau)p$.

Example 2.3. In the argument of [**I**, Section 4] the nice set is a Besov space ball of radius M. It is shown that we can choose $\tau \simeq M^{-3}$, and that the complement of the ball has probability $p \lesssim M^{-k}$ for any $k > 0$.

We can therefore fix e.g. $k = 4$ to get wellposedness up to time T in a set of probability $1 - CM^{-1}$. The union over $M \in \mathbb{Z}_+$ yields a set of probability 1.

Bourgain's globalization argument is not the only possible method. There are also energy methods that rule out blow-up of solutions in finite time for almost all initial data. For example, [40, Theorem 4.2] and [73, Proposition 6.23] reduce the wellposedness of a nonlinear equation to that of the matching linear equation. A more advanced variant is "coming down from infinity" [94].

2.2 Four equations

As discussed above, the measure (2.1) is connected to a parabolic SPDE via Langevin dynamics. It is however not the only possible model.

To be precise, (2.2) corresponds to *overdamped* Langevin dynamics. If we modify the measure to be

$$\frac{1}{\mathcal{Z}} \exp\left(-\frac{1}{\varepsilon}\int_{\Lambda^d} V(\phi)\,\mathrm{d}x\right) \mathrm{dGFF}_{\varepsilon(m^2-\Delta)^{-1}}(\phi)\,\mathrm{dWN}_{\varepsilon}(\partial_t\phi), \tag{2.3}$$

where WN denotes a white noise measure, then the energy yields the *underdamped* Langevin equation

$$\partial_{tt}u(x,t) + \gamma\partial_t u(x,t) + (m^2 - \Delta)u(x,t) = -V'(u(x,t)) + \sqrt{2\gamma\varepsilon}\xi. \tag{2.4}$$

Here $\gamma > 0$ is called a damping parameter. This process is known as hyperbolic or canonical stochastic quantization [113]. The measure of interest is the first component of the product measure over the pair $(u, \partial_t u)$.

The modified energy also gives rise to the *Hamiltonian equation*

$$\partial_{tt} u(x,t) + (m^2 - \Delta)u(x,t) = -V'(u(x,t)). \tag{2.5}$$

This equation lacks both the damping and the stochastic forcing. Yet if the initial data is sampled from the product measure (2.3), the measure is still invariant under the flow. This is because of conservation of the Hamiltonian [4, Chapter 16]. Although this pairing cannot be used to construct the measure, it is useful for analyzing the nonlinear wave equation.

Finally, the Hamiltonian equation can also be studied over a complex state space. This gives rise to a nonlinear Schrödinger equation. It too is related to interesting PDE questions.

We now present local solution theory for these four dynamics in roughly increasing order of difficulty. For notational simplicity, we state the equations in un-renormalized form.

2.2.1 Heat equation

Equation (2.2) specialized to the ϕ^4 potential reads

$$\begin{cases} \partial_t u(x,t) + (m^2 - \Delta)u(x,t) = -u(x,t)^3 + \sqrt{2\varepsilon}\xi, \\ u(0) \sim \phi_d^4(\varepsilon). \end{cases} \tag{2.6}$$

We consider *mild solutions*[a] to the equation, satisfying

$$u(t) = e^{t(\Delta - m^2)}u(0) - \int_0^t e^{(t-s)(\Delta-m^2)}u(s)^3 \,\mathrm{d}s + \sqrt{2\varepsilon}\int_0^t e^{(t-s)(\Delta-m^2)}\,\mathrm{d}\xi(s). \tag{2.7}$$

Here the *heat semigroup* $e^{t\Delta}$ on $\mathbb{T}^d$ has the Fourier space representation

$$\widehat{e^{t\Delta}v}(k) = e^{-|k|^2 t}\hat{v}(k), \quad \text{and hence} \quad \big[\widehat{e^{t(\Delta-m^2)}v}\big](k) = e^{-(m^2+|k|^2)t}\hat{v}(k). \tag{2.8}$$

The heat semigroup has a powerful smoothing property:

Theorem 2.4. *For any $s \geq 0$, we have*

$$\big\|e^{t\Delta}w\big\|_{H^s} \leq (1 + C(s)t^{-s/2})\|w\|_{L^2}.$$

Proof. By Parseval's identity and the estimate $e^{-x} \leq C(s)x^{-s}$ we have

$$\begin{aligned} \big\|e^{t\Delta}w\big\|_{H^s}^2 &= \sum_{k\in\mathbb{Z}^d} (1+|k|^2)^s e^{-2|k|^2 t}\hat{w}(k)^2 \\ &\leq \sum_{k\in\mathbb{Z}^d} \hat{w}(k)^2 + \sum_{k\in\mathbb{Z}^d} |k|^{2s} C(s)(2|k|^2 t)^{-s}\hat{w}(k)^2, \end{aligned} \tag{2.9}$$

from which the claim follows. □

[a] It is an interesting question whether u solves the equation also in weak or strong sense, but we always study mild solutions within this thesis.

This property can be extended to a time-integrated version known as the Schauder estimate [see 14, Theorem 3.3.9]. The heat semigroup also gives us another characterization of the Gaussian free field:

Theorem 2.5. *The Gaussian free field with covariance $\varepsilon(m^2-\Delta)^{-1}$ is the unique invariant measure of the linear equation*

$$\partial_t \text{𝔣}(x,t) + (m^2 - \Delta)\text{𝔣}(x,t) = \sqrt{2\varepsilon}\xi.$$

Proof. By taking Fourier transform of the equation, we see that each Fourier mode satisfies

$$\mathrm{d}\hat{\text{𝔣}}(k,t) + (m^2 + |k|^2)\hat{\text{𝔣}}(k,t)\,\mathrm{d}t = \sqrt{2\varepsilon}\,\mathrm{d}B_t^{(k)}, \tag{2.10}$$

where $B_t^{(k)}$ are independent Brownian motions for each $k \in \mathbb{Z}^d$. This equation defines an Ornstein–Uhlenbeck process, whose stationary measure is Gaussian with covariance $\varepsilon(m^2 + |k|^2)^{-1}$. □

The solution theory of (2.6) on one-dimensional torus is quite straightforward [14, Section 3.4], so let us consider the equation on $\mathbb{T}^2$. In this case the nonlinearity $-u(x,t)^3$ is replaced by $-{:}u(x,t)^3{:}$. We omit the truncation from notation for simplicity. Since already the linear equation is only distributionally solvable, the SPDE is called *singular*.

