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A Note on the Jacobian Problem of Coifman, Lions, Meyer and Semmes

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Abstract

Coifman, Lions, Meyer and Semmes asked in 1993 whether the Jacobian operator and other compensated compactness quantities map their natural domain of definition onto the real-variable Hardy space $\mathcal{H}^1(\mathbb{R}^n)$. We present an axiomatic, Banach space geometric approach to the problem in the case of quadratic operators. We also make progress on the main open case, the Jacobian equation in the plane.

Keywords Jacobian equation · Compensated compactness · Commutators

Mathematics Subject Classification Primary 46N20 · Secondary 47N20 · 47B47 · 42B37

1 Introduction

The real-variable Hardy space $\mathcal{H}^1(\mathbb{R}^n)$ often acts as a good substitute for $L^1(\mathbb{R}^n)$, mirroring many of the ways in which $\text{BMO}(\mathbb{R}^n)$ substitutes $L^\infty(\mathbb{R}^n)$ [18, 53]. In order to define $\mathcal{H}^1(\mathbb{R}^n)$ we fix $\chi \in C_c^\infty(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \chi(x) dx \neq 0$, denote $\chi_t(x) := t^{-n} \chi(x/t)$ for every $x \in \mathbb{R}^n$ and $t > 0$ and set

$$\mathcal{H}^1(\mathbb{R}^n) := \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \sup_{t>0} |f * \chi_t(\cdot)| \in L^1(\mathbb{R}^n) \right\}.$$

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We endow $\mathcal{H}^1(\mathbb{R}^n)$ with the norm $\|f\|_{\mathcal{H}^1} := \left\| \sup_{t>0} |f * \chi_t(\cdot)| \right\|_{L^1}$.

Connections between \mathcal{H}^1 integrability, commutators and weak sequential continuity were explored by Coifman & al. in the highly influential work [8]. Coifman & al. showed that when $n \geq 2$, the Jacobian determinants of mappings in $\dot{W}^{1,n}(\mathbb{R}^n, \mathbb{R}^n)$ and several other compensated compactness quantities belong to $\mathcal{H}^1(\mathbb{R}^n)$. The result was motivated by Müller's higher integrability result on Jacobians: if $u \in W^{1,n}(\Omega, \mathbb{R}^n)$ and $J_u \geq 0$ a.e. in an open set $\Omega \subset \mathbb{R}^n$, then $J_u \log(2 + J_u) \in L^1_{loc}(\Omega)$ [49], in direct analogy to Stein's classical $L \log L$ result on \mathcal{H}^1 functions. For later developments of the \mathcal{H}^1 theory of compensated compactness quantities see e.g. [2, 3, 6, 17, 19, 22, 23, 26, 33, 34, 40–42, 45, 47, 51, 54, 57].

Coifman & al. proceeded to ask whether these nonlinear quantities are surjections onto $\mathcal{H}^1(\mathbb{R}^n)$ [8, p. 258]. The most famous open case is the following:

$$\text{Is } J : \dot{W}^{1,n}(\mathbb{R}^n, \mathbb{R}^n) \rightarrow \mathcal{H}^1(\mathbb{R}^n) \text{ surjective?} \quad (1.1)$$

As a partial result, Coifman & al. showed that $\mathcal{H}^1(\mathbb{R}^n)$ is the smallest Banach space that contains the range of the Jacobian. More precisely,

$$\mathcal{H}^1(\mathbb{R}^n) = \left\{ \sum_{j=1}^{\infty} J u_j : \sum_{j=1}^{\infty} \|D u_j\|_{L^n}^n < \infty \right\} \quad (1.2)$$

[8, Theorem III.2]. Further partial results were presented in [21, 23, 44]. When the domain of definition of J is the *inhomogeneous* Sobolev space $W^{1,n}(\mathbb{R}^n, \mathbb{R}^n)$, the author proved non-surjectivity in [45].

In bounded domains, the Dirichlet problem $Ju = f$ in Ω , $u = \text{id}$ on $\partial\Omega$ has a classical theory starting from the seminal works of Moser and Dacorogna in [11, 48] and reviewed in [10]. In a setting close to ours, when $u \in W^{1,n}_{\text{id}}(\Omega, \mathbb{R}^n)$ has Jacobian $Ju \geq 0$ a.e., Müller's higher integrability result implies that $Ju \in L \log L(\Omega)$. As an analogue of (1.1), Hogan & al. asked whether every non-negative $f \in L \log L(\Omega)$ with mean 1 has a solution $u \in W^{1,n}_{\text{id}}(\Omega, \mathbb{R}^n)$ [26, p. 206]. Counter-examples were recently given by Guerra & al. in [20]. For all $L^{q/n}$ data, $1 < q < n$, "pointwise" (as opposed to distributional) solutions in $W^{1,q}(\Omega, \mathbb{R}^n)$ with arbitrary boundary values in $W^{1-1/q,q}(\Omega, \mathbb{R}^n)$ were constructed in [38]. At any rate, compared to the Dirichlet problem, completely different ideas are needed in the case of \mathbb{R}^n due to the unboundedness of the domain and the absence of boundary conditions.

In [32], Iwaniec conjectured that for every $n \geq 2$ and $p \in [1, \infty)$ the Jacobian operator $J : \dot{W}^{1,np}(\mathbb{R}^n, \mathbb{R}^n) \rightarrow \mathcal{H}^p(\mathbb{R}^n)$ not only is surjective but also has a continuous right inverse $G : \mathcal{H}^p(\mathbb{R}^n) \rightarrow \dot{W}^{1,np}(\mathbb{R}^n, \mathbb{R}^n)$; recall that $\mathcal{H}^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ whenever $1 < p < \infty$. Hytönen proved in [29] the natural analogue of (1.2), so that $L^p(\mathbb{R}^n)$ is, again, the minimal Banach space containing $J(\dot{W}^{1,np}(\mathbb{R}^n, \mathbb{R}^n))$. We discuss Hytönen's contribution in §7.

We next briefly summarise some of the ideas presented in [32]. Whenever $f \in \mathcal{H}^p(\mathbb{R}^n)$ and the equation $Jv = f$ has a solution, it has a *minimum norm solution* $u \in \dot{W}^{1,np}(\mathbb{R}^n, \mathbb{R}^n)$, that is, $Ju = f$ and $\int_{\mathbb{R}^n} |Du|^{np} = \min_{Jv=f} \int_{\mathbb{R}^n} |Dv|^{np}$. Furthermore,

the range of J is dense in $\mathcal{H}^p(\mathbb{R}^n)$. Iwaniec suggested a possible way of finding a continuous right inverse G :

Strategy 1.1 When $n \geq 2$ and $1 \leq p < \infty$, the following claims would yield a continuous right inverse of $J: \dot{W}^{1,np}(\mathbb{R}^n, \mathbb{R}^n) \rightarrow \mathcal{H}^p(\mathbb{R}^n)$:

- (1) Every energy-minimal solution u satisfies $\|Du\|_{L^{np}}^n \lesssim \|Ju\|_{\mathcal{H}^p}$.
- (2) For every $f \in \mathcal{H}^p(\mathbb{R}^n)$ there is a unique energy-minimal solution u_f , modulo rotations.
- (3) There exist rotations $R_f \in SO(n)$ such that $f \mapsto R_f u_f$ is a continuous right inverse of J .

Claim (1) would easily imply the surjectivity of $J: \dot{W}^{1,np}(\mathbb{R}^n, \mathbb{R}^n) \rightarrow \mathcal{H}^p(\mathbb{R}^n)$ (see [44, p. 36]). In [23], (1) was shown to be, in fact, *equivalent* to the surjectivity of $J: \dot{W}^{1,np}(\mathbb{R}^n, \mathbb{R}^n) \rightarrow \mathcal{H}^p(\mathbb{R}^n)$; this amounts to an open mapping theorem for the Jacobian. In [21], in turn, (2) was shown to be false whenever $n = 2$ and $1 \leq p < \infty$. Nevertheless, in [21], Guerra & al. found an explicit class of data whose energy-minimal solution is, indeed, unique up to rotations. Another large class of such data is constructed in Theorem 1.10 below. Despite the falsity of (2), claim (1) and Iwaniec's conjecture itself remain open. In the case $n = 2$, $p = 1$, a novel Banach space geometric approach was presented in [44] to attack claim (1).

In this work, we present a natural abstract framework for the ideas of [44] and study the surjectivity question for rather general quadratic compensated compactness quantities, streamlining the exposition of [44] considerably. We also make further progress on the main special case $J: \dot{W}^{1,2}(\mathbb{R}^2, \mathbb{R}^2) \rightarrow \mathcal{H}^1(\mathbb{R}^2)$. In §7, we discuss how to adapt the methods to the case $n = 2$, $1 < p < \infty$.

1.1 Connection to Commutators

Question (1.1) can be seen as a variant of the classical *factorisation problem* from complex analysis [52, §4.2]. Indeed, an equivalent formulation of (1.1) in terms of differential forms is whether $\mathcal{H}^1(\mathbb{R}^n) = * \wedge_{j=1}^n d\dot{W}^{1,n}(\mathbb{R}^n)$ (since $Ju = *du_1 \wedge \cdots \wedge du_n$). The decomposition (1.2) is called a *weak factorisation* of $\mathcal{H}^1(\mathbb{R}^n)$.

In the plane, one can also deduce (1.2) from the two-sided commutator estimate

$$c \|b\|_{\text{BMO}} \leq \|[b, T]\|_{L^2 \rightarrow L^2} \leq C \|b\|_{\text{BMO}} \quad (1.3)$$

for the *Beurling transform* $T = \mathcal{S}: L^2(\mathbb{C}, \mathbb{C}) \rightarrow L^2(\mathbb{C}, \mathbb{C})$ (a Calderón-Zygmund operator with kernel $K(z) := 1/z^2$; see [1, §4] for its various properties) and $b \in \text{BMO}(\mathbb{C})$. Here, recall (for $n \geq 1$) that

$$\text{BMO}(\mathbb{R}^n) := \left\{ b \in L^1_{loc}(\mathbb{R}^n): \|b\|_{\text{BMO}} := \sup_Q \int_Q |b(x) - b_Q| dx < \infty \right\},$$

where the supremum is taken over cubes $Q \subset \mathbb{R}^n$ and $b_Q := \int_Q b(y) dy$. The closure of $C_c^\infty(\mathbb{R}^n)$ in $\text{BMO}(\mathbb{R}^n)$ is called $\text{CMO}(\mathbb{R}^n)$. By classical results by Fefferman and

by Coifman and Weiss, respectively, we have the dualities $[\mathcal{H}^1(\mathbb{R}^n)]^* = \text{BMO}(\mathbb{R}^n)$ and $[\text{CMO}(\mathbb{R}^n)]^* = \mathcal{H}^1(\mathbb{R}^n)$.

Estimates of the form (1.3) go back to the seminal work [7] of Coifman & al., with Nehari’s theorem on Hankel operators as a precursor. When $j \in \{1, \dots, n\}$ and $T = R_j$ is a Riesz transform, the *upper bound estimate* $\| [b, R_j] \|_{L^2 \rightarrow L^2} \leq C \|b\|_{\text{BMO}}$ [7], the formula

$$\int_{\mathbb{R}^n} b(\omega R_j \gamma + \gamma R_j \omega) = \int_{\mathbb{R}^n} \omega [b, R_j] \gamma \tag{1.4}$$

and $\mathcal{H}^1 - \text{BMO}$ duality imply that $\omega R_j \gamma + \gamma R_j \omega \in \mathcal{H}^1(\mathbb{R}^n)$ for all $\omega, \gamma \in L^2(\mathbb{R}^n)$. The *lower bound estimate* $\| [b, R_j] \|_{L^2 \rightarrow L^2} \geq c \|b\|_{\text{BMO}}$ [7, 35, 55] gives the weak factorisation $\mathcal{H}^1(\mathbb{R}^n) = \{ \sum_{i=1}^\infty (\omega_i R_j \gamma_i + \gamma_i R_j \omega_i) : \sum_{i=1}^\infty (\|\omega_i\|_{L^2}^2 + \|\gamma_i\|_{L^2}^2) < \infty \}$ by simple functional analysis. By a result of Uchiyama, the commutator $[b, R_j]$ is compact if and only if $b \in \text{CMO}(\mathbb{R}^n)$. Note that (1.4) and the compactness of $[b, R_j]$ immediately lead to the weak-to-weak* sequential continuity of the quadratic operator $(\omega, \gamma) \mapsto \omega R_j \gamma + \gamma R_j \omega : L^2(\mathbb{R}^n)^2 \rightarrow \mathcal{H}^1(\mathbb{R}^n)$. These results have been extended in numerous ways, as reviewed in [56]. In §3.1 we briefly discuss the case of commutators with Calderón-Zygmund operators.

