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Single-Field Inflation with the Standard Model Higgs

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Tiivistelmä — Referat — Abstract

We study single scalar field inflation with the standard model Higgs boson as the inflaton. We first review the homogeneous and isotropic description of the universe given by the FLRW model as well as the inflation scenario. Then we study how this scenario can can be achieved by a single scalar field minimally coupled to gravity in the slow-roll approximation. Next we study linear perturbation theory around the FLRW background. Here the perturbations are decoupled into scalar, vector and tensor perturbations which allows to study them separately. The split of physical quantities into perturbations around a background introduces gauge degrees of freedom which we address by reviewing gauge transformation of the scalar and tensor perturbations (the latter which turns out to be gauge-independent). We then use the comoving gauge and define, for the scalar perturbations, the gauge-invariant quantity known as the comoving curvature perturbation $\mathcal{R}$. For scalar perturbations the Einstein Field equation yields the Mukhanov-Sasaki equation, which we solve to first order in the slow-roll approximation in terms of the Mukhanov variable ν. We then quantize this variable using canonical quantization and calculate the power spectrum from vacuum fluctuations. We also carry the same analysis for tensor perturbations. With the power spectra at hand we introduce the spectral parameters and discuss current observations and constraints on such parameters.

In Higgs inflation the Standard Model Higgs boson takes the role of the inflaton. Here the Higgs field is also coupled to the Ricci scalar, giving us a non-minimal coupling to gravity. This coupling can be transformed away using a conformal transformation at the expense of a field re-definition. This enables us to use the results reviewed thus far. At tree level we find the inflationary predictions to be in excellent agreement with current observations. However, quantum corrections complicate this picture. We review the tree level unitarity of the model and examine arguments in favour and against it. We also study how quantum corrections can qualitatively change the shape of the potential and the viability of Higgs inflation in each scenario.

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1. Introduction

Over the last century, our understanding of the universe has been revolutionized with the advent of the theory of relativity and quantum physics, as well as the increasing wealth of precise astronomical observations. The ΛCDM (Lambda cold dark matter) model, sometimes called the standard model of Big Bang cosmology, is the simplest model that is able to explain observations such as the cosmic microwave background radiation (CMB), the large-scale structure in the distribution of galaxies and the accelerating expansion of the universe as observed in the light from distant galaxies and supernovae. Despite its successes, the Big Bang model of cosmology still has unresolved problems. Three of them are: the horizon, flatness (also known as oldness) and relic problems. Cosmic inflation is a widely accepted[1] paradigm in cosmology that alleviate these problems. It also provides a mechanism for the generation of the seeds of structure where all structure originates from quantum fluctuations in the early universe. We say inflation in an scenario because it has generic features despite the fact that the details of the particle physics mechanism responsible for it is unknown. One of the most simple cases is that of a single-scalar field minimally coupled to gravity. The Standard Model of elementary particles is the best description of the sub-atomic particles and known interactions. It provides the only known scalar field: the Higgs field. One could, therefore, ask whether inflation could have taken place due to the Higgs field. This is the question we set

[1]For dissenting opinions, see [1, 2, 3, 4] for a discussion sparked after Planck 2013's results.

to study in this thesis.

In §2, we review the main features of the Friedmann-Lemaître-Robertson-Walker model such as the kinematics and dynamics of the expansion of the universe. We then introduce the theory of inflation and see what conditions are necessary for this stage. To finish this chapter, we introduce the slow-roll approximation and discuss how to track scales throught the history of the universe.

In §3, we discuss how to introduce inhomogeneities and anisotropies in the form of perturbations and discuss the basics of cosmological perturbation theory. We study gauge transformations and introduce gauge-invariant quantities, which will allow us to get information about the perturbations generated during inflation.

In §4, we discuss how scalar and tensor perturbations are generated from vacuum fluctuations. For scalar perturbations, we derive the Mukhanov-Sasaki equation and solve it in the slow-roll approximation. To get the initial conditions, we then proceed discuss the quantization process and proceed to obtain the primordial spectrum of scalar perturbations fluctuations. Due to gauge-invariance, the procedure for tensor perturbations is in fact simpler, so we outline the most important steps and results to get the primordial spectrum of tensor perturbations. We then proceed to discuss the features and predictions of slow-roll inflation such as Gaussian perturbations and a nearly scale-invariant spectrum of perturbations. Lastly, we discuss how observations, in particular the Planck mission, constrains the slow-roll predictions.

Finally, in §5 we introduce the Higgs inflation model, where the Standard Model Higgs boson assumes the role of the inflaton. We review the tools to deal with actions of scalar fields non-minimally coupled to gravity, and discuss conformal transformations to bring the non-standard gravitation theory into the standard (general relativity form) in order to be able to use the single-field slow-roll formalism used in this thesis to obtain the tree-level predictions of the model. Then we proceed to discuss some of the challenges involving the quantum corrections to the model. In particular we discuss the unitarity and qualitative changes to the potential.

1.1 Conventions

We use the $(-+++)$ metric signature where the metric tensor in Minkowski space becomes $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$.

Other notation include:

- $\dot{a}$ denote the derivative of a with respect to cosmic time t
- a' denote the derivative of a with respect to conformal time η
- ∇_μ or $_{;\mu}$ denore covariant derivatives with respect to coordinates x^μ
- ∂_μ or $_,$ denote partial derivatives with respect to coordinate x^μ
- Greek indices $(\alpha, \beta, \gamma ...)$ run from 0 to 3, whereas latin indices (i, j, k,...) run from 1 to 3
- $\pounds$ is the Lie derivative
- $\mathcal{L}$ is the Lagrange density, which we shall sometimes refer simply as "the lagrangian"

We use the *reduced Planck mass* $M_{\rm Pl}^2 = \dfrac{1}{8\pi G}$, which in §5 we will set equal to one. We also use natural units where $c = \hbar = 1$.

2. Homogeneous and Isotropic Universe

2.1 FLRW universe

On cosmological scales, gravity is the dominant of the four known fundamental interactions in nature. To date, our best understanding of the phenomena associated with it comes from Einstein's theory of general relativity, which describes gravitation as the result of the geometry of spacetime. As the name implies, this spacetime is something that merges our notion of space and time into the single entity that goes under said name. One of the most profound implications of this theory is that the universe itself is dynamic and not just an arena where physical phenomena takes place. As observations can tell, the universe is very inhomogeneous in the scales of planets, stars and even galaxies. However, current observations also tell us that when we consider much larger structures in the universe in scales greater than 100 MPc the universe looks statistically homogeneous and isotropic. Therefore as a first approximation, we can assume that the universe is exactly homogeneous and isotropic. The model that describes an exactly homogeneous and isotropic universe is the *Friedmann–Lemaître–Robertson–Walker* (FLRW) model. We begin this chapter by reviewing the kinematics and dynamics of this model, including the concepts from unperturbed cosmology that we will use throughout this thesis. Then we proceed to understand some of the problems of the standard big bang model. This will serve as motivation to introduce the idea of inflation - a period of accelerated expandion- and we will see how it addresses these problems. We finish the chapter with additional ideas from inflation that will use later when we consider perturbations around the FLRW metric.

2.1.1 Geometry

The geometry of spacetime is encoded in the metric. To describe a perfectly homogeneous and isotropic universe, we can foliate spacetime into homogeneous and isotropic hypersurfaces of simultaneity (constant time) so that the metric takes the form

$$\mathrm{d}s^2 = -\,\mathrm{d}t + a(t)\,\mathrm{d}\sigma^2\,. \tag{2.1}$$

The time coordinate is called the *cosmic time*, and the function $a(t)$ is the *scale factor*, which tells how the universe expands (or contracts). In Cartesian coordinates (t, x, y, z) the metric can be written as [5, p. 722]

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + a^2(t)\frac{1}{\left(1+\frac{K}{4}r^2\right)^2}\delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j \tag{2.2}$$

where K is a constant related to the curvature of space (i.e. the curvature of $\mathrm{d}\sigma^2$). In this form, the metric is known as the FLRW metric. If $K = 0$, $\mathrm{d}\sigma^2$ is the same as Euclidean space, and the FLRW metric is said to be *spatially flat*. We will be using this choice throughout this thesis since it simplifies the formalism, and is also supported by observations [7]. In this case, (2.2) becomes

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2 + a^2(t)(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2). \tag{2.3}$$

At any time t, the expansion rate of the universe is characterized by the *Hubble parameter*

$$H(t) \equiv \frac{\dot{a}(t)}{a(t)}. \tag{2.4}$$

It has dimensions of 1/time or velocity/distance and it's present value $H_0 \equiv H(t_0)$ is called the *Hubble constant*. A recent analysis reports its value to be [8]

$$H_0 = 72.5 \pm 2.5\,\mathrm{km\,s^{-1}\,Mpc^{-1}}, \tag{2.5}$$

with 68% confidence level (CL). Other observations such as the Planck mission [7] report a slightly lower value. Taking the time derivative of Hubble parameter, we obtain a relation to the expansion of the universe:

$$\frac{\ddot{a}}{a} = \dot{H} + H^2. \tag{2.6}$$

For our purposes, it will be useful to redefine the time coordinate t by introducing the *conformal time* η defined as

$$\mathrm{d}\eta \equiv \frac{1}{a(t)}\,\mathrm{d}t \quad \text{or} \quad \eta = \int_{t_i}^{t} \frac{\mathrm{d}t'}{a(t')}, \tag{2.7}$$

which describes the comoving distance covered by light between an initial hypersurface at time t_i and the hypersurface at time t. Using this reparameterization the spatially flat FLRW metric (2.3) becomes

$$\mathrm{d}s = a^2(\eta)\Big(-\,\mathrm{d}\eta^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\Big). \tag{2.8}$$

We also have the *conformal Hubble parameter*

$$\mathcal{H}(\eta) \equiv \frac{a'(\eta)}{a(\eta)} = a(t)H(t) = \dot{a}(t), \tag{2.9}$$

together with the relation (2.6)

$$\frac{a''}{a} = \mathcal{H}^2\left(1 + \frac{\mathcal{H}'}{\mathcal{H}^2}\right). \tag{2.10}$$

2.1.2 Dynamics

The evolution of the scale factor is governed by the Einstein field equation

$$G_{\mu\nu} = \frac{1}{M_{\mathrm{Pl}}^2} T_{\mu\nu}, \tag{2.11}$$

where $G_{\mu\nu}$ is the *Einstein tensor* and $T_{\mu\nu}$ the *energy-momentum tensor*. The first one describes the structure of space-time whereas the second the matter content. The Einstein tensor is defined as

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}, \tag{2.12}$$

where $R_{\mu\nu}$ and R are the *Ricci tensor* and the *the Ricci scalar* (also known as *scalar curvature*) which are contractions of *Riemann tensor*

$$R^{\rho}{}_{\sigma\mu\nu} \equiv \Gamma^{\rho}_{\nu\sigma,\mu} - \Gamma^{\rho}_{\mu\sigma,\nu} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma} \tag{2.13}$$

$$R_{\mu\nu} \equiv R^{\lambda}{}_{\mu\lambda\nu} \tag{2.14}$$

$$R \equiv g^{\mu\nu} R_{\mu\nu}. \tag{2.15}$$

Here $\Gamma^{\rho}_{\mu\nu}$ are the connection coefficients. In standard general relativity, the connection is the Levi-Civita connection where the connection coefficients are known as the *Christoffel symbols*. These are given in terms of the metric tensor $g_{\mu\nu}$ as

$$\Gamma^{\rho}_{\mu\nu} = g^{\rho\sigma}(g_{\nu\sigma,\mu} + g_{\sigma\mu,\nu} - g_{\mu\nu,\sigma}). \tag{2.16}$$

To solve the Einstein equation, we introduce the details of the FLRW metric and determine the different components of the Einstein and energy-momentum tensors. The Einstein tensor components are [5, p. 357]

$$G^{0}{}_{0} = -3H^2 - 3\frac{K}{a^2} \tag{2.17a}$$

$$G^{i}{}_{j} = -\left(\frac{2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{K}{a^2}\right)\delta^{i}{}_{j} = -\left(2\dot{H} + H^2\frac{K}{a^2}\right)\delta^{i}{}_{j} \tag{2.17b}$$

$$G^{0}{}_{i} = G^{i}{}_{0} = 0. \tag{2.17c}$$

From the symmetries of the FLRW metric, it follows that the energy momentum tensor takes the perfect fluid form, that is

$$T^{\mu}{}_{\nu} = (\rho + p)u^{\mu}u_{\nu} + \delta^{\mu}{}_{\nu}p. \tag{2.18}$$

Using comoving coordinates, the fluid will be at rest and the four-velocity is then $u^\mu = (1, 0, 0, 0)$. Therefore, the components of $T^\mu{}_\nu$ are

$$T^0{}_0 = -\rho \tag{2.19a}$$

$$T^i{}_j = p\delta^i{}_j \tag{2.19b}$$

$$T^i{}_0 = T^0{}_i = 0. \tag{2.19c}$$

Plugging these results in the Einstein field equation in (2.11), we find the *Friedmann equations*

$$\frac{\dot{a}^2}{a^2} + \frac{K}{a^2} = \frac{\rho}{3M_{\rm Pl}^2} \tag{2.20}$$

$$3\frac{\ddot{a}}{a} = -\frac{1}{2M_{\rm Pl}^2}(\rho + 3p), \tag{2.21}$$

where (2.20) is called the first Friedmann equation (or just the Friedmann equation) and (2.21) is sometimes referred as the second Friedmann equation. In terms of the Hubble parameter H, they are given by

$$H^2 = \frac{1}{3M_{\rm Pl}^2}\rho - \frac{K}{a^2} \tag{2.22}$$

$$\dot{H} = -\frac{1}{2M_{\rm Pl}^2}(\rho + p) + \frac{K}{a^2}. \tag{2.23}$$

2.2 Inflation

We now introduce the idea of inflation [9, 10, 11, 12, 13] where we follow [14]. Broadly speaking, inflation is defined as a period of accelerated expansion of the universe

$$\ddot{a} > 0 \quad \Leftrightarrow \quad \text{(Inflation)}. \tag{2.24}$$

To understand better what is meant by this, let us start with the implications for the kinematics of the FLRW model. Using the Hubble parameter (2.4), we find

$$\ddot{a} = \frac{\mathrm{d}\dot{a}}{\mathrm{d}t} > 0 \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{aH}\right) < 0 \quad \Leftrightarrow \quad \text{(Inflation)}, \tag{2.25}$$

that is, a period of accelerated expansion implies a shrinking comoving Hubble length $aH = \mathcal{H}$. This means that for an observer with coordinates fixed with the expansion, the observable universe becomes smaller during inflation. From the relation between the Hubble parameter and the expansion of the universe in (2.6), we find that the condition for inflation in (2.25) implies that

$$\frac{\dot{H}}{H^2} > -1. \tag{2.26}$$

What are the necessary conditions for this period of accelerated expansion? To answer this, we now look at the dynamics of the FLRW model. From the second Friedmann equation (2.21), the requirement for inflation in (2.24) implies that

$$(\rho + 3p) < 0 \quad \Rightarrow \quad p = -\frac{1}{3}\rho \quad \Leftrightarrow \quad \text{(Inflation)}. \tag{2.27}$$

Thus we require our matter content to have negative pressure since we assume that the energy momentum energy ρ is positive. To better understand what this means, we can adopt the field picture where we postulate the existence of a scalar field during this period of accelerated expansion.

2.2.1 Scalar field in an expanding universe

We now discuss how to construct the simplest model of inflation consisting of a scalar field inmersed in a curved background using classical field theory. To do so, we follow [6, 27]. The dynamical variables of the model are the set of fields $\Phi^i(x^\mu)$. The classical solution to the field theory containing such fields will be those that are the critical points of an action S, generally expressed as an integral over space of a Lagrange density (often referred as “the Lagrangian") $\mathcal{L}(\Phi^i, \nabla_\mu \Phi^i)$,

$$S = \int \mathrm{d}^4x \sqrt{-g}\mathcal{L}(\Phi^i, \nabla_\mu \Phi^i), \tag{2.28}$$

where g is the determinant of the metric tensor. These critical points are given by the Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial \Phi^i} - \nabla_\mu \left(\frac{\partial \mathcal{L}}{\partial [\nabla_\mu \Phi^i]} \right) = 0, \tag{2.29}$$

Which follows from the *principle of stationary action* in which we demand that the action is stationary (i.e. is unchanged) to first order under variations of the fields. For standard general relativity in the absence of matter, the Einstein field equations may be derived from *Einstein-Hilbert action*

$$S_{\rm H} = \frac{M_{\rm Pl}^2}{2} \int {\rm d}^4 x \sqrt{-g} R. \tag{2.30}$$

The simplest form of matter which has a negative pressure is a scalar field, so the simplest inflationary models involve a single scalar field. The field responsible for inflation (and the corresponding spin 0 particle) is called the *inflaton* which we shall denote by φ. The simplest Lagrangian that is consistent with the principle of general covariance is such that the kinetic term of the field has the *canonical form* and the field is *minimally coupled*, i.e.

$$\mathcal{L}_\varphi = -\frac{1}{2} g^{\mu\nu} \nabla_\mu \varphi \nabla_\nu \varphi - V(\varphi), \tag{2.31}$$

where $V(\varphi)$ is the potential of the field φ. To incorporate this field into the model, we add the action corresponding to this Lagrangian to the Einstein-Hilbert action (2.30). Doing so we obtain the field theory of a scalar field minimally coupled to gravity

$$S = \int {\rm d}^4 x \sqrt{-g} \left[\frac{M_{\rm Pl}^2}{2} R - \frac{1}{2} g^{\mu\nu} \nabla_\mu \varphi \nabla_\nu \varphi - V(\varphi) \right]. \tag{2.32}$$

variations with respect to the inverse of the inverse metric $g^{\mu\nu}$ yield the Einstein field equations (2.11) with the momentum tensor given by

$$T_{\mu\nu} = -\frac{\partial \mathcal{L}}{\partial (\nabla^\mu \varphi)} \nabla_\nu \varphi + g_{\mu\nu} \mathcal{L}_\varphi. \tag{2.33}$$

Variations with respect to the scalar field φ yields the Euler-Lagrange equation (2.29) which, after plugging the Lagrangian for the minimally coupled scalar field

(2.31), becomes

$$-\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\varphi) + \frac{\mathrm{d}V}{\mathrm{d}\varphi} = 0. \tag{2.34}$$

Using the spatially flat FLRW metric (2.3), this equation becomes

$$\ddot{\varphi} + 3H\dot{\varphi} - \frac{1}{a^2}\nabla^2\varphi + V_{,\varphi} = 0. \tag{2.35}$$

During inflation the inflation φ, just like space, is almost homogeneous such that $\varphi_{,i} = 0$ and thus the inflaton field equation in an expanding universe is

$$\ddot{\varphi} + 3H\dot{\varphi} + V_{,\varphi} = 0. \tag{2.36}$$

Plugging the Lagrangian (2.31) together with the flat FLRW metric (2.3) in (2.33) we find the components of the energy momentum tensor for the inflaton

$$T^\mu{}_\nu = g^{\mu\lambda}\varphi_{,\lambda}\varphi_{,\nu} - \delta^\mu_\nu\left[\frac{1}{2}g^{\rho\sigma}\varphi_{,\rho}\varphi_{,\sigma} + V(\varphi)\right]. \tag{2.37}$$

Therefore, using the components of energy-momentum tensor for a perfect fluid in (2.19), we find

$$\rho = -T^0{}_0 = \frac{1}{2}\dot{\varphi}^2 + V \tag{2.38}$$

$$p = T^i{}_i = \frac{1}{2}\dot{\varphi}^2 - V, \tag{2.39}$$

where $T^i{}_i$ indicates a diagonal component and not an implicit sum. From these equations, we can derive the relation

$$\rho + 3p = 2(\dot{\varphi} - V), \tag{2.40}$$

from which we can see that the negative pressure condition is obtained when the potential dominates over the kinetic term,

$$\dot{\varphi}^2 < V \qquad \Rightarrow \qquad \text{(Negative pressure condition)}. \tag{2.41}$$

Relative to Minkowski spacetime, we see that the expansion of the universe in the FLRW model has the effect of adding a term $3H\dot{\varphi}$ sometimes called the *Hubble* *friction* into the inflaton field equation (2.36), since it slows down the evolution of φ. The idea behind inflation is that the inflaton is initially far from the minimum of $V(\varphi)$ and then the latter pulls the inflaton towards the minimum. If the potential has a suitably flat shape, the friction term makes $\dot{\varphi}$ small enough to satisfy (2.41), even if it was not satisfied initially. Introducing the equation of state parameter

$$w \equiv \frac{p}{\rho}, \tag{2.42}$$

we see that ρ and p in (2.38) and (2.39) give

$$w = \frac{1 - 2V/\dot{\varphi}^2}{1 + 2V/\dot{\varphi}^2} \qquad \Rightarrow \qquad -1 \leq w \leq 1. \tag{2.43}$$

Therefore, if the kinetic term dominates, $w \approx 1$ whereas if the potential dominates, $w \approx -1$. The Friedmann equations (2.20) and (2.21) (with $K = 0$) give

$$\frac{\dot{a}^2}{a^2} = H^2 = \frac{1}{3M_{\text{Pl}}^2}\left(\frac{1}{2}\dot{\varphi}^2 + V\right) \tag{2.44}$$

$$\frac{\ddot{a}}{a} = \dot{H} + H^2 = -\frac{1}{6M_{\text{Pl}}^2}(\dot{\varphi} - V). \tag{2.45}$$

Taking the time derivative of (2.44) and subtracting $\dot{\varphi}$ times (2.45), we obtain another useful identity:

$$\dot{H} = -\frac{1}{2M_{\text{Pl}}^2}\dot{\varphi}^2 \qquad \text{or} \qquad \mathcal{H}^2 - \mathcal{H}' = \frac{1}{2M_{\text{Pl}}}\varphi'^2. \tag{2.46}$$

2.2.2 Slow-roll inflation

We have seen that the expansion of the universe slows down the evolution of the inflaton φ towards the minimum of the potential. In doing so the system can reach a set of conditions known as the *slow-roll conditions*:

$$\dot{\varphi}^2 \ll V, \qquad |\ddot{\varphi}| \ll 3H|\dot{\varphi}|. \tag{2.47}$$