It is natural to use a fixpoint argument to construct a local solution. Here the nonlinearity causes issues: even though Theorem 1.9 gives that all Wick powers of the GFF belong to $\mathcal{C}^{0-}$, the same is not obvious for Wick powers of u.

This issue can be solved by the so-called *Da Prato–Debussche trick* [40]: we decompose $u = \text{𝔣} + v$, where the two components solve

$$\begin{cases} \partial_t \text{𝔣} + (m^2 - \Delta)\text{𝔣} = \sqrt{2\varepsilon}\xi, \\ \qquad\qquad \text{𝔣}(0) \sim \mathrm{GFF}_{\varepsilon(m^2-\Delta)^{-1}}, \\ \partial_t v + (m^2 - \Delta)v = -{:}(\text{𝔣} + v)^3{:}, \\ \qquad\qquad v(0) = u(0) - \text{𝔣}(0). \end{cases} \tag{2.11}$$

With Besov space multiplication properties and Theorem 2.4 it is then almost surely possible to close a fixpoint argument for v in $\mathcal{C}([0,\tau];\, H^{2-}(\mathbb{T}^2))$, where τ depends on the initial data and the noise. The regularity of v makes ${:}(\text{𝔣} + v)^3{:}$ well-defined in $\mathcal{C}^{0-}$. Full details can be found in [40] or [93].

As discussed on page 21, the ϕ^4_3 model requires further renormalization. The T-regularized equation becomes

$$\partial_t u(x,t) + (m^2 - 3\,\mathbb{E}[\text{𝔣}_T^2] + 2\gamma_T - \Delta)u(x,t) = -u(x,t)^3 + \sqrt{2\varepsilon}\xi_T. \tag{2.12}$$

That is, the renormalization appears as a divergent mass term. The divergence is of order T as $T \to \infty$, whereas in two dimensions it is of order $\log(T)$. The local wellposedness theory was achieved in 2010s with three different approaches:

- **Regularity structures.** Hairer [74, 75] extended the theory of rough paths (see e.g. [89]) by developing a generalization of Taylor expansions. The solution is locally approximated by a sum of increasingly complicated stochastic objects. It is then possible to combine these local approximations into a full solution.

  This theory is very flexible, and applies to many singular SPDEs such as the KPZ equation. It is quite algebraic in nature; there is also the recent work [87] in which trees are replaced by multi-indices, which avoids some combinatorial steps.

- **Paracontrolled distributions.** The approach of Gubinelli, Imkeller, and Perkowski [67] also extends rough paths. It was applied to ϕ^4_3 by Catellier and Chouk [34].

  This approach is more analytic and arguably simpler than regularity structures, but in some cases it cannot reach quite as far. Here the idea is to produce a decomposition for the solution with *paraproducts* from harmonic analysis [18]; see [34, Ansatz 2.11] for an example. Paraproducts are naturally estimated in Besov spaces.

- **Renormalization group.** This approach is based on controlling "effective equations" for discrete (e.g. dyadic) length scales. It was pioneered by Wilson [127] in late 1960s, and applied to ϕ^4_3 by Kupiainen [83].

  The approach was extended to continuous scaling by Polchinski [111]. See also the recent application by Duch [48] and the survey [12].

The boundaries between these approaches are not sharp: For example, article [**II**] builds upon [10], which combines estimates from the paracontrolled approach with the Polchinski-style Theorem 1.5.

2.2.2 Stochastic wave equation

The underdamped equation (2.4) is specialized to the ϕ^4 case as

$$\begin{cases} \partial_{tt}u(x,t) + \gamma\partial_t u(x,t) + (m^2 - \Delta)u(x,t) = -u(x,t)^3 + \sqrt{2\gamma\varepsilon}\xi, \\ u(0) \sim \phi^4_d(\varepsilon), \\ \partial_t u(0) \sim \text{white noise}(\varepsilon). \end{cases} \tag{2.13}$$

The mild solution satisfies

$$u(t) = (\mathcal{C}^\gamma_t + \tfrac{\gamma}{2}\mathcal{S}^\gamma_t)u(0) + \mathcal{S}^\gamma_t(\partial_t u(0)) - \int_0^t \mathcal{S}^\gamma_{t-s}u(s)^3\,\mathrm{d}s + \sqrt{2\gamma\varepsilon}\int_0^t \mathcal{S}^\gamma_{t-s}\,\mathrm{d}\xi(s), \tag{2.14}$$

where $\mathcal{S}^\gamma$ and $\mathcal{C}^\gamma$ are the damped wave propagators

$$\mathcal{S}^\gamma_t = e^{-\frac{t\gamma}{2}}\frac{\sin\Big(t\sqrt{m^2 - \Delta - \frac{\gamma^2}{4}}\Big)}{\sqrt{m^2 - \Delta - \frac{\gamma^2}{4}}}, \quad \mathcal{C}^\gamma_t = e^{-\frac{t\gamma}{2}}\cos\Big(t\sqrt{m^2 - \Delta - \tfrac{\gamma^2}{4}}\Big). \tag{2.15}$$

The Schauder estimate shows that the heat semigroup $e^{t\Delta}$ regularizes its argument by two derivatives. It can be seen from (2.15) that $\mathcal{S}^\gamma$ regularizes only by one derivative. This makes canonical stochastic quantization more challenging; see also [98]. (While $\mathcal{C}^\gamma$ does not improve differentiability at all, it only appears in the linear term, and hence does not cause issues.)

Well-posedness of (2.13) on $\mathbb{T}^2$ was recently shown by Gubinelli, Koch, Oh, and Tolomeo in [68]. The local-in-time fixpoint argument is a modification of the Da Prato–Debussche argument used for the heat equation (2.6). Global well-posedness then follows from Bourgain's argument or energy methods.

Mathematically rigorous results on canonical stochastic quantization are a rather recent advancement. For example, wellposedness of (2.13) on $\mathbb{T}^3$ is still unproven as of this writing. There are some results on related, slightly easier models [99, 105].