In Assumptions 1.2 and 1.3 below, we axiomatise the above-mentioned properties of the Riesz transform R_j and the quantity $\omega R_j \gamma + \gamma R_j \omega$. We study the factorisation problem for quadratic weakly continuous quantities in real-variable function spaces such as $\mathcal{H}^1(\mathbb{R}^n)$, using two-sided commutator estimates and isometric Banach space geometry as some of the main tools. Since our choice of methodology is rather unconventional in the study of solvability of nonlinear PDE’s, we motivate it at length in §1.3. First, however, we specify the mathematical setting of this paper.

1.2 The Mathematical Setting

We fix a real Banach space X with a separable dual X^* and a real or complex Hilbert space H , and we denote the coefficient field of H by $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. The canonical examples are $X = \text{CMO}(\mathbb{R}^n)$, $X^* = \mathcal{H}^1(\mathbb{R}^n)$, $X^{**} = \text{BMO}(\mathbb{R}^n)$ and $H = L^2(\mathbb{R}^n, \mathbb{R}^m)$ or $H = L^2(\mathbb{R}^n, \mathbb{C})$, where $n, m \in \mathbb{N}$.

Assumption 1.2 A bilinear mapping

$$(b, \omega) \mapsto T_b \omega : X^{**} \times H \rightarrow H$$

satisfies the following conditions:

- (i) $c \|b\|_{X^{**}} \leq \|T_b\|_{H \rightarrow H} \leq C \|b\|_{X^{**}}$ for every $b \in X^{**}$,
- (ii) T_b is compact for every $b \in X$.

Assumption 1.3 The bilinear mapping $(b, \omega) \mapsto T_b \omega$ satisfies the following conditions for every $b \in X^{**}$:

- (i) T_b is self-adjoint.

$$(ii) \|T_b\|_{H \rightarrow H} = \sup_{\|\omega\|_H=1} \langle T_b \omega, \omega \rangle.$$

Definition 1.4 Given X, H and $(b, \omega) \mapsto T_b \omega$ we define a norm-to-norm and weak-to-weak* sequentially continuous mapping $Q: H \rightarrow X^*$ by

$$\langle b, Q\omega \rangle_{X-X^*} := \langle T_b \omega, \omega \rangle_H.$$

Henceforth, Assumptions 1.2 and 1.3 will remain in place for the rest of the introduction. Assumption 1.3 is made mainly to make the quadratic operator Q real-valued and ensure that $X^* = \{\sum_{j=1}^\infty Q\omega_j : \sum_{j=1}^\infty \|\omega_j\|_H^2 < \infty\}$. Whenever the map $(b, \omega) \mapsto T_b \omega$ satisfies Assumption 1.2, the modified operator $(b, (\omega, \gamma)) \mapsto \tilde{T}_b(\omega, \gamma) := (T_b^* \gamma, T_b \omega): X^{**} \times (H \times H) \rightarrow H \times H$ satisfies Assumptions 1.2 and 1.3 (see Proposition 2.10). Examples 3.5 and 3.6 illustrate the role of Assumption 1.3 further.

The planar Jacobian arises as follows (see Example 3.5): defining $T_b: L^2(\mathbb{C}, \mathbb{C}) \rightarrow L^2(\mathbb{C}, \mathbb{C})$ by $T_b \omega := (Sb - bS)\overline{S\omega}$, where S is the Beurling transform, we can write $Q\omega = |S\omega|^2 - |\omega|^2 = |u_z|^2 - |u_{\bar{z}}|^2 = Ju$, where $u \in \dot{W}^{1,2}(\mathbb{C}, \mathbb{C})$ is the Cauchy transform of $\omega \in L^2(\mathbb{C}, \mathbb{C})$.

Assumptions 1.2 and 1.3 are not enough to determine whether $Q(H) = X^*$. Indeed, $Q: L^2(\mathbb{R}) \rightarrow \mathcal{H}^1(\mathbb{R})$, $Q(\omega) = \omega^2 - (H\omega)^2$ is non-surjective but (its Gâteaux derivative) $\tilde{Q}: L^2(\mathbb{R}) \times L^2(\mathbb{R}) \rightarrow \mathcal{H}^1(\mathbb{R})$, $\tilde{Q}(\omega, \gamma) = \omega\gamma - H\omega H\gamma$ is surjective (see Example 3.6). We address the following question:

Question 1.5 *Under which extra assumptions is $Q(H) = X^*$?*

Below we mostly study all the operators given by Definition 1.4 in a unified manner, as precise information on $Q(H)$ is valuable whether one intends to prove surjectivity or non-surjectivity. However, in Assumption 1.8 we specify a useful criterion which holds for the operators $\omega \mapsto \omega^2 - (H\omega)^2: L^2(\mathbb{R}) \rightarrow \mathcal{H}^1(\mathbb{R})$ and $\omega \mapsto |S\omega|^2 - |\omega|^2: L^2(\mathbb{C}, \mathbb{C}) \rightarrow \mathcal{H}^1(\mathbb{C})$ but fails for their Gâteaux derivatives.

1.3 Overall Strategy and Aim

We next describe the motivation behind the approach adopted in [44] and this paper. Elementary proofs of various statements are given in §4.

- Whenever $f \in X^*$ and the equation $Q\gamma = f$ has a solution, it has a *minimum norm solution* $\omega \in H$, that is, $Q\omega = f$ and $\|\omega\|_H = \min_{Q\gamma=f} \|\gamma\|_H$.
- We define an *energy functional* $\mathcal{E}: X^* \rightarrow \mathbb{R} \cup \{\infty\}$ by

$$\mathcal{E}(f) := \begin{cases} \min\{\|\omega\|_H^2 : Q\omega = f\}, & f \in Q(H), \\ \infty, & f \notin Q(H). \end{cases}$$

Almost by definition, $\mathcal{E} \geq \|\cdot\|_{X^*}$. Claim (1) in Strategy 1.1 can be stated equivalently as the estimate $\mathcal{E} \lesssim \|\cdot\|_{X^*}$.

- By a nonlinear version of the Banach-Schauder open mapping theorem, recently proved by Guerra & al. in [23], for translation-invariant operators (such as the Jacobian) we have

either $\mathcal{E} \lesssim \|\cdot\|_{X^*}$ in X^* or $\mathcal{E} = \infty$ outside a meagre set.

In particular, the surjectivity of $Q: H \rightarrow X^*$ is *equivalent* to the statement that $Q(H)$ is dense in X^* and every minimum norm solution satisfies $\|\omega\|_H^2 \lesssim \|Q\omega\|_{X^*}$. This gives a metamatematical justification for taking claim (1) as a goal.

- Given a minimum norm solution $\omega \in H$ of $Q\omega = f$, we may use calculus of variations to study ω via perturbed solutions $\omega_\epsilon \in H$ of $Q\omega_\epsilon = f, \|\omega_\epsilon - \omega\|_H \rightarrow 0$. However, it tends to be very hard to construct variations that satisfy the nonlinear constraint $Q\omega_\epsilon = f$. In particular, the first variation $\omega_\epsilon = \omega + \epsilon\varphi, \varphi \in H$, is in general unavailable.
- The constraint $Q\omega_\epsilon = f$ could be relaxed if we found a *Lagrange multiplier*, that is, $b \in X^{**}$ satisfying

$$\frac{d}{d\epsilon} (\langle b, Q(\omega + \epsilon\varphi) \rangle_{X^{**}-X^*} - \|\omega + \epsilon\varphi\|_H^2) \Big|_{\epsilon=0} = 0 \quad \text{for every } \varphi \in H. \tag{1.5}$$

If, furthermore, $\|b\|_{X^{**}} \leq C$ uniformly in f , then setting $\varphi = \omega$ in (1.5) would give the sought inequality $\mathcal{E} \lesssim \|\cdot\|_{X^*}$. However, given an arbitrary minimum norm solution, the construction of a Lagrange multiplier $b \in X^{**}$ is a formidable task—in particular, the standard Liusternik-Shnirelman method is not applicable.

- Nevertheless, many minimum norm solutions do possess a Lagrange multiplier. Indeed, (1.5) says that ω is a critical point of the functional $I_b: H \rightarrow \mathbb{R}$,

$$I_b(\theta) := \langle b, Q\theta \rangle_{X^{**}-X^*} - \|\theta\|_H^2 = \langle T_b\theta, \theta \rangle_H - \|\theta\|_H^2.$$

Now, if $b \in X$ and $\|T_b\|_{H \rightarrow H} = 1$, then $\sup_{\|\theta\|_H=1} I_b(\theta) = 0$ is attained at some $\omega \in \mathbb{S}_H$, so that b is a Lagrange multiplier of ω and $\|\omega\|_H^2 \lesssim \|Q\omega\|_{X^*}$.

- As every $b \in X$ with $\|T_b\|_{H \rightarrow H} = 1$ is a Lagrange multiplier, we use the norm of X given by

$$\|b\|_{X_Q} := \sup_{\|\omega\|_H=1} \langle b, Q\omega \rangle_{X-X^*} = \|T_b\|_{H \rightarrow H},$$

endow X^* with the dual norm

$$\|f\|_{X_Q^*} := \sup_{\|b\|_{X_Q}=1} \langle f, b \rangle_{X^*-X}$$

and denote the resulting Banach spaces by X_Q and X_Q^* . Thus every $b \in \mathbb{S}_{X_Q}$ is a Lagrange multiplier of some $\omega \in \mathbb{S}_H$ with $Q\omega \in \mathbb{S}_{X^*}$.

- As a consequence, the set

$$\mathcal{A} := \{\omega \in \mathbb{S}_H : Q\omega \in \mathbb{S}_{X_Q^*}\} \subset \mathbb{S}_H$$

and its image $Q(\mathcal{A}) \subset \mathbb{S}_{X_Q^*}$ are rather large. The motivation above leads to the following more precise variant of Question 1.5, presented for the planar Jacobian in [44]:

Question 1.6 *Is $Q(\mathcal{A}) = \mathbb{S}_{X_Q^*}$? Equivalently, is $\mathcal{E} = \|\cdot\|_{X^*}$?*

As already noted, there exist some natural cases where the answer to Question 1.6 is negative. A positive answer holds for the simple operator $(b, (\omega, \gamma)) \mapsto (b\omega, -b\gamma) : \mathbb{R} \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$. It is hoped that suitable further properties of T , beyond Assumptions 1.2 and 1.3, will yield an answer to Question 1.6 for natural classes of operators. In this paper we present several partial results.

1.4 Main Results

As a first largeness criterion on $Q(\mathcal{A})$ we mention that

$$Q(\mathcal{A}) \supset \text{ext}(\mathbb{B}_{X_Q^*}), \tag{1.6}$$

where $\text{ext}(\mathbb{B}_{X_Q^*})$ is the set of extreme points of $\mathbb{B}_{X_Q^*}$. Since X_Q^* is a separable dual space, the Bessaga-Pelczynski Theorem (Theorem 2.7) implies that $\overline{\text{co}}(\text{ext}(\mathbb{B}_{X_Q^*})) = \mathbb{B}_{X_Q^*}$. The inclusion (1.6) is contained in Theorem 5.1. We will find more refined information on $Q(\mathcal{A})$ by using the set-valued duality mapping.

Definition 1.7 The duality mapping $D : \mathbb{S}_{X_Q} \rightarrow 2^{\mathbb{S}_{X_Q^*}}$ is defined by

$$D(b) := \{f \in \mathbb{S}_{X_Q^*} : \langle b, f \rangle_{X_Q - X_Q^*} = 1\}.$$

We denote the set of norm-attaining points by $NA_{\|\cdot\|_{X_Q}} := \cup_{b \in \mathbb{S}_{X_Q}} D(b)$.

By the Bishop-Phelps Theorem [15, Theorem 3.54], $NA_{\|\cdot\|_{X_Q}}$ is dense in \mathbb{S}_{X_Q} . Since $Q(\mathcal{A})$ is closed, Question 1.6 reduces to the question whether $D(b) \cap Q\mathcal{A} = D(b)$ for every $b \in \mathbb{S}_{X_Q}$. Before presenting partial results we formulate a useful extra assumption which is satisfied by the planar Jacobian. Its aim is to quantify the symmetries of the class \mathcal{A} (see Remark 5.6).

Assumption 1.8 If $\omega, \gamma \in \mathcal{A}$ satisfy $Q\omega = Q\gamma$, then $Q'_\omega(c\gamma) \neq 0$ for some $c \in \mathbb{S}_{\mathbb{K}}$.

The following result collects partial results on Question 1.6; for the relevant definitions see §2.1.

Theorem 1.9 *Under Assumptions 1.2 and 1.3, the following statements hold:*

- (i) *For every $b \in \mathbb{S}_{X_Q}$, the convex set $D(b)$ has finite affine dimension.*

- (ii) For every $b \in \mathbb{S}_{X_Q}$, $D(b) \cap Q(\mathcal{A})$ contains $\text{ext}(D(b))$ and is path-connected.
 (iii) The norm and relative weak* topologies coincide in $Q(\mathcal{A})$.