If these conditions hold, then we may apply what is known as the *slow-roll approximation* to the inflation field equation (2.36) and the Friedmann equation (2.44) to obtain the *slow-roll equations*

$$H^2 = \frac{V}{3M_{\rm Pl}^2} \tag{2.48}$$

$$3H\dot{\varphi} = -V_{,\varphi}. \tag{2.49}$$

From these two, we can also derive the relation

$$\frac{\dot{\varphi}}{H} = -M_{\rm Pl}^2 \frac{V_{,\varphi}}{V}. \tag{2.50}$$

Taking the derivative of the slow-roll equations with respect to φ, we find

$$\dot{H} = -\frac{1}{6}\frac{V_{,\varphi}^2}{V} \tag{2.51}$$

$$\ddot{\varphi} = \frac{M_{\rm Pl}^2}{3}\left(\frac{V_{,\varphi\varphi}V_{,\varphi}}{V} - \frac{1}{2}\frac{V_{,\varphi}^3}{V^2}\right). \tag{2.52}$$

Notice that everything now depends on φ via $V(\varphi)$ and its derivatives. Next, we define the *slow-roll parameters* [14]

$$\varepsilon_V \equiv \frac{M_{\rm Pl}^2}{2}\left(\frac{V_{,\varphi}}{V}\right)^2 \tag{2.53a}$$

$$\eta_V \equiv M_{\rm Pl}^2 \frac{V_{,\varphi\varphi}}{V} \tag{2.53b}$$

$$\zeta_V \equiv M_{\rm Pl}^4 \frac{V_{,\varphi\varphi\varphi}V_{,\varphi}}{V^2}, \tag{2.53c}$$

where we use the subscript V to denote the quantities constructed from the potential, and to distinguish the conformal time η from the slow-roll parameter in (2.53b). These parameters characterize the shape of the potential and for the slow-roll approximation to be valid, they necessarily satisfy

$$\varepsilon_V < 1 \quad \text{and} \quad |\eta_V| < 1. \tag{2.54}$$

Notice however, that these are necessary but not sufficient conditions since they only restrict the shape of the potential. Because the full scalar wave equation is second order, we can choose $\dot{\varphi}$ such that the slow-roll conditions are not valid. Using the slow roll parameters (2.53a)-(2.53c) and the slow roll equations (2.48)-(2.50), we can form the following quantities which we use in the following chapters:

$$H^{-2}\dot{H} = \mathcal{H}^{-2}(\mathcal{H}' - \mathcal{H}^2) = -\varepsilon_V \tag{2.55}$$

$$(H^{-1}\dot{\varphi})^2 = (\mathcal{H}^{-1}\varphi')^2 = 2M_{\text{Pl}}^2\varepsilon_V \tag{2.56}$$

$$(H\dot{\varphi})^{-1}\ddot{\varphi} = (\mathcal{H}\varphi')^{-1}(\varphi'' - \mathcal{H}\varphi') = \varepsilon_V - \eta_V \tag{2.57}$$

$$H^{-1}\dot{\varepsilon}_V = \mathcal{H}^{-1}\varepsilon_V' = -2\varepsilon_V(\eta_V - 2\varepsilon_V) \tag{2.58}$$

$$H^{-1}\dot{\eta}_V = \mathcal{H}^{-1}\eta_V' = 2\varepsilon_V\eta_V - \zeta_V. \tag{2.59}$$

We can now solve for the background evolution by looking at how $\mathcal{H}$ and a evolve in terms of the slow-roll parameters. Integrating (2.55) we find

$$\frac{\mathrm{d}\mathcal{H}}{\mathcal{H}^2} = (1 - \varepsilon_V)\,\mathrm{d}\eta \quad \Rightarrow \quad \mathcal{H} = \frac{-1}{(1 - \varepsilon_V)\eta}. \tag{2.60}$$

It will also be useful to have $\mathcal{H}'$ and $\mathcal{H}^2$ in terms of the slow-roll parameters. To first order they are given by

$$\mathcal{H}' = \frac{1 + \varepsilon_V}{\eta^2} + \mathcal{O}\Big(\epsilon^2\Big) \tag{2.61}$$

$$\mathcal{H}^2 = \frac{1 + 2\varepsilon_V}{\eta^2} + \mathcal{O}\Big(\epsilon^2\Big), \tag{2.62}$$

where ϵ^2 indicates a second order term in either ε_V or η_V. To solve for $a(\eta)$ we recall from (2.9) that $\mathcal{H} = a'/a$. Therefore, integrating again, we find

$$\frac{\mathrm{d}a}{a} = -\frac{1}{1 - \varepsilon_V}\frac{\mathrm{d}\eta}{\eta} \quad \Rightarrow \quad a \propto (-\eta)^{-1/(1-\varepsilon_V)}. \tag{2.63}$$

Note that here η is negative; as time goes on $\eta \to 0$ and $a \to \infty$, assuming that slow-roll inflation continues indefinitely.

2.2.3 Scales of inflation

To quantify how much the scale factor grows from a time t_i until the end of inflation, we define the *number of e-folds* as [14, p. 43]

$$\frac{a_i}{a_{\text{end}}} \equiv e^{-N} \quad \Leftrightarrow \quad N = \ln\left(\frac{a_{\text{end}}}{a_i}\right). \tag{2.64}$$

Noting that

$$\ln\left(\frac{a_{\rm end}}{a_i}\right) = \ln a\Big|_{a_i}^{a_{\rm end}} = \int_{a_i}^{a_{\rm end}} \frac{{\rm d}a}{a}, \tag{2.65}$$

we now use the Hubble parameter (2.4), to get the relation

$$N = \int_{t_i}^{t_{\rm end}} H\,{\rm d}t = \int_{\varphi_i}^{\varphi_{\rm end}} \frac{H}{\dot{\varphi}}\,{\rm d}\varphi\,. \tag{2.66}$$

Assuming we are in the slow-roll regime, using the slow-roll relation (2.50) we arrive at the number of e-folds in terms of the shape of the potential

$$N(\varphi_i, \varphi_{\rm end}) \simeq \frac{1}{M_{\rm Pl}^2} \int_{\varphi_{\rm end}}^{\varphi_i} \frac{V_{,\varphi}}{V}\,{\rm d}\varphi\,. \tag{2.67}$$

When discussing the evolution of density perturbations (to which we will get later), we will be interested in the history of each comoving distance scale, or each *comoving wave number* k (from Fourier expansion in comoving coordinates)

$$k = \frac{2\pi}{\lambda}, \qquad k^{-1} = \frac{\lambda}{2\pi}. \tag{2.68}$$

An important question is whether a distance scale is larger or smaller than the Hubble length at a given time. A scale is said to be *super-Hubble*, when $k < \mathcal{H}$ ($k^{-1} > \mathcal{H}^{-1}$); *at Hubble crossing* (entering or leaving the Hubble length), when $k = \mathcal{H}$; and *sub-Hubble*, when $k > \mathcal{H}$ ($k^{-1} < \mathcal{H}^{-1}$). Note that large length scales (large k^{-1}) correspond to small k, and vice versa, although we often talk about *scale* k. It is also customary in the literature on inflation to refer to the Hubble length with the word *horizon*, but this can cause confusion since there are at least 2 other concepts the word horizon might refer to; the particle horizon and the event horizon. We therefore avoid this terminology throughout this thesis.

To identify the distance scales *during inflation* with the corresponding distance scales in the *present universe*, we need a complete history from inflation to the present. We follow [14] and divide it into the following periods (assuming instantaneous transition between the regimes):

1. From the time the scale k of interest exits the Hubble length during inflation to the end of inflation (t_k to $t_{\rm end}$).

2. From the end of inflation to the time when thermal equilibrium at high temperature (Hot Big Bang conditions) is achieved, i.e. reheating. We assume that the universe behaves as if matter-dominated, $\rho \ll a^{-3}$, during this period, ($t_{\rm end}$ to $t_{\rm reh}$).

3. From reheating to matter-radiation equality ($t_{\rm reh}$ to $t_{\rm eq}$).

4. The matter era, $\rho \ll a^{-3}$ from $t_{\rm eq}$ to t_0.

Let us find how large is a scale k that exited the Hubble radius at some time t_k during inflation, relates to the present Hubble length. To do this, we consider the ratio

$$\frac{k}{a_0 H_0} = \frac{a_k}{a_{\rm end}}\frac{a_{\rm end}}{a_{\rm reh}}\frac{a_{\rm reh}}{a_0}\frac{H_k}{H_0} = e^{-N(k)}\frac{a_{\rm end}}{a_{\rm reh}}\frac{a_{\rm reh}}{a_0}\frac{H_k}{H_0}, \tag{2.69}$$

where we have used the definition of the number of e-folds. From the adiabatic approximation to the expansion of the universe, we can relate the scale factor of a particular era to the temperature as $g_{*s}(T)a^3T^3 = const$, where g_{*s} is the *effective entropy degrees of freedom.* Also note that as long as all the particle species are in thermal equilibrium, we can approximate $g_{*s} \simeq g_*$, where this latter is the *effective degrees of freedom.* At early times in the history of the universe, the energy density is related to the temperature of the universe as

$$\rho = \frac{\pi^2}{30} g_*(T) T^4, \tag{2.70}$$

This expression only include relativistic particle species, since their energy density is much greater than that of non-relativistic particle species (this is ceases to be true at later times when we enter the matter dominated era). The particles produced from the inflaton will interact, create other particles through particle reactions, and the resulting soup will eventually reach thermal equilibrium (reheating) with some temperature $T_{\rm reh}$ given by

$$\rho_{\rm reh} = \frac{\pi^2}{30} g_*(T_{\rm reh}) T_{\rm reh}^4. \tag{2.71}$$

where $g_*(T_{\rm reh})$ is the *effective number of degrees of freedom* during reheating. Therefore, noting that during inflation $\rho \simeq V$, we have from the end of inflation, until reheating that

$$\frac{a_{\rm end}}{a_{\rm reh}} = \frac{\rho_{\rm end}^{-1/3}}{\rho_{\rm reh}^{-1/3}} = \left[\frac{\pi^2 g_*(T_{\rm reh}) T_{\rm reh}^4}{V_{\rm end}}\right]^{1/3}. \tag{2.72}$$

After the thermalization of at least the baryons, protons and neutrinos, we recover the big bang conditions. Therefore, from reheating until the present time we have that

$$\frac{a_{\rm reh}}{a_0} = \left[\frac{g_{*s}(T_0)}{g_*(T_{\rm reh})}\right]^{\frac{1}{3}} \frac{T_0}{T_{\rm reh}} = \left[\frac{g_{*s}(T_0)}{g_*(T_{\rm reh})}\right]^{\frac{1}{12}} \frac{\rho_{r0}^{1/4}}{\rho_{\rm reh}^{1/4}} \tag{2.73}$$

Finally, recalling that the Hubble parameter is related to the potential in the Friedmann equation as $H \propto \sqrt{V}$ and collecting all the pieces, we have

$$\frac{k}{a_0 H_0} = e^{-N} \left(\frac{g_{*s}^{1/12}(T_0)}{g_*^{1/12}(T_{\rm reh})} \frac{\rho_{r0}^{1/4}}{\sqrt{\rho_{c0}}} \cdot 10^{16}\text{GeV}\right) \left(\frac{V_k^{1/4}}{10^{16}\text{GeV}}\right) \left(\frac{V_k}{V_{\rm end}}\right)^{\frac{1}{4}} \left(\frac{\rho_{\rm reh}^{1/4}}{V_{end}^{1/4}}\right)^{\frac{1}{3}}. \tag{2.74}$$

Note that we have included the energy scale 10^{16} GeV as a reference scale. The first two factors are model-independent numbers, whereas the rest depend on the model in question. Plugging in the known values $g_{*s}(T_0) = 43/11$ for the present day effective entropy degrees of freedom, $g_*(T_{\rm reh}) = 106.75$ for the effective degrees of freedom during reheating, $\rho_{r0} = 4.18 \times 10^{-5} h^{-2} \rho_{c0}$ and $\rho_{c0} = (\sqrt{3} H_0 M_{\rm Pl})^2$ with $h = 0.7$, we get

$$N = 61 - \ln\left(\frac{k}{a_0 H_0}\right) - \frac{1}{3}\ln\left(\frac{\rho_{\rm reh}^{1/4}}{V_{\rm end}^{1/4}}\right) + \ln\left(\frac{V_k^{1/4}}{V_{\rm end}^{1/4}}\right) - \ln\left(\frac{10^{16}\text{GeV}}{V_k^{1/4}}\right) \tag{2.75}$$

We will resume this discussion once we have introduced the Higgs inflation model, and calculated its slow-roll predictions.

3. Cosmological Perturbation Theory

To study small inhomogeneity and anisotropy, we may take a perturbative approach to General Relativity, where we model the *real* or *perturbed* universe after a more symmetrical model called the *background* universe, in which perturbations are introduced to a desired degree of complexity. In the context of cosmological perturbation theory, this background is the FLRW model. We shall be working with linear perturbations only. In this chapter we follow [15] and [17].

3.1 Perturbations introduced

The idea behind cosmological perturbation theory is that a physical quantity represented by the tensor field $\mathbf{T}$ in the perturbed universe may be split into a homogeneous part $\mathbf{T}_0$ and an inhomogeneous part $\delta\mathbf{T}$ as follows,

$$\mathbf{T}(\eta, x^i) = \mathbf{T}_0(\eta) + \delta\mathbf{T}(\eta, x^i), \tag{3.1}$$

where the homogeneous part is called the *background*, and corresponds to the quantity in the FLRW universe. We shall use the subscript 0 to denote background quantities in an index-free discussion and overbars when indices are present. The inhomogeneous part is called the *perturbation* and it is the term that describes the inhomogeneities in our model. Since we work with linear perturbation theory, the inhomogeneous part is composed of a linear term only. Also, we consider terms composed by two or more perturbations to be small corrections that can be ignored.

3.1.1 Metric perturbations

The metric of the perturbed universe $g_{\mu\nu}$ may be written as the sum of the spatially flat FLRW metric[1] $\bar{g}_{\mu\nu}$ as the background, and a perturbation $\delta g_{\mu\nu}$ as follows,

$$g_{\mu\nu}(x^\mu) = \bar{g}_{\mu\nu}(\eta) + \delta g_{\mu\nu}(\eta, x^i) = a^2(\eta)(\eta_{\mu\nu} + h_{\mu\nu}), \tag{3.2}$$

where on the last equality, we have set

$$h_{\mu\nu}(x^\mu) \equiv \frac{1}{a^2(\eta)}\delta g_{\mu\nu}(x^\mu). \tag{3.3}$$

Depending on the situation, we may use either $h_{\mu\nu}$ or $\delta g_{\mu\nu}$, but refer to either as the *metric perturbation*. For quantities in the perturbed universe, we raise and lower indices with the perturbed metric $g_{\mu\nu}$ in (3.2). The quantity $h_{\mu\nu}$ is not a tensor in the perturbed spacetime, but we define

$$h^\mu{}_\nu \equiv \eta^{\mu\rho}h_{\rho\nu}, \qquad h^{\mu\nu} \equiv \eta^{\mu\rho}\eta^{\sigma\nu}h_{\rho\sigma}, \quad \text{and} \quad h \equiv \eta^{\mu\nu}h_{\mu\nu}. \tag{3.4}$$

From the definition of the inverse metric,

$$g^{\mu\rho}g_{\rho\nu} = g_{\sigma\nu}g^{\sigma\mu} = \delta^\mu{}_\nu, \tag{3.5}$$

we find that to first order it is given by

$$g^{\mu\nu} = \frac{1}{a^2}(\eta^{\mu\nu} - h^{\mu\nu}). \tag{3.6}$$

Since the metric of the perturbed universe is symmetric, we know that $h_{\mu\nu}$ has 10 independent components which we may label as the temporal, mixed and spatial

[1] For the details of cosmological perturbation theory with the spatialy curved FLRW metric, see [19].

parts h_{00}, h_{0i} and h_{ij} respectively. Using the conventions in [15], we define

$$h_{\mu\nu} = \begin{pmatrix} h_{00} & h_{0j} \\ h_{i0} & h_{ij} \end{pmatrix} \equiv \begin{pmatrix} -2A & -B_j \\ -B_i & -2D\delta_{ij} + 2E_{ij} \end{pmatrix}, \tag{3.7}$$

where

$$D \equiv -\frac{1}{6}h \tag{3.8}$$

carries the trace of h_{ij} and E_{ij} is the traceless part. $A \equiv A(\eta, x^i)$ is called the *lapse function* and $B_i \equiv B_i(\eta, x^i)$ the *shift vector*. A and D are scalars, B_i a vector, and E_{ij} a tensor. From (3.4), we find that the inverse metric perturbations are given by

$$h^{\mu\nu} = \begin{pmatrix} -2A & B_j \\ B_i & -2D\delta_{ij} + 2E_{ij} \end{pmatrix}. \tag{3.9}$$

From (3.4), we see that we raise and lower the indices of the spatial part of the metric perturbations h_{ij} with the euclidean metric δ_{ij}, which allow us to write $B^i = \delta^{ij}B_j = B_i$ and similarly for E_{ij}. The metric of the linearly perturbed universe is then

$$g_{\mu\nu} = a^2(\eta) \begin{pmatrix} -(1+2A) & -B_j \\ -B_i & (1-2D)\delta_{ij} + 2E_{ij} \end{pmatrix}. \tag{3.10}$$

This is the most general perturbed metric tensor in the linear approximation. However, we may now proceed to apply a procedure known as the *SVT (Scalar-Vector-Tensor) decomposition* to further decompose the metric perturbation B_i and E_{ij} into their irreducible parts. The shift vector B_i may be decomposed into a scalar (or curl-free) part B and a divergence-free vector part B_i^V,

$$B_i = -B_{,i} + B_i^V, \qquad \text{where} \qquad \delta^{ij}B_{i,j}^V = \nabla \cdot \vec{B}^V = 0. \tag{3.11}$$

Since we are working with the spatially flat FLRW metric, we can see this is just the Helmholtz decomposition. Similarly, E_{ij} can be divided into three parts

$$E_{ij} = E_{ij}^S + E_{ij}^V + E_{ij}^T, \tag{3.12}$$

where E_{ij}^S and E_{ij}^V can be expressed in terms of a scalar field E and a divergence-free vector field E_i respectively

$$E_{ij}^S = E_{,ij} - \frac{1}{3}\delta_{ij}\delta^{kl}E_{,kl} \tag{3.13}$$

$$E_{ij}^V = -\frac{1}{2}(E_{i,j} + E_{j,i}), \tag{3.14}$$

where

$$\delta^{ij}E_{i,j} = \vec{\nabla}\cdot\vec{E} = 0. \tag{3.15}$$

The tensor part E_{ij}^T is traceless and transverse, that is

$$\delta^{ik}E_{ij,k}^T = \delta^{ij}E_{ij}^T = 0, \tag{3.16}$$

and is symmetric by construction. Then, the SVT decomposition of E_{ij} is

$$E_{ij} = E_{,ij} - \frac{1}{3}\delta_{ij}\delta^{kl}E_{,kl} - E_{(i,j)} + E_{ij}^T, \tag{3.17}$$

where $E_{(i,j)}$ denote the symmetrization of $E_{i,j}$, that is,

$$E_{(i,j)} = \frac{1}{2}(E_{i,j} + E_{j,i}). \tag{3.18}$$

As we can see, the metric perturbations can be divided into four scalars A, B, D, and E; two vectors B_i^V, E_i and one tensor E_{ij}^T. The last one, is a purely tensorial contribution and it cannot be constructed from any scalar or vector. The purpose of this separation is to take advantage of the fact that in linear perturbation theory, the scalar, vector, and tensor parts do not couple to each other and evolve independently, which allows us to study one in the absence of the others. We shall neglect vector perturbations, since they can be shown to decay in an expanding universe. For details, see [16]. Scalar perturbations turn out to be responsible for the structure formation of the universe whereas tensor perturbations correspond to gravitational waves, as we shall see in the next chapter.

Scalar perturbations

If we consider scalar perturbations only, then we may select the scalar components of B_i and E_{ij} so that the metric perturbations become

$$h_{\mu\nu} = \begin{pmatrix} -2A & B_{,j} \\ B_{,i} & -2\psi\delta_{ij} + 2E_{,ij} \end{pmatrix}, \tag{3.19}$$

where we have defined the *curvature perturbation*

$$\psi \equiv D + \frac{1}{3}\nabla^2 E, \tag{3.20}$$

so that the metric of the perturbed spacetime is

$$g_{\mu\nu} = a^2(\eta) \begin{pmatrix} -(1+2A) & B_{,j} \\ B_{,i} & (1-2\psi)\delta_{ij} + 2E_{,ij} \end{pmatrix}. \tag{3.21}$$

Tensor perturbations

If we consider tensor perturbations only, then we only have one degree of freedom, since the only perturbation with tensor components is E_{ij}

$$h_{ij} = 2E_{ij} = 2E^T_{ij}, \tag{3.22}$$

and the perturbed metric is simply

$$g_{\mu\nu} = a^2(\eta) \begin{pmatrix} -1 & 0 \\ 0 & \delta_{ij} + 2E^T_{ij} \end{pmatrix}. \tag{3.23}$$

3.1.2 Perturbations in the Einstein tensor

Our next task is to calculate the perturbation in the Einstein tensor $\delta G^\mu{}_\nu$ given according to (3.1) as

$$\delta G^\mu{}_\nu = G^\mu{}_\nu - \bar{G}^\mu{}_\nu. \tag{3.24}$$

This is done by using the perturbed metric (3.10) to calculate to first order in the perturbations, the perturbed Christoffel symbols and scalar curvature in (2.16) and

(2.15) respectively, so that then we may use them to calculate the perturbations in the Einstein tensor (2.12). This is done in [15] where the components are shown to be given by

$$\delta G^0{}_0 = a^{-2}\Big[-2\nabla^2 D + 6\mathcal{H}D' + 6\mathcal{H}^2 A - 2\mathcal{H}B_{i,i} - E_{ik,ik}\Big] \tag{3.25a}$$

$$\delta G^0{}_i = a^{-2}\Big[-2D'_{,i} - 2\mathcal{H}A_{,i} - B_{[i,k]k} - E'_{ik,k}\Big] \tag{3.25b}$$

$$\delta G^i{}_0 = a^{-2}\Big[2D'_{,i} + 2\mathcal{H}A_{,i} + B_{[i,k]k} + 2(\mathcal{H}' - \mathcal{H}^2)B_i + E'_{ik,k}\Big] \tag{3.25c}$$

$$\begin{aligned}\delta G^i{}_j = a^{-2}\Big[&2D'' - \nabla^2(D-A) + 2\mathcal{H}(A' - 2D') + 2(2\mathcal{H}' + \mathcal{H}^2)A \\ &-B'_{k,k} - 2\mathcal{H}B_{k,k} - E_{kl,kl}\Big]\delta^i{}_j + a^{-2}\Big[(D-A)_{,ij} + B'_{(i,j)} + 2\mathcal{H}B_{(i,j)} \\ &+E''_{ij} - \nabla^2 E_{ij} + 2E_{k(i,j)k} + 2\mathcal{H}E'_{ij}.\Big]\end{aligned} \tag{3.25d}$$

If we focus on scalar perturbations, the perturbation in the Einstein tensor becomes

$$\delta G^0{}_0 = a^{-2}\Big[-2\nabla^2\psi + 6\mathcal{H}D' + 6\mathcal{H}^2 A + 2\mathcal{H}\nabla^2 B\Big] \tag{3.26a}$$

$$\delta G^0{}_i = a^{-2}[-2\psi' - 2\mathcal{H}A]_{,i} \tag{3.26b}$$

$$\delta G^i{}_0 = a^{-2}\Big[2\psi' + 2\mathcal{H}A - 2(\mathcal{H}' - \mathcal{H}^2)B\Big]_{,i} \tag{3.26c}$$

$$\begin{aligned}\delta G^i{}_j = a^{-2}\Big[&2D'' - \nabla^2(D-A) + \mathcal{H}(2A' + 4D') + 4\mathcal{H}'A \\ &+ 2\mathcal{H}^2 A + \nabla^2 B' + 2\mathcal{H}\nabla^2 B - \frac{1}{3}\nabla^2\Big(\nabla^2 E + E'' + 2\mathcal{H}E'\Big)]\delta^i{}_j \\ &+ (D - A - B' - 2\mathcal{H}B + E'' + \frac{1}{3}\nabla^2 E + 2\mathcal{H}E')_{,ij}.\end{aligned} \tag{3.26d}$$

For tensor perturbations, the only metric perturbation carrying a tensor part is E_{ij}. Since E^T_{ij} is transverse, the nonvanishing part is the spatial part

$$\delta G^i{}_j = a^{-2}\Big[E_{ij} + 2\mathcal{H}E'_{ij} - \nabla^2 E_{ij}\Big], \tag{3.27}$$

where we have dropped the T in E^T_{ij} for simplicity of notation.