2.2.3 Deterministic wave equation

If we set $\gamma = 0$ in (2.13) above, we get

$$\begin{cases} \partial_{tt} u(x,t) + (m^2 - \Delta) u(x,t) = -u(x,t)^3, \\ u(0) \sim \phi_d^4(\varepsilon), \\ \partial_t u(0) \sim \text{white noise}(\varepsilon), \end{cases} \tag{2.16}$$

where $\varepsilon > 0$ is chosen freely. A mild solution to this equation then satisfies

$$u(t) = \mathcal{C}_t u(0) + \mathcal{S}_t(\partial_t u(0)) - \int_0^t \mathcal{S}_{t-s} u(s)^3 \, \mathrm{d}s, \tag{2.17}$$

where

$$\mathcal{S}_t = \frac{\sin(t\sqrt{m^2 - \Delta})}{\sqrt{m^2 - \Delta}}, \quad \mathcal{C}_t = \cos(t\sqrt{m^2 - \Delta}). \tag{2.18}$$

This equation is interesting already from purely PDE perspective. Wave equations feature finite speed of propagation, meaning that $u(x,t)$ depends on the initial data only inside a ball $B(x, ct)$, where c represents "speed of sound".

The $-u^3$ nonlinearity adds *dispersion*: roughly stated, mass will not stay concentrated for very long [119, Section 2.3]. This gives rise to hard-to-analyze nonlinear effects. From PDE perspective, the roughness of initial data presents an additional challenge.

Global wellposedness on $\mathbb{T}^2$ was shown by Oh and Thomann [103], although Bourgain stated the result without proof much earlier [24, Theorem 111]. Oh and Thomann prove the result for more general nonlinearities; a simplified proof of the $-u^3$ case is included in [**I**, Section 4].

Global wellposedness on $\mathbb{T}^3$ was recently shown in the breakthrough article of Bringmann, Deng, Nahmod and Yue [32]. The bulk of the 279-page paper is devoted to estimating stochastic cancellations within the ansatz [32, Eq. (1.32)]. This is possible with the recent theory of random tensors [43].

2.2.4 Schrödinger equation

The complex ϕ^4 measure (again with arbitrary $\varepsilon > 0$) is formally invariant under the nonlinear Schrödinger equation

$$\begin{cases} -i\partial_t u(x,t) - \Delta u(x,t) = -u(x,t)^3, \\ u(0) \sim \phi^4_{2,\mathbb{C}}(\varepsilon). \end{cases} \tag{2.19}$$

This equation has applications e.g. in plasma waves, nonlinear optics, and Bose–Einstein condensates (see e.g. the introduction of [96]). The study of the Gibbs measure by Lebowitz, Rose, and Speer [85] is what originally prompted Bourgain to develop techniques for random dispersive PDEs [21, 22].[b]

This equation is classically studied with $m^2 = 0$. The lack of a mass term causes some complications for defining the GFF, but this is circumvented by the conservation of L^2 norm under (2.19).

Already the linear Schrödinger equation shows dispersion. If v solves the linear equation with initial data u_0, then $v(t) = e^{i\Delta t}u_0$, meaning that the *spatial* Fourier transform satisfies

$$\hat{v}(k,t) = \exp(-ik^2 t)\hat{u}_0(k), \quad k \in \mathbb{Z}^d. \tag{2.20}$$

This shows that a wave of frequency k moves at speed $|k|^2$.

The linear propagator (2.20) does not regularize its argument at all. This rules out a fixpoint argument in a Besov space. The key insight is that the *spacetime* Fourier transform of the linear equation satisfies

$$(k^2 + \theta)\hat{v}(k,\theta) = 0. \tag{2.21}$$

Therefore $\hat{v}(k,\theta)$ is supported on the parabola $\{k \in \mathbb{Z}, \theta \in \mathbb{R} \colon k^2 = -\theta\}$. For short timescales, the support of the solution to (2.19) concentrates near this set, which is captured by the *Fourier restriction norm*

$$\|u\|_{X^{s,b}} := \left\| (1+|k|^2)^{s/2} \, (1+|\theta + k^2|^2)^{b/2} \, \hat{u}(k,\theta) \right\|_{L^2(\mathbb{Z}^d \times \mathbb{R})}. \tag{2.22}$$

The fixpoint argument can then be done in the $X^{s,b}$ space defined by this norm. See e.g. [119, Section 2.6] for properties and embeddings of these spaces.

Bourgain also developed the two-dimensional solution theory [23]. This case is also presented in the exposition [102] and the recent generalization [44] based on random tensors.

The solution theory on $\mathbb{T}^3$ is still open. As noted in [32, Section 1.5.2], certain scaling heuristics suggest that this question is much harder than the other equations discussed above.

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[b] The original works considered a *focusing* cubic nonlinearity with the opposite sign. The local wellposedness theory is similar for both nonlinearities, but most globalization results and the connection with the ϕ^4 measure are based on the *defocusing* nonlinearity shown here.

2.3 Article [I]

While we discussed stochastic quantization on $\mathbb{T}^d$ above, it is also possible to pass to the infrared limit. Global wellposedness and invariance of measure for the ϕ^4_2 stochastic heat equation on $\mathbb{R}^2$ was shown by Mourrat and Weber [93]. The ϕ^4_3 case was later proved by Gubinelli and Hofmanová with a different argument [66]; see also the recent preprint [49].

The idea in [93] is to start from suitable initial data on $\mathbb{R}^2$ and estimate a solution restricted to a torus of side length L. The estimates are done in a weighted Besov norm and are uniform in L. There is a compact embedding of polynomially weighted Besov spaces

$$B^{-\varepsilon}_{p,r}((1+|x|^2)^{-\alpha_1/2}) \hookrightarrow B^{-2\varepsilon}_{p,r}((1+|x|^2)^{-\alpha_2/2}) \tag{2.23}$$

when $\alpha_2 > \alpha_1 > d$. This is used to find a subsequence of periodized solutions that converges to a full-space solution as $L \to \infty$. The solution is then found to be unique with a Grönwall argument.

Article [**I**] extends the ϕ^4_2 infrared limit to hyperbolic setting, namely the deterministic wave equation (2.16). With the wave equation it is possible to use finite speed of propagation, which greatly simplifies the argument:

- As in [93], periodic solutions are estimated in a weighted norm [**I**, Section 4].
- Periodized solutions form almost surely a Cauchy sequence that converges to a full-space limit [**I**, Section 5.1].
- The σ-algebra of distributions on $\mathbb{R}^2$ is generated by restrictions to bounded subsets of $\mathbb{R}^2$, so the invariance of measure is inherited from the periodic case [**I**, Section 5.2].