Under the further Assumption 1.8, the following statements hold:

- (iv) For every b in a dense, relatively open subset of \mathbb{S}_{X_Q} , $D(b) = \{Q\omega\}$ for some $\omega \in \mathcal{A}$ and $\|\cdot\|_{X_Q}$ is Fréchet differentiable at b .
 (v) $D: \mathbb{S}_{X_Q} \rightarrow 2^{\mathbb{S}_{X_Q^*}}$ is a cusco map, and $b \mapsto D(b) \cap Q(\mathcal{A}): \mathbb{S}_{X_Q} \rightarrow 2^{\mathbb{S}_{X_Q^*}}$ is an usco map.
 (vi) The norm and relative weak* topologies also coincide in $NA_{\|\cdot\|_{X_Q}}$.

The proof is presented in §6. The parts (iv)–(vi) are new also in the case of the Jacobian. The second main result concerns uniqueness of minimum norm solutions and is proved in §5.

Theorem 1.10 Suppose Assumptions 1.2, 1.3 and 1.8 hold, and let $f \in \text{ext}(\mathbb{S}_{X_Q^*})$. Then the minimum norm solution $\omega \in \mathcal{A}$ of $Q\omega = f$ is unique up to multiplication by $c \in \mathbb{S}_{\mathbb{K}}$.

In view of (1.6) it is natural to ask whether X_Q^* is strictly convex, that is, $\text{ext}(\mathbb{B}_{X_Q^*}) = \mathbb{S}_{X_Q^*}$. In the case of the planar Jacobian, the answer is negative, as an immediate consequence of Theorems 1.9 and 1.10 and the non-uniqueness of general minimum norm solutions:

Corollary 1.11 In the case of $\omega \mapsto Q\omega = |\mathcal{S}\omega|^2 - |\omega|^2: L^2(\mathbb{C}, \mathbb{C}) \rightarrow \mathcal{H}^1(\mathbb{C})$ we have $\text{ext}(\mathbb{B}_{X_Q^*}) \subsetneq \mathbb{S}_{X_Q^*}$.

It is unclear to the author whether an analogue of Corollary 1.11 holds for the Gâteaux derivative $(\omega, \gamma) \mapsto Q'_\omega \gamma: L^2(\mathbb{C}, \mathbb{C})^2 \rightarrow \mathcal{H}^1(\mathbb{C})$.

2 Preliminaries

The main tools of this work come from isometric Banach space geometry. We collect some notions and results and refer to [14] and [15] for most of the proofs.

2.1 Smoothness Properties of Norms and Duality Mappings

In this subsection, X is a real Banach space. Suppose $U \subset Z$ is open and $g: U \rightarrow Z$ is convex, where Z is a real Banach space. We say that g is Gâteaux differentiable at $x \in U$ if there exists $L \in \mathcal{B}(X, Z)$ such that

$$\lim_{t \rightarrow 0} \left\| \frac{g(x + th) - g(x)}{t} - Lh \right\| = 0 \quad \text{for every } h \in X. \quad (2.1)$$

The operator L is then denoted by g'_x and called the Gâteaux derivative of g at x . If the limit in (2.1) is uniform in $h \in \mathbb{S}_X$, then g is said to be Fréchet differentiable at x .

Note that the convex function $g(x) = \|x\|_X$ is Gâteaux (Fréchet) differentiable at $x \in \mathbb{S}_X$ if and only if it is Gâteaux (Fréchet) differentiable at λx for every $\lambda \in \mathbb{R} \setminus \{0\}$. We will use the following theorem of Asplund and Lindenstrauss (see [15, Theorem 8.21]) on the norm of X .

Theorem 2.1 *Suppose X^* is separable and $g: X \rightarrow \mathbb{R}$ is a continuous convex function. Then g is Fréchet differentiable in a dense G_δ subset of X .*

Recall that the duality mapping $D: \mathbb{S}_X \rightarrow 2^{\mathbb{S}_{X^*}}$ is defined by

$$D(x) := \{x^* \in \mathbb{S}_{X^*} : \langle x^*, x \rangle_{X^*-X} = 1\}.$$

If $g(x) = \|x\|_X$ is Gâteaux differentiable at $x \in \mathbb{S}_X$, then $D(x) = \{g'_x\}$.

Definition 2.2 Suppose Y and Z are Hausdorff topological spaces. A mapping $F: Y \rightarrow 2^Z \setminus \{\emptyset\}$ is said to be *upper semicontinuous* at $y \in Y$ if for any open set $V \supset F(y)$, there exists a neighbourhood U of y in Y such that $F(U) \subset V$.

We recall characterisations of the norm-to-norm upper semicontinuity of the duality mapping [16, 28].

Theorem 2.3 *Let $x \in \mathbb{S}_X$. The following statements are equivalent.*

- (i) *D is norm-to-norm upper semicontinuous at x and $D(x)$ is norm compact.*
- (ii) *For every sequence of points x_j^* in \mathbb{S}_{X^*} such that $\langle x_j^*, x \rangle \rightarrow 1$, there exists a subsequence convergent to some $x^* \in D(x)$.*
- (iii) *The weak* and norm topologies agree on \mathbb{S}_{X^*} at points of $D(x)$.*

The norm-to-norm upper semicontinuity of D can also be used to characterise Fréchet differentiability of the norm (see [14, Corollary 7.16]).

Theorem 2.4 *The norm $\|\cdot\|_X$ is Fréchet differentiable at $x \in \mathbb{S}_X$ if and only if $D(x)$ is a singleton and D is norm-to-norm upper semicontinuous at x .*

Definition 2.5 Let Y and Z be Hausdorff topological spaces. A set-valued map $F: Y \rightarrow 2^Z \setminus \{\emptyset\}$ is said to be *usco* (upper semicontinuous nonempty compact-valued) if it is norm-to-norm upper semicontinuous and $F(y)$ is compact for every $y \in Y$. If, furthermore, $F(y)$ is convex for every $y \in Y$, then F is said to be *cusco*.

Usco and cusco maps and related notions are treated extensively in [27].

2.2 Extreme Points of the Unit Ball of a Separable Dual

When C is a convex subset of a real Banach space Y , a point $y \in C$ is an *extreme point* of C if there exists no proper line segment that contains y and lies in C .

Definition 2.6 A Banach space is said to have the *Krein-Milman property* if every closed, bounded, convex set is the closed convex hull of its extreme points.

We recall a result of C. Bessaga and A. Pełczyński (see e.g. [12, p. 198]).

Theorem 2.7 *Every separable dual space has the Krein-Milman property.*

Recall that in Assumptions 1.2 and 1.3, the dual space X^* is separable. The real-variable Hardy space $\mathcal{H}^1(\mathbb{R}^n) = (\text{CMO}(\mathbb{R}^n))^*$ satisfies this criterion (see [44, Proposition 2.15]). We also recall Milman's theorem on extreme points which is formulated as follows (see [14, Theorem 3.66]).

Theorem 2.8 *Let K be a non-empty subset of a Hausdorff locally convex space such that $\overline{\text{co}}(K)$ is compact. Then every extreme point of $\overline{\text{co}}(K)$ lies in \overline{K} .*

2.3 Self-adjoint Variants of Non-self-adjoint Operators

When Assumption 1.2 holds but Assumption 1.3 does not, we use a natural self-adjoint modification of T_b . Its existence and uniqueness are guaranteed by the following simple lemma.

Lemma 2.9 *Suppose $A: H \rightarrow H$ is \mathbb{K} -linear. Then there exists a unique self-adjoint \mathbb{K} -linear operator $B: H \times H \rightarrow H \times H$ such that*

$$\langle B(x, y), (x, y) \rangle_{H \times H} = 2\text{Re}\langle Ax, y \rangle_H \quad \text{for all } x, y \in H.$$

*The operator B is of the form $B(x, y) = (A^*y, Ax)$ and satisfies $\|B\|_{H \times H \rightarrow H \times H} = \|A\|_{H \rightarrow H}$.*

Proposition 2.10 *If a bilinear mapping $T: X^{**} \times H \rightarrow H$ satisfies Assumption 1.2, then the modified operator*

$$(b, \omega, \gamma) \mapsto \tilde{T}_b(\omega, \gamma): X^{**} \times (H \times H) \rightarrow H \times H, \quad \tilde{T}_b(\omega, \gamma) := (T_b^* \gamma, T_b \omega)$$

satisfies Assumptions 1.2 and 1.3.

If T satisfies Assumptions 1.2 and 1.3, then \tilde{T} is related to the Gâteaux derivative of Q by

$$\langle \tilde{T}_b(\omega, \gamma), (\omega, \gamma) \rangle_{H \times H} = \langle b, Q'_\omega \gamma \rangle_{X^{**} - X^*} \quad \text{for all } \omega, \gamma \in H, b \in X^{**}. \quad (2.2)$$

We illustrate Proposition 2.10 in Examples 3.5 and 3.6.

3 Operators Satisfying Assumptions 1.2 and 1.3

In this section we discuss two classes of operators which satisfy assumption 1.2: commutators of Calderón-Zygmund operators with BMO functions and paracommutators. Specific examples are then given in §3.3.

3.1 Commutators of Calderón–Zygmund Operators and BMO Functions

Coifman & al. showed in [7] that when T is a Calderón-Zygmund operator with a suitable smooth kernel Ω , we have $\|[b, T]\|_{L^2 \rightarrow L^2} \lesssim \|b\|_{\text{BMO}}$ for all $b \in \text{BMO}(\mathbb{R}^n)$.

Specifically, Ω was assumed to be homogeneous of degree zero with vanishing mean over S^{n-1} and to satisfy $|\Omega(x) - \Omega(y)| < |x - y|$ for all $x, y \in S^{n-1}$. They also showed that if $[b, R_j]: L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ is bounded for every $j \in \{1, \dots, n\}$, then $b \in \text{BMO}(\mathbb{R}^n)$ and $\|b\|_{\text{BMO}} \lesssim \max_{1 \leq j \leq n} \|[b, R_j]\|_{L^2 \rightarrow L^2}$. Uchiyama [55] and Janson [35] showed independently that in order to obtain $b \in \text{BMO}(\mathbb{R}^n)$, it suffices to show boundedness of $[b, T]$ in $L^2(\mathbb{R}^n)$ for only one of the kernels $\Omega \not\equiv 0$ in the result of [7]. Uchiyama also showed that $[b, T]$ is compact if and only if $b \in \text{CMO}(\mathbb{R}^n)$. Thus $T_b \omega := [b, T]\omega$ satisfies Assumption 1.2.

The two-sided estimate (1.3) has been extended in numerous ways (to multi-parameter and weighted spaces etc.); see e.g. [25, 29, 56] and the references contained therein. In the case of commutators with Calderón-Zygmund operators, Hytönen recently proved the estimate $\|[b, T]\|_{L^2 \rightarrow L^2} \simeq \|b\|_{\text{BMO}}$ under the very weak assumption that the kernel is "non-degenerate". We recall relevant definitions from [29].

Definition 3.1 A measurable function $K: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ is called an ω -Calderón-Zygmund kernel if $K(x, y) \leq c_K |x - y|^{-n}$ whenever $x \neq y$ and

$$|K(x, y) - K(x', y)| + |K(y, x) - K(y, x')| \leq \frac{1}{|x - y|^n} \omega \left(\frac{|x - x'|}{|x - y|} \right)$$

whenever $|x - x'| < |x - y|/2$, where $\omega: [0, 1) \rightarrow [0, \infty)$ is increasing.

Definition 3.2 A measurable function $K: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ is called a *non-degenerate Calderón-Zygmund kernel* if K satisfies at least one of the following two conditions:

- (1) K is an ω -Calderón-Zygmund kernel with $\omega(t) \rightarrow 0$ as $t \rightarrow 0$ and for every $y \in \mathbb{R}^n$ and $r > 0$, there exists $x \in \mathbb{C}B(y, r)$ such that $|K(x, y)| \leq cr^{-d}$.
- (2) K is an homogeneous Calderón-Zygmund kernel with $\Omega \in L^1(S^{n-1}) \setminus \{0\}$.

For compactness of commutators of CMO functions and Calderón-Zygmund operators on (possibly weighted) $L^2(\mathbb{R}^n)$ we refer to [4, 5, 24, 30, 39, 55]. In particular, when an homogeneous kernel $\Omega \in L^1(S^{n-1}) \setminus \{0\}$ has vanishing mean and satisfies

$$\sup_{\zeta \in S^{n-1}} \int_{S^{n-1}} |\Omega(\eta)| \left(\ln \frac{1}{|\eta \cdot \zeta|} \right)^\theta d\eta < \infty$$

for some $\theta > 2$, the corresponding commutator satisfies Assumption 1.2 [4, 29]. Assumption 1.2 is verified for commutators with the Cauchy integral operator in [43].

In many natural instances, an operator T_b satisfying Assumption 1.2 is not precisely a commutator but, for instance, the composition of a commutator with a Calderón-Zygmund operator. A natural general framework for such operators is provided by *paracommutators*.