3.1.3 Matter perturbations

The energy-momentum tensor of the perturbed universe is given according to (3.1) as the sum

$$T^{\mu}{}_{\nu} = \bar{T}^{\mu}{}_{\nu} + \delta T^{\mu}{}_{\nu}, \tag{3.28}$$

where $\bar{T}^{\mu}{}_{\nu}$ is the energy-momentum tensor of the FLRW model, given in (2.19). We then proceed to divide the perturbations into perfect fluid plus non-perfect fluid degrees of freedom (5 each). The perfect fluid degrees of freedom in $\delta T^{\mu}{}_{\nu}$ are those that keep the perturbed energy momentum $T^{\mu}{}_{\nu}$ in the perfect fluid form

$$T^{\mu}{}_{\nu} = (\rho + p)u^{\mu}u_{\nu} + p\delta^{\mu}{}_{\nu}. \tag{3.29}$$

Thus they may be taken as the density, pressure, and four-velocity perturbations $\delta\rho$, δp and δu^i such that

$$\rho = \bar{\rho} + \delta\rho, \quad p = \bar{p} + \delta p, \quad \text{and} \quad u^i = \bar{u}^i + \delta u^i = \delta u^i \equiv \frac{1}{a}v_i, \tag{3.30}$$

where we have introduced the quantity

$$v_i \equiv au^i, \tag{3.31}$$

which we shall refer as the velocity perturbation. It is equal to the coordinate velocity, since to first order,

$$\frac{\mathrm{d}x^i}{\mathrm{d}\eta} = \frac{u^i}{u^0} = \frac{u^i}{\bar{u}^0} = au^i = v_i. \tag{3.32}$$

Note that δu^0 is not an independent degree of freedom, since it is constrained by the normalization

$$u_\mu u^\mu = -1. \tag{3.33}$$

Splitting the four-velocity u^μ into the background and the perturbation, using the perturbed metric in (3.10), and dropping second order terms in the perturbations, we find that the 0-component is

$$u_0 = g_{0\mu}u^\mu = -a - a^2\delta u^0 - 2aA. \tag{3.34}$$

Since $\bar{u}_0 = -a$ (fluid at rest, recall that we are working with proper time η instead of t), we find that

$$\delta u_0 = -a^2 \delta u^0 - 2aA. \tag{3.35}$$

Likewise, for the spatial components, we find

$$\delta u_i = u_i = g_{i\mu} u^\mu = -aB_i + av_i. \tag{3.36}$$

It only remains to determine δu^0, which we get from the normalization condition (3.33) as

$$\delta u^0 = \frac{1}{a} A. \tag{3.37}$$

Therefore, the linearly perturbed four-velocity is given by

$$u^\mu = \frac{1}{a}(1 - A, v_i) \quad \text{and} \quad u_\mu = a(-1 - A, v_i - B_i). \tag{3.38}$$

Plugging these into the perturbed energy momentum tensor $T^\mu{}_\nu$ in (3.28), we get (after separating the background terms from the perturbations)

$$\delta T^0{}_0 = -\delta\rho \tag{3.39}$$

$$\delta T^0{}_i = (\bar{\rho} + \bar{p})(v_i - B_i) \tag{3.40}$$

$$\delta T^i{}_0 = -(\bar{\rho} + \bar{p}) v_i. \tag{3.41}$$

There are five remaining degrees of freedom in the spatial part of the energy-momentum perturbation, corresponding to deviations from the perfect fluid form, we write them as

$$\delta T^i{}_j = \delta p \delta^i{}_j + \Sigma_{ij}, \tag{3.42}$$

where Σ_{ij} is called the *anisotropic stress* (or *anisotropic pressure*). Σ_{ij} is symmetric and traceless, so that we may write the pressure perturbation as the trace of the spatial part of the energy-momentum perturbation and the anisotropic pressure as the traceless part, that is

$$\delta p \equiv \delta T^k{}_k \quad \text{and} \quad \Sigma_{ij} \equiv \delta T^i{}_j - \frac{1}{3}\delta^i{}_j \delta T^k{}_k. \tag{3.43}$$

For a perfect fluid, the anisotropic pressure vanishes.

Separation into scalar, vector, and tensor parts

The perturbations in the energy momentum tensor are built out of two scalar perturbations δp and $\delta\rho$; the three-vector v_i, and the traceless three-tensor Σ_{ij}. Just as we did with the metric perturbations, we may decompose v_i and Π_{ij} into their irreducible parts. For v_i we just apply Helmholtz's theorem

$$v_i = v_i^S + v_i^V = -v_{,i} + v_i^V \quad \text{where} \quad \delta^{ij} v_{i,j}^V = 0, \tag{3.44}$$

and for Σ_{ij}, we have

$$\Sigma_{ij} = \Sigma_{ij}^S + \Sigma_{ij}^V + \Sigma_{ij}^T, \tag{3.45}$$

where

$$\Sigma_{ij}^S = \Sigma_{,ij} - \frac{1}{3}\delta_{ij}\nabla^2\Sigma \tag{3.46}$$

$$\Sigma_{ij}^V = -\frac{1}{2}(\Sigma_{i,j} + \Sigma_{j,i}) \tag{3.47}$$

$$\delta^{ik}\Sigma_{ij,k}^T = 0. \tag{3.48}$$

That is, Σ_{ij}^T is transverse. Then for scalar perturbations, the perturbations in the energy-momentum tensor become

$$\delta T^\mu{}_\nu = \begin{pmatrix} -\delta\rho & -(\bar\rho + \bar p)(v - B)_{,j} \\ (\bar\rho + \bar p)v_{,i} & \delta p \delta^i{}_j + \Sigma_{,ij} - \frac{1}{3}\delta_{ij}\nabla^2\Sigma. \end{pmatrix}. \tag{3.49}$$

3.1.4 Perturbations of the energy-momentum tensor of a single scalar field

During inflation, the inflationary energy is the dominant contribution to the energy-momentum tensor of the perturbed universe. According to (3.1), the perturbed inflaton field is given by

$$\varphi(\eta, x^i) = \bar\varphi(\eta) + \delta\varphi(\eta, x^i). \tag{3.50}$$

The perturbations in the energy-momentum tensor are given by (3.1) as

$$\delta T^{\mu}{}_{\nu} = T^{\mu}{}_{\nu} - \bar{T}^{\mu}{}_{\nu}. \tag{3.51}$$

To calculate these, use the perturbed metric (3.10) and the perturbed inflation field (3.50) for the energy-momentum tensor for a single scalar field $T^{\mu}{}_{\nu}$ in (2.37), and use the background values for $\bar{T}^{\mu}{}_{\nu}$ in (2.19). After some work and dropping terms of second order in perturbations, we find that the different components are

$$\delta T^{0}{}_{0} = -a^{-2}\Big(-A\bar{\phi}'^{2} + \bar{\varphi}'\delta\varphi' + a^{2}V_{,\varphi}\delta\varphi\Big) \tag{3.52a}$$

$$\delta T^{0}{}_{j} = -a^{-2}\bar{\varphi}'\delta\varphi_{,j} \tag{3.52b}$$

$$\delta T^{i}{}_{0} = -a^{-2}\Big(\bar{\varphi}'^{2}B_{i} - \bar{\varphi}'\delta\varphi_{i}\Big) \tag{3.52c}$$

$$\delta T^{i}{}_{j} = -a^{-2}\Big[A\bar{\varphi}'^{2} - \bar{\varphi}'\delta\varphi' + a^{2}V_{,\varphi}\delta\varphi\Big]\delta^{i}{}_{j}. \tag{3.52d}$$

By comparing (3.52d) and (3.42) we see that to first order, the anisotropic pressure Σ_{ij} vanishes, and thus scalar fields do not source tensor perturbations.

3.2 Gauge freedom

In general relativity, we have the freedom to choose a coordinate system that best suits the problem, since the theory is manifestly covariant. In this context, we have chosen to use perturbation theory to split physical quantities according to (3.1). However, this is not a covariant procedure, meaning that it introduces a gauge dependence that affects the perturbations and hence, our calculations of the perturbed universe. Therefore, we would like to remove these gauge degrees of freedom. To do so, we must study how the different tensorial quantities transform under a gauge transformation.

We shall adopt the *active approach* to gauge transformations developed in[18], where the gauge transformation rule of a tensor field $\mathbf{T}$ is given by the exponential map[2]

$$\widetilde{\mathbf{T}} = e^{-\pounds_\xi}\mathbf{T}, \tag{3.53}$$

where the tilde indicates the tensor in the new gauge and $\pounds_\xi$ is the Lie derivative with respect to ξ^λ, the vector that generates the transformation. The exponential in (3.53) is to be understood as a series, which to to first order is

$$e^{\pounds_\xi} = 1 - \pounds_\xi. \tag{3.54}$$

Therefore, using (3.1) the perturbed tensor transforms as

$$\widetilde{\mathbf{T}}_0 + \delta\widetilde{\mathbf{T}} = (1 - \pounds_\xi)(\mathbf{T}_0 + \delta\mathbf{T}) = \mathbf{T}_0 + \delta\mathbf{T} - \pounds_\xi \mathbf{T}_0, \tag{3.55}$$

where the background quantity remains unchanged by construction, but the perturbation transforms as

$$\widetilde{\delta\mathbf{T}} = \delta\mathbf{T} - \pounds_\xi \mathbf{T}_0. \tag{3.56}$$

3.2.1 Gauge transformations of the metric perturbations

Applying (3.56) to $\delta g_{\mu\nu}$, we have

$$\widetilde{\delta g_{\mu\nu}} = \delta g_{\mu\nu} - \pounds_\xi \bar{g}_{\mu\nu} \tag{3.57}$$

$$= \delta g_{\mu\nu} - \bar{g}_{\mu\nu,\rho}\xi^\rho - \bar{g}_{\rho\nu}\xi^\rho{}_{,\mu} - \bar{g}_{\mu\rho}\xi^\rho{}_{,\nu}. \tag{3.58}$$

Let us calculate explicitly, the gauge transformation behaviour of δg_{00}. Using the metric of the flat FLRW universe (2.8) and recalling that $\delta g_{\mu\nu} = a^2 h_{\mu\nu}$, we have

$$\widetilde{\delta g_{00}} = \delta g_{00} - \bar{g}_{00,\rho}\,\xi^\rho - \bar{g}_{\rho 0}\,\xi^\rho{}_{,0} - \bar{g}_{0\rho}\,\xi^\rho{}_{,0} \tag{3.59}$$

$$a^2\widetilde{h_{00}} = a^2 h_{00} + 2aa'\xi^0 + 2a^2\xi^0{}_{,0}. \tag{3.60}$$

[2] Note that we have used a minus sign in the exponential map in (3.53) as opposed to a positive sign as in [17]. This is done so that the signs in the transformation rules conform to the other references we are following, which are derived using the *passive view* instead.

Plugging h_{00} from (3.7) and rearranging some terms, we get the gauge transformation rule for the lapse function A

$$\widetilde{A} = A + \mathcal{H}\xi^0 + \xi^0{}_{,0}. \tag{3.61}$$

A similar calculation for the mixed part δg_{0i} gives the gauge transformation rule of the shift vector B_i

$$\widetilde{B}_i = B_i + \xi^i{}_{,0} - \xi^0{}_{,i}. \tag{3.62}$$

For the spatial part of (3.58), we study the trace and traceless parts separately. From the trace we get

$$\widetilde{D} = D + \mathcal{H}\xi^0 + \frac{1}{3}\xi^k{}_{,k}, \tag{3.63}$$

and from the traceless part $\widetilde{\delta g_{ij}} - \frac{1}{3}\widetilde{\delta g_{kk}}$ we get

$$\widetilde{E}_{ij} = E_{ij} - \xi_{(i,j)} + \frac{1}{3}\delta_{ij}\xi^k{}_{,k}. \tag{3.64}$$

We still need the transformation properties of the irreducible parts B, B_i^V, E, and E_i. To begin, we decompose ξ^μ into its irreducible parts

$$\xi^\mu = \left(\xi^0; \xi^i\right) = \left(\xi^0; -\xi_,{}^i + \xi^{V\,i}\right) \quad \text{where} \quad \delta^{ij}\xi^V_{i,j} = \xi^{V\,i}_{i,} = \xi^{V\,i}{}_{,i} = 0. \tag{3.65}$$

To get the transformation rule for B, we first note that in (3.11), the shift vector B_i is decomposed into a scalar B and divergence-free part B_i^V, which allows us to isolate B by taking the divergence of (3.62) to get

$$-\nabla^2\widetilde{B} = -\nabla^2 B - \nabla^2\xi^0 - \nabla^2\xi', \tag{3.66}$$

from which it follows that

$$\widetilde{B} = B + \xi^0 + \xi'. \tag{3.67}$$

Plugging this into the transformation rule of B_i in (3.62), we get

$$\widetilde{B}_i^V = B_i^V + \xi_i^V. \tag{3.68}$$

Similarly for E and E_i, we look for a way of isolating them from the overall behavior of E_{ij}. Looking at (3.16) we see that we can apply the operator $\partial^i\partial^j$ to (3.17) to get

$$E_{ij}{}^{,ij} = E_{,ij}{}^{ij} - \frac{1}{3}\delta_{ij}\nabla^2 E^{,ij} - E_{(i,j)}{}^{,ij} - E^T{}_{ij}{}^{,ij} \tag{3.69}$$

$$= \nabla^2\nabla^2 E - \frac{1}{3}\nabla^2\nabla^2 E \tag{3.70}$$

$$= \frac{2}{3}\nabla^2\nabla^2 E. \tag{3.71}$$

Next, applying $()^{,ij}$ to (3.64) and using (3.71), we get

$$\widetilde{E} = E + \xi. \tag{3.72}$$

For E_i, take the divergence of (3.64) and the above result to obtain

$$\widetilde{E}_i = E_i + \xi_i^V. \tag{3.73}$$

Lastly, we need the gauge transformation rule for E^T_{ij}. To get it, plug the gauge transformation rules for E, E_i in (3.72) and (3.73) into E_{ij} written in terms of its irreducible parts in (3.17). After some work, we find that

$$\widetilde{E}^T_{ij} = E^T_{ij}, \tag{3.74}$$

that is, E^T_{ij} is a *gauge-independent* quantity in first order perturbation theory. The last quantity we need to calculate its gauge transformation property is the curvature perturbation ψ. Plugging (3.63) and (3.72) into (3.20), we find

$$\widetilde{\psi} = \psi + \mathcal{H}\xi^0, \tag{3.75}$$

where we have also used the decomposition of ξ^i in (3.65).

3.2.2 Gauge transformations of the energy-momentum tensor perturbations

To obtain the gauge transformation of the energy-momentum tensor perturbations $\delta T^{\mu}{}_{\nu}$, we apply the general gauge transformation rule (3.56) to obtain

$$\widetilde{\delta T}^{\mu}{}_{\nu} = \delta T^{\mu}{}_{\nu} - \bar{T}^{\mu}{}_{\nu,\lambda}\,\xi^{\lambda} + \bar{T}^{\lambda}{}_{\nu}\,\xi^{\mu}{}_{,\lambda} - \bar{T}^{\mu}{}_{\lambda}\,\xi^{\lambda}{}_{,\nu}. \tag{3.76}$$

Then, analyzing the temporal, mixed, and spatial (trace and traceless part) we get respectively

$$\widetilde{\delta\rho} = \delta\rho - \bar{\rho}'\xi^0 \tag{3.77}$$

$$\widetilde{v}_i = v_i + \xi^i{}_{,0} \tag{3.78}$$

$$\widetilde{\delta p} = \delta p - \bar{p}'\xi^0 \tag{3.79}$$

$$\widetilde{\Sigma}_{ij} = \Sigma_{ij}. \tag{3.80}$$

Therefore the anisotropic stress is gauge-independent. For scalar perturbations, we have $\xi^i = -\xi_{,i}$ and the gauge transformations

$$\widetilde{v} = v + \xi' \tag{3.81}$$

$$\widetilde{\Sigma} = \Sigma. \tag{3.82}$$

In the case of inflation, the energy-momentum tensor is dominated by the inflaton φ. Applying (3.56) to (3.50) we get

$$\widetilde{\delta\varphi} = \delta\varphi - \pounds_{\xi}\bar{\varphi} = \delta\varphi - \xi^0\bar{\varphi}'. \tag{3.83}$$

3.2.3 Comoving gauge

The two scalar gauge functions ξ^0 and ξ, which specify a choice of time-slicing and threading respectively, allow two of the perturbations to be eliminated. Such choice will fix a gauge. We shall use the *comoving gauge*, which is defined by requiring both the slicing and threading to be comoving. We say that the slicing is comoving if the time slices are orthogonal to the fluid four-velocity u^μ, which requires that $v_i - B_i = 0$. For scalar perturbations the comoving slicing condition is

$$v = B \qquad \text{(Comoving slicing)}. \tag{3.84}$$

The threading (spatial coordinates x^i) is said to be comoving, if it is the worldline of an observer comoving with the fluid, i.e., chosen so that the velocity perturbation vanishes, $v^i = 0$. For scalar perturbations, we then have

$$v = 0 \qquad \text{(Comoving threading)}. \tag{3.85}$$

Therefore, we go to the comoving gauge by setting

$$v^C = B^C = 0 \qquad \text{(Comoving gauge)}. \tag{3.86}$$

To go from an arbitrary gauge to the comoving one, we see from the gauge transformations rules of B and v in (3.67) and (3.81) that this is achieved by setting

$$\xi' = -v \tag{3.87}$$

$$\xi^0 = v - B. \tag{3.88}$$

We next define the *comoving curvature perturbation* $\mathcal{R}$ as the curvature perturbation (3.89) in the comoving gauge, that is,

$$\mathcal{R} \equiv -\psi^C. \tag{3.89}$$

Using the gauge transformation rule of ψ in (3.75) and the comoving gauge condition in (3.88), we have the following

$$\psi^C = \psi + \mathcal{H}(v - B) = -\mathcal{R}. \tag{3.90}$$

This combination of metric and matter perturbations turns out to be *gauge invariant*, that is, remains unchanged under a gauge transformation, as we can see from the transformation rules for B and v in (3.67) and (3.81) respectively. This is an important property that will enable us avoid the pitfall of gauge modes, that is, modes that can be eliminated via a gauge transformation and which therefore, do not correspond to physical observables. This quantity has also the property that, in the case of inflation driven by a single scalar field, it remains constant at scales larger than the Hubble radius ($k \ll \mathcal{H}$) [16], which will justify the computation of the perturbations at Hubble crossing and ignoring the evolution on these scales.