Just a few weeks before arXiv publication of [**I**], Oh, Tolomeo, Wang, and Zheng [104] published a similar result on the stochastic wave equation (2.13). The basic ideas in the two articles are similar, but [104] addresses additional complications from the stochastic term and includes convergence in a Wasserstein metric.

The methods of these articles could feasibly be combined with [32] to prove infrared limit for the three-dimensional wave dynamics (2.16). There is naturally much technical work in the details.

Infrared limit for the one-dimensional nonlinear Schrödinger equation (2.19) was constructed by Bourgain [25], but the higher-dimensional cases are still open. The difficulty lies in controlling the propagation of randomness since there is neither finite speed of propagation nor smoothing.

[**I**, Section 6] shows a weaker infrared limit of (2.19) on $\mathbb{R}^2$. Instead of proving invariance of measure as in Definition 2.1, we only show the law of the solution to be $\phi^4_{2,\mathbb{C}}$ at any fixed time. This weaker notion was used by Albeverio and Cruzeiro [3] in the context of stochastic fluid dynamics, and by Oh and Thomann for the nonlinear Schrödinger equation on $\mathbb{T}^2$ [102].

The lack of finite speed of propagation can be counteracted at the cost of derivatives [**I**, Lemma 6.4]. As a result, periodic solutions of (2.19) are shown to converge almost surely to a limit in a weighted $H^{-\alpha}$ space for some $\alpha > 4$. This space is of course much larger than $H^{-\varepsilon}$ where the measure is supported, and we cannot prove uniqueness of solutions with this method.

2.4 Further questions

Uniqueness of the invariant measure has also been studied. The ϕ^4_d measure is the unique invariant measure for stochastic heat dynamics on $\mathbb{T}^2$ and $\mathbb{T}^3$ [76, 121]. The strong Feller property is not applicable to wave dynamics, but Tolomeo [120] recently developed an ergodicity and uniqueness result for the stochastic wave equation on $\mathbb{T}^2$.

An interesting question is whether uniqueness holds in the infrared limit, since the limit constructions generally involve a passing to subsequence. The answer is negative: it is known that if one parametrizes the ϕ^4 potential as $\lambda\phi^4$ for $\lambda > 0$, the ϕ^4_2 measure is unique only for sufficiently small λ [62, 63]. The proof uses techniques developed for the Ising model. The same holds for ϕ^4_3 [37, 49, 57].

Article [**III**] considers the massless sine-Gordon model. The massive ($m^2 > 0$) model has an invariant measure that can be constructed with stochastic quantization. Equation (2.6) is modified to be

$$\partial_t u(x,t) + (m^2 - \Delta)u(x,t) = -\gamma \sin(\beta u(x,t)) + \sqrt{2}\xi, \tag{2.24}$$

where $\beta > 0$ is explained in Section 1.3. In one dimension $\gamma > 0$ is a free parameter, but in two dimensions it is the divergent renormalization constant given by (1.18). In the rest of this section we consider only the two-dimensional model.

Local solution theory (including the $m^2 = 0$ case) was developed by Chandra, Hairer and Shen in the two articles [38, 77]. The renormalization thresholds described in Section 1.3 are reflected in the SPDE, which is locally wellposed for $\beta^2 < 8\pi$. Global solution theory is currently known for $\beta^2 < 6\pi$, due to Bringmann and Cao [31]. There are also recent results on the infrared limit [65, 69] for a restricted range of β.

The first results on the canonical stochastic quantization[c]

$$\partial_{tt} u(x,t) + \partial_t u(x,t) + (m^2 - \Delta)u(x,t) = -\gamma{:}\sin(\beta u(x,t)){:} + \sqrt{2}\xi. \tag{2.25}$$

have appeared quite recently [100, 101, 129]. The issues due to weaker regularization from the wave propagator apply: for example, the Da Prato–Debussche trick fails already at $\beta^2 = 2\pi$ instead of $\beta^2 = 4\pi$.

Finally, stochastic quantization can be applied to many more models. The sine-Gordon model is related to the imaginary Gaussian multiplicative chaos of form

[c] The coupling parameter γ in (2.25) and the damping parameter in (2.13) are unrelated. Unfortunately the stochastic quantization literature quite commonly uses the letter γ for both purposes. One could also parametrize the damping in (2.25).

$e^{i\beta\phi}$ for a Gaussian field ϕ. The non-imaginary setting $e^{\beta\phi}$ is related to the Liouville conformal field theory (see e.g. the recent survey [72]). There has also been recent interest towards the Yang–Mills model, which is connected to the Standard Model of particle physics [30, 35, 36].

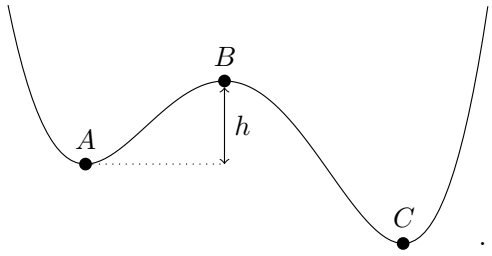
The local solution theory outlined in Section 2.2 depends on understanding the interplay between deterministic and probabilistic elements. The study of dispersive PDEs is a vast field on its own right. It is possible to modify the equations by incorporating, for example, higher-order nonlinearities [33] or a fractional Laplacian [88].

Chapter 3

Metastability

3.1 Motivation

A physical system is said to be in a *metastable* state if its configuration is locally energetically optimal, but a sufficient perturbation could push it to an even lower-energy state. A simple one-dimensional system could have a potential like



Here the state A has locally minimal energy, but if the system passes the barrier B of height h, it can "fall down" to the globally optimal state C. If there is some random fluctuation in the system, it is interesting to estimate the (random) transition time from A to C.

Concretely, the system could be a chemically reactive molecule and the fluctuation come from thermal motion of atoms. As there are many degrees of freedom, the potential landscape would consist of high-dimensional "hills" and "valleys", as opposed to the simple picture above.

Computation of chemical reaction rates motivated the study of metastability in the late 19^{th} to early 20^{th} century by Van 't Hoff [81, p. 114], Arrhenius [5], Eyring [51], Kramers [82], and others; see also the historical survey [79].

