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Orponen, Tuomas

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ON THE DISTANCE SETS OF AD-REGULAR SETS

TUOMAS ORPONEN

ABSTRACT. I prove that if $\emptyset \neq K \subset \mathbb{R}^2$ is a compact s-Ahlfors-David regular set with $s \geq 1$, then

$$\dim_p D(K) = 1,$$

where $D(K) := \{|x - y| : x, y \in K\}$ is the distance set of K, and $\dim_p$ stands for packing dimension.

The same proof strategy applies to other problems of similar nature. For instance, one can show that if $\emptyset \neq K \subset \mathbb{R}^2$ is a compact s-Ahlfors-David regular set with $s \geq 1$, then there exists a point $x_0 \in K$ such that $\dim_p K \cdot (K - x_0) = 1$. Specialising to product sets, one derives the following sum-product corollary: if $A \subset \mathbb{R}$ is a non-empty compact s-Ahlfors-David regular set with $s \geq 1/2$, then

$$\dim_p[A(A - a_1) + A(A - a_2)] = 1$$

for some $a_1, a_2 \in A$. In particular, $\dim_p[AA + AA - AA - AA] = 1$. In all of the results mentioned above, compactness can be relaxed to boundedness and $\mathcal{H}^s$-measurability, if packing dimension is replaced by upper box dimension.

1. INTRODUCTION

Given a planar set K, the *distance set problem* asks for a relationship between the size of K, and the size of the distance set

$$D(K) := \{|x - y| : x, y \in K\}.$$

For finite sets K, the problem is due to P. Erdős from 1946, and the *Erdős distance conjecture* states that the cardinality of $D(K)$ should satisfy $|D(K)| \gtrsim |P|/\sqrt{\log |P|}$. L. Guth and N. Katz [4] nearly resolved the question in 2011 by showing that $|D(K)| \gtrsim |P|/\log |P|$.

The "continuous" version of the distance set problem was proposed by K. Falconer [3] in 1985. The *Falconer distance conjecture* claims that if $K \subset \mathbb{R}^2$ is a Borel set with $\dim K > 1$, then $D(K)$ has positive length. As far as I know, the current records in this setting are the following theorems of T. Wolff [9] from 1999 and J. Bourgain [2] from 2003:

Theorem 1.1 (Wolff). *If $K \subset \mathbb{R}^2$ is Borel with $\dim K > 4/3$, then $D(K)$ has positive length.*

Theorem 1.2 (Bourgain). *If $K \subset \mathbb{R}^2$ is Borel with $\dim K \geq 1$, then $\dim_H D(K) \geq 1/2 + \epsilon$ for some (small) absolute constant $\epsilon > 0$.*

In Bourgain's result, $\dim_H$ stands for Hausdorff dimension. The following theorem is the main result of this note:

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Theorem 1.3. *Assume that $\emptyset \neq K \subset \mathbb{R}^2$ is a compact s-Ahlfors-David-regular set with $s \geq 1$. Then*
$$\dim_{\mathrm{p}} D(K) = 1,$$
where $D(K) = \{|x-y| : x, y \in K\}$ is the distance set of K.

The proof of Theorem 1.3 can be easily modified to give various other results of similar nature. I have Alex Iosevich to thank for the following suggestions:

Theorem 1.4. *Assume that $\emptyset \neq K \subset \mathbb{R}^2$ is a compact s-Ahlfors-David-regular set with $s \geq 1$. Then, there exists a point $x_0 \in K$ such that*
$$\dim_{\mathrm{p}} K \cdot (K - x_0) = \dim_{\mathrm{p}}\{x_1 \cdot (x_2 - x_0) : x_1, x_2 \in K\} = 1.$$

Corollary 1.5. *Assume that $\emptyset \neq A \subset \mathbb{R}$ is a compact Ahlfors-David regular set with $\dim_{\mathrm{H}} A \geq 1/2$. Then, there exist points $a_1, a_2 \in A$ such that*
$$\dim_{\mathrm{p}}[A(A - a_1) + A(A - a_2)] = 1.$$

Theorem 1.3 is proved in Section 4, after the necessary tools have been developed in Section 3. The small modifications needed to prove Theorem 1.4 and Corollary 1.5 are discussed in Section 5. We conclude the introduction by defining the some basic concepts.

Definition 1.6 (Packing and box dimensions)**.** Above, $\dim_{\mathrm{p}}$ stands for packing dimension. To define it, we first write
$$\overline{\dim}_{\mathrm{B}} A := \limsup_{\delta \to 0} \frac{\log N(A, \delta)}{-\log \delta}$$
for bounded sets $A \subset \mathbb{R}^d$, where $N(A, \delta)$ is the least number of δ-balls required to cover A. The dimension $\overline{\dim}_{\mathrm{B}} A$ is commonly known as upper box dimension or upper Minkowski dimension. Then, $\dim_{\mathrm{p}}$ is defined by
$$\dim_{\mathrm{p}}(B) = \inf \left\{ \sup \overline{\dim}_{\mathrm{B}} F_i : B \subset \bigcup_i F_i, \ F_i \text{ closed} \right\}.$$
Since $\overline{\dim}_{\mathrm{B}} A = \overline{\dim}_{\mathrm{B}} \bar{A}$ for all bounded sets A, one has $\dim_{\mathrm{p}} A \leq \overline{\dim}_{\mathrm{B}} A$ for all bounded sets A. The converse need not hold even for compact sets, since
$$\dim_{\mathrm{p}}\{0, 1, \tfrac{1}{2}, \tfrac{1}{3}, \ldots\} = 0 < 1/2 = \overline{\dim}_{\mathrm{B}}\{0, 1, \tfrac{1}{2}, \tfrac{1}{3}, \ldots\}.$$

Definition 1.7 (AD-regular sets and measures)**.** A Borel measure μ on $\mathbb{R}^d$ is said to be (s, A)-Ahlfors-David regular – or (s, A)-AD-regular in short – if
$$\frac{r^s}{A} \leq \mu(B(x, r)) \leq Ar^s$$
for all $x \in \operatorname{spt} \mu$ and $0 < r \leq \operatorname{diam}(\operatorname{spt} \mu)$. An $\mathcal{H}^s$-measurable set $K \subset [0, 1]$ is called (s, A)-AD regular, if $0 < \mathcal{H}^s(K) < \infty$, and the restriction $\mu := \mathcal{H}^s|_K$ of $\mathcal{H}^s$ to K is (s, A)-AD regular.