So far, we have developed the comoving gauge in the case of matter described in fluid terms. Since we work with inflation driven by a single-scalar field φ in the next chapter, we would now like to see how these results correspond to the field picture. The concept of velocity is not the most appropriate in this picture, but we note that from the relation (3.41), we get

$$\delta T^i{}_0 = -(\rho + p) v_i \qquad \Rightarrow \qquad v_{,i} = \frac{\delta T^i{}_0}{\rho + p}, \tag{3.91}$$

where we have selected scalar part v and dropped the vector perturbation. Using the backround values of ρ and p in (2.38),(2.39); and $\delta T^i{}_0$ for a single scalar field in (3.52c), we have

$$v_{,i} = B_{,i} + \frac{\delta\varphi_{,i}}{\varphi'}, \tag{3.92}$$

therefore, in the comoving gauge we have

$$v^C = 0 \qquad \Rightarrow \qquad \delta\varphi^C = 0. \tag{3.93}$$

That is, the inflaton field remains unperturbed. We can therefore define the comoving gauge in the field picture as

$$\delta\varphi^C = B^C = 0. \tag{3.94}$$

4. The Generation of Perturbations

Having studied how to describe small perturbations, our next task is to study how inflation produces perturbations from a homogeneous and isotropic state. In this chapter, we review the calculation of the power spectrum of scalar and tensor perturbations from first principles. As noted in the previous chapter, the different kinds of perturbations do not couple to each other in linear perturbation theory, which allows us to study them separately. We will start with the case of scalar perturbations and then move to tensor perturbations in §4.2. After this, we discuss in §4.3 what the observations of anisotropies in the CMB, in particular by the Planck mission, tells us about these primordial perturbations and inflation in general. We work with single field slow-roll models of inflation defined by the action

$$S = \int \mathrm{d}^4x \left[\frac{M_{\mathrm{Pl}}^2}{2}R - \frac{1}{2}g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu} - V(\varphi)\right], \tag{4.1}$$

in which we introduce the metric and matter perturbations $\varphi = \bar{\varphi} + \delta\varphi$. To do this, we may expand the action above to second order in the perturbations. This is the approach taken in [16] (see also [19]). We may also use the Einstein field equation for the perturbations,

$$\delta G^{\mu}{}_{\nu} = \frac{1}{M_{\mathrm{Pl}}^2}\delta T^{\mu}{}_{\nu}, \tag{4.2}$$

together with the Klein-Gordon equation for the scalar field in the perturbed universe,

$$\frac{1}{\sqrt{-g}}\partial_\mu\Big(\sqrt{-g}g^{\mu\nu}\partial_\nu\varphi\Big) - V_{,\varphi} = 0. \tag{4.3}$$

This is the approach taken in [15] and the one we shall follow. In (3.94) we found that one of the conditions that define the comoving gauge is that the scalar field is unperturbed (i.e. $\delta\varphi = 0$). Since we will use this gauge, we only need to introduce the metric perturbations into the Klein-Gordon equation. Using the linearly perturbed metric we found in (3.10), we find that the square root of the determinant is

$$\sqrt{-g} = a^4(1 + A - 3D), \tag{4.4}$$

so that to first order in the metric perturbations, the Klein-Gordon equation becomes

$$\bar{\varphi}'' + 2\mathcal{H}\bar{\varphi}' - \nabla^2\bar{\varphi} - (A' + 3D' - B_{i,i})\bar{\varphi}' = -a^2(1 + 2A)V_{\bar{\varphi}}. \tag{4.5}$$

Subtracting the background field equation (2.35), the Klein-Gordon equation for the metric perturbations is just

$$(A' + 3D' - B_{i,i})\bar{\varphi}' = 2a^2 A V_{\bar{\varphi}}. \tag{4.6}$$

We now turn to scalar perturbations.

4.1 Scalar perturbations

Plugging the perturbation in the Einstein tensor and the energy-momentum field of a single scalar field given by (3.26) and (3.52), respectively, into the Einstein field equation for the perturbations (4.2), we get

$$3\mathcal{H}(\mathcal{H}A + D') - \nabla^2(\psi - \mathcal{H}B) = \frac{1}{2M_{\rm Pl}^2}\varphi'^2 A \tag{4.7}$$

$$\psi' + \mathcal{H}A = 0 \tag{4.8}$$

$$\psi' + \mathcal{H}A - (\mathcal{H}' - \mathcal{H}^2)B = \frac{1}{2M_{\rm Pl}^2}\varphi'^2 B \tag{4.9}$$

$$(2\mathcal{H}' + \mathcal{H}^2)A + \mathcal{H}A' - \psi'' + 2\mathcal{H}\psi' = -\frac{1}{2M_{\rm Pl}^2}\varphi'^2 A, \tag{4.10}$$

where we have dropped the overbars on the background quantities, since all perturbations have been introduced and no confusion should arise. In the previous chapter, we introduced the gauge-invariant curvature perturbation $\mathcal{R}$ defined in (3.89) which has the property that it is conserved on super Hubble scales. This will allow us to compute the power spectrum of curvature perturbations at the moment of Hubble crossing ($k = \mathcal{H}$) without worrying about what happens at sub-Hubble scales, during and after reheating until k re-enters the Hubble length for a given $\mathcal{R}$-mode. It turns out that in the comoving gauge, the remaining metric perturbations A^C and D^C will be related to $\mathcal{R}$. Switching to the comoving gauge defined in (3.94), the field equations become

$$3\mathcal{H}(\mathcal{H}A^C + D^{C\prime}) + \nabla^2\mathcal{R} = \frac{1}{2M_{\rm Pl}^2}\varphi'^2 A^C \tag{4.11}$$

$$-\mathcal{R} + \mathcal{H}A^C = D^C \tag{4.12}$$

$$(2\mathcal{H}' + \mathcal{H}^2)A^C + \mathcal{H}A^{C\prime} - \mathcal{R}'' - 2\mathcal{H}\mathcal{R}' = -\frac{1}{2M_{\rm Pl}^2}\varphi'^2 A^C, \tag{4.13}$$

where the equations (4.8) and (4.9) became (4.12). The field perturbation equation (4.6) becomes

$$-2a^2 A^C V_\varphi + \varphi'(A^{C\prime} + 3D^{C\prime}) = 0, \tag{4.14}$$

from which we get the relation between D^C and $\mathcal{R}$

$$D^{C\prime} = \frac{1}{3\mathcal{H}}\left[2a^2\frac{V_\varphi}{\varphi'}\mathcal{R}' - \mathcal{R}'' + \frac{\mathcal{H}'}{\mathcal{H}}\mathcal{R}'\right]. \tag{4.15}$$

Equating (4.11) with (4.13) and using (4.15) we get a single equation with the comoving curvature perturbation $\mathcal{R}$ only, known as the *Mukhanov-Sasaki* equation[24, 22]

$$\mathcal{R}'' + 2\frac{z'}{z}\mathcal{R}' - \nabla^2\mathcal{R} = 0, \qquad z \equiv \frac{a\varphi'}{\mathcal{H}}. \tag{4.16}$$

Introducing the *Mukhanov variable*

$$\nu(\eta, x^i) \equiv z(\eta)\mathcal{R}(\eta, x^i), \tag{4.17}$$

the Mukhanov-Sasaki equation becomes

$$\nu'' - \frac{z''}{z}\nu - \nabla^2\nu = 0. \tag{4.18}$$

Note that it has the same form as the Klein-Gordon equation in Minkowski space with the effective η-dependent mass term

$$m^2_{\text{eff}}(\eta) \equiv -\frac{z''}{z}. \tag{4.19}$$

This can be seen by recalling that in Minkowski space, the d'Alembertian is

$$\Box\nu \equiv g^{\alpha\beta}\nu_{;\alpha\beta} = \eta^{\alpha\beta}\nu_{,\alpha\beta} = -\nu'' + \nabla^2\nu. \tag{4.20}$$

Therefore, (4.18) becomes

$$\Big(\Box - m^2_{\text{eff}}(\eta)\Big)\nu = 0. \tag{4.21}$$

4.1.1 General Solution in Fourier space

To obtain a general solution to (4.18), we Fourier expand the Mukhanov variable ν in terms of a comoving wavenumber k

$$\nu(\eta, \mathbf{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^{2/3}}\nu_{\mathbf{k}}(\eta)e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{4.22}$$

so that in Fourier space, the Mukhanov-Sasaki equation (4.18) is given by

$$\nu''_{\mathbf{k}} + \omega^2_k(\eta)\nu_{\mathbf{k}} = 0 \qquad \omega^2_k(\eta) \equiv k^2 + m^2_{\text{eff}}(\eta). \tag{4.23}$$

Perturbations are real field variables, so we impose the *reality condition*

$$\nu^*(\eta, \mathbf{x}) = \nu(\eta, \mathbf{x}) \qquad \Rightarrow \qquad \nu^*_{\mathbf{k}} = \nu_{-\mathbf{k}}. \tag{4.24}$$

Because ω_k^2 depends only on the magnitude of the wavevector ($k \equiv |\mathbf{k}|$), the general solution of (4.23) may be written as

$$\nu_{\mathbf{k}}(\eta) = a^-_{\mathbf{k}} v^*_k(\eta) + a^+_{-\mathbf{k}} v_k(\eta), \tag{4.25}$$

where $v_k(\eta)$ and $v^*_k(\eta)$ are two linearly independent solutions of (4.23) and $a^\pm_{\mathbf{k}}$ two complex constants of integration. The index $-\mathbf{k}$ in the second term is chosen for convenience. The reality condition (4.24) relates the constants of integration $(a^-_{\mathbf{k}})^* = a^+_{\mathbf{k}}$. To see that $v_k(\eta)$ and $v^*_k(\eta)$ are linearly independent, we can study the Wronskian constructed from them as follows

$$W[v_k, v^*_k] \equiv \begin{vmatrix} v_k & v^*_k \\ v'_k & v^{*\prime}_k \end{vmatrix} = v_k v^{*\prime}_k - v^*_k v'_k. \tag{4.26}$$

Taking the η derivative of the Wronskian in (4.26), we get

$$W' = v_k v^{*\prime\prime}_k - v^*_k v''_k \tag{4.27}$$

$$= v_k(-\omega^{*2}_k v^*_k) - v^*_k(-\omega^2_k v_k) \tag{4.28}$$

$$= 0, \tag{4.29}$$

where we have used the fact that v_k and v^*_k are solutions of (4.23) and $\omega^{*2}_k = \omega^2_k$. Thus we see that the Wronskian vanishes, which implies that v_k and v^*_k are linearly independent [25, p. 360]. Finally, with (4.25), the Fourier expansion of ν in (4.22) becomes

$$\begin{aligned}\nu(\eta, \mathbf{x}) &= \int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}} \Big[a^-_{\mathbf{k}} v^*_k(\eta) + a^+_{-\mathbf{k}} v_k(\eta)\Big] e^{i\mathbf{k}\cdot\mathbf{x}} \\ &= \int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}} \Big[a^-_{\mathbf{k}} v^*_k(\eta) e^{i\mathbf{k}\cdot\mathbf{x}} + a^+_{\mathbf{k}} v_k(\eta) e^{-i\mathbf{k}\cdot\mathbf{x}}\Big],\end{aligned} \tag{4.30}$$

where on the last line, we have relabeled $-\mathbf{k} \leftrightarrow \mathbf{k}$ for the second term.

4.1.2 Solution to 1st order in the slow-roll approximation

Although the Mukhanov-Sasaki equation takes the simple form of a harmonic oscillator in Fourier space, it has an η-dependent effective frequency $\omega_k^2(\eta)$ which makes it difficult to obtain a general solution (i.e. specify the mode function $v_k(\eta)$ unambiguously), however, we are interested in slow-roll inflation, therefore, it will suffice to obtain a solution in the slow-roll approximation. The slow-roll parameters enter (4.23) through the effective mass $m_{\text{eff}}(\eta)$ which depends on the background dynamics. This mass term can be calculated to first order in the slow-roll parameters as follows. Taking the time derivative of z,

$$z' = \frac{a'\bar{\varphi}'}{\mathcal{H}} - \frac{a\mathcal{H}'\bar{\varphi}'}{\mathcal{H}^2} + \frac{a\bar{\varphi}''}{\mathcal{H}}, \tag{4.31}$$

we can form the term

$$\frac{z'}{\mathcal{H}z} = \left(1 - \frac{\mathcal{H}'}{\mathcal{H}}^2\right) + \frac{\bar{\varphi}''}{\mathcal{H}\bar{\varphi}'} = 1 + 2\varepsilon_V - \eta_V, \tag{4.32}$$

where we have used the slow-roll relations (2.55) and (2.57) in the first and second terms respectively to introduce the slow-roll parameters. Applying $\mathrm{d}/\mathrm{d}\eta$ both sides of this equation we find

$$\frac{z''}{\mathcal{H}z} - \frac{\mathcal{H}'z'}{\mathcal{H}^2 z} - \frac{z'^2}{\mathcal{H}z^2} = 2\varepsilon_V' - \eta_V' = \mathcal{O}\left(\epsilon^2\right), \tag{4.33}$$

that is, we find that the expression on the left hand side vanishes to first order in slow-roll since ε_V' and η_V' are second order quantities as can be seen in (2.58) and (2.59). This expression can now be used to solve for z''/z and get

$$m_{\text{eff}}^2 = -\frac{z''}{z} = -\left[\frac{z'^2}{z^2} + \frac{\mathcal{H}'}{\mathcal{H}}\frac{z'}{z}\right] = -\mathcal{H}^2\left(\frac{z'}{\mathcal{H}z} + \frac{\mathcal{H}'}{\mathcal{H}^2}\right)\frac{z'}{\mathcal{H}z} \tag{4.34}$$

$$= -\frac{2 + 9\varepsilon_V - 3\eta_V}{\eta^2}. \tag{4.35}$$

where we have used (2.62) for $\mathcal{H}^2$. Therefore, to first order in the slow-roll approximation, we have the Mukhanov-Sasaki equation in Fourier space:

$$\nu_{\mathbf{k}}'' + \left[k^2 + m_{\text{eff}}^2(\eta)\right]\nu_{\mathbf{k}} = 0, \qquad \text{where} \qquad m_{\text{eff}}^2 = -\frac{2 + 9\varepsilon_V - 3\eta_V}{\eta^2}. \tag{4.36}$$

Making the substitutions $\nu_{\mathbf{k}} = \sqrt{-\eta}y$ and $k\eta = -x$, this equation becomes

$$x^2 \frac{\mathrm{d}^2 y(x)}{\mathrm{d}x^2} + x\frac{\mathrm{d}y(x)}{\mathrm{d}x} + (x^2 - \gamma^2)y(x) = 0, \tag{4.37}$$

where

$$\gamma^2 = \sqrt{\frac{9}{4} + 3\varepsilon_V - 3\eta_V} \qquad \Rightarrow \qquad \gamma = \frac{3}{2} + 3\varepsilon_V - \eta_V + \mathcal{O}\Big(\epsilon^2\Big). \tag{4.38}$$

This is Bessel's ordinary differential equation in standard form whose solution can be formulated as a linear combination of the Hankel functions of the first and second kind $H_\gamma^{(1)}(x)$ and $H_\gamma^{(2)}(x)$ [25, ch. 14]. However, since $x \equiv -k\eta$ is a real variable, these two are the complex conjugate of each other, i.e. $H_\gamma^{(2)}(x) = H_\gamma^{(1)}(x)^*$. Their asymptotic behaviour is

$$H_\gamma^{(1)}(x) \propto \begin{cases} \sqrt{\dfrac{2}{\pi x}} e^{i\left[x - \left(\gamma + \frac{1}{2}\right)\frac{\pi}{2}\right]} & \text{for} \quad x \equiv -k\eta \to \infty \\ \sqrt{\dfrac{2}{\pi}} e^{-i\frac{\pi}{2}} 2^{\gamma - \frac{3}{2}} \dfrac{\Gamma(\gamma)}{\Gamma(\frac{3}{2})} x^{-\gamma} & \text{for} \quad x \equiv -k\eta \to 0, \end{cases} \tag{4.39}$$

where Γ is the gamma function, and $\Gamma(\frac{3}{2}) = \sqrt{\pi}/2$. Early times correspond to $x = -k\eta \to \infty$ and late times to $-k\eta \to 0$. For our purposes, we may ignore the phase factors

$$e^{-i(\gamma + \frac{1}{2})\frac{\pi}{2}} \qquad \text{and} \qquad e^{-i\frac{\pi}{2}}, \tag{4.40}$$

since they will not contribute to the calculation of observables such as the power spectrum. Therefore, the two independent solutions are

$$v_k(\eta) = C_{\mathbf{k}} \sqrt{-\eta} H_\gamma^{(1)}(-k\eta) \tag{4.41}$$

and its complex conjugate $v_k^*(\eta)$. The piece we are still missing is the constant $C_{\mathbf{k}}$.

4.1.3 Quantization

We found that Mukhanov-Sasaki equation (4.18) has the same form as the equation of motion for a free, real scalar field in Minkowski spacetime, for which the quantization procedure is standard. The Lagrangian is

$$\mathcal{L} = -\frac{1}{2}\eta^{\alpha\beta}\nu_{,\alpha}\nu_{,\beta} - \frac{1}{2}m_{\text{eff}}^2(\eta)\nu^2, \tag{4.42}$$

where the effective mass to first order in slow-roll is given in (4.35). We now proceed to quantize in the Heisenberg picture. The dynamical variable is the Mukhanov variable ν and the canonical momentum is given by

$$\pi = \frac{\partial\mathcal{L}}{\partial(\partial_\eta\nu)} = \frac{\partial\mathcal{L}}{\partial\nu'} = \nu'. \tag{4.43}$$

The Hamiltonian density is

$$\mathcal{H} = \pi\nu' - \mathcal{L} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\nu)^2 + \frac{1}{2}m_{\text{eff}}^2, \tag{4.44}$$

and the Hamiltonian of the system is

$$H = \int \mathrm{d}^3x\,\mathcal{H} = \frac{1}{2}\int \mathrm{d}^3x\left(\pi^2 + (\nabla\nu)^2 + m_{\text{eff}}^2\,\nu^2\right). \tag{4.45}$$

The first step is to promote the field variables ν and π to Hermitian operators $\hat{\nu}$ and $\hat{\pi}$

$$\nu \to \hat{\nu} = \hat{\nu}^\dagger, \qquad \pi \to \hat{\pi} = \hat{\pi}^\dagger, \tag{4.46}$$

and impose the equal-time commutation relations between them

$$\begin{aligned} &[\hat{\nu}(\eta,\mathbf{x}), \hat{\pi}(\eta,\mathbf{y})] = i\delta(\mathbf{x}-\mathbf{y}) \\ &[\hat{\nu}(\eta,\mathbf{x}), \hat{\nu}(\eta,\mathbf{y})] = [\hat{\pi}(\eta,\mathbf{x}), \hat{\pi}(\eta,\mathbf{y})] = 0. \end{aligned} \tag{4.47}$$

The quantized Hamiltonian of the system is then

$$\hat{H} = \frac{1}{2}\int \mathrm{d}^3x\left(\hat{\pi}^2 + (\nabla\hat{\nu})^2 + m_{\text{eff}}^2\,\hat{\nu}^2\right), \qquad \text{where} \qquad \hat{H}^\dagger = \hat{H}, \tag{4.48}$$

which we can use to derive Hamilton's equations of motion

$$\hat{\nu}' = \frac{\delta H}{\delta\pi} = \hat{\pi} \tag{4.49}$$

$$\hat{\pi}' = -\frac{\delta H}{\delta\nu} = \nabla^2\hat{\nu} - m_{\text{eff}}^2\,\hat{\nu} \tag{4.50}$$

$$\Rightarrow \quad \left[\Box - m_{\text{eff}}^2(\eta)\right]\hat{\nu}(\eta,\mathbf{x}) = 0, \tag{4.51}$$

where we have used the d'Alembertian in Minkowski space (4.20). Therefore, we see that the quantum operator $\hat{\nu}$ obeys the Mukhanov-Sasaki equation, this motivates using the results of §4.1.1 where we use the mode expansion in (4.30) and its time derivative for $\hat{\nu}(\eta, \mathbf{x})$ and $\hat{\pi}(\eta, \mathbf{x})$,

$$\hat{\nu}(\eta, \mathbf{x}) = \int \frac{\mathrm{d}^3 k}{(2\pi)^{3/2}} \hat{\nu}_{\mathbf{k}}(\eta) e^{i\mathbf{k}\cdot\mathbf{x}} = \int \frac{\mathrm{d}^3 k}{(2\pi)^{3/2}} \left[v_k^*(\eta) e^{i\mathbf{k}\cdot\mathbf{x}} \hat{a}_{\mathbf{k}}^- + v_k(\eta) e^{-i\mathbf{k}\cdot\mathbf{x}} \hat{a}_{\mathbf{k}}^+ \right], \quad (4.52)$$

$$\hat{\pi}(\eta, \mathbf{x}) = \int \frac{\mathrm{d}^3 k}{(2\pi)^{3/2}} \hat{\pi}_{\mathbf{k}}(\eta) e^{i\mathbf{k}\cdot\mathbf{x}} = \int \frac{\mathrm{d}^3 k}{(2\pi)^{3/2}} \left[v_k'^*(\eta) e^{i\mathbf{k}\cdot\mathbf{x}} \hat{a}_{\mathbf{k}}^- + v_k'(\eta) e^{-i\mathbf{k}\cdot\mathbf{x}} \hat{a}_{\mathbf{k}}^+ \right]. \quad (4.53)$$

where we have replaced the constants of integration $a_{\mathbf{k}}^{\pm}$ with quantum operators $\hat{a}_{\mathbf{k}}^{\pm}$ which the Hermicity of $\hat{\nu}$ relates as $\hat{a}_{\mathbf{k}}^+ = (\hat{a}_{\mathbf{k}}^-)^\dagger$. The mode functions $v_k(\eta)$ satisfy

$$v_k'' + \omega_k^2(\eta) v_k = 0, \qquad \omega_k^2(\eta) \equiv k^2 + m_{\text{eff}}^2(\eta), \quad (4.54)$$

and the Fourier modes $\hat{\nu}_{\mathbf{k}}$ and $\hat{\pi}_{\mathbf{k}}$ are given by

$$\hat{\nu}_{\mathbf{k}}(\eta) = \int \frac{\mathrm{d}^3 k}{(2\pi)^{3/2}} \hat{\nu}(\eta, \mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} = v_k^*(\eta) \hat{a}_{\mathbf{k}}^- + v_k(\eta) \hat{a}_{-\mathbf{k}}^+, \quad (4.55)$$

$$\hat{\pi}_{\mathbf{k}}(\eta) = \int \frac{\mathrm{d}^3 k}{(2\pi)^{3/2}} \hat{\pi}(\eta, \mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}} = v_k'^*(\eta) \hat{a}_{\mathbf{k}}^- + v_k'(\eta) \hat{a}_{-\mathbf{k}}^+. \quad (4.56)$$

They are not Hermitian but satisfy

$$\hat{\nu}_{\mathbf{k}} = \hat{\nu}_{-\mathbf{k}}^\dagger, \qquad \hat{\pi}_{\mathbf{k}} = \hat{\pi}_{-\mathbf{k}}^\dagger. \quad (4.57)$$

Choosing the normalization condition

$$W[v_k^*, v_k] = v_k' v_k^* - v_k^{*\prime} v_k = i, \quad (4.58)$$

we see from the canonical commutation relations (4.47), that the Fourier modes $\hat{\nu}_{\mathbf{k}}$ and $\hat{\pi}_{\mathbf{k}}$ obey the commutation relations

$$\begin{aligned} [\hat{\nu}_{\mathbf{k}}, \hat{\pi}_{\mathbf{k}'}] &= i\delta(\mathbf{k} - \mathbf{k}') \\ [\hat{\nu}_{\mathbf{k}}, \hat{\nu}_{\mathbf{k}'}] &= [\hat{\pi}_{\mathbf{k}}, \hat{\pi}_{\mathbf{k}'}] = 0. \end{aligned} \quad (4.59)$$

Similarly, inverting (4.55), (4.56) and using (4.59), we find that the operators $\hat{a}_{\mathbf{k}}^{\pm}$ satisfy the commutation relations

$$\left[\hat{a}_{\mathbf{k}}^-, \hat{a}_{\mathbf{k}'}^+\right] = \delta(\mathbf{k} - \mathbf{k}'), \qquad \left[\hat{a}_{\mathbf{k}}^-, \hat{a}_{\mathbf{k}}^-\right] = \left[\hat{a}_{\mathbf{k}}^+, \hat{a}_{\mathbf{k}'}^+\right] = 0, \quad (4.60)$$

which is the usual algebra of ladder operators. This allow us to interpret $\hat{a}^{\pm}_{\mathbf{k}}$ as operator that create (+) and annihilate (−) excitations (particle states) of the quantum field $\hat{\nu}$, and use them to construct the basis of quantum states. However, such states will acquire an unambiguous physical interpretation only after the particular mode functions $v_k(\eta)$ have been specified. The normalization condition in (4.58) is not sufficient and thus we need to specify an additional initial condition for $v_k(\eta)$. In the case of the simple harmonic oscillator with time-dependent frequency $\omega(t)$ (and in our case, for quantum fields in curved spacetime) there is no unique choice for the mode function $v_k(\eta)$. Hence, no unique decomposition of $\hat{\nu}$ into creation and annihilation operators and no unique notion of the vacuum. Different choices for the mode functions $v_k(\eta)$ give different vacuum solutions (See [26, 27] for a discussion). One choice is the *Minkowski initial condition*,

$$\lim_{-\eta\to\infty} v_k(\eta) = \frac{1}{\sqrt{2k}} e^{-ik\eta}, \tag{4.61}$$

which results in selecting a vacuum known as *the Bunch-Davies vacuum*. This condition is motivated by the observation that at very early times ($-\eta \to \infty$), all scales were much smaller than the Hubble length

$$\frac{k}{\mathcal{H}} \approx |k\eta| \gg 1 \qquad \text{(sub-Hubble scales)}. \tag{4.62}$$

Therefore, using the first order slow-roll approximation for the equation of motion (4.36), we see that the effective frequency becomes time independent in the remote past

$$\omega_k^2(\eta) \equiv k^2 - \frac{2 + 9\varepsilon_V - 3\eta_V}{\eta^2} \approx k^2 \qquad \text{(sub-Hubble scales)}, \tag{4.63}$$

and the equation for the modes (4.54) reduces to

$$v_k'' + k^2 v_k = 0, \tag{4.64}$$

which is the same equation as that of the simple harmonic oscillator with time independent frequency, where the unique normalized mode function is known to be

$$v_k(\eta) = \frac{1}{\sqrt{2k}} e^{-ikr}. \tag{4.65}$$

In this case, the additional condition that allows us to unambiguously define the mode function $v_k(\eta)$ was that it should be the one that makes the vacuum state $|0\rangle$ the ground state of the Hamiltonian. Therefore, applying the initial condition in (4.61) to $v_k(\eta)$ in (4.41), we have

$$v_k(\eta) = \frac{\sqrt{-\pi\eta}}{2} H_\gamma^{(1)}(-k\eta), \tag{4.66}$$

which we can apply at all times during inflation in the slow-roll approximation.