This work was concurrent with the development of statistical mechanics and theory of phase transitions. Nowadays high-dimensional potential landscapes also appear in statistical inference and machine learning.

The so-called *Eyring–Kramers law* states that the mean transition time is proportional to $\exp(h/T)$, where h is the potential barrier height and T the temperature (in compatible units). In applications, the subexponential prefactor would be measured experimentally; as we discuss below, it is related to local geometry of the potential.

The precise value of the subexponential factor also depends on what counts as a transition. *Transition state theory* (TST) estimates the time needed to reach

the top of the potential barrier (B in the preceding picture).[a] It then postulates that the reaction always continues from there towards C.

However, it is possible that a path started from A reaches B but falls back into A. The proportion of complete $A \to C$ transitions to the $A \to B$ hits is called the *transmission coefficient* or *dynamical correction*. When this correction is applied, we get the true transition time between neighbourhoods of A and C. This sense of transition is the one usually associated with an Eyring–Kramers law in mathematics.

3.2 Mathematical techniques

Since the work of Kramers [82], there has been extensive work to rigorously prove and generalize the theory of transition times. We now outline the ideas of the most successful approaches; see also [29, Chapter 1] for additional references.

In this section we still consider finite-dimensional systems. The prototypical equation to be studied is the overdamped Langevin equation

$$\mathrm{d}X_t = -\nabla F(X_t)\,\mathrm{d}t + \sqrt{2\varepsilon}\,\mathrm{d}B_t, \tag{3.1}$$

where $X_t \colon \mathbb{R} \to \mathbb{R}^d$, the function F gives a suitably regular potential, and B_t is a d-dimensional Brownian motion. Typical assumptions on F include sufficiently many continuous derivatives and some growth bounds. We will make the further simplifying assumption that F has exactly two minima as on page 33.

3.2.1 Large deviations

The first rigorous approach was the *large deviations* method of Freĭdlin and Wentzell [54]. Their large deviation principle implies (see e.g. [29, Section 6.5]) that the hitting time τ_C of a small ball C around a potential minimum satisfies

$$\lim_{\varepsilon\to 0} \mathbb{P}_A\Big(\exp((h-\kappa)/\varepsilon) \le \tau_C \le \exp((h+\kappa)/\varepsilon)\Big) = 1. \tag{3.2}$$

Here $\mathbb{P}_A$ is probability with respect to initial state in a small ball A around the other minimum, h is the height of the potential barrier as on page 33, and $\kappa > 0$ is arbitrarily small.

The same approach also yields estimates of the form

$$\mathbb{P}_x\Big(\sup_{0\le t\le T} \big|X_t^\varepsilon - X_t^0\big| > \delta\Big) \le \exp(-K/\varepsilon), \tag{3.3}$$

where X_t^0 is the deterministic flow started from x, and $K > 0$ depends on all parameters except ε. In other words, the probability of straying from the energy-minimizing path between the wells decays rapidly as $\varepsilon \to 0$.

This approach is good in that it gives pathwise estimates. Its downside is that it gives no information on the subexponential prefactor of the Eyring–Kramers law. Moreover, the constants like K in (3.3) are quite implicitly defined.

[a] In chemistry vocabulary, *transition state* is an intermediate product of a chemical reaction. Such a molecule is always unstable.

3.2.2 Potential theory

These issues are addressed by the newer approach based on *potential theory*. It was introduced by Bovier, Eckhoff, Gayrard, and Klein in the early 2000s [26–28]. This method is based on the connection between Markov diffusion processes and elliptic PDEs. By some manipulation of the PDEs (see [17, Section 3.1] for an excellent summary, and [29, Parts III and IV] for details), we end up with the following formula:

$$\mathbb{E}_x \tau_C = \frac{\int_{\mathbb{R}^d \setminus C} h_{A,C}(z) \exp(-F(z)/\varepsilon) \, \mathrm{d}z}{\operatorname{cap}(A, C)} (1 + o(1)). \tag{3.4}$$

Here C is a neighbourhood of the global potential minimum and τ_C its hitting time, the set A is an ε-ball around x, and the *equilibrium potential* $h_{A,C}$ and *capacity* are defined as follows:

Definition 3.1. The *equilibrium potential* between disjoint sets $A, C \subset \mathbb{R}^d$ is

$$h_{A,C}(z) = \mathbb{P}_z(\tau_A < \tau_C),$$

where τ_A and τ_C are the hitting times of A and C respectively. Moreover, the *capacity* between A and C is

$$\operatorname{cap}(A, C) = \inf_g \varepsilon \int_{\mathbb{R}^d} |\nabla g(z)|^2 \exp(-F(z)/\varepsilon) \, \mathrm{d}z,$$

where the infimum is over $g \in H^1(\mathbb{R}^d)$ such that $g \geq 1$ on A and $g = 0$ on C.

In the special case where the two potential minima $z_A, z_C \in \mathbb{R}^d$ are at the same height and there is exactly one saddle point $z_B \in \mathbb{R}^d$, the transition time takes the form

$$\mathbb{E}_{z_A} \tau_C = \frac{2\pi}{\mu} \sqrt{\frac{|\det \operatorname{Hess} F(z_B)|}{\det \operatorname{Hess} F(z_A)}} \exp\left(\frac{F(z_B) - F(z_A)}{\varepsilon} \right) (1 + o(1)). \tag{3.5}$$

Here $-\mu$ is the single negative eigenvalue of the Hessian matrix $\operatorname{Hess} F(z_B)$. This formula requires that the Hessian matrices have no vanishing eigenvalues; there are extensions that permit degeneracies [16, 17].

The prefactor of (3.5) exhibits a twofold dependency on the geometry of the potential: the capacity is mostly determined by F near the saddle, whereas the numerator of (3.4) depends mostly on F near the minima.

The benefit of the potential-theoretic approach is that both the numerator and denominator of (3.4) can often be estimated explicitly. In many physical applications there is a quadratic term that gives rise to a Gaussian.