2. Acknowledgements

I am grateful to Alex Iosevich for pointing out that Theorem 1.4 and Corollary 1.5 follow from the proof strategy employed in Theorem 1.3. In fact, an earlier version of the manuscript claimed even stronger versions of Theorem 1.4 and Corollary 1.5, but Pablo Shmerkin pointed out an error in the argument; I'm very thankful for his careful reading!

3. Preliminaries on entropy and projections

Many of the arguments in this section are repeated from [8], where, further, the discussion closely followed that of M. Hochman's paper [5].

Definition 3.1 (Measures and their blow-ups in $\mathbb{R}^d$)**.** Let $\mathcal{P}(\Omega)$ stand for the space of Borel probability measures on Ω. In what follows, Ω will be $\mathbb{R}^d$, or a cube in $\mathbb{R}^d$, and $d \in \{1,2\}$. If $Q = r[0,1)^d + a$ is a cube in $\mathbb{R}^d$, let $T_Q(x) := (x-a)/r$ be the unique homothety taking Q to $[0,1)^d$. Given a measure $\mu \in \mathcal{P}(\mathbb{R}^d)$ and a cube Q as above, with $\mu(Q) > 0$, define the measures

$$\mu_Q := \frac{1}{\mu(Q)}\mu|_Q \quad \text{and} \quad \mu^Q := T_{Q\sharp}(\mu_Q),$$

where $\mu|_Q$ is the restriction of μ to Q, and $T_{Q\sharp}$ is the push-forward under T_Q. So, μ^Q is a "blow-up" of μ_Q into $[0,1)^d$.

Definition 3.2 (Entropy)**.** Let $\mu \in \mathcal{P}(\Omega)$, and let $\mathcal{F}$ be a countable μ-measurable partition of Ω. Set

$$H(\mu, \mathcal{F}) := -\sum_{F \in \mathcal{F}} \mu(F) \log \mu(F),$$

where the convention $0 \cdot \log 0 := 0$ is used. If $\mathcal{E}$ and $\mathcal{F}$ are two μ-measurable partitions, one also defines the conditional entropy

$$H(\mu, \mathcal{E}|\mathcal{F}) := \sum_{F \in \mathcal{F}} \mu(F) \cdot H(\mu_F, \mathcal{E}),$$

where $\mu_F := \mu|_F/\mu(F)$, if $\mu(F) > 0$.

The notion of conditional entropy is particularly useful, when $\mathcal{E}$ *refines* $\mathcal{F}$, which means that every set in $\mathcal{E}$ is contained in a (unique) set in $\mathcal{F}$:

Proposition 3.3 (Conditional entropy formula)**.** *Assume that $\mathcal{E}, \mathcal{F}$ are partitions as in Definition 3.2, and $\mathcal{E}$ refines $\mathcal{F}$. Then*

$$H(\mu, \mathcal{E}|\mathcal{F}) = H(\mu, \mathcal{E}) - H(\mu, \mathcal{F}).$$

In particular, $H(\mu, \mathcal{E}) \geq H(\mu, \mathcal{F})$.

Proof. For $F \in \mathcal{F}$, let $\mathcal{E}(F) := \{E \in \mathcal{E} : E \subset F\}$. A direct computation gives

$$\begin{aligned}
H(\mu, \mathcal{E}|\mathcal{F}) &= -\sum_{F \in \mathcal{F}} \mu(F) \cdot \sum_{E \in \mathcal{E}} \mu_F(E) \log \mu_F(E) \\
&= -\sum_{F \in \mathcal{F}} \sum_{E \in \mathcal{E}(F)} \mu(E) \log \frac{\mu(E)}{\mu(F)} \\
&= -\left(\sum_{E \in \mathcal{E}} \mu(E) \log \mu(E) - \sum_{F \in \mathcal{F}} \log \mu(F) \sum_{E \in \mathcal{E}(F)} \mu(E) \right) \\
&= H(\mu, \mathcal{E}) + \sum_{F \in \mathcal{F}} \mu(F) \log \mu(F) = H(\mu, \mathcal{E}) - H(\mu, \mathcal{F}),
\end{aligned}$$

as claimed. □

The partitions $\mathcal{E}, \mathcal{F}$ used below will be the dyadic partitions of $\mathbb{R}^d$: $\mathcal{E}, \mathcal{F} = \mathcal{D}_n$. The lemma below contains three more useful and well-known – or easily verified – properties of entropy. The items are selected from [5, Lemma 3.1] and [5, Lemma 3.2].

Lemma 3.4. *Let $\mathcal{E}, \mathcal{F}$ be countable μ-measurable partitions of Ω.*

(i) *The functions $\mu \mapsto H(\mu, \mathcal{E})$ and $\mu \mapsto H(\mu, \mathcal{E}|\mathcal{F})$ are concave.*

(ii) *If $\operatorname{spt}\mu \subset B(0,R)$, and $f, g\colon B(0,R) \to \mathbb{R}$ are functions so that $|f(x) - g(x)| \leq R2^{-n}$ for $x \in B(0,R)$, then*
$$|H(f_\sharp\mu, \mathcal{D}_n) - H(g_\sharp\mu, \mathcal{D}_n)| \leq C,$$
where $C > 0$ only depends on R.