3.2 Paracommutators

We briefly recall basic definitions and results from the theory of paracommutators and refer to [36]. Paracommutators are operators $T_b(A)$ of the form

$$\widehat{T_b(A)\omega}(\xi) := (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{b}(\xi - \eta) A(\xi, \eta) \hat{\omega}(\eta) d\eta, \quad \xi \in \mathbb{R}^n, \quad (3.1)$$

where $A: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ and $\omega: \mathbb{R}^n \rightarrow \mathbb{R}$ are functions and $b: \mathbb{R}^n \rightarrow \mathbb{R}$ is called the *symbol of $T_b(A)$* .

As the most basic example, $A(\xi, \eta) \equiv 1$ yields (under suitable integrability assumptions) the multiplication operator $T_b\omega = b\omega$. The commutators $T_b(A) = [b, R_j]$ arise via $A(\xi, \eta) = \xi_j/|\xi| - \eta_j/|\eta|$. More generally, if T is a Calderón-Zygmund singular integral operator with Fourier symbol m and $A(\xi, \eta) = m(\xi) - m(\eta)$, then $T_b(A) = [b, T]$. Several further examples are presented in [36, pp. 469-473] and [51, pp. 513-519].

In [36], Janson and Peetre gave conditions on the function A under which the boundedness of $T_b(A): L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ is equivalent to the condition $b \in \text{BMO}(\mathbb{R}^n)$. In order to state the result we need some definitions. We do not motivate them here but refer to [36] for more information.

Definition 3.3 Let $U, V \subset \mathbb{R}^n$. The space $M(U \times V)$ consists of functions $\varphi \in L^\infty(U \times V)$ that admit a representation

$$\varphi(\xi, \eta) = \int_X \alpha(\xi, x) \beta(\eta, x) d\mu(x) \quad (3.2)$$

for some σ -finite measure space (X, μ) and measurable functions $\alpha: U \times X \rightarrow \mathbb{R}$ and $\beta: V \times X \rightarrow \mathbb{R}$ which satisfy

$$\int_X \|\alpha(\cdot, x)\|_{L^\infty(U)} \|\beta(\cdot, x)\|_{L^\infty(V)} d\mu(x) < \infty. \quad (3.3)$$

Furthermore, $M(U \times V)$ is a Banach algebra in the norm given by minimising the left hand side of (3.3) over all representations (3.2).

For every $j \in \mathbb{Z}$ we denote $\Delta_j := \{\xi \in \mathbb{R}^n : 2^j \leq |\xi| < 2^{j+1}\}$. We list some of the assumptions of [36] and retain their numbering.

(A0) There exists $r > 1$ such that $A(r\xi, r\eta) = A(\xi, \eta)$ for all $\xi, \eta \in \mathbb{R}^n$.

(A1) $\|A\|_{M(\Delta_j \times \Delta_k)} \leq C$ for all $j, k \in \mathbb{Z}$.

(A2) There exist $A_1, A_2 \in M(\mathbb{R}^n \times \mathbb{R}^n)$ and $\delta > 0$ such that

$$\begin{aligned} A(\xi, \eta) &= A_1(\xi, \eta) \quad \text{when } |\eta| < \delta |\xi|, \\ A(\xi, \eta) &= A_2(\xi, \eta) \quad \text{when } |\xi| < \delta |\eta|. \end{aligned}$$

(A3(α)): There exist α, δ > 0 such that whenever r < δ |ξ₀|, we have

$$\|A\|_{M(B(\xi_0,r),B(\xi_0,r))} \leq C \left(\frac{r}{|\xi_0|} \right)^\alpha.$$

(A5) For every ξ₀ ≠ 0 there exist δ > 0 and η₀ ∈ ℝⁿ such that 1/A(ξ, η) ∈ M(U × V), where U = {ξ : |ξ|/|ξ| - ξ₀/|ξ₀| < δ and |ξ| > |ξ₀|} and V = B(η₀, δ |ξ₀|).

We collect the results of [36] that are most relevant to this paper.

Theorem 3.4 Suppose A : ℝⁿ × ℝⁿ → ℝ satisfies (A0), (A1), (A2), (A3(α)) and A(5), where α > 0. Then T_b(A) satisfies Assumption 1.2.

By a result of Peng, under the assumptions (A0), (A1), (A3(α)) and (A5), compactness of T_b : L²(ℝⁿ) → L²(ℝⁿ) conversely implies b ∈ CMO(ℝⁿ) [50]. We also mention that under (A0)–(A5), Assumption 1.3 is not satisfied in general. Indeed, denoting $\tilde{b}(x) := b(-x)$ and abusing notation, $\langle T_b(A(\xi, \eta))\omega, \gamma \rangle = \langle \omega, T_{\tilde{b}}(A(\eta, \xi))\gamma \rangle$ for all ω, γ ∈ L²(ℝⁿ).

If the assumptions of Theorem 3.4 and Assumption 1.3 hold, we may use Definition 1.4 to define a quadratic operator Q : L²(ℝⁿ) → ℒ¹(ℝⁿ). Its form is

$$Q_{\tilde{A}}(\omega, \gamma)(x) = \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \hat{\omega}(\xi) \hat{\gamma}(\eta) \tilde{A}(\xi, \eta) e^{-i(\xi+\eta)\cdot x} d\xi d\eta, \tag{3.4}$$

where $\tilde{A}(\eta, -\xi) = A(\xi, \eta)$. Conversely, Q_{\tilde{A}}, defined via (3.4), gives rise to a para-commutator T_b(A) defined by (3.1). These and many other relations between T_b(A) and Q_{\tilde{A}} are explored in [51].

3.3 Specific Operators

We list some concrete quadratic operators which arise via Assumption 1.2; numerous further examples are presented in [36, 51].

Example 3.5 Let u ∈ $\dot{W}^{1,2}(\mathbb{C}, \mathbb{C})$. When $u_z := 2^{-1}(\partial_x - i\partial_y)(u_1 + iu_2)$ and $u_{\bar{z}} := 2^{-1}(\partial_x + i\partial_y)(u_1 + iu_2)$ are the Wirtinger derivatives of u and S : L²(ℂ, ℂ) → L²(ℂ, ℂ) is the Beurling transform, we have S u_{\bar{z}} = u_z. Therefore, the Jacobian Ju = |u_z|² - |u_{\bar{z}}|² corresponds to Qω := |Sω|² - |ω|² via the isometric isomorphism u ↦ u_{\bar{z}} =: ω : $\dot{W}^{1,2}(\mathbb{C}, \mathbb{C}) \rightarrow L^2(\mathbb{C}, \mathbb{C})$ whose inverse is the Cauchy transform C. (We identify elements of $\dot{W}^{1,2}(\mathbb{C}, \mathbb{C})$ that differ by a constant and set ||u||_{\dot{W}^{1,2}} := ||u_{\bar{z}}||_{L^2} = ||Du||_{L^2} / 2.) The quadratic operator Q arises via the formula ⟨b, Qω⟩_{BMO-ℒ¹} = ⟨T_b ω, ω⟩_{L²}, where T_b ω := (Sb - bS)S̄ω satisfies Assumption 1.2 for H = L²(ℂ, ℂ) and X = CMO(ℂ) (see [44]).

It is instructive to also consider the planar Jacobian in real variable notation. Define

$$T_b := R_1[b, R_2] - R_2[b, R_1] : L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2). \tag{3.5}$$

Now $T_b^* = -T_b \neq 0$ and thus T_b is not self-adjoint. However, the operator $\tilde{T}_b: L^2(\mathbb{R}^2, \mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2, \mathbb{R}^2)$, defined in Proposition 2.10, generates the Jacobian:

$$\begin{aligned} \int_{\mathbb{R}^2} \tilde{T}_b(\omega_1, \omega_2) \cdot (\omega_1, \omega_2) &= 2\langle b, R_1\omega_1 R_2\omega_2 - R_2\omega_1 R_1\omega_2 \rangle_{\text{BMO}-\mathcal{H}^1} \\ &= 2\langle b, \partial_1 v_1 \partial_2 v_2 - \partial_2 v_1 \partial_1 v_2 \rangle_{\text{BMO}-\mathcal{H}^1}, \end{aligned}$$

where $v = \Lambda\omega := (-\Delta)^{-1/2}\omega \in \dot{W}^{1,2}(\mathbb{R}^2)$. This example serves to further motivate Assumption 1.3.

Example 3.6 On the real line, two quadratic \mathcal{H}^1 -integrable quantities arise as follows. Isomorphically, $\mathcal{H}^p(\mathbb{R}) \cong \mathcal{H}^p(\mathbb{R}, \mathbb{C}) := \{g + iHg : g, Hg \in L^p(\mathbb{R})\}$ for all $p \in (0, \infty)$, where H is the Hilbert transform. Given $\omega \in L^2(\mathbb{R})$ we set

$$(\omega + iH\omega)^2 = \omega^2 - (H\omega)^2 + 2i\omega H\omega =: Q_1\omega + iQ_2\omega.$$

As is well-known, $Q_1, Q_2: L^2(\mathbb{R}) \rightarrow \mathcal{H}^1(\mathbb{R})$ are not surjective but their Gâteaux derivatives $(Q_1)'_\omega\gamma = \omega\gamma - H\omega H\gamma$ and $(Q_2)'_\omega\gamma = \omega H\gamma + \gamma H\omega$ are—in other words,

$$\{\alpha^2 : \alpha \in \mathcal{H}^2(\mathbb{R}, \mathbb{C})\} \subsetneq \{\alpha\beta : \alpha, \beta \in \mathcal{H}^2(\mathbb{R}, \mathbb{C})\} = \mathcal{H}^1(\mathbb{R}, \mathbb{C}). \tag{3.6}$$

We give a proof of (3.6) in Appendix A for the reader’s convenience.

The operator Q_1 arises via the bounded linear operator $T_b: L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$, $T_b := [H, b]H$ by the formula $\langle b, Q_1\omega \rangle_{\text{BMO}-\mathcal{H}^1} := \int_{\mathbb{R}} \omega T_b\omega$. Now T_b satisfies Assumptions 1.2 and 1.3 but $Q_1: L^2(\mathbb{R}) \rightarrow \mathcal{H}^1(\mathbb{R})$ is not surjective. This illustrates the fact that the modified operator \tilde{T}_b (see Proposition 2.10) might be needed to ensure surjectivity even if T_b satisfies Assumptions 1.2 and 1.3.

Example 3.7 In [57], Wu considered higher-dimensional variants of the operators Q_j of Example 3.6 in Clifford algebras. When $\Omega = \omega + \sum_{j=1}^n R_j\omega\mathbf{e}_j \in L^2(\mathbb{R}^n, \mathbb{C}_{(n)})$ and $\Gamma = \gamma + \sum_{j=1}^n R_j\gamma\mathbf{e}_j \in L^2(\mathbb{R}^n, \mathbb{C}_{(n)})$, their product can be written as

$$\begin{aligned} \Omega\Gamma &= \omega\gamma - \sum_{j=1}^n R_j\omega R_j\gamma + \sum_{j=1}^n (\omega R_j\gamma + \gamma R_j\omega)\mathbf{e}_j \\ &\quad + \sum_{j < k} (R_j\omega R_k\gamma - R_k\omega R_j\gamma)\mathbf{e}_{jk}. \end{aligned} \tag{3.7}$$

Each component of $\Omega\Gamma$ then belongs to $\mathcal{H}^1(\mathbb{R}^n)$ [57, Theorem 12.3.1]. Note that when $n = 2$, the last term of (3.7) is a Jacobian in disguise: $R_1\omega R_2\gamma - R_2\omega R_1\gamma = Jv$ for $v = (\Lambda^{-1}\omega, \Lambda^{-1}\gamma) \in \dot{W}^{1,2}(\mathbb{R}^2, \mathbb{R}^2)$. In analogy to (3.6), Wu conjectured that $\mathcal{H}^1(\mathbb{R}^n) = \{\omega\gamma - \sum_{j=1}^n R_j\omega R_j\gamma : \omega, \gamma \in L^2(\mathbb{R}^n)\}$ [57, p. 237].

Wu also studied more general combinations of Riesz transforms that arise via $A(\xi, \eta) = 1 - (\xi \cdot \eta)^m / (|\xi||\eta|)^m$, $m \in \mathbb{N}$, and showed that each such A satisfies the

assumptions of Theorem 3.4. As an example, $T_b := [R, b] \cdot R : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ generates

$$Q\omega = |\omega|^2 - |R\omega|^2 = |\Lambda u|^2 - |\nabla u|^2,$$

where $\Lambda u = f$. In this case, $A(\xi, \eta) = 1 - \eta \cdot \xi / (|\eta| |\xi|)$. Thus Wu’s conjecture says that the Gâteaux derivative $Q'_\omega \gamma = 2(\omega\gamma - R\omega \cdot R\gamma) = 2(\Lambda u \Lambda v - \nabla u \cdot \nabla v)$ gives a surjective bilinear operator $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \rightarrow \mathcal{H}^1(\mathbb{R}^n)$.