4.1.4 Vacuum fluctuations during inflation

With the mode function completely fixed, we now move to define the *vacuum state* $|0\rangle$ as the state with no particles, that is

$$\hat{a}_{\mathbf{k}}^- |0\rangle = 0, \qquad \text{for all } \mathbf{k}. \tag{4.67}$$

States with particles of definite $\mathbf{k}$ are formed by operating on the vacuum with the creation operator $\hat{a}_{\mathbf{k}}^+$. For instance, a state with one particle whose momentum is $\mathbf{k}$ is given by

$$|\mathbf{k}\rangle = \hat{a}_{\mathbf{k}}^+ |0\rangle. \tag{4.68}$$

We denote the hermitian conjugate of the vacuum state by $\langle 0|$. Thus

$$\langle 0| \hat{a}_{\mathbf{k}}^- = \langle \mathbf{k}| \qquad \text{and} \qquad \langle 0| \hat{a}_{\mathbf{k}}^+ = 0 \tag{4.69}$$

The vacuum is normalized such that $\langle 0|0\rangle = 1$ and the normalization of $|\mathbf{k}\rangle$ is dictated by the commutation relation of $\hat{a}_{\mathbf{k}}^\pm$ in (4.60), which gives

$$\langle \mathbf{k}|\mathbf{k}'\rangle = \delta(\mathbf{k} - \mathbf{k}'). \tag{4.70}$$

We assume that the universe is in the vacuum state of ν during inflation. From (4.48), we can see that the Hamiltonian of the system depends on $\hat{\pi}$, which implies that it does not commute with the field operator $\hat{\nu}$. As a result, the Hamiltonian and the field operator do not share a complete set of eigenstates. So, in general an eigenstate of the Hamiltonian is not an eigenstate of the field operator. Eigenstates of the Hamiltonian operator are the energy eigenstates, and the state with the smallest energy is called the vacuum state. Since the vacuum is not an eigenstate of the field operator, the eigenvalues of the field operator are not well defined, instead we have only a distribution of values. In other words, the scalar field has *vacuum fluctuations.* It can be shown that these fluctuations are Gaussian, which means that they are completely characterized by the power spectrum, which is the quantity that gives the variance of ν as

$$\Big\langle \nu^2(\eta, \mathbf{x}) \Big\rangle = \int_0^\infty \mathrm{d}\ln k\, \mathcal{P}_\nu(\eta, k). \tag{4.71}$$

Using the mode expansion of $\hat{\nu}$ in (4.52) and the definition of the vacuum state in (4.67), we can calculate the power spectrum of the vacuum fluctuations. We first note that the expectation value of $\hat{\nu}$ vanishes

$$\langle 0|\hat{\nu}(\eta, \mathbf{x})|0\rangle = 0, \tag{4.72}$$

and the variance is given by

$$\langle 0|\hat{\nu}^2(\eta, \mathbf{x})|0\rangle = \int \frac{\mathrm{d}^3k\, \mathrm{d}^3k'}{(2\pi)^3} \langle 0|\hat{\nu}_{\mathbf{k}}\hat{\nu}_{\mathbf{k}'}|0\rangle\, e^{i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}}, \tag{4.73}$$

where

$$\begin{aligned}\langle 0|\hat{\nu}_{\mathbf{k}}\hat{\nu}_{\mathbf{k}'}|0\rangle &= \langle 0| \left(\hat{a}^-_{\mathbf{k}} v_k + \hat{a}^+_{-\mathbf{k}} v^*_k\right)\left(\hat{a}^-_{\mathbf{k}'} v_{k'} + \hat{a}^+_{-\mathbf{k}'} v^*_{k'}\right)|0\rangle \\ &= v_k v^*_{k'} \langle 0|\, \hat{a}^-_{\mathbf{k}} \hat{a}^+_{-\mathbf{k}'}\, |0\rangle \\ &= |v_k|^2 \delta^{(3)}(\mathbf{k} + \mathbf{k}'). \end{aligned} \tag{4.74}$$

Therefore

$$\langle 0|\hat{\nu}^2(\eta, \mathbf{x})|0\rangle = \int \frac{\mathrm{d}^3k}{(2\pi)^3}|v_k|^2 = \int_0^\infty \frac{\mathrm{d}k}{k}\frac{k^3}{2\pi^2}|v_k|^2 \tag{4.75}$$

where on the last equality, we have integrated over the angles. Comparing with the definition of the power spectrum in (4.71), we see that the power spectrum of the vacuum fluctuations is given by

$$\mathcal{P}_{\nu}(\eta,k) \equiv \frac{k^3}{2\pi^2}|v_k(\eta)|^2, \tag{4.76}$$

where the mode function is given by (4.66). At late times, we have

$$\lim_{-\eta\to 0}\mathcal{P}_{\nu}(\eta,k) = \frac{1}{(2\pi)^2}2^{2\gamma-3}\left[\frac{\Gamma(\gamma)}{\Gamma(\frac{3}{2})}\right]^2 k^{3-2\gamma}(-\eta)^{1-2\gamma}, \tag{4.77}$$

At this point we assume that the quantum fluctuations become classical variables so that we replace an expectation value of a quantum state with the ensemble average of a classical distribution. The question of how the classical fluctuations emerge from a quantum system remain remains an open problem (see [28] for a discussion). For our purposes, quantum mechanics generates the initial perturbations and solves the problem of how perturbations can emerge from a state which is homogeneous and isotropic. As a result of the indeterministic origin of the fluctuations, we cannot predict the specific member of the ensemble which is realized in the universe, we can only calculate the statistical distribution of perturbations. As noted, this distribution is Gaussian, so all Fourier modes acquire their values as independent random variables (except for the reality condition in (4.24)) with a Gaussian probability distribution.

4.1.5 The primordial power spectrum

In (4.77), we see that the power spectrum of ν has an explicit time dependence

$$\mathcal{P}_{\nu}(\eta,k) \propto (-\eta)^{1-2\gamma} = (-\eta)^{-2-6\varepsilon_V+2\eta_V}. \tag{4.78}$$

However, we mentioned before that the comoving curvature perturbation $\mathcal{R}$, stays constant for $k \ll \mathcal{H}$, thus recalling from (4.17) that the Mukhanov variable is

$\nu = z\mathcal{R}$, we find that the power spectrum of ν and $\mathcal{R}$ are related as

$$\mathcal{P}_{\mathcal{R}}(\eta, k) = \frac{1}{z^2}\mathcal{P}_{\nu}(\eta, k). \tag{4.79}$$

For $k \ll \mathcal{H}$, we have

$$\lim_{-\eta\to 0} \mathcal{P}_{\mathcal{R}}(\eta, k) = \frac{1}{(2\pi)^2} 2^{2\gamma-3}\left[\frac{\Gamma(\gamma)}{\Gamma(\frac{3}{2})}\right] k^{3-2\gamma}\frac{(-\eta)^{1-2\gamma}}{z^2}, \tag{4.80}$$

which appears to still have a time dependence

$$\lim_{-\eta\to 0} \mathcal{P}_{\mathcal{R}}(\eta, k) \propto \frac{(-\eta)^{1-2\gamma}}{z(\eta)^2}, \tag{4.81}$$

but it turns out that it is constant. To see this, recall (4.32) and $\mathcal{H}$ in terms of slow-roll parameters (2.60)

$$\frac{z'}{z} = \frac{1+\varepsilon_V}{(-\eta)}(1 + 2\varepsilon_V - \eta_V) = \frac{1}{(-\eta)}\left(\frac{1}{2} + \gamma\right). \tag{4.82}$$

Solving for $z(\eta)$ we find that the explicit time dependence is given by

$$z \propto (-\eta)^{\frac{1}{2}-\gamma}. \tag{4.83}$$

Plugging this into (4.81), we see that indeed, the power spectrum is constant for $k \ll \mathcal{H}$; it is said that the fluctuations *freeze* as they get stretched beyond the Hubble scale. We may choose to evaluate the power spectrum for each k at the time $\eta = \eta_\star$ of Hubble crossing, i.e. $k = \mathcal{H}_\star$, where $\mathcal{H}_\star = \mathcal{H}(\eta_\star)$. However, note that our expression in (4.80) is the asymptotic behaviour of the power spectrum and does not give the spectrum at this moment of $k = \mathcal{H}_\star$; this is given by the full form of the Hankel function. Using the Hubble parameter to first order in the slow-roll parameters (2.60) and $z = a\varphi'/\mathcal{H}$ from (4.16), the power spectrum becomes

$$\mathcal{P}_{\mathcal{R}}(k) = 2^{2\gamma-3}\left[\frac{\Gamma(\gamma)}{\Gamma\left(\frac{3}{2}\right)}\right](1-\varepsilon_V)^{2\gamma-1}\left(\frac{\mathcal{H}}{2\pi}\right)^2\left(\frac{\mathcal{H}}{a\varphi'}\right)^2\left(\frac{k}{\mathcal{H}}\right)^{3-2\gamma}. \tag{4.84}$$

This has the scale dependence

$$\mathcal{P}_{\mathcal{R}}(k) \propto k^{3-2\gamma} = k^{-6\varepsilon_V+3\eta_V} = k^{n_s-1} \tag{4.85}$$

where we have defined the *scalar spectral index* or *tilt*

$$n_s - 1 \equiv \frac{\mathrm{d}\ln \mathcal{P}_{\mathcal{R}}}{\mathrm{d}\ln k} = -6\varepsilon_V + 2\eta_V. \tag{4.86}$$

The inclusion of -1 in the definition is just customary for historical reasons, and is related to other ways of defining the power spectrum of perturbation. The derivative $\mathrm{d}/\mathrm{d}\ln k$ is calculated by noting that to first order in the slow-roll parameters for a scale at the time of Hubble crossing $k = \mathcal{H}_\star$,

$$\frac{\mathrm{d}\ln k}{\mathrm{d}\eta} = \frac{\mathrm{d}\ln \mathcal{H}}{\mathrm{d}\eta} = \frac{\mathcal{H}'}{\mathcal{H}} = (1-\varepsilon_V)\mathcal{H}, \tag{4.87}$$

where we have used the slow-roll identity $\mathcal{H}^{-2}\mathcal{H}' = (1-\varepsilon_V)$ we found in (2.55), therefore

$$\frac{\mathrm{d}}{\mathrm{d}\ln k} = \frac{1}{1-\varepsilon_V}\frac{1}{\mathcal{H}}\frac{\mathrm{d}}{\mathrm{d}\eta} = \frac{1}{1-\varepsilon_V}\left(-M_{\mathrm{Pl}}^2\frac{V_{,\varphi}}{V}\right)\frac{\mathrm{d}}{\mathrm{d}\varphi} \simeq -M_{\mathrm{Pl}}^2\frac{V_{,\varphi}}{V}\frac{\mathrm{d}}{\mathrm{d}\varphi}, \tag{4.88}$$

where we have used the slow-roll identity $\mathcal{H}^{-1}\varphi' = M_{\mathrm{Pl}}^2 V_{,\varphi}/V$ from (2.50) and calculated to leading order in ε_V on the last equality. Evaluating (4.84) for each k at the time of Hubble exit, we get

$$\mathcal{P}_{\mathcal{R}}(k=\mathcal{H}_\star) = \left[2^{2\gamma-3}\left(\frac{\Gamma(\gamma)}{\Gamma\left(\frac{3}{2}\right)}\right)^2(1-\varepsilon_V)^{2\gamma-1}\left(\frac{\mathcal{H}}{2\pi}\right)^2\left(\frac{\mathcal{H}}{a\varphi'}\right)^2\right]_{\eta=\eta_\star}. \tag{4.89}$$

Switching back to cosmic time t, we see that this in agreement with [29]. This is our final result, the *primordial power spectrum*. It depends only on k. Statistical homogeneity and isotropy of the perturbations, inherited from the symmetry of the background, is a strong feature of inflation. This derivation was carried out by calculating the effective mass $m_{\mathrm{eff}} = -z''/z$ from the Mukhanov-Sasaki equation in (4.18) to first order in the slow-roll parameters. A further approximation can be done by ignoring the background dynamics (setting $\varepsilon_V, \eta_V \to 0$), so that the Mukhanov-Sasaki equation at first order in slow-roll in (4.36) becomes

$$\nu_{\mathbf{k}}'' + \left(k^2 - \frac{2}{\eta^2}\right)\nu_{\mathbf{k}} = 0. \tag{4.90}$$

This limit is sometimes referred as the *de Sitter limit*. In the Mukhanov-Sasaki equation to first order in slow-roll in the form of Bessel's equation in (4.37) this implies that $\gamma = 3/2$ so that we can find that

$$\mathcal{P}_{\mathcal{R}}(k = \mathcal{H}_\star) = \left[\left(\frac{\mathcal{H}}{\varphi'}\right)^2 \left(\frac{\mathcal{H}}{2\pi a}\right)^2\right]_{\eta=\eta_\star}. \tag{4.91}$$

This the main result for quantum fluctuations during inflation. The problem has now been completely reduced to the evolution of the background scalar field φ and the background Hubble parameter $\mathcal{H}$. To complete the information about the curvature perturbation $\mathcal{R}$, we now need to specify the inflaton potential $V(\varphi)$, solve for the background dynamics to get $\mathcal{H}$ and plug it in (4.91). We will come to this in chapter 5 where we introduce the details of Higgs inflation. That, in turn, is the starting point for calculating structure formation and the CMB anisotropy. Turning this around, observations of large-scale structure and the CMB can be used obtain information about quantum processes in the primordial universe. We will discuss this on §4.3. Our next task is calculation of the power spectrum for gravitational waves, the subject of the next section.

4.2 Tensor perturbations

In this section we review another important prediction of any inflationary model, namely the production of a stochastic background of gravitational waves. The main mechanisms of production, ranging from quantum fluctuations of the gravitational field to other mechanisms that can take place during or after inflation are reviewed in [30]. If we consider tensor perturbations only, the Einstein equation for the perturbations gives

$$E^{T\prime\prime}_{ij} + 2\mathcal{H}E^{T\prime}_{ij} - \nabla^2 E^T_{ij} = 0, \tag{4.92}$$

where we have used the perturbation in the Einstein tensor in (3.27). From this equation we see that tensor perturbations solve a wave equation, and hence they can be interpreted as gravitational waves. The absence of matter perturbations follows from the fact that a single scalar field do not source tensor perturbations, as we saw in (3.52). We also recall that in (3.74) we found that E_{ij}^T is a gauge independent quantity, and so we do not have to worry about choosing a gauge, or eliminating gauge degrees of freedom. In Fourier space we may write E_{ij}^T as a sum of the two gravitational polarization amplitudes h_+ and $h_\times$ as

$$E_{ij}^T = \int \frac{\mathrm{d}^3k}{(2\pi)^{3/2}} \sum_{p=+,\times} e_{ij}(\mathbf{k},p) h_{\mathbf{k},p}(\eta) e^{i\mathbf{k}\cdot\mathbf{x}} \tag{4.93}$$

where p labels the polarization and e_{ij} is the circular polarization tensor. For $\mathbf{k}$ in the $\hat{z}$ direction they are

$$e_{ij}(\hat{z},+) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad e_{ij}(\hat{z},\times) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i & 0 \\ -i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.94}$$

while for $\mathbf{k}$ in any other direction, e_{ij} is defined by applying a standard rotation to the i and j indices. They satisfy the relation

$$\sum_{i,j,p} e_{ij}(\mathbf{k},p) e_{ij}^*(\mathbf{k},p) = 2. \tag{4.95}$$

Plugging the (1,1) and (1,2) components of E_{ij}^T into (4.92) we find

$$h''_{\mathbf{k},p} + 2\mathcal{H}h'_{\mathbf{k},p} + k^2 h_{\mathbf{k},p} = 0, \qquad p = +,\times. \tag{4.96}$$

Making the field redefinition[1]

$$\mu_{\mathbf{k},p} = \frac{aM_{\mathrm{Pl}}}{\sqrt{2}} h_{\mathbf{k},p}, \tag{4.97}$$

[1]The prefactor $\frac{M_{\mathrm{Pl}}}{\sqrt{2}}$ provides the normalization that becomes important when we calculate the power spectrum of tensor perturbation and other related quantities. To derive it, we would have to expand the Einstein-Hilbert action to second order in tensor perturbations where we would find that the kinetic term will be multiplied by a factor of $\frac{M_{\mathrm{Pl}}^2}{2}$. A canonical scalar field has a prefactor equal to a half, so the additional $\frac{2}{M_{\mathrm{Pl}}^2}$ must be absorbed into a redefinition of the field. This calculation in carried in detail in [16, 19].

this equation becomes

$$\mu''_{\mathbf{k},p} + \left(k^2 - \frac{a''}{a}\right)\mu_{\mathbf{k},p} = 0, \qquad p = +, \times. \tag{4.98}$$

This equation has the same form of the Mukhanov-Sasaki equation in Fourier space (4.23). To first order in slow-roll, the effective mass is given by

$$m^2_{\text{eff}} = -\frac{a''}{a} = \mathcal{H}' + \mathcal{H}^2 = \frac{2 + \varepsilon_V}{\eta^2}. \tag{4.99}$$

Making the substitutions $\mu_{\mathbf{k}} = \sqrt{-\eta}y(x)$ where $x = -k\eta$, we see that (4.98) is again Bessel's equation. The solution is a linear combination of the Hankel functions, this time of order

$$\lambda^2 = \frac{9}{4} + 3\varepsilon_V \qquad \Rightarrow \qquad \lambda = \frac{3}{2} + \varepsilon_V. \tag{4.100}$$

Since the equation of motion for $\mu_{\mathbf{k}}$ in (4.98) has the same form as that of $\nu_{\mathbf{k}}$ in (4.36), the quantization of $\mu_{\mathbf{k}}$ to obtain initial conditions due to quantum fluctuations is done in the same way as we did for the scalar perturbations. The correctly normalized mode function is

$$w_k(\eta) = \frac{\sqrt{-\eta\pi}}{2} H^{(1)}_{\lambda}(-k\eta), \tag{4.101}$$

and $\mu_{\mathbf{k}}$ acquires the spectrum

$$\mathcal{P}_{\mu}(\eta, k) = \frac{k^3}{2\pi^2}|w_k|^2. \tag{4.102}$$

Therefore, well after Hubble crossing, we have from (4.77)

$$\lim_{-\eta\to 0} \mathcal{P}_{\mu}(\eta, k) = \frac{2^{2\lambda-3}}{(2\pi)^2}\left[\frac{\Gamma(\lambda)}{\Gamma(\frac{3}{2})}\right]^2 (-\eta)^{1-2\lambda}k^{3-2\lambda}. \tag{4.103}$$

Switching to coordinate time t, we see that this result agrees with [29]. However, note that this is the power spectrum for single polarization using the field redefinition in (4.97). The power spectrum of gravitational waves is defined by taking into account both polarizations as

$$\mathcal{P}_T(\eta, k) \equiv \frac{k^3}{2\pi^2}\left|E^T_{ij}(\eta, k)\right|^2 = \frac{k^3}{2\pi^2}(2|h_{\mathbf{k},+}|^2 + 2|h_{\mathbf{k},\times}|^2), \tag{4.104}$$

where we have used the expansion of $E^T_{ij}(\eta, k)$ into the two independent degrees of freedom in (4.93) and the relation in (4.95) for the polarization tensors. Using the field redefinition in (4.97)

$$\mathcal{P}_T(\eta, k) = \frac{k^3}{2\pi^2} \cdot \frac{4}{a^2 M_{\rm Pl}^2}(|\mu_{\mathbf{k},+}|^2 + |\mu_{\mathbf{k},\times}|^2) \tag{4.105}$$

$$= \frac{k^3}{2\pi^2} \cdot \frac{8}{a^2 M_{\rm Pl}^2}|w_k|^2 \tag{4.106}$$

$$= \frac{8}{a^2 M_{\rm Pl}^2}\mathcal{P}_\mu(\eta, k). \tag{4.107}$$

Therefore, we have for the spectrum of gravitational waves

$$\lim_{-\eta\to 0} \mathcal{P}_T = \frac{2^{2\lambda}}{(2\pi M_{\rm Pl})^2}\left[\frac{\Gamma(\lambda)}{\Gamma(\frac{3}{2})}\right]^2 k^{3-2\lambda}\frac{(-\eta)^{1-2\lambda}}{a(\eta)^2}. \tag{4.108}$$