(iii) *If every set in $\mathcal{E}$ meets at most $C \geq 1$ sets in $\mathcal{F}$ and vice versa, then*
$$|H(\mu, \mathcal{E}) - H(\mu, \mathcal{F})| \lesssim_C 1.$$

Let $\mathcal{D}_n$ be the family of dyadic cubes of side-length 2^{-n} in $\mathbb{R}^d$ (the notation will be used in both $\mathbb{R}^2$ and $\mathbb{R}$). For $n \in \mathbb{N}$, write H_n for the *normalised scale 2^{-n}-entropy*
$$H_n(\mu) := \frac{1}{\log 2^n} \cdot H(\mu, \mathcal{D}_n) = \sum_{Q \in \mathcal{D}_n} \mu(Q) \cdot \left(\frac{\log \mu(Q)}{\log 2^{-n}}\right).$$

Now, all the definitions and tools are in place to state and prove the key auxiliary result from Hochman's paper, namely [5, Lemma 3.5], in slightly modified form:

Lemma 3.5. *Let $\mu \in \mathcal{P}([0,1]^2)$ and $m, n \in \mathbb{N}$ with $m < n$. Then, for any continuous mapping $f\colon [0,1]^2 \to \mathbb{R}$,*
$$H_n(f_\sharp\mu) \geq \frac{1}{n} \sum_{k=0}^{\lfloor n/m \rfloor - 1} \sum_{Q \in \mathcal{D}_{km}} \mu(Q) \cdot H(f_\sharp\mu_Q, \mathcal{D}_{(k+1)m} | \mathcal{D}_{km}).$$
The sum $Q \in \mathcal{D}_{km}$ only runs over those Q with $\mu(Q) > 0$.

Proof. Write $n = k_0 m + r$, where $0 \leq r < m$, and $k_0 = \lfloor n/m \rfloor$. Then
$$H(f_\sharp\mu, \mathcal{D}_n) \geq H(f_\sharp\mu, \mathcal{D}_{k_0 m}) = \sum_{k=0}^{k_0 - 1} H(f_\sharp\mu, \mathcal{D}_{(k+1)m} | \mathcal{D}_{km}) + H(f_\sharp\mu, \mathcal{D}_0)$$
$$\geq \sum_{k=0}^{k_0 - 1} H(f_\sharp\mu, \mathcal{D}_{(k+1)m} | \mathcal{D}_{km})$$
by repeated application of Proposition 3.3. Next, since $f_\sharp\colon \mathcal{P}([0,1]^2) \to \mathcal{P}(\mathbb{R})$ is linear (even if f is not), one has
$$f_\sharp\mu = f_\sharp\left(\sum_{Q \in \mathcal{D}_{km}} \mu|_Q\right) = \sum_{Q \in \mathcal{D}_{km}} f_\sharp(\mu|_Q) = \sum_{Q \in \mathcal{D}_{km}} \mu(Q) \cdot f_\sharp\mu_Q,$$
so, by Jensen's inequality and the concavity of (conditional) entropy,
$$H(f_\sharp\mu, \mathcal{D}_{(k+1)m} | \mathcal{D}_{km}) \geq \sum_{Q \in \mathcal{D}_{km}} \mu(Q) \cdot H(f_\sharp\mu_Q, \mathcal{D}_{(k+1)m} | \mathcal{D}_{km}).$$
Dividing by n completes the proof. □

Remark 3.6. In case f is linear, say $f = \pi_e$ for some $e \in S^1$, where π_e is the orthogonal projection $\pi_e(x) = x \cdot e$, the lemma can be taken a step further. Observe that

$$H(\pi_{e\sharp}\mu_Q, \mathcal{D}_{(k+1)m}|\mathcal{D}_{km}) = H(\pi_{e\sharp}\mu^Q, \mathcal{D}_m|\mathcal{D}_0) = H(\pi_{e\sharp}\mu^Q, \mathcal{D}_m) - H(\pi_{e\sharp}\mu^Q, \mathcal{D}_0),$$

by the linearity of π_e and Proposition 3.3. Here $H(\pi_{e\sharp}\mu^Q, \mathcal{D}_0) \leq 3$, because $\pi_{e\sharp}\mu^Q$ is supported in an interval of length $\sqrt{2}$. So,

$$H(\pi_{e\sharp}\mu_Q, \mathcal{D}_{(k+1)m}|\mathcal{D}_{km}) \geq m \cdot H_m(\pi_{e\sharp}\mu^Q) - 3. \tag{3.7}$$

3.1. An entropy version of Marstrand's theorem.

To estimate the quantity on the right hand side of (3.7) from below, one needs the following Marstrand type projection result:

Proposition 3.8. *Assume that $\mu \in \mathcal{P}([0,1]^2)$ satisfies the linear growth condition $\mu(B(x,r)) \leq Ar$ for $x \in \mathbb{R}^2$, $r > 0$ and some $A \geq 1$. Then*

$$\int_{S^1} H_m(\pi_{e\sharp}\mu)\,d\sigma(e) \geq s - ACm \cdot 2^{(s-1)m}, \qquad 0 < s < 1,$$

where σ is the unit-normalised length measure on S^1, and $C > 0$ is an absolute constant.

Proof. Fix $m \in \mathbb{N}$. It follows from the linear growth condition for μ that

$$\int_{S^1} 2^m \sum_{Q \in \mathcal{D}_m} [\pi_{e\sharp}\mu(Q)]^2\,d\sigma(e) \lesssim Am. \tag{3.9}$$

This is standard, so I only sketch the details: observe that for any $\nu \in \mathcal{P}([0,1]^2)$

$$\int_{S^1} \|\pi_{e\sharp}\nu\|_2^2\,d\sigma(e) = \int_{S^1}\int_{\mathbb{R}} |\hat{\nu}(te)|^2\,dt\,d\sigma(e)$$
$$\sim \int_{\mathbb{R}^2} |\hat{\nu}(\xi)|^2|\xi|^{-1}\,d\xi \sim \iint \frac{d\nu x\,d\nu y}{|x-y|} =: I_1(\nu).$$

Apply this with $\nu := \mu * \psi_m$, where $\psi_m(x) := 2^{2m}\psi(2^m x)$ and ψ is a radial bump function with $\chi_{B(0,5)} \leq \psi \leq \chi_{B(0,10)}$. Using the linear growth condition for μ, it is easy to verify that $I_1(\mu * \psi_m) \lesssim Am$, for $A, m \geq 1$. Further, since ψ is radial, the projection $\pi_{e\sharp}(\mu * \psi_m)$ has the form $(\pi_{e\sharp}\mu) * \phi_m$, where ϕ_m is a bump in $\mathbb{R}$ at scale 2^{-m}, independent of e. Finally, the left hand side of (3.9) is controlled by an absolute constant times $\|(\pi_{e\sharp}\mu) * \phi_m\|_2^2$. The inequality now follows by combining all the observations.