The planar Monge-Ampère equation arises as follows. Define $T_b : L^2(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)$ by

$$T_b \omega := \frac{[R_{11}, b]R_{22}\omega + [R_{22}, b]R_{11}\omega - 2[R_{12}, b]R_{12}\omega}{2}.$$

Then $Q : L^2(\mathbb{R}^2) \rightarrow \mathcal{H}^1(\mathbb{R}^2)$ is of the form $Q\omega = R_{11}\omega R_{22}\omega - R_{12}\omega R_{21}\omega$. Using the isomorphism $-\Delta : \dot{W}^{2,2}(\mathbb{R}^2) \rightarrow L^2(\mathbb{R}^2)$ we may write the Hessian as $\mathcal{H} = Q \circ (-\Delta)^{-1} : \dot{W}^{2,2}(\mathbb{R}^2) \rightarrow \mathcal{H}^1(\mathbb{R}^2)$. In this case, $A(\xi, \eta) = 1 - (\xi \cdot \eta)^2 / (|\xi|^2 |\eta|^2)$.

4 Banach Space Geometric Considerations

This subsection is devoted to proving various claims presented in §1.3. We assume that T satisfies Assumptions 1.2 and 1.3.

4.1 Minimum Norm Solutions

We begin by noting that if the equation $Q\omega = f$ has a solution, then the direct method gives a minimum norm solution.

Proposition 4.1 *Suppose $f \in X_Q^*$, $\gamma \in H$ and $Q\gamma = f$. Then there exists $\omega \in H$ with $Q\omega = f$ and $\|\omega\|_H = \min_{Q\gamma=f} \|\gamma\|_H$.*

Proof Choose a minimising sequence so that $Q\omega_j = f$ for every $j \in \mathbb{N}$ and $\lim_{j \rightarrow \infty} \|\omega_j\|_H = \inf_{Q\gamma=f} \|\gamma\|_H$. Since \mathbb{B}_H is sequentially weakly compact and Q is weak-to-weak* continuous, we have $\omega_j \rightharpoonup \omega$ and $Q\omega_j \xrightarrow{*} Q\omega$ for a subsequence, and ω is the sought minimum norm solution. \square

We then show that surjectivity of $Q : H \rightarrow X_Q^*$ would follow from the weak* density of the range $Q(H)$ in X_Q^* combined with a suitable a priori estimate.

Proposition 4.2 *Suppose that $\overline{Q(H)}^{w-*} = X_Q^*$ and that $\|\omega\|_H^2 \lesssim \|Q\omega\|_{X_Q^*}$ for every minimum norm solution. Then $Q(H) = X_Q^*$.*

Proof Let $f \in X^*$ and choose minimum norm solutions $\omega_j \in H$ with $Q\omega_j \xrightarrow{*} f$. For large enough $j \in \mathbb{N}$ we have $\|\omega_j\|_H^2 \lesssim \|f\|_{X^*} + 1$. After passing to a subsequence, a weak limit ω of $(\omega_j)_{j=1}^\infty$ satisfies $Q\omega = f$. \square

We also mention a version of the Banach-Schauder open mapping theorem for multilinear (and more general) operators from [23]. We say that Q enjoys *generalised translation invariance* if there exist isometric isomorphisms $\sigma_j : H \rightarrow H$ and $\sigma_j : X_Q^* \rightarrow X_Q^*$, $j \in \mathbb{N}$, such that $Q \circ \sigma_j = \sigma_j \circ Q$ for all $j \in \mathbb{N}$ and $\sigma_j f \xrightarrow{*} 0$ as $j \rightarrow \infty$ for all $f \in X_Q^*$. The example one should have in mind is $\sigma_j \omega(x) = \omega(x - je)$ and $\sigma_j f(x) := f(x - je)$ for a non-zero vector e .

Theorem 4.3 *Suppose Q enjoys generalised translation invariance. Then one of the following claims holds:*

- (1) $Q(H) = X_Q^*$ and every minimum norm solution satisfies $\|\omega\|_H^2 \lesssim \|Q\omega\|_{X_Q^*}$.
- (2) $Q(H)$ is meagre in X_Q^* .

Theorem 4.3 is primarily a practical tool for showing *non-surjectivity* of $Q : H \rightarrow X_Q^*$; disproving the *a priori* estimate $\|\omega\|_H^2 \lesssim \|Q\omega\|_{X_Q^*}$ is a markedly less daunting task than disproving surjectivity [23].

4.2 Definition and Existence of Lagrange Multipliers

When $\omega \in H$ is a minimum norm solution, it is natural to look for a Lagrange multiplier of ω , as discussed in §1.3. The Lagrange multiplier condition (1.5) can be written more concisely as follows: $b \in X_Q^{**}$ is a Lagrange multiplier of $\omega \in H$ if

$$2\text{Re}\langle \omega, \varphi \rangle_H = \langle b, Q'_\omega \varphi \rangle_{X_Q^{**}-X_Q^*} \quad \text{for every } \varphi \in H. \tag{4.1}$$

Yet more concisely, $(Q'_\omega)^* b = 2\omega$. The existence of b then means that $\omega \in \text{ran}((Q'_\omega)^*) = \ker(Q'_\omega)^\perp$.

The standard tool for showing the existence of a Lagrange multiplier in a Banach space is the Liusternik-Schnirelman theorem which, in this setting, says that if Q'_ω maps H onto X_Q^* , then ω possesses a Lagrange multiplier. However, in the cases of interest to us, the Liusternik-Schnirelman theorem is not available, as shown below.

Proposition 4.4 *Suppose X_Q^* is not isomorphic to a Hilbert space. Then we have $\{Q'_\omega \gamma : \gamma \in H\} \subsetneq X_Q^*$ for every $\omega \in H$.*

Proof Seeking a contradiction, suppose $Q'_\omega H = X_Q^*$. Thus $Q'_\omega : \ker(Q')^\perp \rightarrow X_Q^*$ is an isomorphism from a Hilbert space onto X_Q^* . □

Proposition 4.5 *Every $b \in \mathbb{S}_{X_Q}$ is a Lagrange multiplier of some $\omega \in \mathcal{A}$.*

Proof Choose a maximizing sequence: $\langle T_b \omega_j, \omega_j \rangle_H \rightarrow 1$. For a subsequence, $\omega_j \rightarrow \omega$ in H , and so $T_b \omega_j \rightarrow T_b \omega$, giving $1 = \langle T_b \omega, \omega \rangle_H = \langle b, Q\omega \rangle_{X_Q-X_Q^*} \leq \|Q\omega\|_{X_Q^*} \leq \|\omega\|_H^2 \leq 1$, so that $\omega \in \mathcal{A}$. □

Definition 4.6 When Y is a real Banach space, a set $A \subset \mathbb{S}_{Y^*}$ is called a *James boundary of Y* if for every $y \in \mathbb{S}_Y$ there exists $y^* \in A$ such that $\langle y^*, y \rangle_{Y^*-Y} = 1$.

Proposition 4.5 then says that $Q(\mathcal{A})$ is a James boundary of X_Q . By Godefroy’s Theorem (see [15, Theorem 3.46]), if $A \subset \mathbb{S}_{Y^*}$ is a separable James boundary of Y , then $\overline{\text{co}}(A) = \mathbb{B}_{Y^*}$.

Corollary 4.7 $\overline{\text{co}}(Q(\mathcal{A})) = \mathbb{B}_{X_Q^*}$.

Corollary 4.8 $\|f\|_{X_Q^*} = \inf\{\sum_{j=1}^\infty \|\omega_j\|_H^2 : f = \sum_{j=1}^\infty Q\omega_j\}$ for all $f \in X_Q^*$.

4.3 Further Properties of Lagrange Multipliers

The Lagrange multiplier condition (4.1) has several useful equivalent characterisations, and we collect two of them in the following proposition.

Proposition 4.9 *Whenever $b \in \mathbb{S}_{X_Q^{**}}$ and $\omega \in \mathbb{S}_H$, the following conditions are equivalent:*

- (i) b is a Lagrange multiplier of ω , i.e., (4.1) holds.
- (ii) $\langle b, Q\omega \rangle_{X_Q^{**}-X_Q^*} = 1$.
- (iii) $\omega \in \ker(I - T_b)$.

If (i)–(iii) hold, then ω is a minimum norm solution and belongs to \mathcal{A} .

Proof We first prove (i) \Leftrightarrow (ii). Suppose (ii) holds and fix $\varphi \in H$ and small $\delta > 0$. The function $I : [-\delta, \delta] \rightarrow \mathbb{R}, I(\epsilon) := \langle b, Q(\omega + \epsilon\varphi) \rangle_{X_Q^{**}-X_Q^*} / \|\omega + \epsilon\varphi\|_H^2$ is maximised at $\epsilon = 0$, and therefore $I'(0) = \langle b, Q'_\omega\varphi \rangle_{X_Q^{**}-X_Q^*} - 2\text{Re}\langle \omega, \varphi \rangle_H = 0$, giving (i). The direction (i) \Rightarrow (ii) is proved by setting $\varphi = \omega$. We then prove (ii) \Leftrightarrow (iii). First, (ii) gives $\langle \omega, T_b\omega \rangle_H = 1$. Since $\|T_b\|_{H \rightarrow H} = 1$ and H is strictly convex, we conclude that (iii) holds. On the other hand, if (iii) holds, then $\langle b, Q\omega \rangle_{X_Q^{**}-X_Q^*} = \langle \omega, T_b\omega \rangle_H = \langle \omega, \omega \rangle_H = 1$. □

In many cases, conditions (i)–(iii) can be supplemented by Euler-Lagrange equations for a suitable potential. In the case of the planar Jacobian, denoting $u_{\bar{z}} = \omega$ as before, (i)–(iii) are equivalent to $u_{z\bar{z}} = (bu_{\bar{z}})_z - (bu_z)_{\bar{z}}$ [44, Proposition 4.8].

Lemma 4.10 $(X^{**}, \|T_\cdot\|_{H \rightarrow H}) = (X_Q)^{**}$ isometrically.

Proof Let $b \in X^{**}$. Since $Q(\mathbb{B}_H) \subset \mathbb{B}_{X_Q^*} = \overline{\text{co}}(Q\mathcal{A})$, we get

$$\|T_b\|_{H \rightarrow H} = \sup_{\|f\|_H=1} \langle b, Q\omega \rangle_{X^{**}-X^*} = \sup_{f \in \text{co}(Q\mathcal{A})} \langle b, f \rangle_{X^{**}-X^*} = \|b\|_{(X^*, \|\cdot\|_{X_Q^*})^*}.$$

□

Proposition 4.11 *Suppose $b \in \mathbb{S}_{X_Q^{**}}$. The following conditions are equivalent.*

- (i) b is a Lagrange multiplier of some $\omega \in \mathcal{A}$.
- (ii) b is norm-attaining.

Proof If b is a Lagrange multiplier of $\omega \in \mathcal{A}$, then $T_b\omega = \omega$ and therefore $\langle b, Q\omega \rangle_{X_Q^{**}-X_Q^*} = \langle T_b\omega, \omega \rangle_H = 1$.

Conversely, if b is norm-attaining, denote $D^{-1}(b) := \{f \in \mathbb{S}_{X_Q^*} : \langle b, f \rangle_{X_Q^{**}-X_Q^*} = 1\}$. Since $D^{-1}(b)$ is closed, bounded and convex, Theorem 2.7 implies that $D^{-1}(b)$ contains an extreme point f . The definition of $D^{-1}(b)$ implies that f is also an extreme point of $\mathbb{S}_{X_Q^*}$, and Lemma 5.1 then gives $f \in Q(\mathcal{A})$. \square

We also characterise elements of \mathbb{S}_H that possess a Lagrange multiplier.

Proposition 4.12 *Let $\omega \in \mathbb{S}_H$. The following conditions are equivalent.*

- (i) ω has a Lagrange multiplier $b \in \mathbb{S}_{X_Q^{**}}$.
- (ii) $\omega \in \mathcal{A}$.

Proof If $\omega \in \mathbb{S}_H$ has a Lagrange multiplier $b \in \mathbb{S}_{X_Q^{**}}$, then $\langle b, Q\omega \rangle_{X_Q^{**}-X_Q^*} = \langle T_b\omega, \omega \rangle_H = \|\omega\|_H^2 = 1$ so that $\omega \in \mathcal{A}$.

Conversely, let $\omega \in \mathcal{A}$. Since $Q\omega \in \mathbb{S}_{X_Q^*}$, there exists $b \in \mathbb{S}_{X_Q^{**}}$ such that $\langle b, Q\omega \rangle_{X_Q^{**}-X_Q^*} = 1$. Now b is a Lagrange multiplier of ω . \square

Using the results above we collect many equivalent formulations of Question 1.6.