During slow-roll, the scale factor is given by $a \propto (-\eta)^{-1/(1-\varepsilon_V)}$, and thus we see that at late times

$$\lim_{-\eta\to 0} \mathcal{P}_T \propto \frac{(-\eta)^{1-2\gamma}}{a^2} = (-\eta)^0, \tag{4.109}$$

that is, spectrum freezes well after entering the Hubble radius. Using $\mathcal{H}$ during slow roll in (2.60), we have at late times

$$\mathcal{P}_T(k) = \frac{2^{2\lambda}}{M_{\rm Pl}^2}\left[\frac{\Gamma(\lambda)}{\Gamma(\frac{3}{2})}\right]^2 (1-\varepsilon_V)^{2\lambda-1}\left(\frac{\mathcal{H}}{2\pi a}\right)^2\left(\frac{k}{\mathcal{H}}\right)^{3-2\lambda}, \tag{4.110}$$

which has a scale dependence,

$$\mathcal{P}_T \propto k^{3-2\lambda} = k^{-2\varepsilon_V} \equiv k^{n_t}, \tag{4.111}$$

where we have introduced the *tensor spectral index* or *tilt*

$$n_t \equiv \frac{\mathrm{d}\ln \mathcal{P}_t}{\mathrm{d}\ln k} = -2\varepsilon_V. \tag{4.112}$$

Notice that we have not included a -1 in the definition of n_t as we did for n_s in (4.86), this is the common convention. Evaluating (4.110) at the time when the scale k enters the Hubble radius, we arrive at the *primordial spectrum of tensor perturbations*

$$\mathcal{P}_T(k = \mathcal{H}_\star) = \left[\frac{2^{2\lambda}}{M_{\rm Pl}^2}\left(\frac{\Gamma(\lambda)}{\Gamma(\frac{3}{2})}\right)^2 (1-\varepsilon_V)^{2\lambda-1}\left(\frac{\mathcal{H}}{2\pi a}\right)^2\right]_{\eta=\eta_\star}. \tag{4.113}$$

Next we consider the de Sitter limit as we did for the case of scalar perturbations in the previous section. Setting $\varepsilon_V \to 0$, we have from (4.100) that (4.113) becomes

$$\mathcal{P}_T(k = \mathcal{H}_\star) = \frac{8}{M_{\rm Pl}^2}\left(\frac{\mathcal{H}}{2\pi a}\right)^2_{\eta=\eta_\star}. \tag{4.114}$$

4.3 Contact with observations

To compare the theoretical predictions of inflation with observations, it is useful to introduce a phenomenological parameterization of the primordial spectra of scalar and tensor perturbations as [32]

$$\mathcal{P}_\mathcal{R}(k) = A_s(k_*)\left(\frac{k}{k_*}\right)^{n_s(k_*)-1+\frac{1}{2}\frac{{\rm d}n_s(k_*)}{{\rm d}\ln k}\ln\left(\frac{k}{k_*}\right)+\cdots} \tag{4.115}$$

$$\mathcal{P}_T(k) = A_t(k_*)\left(\frac{k}{k_*}\right)^{n_t(k_*)+\frac{1}{2}\frac{{\rm d}n_t(k_*)}{{\rm d}\ln k}\ln\left(\frac{k}{k_*}\right)+\cdots}, \tag{4.116}$$

where A_s (A_t) is the amplitude of the scalar (tensor) perturbation, n_s (n_t) is the scalar (tensor) spectral indices, and ${\rm d}n_s/{\rm d}\ln k$ (${\rm d}n_t/{\rm d}\ln k$) is the *running of the scalar (tensor) spectral index*. These quantities are evaluated at some reference point k_* called the *pivot scale*. Lastly, the dots in the exponentials indicate higher order terms such as the running of the running of the indices and so on. If n_s and n_t are independent of k, we say that their respective power spectrum is *scale-free*. If $n_s = 1$ and $n_t = 0$, the corresponding power spectrum is constant,

$$\mathcal{P}_\mathcal{R}(k) = const, \qquad \mathcal{P}_T(k) = const, \tag{4.117}$$

and is said to be *scale-invariant* (a special case of scale-free spectrum). A scale-invariant spectrum is also called a *Harrison–Zeld'ovich* spectrum. From (4.84) and (4.110) we see that to zeroth order in the slow-roll parameters, the power spectrum of scalar and tensor perturbations is scale-invariant. It is then said that inflation produces nearly scale-invariant or nearly Harrison– Zeld'ovich spectra. The Planck full mission temperature and large angular scale polarization data rule out an exactly scale-invariant spectrum of curvature perturbations at 5.6σ [32] and at least the first two terms A_s and n_s in the expansion 4.115 are needed to fit the data. When we calculated the primordial spectrum of the scalar and tensor perturbations, we found their respective spectral indices n_s and n_t to be

$$n_s - 1 = \frac{\mathrm{d}\ln\mathcal{P}_\mathcal{R}}{\mathrm{d}\ln k} = 2\eta_V - 6\varepsilon_V, \qquad n_t \equiv \frac{\mathrm{d}\ln\mathcal{P}_T}{\mathrm{d}\ln k} = -2\varepsilon_V. \tag{4.118}$$

Recall that the presence of a -1 in n_s (or its absence in n_t) is conventional. To leading order in the slow-roll parameters, using (4.91), we find the *amplitude of scalar perturbation*

$$A_s \equiv \mathcal{P}_\mathcal{R}(k_*) = \frac{1}{12\pi^2 M_{\rm Pl}^6}\frac{V^3}{V_\varphi^2} = \frac{1}{24\pi^2 M_{\rm Pl}^4}\frac{V}{\varepsilon_V}, \tag{4.119}$$

where we have used the slow-roll equations in (2.48) and (2.49) on the first equality and the slow-roll parameter (2.53a) in the second. The next term in the expansion of $\mathcal{P}_\mathcal{R}$ is the running of the scalar spectral index $\mathrm{d}n_s/\mathrm{d}\ln k$ which contains the scale dependence of n_s. To get it, we first calculate the derivative of the slow-roll parameters using (4.88) and find

$$\frac{\mathrm{d}\varepsilon_V}{\mathrm{d}\ln k} = 4\varepsilon_V - 2\varepsilon_V\eta_V, \qquad \frac{\mathrm{d}\eta_V}{\mathrm{d}\ln k} = 2\varepsilon_V\eta_V - \zeta_V. \tag{4.120}$$

Note these quantities are second order in the slow-roll parameters. The running of the scalar spectral index is then

$$\frac{\mathrm{d}n_s}{\mathrm{d}\ln k} = 16\varepsilon_V\eta_V - 24\varepsilon_V^2 - 2\zeta_V. \tag{4.121}$$

In the case of the tensor perturbations, we proceed analogously. The *amplitude of tensor perturbations* is

$$A_t(k_*) \equiv \mathcal{P}_T(k_*) = \frac{2V}{3\pi^2 M_{\rm Pl}^4}, \tag{4.122}$$

which we use to define the *tensor-to-scalar ratio*:

$$r(k_*) \equiv \frac{A_t(k_*)}{A_s(k_*)}. \tag{4.123}$$

This quantity yields the amplitude of the gravitational waves with respect to that of the scalar perturbations at some fixed pivot scale k_*. From (4.91) and (4.114), we find that it depends on the evolution of the inflaton field as

$$r = \frac{8}{M_{\rm Pl}^2}\left(\frac{\varphi'}{\mathcal{H}}\right)^2 = 16\varepsilon_V, \tag{4.124}$$

where we have used the slow-roll relation in (2.50). The tensor spectral parameter and the tensor to scalar ratio are both proportional to the slow roll parameter ε_V, so they are not independent of each other. This provides what is called the *consistency relation*:

$$r = -8n_t. \tag{4.125}$$

Violation of this relation could imply that there was more than one field driving the inflation or a single field with non-canonical kinetic term [31]. For the base ΛCDM model with a power-law power spectrum of curvature perturbations as in (4.115), the Planck 2015 full mission temperature and polarization data measures the scalar spectral index, n_s and the amplitude of scalar perturbations A_s at 68% confidence limit (CL) to be [31]

$$n_s = 0.9655 \pm 0.0062. \qquad (68\%\,\text{CL, Planck TT} + \text{lowP}), \tag{4.126}$$

$$A_s = (2.198^{+0.076}_{-0.085}) \cdot 10^{-9} \qquad (68\%\,\text{CL, Planck TT} + \text{lowP}), \tag{4.127}$$

where "Planck TT" is the best-fit CMB temperature power spectrum and "lowP" is the Planck polarization data in the low-ℓ (multipole) likelihood. From (4.119) we see that we can relate the energy scale of inflation at the time when the pivot scale exits the Hubble radius directly to the slow-roll parameter ε_V as

$$\frac{V}{\varepsilon_V} = 24\pi^2 A_s M_{\rm Pl}^4 \simeq (0.0269 M_{\rm Pl})^4. \tag{4.128}$$

This expression is known as the *CMB normalization*. Since $\varepsilon_V \ll 1$ during inflation, we get an upper limit on the energy scale during inflation

$$V^{1/4} < 6.56 \times 10^{16}\text{GeV}, \tag{4.129}$$

where we have set $\varepsilon_V \approx 1$ corresponding to the end of inflation. The Planck mission also found no evidence for a running of the spectral index. When included in the parameterization (4.115) (ΛCDM + $\mathrm{d}n_s/\mathrm{d}\ln k$), its value is constrained to

$$\frac{\mathrm{d}n_s}{\mathrm{d}\ln k} = -0.0084 \pm 0.0082 \qquad (68\%\,\mathrm{CL},\ \mathrm{Planck\ TT + lowP}). \tag{4.130}$$

Assuming the ΛCDM model plus the addition of tensor perturbations and running of n_s ($\Lambda\mathrm{CDM} + r + \mathrm{d}n_s/\mathrm{d}\ln k$), and the standard slow-roll consistency relation (4.125), the Planck mission gives the following constraints:

$$r_{0.002} < 0.18 \qquad (95\%\,\mathrm{CL},\ \mathrm{Planck\ TT + lowP}) \tag{4.131}$$

$$\frac{\mathrm{d}n_s}{\mathrm{d}\ln k} = -0.013^{+0.010}_{-0.009} \qquad (68\%\,\mathrm{CL},\ \mathrm{Planck\ TT + lowP}) \tag{4.132}$$

with $n_s = 0.9667 \pm 0.0066\,(68\%\,\mathrm{CL})$ (the subscript 0.002 on r is to indicate that it is evaluated at the pivot scale $k_* = 0.002\,\mathrm{Mpc}^{-1}$). At the standard pivot scale, $k_* = 0.05\,\mathrm{Mpc}^{-1}$, the bound is stronger ($r < 0.17$ at $95\,\%$ CL). However, a tighter constraint on gravitational waves comes from measurements of B-mode polarization by the BICEP2 and Keck array collaborations [33] where they provide the upper limit $r < 0.09$ at 95 % CL and when combining these B-mode results with the (more model-dependent) constraints from Planck analysis of CMB temperature plus baryonic acoustic oscillations (BAO) and other data, the analysis yields a combined limit $r_{0.05} < 0.07$ at 95% CL. A constraint on r corresponds to an upper limit to the inflationary energy scale which can be seen by using the link between ε_V and r in 4.124), and the measurement of A_s to get the expression

$$V_* = \frac{3\pi^2 A_s}{2}\, r\, M_{\mathrm{Pl}}^4 = (1.88 \times 10^{16}\,\mathrm{GeV})^4 \frac{r_*}{0.10}. \tag{4.133}$$

Among the models considered in [32], the R^2 inflationary model [9] is the most preferred. Due to high tensor- to-scalar ratio predictions, monomials ($V \propto \varphi^n$) with $n > 2$ are strongly disfavoured with respect to R^2 inflation for Planck TT+lowP in combination with BAO data. By combining with the BICEP2/Keck Array-Planck

(BKP) likelihood [34], this trend is confirmed and natural inflation is also disfavoured. As we will see in the next chapter, the predictions from Higgs-inflation model will also agree well with these results.

5. Higgs inflation

5.1 Introduction

As we have seen in the previous chapters, the inflationary scenario gives an explanation for the observed homogeneity and isotropy at large scale. It also provides a mechanism to explain how the seeds for structure formation arise from quantum fluctuations. This mechanism and its predictions are generic in the sense that they do not depend on a particular model. However which particular realization of inflation takes place in nature is not known. We also discussed the results of the Planck mission, which favours slow-roll inflation with a single field. The Large Hadron Collider (LHC) experiments have confirmed the existence of the Higgs boson [38], with mass value $m_H = 125.09 \pm 0.21\,(\text{stat.}) \pm 0.11\,(\text{syst.})$ GeV, and at the same time found no sign of physics beyond the Standard Model. It turns out that the measured value of m_H theoretically allows for the extension of the theory from the electroweak scale all the way to the Planck scale[1] while staying in the perturbative regime. Since the Higgs boson is the only scalar particle in the SM, one could ask whether it can take the role of the inflaton so that the inflationary phase of the universe is a consequence of the Standard Model. At a first glance this does not

[1]Leaving the issue of the Landau pole aside, this extension can go beyond the the Planck scale. However at this scale the quantum effects of gravity become strong and present descriptions particle interactions in terms of quantum field theory break down because of the non-renormalizability of gravity.

seem to be possible because the Higgs boson has a large coupling constant $\lambda \sim 0.1$ at the electroweak level and it is known that to generate the observed amplitude of perturbations in the chaotic inflation scenario, we need a scalar field with the quartic self-coupling $\lambda \sim 10^{-13}$ at the inflationary scale, or mass $m \sim 10^{13}$ GeV [12]. However, it has been known since the late 1980s that when a scalar field is coupled to the Ricci scalar, we can relax the requirement on the smallness of λ in exchange of a large non-minimal coupling constant [41, 42, 43, 44]. This possibility leads to the question on whether the Higgs boson could be the inflaton. This was the proposal made in [39] in which we have the Lagrangian of the Standard Model with a non-minimal coupling to gravity:

$$\mathcal{L} = \frac{1}{2}M^2 R + \xi \Phi^\dagger \Phi R + \mathcal{L}_{\rm SM}, \tag{5.1}$$

where $\mathcal{L}_{\rm SM}$ is the Standard Model lagrangian, M some mass parameter, and Φ the Higgs doublet. This inflationary model is called *Higgs inflation*, and it is a minimal proposal in the sense that it does not introduce any new degree of freedom besides those already existing in the Standard Model. The non-minimal coupling term is in fact necessary since the renormalization of a quantized scalar field in a curved (classical) background introduces divergent counter terms of this form [26]. In this chapter, we will study the cosmological predictions of this proposal, starting by reviewing the technique of conformal transformations, which allows us to transform the action of the theory in standard general relativity form at the cost of a field redefinition. This will allow us to use the single-field slow-roll formalism we have developed in the previous chapters. Then we will discuss two aspects of the quantum corrections to the model, namely the unitarity of Higgs inflation and how loop corrections can change the shape of the potential and affect the viability of Higgs inflation.

5.2 Non-minimal coupling

The simplest way to construct an action describing a scalar field in a curved spacetime is to start with the action in Minkowski space and replace ordinary derivatives ∂_μ with covariant ones ∇_μ, the Minkowski metric $\eta_{\mu\nu}$ with a curved spacetime metric $g_{\mu\nu}$, and the volume element d^4x with the Lorentz invariant volume element $\mathrm{d}^4x\,\sqrt{-g}$. This is called the *minimal coupling* prescription since it gives the simplest coupling of the scalar field to gravity that is consistent with the principle of general covariance. However, in principle, the action of a single scalar field in a curved background can also include additional terms which couple the field directly to the Ricci scalar. Such couplings are called *non-minimal* and they violate the strong equivalence principle which states that in a small enough region of spacetime, all local effects of gravity must disappear. The action describing these non-minimal coupling is [43]

$$S = \int \mathrm{d}^4x\,\sqrt{-g}\Big[f(\varphi)R - \frac{1}{2}\varphi^{;\mu}\varphi_{;\mu} - V(\varphi)\Big], \tag{5.2}$$

where $f(\varphi)$ is a general function of the scalar field φ and $V(\varphi)$ a general potential. To calculate the cosmological predictions from a theory whose action is of the form of (5.2) would require calculating new single-field slow-roll equations since we worked assuming standard general relativity. To avoid this, we can use the technique of conformal transformations [2]. We already encountered a particular kind of this transformation when we introduced the conformal time η back in (2.7). In that case, the transformation only involved a reparameterization of the time coordinate. This time however, we will see that conformal transformations can also incorporate matter degrees of freedom. The transformation consists on the mapping of the original metric $g_{\mu\nu}$ into a new metric $\hat{g}_{\mu\nu}$ according to

$$\hat{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \tag{5.3}$$

[2]See [45] for a review

where a hat indicates quantities in the new (conformal) frame. The function Ω may depend on both the spacetime curvature and on the matter fields. A careful choice of Ω can restore general relativity so that a non-standard theory of gravity is mapped into standard general relativity at the cost of field redefinitions, which result in more complicated behaviour in the matter sector of the theory. Each choice of field definition is called a *frame* and the generalized action (5.2) where the non-standard interaction is explicit is called the *Jordan frame*, whereas the frame where a non-standard theory of gravity takes the form of general relativity (if it exists) is called the *Einstein frame*. Let us now show that we can put the action (5.2) into the standard form with the conformal transformation (5.3) where

$$\Omega^2 = 2f(\varphi). \tag{5.4}$$

The transformations properties of the other quantities derived from the metric in the action are [6, p. 467]

$$g^{\mu\nu} \to \hat{g}^{\mu\nu} = \frac{1}{\Omega^2} g^{\mu\nu} \tag{5.5}$$

$$\sqrt{-g} \to \sqrt{-\hat{g}} = \Omega^4 \sqrt{-g} \tag{5.6}$$

$$R \to \hat{R} = \frac{1}{\Omega^2} - \frac{6g^{\mu\nu}}{\Omega^3} \nabla_\mu \nabla_\nu \Omega. \tag{5.7}$$

However, we are interested in rewriting in all quantities in terms of the transformed metric $\hat{g}_{\mu\nu}$ instead, which are given by

$$g^{\mu\nu} = \frac{1}{\Omega^2} g^{\mu\nu} \tag{5.8}$$

$$\sqrt{-g} = \frac{1}{\Omega^4} \sqrt{-\hat{g}} \tag{5.9}$$

$$R = \Omega^2 \hat{R} + 6\Omega \hat{g}^{\mu\nu} (\hat{\nabla}_\mu \hat{\nabla}_\nu \Omega) - 12 \hat{g}^{\mu\nu} (\hat{\nabla}_\mu \Omega)(\hat{\nabla}_\nu \Omega), \tag{5.10}$$

where $\hat{\nabla}$ is the covariant derivative built form the conformal metric. Using these and noting that for a scalar field we have $\nabla_\mu\Omega = \partial_\mu\Omega$, the action (5.2) becomes

$$\begin{aligned} S = \int \mathrm{d}^4x\, \frac{\sqrt{-\hat{g}}}{\Omega^4}\Big[\frac{1}{2}\Omega^4 R + 3\Omega^3\hat{g}^{\mu\nu}\hat{\nabla}_\mu\hat{\nabla}_\nu\Omega \\ -6\Omega^2\hat{g}^{\mu\nu}\hat{\nabla}_\mu\Omega\hat{\nabla}_\nu\Omega - \frac{1}{2}\Omega^2\hat{g}^{\mu\nu}\hat{\nabla}_\mu\varphi\hat{\nabla}_\nu\varphi - V(\varphi)\Big] \end{aligned} \tag{5.11}$$

$$\begin{aligned} = \int \mathrm{d}^4x\,\sqrt{-\hat{g}}\Big[\frac{1}{2} - \frac{1}{2\Omega^2}\hat{g}^{\mu\nu}\hat{\nabla}_\mu\varphi\hat{\nabla}_\nu\varphi - U(\varphi) \\ -3\hat{g}^{\mu\nu}\left(2\frac{\hat{\nabla}_\mu\Omega\hat{\nabla}_\nu\Omega}{\Omega^2} - \frac{\hat{\nabla}_\mu\hat{\nabla}_\nu\Omega}{\Omega}\right)\Big], \end{aligned} \tag{5.12}$$

where we have introduced the rescaled potential

$$U(\varphi) \equiv \frac{V(\varphi)}{\Omega^4(\varphi)}. \tag{5.13}$$

The last expression inside the parentheses can be rewritten as

$$2\frac{\hat{\nabla}_\mu\Omega\hat{\nabla}_\nu\Omega}{\Omega^2} - \frac{\hat{\nabla}_\mu\hat{\nabla}_\nu\Omega}{\Omega} = \frac{\hat{\nabla}_\mu\Omega\hat{\nabla}_\nu\Omega}{\Omega^2} - \hat{\nabla}_\mu\hat{\nabla}_\nu\log\Omega. \tag{5.14}$$

Plugging this into the equation above, we see that the last term

$$3\hat{g}^{\mu\nu}\hat{\nabla}_\mu\hat{\nabla}_\nu\log\Omega = 3\hat{\nabla}_\mu(\hat{\nabla}^\mu\log\Omega) \tag{5.15}$$

is a total derivative which represents a surface integral that can be dropped from the integration. Also, using the chain rule, we can write the covariant derivative of the conformal factor as $\hat{\nabla}_\mu\Omega(\varphi) = \Omega'\hat{\nabla}_\mu\varphi$, where $\Omega' \equiv \mathrm{d}\Omega/\mathrm{d}\varphi$, therefore we have

$$\frac{\hat{\nabla}_\mu\Omega\hat{\nabla}_\nu\Omega}{\Omega^2} = \frac{(\Omega')^2}{\Omega^2}\hat{\nabla}_\mu\varphi\hat{\nabla}_\nu\varphi = \frac{(\Omega^2)'^2}{4\Omega^4}\hat{\nabla}_\mu\varphi\hat{\nabla}_\nu\varphi. \tag{5.16}$$

Putting all together, the action is now

$$S = \int \mathrm{d}^4x\,\sqrt{-\hat{g}}\left[\frac{1}{2}\hat{R} - \frac{1}{2}\left(\frac{\Omega^2 + \frac{3}{2}(\Omega^2)'^2}{\Omega^4}\right)\hat{g}^{\mu\nu}\hat{\nabla}_\mu\varphi\hat{\nabla}_\nu\varphi - U(\varphi)\right]. \tag{5.17}$$

Thus we see that we traded a non-minimal coupling of φ to gravity with a non-canonical kinetic field. We can put this term again in canonical form with the following field redefinition

$$\frac{\mathrm{d}\chi}{\mathrm{d}\varphi} = \frac{\sqrt{\Omega^2 + \frac{3}{2}(\Omega^2)'^2}}{\Omega^2}, \tag{5.18}$$

and arrive at the action in the Einstein frame

$$S_E = \int \mathrm{d}^4x\,\sqrt{-\hat{g}}\Big[\frac{1}{2}\hat{R} - \frac{1}{2}\hat{\nabla}^\mu\chi\hat{\nabla}_\mu\chi - U(\chi)\Big], \tag{5.19}$$

where we have introduced a subscript E in the action to indicate that it is given in the Einstein Frame. Notice that we have written the rescaled potential U as a function of φ in (5.17) and then as a function of χ in (5.19). This is possible since φ and χ are uniquely related through (5.18) and we will be writing both U and Ω^2 as either functions of φ or χ, depending on convenience. We can now use these results to carry the analysis of the Higgs inflation model in the Einstein frame with the single-field slow-roll inflation formalism we have developed in the previous chapters.