Let

$$C_e := 2^m \sum_{Q \in \mathcal{D}_m} [\pi_{e\sharp}\mu(Q)]^2.$$

Then, for $s < 1$ fixed,

$$\pi_{e\sharp}\mu\left(\bigcup\{Q \in \mathcal{D}_m : \pi_{e\sharp}\mu(Q) \geq 2^{-ms}\}\right) \leq C_e 2^{(s-1)m},$$

and so

$$\int_{S^1} \pi_{e\sharp}\mu\left(\bigcup\left\{Q \in \mathcal{D}_m : \frac{\log \pi_{e\sharp}\mu(Q)}{\log 2^{-m}} \leq s\right\}\right) d\sigma(e) \lesssim Am \cdot 2^{(s-1)m}. \tag{3.10}$$

Inspired by (3.10), let

$$\mathcal{D}_m^{e-\text{bad}} := \left\{Q \in \mathcal{D}_m : \frac{\log \pi_{e\sharp}\mu(Q)}{\log 2^{-m}} \leq s\right\},$$

and denote by β_e the $\pi_{e\sharp}\mu$-measure of the e-bad intervals. Then,

$$\int_{S^1} H_m(\pi_{e\sharp}\mu)\,d\sigma(e) \geq \int_{S^1} \sum_{Q \in \mathcal{D}_m \setminus \mathcal{D}_m^{e-bad}} \pi_{e\sharp}\mu(Q) \left(\frac{\log \pi_{e\sharp}\mu(Q)}{\log 2^{-m}}\right)$$
$$\geq \int_{S^1} s(1-\beta_e)\,d\sigma(e) \geq s - ACm \cdot 2^{(s-1)m},$$

as claimed. □

Corollary 3.11. *Let μ be as in Proposition 3.8, and let S_{2^m} be a collection of vectors with $|S_{2^m}| \sim 2^m$ such that every vector $e \in S^1$ is at distance $\lesssim 2^{-m}$ from one of the vectors in S_{2^m}. Then,*

$$\sum_{e \in S_{2^m}} p_e \cdot H_m(\pi_{e\sharp}\mu) \geq s - AC(m \cdot 2^{(s-1)m} + 1/m), \quad 0 < s < 1,$$

where $p_e \sim 2^{-m}$ depends only on S_{2^m}, and the $C \geq 1$ only depends on the constants behind the $\sim$ and $\lesssim$ notation in the hypothesis.

Proof. For each $e \in S_{2^m}$, let

$$J_e := \{\xi \in S^1 : |\xi - e| \leq C2^{-m}\}.$$

If C is large enough, S^1 is contained in the union of the arcs J_e. Let $p_e := \sigma(J_e) \sim 2^{-m}$, and note that

$$|H_m(\pi_{e_1\sharp}\mu) - H_m(\pi_{e_2\sharp}\mu)| \lesssim \frac{1}{m}$$

by Lemma 3.4(ii), whenever $e_1, e_2 \in J_e$ for a fixed $e \in S_{2^m}$. Then,

$$\sum_{e \in S_{2^m}} p_e \cdot H_m(\pi_{e\sharp}\mu) = \sum_{e \in S_{2^m}} \int_{J_e} H_m(\pi_{e\sharp}\mu)\,d\sigma(\xi)$$
$$\geq \sum_{e \in S_{2^m}} \int_{J_e} (H_m(\pi_{\xi\sharp}\mu) - C/m)\,d\sigma(\xi)$$
$$\geq \int_{S^1} H_m(\pi_{\xi\sharp}\mu)\,d\sigma(\xi) - \sum_{e \in S_{2^m}} \frac{Cp_e}{m}$$
$$\geq s - AC(m \cdot 2^{(s-1)m} + 1/m),$$

where the constant C possibly changed between the last two lines. The proof is complete. □

To sum up the progress so far, Lemma 3.5 shows that, for a rather arbitrary function f, we can lower bound the entropy $H_n(f_\sharp\mu)$ by a linear combination of "partial entropies" of the form $H(f_\sharp\mu_Q, \mathcal{D}_{(k+1)m}|\mathcal{D}_{km})$. Further, if f were the orthogonal projection π_e, a combination of equation (3.7) and Corollary 3.11 implies that the terms $H(\pi_{e\sharp}\mu_Q, \mathcal{D}_{(k+1)m}|\mathcal{D}_{km})$ are rather large on average. Next, we record a fairly standard "error estimate", saying that if f and π_e are close, then $H(\pi_{e\sharp}\mu_Q, \mathcal{D}_{(k+1)m}|\mathcal{D}_{km})$ and $H(f_\sharp\mu_Q, \mathcal{D}_{(k+1)m}|\mathcal{D}_{km})$ do not differ much, either:

Lemma 3.12. *Assume that $\mu \in \mathcal{P}([0,1])^2$ is supported on a square $Q \in \mathcal{D}_{km}$, $km \in \mathbb{N}$. Assume that $f\colon \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable, and $|\nabla f(x) - e| \leq 2^{-m}$ for all $x \in Q$, and for some fixed vector $e \in S^1$. Then*

$$|H(\pi_{e\sharp}\mu, \mathcal{D}_{(k+1)m}|\mathcal{D}_{km}) - H(f_\sharp\mu, \mathcal{D}_{(k+1)m}|\mathcal{D}_{km})| \leq C$$

for some absolute constant $C \geq 1$.