Corollary 4.13 *If $Q(H)$ is dense in X_Q^* , the following conditions are equivalent:*

- (i) $Q(\mathcal{A}) = \mathbb{S}_{X_Q^*}$.
- (ii) $\mathcal{E} = \|\cdot\|_{X_Q^*}$.
- (iii) Every minimum norm solution satisfies $\|\omega\|_H^2 = \|Q\omega\|_{X_Q^*}$.
- (iv) Every minimum norm solution has a Lagrange multiplier in $\mathbb{S}_{X_Q^{**}}$.

5 The Proof of Theorem 1.10

We get the proofs of Theorems 1.9 and 1.10 underway by proving (1.6).

Lemma 5.1 *If Assumptions 1.2 and 1.3 holds, then $Q(\mathcal{A}) \supset \text{ext}(\mathbb{B}_{X_Q^*})$. Furthermore, $Q(\mathcal{A})$ is closed in the relative weak* topology of $\mathbb{S}_{X_Q^*}$. In particular, it is norm closed.*

Proof We first show that $Q(\mathcal{A})$ is relatively weak* (sequentially) closed in $\mathbb{S}_{X_Q^*}$. Since X_Q is separable, it suffices to consider sequences instead of nets. Suppose $\omega_j \in \mathcal{A}$ for every $j \in \mathbb{N}$ and $Q\omega_j \xrightarrow{*} f \in \mathbb{S}_{X_Q^*}$. We claim that $f = Q\omega$ for some $\omega \in \mathcal{A}$.

Passing to a subsequence, $\omega_j \rightarrow \omega \in \mathbb{B}_H$, since \mathbb{B}_H is weakly compact. Thus $Q\omega_j \xrightarrow{*} Q\omega$, and so it suffices to show that $\omega \in \mathcal{A}$. The chain of inequalities $\|\omega\|_H^2 \leq \liminf_{j \rightarrow \infty} \|\omega_j\|_H^2 = 1 = \|Q\omega\|_{X_Q^*} \leq \|\omega\|_H^2$ implies that $\omega \in \mathcal{A}$.

We then show that every extreme point of $\mathbb{B}_{X_Q^*}$ lies in $Q(\mathcal{A})$. In Theorem 2.8, choose K to be the weak* closure of $Q(\mathcal{A})$ in $\mathbb{B}_{X_Q^*}$. By Corollary 4.7, $\overline{\text{co}}^{w*}(K) = \mathbb{B}_{X_Q^*}$. Theorem 2.8 implies that $\text{ext}(\mathbb{B}_{X_Q^*}) \subset K$. Let now $f \in \text{ext}(\mathbb{B}_{X_Q^*})$ and choose $\omega_j \in \mathcal{A}$ such that $Q\omega_j \xrightarrow{*} f$. Since $Q(\mathcal{A})$ is relatively weak* closed in $\mathbb{S}_{X_Q^*}$, we obtain $f \in Q(\mathcal{A})$. \square

We recall the statement of Theorem 1.10.

Claim 5.2 *Suppose Assumptions 1.2, 1.3 and 1.8 hold, and let $f \in \text{ext}(\mathbb{B}_{X_Q^*})$. Then the minimum norm solution $\omega \in \mathcal{A}$ of $Q\omega = f$ is unique up to multiplication by $c \in \mathbb{S}_{\mathbb{K}}$.*

Proof Seeking a contradiction, suppose $f \in \text{ext}(\mathbb{B}_{X_Q^*})$ and $Q\omega = Q\gamma = f$, where $\omega, \gamma \in \mathcal{A}$ are not equal up to a rotation. Thus we may write $\gamma = s\omega + t\omega^\perp$, where $s, t \in \mathbb{K}$ with $|s|^2 + |t|^2 = 1, t \neq 0$ and $\omega^\perp \in \mathbb{S}_H \cap \{\omega\}^\perp$. We intend to show that

$$Q(\omega^\perp) = Q\omega. \tag{5.1}$$

Once (5.1) is proved, Assumption 1.8 yields $c \in \mathbb{S}_{\mathbb{K}}$ such that $Q'_\omega(c\omega^\perp) \neq 0$. Denoting $\psi^\pm := (\omega \pm c\omega^\perp)/\sqrt{2} \in \mathbb{S}_H$ we therefore get $Q\omega = (Q\psi^+ + Q\psi^-)/2$ but $\mathbb{B}_{X_Q^*} \ni Q\psi^\pm = Q\omega \pm Q'_\omega(c\omega^\perp)/2 \neq Q\omega$, thereby obtaining the sought contradiction.

Formula (5.1) clearly holds if $s = 0$, so assume $s \neq 0$. We denote $\tilde{\omega}^\perp := (|st|/s\bar{t})\omega^\perp$ and $\tilde{\gamma} := (\bar{s}/|s|)\gamma = |s|\omega + |t|\tilde{\omega}^\perp \in \mathcal{A}$. Now $Q\omega = Q\tilde{\gamma} = |s|^2 Q\omega + |t|^2 Q\tilde{\omega}^\perp + |s||t| Q'_\omega\tilde{\omega}^\perp$, which yields

$$Q\tilde{\omega}^\perp = Q\omega - \frac{|s|}{|t|} Q'_\omega\tilde{\omega}^\perp. \tag{5.2}$$

Since $Q\tilde{\omega}^\perp = Q\omega^\perp$, it therefore suffices to show that $Q'_\omega\tilde{\omega}^\perp = 0$.

Whenever $\delta, \epsilon \in \mathbb{R}$ with $\delta^2 + \epsilon^2 = 1$, we have

$$Q(\delta\omega + \epsilon\tilde{\omega}^\perp) = \delta^2 Q\omega + \delta\epsilon Q'_\omega\tilde{\omega}^\perp + \epsilon^2 Q\tilde{\omega}^\perp = Q\omega + (\delta\epsilon - \epsilon^2 |s||t|^{-1}) Q'_\omega\tilde{\omega}^\perp. \tag{5.3}$$

By choosing any $r > 0$ and letting $\delta_\pm = \epsilon(|s|/|t| \pm r)$ with $\delta_\pm^2 + \epsilon^2 = 1$ we get

$$\delta_+\epsilon - \epsilon^2 |s|/|t| = -(\delta_-\epsilon - \epsilon^2 |s|/|t|) \neq 0, \tag{5.4}$$

$$Q(\delta_+\omega + \epsilon_1\tilde{\omega}^\perp) + Q(\delta_-\omega + \epsilon_2\tilde{\omega}^\perp) = 2Q\omega. \tag{5.5}$$

Since $Q\omega \in \text{ext}(\mathbb{B}_{X_Q^*})$, we conclude from (5.3) and (5.5) that $Q'_\omega\tilde{\omega}^\perp = 0$, and now (5.2) and the definition of $\tilde{\omega}^\perp$ imply (5.1). □

Corollary 5.3 *Suppose Assumptions 1.2 and 1.3 and 1.8 hold and $b \in \mathbb{S}_{X_Q}$ is a point of Fréchet differentiability. Then $\ker(I - T_b)$ is one-dimensional.*

Proposition 5.4 *The operator $\omega \mapsto Q\omega = |S\omega|^2 - |\omega|^2 : L^2(\mathbb{C}, \mathbb{C}) \rightarrow \mathcal{H}^1(\mathbb{C})$ satisfies Assumption 1.8, whereas $(\omega, \gamma) \mapsto \tilde{Q}(\omega, \gamma) := G'_\omega\gamma : L^2(\mathbb{C}, \mathbb{C})^2 \rightarrow \mathcal{H}^1(\mathbb{C})$ does not.*

Proof We prove the first statement by contradiction. Suppose $\omega, \gamma \in \mathcal{A}$ satisfy $Q\omega = Q\gamma$ and $Q'_\omega(c\gamma) = 2\text{Re}(S\omega\overline{S(c\gamma)} - \omega\overline{c\gamma}) = 0$ for every $c \in \mathbb{S}_{\mathbb{K}}$. Thus $S\omega\overline{S\gamma} - \omega\overline{\gamma} =$

0. Fix $\delta > 0$ and denote $X_\delta := \{z \in \mathbb{C} : |\omega(z)| \in [\delta, 1/\delta]\}$. In X_δ we use the formula $\mathcal{S}\omega\overline{\mathcal{S}\gamma} - \omega\overline{\gamma} = 0$ to write

$$|\mathcal{S}\omega|^2 - |\omega|^2 = |\mathcal{S}\gamma|^2 - |\gamma|^2 = -\frac{|\mathcal{S}\gamma|^2}{|\omega|^2}(|\mathcal{S}\omega|^2 - |\omega|^2),$$

so that $|\mathcal{S}\omega|^2 - |\omega|^2 = 0$ in X_δ . Thus

$$\int_{\omega \neq 0} \left| |\mathcal{S}\omega|^2 - |\omega|^2 \right| = \lim_{\delta \rightarrow 0} \int_{X_\delta} \left| |\mathcal{S}\omega|^2 - |\omega|^2 \right| = 0,$$

and therefore $\int_{\omega \neq 0} |\mathcal{S}\omega|^2 = 1 - \int_{\omega \neq 0} |\omega|^2 = 1 - \int_{\omega \neq 0} |\omega|^2 = 0$. We conclude that $|\mathcal{S}\omega|^2 - |\omega|^2 = 0$, which yields the desired contradiction.

We show a strengthened version of the second statement: whenever $(\omega, \gamma) \in \mathcal{A}$, there exists $(\varphi, \psi) \in \mathcal{A}$ such that $\tilde{Q}(\omega, \gamma) = \tilde{Q}(\varphi, \psi)$ but $\tilde{Q}'_{(\omega, \gamma)}[c(\varphi, \psi)] = 0$ for all $c \in S^1$. Given (ω, γ) , we choose $(\varphi, \psi) = (-\overline{\mathcal{S}\gamma}, \overline{\mathcal{S}\omega})$ and use the facts that $\mathcal{S} : L^2(\mathbb{C}, \mathbb{C}) \rightarrow L^2(\mathbb{C}, \mathbb{C})$ is an isometry and $\mathcal{S}\overline{\mathcal{S}\eta} = \eta$ for all $\eta \in L^2(\mathbb{C}, \mathbb{C})$. \square

An almost identical proof applies to Example 3.6, but instead of $(\varphi, \psi) = (-\overline{\mathcal{S}\gamma}, \overline{\mathcal{S}\omega})$ we set $(\varphi, \psi) = (-H\omega, H\gamma)$.

Proposition 5.5 *The operator $\omega \mapsto \omega^2 - (H\omega)^2 : L^2(\mathbb{R}) \rightarrow \mathcal{H}^1(\mathbb{R})$ satisfies Assumption 1.8, whereas $(\omega, \gamma) \mapsto \omega\gamma - H\omega H\gamma : L^2(\mathbb{R})^2 \rightarrow \mathcal{H}^1(\mathbb{R})$ does not.*

Remark 5.6 The proof of Proposition 5.4 uses a central difference of the operator $\omega \mapsto Q\omega = |\mathcal{S}\omega|^2 - |\omega|^2$ and its Gâteaux derivative \tilde{Q} . Given $\omega \in \mathcal{A}$, the set $\{\varphi \in \mathcal{A} : Q\varphi = Q\omega\}$ sometimes consists only of rotations of ω , whereas all the corresponding sets $\{(\varphi, \psi) \in \tilde{\mathcal{A}} : \tilde{Q}(\varphi, \psi) = \tilde{Q}(\omega, \gamma)\}$ are invariant under the linear transformation $L(\omega, \gamma) := (-\overline{\mathcal{S}\gamma}, \overline{\mathcal{S}\omega})$. Note that $L \circ L = -\text{id}$. Similar remarks apply to the sets of minimum norm solutions for a given datum $f \in \mathcal{H}^1(\mathbb{C})$.

6 The proof of Theorem 1.9

6.1 Claims (i)–(iii)

In this subsection, Assumptions 1.2 and 1.3 are in effect. Before recalling Claims (i)–(iii) we note the following consequence of Corollary 4.13.

Lemma 6.1 *Let $b \in \mathbb{S}_{X_Q}$. Then $\{\omega \in \mathcal{A} : Q\omega \in D(b)\} = \ker(I - T_b) \cap \mathbb{S}_H$.*

Claim 6.2 *For every $b \in \mathbb{S}_{X_Q}$, $D(b) \cap Q(\mathcal{A})$ contains $\text{ext}(D(b))$ and is path-connected.*

Proof Lemma 5.1 directly implies that $D(b) \cap Q(\mathcal{A})$ contains $\text{ext}(D(b))$. The path-connectedness of $D(b) \cap Q(\mathcal{A})$ follows from Lemma 6.1, the path-connectedness of $\ker(I - T_b) \cap \mathbb{S}_H$ and the continuity of Q . \square

Claim 6.3 For every $b \in \mathbb{S}_{X_Q}$, the convex set $D(b)$ has finite affine dimension.