5.3 Higgs inflation: Tree level analysis

Consider the Lagrangian of the Standard Model (5.1) with a nonminimal coupling of the Higgs field to the Ricci scalar,

$$S_J = \int \mathrm{d}^4x\,\sqrt{-g}\Big(\mathcal{L}_{SM} + \frac{1}{2}MR^2 + \xi\Phi^\dagger\Phi R\Big), \tag{5.20}$$

where the subscript J indicates that we are in the Jordan frame. We work with the unitary gauge where the Higgs doublet is given by

$$\Phi = \begin{pmatrix} h/\sqrt{2} \\ 0 \end{pmatrix}, \tag{5.21}$$

such that $\xi\Phi^\dagger\Phi R = \xi h^2 R/2$ where h is the Higgs field. Consider now the Higgs-gravity sector of the theory and neglect all other matter fields and interactions[3], then the action of the theory becomes

$$S_J = \int \mathrm{d}^4x\,\sqrt{-g}\Big[\frac{1}{2}\Big(M^2 + \xi h^2\Big)R - \frac{1}{2}g^{\mu\nu}h_{,\mu}h_{,\nu} - \frac{\lambda}{4}(h^2 - v^2)^2\Big], \tag{5.22}$$

[3]The interactions of the Higgs with the other particles as well as their conformal transformations are considered in [40]

where λ is the Higgs self coupling parameter and v is expectation value of the Higgs field, which is fixed by the Fermi coupling G_F: $v = (\sqrt{2}G_F)^{-1/2} = 246$ GeV. The mass of the Higgs boson in given by $m_h = \sqrt{2\lambda}v$. The experimentally determined value of the Higgs mass $m_h = 125$ GeV implies that $\lambda = 0.13$ at the electroweak level. At the inflationary level we assume $1 \ll \xi$ and negligible vacuum contribution of the non-minimal coupling to the Planck mass, $\xi v \ll 1$. To study the inflationary predictions of this model, we shall work in the Einstein frame which enables us to use the results from the previous chapters[4]. Comparing this action with the generalized action in (5.2), we have

$$f(h) = \frac{M^2 + \xi h^2}{2}, \qquad V(h) = \frac{\lambda}{4}\left(h^2 - v^2\right)^2. \tag{5.23}$$

The value $\xi = 0$ corresponds to the minimally coupled case since we recover the Einstein-Hilbert term.[5] The Planck scale today is $\sqrt{M^2 + \xi v^2} = 2.4 \times 10^{18}$ GeV. Since $v = 246$ GeV, we have $M \approx M_{\rm Pl} = 1$. To remove the non-minimal coupling, we apply a conformal transformation (5.3) with

$$\Omega^2 = 2f(h) = 1 + \xi h^2. \tag{5.24}$$

The Einstein frame potential is then

$$U(\chi) = \frac{\lambda}{4}\frac{h^4}{(1+\xi h^2)^2}, \tag{5.25}$$

where we have dropped the Higgs vacuum expectation value contribution. To remove the non-canonical kinetic term, we make the field redefinition (5.18) with (5.24) to find

$$\frac{\mathrm{d}\chi}{\mathrm{d}h} = \frac{\sqrt{1 + (1+6\xi)\xi h^2}}{1+\xi h^2}. \tag{5.26}$$

[4] See [43] for the analysis in the Jordan frame

[5] Another special case is the *conformal coupling* $\xi = 1/6$ which acquires its name from the fact that when $M = 0$ and $V = 0$ the action is invariant under transformations of the form 5.3 [27, p. 57].

This relation can be integrated to find

$$\sqrt{\xi}\chi(h) = \sqrt{1+6\xi}\sinh^{-1}\left(\sqrt{(1+6\xi)\xi}h\right) - \sqrt{6\xi}\sinh^{-1}\left(\frac{\sqrt{6}\xi h}{\sqrt{1+\xi h^2}}\right). \tag{5.27}$$

It is useful to present this relation in a an asymtotic form for three different ranges of h.

(1) Small field regime $0 \ll h \ll \frac{1}{\xi}$: In this interval one has

$$\chi \simeq h, \qquad U(\chi) \simeq \frac{\lambda}{4}h^4. \tag{5.28}$$

Here the non-minimum coupling becomes negligible and the potential turns into the usual Standard Model quartic potential. Thus the energy scale $1/\xi$ corresponds to the scale where deviations from the Standard Model become significant.

(2) Intermediate field regime $\frac{1}{\xi} \ll h \ll \frac{1}{\sqrt{\xi}}$: Here one has

$$\chi \simeq \frac{\sqrt{6}}{2}\xi h^2, \qquad U(\chi) \simeq \frac{\lambda}{6\xi^2}\left(\frac{h}{1+\frac{\sqrt{6}}{3}h}\right)^2. \tag{5.29}$$

This is the region relevant for reheating.

(3) Large field regime $\frac{1}{\sqrt{\xi}} \ll h$ (or equivalently, $1 \ll \chi$):

$$\chi \simeq \frac{\sqrt{6}}{2}\ln\left(\xi h^2\right), \qquad U(\chi) \simeq \frac{\lambda}{4\xi^2}\left(1 - e^{-\frac{\sqrt{6}}{3}\chi}\right)^2. \tag{5.30}$$

Here we can see that the potential is exponentially flat since for large values of the field we have $U \to U_0 \equiv \lambda/4\xi^2$. This enables the Higgs field to drive inflation at the plateau. Let us now turn to the slow-roll predictions of this model. Since we are working in the Einstein frame, the slow-roll parameters are those we found in (2.53)

$$\epsilon_U = \frac{1}{2}\left(\frac{U_{,\chi}}{U}\right)^2, \qquad \eta_U = \frac{U_{,\chi\chi}}{U}, \qquad \zeta_U = \frac{U_{,\chi\chi\chi}U_{,\chi}}{U^2}, \tag{5.31}$$

The spectral index n_s, tensor-to-scalar ratio r and running of the spectral index α_s are then

$$n_s = 1 - 6\epsilon_U + 2\eta_U, \qquad r = 16\epsilon_U, \qquad \alpha_s = 16\epsilon_U\eta_U - 24\epsilon_U^2 - 2\zeta_U. \tag{5.32}$$

Using the asymptotic relationship between h and χ as well as the potential in (5.30) we find

$$\epsilon_U = \frac{4}{3}\frac{1}{\xi^2 h^4}, \qquad \eta_U = -\frac{4}{3}\left(\frac{1}{\xi h^2} - \frac{1}{\xi^2 h^4}\right), \qquad \zeta_U = \frac{16}{9}\left(\frac{1}{\xi h^2} - \frac{3}{\xi^3 h^6}\right). \tag{5.33}$$

Inflation ends when the slow-roll conditions fail, i.e., when $\epsilon_U \approx 1$ or $|\eta_U| \approx 1$. This happens for

$$\xi h_{\text{end}}^2 = \sqrt{\frac{4}{3}}. \tag{5.34}$$

In the Einstein frame this value is $\chi_{\text{end}} = 0.940$. Plugging this into the Higgs inflation potential (5.25) we find its value at the end of inflation

$$U_{\text{end}} = 4U_0(7 - 4\sqrt{3}). \tag{5.35}$$

To determine the CMB parameters and compare the predictions of Higgs Inflation with the observed values, we need the value of the field h at Hubble crossing for the pivot scale. Parameters at this moment will be denoted by an asterisk " $*$ ". To get this we recall from (2.67) that the number of e–folds is related to the shape of the potential as

$$N_* = \int_{\chi_{\text{end}}}^{\chi_*} \mathrm{d}\chi \, \frac{U}{U_{,\chi}} = \int_{h_{\text{end}}}^{h_*} \mathrm{d}h \, \frac{U}{U_{,h}} \left(\frac{\mathrm{d}\chi}{\mathrm{d}h}\right)^2 \tag{5.36}$$

$$= \frac{1+6\xi}{8}(h_*^2 - h_{\text{end}}^2) - \frac{3}{4}\ln\frac{1+\xi h_*^2}{1+\xi h_{\text{end}}^2} \tag{5.37}$$

$$\approx \frac{3}{4}\xi h_*^2, \tag{5.38}$$

where we have used (5.26) for the chain rule, dropped the logarithmic term and noted that $\xi \gg 1$. Also, during the stage when perturbations are generated we have $h_*^2 \gg h_{\text{end}}^2$. From this we find that

$$\xi h_*^2 = \frac{4}{3}N_*. \tag{5.39}$$

Using this value into the the slow-roll parameters in (5.33) we find to leading order in N

$$n_s = 1 - \frac{2}{N}, \qquad r = \frac{12}{N^2}, \qquad \alpha_s = -\frac{2}{N^2}. \tag{5.40}$$

Now we need the number of e–folds during inflation. To find this we need to track the pivot scale from the moment of Hubble crossing until today. In particular, we will need details from the time the pivot scale crossed the Hubble scale to the end of reheating, which is defined as the moment when the energy starts to scale like radiation. The details of the preheating and reheating stage for the Standard Model Higgs field non-minimally coupled to gravity have been studied in [46, 47, 48, 49, 50, 51, 52]. It is found that at this stage where the Higgs field is initially an oscillating coherent condensate, the universe does not reheat inmediately during perturbative decay and instead undergoes a complex process where perturbative and non-perturbative effects take place leaving the universe filled with the remnant condensate of the Higgs and a non-thermal distribution of the fermions and bosons from the Standard Model. From there on until thermalization, the evolution of the system is highly non-linear and non-perturbative. Lattice simulations of these processes are carried in [51, 52]. The analysis of this stage gives $\Delta N_{\rm reh} = 4$. Note however that the preheating dynamics can be much more violent than previously thought [53]. Using this value and the expression for the number of e-folds we found in (2.75) from the time the pivot scale $k_*/a_0 = 0.05\,\mathrm{Mpc}^{-1}$ exited the Hubble ratius until today, we find

$$N_* = 61 - \ln\left(\frac{k_*}{a_0 H_0}\right) - \Delta N_{\rm reh} + \ln\left(\frac{U_{k_*}^{1/4}}{U_{\rm end}^{1/4}}\right) - \ln\left(\frac{10^{16}\mathrm{GeV}}{U_{k_*}^{1/4}}\right) \tag{5.41}$$

$$= 52 + \frac{1}{4}\ln\left(\frac{r_*}{0.07}\right) \tag{5.42}$$

where we have used $H_0 = 72.5\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ in the first term and the CMB constraint on r (4.133) in the last term and written in the value 0.07 of r_* allowed by CMB observations as a point of comparison. We have also dropped the third term since it is less than one[6]. The pivot scale thus corresponds to $N_* = 52$ (or $\xi h_*^2 = 70$).

[6]To see this, recall that we can approximate the value of the potential at the moment of Hubble crossing as $U_{k_*} \approx U_0 = \lambda/4\xi^2$. Using this consideration with (5.35) we get $\Delta N \approx 0.3$.

Knowing the number of e–folds allows us to find a relationship between λ and ξ from the normalization condition (4.128)

$$\frac{\xi}{\sqrt{\lambda}} = \frac{1}{12\pi}\sqrt{\frac{2}{A_s}}N_* = 40000. \tag{5.43}$$

Next, using $N_* = 52$ we get the tree level predictions from Higgs Inflation [39, 40]

$$n_s = 0.96, \qquad r = 4.4 \times 10^{-3}, \qquad \alpha_s = -7.4 \times 10^{-4}, \tag{5.44}$$

which are in good agreement with the current observations from the Planck mission, discussed in §4.3.

5.4 Beyond tree level

In the previous section, we found that the potential for Higgs inflation is exponentially flat and its predictions at tree level agree well with the CMB predictions discussed in (§4.3). Beyond the tree-level, the analysis is more complex, and an active area of research. The non-minimal coupling to gravity makes Higgs inflation non-renormalizable, and while we can adopt an effective field theory (EFT) approach, we still need to verify whether the theory is self-consistent. otherwise the conclusions from the previous section would be invalid or questionable to say the least. We explore the unitarity of Higgs inflation in §5.4.1.

We also need to take into account how quantum corrections may change the tree-level results. Corrections to the inflationary potential depend of the Higgs and top quark masses measured at the electroweak (low energy) level which provide a non-trivial connection between particle physics measurements and cosmological observations. Since the action (5.20) for Higgs inflation is non-renormalizable, the question of how to calculate the loop corrections does not have a unique answer since this means, in principle, that an infinite number of counter-terms should be fixed in order to obtain predictions from the theory. This means that observations from scattering experiments at energies below $1/\xi$ around the electroweak scale does not allow us to distinguish between different potentials in the inflationary regime. Therefore additional principles (or assumptions about the UV completion) should be employed to make the theory predictable since we are interested in large background fields and this takes us beyond the limits of ordinary effective field theory [67].

It has been shown that small, mid and large field values Higgs inflation is renormalizable when adopting an effective feld theory approach [57, 55, 66, 56, 82, 93, 94]. However it is not clear how these regimes should be patched. At those boundaries the non-renormalizable operators, also known as threshold corrections, are expected to become important and can have an effect on the renormalization group equations and the inflationary parameters. However it turns out that in most cases the loop corrections do not significantly change the shape of the tree-level potential and instead they reduce the value of λ for large fields [92, 89]. More specifically, the authors in [92] demonstrate that if inflation takes place on the plateau, then as long as the UV corrections do not affect the potential at-tree level but only as corrections to the renormalization group equations, the inflationary predictions are (almost) unaffected. More specifically they show that when UV corrections[7]enter the renormalization group equations, the slow-roll parameters and the number of e-folds depend on the beta-functions and differ from the tree-level expressions. However when the spectral index and the tensor-to-scalar ration are calculated, they show that to leading order in the $1/N$, N being the number of-efolds, all dependence on the beta-functions cancel exactly and the results are identical to the tree-level case. This means that the results are independent from the running and thus insensitive to UV physics that change the running, and also independent of the electroweak boundary conditions on the couplings. These results are also shown to hold for a larger class of inflationary models known as α- and ξ-attractors[89], Higgs inflation being a particular case of the latter. However as we shall see in §5.4.2, for certain fine-tuned parameter values, the running of the coupling constants can cause a qualitative change in the shape of the potential, generating an inflection point (or an almost inflection point, called a critical point) [82, 83, 84, 85, 92, 88, 86], a hilltop [96, 97], or a degenerate vacuum [107, 108].

5.4.1 Unitarity of Higgs Inflation

When coupling general relativity with particle physics models, the energy scale where unitarity is violated strongly depend on the matter content of the theory [68]. The original Higgs inflation original proposal is motivated by the fact that no physics beyond the Standard Model has been observed and we could, in principle, extend the validity of the Standard Model all the way to the Planck scale where we can expect quantum gravitational effects to become relevant. However it is known that this requirement results in the bound $-0.81 \leq \xi \leq 0.64$ on the non-minimal coupling [69] which is small compared to the relatively large value of $\xi = \mathcal{O}(10^4)$ required by Higgs Inflation. This casts some doubt in the consistency of the theory since this would imply that new physics would be required to restore unitary and it could also change the inflationary predictions. The useful criterion to determine the validity of perturbation theory is the tree-level unitarity [58], where we consider the high-energy level behaviour of the tree-level aplitude of N-particle scattering $\mathcal{M}_N$ processes. In renormalizable theories, which might be regarded as fundamental theories valid at arbitrary energy levels[8], these amplitudes grow as

$$\mathcal{M}_N \propto E^{4-N}, \tag{5.45}$$

[7]The choice of UV completion can motivated either by assuming an approximate shift symmetry in the Einstein frame, or demanding that UV physics only enters when required by the self-consistency of the theory. However the authors point that the results for n_s and r will not depend on this choice.

[8]Once again we leave aside the issue of Landau poles, which if present, occur at very high energies and are irrelevant to our discussion.

where E is the energy of the process. If instead the tree amplitudes grow with energy or fall slower than (5.45), the perturbation theory fails at some energy Λ, which can be called an ultra-violet cutoff. Whether the theory gets inconsistent at energies higher than Λ, or just enters into a strongly interacting phase, can not be deduced a priori. In any event, the theory is only predictive with the use of traditional perturbative methods at energies $E < \Lambda$. Thus we arrive at the following definition of the cutoff scale Λ: compute all tree amplitudes with N-particles and find the energy Λ_N at which the unitarity bound in each of them is violated. Then define the cutoff as the minimum of these Λ_N. Note that this actually gives us the worst case scenario since we are not allowing the possibility of cancellations which may raise the value of this cutoff. However, this will suffice for our purposes. Instead of actually calculating the N-particle amplitudes, the cutoff can be estimated by power counting of the operators[9] as was first done in [59, 60]. Let us start by looking at the estimate in the Jordan frame, where the metric is expanded around the Minkowski background and the scalar field around the electroweak vacuum expectation value, i.e.,

$$\Phi = h + v \tag{5.46}$$

$$g_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu}, \tag{5.47}$$

where $\gamma_{\mu\nu}$ is a canonically normalized kinetic term. We then proceed to insert this into the Jordan frame action (5.22) and look for operators with dimension larger than four, divided by some scale Λ_N. Then the leading interaction term is of the form

$$\xi h^2 \eta^{\mu\nu} \partial^2 \gamma_{\mu\nu} + \cdots , \tag{5.48}$$

where the dots stand for the other tensor structures conforming the linearized approximation of the Ricci scalar. This is an operator of dimension five from which we

[9]See [60] for the formalism and its applications to loop corrections in theories of slow-roll inflation.

can read $\Lambda \sim 1/\xi$, which is low since $1/\xi < 1/\sqrt{\xi}$. Thus on dimensional grounds this term makes the tree-level scattering amplitude grow at high energies, and eventually violates the unitarity bound. If true, this would make Higgs inflation *unnatural*, which in this context means that if this conclusion is true, then the ability of Higgs inflation to drive inflation with the degrees of freedom of the Standard Model alone is lost since we now require knowledge of the UV completion of the theory beyond $\Lambda = 1/\xi$. Now let's have a look at the Einstein frame. In this frame all non-linearities are moved into the scalar sector with the conformal transformation (5.24) and the field redefinition in (5.26). For small values of the Higgs field we have

$$h \simeq \chi\Big[1 - \xi^2\chi^2\Big], \tag{5.49}$$

where we have neglected higher order terms. Substituting this into the potential (5.23), the factor of $1 - \xi^2\chi^2$ induces an effective operator fo dimension six proportional to $\xi^2\chi^6$ which once again shows an effective potential[10] $\Lambda = 1/\xi$. However it was pointed out in [61, 62, 63] that the dimension 6 terms in the small field expansion (5.47) of the potential only give rise to unitarity problems in the large field regime $h \equiv \langle\Phi_J\rangle \gg 1/\xi$ or $\chi \equiv \langle\Phi_E\rangle \geq 1$, where the small field expansion is no longer valid. Instead, one should perform a perturbative expansion around large Higgs vacuum expectation values. In particular, [61] claimed that Higgs inflation is consistent all the way to the Planck scale, and the apparent breakdown of the calculation of Higgs-Higgs scattering via s-channel graviton exchange in the Jordan frame at $E \gtrsim \Lambda$ does not signify a new physics scale but rather the failure in the Jordan frame as a calculation method since such breakdown did not happen in the Einstein frame. However this was countered in [64, 65] where it was shown that the unitarity bound $1/\xi$ appears once again in both the Einstein and Jordan frames when Goldstone bosons are taken into account and in particular, [64] showed that even in the unitary gauge, which we used to get the tree level predictions in the

[10] $\mathcal{O}(1)$ factors like $\sqrt{\lambda}$ are neglected.

preceding section, the cutoff appears in the Higgs-gauge interactions.