Proof. First note that

$$H(\pi_{e\sharp}\mu, \mathcal{D}_{(k+1)m}|\mathcal{D}_{km}) = H(\pi_{e\sharp}\mu, \mathcal{D}_{(k+1)m}) - H(\pi_{e\sharp}\mu, \mathcal{D}_{km})$$

by Proposition 3.3, and the second term is bounded by a constant, since the support of $\pi_{e\sharp}\mu_Q$ is contained in a constant number of cubes (rather: intervals) in $\mathcal{D}_{km}$. Since f is 2-Lipschitz on Q, similar considerations apply to $H(f_\sharp\mu, \mathcal{D}_{(k+1)m}|\mathcal{D}_{km})$. Consequently, it suffices to show that

$$|H(\pi_{e\sharp}\mu, \mathcal{D}_{(k+1)m}) - H(f_\sharp\mu_Q, \mathcal{D}_{(k+1)m})| \leq C.$$

To this end, by Lemma 3.4 (iii) and the assumption on the support of μ, one needs to prove that any set of the form $f^{-1}(I) \cap Q$ with $I \in \mathcal{D}_{(k+1)m}$ can be covered by a constant number of sets $\pi_e^{-1}(I') \cap Q$, $I' \in \mathcal{D}_{(k+1)m}$ and vice versa. Fix $I \in \mathcal{D}_{(k+1)m}$ and consider $x, y \in f^{-1}(I) \cap Q$. Then, for some ξ on the line segment connecting x and y, one has

$$|\pi_e(x) - \pi_e(y)| = |(x-y)\cdot(e - \nabla f(\xi))| + |f(x) - f(y)| \lesssim 2^{-(k+1)m}.$$

This proves that $f^{-1}(I)$ is contained in a bounded number of sets $\pi_e^{-1}(I')$, $I' \in \mathcal{D}_{(k+1)m}$, and the converse inclusion is verified similarly. □

4. Proof of the main theorem

We are prepared to prove Theorem 1.3. We first do so for upper box dimension instead of packing dimension. In other words, we prove the following theorem:

Theorem 4.1. *Assume that $\emptyset \neq K \subset \mathbb{R}^2$ is a bounded $\mathcal{H}^s$-measurable s-AD-regular set with $s \geq 1$. Then $\overline{\dim}_{\mathrm{B}} D(K) = 1$.*

The proof of Theorem 4.1 contains all the main ideas needed for Theorem 1.3. The passage from box dimension to packing dimension only requires a fairly standard argument using Baire's theorem. The details can be found in Section 4.1.

Proof of Theorem 4.1. Let $s \geq 1$, and let $K \subset B(0,1)$ be an s-AD-regular set with $\mathcal{H}^s(K) > 0$. If $s = 1$ and K contains a non-trivial rectifiable part, then a result of Besicovitch and Miller [1] from 1948 tells us immediately that $D(K)$ has positive length: in particular $\dim_{\mathrm{p}} D(K) = \overline{\dim}_{\mathrm{B}} D(K) = 1$. So, one may assume that either $s > 1$, or K is purely 1-unrectifiable. In both cases, the following holds for $\mathcal{H}^s$ almost all $x \in K$: the set of directions

$$R_x := \left\{ \frac{y-x}{|y-x|} : y \in K \setminus \{x\} \right\}$$

is dense in S^1. For $s > 1$, this is a consequence of Marstrand's classical slicing result, see [6] or the exposition in Mattila's book [7, Chapter 10]. For $s = 1$ the claim follows from unrectifiability, see (the proof of) [7, Lemma 15.13]. For convenience, assume that $0 \in K$, and R_0 is dense in S^1.

Let $m \in \mathbb{N}$ be a large integer to be specified later, and choose a 2^{-m}-net of vectors $S_{2^m} \subset R_0$. Also, for each $e \in S_{2^m}$, fix a point $x_e \in K \setminus \{0\}$ such that $x_e/|x_e| = e$. With each such x_e, associate the mapping $f_e(y) := |y - x_e|^2/(2|x_e|)$. The point of this definition is that

$$\nabla f_e(y) = \frac{y - x_e}{|x_e|}, \tag{4.2}$$

so if y ranges in a small square Q_0 with $\mathrm{dist}(0, Q_0) \ll |x_e|$ and $\ell(Q_0) \ll |x_e|$, then $\nabla f_e(y)$ deviates only a bit from the vector $-x_e/|x_e| \in -S_{2^m}$.

Now fix a small dyadic square Q_0 such that $\overline{Q_0}$ contains the origin and such that $\mathcal{H}^s(K \cap Q_0) \geq \ell(Q_0)^s/(10A)$; this is possible since $0 \in K$. Fixing also $0 < t < 1$, the claim is that that if m is large enough (depending on t and the AD-regularity constant A of K), and Q_0 is small enough (depending on the lengths of the vectors x_e selected after m was chosen), we can prove that

$$\sum_{e \in S_{2m}} p_e \cdot N(f_e(K \cap Q_0), \delta) \geq \delta^{-t} \tag{4.3}$$

for all small enough $\delta > 0$ (depending on $\ell(Q_0)$). The constants $p_e \sim 2^{-m}$ were defined in Corollary 3.11. Inequality (4.3) clearly implies Theorem 4.1.