The derivation in [27] assumes reversibility of the diffusion process and ellipticity of the generator. Both of these are satisfied by (3.1), but neither is satisfied by the underdamped system

$$\begin{cases} \mathrm{d}q(t) = p(t) \, \mathrm{d}t, \\ \mathrm{d}p(t) = -\nabla F(q_t) \, \mathrm{d}t - \gamma p(t) \, \mathrm{d}t + \sqrt{2\gamma\varepsilon} \, \mathrm{d}B_t, \end{cases} \tag{3.6}$$

where $q(t), p(t) \in \mathbb{R}^d$ and $\gamma > 0$. These assumptions have been progressively lifted in some recent works [e.g. 19, 84]. As a result, an Eyring–Kramers law for (3.6) was proved in the recent preprint [86] by Lee, Ramil, and Seo. The expected transition time is

$$\frac{4\pi}{-\gamma + \sqrt{\gamma^2 + 4\mu}} \sqrt{\frac{|\det \operatorname{Hess} F(z_B)|}{\det \operatorname{Hess} F(z_A)}} \exp\left(\frac{F(z_B) - F(z_A)}{\varepsilon} \right) (1 + o(1)). \tag{3.7}$$

The overdamped equation (3.1) is the limit of (3.6) when $\gamma \to \infty$ and time is scaled by $1/\gamma$; then (3.7) matches (3.5). On the other hand, in the formal $\gamma = 0$ limit the transition times differ by a factor of $\sqrt{\mu}$.

Another recent development is the connection of (3.4) to quantities of geometric function theory: geodesic distance and minimal cut [7]. This is motivated by the study of degenerate saddles [16] and the topology of more complicated potential landscapes [6].

3.2.3 Other methods

A third common method is to study the system on *spectral* side. The generator of a reversible Markov diffusion has non-negative eigenvalues, including 0. The system is metastable if it has exponentially small (in ε) eigenvalues corresponding to the potential minima, whereas the remaining eigenvalues are uniformly bounded from below [42].

The spectral question is tightly connected to the potential-theoretic approach. With tools from the latter, it can be shown that the *spectral gap* between 0 and the next eigenvalue is given by the inverse of the Eyring–Kramers law [28].

The spectral approach leads to a natural proof that the transition times are *asymptotically* exponentially distributed as $\varepsilon \to 0$ [29, Section 8.4.4]. See also e.g. [91, 122].

The spectrum has also been studied with semiclassical analysis of the Witten complex; see e.g. [45, 80] for examples. More references and a comparison of the different approaches can be found in the survey [13].

3.3 Extension to SPDEs

The stochastic ordinary differential equation (3.1) can be naturally extended to the SPDE

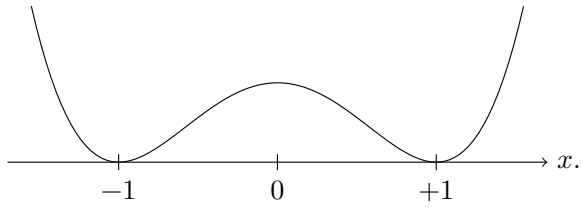
$$\partial_t u(x,t) - \Delta u(x,t) = -\nabla \tilde{F}(u(x,t)) + \sqrt{2\varepsilon}\xi, \quad x \in \mathbb{T}^d, \tag{3.8}$$

if the potential F in (3.1) is chosen to be the sum of $\tilde{F}$ and a discrete gradient term that gives rise to the Laplacian. This is exactly the stochastic quantization equation discussed in Section 2.2.1.

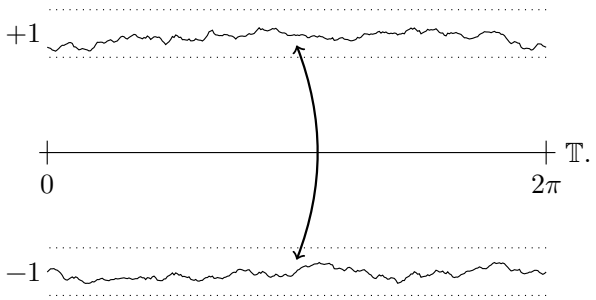
In particular, if we flip the sign of the mass term in (2.6), we get

$$\partial_t u(x,t) - \Delta u(x,t) = -u(x,t)^3 + u(x,t) + \sqrt{2\varepsilon}\xi. \tag{3.9}$$

This is called the *stochastic Allen–Cahn equation*. In the notation of (3.8), the potential is $\tilde{F}(u) = (u^2 - 1)^2$, which yields two symmetric potential wells:



This equation has two stable minima, namely the constant functions at ± 1. Metastable transitions are then defined as transitions between neighbourhoods of these functions (here dotted lines indicate L^∞ balls):



How do the techniques from Section 3.2 adapt to this SPDE?

The Freĭdlin–Wentzell approach works quite naturally. Although $u(t)$ is now a point in an infinite-dimensional space, the abstract large deviations setup works unchanged. Exponential bounds as in (3.2) were found for the one-dimensional equation already in 1980s [52, 53]. Large deviations for the two- and three-dimensional versions were studied more recently by Hairer and Weber [78].

The potential-theoretic approach was adapted to the one-dimensional SPDE setup by Berglund and Gentz in the early 2010s [17]. As in Section 1.2, they truncate the Fourier series of u to modes of size at most N, and then compute uniform-in-N bounds for transition times of the finite-dimensional system. There are two technical issues:

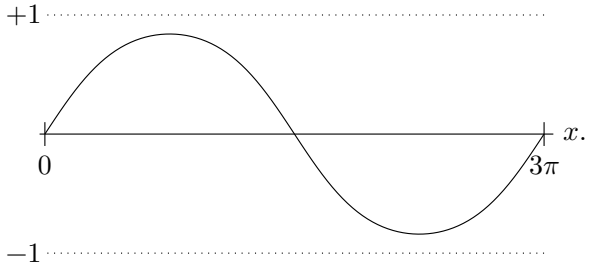
- The derivation of (3.4) uses a Harnack inequality that depends on N. Without it, the expectation cannot be taken with respect to fixed initial data; instead the data must be sampled from a probability measure concentrated near a minimum. It is however possible to pass to deterministic data by a post-processing argument.

- The argument does not give a transition time τ for the full equation (3.9), but rather a sequence of transition times τ_N for truncated equations. While $\mathbb{E}\,\tau_N$ is bounded uniformly in N, the convergence of $\tau_N \to \tau$ as random variables also requires some post-processing steps.