In [62, 63] the expansion of h around a large expectation value $h \gg 1/\xi$ (here $h \equiv \langle \Phi_J \rangle$ and Φ_J Is the Higgs field in the unitary gauge) is considered. It was then shown that the cutoff scale in the Einstein frame is $\Lambda = 1$ in the inflationary regime, but undergoes a transition to $1/\xi$ for small field values $h \ll 1/\xi$. The authors then argued that this post-inflation unitary bound does not affect our ability to describe physical processes during inflation. This last consideration, together with the observation that during the inflationary evolution of the Universe, the system is not described by its perturbations about the vacuum solution, but rather by excitations above some classical background, were brought together in [67]. It was pointed out that to begin with, the cutoff scale in an effective field theory is not just a number but rather a function depending on the background bosonic field(s) in the theory. Therefore to define the region of validity for the theory, we should fix the background and consider the asymptotic high energy behaviour of tree N-particle amplitudes. In this way, the fields are naturally divided in the slowly varying classical part and quantum excitations

$$\Phi(t, \mathbf{x}) = \bar{\Phi}(t) + \delta\Phi(t, \mathbf{x}), \tag{5.50}$$

where Φ now stands for the generic set of fields in the theory (in this case the metric $g_{\mu\nu}$ and the Higgs field h). The perturbations relevant for the cutoff determination have high frequencies corresponding to short time scales. These are much shorter than the typical time scale of the background evolution and thus the background can be approximated as static with a good accuracy. Let us now sketch the steps for this estimate of the cutoff in the Jordan frame [11] by power counting of the operators, present in the expansion of the action in $\delta\Phi$. Expanding the metric and the Higgs

[11] The calculation in the Einstein frame is a bit more involved and is carried out in [67]. Note however that the effect of fermions and bosons on the cutoff is most easily seen in the Einstein frame.

field around their background values,

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \gamma_{\mu\nu}, \qquad h = \bar{h} + \delta h, \tag{5.51}$$

Then plugging in (5.22) and keeping only the terms with two derivatives of the excitations as they determine the unitarity violation scale, we find that the quadratic Lagrangian for the excitations have the form

$$\mathcal{L}^{(2)} = -\frac{1+\xi\bar{h}^2}{8}\left(\gamma^{\mu\nu}\Box\gamma_{\mu\nu} + 2\partial_\nu\gamma^{\mu\nu}\partial^\rho\gamma_{\mu\rho} - 2\partial_\nu\gamma^{\mu\nu}\partial_\mu\gamma - \gamma\Box\gamma\right) \\ + \frac{1}{2}(\partial_\mu\delta h)^2 + \xi\bar{h}\Big(\Box\gamma - \partial_\lambda\partial_\rho\gamma^{\lambda\rho}\Big)\delta h. \tag{5.52}$$

Making the change of variables

$$\delta h = \sqrt{\frac{1+\xi\bar{h}^2}{1+(\xi+6\xi^2)\bar{h}^2}}\,\delta\hat{h} \tag{5.53}$$

$$\gamma_{\mu\nu} = \frac{1}{\sqrt{1+\xi\bar{h}^2}}\hat{\gamma}_{\mu\nu} - \frac{2\xi\bar{h}\bar{g}_{\mu\nu}}{\sqrt{(1+\xi\bar{h}^2)\Big[1+(\xi+6\xi^2)\bar{h}^2\Big]}}\,\delta\hat{h}, \tag{5.54}$$

we can diagonalize the kinetic term. Then the leading operator is the cubic scalar-graviton interaction $\xi(\delta h)^2\Box\gamma$, which is written in terms of the canonical fields $\delta\hat{h}$ and $\hat{\gamma}_{\mu\nu}$. From this we obtain the dimensional coefficient corresponding to the cut-off scale

$$\Lambda^J_{g-s}(\bar{h}) = \frac{1+(\xi+6\xi^2)\bar{h}^2}{\xi\sqrt{1+\xi\bar{h}^2}} \simeq \begin{cases} \dfrac{1}{\xi}, & \bar{h} \ll \dfrac{1}{\xi}; \\ \bar{h}^2\xi, & \dfrac{1}{\xi} \ll \bar{h} \ll \dfrac{1}{\sqrt{\xi}}; \\ \sqrt{\xi}\bar{h}, & \bar{h} \gg \dfrac{1}{\sqrt{\xi}}. \end{cases} \tag{5.55}$$

The subscript $g-s$ indicates that this cutoff corresponds to the scalar-gravity sector. Notice that that for the field values $\bar{h} \ll 1/\xi$, the cutoff from the preceding discussion is recovered. However for $1/\xi < \bar{h} < 1/\sqrt{\xi}$ (relevant for reheating) the cutoff is still below the Planck mass but starts to grow quadratically and for large values the cutoff grows linearly and coincides with the cutoff in the gravitational sector defined by the effective Planck mass $\sqrt{1+\xi\bar{h}^2}$. Therefore during the whole evolution of the Universe the relevant energy scales are parametrically below the background dependent cutoff. So far this result applies to the pure inflaton model. The effects on the cutoff by the incorporation of fermions and gauge bosons is also studied in [67] where it is shown (in the Einstein frame) that fermions do not change the value of the cutoff. However gauge bosons are found to lower the value of the cutoff. The expression for the Jordan frame is given by [40]

$$\Lambda^J_{\text{gauge}}(\bar{h}) \sim \frac{\sqrt{1+\xi(1+6\xi)\bar{h}^2}}{\sqrt{6}\xi} \sim \begin{cases} \frac{1}{\xi}, & \bar{h} \ll \frac{1}{\xi}; \\ \bar{h}, & \bar{h} \gg \frac{1}{\xi}. \end{cases} \tag{5.56}$$

The value is lower that the one obtained in the scalar-gravity sector, in particular we see that it doesn't grow quadratically in the reheating region as in the previous case and some tension may arise at the beginning of reheating where $\bar{h} \sim 1/\sqrt{\xi}$. Althought this makes a careful calculation more complicated, it is not expected to alter the results singnificantly at the end of this epoch which in turn should not affect the CMB predictions since this effect would enter through the number of e-folds which is logarithmically weak.[12]

With the discovery of the Higgs boson, the non-minimal coupling ξ, so far an unconstrained parameter of nature, was found to be constrained to $|\xi| < 2.6 \times 10^{15}$ [71]. The coupling was also probed through its contribution to weak boson scattering in [76, 77] for both the electroweak and inflationary regimes respectively. In particular, they analyse the perturbative unitarity bound of the SM + GR effective field theory quantitatively and find results consistent with [67]. However objections to the unitarity of Higgs inflation a remain [70, 72]. For instance, the authors in [70] argue that while the cutoff is depends on the background fields, the fact that there

[12]Recall we found in (2.75) that the number of e–folds is

$$N = 61 - \ln\left(\frac{k}{a_0 H_0}\right) - \frac{1}{3}\ln\left(\frac{\rho_{\text{reh}}^{1/4}}{V_{\text{end}}^{1/4}}\right) + \ln\left(\frac{V_k^{1/4}}{V_{\text{end}}^{1/4}}\right) - \ln\left(\frac{10^{16}\text{GeV}}{V_k^{1/4}}\right).$$

.

is a violation of unitarity at the scale $\Lambda \sim 1/\xi$ when the fields are expanded around small field values, implies that in today's universe, new physics must appear, characterized by a mass scale $m \sim 1/\xi$, in order to fix the unitarity problem. In addition to this, they comment that since the scale of unitarity violation increases with large background field values, either this new physics remains at this mass scale during inflation or any mechanism that lifts the mass scale of the new physics would likely mix the new degrees of freedom with the Higgs boson. Therefore they conclude that either way the new physics will become a part of the inflationary dynamics, not just degrees of freedom required to unitarize the model.

In recent years there have been new arguments in favour of the unitarity of Higgs inflation. It has been argued that writing the terms of the action in terms of gauge–independent variables it can be shown that the theory remains unitary all the way to the Planck scale but only tree-level 2-to-2 Higgs scattering has been considered so far [73, 74, 75]. The unitarity of Higgs inflation was also reconsidered in [78] where they use resummation of a certain class of one-loop diagrams and show that the relevant dressed amplitude fulfills what is know as the *Cutkosky rule* (a rule implied by pertubativity) exactly, which the authors present as an example of a self-healing mechanism and a strong indicator that unitarity is restored order by order in perturbation theory. Additionally, [79] points out that the use of dimensional analysis is wrong for large n because it neglects the phase-space volume contribution to the scattering amplitudes, which is precisely the same fact that helped unitarity in the large n-scatterings in electroweak theory [80]. Let us examine this argument in more detail. As we saw in preceding discussion using dimensional analysis, the process consisted of diagonalizing the graviton-Higgs system and then check the scales E_N suppressing the non-renormalizable operators constructed with N powers of the diagonalised field fluctuations. This procedure is however background dependent and so are the vertices E_N. The diagonalization of the Higgs-gravity system does not have an analytical form. For this reason, the usual approach has been to consider the perturbative cutoffs Λ_N constructed on the vertices E_N, only around the electroweak scale ($\delta h_{\rm EW} = v \ll 1/\xi$) and large Higgs vacuum expectation values ($\delta h_{\rm Inf} \gg 1/\sqrt{\xi}$) scales. Within the above approximation, one finds that cutoff in dimensional analysis, i.e. for $\Lambda_N \sim E_N$, is obtained in the $N \to \infty$ limit. In particular, since cut-off are too background dependent, one readily finds $\Lambda_{\rm EW} < \Lambda_{\rm Inf}$. As the potential energy of the Higgs boson becomes larger than $\Lambda_{\rm Inf}$ for $h \ll h_{\rm Inf}$, extrapolating the behaviour of the cut-off in the transition region $v < h < h_{\rm Inf}$, standard lore was to advocate a perturbative unitarity violation there. However, when one takes into account the phase-space contribution to the scattering amplitudes, one finds that it grows linearly with N making the true cutoff Λ_N also growing with N, i.e. $\Lambda_N \sim N E_N$, which contrary to the standard lore, shows that larger and larger dimensional operators correspond to more and more unitary scatterings. Therefore they conclude that no violation of (perturbative) unitarity might ever happen during the whole evolution of the universe.

5.4.2 The effect of quantum corrections to the inflationary potential

To understand the qualitative changes to the potential we note that for field values $v \ll h$ we can approximate the potential as $V \simeq \lambda(h)h^4/4$ with the loop corrections being approximated by the field-dependence of $\lambda(h)$. While the effective potential is gauge-dependent (recall we are working in the unitary gauge), the presence of extrema and field values there are not. Additionally, in Slow-roll inflation, derivatives of the potential are small and thus so will be the gauge-dependence, particularly for inflation near an extremum [90, 100, 91]. We mainly follow [109] (see also[83, 86, 88]) and use the approximation

$$\lambda(h) = \lambda_0 + \frac{b}{4}\ln^2\left(\frac{\xi h^2}{\kappa^2(1+\xi h^2)}\right). \tag{5.57}$$

Here λ_0, b and κ are are some functions of the top quark (pole) mass, Higgs mass, and the strong coupling constant. This approximation assumes the choice of prescription I of [54] in which we set the renormalization scale in the $\overline{\text{MS}}$ renormalization scheme $\mu^2 \sim 1 + \xi h^2$. We also note that this approximation is valid at least near the minimum of $\lambda(h)$, where the logarithm is small (in particular it holds for $\xi h^2 \gtrsim 1$ and $\kappa \lesssim 1$). Using this approximation,the potential in the Einstein frame becomes

$$U(\chi) = \frac{\lambda(h)h^4}{4\Omega^4(h)} = \frac{1}{4}\left[\lambda_0 + \frac{b}{4}\ln^2\left(\frac{\xi h^2}{\kappa^2(1+\xi h^2)}\right)\right]\left(\frac{\xi h^2}{1+\xi h^2}\right)^2 \tag{5.58}$$

$$\equiv \frac{\lambda_0\kappa^4}{4\xi^2}(1 + c\ln^2 x)x^2, \tag{5.59}$$

where $h \equiv h(\chi)$ is given by (5.26). We have also defined

$$c \equiv \frac{b}{4\lambda_0}; \qquad x \equiv \frac{\xi h^2}{\kappa^2(1+\xi h^2)}. \tag{5.60}$$

With these, the loop corrected λ in (5.57) can be written as

$$\lambda(h) = \lambda_0(1 + c\ln^2 x). \tag{5.61}$$

The value of b is stable against the change of the Standard Model parameters, including those in the vicinity of the critical point and can therefore be fixed as $b = 2.3 \times 10^{-5}$ [83]. For $\kappa < 1$, the parameter λ_0 is the value of λ at its minimum, and κ sets the location of the minimum. For $\kappa \geq 1$, the minimum moves to infinite field values. We will focus in the case where the Standard Model is absolutely stable and take $\lambda_0 > 0$. If we could relate the electroweak scale parameters to the inflationary parameters, the values of λ_0 and κ would be fixed; we take them instead as free parameters. Notice that $\lambda \to \lambda_0 + b\ln^2\kappa$ for large ξh^2 and U is asymptotically flat. Plugging (5.58) in the slow-roll relations (5.31) and using (5.26) to convert to derivatives with respect to h, we can find that ϵ is given to lowest order [109]:

$$\epsilon = \frac{8}{(1+[1+6\xi]\xi h^2)h^2}\left(\frac{1+c\ln x+c\ln^2 x}{1+c\ln^2 x}\right)^2 \tag{5.62}$$

If $c < 4$ ($\lambda > b/16$) the only minimum of the potential is at $h = 0$ and the plateau inflation scenario is realized. If $c = 4$ a new feature appears: the first and the second derivatives of the potential vanish at some point and we get an inflection point. For $c > 4$ a new minimum appears at large field values, and there is now a hilltop between this false vacuum and the electroweak vacuum (the inflection point requires $\kappa < e^{1/4}$ while the hilltop can exist for $\kappa < e^{1/4} \dots e^{1/2}$ depending on the value of c). From (5.62) We can see that there are two ways for ϵ to be small: Either $(1 + [1 + 6\xi]\xi h^2)h^2 \gg 1$ or $\left|1 + c\ln x + c\ln^2 x\right| \ll 1$. The first possibility corresponds to plateau inflation where the values of the inflationary parameters are given by the tree-level results in §5.3 and are practically independent of the Standard Model parameters. The second case corresponds to critical point inflation and hilltop inflation (in false vacuum and hillclimbing inflation the potential has a qualitatively similar shape as in hilltop inflation).

Critical Point Inflation

We now turn to the possibility that ϵ is small because the loop corrections are significant, i.e., $\left|1 + c\ln x + c\ln^2 x\right| \ll 1$. This is possible only if $c \gtrsim 4$. In the case when $1 + c\ln x + x\ln^2 x = 0$, which is possible for $c \geq 4$, the potential (5.58) has another extremum, besides $h = 0$. It can be shown that there is an inflection point at $\ln x = -\frac{1}{2}$ if and only if $c = 4$ [109]. The possibility of Higgs inflation at the critical point was first studied in [82] and examined in more detail in [83, 84, 85, 92, 87, 109]. In general it is found that at the critical point, n_s and r get a strong dependence on the precise values of the inflationary masses of the top quark and the Higgs boson and precise predictions should be complemented by a particular UV completion. However these unknown effects can be parameterized with effective jumps in the couplings at the matching [83, 84]. These can then be used, along with the perturbative running at low and high energy regimes, to relate the measured couplings in the Standard Model to the effective Higgs potential during inflation and study the impact when Standard model parameters are varied within their experimentally allowed valued. It is found that inflation at the critical point can increase the value of the tensor-to-scalar ratio significantly relative to its tree-level value. This was also a source of interest since it accommodated the claimed detection of inflationary gravitational waves with $r \approx 0.2$ by the BICEP2 telescope. However this signal has since been shown to be due to foregrounds [95]. It is also found that the running of the spectral index α_s cannot be strongly negative. In particular, [87] finds that a detection of negative running at level $\alpha_s \lesssim -0.01$ would rule out Higgs inflation irrespectively of the details of the matching.

Hilltop inflation

Inflation can also take place when the potential has a local maximum [96, 97]. This maximum is referred to as *the hilltop*. There the slow-roll parameter ϵ is zero and if the potential is flat enough in the vicinity, the slow-roll conditions can also be satisfied such that we may have slow-roll inflation near the hilltop. If $c > 4$, the potential in (5.58) develops a false vacuum and the set up is sensitive to initial conditions since the field has to start close to the hilltop and roll down to the side of the electroweak vacuum after inflation. However the approximation for $\lambda(h)$ in (5.57) is optimized around a minimum and may not be valid nead the hilltop. Nevertheless it does yield a qualitatively correct shape for the potential. Higgs inflation at the hilltop have been studied numerically in [92, 109, 111]. Here too can the effect of the non-renormalizable physics be parameterized with effective jumps between the matchings. However, one crucial difference here is the existence of the hilltop has to be used as an initial condition in the analysis to fix the couplings at the hilltop and only compare the results for with the Standard Model afterwards. In principle, the hilltop can be formed in the small, intermediate or large field values. However, numerical studies carried in [109, 111] show that inflation in agreement with current observations is only possible in the third case where we have $n_s \leq 0.96$, and when n_s is constrained to the observational 95% confidence interval (4.126), they find $1.3 \times 10^{-4} < r < 1.2 \times 10^{-3}$ and $-0.93 \times 10^{-3} < \alpha_s < -0.76 \times 10^{-3}$. Thus comparing with the level results in (5.44) we see that n_s is at most equal to the tree-level results whereas r is always smaller.

False vacuum

The Higgs field can also drive inflation in a false vacuum through tunneling to the electroweak one [100, 101, 102, 103, 104, 105, 106]. Here the non-minimal coupling is not required for the false vacuum to arise, since it can be generated by the loop corrections of the Standard Model alone. However, physics beyond the Standard Model is needed to lower the potential barrier and allow for a graceful exit. Possible mechanisms for this include an extra scalar field, which does not directly couple to the Higgs field [102], and hybrid inflation scenario [101, 104]. In the former case, non-minimal coupling of the extra field plays an important role. In [106] a non-minimal coupling was also added for the Higgs field to make the minimum more shallow. Due to the extra degrees of freedom, this scenario is the a departure from the minimalistic proposal of Higgs inflation and we will not discuss it any further.

Degenerate vacuum

Besides a false vacuum, the Standard Model can also generate another qualitative change in the potential. The observed mass of the Higgs boson $m_H = 125.09 \pm 0.21 \pm 0.11$ GeV turns out to be almost 1σ from the value $m_H = 135 \pm 9$ GeV predicted by the Multiple Point Principle [112] which requires that there exist another vacuum in the Higgs potential around the Planck scale, in addition to the electroweak one. The Hillclimbing scenario [107] is a recent proposal that allows a scalar field to drive inflation without introducing additional degrees of freedom in the presence of the degenerate vacuum. In general, such vacuum spoils the monotonicity of the Higgs potential, which is necessary for successful Inflation. However, if $\Omega \ll 1$ instead of $\Omega \gg 1$, which is what we have employed, it turns out that the Einstein-frame potential U is lifted and and made monotonic, even around a local minimum of the Jordan-frame potential V. As is pointed out in [107], inflation at $\Omega \ll 1$ means that the Jordan-frame potential $V = \Omega^2 U$ increases in time, that is, the inflaton climbs up the Jordan-frame potential hill. This is the crucial feature that allows for successful inflation with inflaton potentials having multiple vacua. The possibility that the Higgs field drives inflation by climbing the denerate vacuum is studied in [108] where two particular models for the conformal factor Ω are considered and it is shown that the resulting inflationary predictions come well within the region favoured by the CMB observations, while showing a sizable deviation from those of the ordinary Higgs inflation. To date, a numerical analysis using jumps in the running of the couplings remains to be carried out.

6. Conclusions

In this thesis we have discussed single-field inflation with the Standard Model Higgs boson as the inflaton. We started by reviewing the basics of the unperturbed, homogeneous and isotropic FLRW model of the universe and introduced the framework of inflation, which is a period of accelerated expansion in the early universe. We found that during such expansion the comoving Hubble length shrinks and we require our matter content to have negative pressure for this stage to take place. These conditions can be met by a single scalar field minimally coupled to gravity slowly-rolling down the potential. Lastly, we discussed how we relate scales during inflation with the ones observed in the present day.

In order to depart from the exact homogeneous and isotropic model, we studied how to describe small inhomogeneities and anisotropies by splitting physical quantities into a background part, described by the FLRW model and a perturbed part. In linear perturbation theory different scales evolve independently. Moreover, the scalar and tensor perturbations do not couple to each other and evolve independently, which enable us to study them separately. The splitting of quantities into a background and perturbation introduces gauge degrees of freedom, that is, degrees of freedom that could be added or removed with a choice of gauge. However we saw that tensor perturbations are gauge independent. However by studying gauge transformations it is possible to construct gauge-invariant quantities, i.e., quantities that depend on the choice of gauge, but remain invariant under the gauge transformation. We considered the comoving gauge and the comoving curvature perturbation $\mathcal{R}$ which has the property that it remains constant at super Hubble scales. This allows calculation of the power spectrum of perturbations.

In the case of the scalar perturbations, we found that the Einstein equation for the perturbations yield the Mukhanov-Sasaki equation which we then proceeded to solve in the slow-roll approximation. The initial conditions for the solution came from the quantization of the system, where the scalar field has vacuum fluctuations that are Gaussian and are characterized by the power spectrum. We found that this spectrum is conserved on super Hubble scales and is nearly scale invariant. A similar result followed for tensor perturbations, whose calculation was simplified by the fact that they are gauge-independent quantities. Lastly, we discussed the constraints imposed by the Planck mission results which favours single-field slow-roll models with low tensor-to-scalar ratio.

We then considered Higgs inflation where the Higgs field plays the role of the inflaton field. Here however the Higgs field is non-minimally coupled to gravity. Nevertheless it is possible to use a conformal transformation to put the action of Higgs inflation in the form of standard general relativity with a minimally coupled field at the expense of a field redefinition. This way we can use the single-field slow-roll formalism we reviewed §2. At tree-level the cosmological predictions agree well with the Planck constraints. However quantum corrections complicate the picture. In particular we focused on the unitarity of Higgs inflation as well as how loop corrections can qualitatively change the shape of the potential. In general, perturbation theory can only be trusted for energies below the unitarity cutoff. In the case of Higgs inflation, power counting arguments of quantum excitations around the vacuum expectation value suggest that field cutoff of the theory is around the energy scale $1/\xi$ which is lower than the inflationary scale $1/\sqrt{\xi}$. However if power counting is carried out above some classical background, as opposed to the vacuum solution, the cutoff is shown to be field-dependent and and all relevant physical scales could remain below this cutoff. However new physics could still enter at this scale and affect the inflationary predictions. It is also possible that there may be a self-healing mechanism that keeps the theory below the unitarity cutoff.

Higgs inflation is a non-renormalizable theory. However it can be studied by adopting an effective field approach. Moreover in the three field regimes we discussed, namely small, mid and large field, it turns out the theory is renormalizable in the effective field theory sense. However it is not clear how should these regimes be patched in order to allow a running through all the regimes. In general this would require knowledge of the the UV completion of the theory. However it turns out that except for fine-tuned conditions, inflation at the plateau is largely unaffected by the loop corrections. However, if the conditions are fine-tuned. then the potential may develop new features such as critical point, a hilltop, a false vacuum or a degenerate vacuum. In the case of critical point and Hilltop inflation numerical studies show that Higgs inflation is still viable and each case produce distinct predictions, allowing some scenarios to be ruled out by the next generation of CMB observations.

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