To prove (4.3), let

$$\mu := (\mathcal{H}^s|_K)_{Q_0} = \frac{1}{\mathcal{H}^s(K \cap Q_0)} \mathcal{H}^s|_{K \cap Q_0}.$$

Recalling that $\mathcal{H}^s(K \cap Q_0) \geq \ell(Q_0)^s/(10A)$, the measure μ is a well-defined probability measure supported on Q_0. Apply Lemma 3.5 to the mappings f_e to deduce that

$$\sum_{e \in S_{2m}} p_e \cdot H_n(f_{e\sharp}\mu) \geq \frac{1}{n} \sum_{k=0}^{\lfloor n/m \rfloor} \sum_{Q \in \mathcal{D}_{km}} \mu(Q) \left[\sum_{e \in S_{2m}} p_e \cdot H(f_{e\sharp}\mu_Q, \mathcal{D}_{(k+1)m} | \mathcal{D}_{km}) \right].$$

To get our hands on π_e instead of f_e, we apply Lemma 3.12, which says that

$$H(f_{e\sharp}\mu_Q, \mathcal{D}_{(k+1)m} | \mathcal{D}_{km}) \geq H(\pi_{e'\sharp}\mu_Q, \mathcal{D}_{(k+1)m} | \mathcal{D}_{km}) - C, \tag{4.4}$$

as soon as $|\nabla f_e(y) - e'| \leq 2^{-m}$ for $y \in Q_0$. With $e' = -e$, equation (4.2) guarantees that this can be arranged by choosing $\ell(Q_0)$ small enough depending only on $|x_e|$ and m (we will have no needs for the precise bounds on $\ell(Q_0)$). After applying (4.4) and (3.7), we end up with

$$\begin{aligned}\sum_{e \in S_{2m}} p_e \cdot H_n(f_{e\sharp}\mu) &\geq \frac{1}{n} \sum_{k=0}^{\lfloor n/m \rfloor} \sum_{Q \in \mathcal{D}_{km}} \mu(Q) \left[\sum_{e \in -S_{2m}} p_e \cdot (m \cdot H_m(\pi_{e\sharp}\mu^Q) - 3 - C) \right] \\ &\geq \left[\frac{m}{n} \sum_{k=0}^{\lfloor n/m \rfloor} \sum_{Q \in \mathcal{D}_{km}} \mu(Q) \sum_{e \in -S_{2m}} p_e \cdot H_m(\pi_{e\sharp}\mu^Q) \right] - \frac{3C}{m},\end{aligned} \tag{4.5}$$

assuming that $C \geq 3$ in (4.4) and recalling that $p_e \sim 1/|S_{2^m}|$.

Next, a straightforward calculation shows that that for $Q \subset Q_0$ with $\mu(Q) > 0$, the measures μ^Q satisfy the growth condition $\mu^Q(B(x,r)) \leq C_Q r^s$, $0 < r \leq 1$, for some constant $C_Q \lesssim A\ell(Q)^s/\mathcal{H}^s(K \cap Q)$.[1] Here A is the AD-regularity constant of K. In particular, since $s \geq 1$, they also satisfy the linear growth condition

$$\mu^Q(B(x,r)) \leq \left(\frac{AC\ell(Q)^s}{\mathcal{H}^s(K \cap Q)} \right) r, \qquad 0 < r \leq 1.$$

[1] One cannot quite say that the measures μ^Q are (s, C_Q)-AD-regular, because the lower bound on the measure of balls may fail close to the boundary of $[0,1)^2$.

Let k_0 be the smallest integer such that $2^{-k_0 m} \leq \ell(Q_0)$ (this implies that if $k \geq k_0$ and $Q \in \mathcal{D}_{km}$ satisfies $\mu(Q) > 0$, then $Q \subset Q_0$, and hence the previous discussion is valid). Applying Corollary 3.11 with $-S_{2^m}$ in place of S_{2^m}, and some $t' \in (t, 1)$, the quantity in brackets on line (4.5) is bounded from below as follows:

$$\frac{m}{n} \sum_{k=k_0}^{\lfloor n/m \rfloor} \sum_{\substack{Q \in \mathcal{D}_{km} \\ \mu(Q)>0}} \mu(Q) \left(t' - \left(\frac{AC\ell(Q)^s}{\mathcal{H}^s(K \cap Q)} \right) (m \cdot 2^{(t'-1)m} + \frac{1}{m}) \right)$$

$$= t' \cdot \frac{m}{n} \cdot (\lfloor n/m \rfloor - k_0) - \frac{ACm}{\mathcal{H}^s(K \cap Q_0) n} \sum_{k=k_0}^{\lfloor n/m \rfloor} \sum_{\substack{Q \in \mathcal{D}_{km} \\ \mu(Q)>0}} \ell(Q)^s \left(m \cdot 2^{(t'-1)m} + \frac{1}{m} \right).$$

To proceed further, observe that, for any fixed generation of squares Q with $\ell(Q) = r \leq \ell(Q_0)$, there are at most $A^3 C \mathcal{H}^s(K \cap Q_0)/r^s$ squares $Q \subset Q_0$ such that $\mu(Q) > 0$. Indeed, by the (s, A)-AD regularity of K, each square Q with $\mu(Q) > 0$ is adjacent to a "good" square Q' with $\ell(Q') = r$ and $\mathcal{H}^s(K \cap Q') \geq r^s/(10A)$. It is possible that such a Q' lies outside Q_0, but it is certainly contained in $2Q_0$; thus, recalling that $\mathcal{H}^s(K \cap Q_0) \geq \ell(Q_0)^s/(10A)$, the number of "good" squares is bounded by

$$\frac{10A\mathcal{H}^s(K \cap 2Q_0)}{r^s} \leq \frac{100A^2 \ell(Q_0)^s}{r^s} \leq \frac{1000A^3 \mathcal{H}^s(K \cap Q_0)}{r^s}.$$

Since each of the "good" square Q' is adjacent to at most eight squares Q with $\mu(Q) > 0$, the number of these Q is bounded by $8000A^3\mathcal{H}^s(K \cap Q_0)/r^s$.

Combining everything so far, one has the estimate

$$\sum_{e \in S_{2^m}} p_e \cdot H_n(f_{e\sharp}\mu) \geq t' \cdot \frac{m}{n} \cdot (\lfloor n/m \rfloor - k_0) - \frac{A^4 C m}{n} \cdot \lfloor n/m \rfloor \cdot (m \cdot 2^{(t'-1)m} + 1/m) - \frac{3C}{m},$$

valid for any $t \leq t' < 1$ and n so large that $n/m > k_0$. Specialising to $t' := (1+t)/2$, say, and choosing $m \geq m(A, t)$, where $m(A, t) \in \mathbb{N}$ depends only on A and t, one obtains

$$\sum_{e \in S_{2^m}} p_e \cdot H_n(f_{e\sharp}\mu) \geq t$$

for large enough n (so large that $(m/n) \cdot (\lfloor n/m \rfloor - k_0)$ is close enough to 1). Via the following lemma, this implies (4.3). The proof of Theorem 4.1 is complete. □

Lemma 4.6. *Let $\nu \in \mathcal{P}(\mathbb{R}^d)$, and assume that $H_n(\nu) \geq s$. Then*

$$|\{Q \in \mathcal{D}_n : \nu(Q) > 0\}| > 2^{nt}$$

for any $t < s - 1/(n \log 2)$. In particular, $N(\operatorname{spt} \nu, 2^{-n}) \gtrsim 2^{nt}$ for such t.