There is also an interesting bifurcation related to the equation (3.9) itself. When the torus has period $L < 2\pi$, the unique saddle point is the constant 0 function. At $L = 2\pi$ another saddle, corresponding to a non-constant function, appears. It is explicitly given by

$$u_{\mathrm{st}}(x) = \sqrt{\frac{2m}{m+1}}\,\mathrm{sn}\Big(\frac{x}{\sqrt{m+1}}, m\Big), \tag{3.10}$$

where sn is a Jacobi elliptic function and $m \in (0,1)$ satisfies the periodic boundary condition [90, Eq. (7)]. For example, for $L = 3\pi$ this transition state looks like



Since the potential is invariant under x translation, any shift of this function is also stationary: this means that the saddle is not a point but a one-dimensional manifold in Fourier space. Further bifurcations happen when L is a multiple of 2π, but the first non-constant saddle stays energetically optimal.

Expected transition times for any L were computed in [17, Theorem 2.6]; the prefactor in (3.5) has a limit as a functional determinant when the truncation is removed. The computation was extended to the renormalized equation on $\mathbb{T}^2$ by Berglund, Di Gesù, and Weber [15] under the assumption $L < 2\pi$, with the post-processing steps carried out by Tsatsoulis and Weber in [122].

Given the connection between heat and wave equations outlined in Section 2.2, it is natural to ask about how the transition rate changes under wave dynamics. Newhall and Vanden-Eijnden [97] considered the one-dimensional wave equation

$$\partial_{tt} u(x,t) - \Delta u(x,t) = -u(x,t)^3 + u(x,t) \tag{3.11}$$

with initial data sampled from the invariant measure of noise strength ε. They consider Dirichlet or Neumann boundary conditions, but the argument also applies to a periodic domain of size $L < 2\pi$.

They did not use any of the approaches mentioned above, but instead adapted the classical transition state theory to the SPDE setup, following earlier work in [117, 124]. Under the invariant measure, the expected frequency[b] of hitting the dividing surface $\{\overline{u} = 0\}$ is given by the simple formula

$$2 \int_{\{\overline{u}=0\}} \max(0, \overline{\partial_t u})\, \mathrm{d}\mu_\varepsilon(u, \partial_t u), \tag{3.12}$$

[b] Note that transition frequency is inverse of the transition time τ considered for (3.8). In general $\mathbb{E}(1/\tau) \neq 1/(\mathbb{E}\,\tau)$, but this is expected to be asymptotically the case in the $\varepsilon \to 0$ limit due to the exponentially distributed jumps.

where μ_ε is the invariant product measure and $\overline{u}$ indicates spatial mean of u. In the ϕ^4 case it factorizes into three quantities:

$$\frac{\left[\mathbb{E}_{\mathrm{WN}(\varepsilon)} \max(0, \overline{\partial_t u})\right] \int_{\{\overline{u}=0\}} \exp\left(-\frac{1}{4\varepsilon} \int_{\mathbb{T}} u(x)^4 \,\mathrm{d}x\right) \mathrm{dGFF}_{\varepsilon(-1-\Delta)^{-1}}(u)}{\mathcal{Z}(\varepsilon)}. \tag{3.13}$$

The Gaussian in the numerator is well-defined since the zero mode is fixed and $L < 2\pi$. In the partition function $\mathcal{Z}(\varepsilon)$ it is possible to re-centre the field to each of the potential wells. If we set $u = 1 + \phi$, then $u^4/4$ is replaced by $\phi^4/4 + \phi^3$ and the covariance becomes $(2 - \Delta)^{-1}$.

Newhall and Vanden-Eijnden evaluate (3.13) with a spatial discretization scheme. They find that the TST frequency is given by the inverse of the expression (3.7) with $\gamma = 0$. Compared to the result on the parabolic stochastic Allen–Cahn equation, this result is however weaker:

- It only gives the TST hitting frequency of the separating surface, without any dynamical correction to account for incomplete transitions.
- It takes initial data from the full invariant measure and not from a measure concentrated near one of the minima.

The same result also applies to the double-well stochastic wave equation (2.13). This is because the TST hitting frequency only depends on the invariant measure, and the damping γ should appear through the dynamical correction.

3.4 Articles [II] and [III]

Article [**II**] continues from Newhall and Vanden-Eijnden's work [97] in that it derives the TST hitting frequency of the two- and three-dimensional, renormalized double-well wave equations on periodic domains.[c]

Remark 3.2. Article [**II**] states results with inverse temperature $\beta = 1/\varepsilon$. To avoid confusion with the parameter β of the sine-Gordon model, we state the results here with ε.

Instead of the spatial discretization of [97], the approximation scheme in [**II**] is Fourier-based as in Berglund *et al.* [15, 17]. A sketch of the argument is:

- The hypersurface integral in (3.13) follows immediately from the existing results on ϕ^4 measures, adapted to fixed zero mode [e.g. 10].
- After performing the $u = 1+\phi$ translation, we need to estimate the partition function $\mathcal{Z}(\varepsilon)$ with the cubic correction term. This is first done in a region away from the saddle [**II**, Section 3.1]. The computation is a straightforward modification of [10, Section 2].

[c] As mentioned in Section 2.2.2, global wellposedness of the three-dimensional stochastic wave equation is not yet proved, so the result on it should be considered conditional.

- The Wick ordering is dependent on the covariance, which is changed in the translation. Therefore some lower-order correction terms are also needed. They are handled at the end of [**II**, Section 3.1].

- Concentration of measure is then used to handle the negligible contribution to $\mathcal{Z}(\varepsilon)$ near the saddle [**II**, Section 3.2].

The cited sections refer to the two-dimensional case. The ϕ^4_3 case in [**II**, Section 5] follows the same overall argument. The computations are technically much more demanding due to the roughness of solutions. In particular, the Wick ordering correction terms require some nontrivial stochastic cancellations [**II**, Section 5.3]. The argument reuses results from [10], but some of the estimates are novel.

Since TST overestimates the frequency of transitions between wells, a full Eyring–Kramers law also includes the transmission coefficient. By the finite-dimensional result (3.7) its $\varepsilon \to 0$ limit should equal

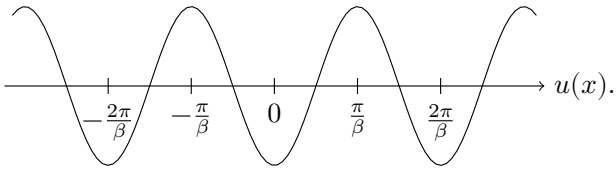
$$\frac{-\gamma + \sqrt{\gamma^2 + 4}}{2}. \tag{3.14}$$

When $\gamma = 0$, the dynamical correction should hence approach 1 as $\varepsilon \to 0$. This is shown for the two-dimensional deterministic equation in [**II**, Section 4]; this result is novel also for the one-dimensional case.