Proof Since $D(b) \cap Q(\mathcal{A})$ contains $\text{ext}(D(b))$, it suffices to show that $D(b) \cap Q(\mathcal{A})$ has finite affine dimension. By Lemma 6.1, $D(b) \cap Q(\mathcal{A}) \subset Q(\ker(I - T_b))$. Since K_b is compact and self-adjoint, $\ker(I - T_b)$ is finite-dimensional, and so $Q(\ker(I - T_b))$ has finite affine dimension. Precisely, when $\{\omega_1, \dots, \omega_n\}$ is an orthonormal basis of $\ker(I - T_b)$ and $Q\omega \in D(b) \cap Q(\mathcal{A})$, we may write $Q\omega \in \text{span} \cup_{j,k=1}^n \{Q'_{\omega_j} \omega_k\}$ (when $\mathbb{K} = \mathbb{R}$) or $Q\omega \in \text{span} \cup_{j,k=1}^n \{\text{Re}(Q'_{\omega_j} \omega_k), \text{Re}(Q'_{\omega_j}(i\omega_k))\}$ (when $\mathbb{K} = \mathbb{C}$). \square

Claim 6.4 The norm and relative weak* topologies coincide in $Q(\mathcal{A})$.

Proof Suppose $\omega_j, \omega \in \mathcal{A}$ and $Q\omega_j \xrightarrow{*} Q\omega$. Seeking a contradiction, suppose that for a subsequence, $\liminf_{j \rightarrow \infty} \|Q\omega_j - Q\omega\|_{X_Q^*} > 0$. By passing to a further subsequence, $\omega_j \rightharpoonup \gamma \in \mathbb{B}_H$. Now $Q\omega_j \xrightarrow{*} Q\gamma$ and thus $Q\gamma = Q\omega$. This implies that $1 = \|Q\gamma\|_{X_Q^*} \leq \|\gamma\|_H \leq 1$. Thus $\omega_j \rightharpoonup \gamma$ and $\|\omega_j\|_H \rightarrow \|\gamma\|_H$, giving $\|\omega_j - \gamma\|_H \rightarrow 0$ and so $\|Q\omega_j - Q\gamma\|_H \rightarrow 0$, and the latter yields the sought contradiction. \square

6.2 Claims (iv)–(vi)

We recall and prove Claims (iv)–(vi) below. In this subsection, we assume that Assumptions 1.2, 1.3 and 1.8 hold.

Claim 6.5 For every b in a relatively open dense subset of \mathbb{S}_{X_Q} , $D(b) = \{Q\omega\}$ for some $\omega \in \mathcal{A}$ and $\|\cdot\|_{X_Q}$ is Fréchet differentiable at b .

Proof By Theorem 2.1, $\|\cdot\|_{X_Q}$ is Fréchet differentiable in a dense subset of \mathbb{S}_{X_Q} . Seeking a contradiction, suppose b is a point of Fréchet differentiability but the points $b_j \rightarrow b, b_j \in \mathbb{S}_{X_Q}$, are not. Thus the subspaces $\ker(I - T_{b_j})$ are at least two-dimensional but, by Corollary 5.3, $\dim(\ker(I - T_{b_j})) = 1$.

For every $j \in \mathbb{N}$ choose $\omega_j, \gamma_j \in \ker(I - T_{b_j}) \cap \mathbb{S}_H$ such that $\langle \omega_j, \gamma_j \rangle_H = 0$. Since \mathbb{B}_H is weakly compact, we may assume that $\omega_j \rightharpoonup \omega$ and $\gamma_j \rightharpoonup \gamma$ in H . Then $Q\omega_j \xrightarrow{*} Q\omega$ and $Q\gamma_j \xrightarrow{*} Q\gamma$ in X_Q^* .

Since $b_j \rightarrow b$ in X_Q and $Q\omega_j \xrightarrow{*} Q\omega$ and $Q\gamma_j \xrightarrow{*} Q\gamma$ in X_Q^* , we conclude that

$$\begin{aligned} \langle Q\omega, b \rangle_{X_Q^* - X_Q} &= \lim_{j \rightarrow \infty} \langle Q\omega_j, b_j \rangle_{X_Q^* - X_Q} = 1, \\ \langle Q\gamma, b \rangle_{X_Q^* - X_Q} &= \lim_{j \rightarrow \infty} \langle Q\gamma_j, b_j \rangle_{X_Q^* - X_Q} = 1. \end{aligned}$$

Since $D(b)$ is a singleton by assumption, we obtain $Q\omega = Q\gamma$. By Corollary 5.3,

$$\gamma = c\omega \quad \text{for some } c \in \mathbb{S}_{\mathbb{K}}. \tag{6.1}$$

Furthermore,

$$1 = \lim_{j \rightarrow \infty} \|\omega_j\|_H^2 \geq \|\omega\|_H^2 \geq \|Q\omega\|_{X_Q^*} \geq \langle Q\omega, b \rangle_{X_Q^* - X_Q} = 1.$$

We thus have $\omega_j \rightarrow \omega$ and $\|\omega_j\|_H \rightarrow \|\omega\|_H$, whereby $\|\omega_j - \omega\|_H \rightarrow 0$. Similarly, $\|\gamma_j - \gamma\|_H \rightarrow 0$. We conclude, via (6.1), that

$$0 = \langle \omega_j, \gamma_j \rangle \rightarrow \langle \omega, \gamma \rangle = \langle \omega, c\omega \rangle = \bar{c},$$

which yields a contradiction. □

Remark 6.6 Note that a straightforward modification of the proof above proves the relative openness in \mathbb{S}_{X_Q} of $\{b \in \mathbb{S}_{X_Q} : \dim(\ker(I - K_b)) \leq n\}$ for every $n \in \mathbb{N}$.

For the following claim recall Definition 2.5.

Claim 6.7 $D : \mathbb{S}_{X_Q} \rightarrow 2^{\mathbb{S}_{X_Q^*}}$ is ausco map, and $b \mapsto D(b) \cap Q(\mathcal{A}) : \mathbb{S}_{X_Q} \rightarrow 2^{\mathbb{S}_{X_Q^*}}$ is anusco map.

Proof We first show that D isusco. Note that D is point-to-compact since for every $b \in \mathbb{S}_{X_Q}$, the closed, bounded set $D(b)$ is contained in a finite-dimensional subspace of X_Q^* . In order to show that D is norm-to-norm upper semicontinuous, suppose $b \in \mathbb{S}_{X_Q}$ and $f_j \in \mathbb{S}_{X_Q^*}$ satisfy $\lim_{j \rightarrow \infty} \langle f_j, b \rangle_{X_Q^*-X_Q} = 1$. By Theorem 2.3 it suffices to show that $f_j \rightarrow f \in D(b)$ for a subsequence.

Denote $\dim(\ker(I - T_b)) = n \in \mathbb{N}$. Then $\dim(\ker(I - T_{b_j})) \leq n$ from some index on by Remark 6.6. Thus we may write every f_j as a convex combination $f_j = \sum_{k=1}^N \lambda_j^k Q\omega_j^k$. By passing to a subsequence, for every $k \in \{1, \dots, N\}$ we have $\lambda_j^k \rightarrow \lambda^k \in [0, 1]$ and $\omega_j^k \rightarrow \omega^k \in \mathbb{B}_H$. In particular, $\sum_{k=1}^N \lambda^k = 1$.

Suppose now $\lambda^k > 0$. Then necessarily $\langle Q\omega_j^k, b \rangle_{X_Q^*-X_Q} \rightarrow 1$ and so $\|\omega_j^k\|_H \rightarrow 1$, giving $\|\omega_j^k - \omega^k\|_H \rightarrow 0$ and $\|Q\omega_j^k - Q\omega^k\|_{X_Q^*} \rightarrow 0$. We conclude that $f_j \rightarrow \sum_{k=1}^N \lambda^k Q\omega^k \in D(b)$, and thus D isusco.

We then show that $b \mapsto D(b) \cap Q(\mathcal{A})$ isusco. The compactness of $D(b) \cap Q(\mathcal{A})$ follows from the compactness of $D(b)$ and Lemma 5.1. We prove the norm-to-norm upper semicontinuity of $b \mapsto D(b) \cap Q(\mathcal{A})$ by contradiction. Suppose $b_j \rightarrow b$ in $\mathbb{S}_{X_Q^*}$ and there exist $\epsilon > 0$ and points $\omega_j \in \mathcal{A}$ such that $Q\omega_j \in (D(b_j) \cap Q(\mathcal{A})) \setminus \mathcal{B}(D(b), \epsilon)$. For a subsequence, $Q\omega_j \xrightarrow{*} f \in \mathbb{B}_{X_Q^*}$. Now $\langle b, f \rangle_{X_Q^*-X_Q} = \lim_{j \rightarrow \infty} \langle b_j, Q\omega_j \rangle_{X_Q^*-X_Q} = 1$ gives $f \in \mathbb{S}_{X_Q^*}$. Lemma 5.1 now yields $f \in D(b) \cap Q(\mathcal{A})$, and so we have found the sought contradiction. □

Claim 6.8 The norm and relative weak* topologies also coincide in $NA_{\|\cdot\|_{X_Q}}$.

Proof The result follows directly from Claim (v) and Theorem 2.3. □

7 The Jacobian Equation with L^p Data

Many of the ideas of this paper can be adapted to study the range of the operator $J : \dot{W}^{1,2p}(\mathbb{R}^2, \mathbb{R}^2) \rightarrow L^p(\mathbb{R}^2)$ when $1 < p < \infty$. In this section we outline such an approach and list those results whose proof extends to this new setting in a straightforward manner.

We can again use basic properties of the Beurling transform to write

$$\langle b, Q\omega \rangle_{L^{p'}-L^p} = \langle T_b\omega, \omega \rangle_{L^{(2p)'}-L^{2p}},$$

$$T_b\omega := \overline{(\mathcal{S}b - b\mathcal{S})\mathcal{S}\omega}, \quad Q\omega := |\mathcal{S}\omega|^2 - |\omega|^2.$$

Note that $\|T_b\|_{L^{2p} \rightarrow L^{(2p)'}} \lesssim_p \|b\|_{L^{p'}}$ for all $b \in L^{p'}(\mathbb{C})$ by a simple application of Hölder’s inequality and the boundedness of $\mathcal{S}: L^p(\mathbb{C}, \mathbb{C}) \rightarrow L^p(\mathbb{C}, \mathbb{C})$. The lower bound estimate

$$\|T_b\|_{L^{2p} \rightarrow L^{(2p)'}} \gtrsim_p \|b\|_{L^{p'}}, \tag{7.1}$$

however, is far from trivial and cannot be proved by simply following the proof of estimates (1.3).

Hytönen proved (7.1) rather recently in [29]. More generally, on \mathbb{R}^n , Hytönen showed that the commutator of a degenerate Calderón-Zygmund operator with $b \in L^1_{loc}(\mathbb{R}^n)$ defines a bounded operator from $L^{q_1}(\mathbb{R}^n)$ into $L^{q_2}(\mathbb{R}^n)$, $q_1 > q_2 > 1$, if and only if $b = a + c$ with $a \in L^r(\mathbb{R}^n)$, $1/r = 1/q_2 - 1/q_1$ and $c \in \mathbb{R}$. In notable contrast to the case of $[b, T]: L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$, the possible cancellations of b do not play an important role; recall, for instance, that there exist $b \notin \text{BMO}(\mathbb{R}^n)$ such that $|b| \in \text{BMO}(\mathbb{R}^n)$.

As a corollary of (7.1), Hytönen obtained an analogue of (1.2): for every $f \in L^p(\mathbb{C})$ there exist $u_j \in \dot{W}^{1,2p}(\mathbb{C}, \mathbb{C})$ such that $f = \sum_{j=1}^\infty Ju_j$ and $\sum_{j=1}^\infty \|Du_j\|_{L^{2p}}^2 \lesssim \|f\|_{L^p}$ [29]. Hytönen also showed an analogous result in higher dimensions by different methods.

In order to adapt the methods of this paper to the L^p case, another crucial ingredient is the compactness of $[b, T]: L^{2p}(\mathbb{R}^2, \mathbb{R}^2) \rightarrow L^{(2p)' }(\mathbb{R}^2, \mathbb{R}^2)$ for all $b \in L^{p'}(\mathbb{R}^2)$. Hytönen & al. recently proved this property for a large class of degenerate Calderón-Zygmund kernels in [31]. We formulate their two main results.

Theorem 7.1 *Let $1 < q_2 < q_1 < \infty$ and $1/r = 1/q_2 - 1/q_1$. Suppose T is a non-degenerate Calderón-Zygmund operator which satisfies one of the following two:*

- (1) *condition (i) of Definition 3.2 with the Dini condition $\int_0^1 t^{-1}\omega(t)dt < \infty$,*
- (2) *condition (ii) of Definition 3.2 with $\Omega \in L^v(S^{n-1})$ for some $v \in (1, \infty]$.*

Then $[b, T]$ is compact from $L^{q_1}(\mathbb{R}^n)$ to $L^{q_2}(\mathbb{R}^n)$ for all $b \in L^r(\mathbb{R}^n)$.