Remark 4.7. Note that the converse of the lemma is false: a large covering number certainly does not guarantee large entropy.

Proof of Lemma 4.6. Assume that $|\{Q \in \mathcal{D}_n : \nu(Q) > 0\}| \leq 2^{nt}$ for some t, and let $\mathcal{D}_n^{\lambda-\mathrm{bad}}$, $\lambda \geq 0$, be the cubes $Q \in \mathcal{D}_n$ such that $\nu(Q) \leq 2^{-\lambda n}$. Then

$$\sum_{Q \in \mathcal{D}_n^{\lambda-\mathrm{bad}}} \nu(Q) \leq 2^{(t-\lambda)n}, \qquad \lambda \geq t,$$

so that

$$s \leq H_n(\nu) = \int_0^\infty \nu\left(\bigcup\left\{Q : \frac{\log \nu(Q)}{\log 2^{-n}} \geq \lambda\right\}\right) d\lambda$$
$$\leq t + \int_t^\infty \nu\left(\bigcup\left\{Q : \nu(Q) \leq 2^{-\lambda n}\right\}\right) d\lambda$$
$$\leq t + \int_t^\infty 2^{(t-\lambda)n}\, d\lambda = t + \frac{1}{n\log 2}.$$

This proves the lemma. □

4.1. **Packing dimension.** The purpose of this section is to complete the proof of Theorem 1.3. This builds heavily on the proof of Theorem 4.1 from the previous section. The inequality (4.3) certainly implies that

$$\max_{e \in S_{2^m}} \overline{\dim}_{\mathrm{B}} f_e(K \cap Q_0) \geq t. \tag{4.8}$$

The choice of the square Q_0 was somewhat arbitrary, and the same proof gives the following just as well: if $Q \subset Q_0$ is any square with $\mathcal{H}^s(K \cap Q) \geq \ell(Q)^s/(10A)$, then (4.8) holds with Q in place of Q_0. Now, we claim

$$\max_{e \in S_{2^m}} \dim_{\mathrm{p}} f_e(K \cap \overline{Q_0}) \geq t,$$

assuming that K is compact and purely unrectifiable. This implies that $\dim_{\mathrm{p}} D(K) = 1$, since $t < 1$ is arbitrary. The pure unrectifiability will only be used here to infer that boundaries of squares are $\mathcal{H}^s|_K$-null. Since K is compact and the mappings f_e are continuous, it suffices to prove that

$$\max_{e \in S_{2^m}} \dim_{\mathrm{p}} \overline{f_e(K \cap \operatorname{int} Q_0)} \geq t. \tag{4.9}$$

Note that $K \cap \operatorname{int} Q_0 \neq \emptyset$, since otherwise $\mathcal{H}^s(K \cap \partial Q_0) > 0$. Recall that

$$\dim_{\mathrm{p}}(B) = \inf\left\{\sup \overline{\dim}_{\mathrm{B}} F_i : B \subset \bigcup_i F_i,\ F_i \text{ closed}\right\}.$$

So, if (4.9) fails, one may find $t' < t$ such that each set $\overline{f_e(K \cap \operatorname{int} Q_0)}$, $e \in S_{2^m}$, can be covered by closed sets F_i^e, $i \in \mathbb{N}$, satisfying the uniform bound $\overline{\dim}_{\mathrm{B}} F_i^e \leq t'$.

Write $S_{2^m} := \{e_1, \ldots, e_N\}$. We first study $\overline{f_{e_1}(K \cap \operatorname{int} Q_0)}$. Since $\overline{f_{e_1}(K \cap \operatorname{int} Q_0)}$ is compact and not empty, Baire's theorem says that one of the sets $F_i^{e_1}$, say $F_{i_1}^{e_1}$, has non-empty interior in the relative topology of $\overline{f_{e_1}(K \cap \operatorname{int} Q_0)}$. In other words, there exists an open set $U^{e_1} \subset \mathbb{R}$ such that

$$\emptyset \neq U^{e_1} \cap \overline{f_{e_1}(K \cap \operatorname{int} Q_0)} \subset F_{i_1}^{e_1}.$$

Now, find a dyadic square $Q_1 \subset \operatorname{int} Q_0$ with $\mathcal{H}^s(K \cap Q_1) \geq \ell(Q_1)^s/(10A)$ such that

$$\overline{f_{e_1}(K \cap Q_1)} \subset U^{e_1} \cap \overline{f_{e_1}(K \cap \operatorname{int} Q_0)} \subset F_{i_1}^{e_1}. \tag{4.10}$$

This is easy: since U^{e_1} is open, one can find $x \in K \cap \operatorname{int} Q_0$ such that $f(x) \in U^{e_1} \cap f_{e_1}(K \cap \operatorname{int} Q_0)$. Then, the AD-regularity of K shows that the desired square $Q_1 \subset \operatorname{int} Q_0$ can be found inside $B(x, r) \subset \operatorname{int} Q_0$ for some small enough radius $r > 0$.

The next step is to apply (4.8) (rather the discussion just below it) to the square Q_1: there exists some index $j \in \{1, \ldots, N\}$ such that

$$\overline{\dim}_{\mathrm{B}} f_{e_j}(K \cap Q_1) \geq t.$$

Certainly $j \neq 1$ by (4.10) and the uniform bound $\overline{\dim}_{\mathrm{B}} F_i^e \leq t' < t$.