The idea is to use the "inertia" from the $\partial_t u$ component. After reaching the saddle, $\overline{u}$ typically has velocity of order $\varepsilon^{1/2}$ towards the other well. The oscillatory modes are of the same order, but they influence $\overline{u}$ only through their second and third powers (of orders ε and $\varepsilon^{3/2}$). Hence they can be neglected and the transition is controlled by an ordinary differential equation for $\overline{u}$.

The same result is expected in three dimensions, but again the renormalized equation is harder to analyze. On the other hand the stochastic equation would need a more delicate estimate, since the forcing on $\overline{u}$ is of the same order $\varepsilon^{1/2}$.

Article [**III**] is more in line with the "classical" parabolic metastability theory of [15, 17]. It considers the massless dynamical sine-Gordon model (2.24) on $\mathbb{T}$, the potential of which looks like



Although there are infinitely many potential wells, symmetry permits us to reduce the evolution to three wells: once the solution has moved to either side of the initial well, it can be shifted back.

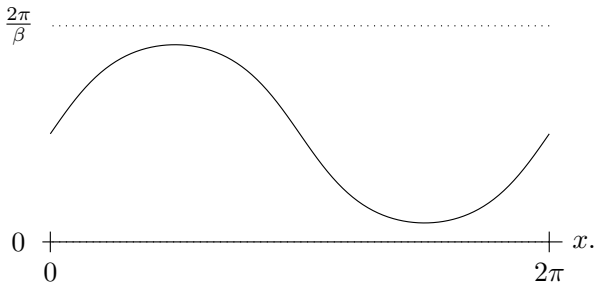
This model is close to the ϕ^4_1 model in that the potential is locally quadratic at saddles and minima. There is also a similar appearance of additional saddles: With sine-Gordon we can fix the domain size $L = 2\pi$ and change the parameters

γ and β. Bifurcations occur when $\gamma\beta$ is square of an integer, and the non-constant stationary state appearing at $\gamma\beta = 1$ is energetically optimal.

The transition state is again given by Jacobi elliptic functions. As derived in [**III**, Eq. (2.15)], the explicit formula is

$$u_{\mathrm{st}}(x) = \frac{1}{\beta}\left[\pi + 2\arcsin(\sqrt{m}\,\mathrm{cd}(\sqrt{\gamma\beta}x, m))\right], \tag{3.15}$$

where m is again chosen to satisfy the periodic boundary condition. Visually (here $\gamma = 1$, $\beta = 3$) the shape is very similar to that shown on page 38:



On the surface, the main contribution of [**III**] is to be a preparatory step: it raises the question of metastability theory of the two-dimensional dynamical sine-Gordon model, and provides a concrete example of a system with infinitely many potential wells.

However, the article also serves a pedagogic purpose: It expands on the exposition in [17], which focuses more on Neumann boundary conditions and gives very few details on the periodic case. The case where the saddle is a manifold and not a set of separated points is quite non-trivial to compute.

3.5 Further questions

Let us finish by addressing three open questions that follow immediately from the articles [**II**] and [**III**], and two further related research questions.

Following the progress of [15, 17], the question of sharp metastable transition times for the three-dimensional Allen–Cahn equation is still open. Given that the TST results in [**II**] are identical between two and three dimensions, the same is expected from the Eyring–Kramers law of the Allen–Cahn equation. The main obstacle appears to be the technical difficulty of the ϕ^4_3 dynamics.

Similar issues restrict the transmission coefficient computation in [**II**, Section 4] to the two-dimensional wave equation. The computation in three dimensions could possibly be solvable with techniques from [32].

Metastable dynamics of the one-dimensional Allen–Cahn equation on a larger domain ($L \geq 2\pi$) were studied in [17]. In one dimension the transition state can be derived explicitly with an argument from classical mechanics ([17, Section 2.2]

and [**III**, Section 2.3]). This argument is not extensible to higher dimensions. A characterization of higher-dimensional stable manifolds is probably related to other geometric questions around the Allen–Cahn equation.

This problem is also related to metastability of the two-dimensional sine-Gordon model. As the parameter γ in (2.24) is replaced by the divergent renormalization constant (1.18), the transition state seems to become non-trivial for any $\beta > 0$. It is an interesting question how the multiplicative renormalization, compared to the additive one of ϕ^4_2, changes the results and the computations.

Finally, article [**II**] gives only a TST result on the stochastic wave equation. The conjectured transmission coefficient (3.14) is still unproven. The stochastic term could possibly be incorporated into the argument of [**II**, Section 4]. Alternatively, it could be possible to redo the potential-theoretic approximation argument of [15, 17] with the recent underdamped result [86].

We have in this chapter assumed the potential to be symmetric in some way. It is of course possible to study a more complicated landscape of hills and valleys. The distribution of exit points from a potential minimum was studied already by Freĭdlin and Wentzell [54]. There are also some results on the convergence of the rescaled system towards a Markov jump process [59, 116]. See e.g. [60] for a recent combination of these questions.

These results appear to be presently limited to finite-dimensional systems. It would be interesting to show convergence of metastable SPDE dynamics to a jump process. It is not obvious even how the convergence should be defined.

Finally, metastability is in some sense dual to the question of convergence to equilibrium: Starting from arbitrary initial data, how fast does the stochastic flow approach the stationary measure?

This question is also related to the spectrum and the spectral gap of the generator; see for example [9, Chapter 4] and [125, Section 7.1]. Again, this work has mostly focused on finite-dimensional systems, so the extension to SPDEs would be an interesting research problem.

As a practical application, the finite-dimensional underdamped dynamics (3.6) has faster convergence to equilibrium than the overdamped dynamics (3.1); see e.g. [50]. This property underlies the Hamiltonian Monte Carlo methods used for efficient numerical sampling from high-dimensional probability distributions [47]. The same is expected for the SPDEs, which would make canonical stochastic quantization the preferred way of numerically sampling from QFT measures.

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