In the next subsection we put the operator $Q = |\mathcal{S}\cdot|^2 - |\cdot|^2 : L^{2p}(\mathbb{C}, \mathbb{C}) \rightarrow L^p(\mathbb{C})$ into a general Banach-space geometric framework.

7.1 Mathematical Setting

We assume that X is a separable, reflexive Banach space and that Y is a separable, reflexive, smooth, strictly convex (i.e. $\text{ext}(\mathbb{B}_Y) = \mathbb{S}_Y$) Banach space with the Kadec-Klee property (i.e. weak and norm topologies coincide on \mathbb{S}_Y). Assumptions 1.2 and 1.3, Definition 1.4 and Proposition 2.10 have natural analogues:

Assumption 7.2 A bilinear mapping

$$(b, \omega) \mapsto T_b \omega: X \times Y \rightarrow Y^*$$

satisfies the following conditions:

- (i) $c \|b\|_X \leq \|T_b\|_{Y \rightarrow Y^*} \leq C \|b\|_X$ for every $b \in X$,
- (ii) T_b is compact for every $b \in X$.

Assumption 7.3 The following conditions hold for every $b \in X$:

- (1) $T_b^* = T_b$.
- (2) $\|T_b\|_{Y \rightarrow Y^*} = \sup_{\|\omega\|_Y=1} \langle T_b \omega, \omega \rangle_{Y^* - Y}$.

Definition 7.4 Suppose Assumptions 7.2 and 7.3 hold. Define the norm-to-norm and weak-to-weak sequentially continuous map $Q: Y \rightarrow X^*$ by

$$\langle b, Q\omega \rangle_{X - X^*} := \langle T_b \omega, \omega \rangle_{Y^* - Y}.$$

Proposition 7.5 If T satisfies Assumption 7.2, then $\tilde{T}_b: Y \times Y \rightarrow Y^* \times Y^*$, $\tilde{T}_b(\omega, \gamma) := (T_b^* \gamma, T_b \omega)$, satisfies Assumptions 7.2 and 7.3.

In Proposition 7.5 we have endowed $Y \times Y$ with the norm $\|(\omega, \gamma)\|_{Y \times Y} := (\|\omega\|_Y^2 + \|\gamma\|_Y^2)^{1/2}$ and similarly for $Y^* \times Y^*$, so that $Y \times Y$ and $Y^* \times Y^*$ are smooth and strictly convex since Y and Y^* are. Denoting $D(\omega) = \{\omega^*\}$ for all $\omega \in \mathbb{S}_Y$ we have $D(\omega, \gamma) = \{(\omega^*, \gamma^*)\}$ for all $\omega, \gamma \in \mathbb{S}_Y$.

Remark 7.6 Assumption 7.3 reveals an important difference to the case of \mathcal{H}^1 data. In the case of $J: \dot{W}^{1,2}(\mathbb{C}, \mathbb{C}) \rightarrow \mathcal{H}^1(\mathbb{C})$, the associated linear operators $T_b: L^2(\mathbb{C}, \mathbb{C}) \rightarrow L^2(\mathbb{C}, \mathbb{C})$ are self-adjoint and therefore the operator norm and numerical radius of T_b coincide, $\|T_b\|_{L^2 \rightarrow L^2} = \sup_{\|\omega\|_{L^2}=1} |\langle T_b \omega, \omega \rangle|$. The standard proof (see e.g. [9, p. 34]) employs the fact that as a Hilbert space, $L^2(\mathbb{C}, \mathbb{C})$ satisfies the parallelogram law $\|\omega + \gamma\|_{L^2}^2 + \|\omega - \gamma\|_{L^2}^2 = 2(\|\omega\|_{L^2}^2 + \|\gamma\|_{L^2}^2)$. The analogue $\|T_b\|_{L^{2p} \rightarrow L^{(2p)'}} = \sup_{\|\omega\|_{L^{2p}}=1} |\langle T_b \omega, \omega \rangle_{L^{(2p)'} - L^{2p}}|$ seems to fail (even though the left and right hand sides are comparable), owing to the fact that the inequality $\|\omega + \gamma\|_{L^{2p}}^2 + \|\omega - \gamma\|_{L^{(2p)'}}^2 \leq 2(\|\omega\|_{L^{2p}}^2 + \|\gamma\|_{L^{(2p)'}}^2)$ fails (see [37, Theorem 2]).

In order to use our full operator theoretic framework to study the planar Jacobian, one then needs to make the concession that $Q\omega := |\mathcal{S}\omega|^2 - |\omega|^2$ is replaced by $\tilde{Q}(f, g) = Q'_f g = J(u_1, v_2) + J(v_1, u_2)$, where $u = u_1 + iu_2 = \mathcal{C}f$ and $v = v_1 + iv_2 = \mathcal{C}g$. It is of course equivalent to studying the range of $(u, v) \mapsto Ju + Jv: \dot{W}^{1,2p}(\mathbb{C}, \mathbb{C})^2 \rightarrow L^p(\mathbb{C})$.

We define norms on X and X^* by $\|b\|_{X_Q} := \sup_{\|\omega\|_X=1} \langle b, Q\omega \rangle_{X - X^*} = \|T_b\|_{Y \rightarrow Y^*}$ and $\|f\|_{X_Q^*} := \sup_{\|b\|_{X_Q}=1} \langle f, b \rangle_{X^* - X}$. We also denote $\mathcal{A} := \{\omega \in \mathbb{S}_Y: Q\omega \in \mathbb{S}_{X_Q^*}\}$.

In the current setting, the Lagrange multiplier condition

$$\left. \frac{d}{d\epsilon} (\langle b, Q(\omega + \epsilon\varphi) \rangle_{X_Q - X_Q^*} - \|\omega + \epsilon\varphi\|_Y^2) \right|_{\epsilon=0} = 0 \quad \text{for every } \varphi \in Y$$

is written concisely as $T_b\omega = \omega^*$, where ω^* is the unique element of Y^* such that $\langle \omega, \omega^* \rangle_{Y-Y^*} = \|\omega\|_Y \|\omega^*\|_{Y^*}$. (In particular, $D(\omega) = \{\omega^*\}$ for all $\omega \in \mathbb{S}_Y$.) This is seen by following the proof of Proposition 4.9.

7.2 Results

We list results of this paper that allow a straightforward adaptation to the current setting: Theorem 4.3; Corollaries 4.7, 4.8, 4.13 and 5.3; Propositions 2.10, 4.1, 4.2, 4.5, 4.9 and 4.12; Lemmas 2.9, 4.10 and 5.1.

Theorem 1.9 (i) has at least the weaker variant that $D: \mathbb{S}_{X_Q} \rightarrow \mathbb{S}_{X_Q^*}$ is point-to-compact. It is natural to ask whether, again, $\text{co}(\text{ext}(D(b))) = D(b)$ for all $b \in \mathbb{S}_{X_Q}$. Since $\text{ext}(\mathbb{B}_{X_Q^*}) \subset Q\mathcal{A}$, we could then write each $f \in \mathbb{S}_{X_Q^*}$ as a convex combination of elements $Q\omega$, $\omega \in \mathcal{A}$. By a standard Baire category argument, one would then obtain an upper bound for the number of terms in these sums [23]:

Proposition 7.7 *Suppose $\text{co}(\text{ext}(D(b))) = D(b)$ for all $b \in \mathbb{S}_{X_Q}$. Then there exists $N \in \mathbb{N}$ such that every $f \in X_Q^*$ can be written as*

$$f = \sum_{j=1}^N Q\omega_j, \quad \sum_{j=1}^N \|\omega_j\|_Y^2 \lesssim \|f\|_{X_Q^*}.$$

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Appendix A. A Proof of (3.6)

While formula (3.6) is well-known and classical, the author has been unable to find a proof in the literature, and therefore one is sketched in this appendix. Note that (3.6) is equivalent to the following proposition.

Proposition A.1 *Every $f \in \mathcal{H}^1(\mathbb{R})$ can be written as*

$$f = \omega\gamma - H\omega H\gamma$$

for some $\omega, \gamma \in L^2(\mathbb{R})$. However, there exists $f \in \mathcal{H}^1(\mathbb{R})$ that cannot be written as $f = \omega^2 - (H\omega)^2$ for any $\omega \in L^2(\mathbb{R})$.

The relevant background can be found e.g. in [13] or [46]. Whenever $0 < p < \infty$, we denote

$$\|U\|_p := \sup_{0 < y < \infty} \left(\int_{-\infty}^{\infty} |U(x + iy)|^p dx \right)^{\frac{1}{p}}.$$

The analytic Hardy space $\mathcal{H}^p(\mathbb{C}_+)$ consists of analytic functions $U : \mathbb{C}_+ \rightarrow \mathbb{C}$ with $\|U\|_p < \infty$.

Proof of Proposition A.1 Let $f \in \mathcal{H}^1(\mathbb{R})$; then the Hilbert transform $Hf \in L^1(\mathbb{R})$. We extend $f + iHf$ analytically into \mathbb{C}_+ by using the Poisson kernel P_y and the conjugate Poisson kernel Q_y ,

$$P_y(x) := \frac{1}{\pi} \frac{y}{x^2 + y^2}, \quad Q_y(x) := \frac{1}{\pi} \frac{x}{x^2 + y^2}.$$

We denote

$$U_1(x + iy) + iU_2(x + iy) := f * P_y(x) + i f * Q_y(x);$$

then $U = U_1 + iU_2$ belongs to $\mathcal{H}^1(\mathbb{C}_+)$ and, furthermore, $\lim_{y \searrow 0} \|U_1(\cdot, y) - f\|_{L^1} = 0$ and $\lim_{y \searrow 0} \|U_2(\cdot, y) - Hf\|_{L^1} = 0$.

We form the Blaschke product B of U and write $U = B\Phi$, where $\Phi \in \mathcal{H}^1(\mathbb{C}_+)$ is zero-free. Since \mathbb{C}_+ is simply connected, Φ has an analytic square root $\Phi^{1/2}$. Set

$$V := B\Phi^{1/2} \in \mathcal{H}^2(\mathbb{C}_+) \quad \text{and} \quad W := \Phi^{1/2} \in \mathcal{H}^2(\mathbb{C}_+).$$

Then there exist $\omega, \gamma \in L^2(\mathbb{R})$ such that

$$\begin{aligned} V_1(\cdot + iy) + iV_2(\cdot + iy) &\rightarrow \omega + iH\omega, \\ W_1(\cdot + iy) + iW_2(\cdot + iy) &\rightarrow \gamma + iH\gamma \end{aligned}$$

in $L^2(\mathbb{R})$ as $y \searrow 0$. As a consequence,

$$\mathbf{Re}[VW] = V_1W_1 - V_2W_2 \rightarrow \omega\gamma - H\omega H\gamma$$

in $L^1(\mathbb{R})$ as $y \searrow 0$. On the other hand,

$$\mathbf{Re}[VW] = \mathbf{Re} U \rightarrow f$$

in $L^1(\mathbb{R})$ as $y \searrow 0$. Thus $f = \omega\gamma - H\omega H\gamma$.

We then find $f \in \mathcal{H}^1(\mathbb{R})$ that cannot be written as $f = \omega^2 - (H\omega)^2$ for any $\omega \in L^2(\mathbb{R})$. Choose $U \in \mathcal{H}^1(\mathbb{C}_+)$ that has at least one zero of odd order. (Given $W \in \mathcal{H}^1(\mathbb{C}_+)$ one may set $U(z) = [(z - z_0)/(z - \bar{z}_0)]W(z)$, where $z_0 \in \mathbb{C}_+$ and

$W(z_0) \neq 0$.) Then U is not the square of any analytic function. We write $f := \lim_{y \searrow 0} U_1 \in \mathcal{H}^1(\mathbb{R})$ where the limit holds in $L^1(\mathbb{R})$.

Seeking a contradiction, assume that $f = \omega^2 - (H\omega)^2$ for some $\omega \in L^2(\mathbb{R})$. Then there exists $V = V_1 + iV_2 \in \mathcal{H}^2(\mathbb{C}_+)$ such that $\lim_{y \searrow 0} \|V_1(\cdot + iy) - \omega\|_{L^2} = 0$. Now $V^2 \in \mathcal{H}^1(\mathbb{C}_+)$ and

$$\operatorname{Re}[V^2] = V_1^2 - V_2^2 \rightarrow \omega^2 - (H\omega)^2 = f$$

in $L^1(\mathbb{R})$ as $y \searrow 0$. Thus $U - V^2 \in \mathcal{H}^1(\mathbb{C}_+)$ has vanishing boundary values at $y = 0$, and so $U = V^2$, which yields a contradiction. \square

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