Assume for instance that $j = 2$. The set $K \cap \operatorname{int} Q_1$ is non-empty, and $\overline{f_{e_2}(K \cap \operatorname{int} Q_1)}$ is covered by the sets $F_i^{e_2}$. By Baire's theorem again, there is an index i_2 and an open set U^{e_2} such that

$$\emptyset \neq U^{e_2} \cap \overline{f_{e_2}(K \cap \operatorname{int} Q_1)} \subset F_{i_2}^{e_2}.$$

And, as before, one can find a square $Q_2 \subset \operatorname{int} Q_1$ such that $\mathcal{H}^s(K \cap Q_2) \geq \ell(Q_2)^s/(10A)$ and

$$\overline{f_{e_2}(K \cap Q_2)} \subset U^{e_2} \cap \overline{f_{e_2}(K \cap \operatorname{int} Q_1)} \subset F_{i_2}^{e_2}. \tag{4.11}$$

Once again, there exists an index $j \in \{1, \ldots, N\}$ such that

$$\overline{\dim}_{\mathrm{B}} f_{e_j}(K \cap Q_2) \geq t.$$

But now $j \notin \{1, 2\}$ by (4.10) and (4.11) combined.

Repeating the same argument for some N steps eventually produces a square $Q_N \subset Q_0$ with $\mathcal{H}^s(K \cap Q_N) \geq \ell(Q_N)^s/(10A)$, and $f_{e_j}(K \cap Q_N) \subset F_{i_j}^{e_j}$ for all $1 \leq j \leq N$, and for some $i_j \in \mathbb{N}$. This contradicts the fact that

$$\max_j \overline{\dim}_{\mathrm{B}} f_{e_j}(K \cap Q_N) \geq t,$$

and the proof of Theorem 1.3 is complete.

5. Further results

In this section, I discuss the proofs of Theorem 1.4 and Corollary 1.5. One first proves a box dimension variant of Theorem 1.4:

Theorem 5.1. *Assume that $\emptyset \neq K \subset \mathbb{R}^2$ is a bounded $\mathcal{H}^s$-measurable s-AD-regular set with $s \geq 1$. Then, there exists a point $x_0 \in K$ such that*

$$\overline{\dim}_{\mathrm{B}} K \cdot (K - x_0) = 1.$$

Proof. The proof is extremely similar to that of Theorem 4.1, even a bit easier. Once again, the result is elementary if $s = 1$ and K contains a non-trivial rectifiable part: in this case $\mathcal{H}^1(K \cdot (K - x_0)) > 0$ for any $x_0 \in K$. Indeed, either K lies on a line spanned by some vector $e \in S^1$, and then $K \cdot (K - x_0) \supset x \cdot (K - x_0)$ contains an affine copy of K for any $x \in K \setminus \{0\}$. Or else one can find two linearly independent vectors $x_1, x_2 \in K$, in which case one of the orthogonal projections $\pi_{x_1/|x_1|}(K - x_0)$ or $\pi_{x_2/|x_2|}(K - x_0)$ has positive length, since otherwise $K - x_0$ (hence K itself) would be purely 1-unrectifiable. Then, it suffices to note that $K \cdot (K - x_0)$ contains an affine copy of both of these projections.

So, one may assume that either $s = 1$ and K is purely 1-unrectifiable, or $s > 1$. In both cases, there exists $x_0 \in K$ such that

$$R_{x_0} := \left\{ \frac{y - x_0}{|y - x_0|} : y \in K \setminus \{x_0\} \right\}$$

is dense in S^1.[2] Then, one fixes $t \in (0,1)$ and $m \in \mathbb{N}$ and picks a set of vectors $S_{2^m} \subset R_{x_0}$ as in Corollary 3.11. For each $e \in S_{2^m}$, one locates a point $x_e \in K \setminus \{x_0\}$ such that $(x_e - x_0)/|x_e - x_0| = e$, and then one considers the the family of orthogonal projections π_e, $e \in S_{2^m}$. Since

$$\pi_e(K) = K \cdot e = \frac{K \cdot (x_e - x_0)}{|x_e - x_0|} \subset \frac{K \cdot (K - x_0)}{|x_e - x_0|},$$

it suffices to prove that

$$\sum_{e \in S_{2^m}} p_e \cdot N(\pi_e(K), \delta) \geq \delta^{-t} \tag{5.2}$$

for all small enough $\delta > 0$, if $m = m(A,t) \in \mathbb{N}$ was chosen large enough. This time one does not even need to restrict K to a small square about the origin, and the rest of the proof runs exactly in the same was as that of Theorem 4.1. □

The argument for the packing dimension version of Theorem 1.4 is the same as given in Section 4.1, and there is no point in repeating the details.

Proof of Corollary 1.5. Assume that $1/2 \leq s \leq 1$, and $A \subset \mathbb{R}$ is a compact AD-regular set with $\dim_{\mathrm{H}} A = s$. Then $A \times A \subset \mathbb{R}^2$ is a compact $2s$-AD-regular set, since the product measure $\mathcal{H}^s|_A \times \mathcal{H}^s|_A$ is equivalent to $\mathcal{H}^{2s}|_{A \times A}$ with uniform constants. It now follows from Theorem 1.4 that there exists a point $(a_1, a_2) \in A \times A$ such that

$$1 = \dim_{\mathrm{p}}(A \times A) \cdot (A \times A - (a_1, a_2)) = \dim_{\mathrm{p}}[A(A - a_1) + A(A - a_2)],$$

as claimed. □

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University of Helsinki, Department of Mathematics and Statistics
E-mail address: tuomas.orponen@helsinki.fi

[2] In the proof for distance sets, one could assume that $x_0 = 0$ at this stage. Here one cannot, since there is no obvious reason why the problem would be translation invariant. The erroneous "without loss of generality $x_0 = 0$" statement made its way to an earlier version of the manuscript, and I am grateful to Pablo Shmerkin for pointing out the issue. If $\overline{R_0}$ happens to have non-empty interior, then the proof could be modified to show that $\dim_{\mathrm{p}} K \cdot K = 1$.