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NACHMAN'S RECONSTRUCTION METHOD FOR THE CALDERÓN PROBLEM WITH DISCONTINUOUS CONDUCTIVITIES

GEORGE LYTTLE, PETER PERRY, AND SAMULI SILTANEN

ABSTRACT. We show that Nachman's integral equations for the Calderón problem, derived for conductivities in $W^{2,p}(\Omega)$, still hold for L^∞ conductivities which are 1 in a neighborhood of the boundary. We also prove convergence of scattering transforms for smooth approximations to the scattering transform of L^∞ conductivities. We rely on Astala-Päivärinta's formulation of the Calderón problem for a framework in which these convergence results make sense.

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1. INTRODUCTION

Calderón's inverse conductivity problem [5] is to reconstruct the conductivity σ of a conducting body Ω from boundary measurements. The electrical potential u obeys the equation

$$(1.1) \quad \begin{aligned} \nabla \cdot (\sigma \nabla u) &= 0 \\ u|_{\partial\Omega} &= f \end{aligned}$$

where $f \in H^{1/2}(\partial\Omega)$ is the potential on the boundary. The induced current through the boundary is given by the Dirichlet to Neumann map

$$(1.2) \quad \Lambda_\sigma f = \sigma \frac{\partial u}{\partial \nu} \Big|_{\partial\Omega}$$

where ν is the outward normal. The Dirichlet-to-Neumann map is known to be bounded from $H^{1/2}(\partial\Omega)$ to $H^{-1/2}(\partial\Omega)$.

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The identifiability problem for Calderón's inverse problem has been extensively studied, culminating in Astala-Päiväranta's proof [3] that the operator Λ_σ determines σ uniquely provided only that $\sigma \in L^\infty$ has essential infimum bounded below by a strictly positive constant. On the other hand, computational methods to recover the conductivity are based on Nachman's [9] integral equations derived in his analysis of Calderón's problem via two-dimensional inverse scattering theory for Schrödinger's equation at zero energy. The Schrödinger potential corresponding to conductivity σ is $q = (\Delta\sqrt{\sigma})/\sqrt{\sigma}$, so that Nachman must assume $\sigma \in W^{2,p}(\Omega)$ for some $p > 1$ in addition to the essential lower bound.

Numerical algorithms based on Nachman's reconstruction algorithm routinely incorporate a high-frequency cutoff on the scattering transform to make efficient and tractable code [12]. These codes can be used to implement efficient reconstructions for applications in medical imaging [7]. Surprisingly, numerical algorithms based on Nachman's integral equations remain effective for *discontinuous* conductivities provided that such a suitable high-frequency cutoff is imposed on the scattering transform, as shown in recent numerical studies [4].

For smooth conductivities, the high-frequency cutoff can be understood as a regularization technique for the inversion process [8]. This paper is the first in a series designed to explain, from the analytical point of view, why this is also the case for non-smooth conductivities.

Nachman's method uses the given Dirichlet-to-Neumann map to compute the *scattering transform* \mathbf{t} of the Schrödinger potential q , regarded as a potential on \mathbb{R}^2 by extending to 0 outside Ω , and recover σ from the scattering transform. The potential q has $\sqrt{\sigma}$ as its ground state solution if we similarly extend $\sigma(x)$ by setting $\sigma(x) = 1$ on $\mathbb{R}^2 \setminus \Omega$. Known $\bar{\partial}$ -methods in two-dimensional Schrödinger scattering allow one to recover the scattering eigenfunctions (and hence, the ground state) from the scattering data by solving a $\bar{\partial}$ problem.

More precisely, given the potential q , one computes the scattering transform \mathbf{t} from Faddeev's [6] complex geometric optics (CGO) solutions of the the Schrödinger equation:

$$(1.3) \quad \begin{aligned} (-\Delta + q)\psi &= 0, \\ \lim_{|x| \rightarrow \infty} \psi(x, k)e^{-ikx} - 1 &= 0, \end{aligned}$$

where $k = k_1 + ik_2$ is a complex parameter and kx denotes complex multiplication of k with $x = x_1 + ix_2$. These exponentially growing solutions determine the scattering transform \mathbf{t} through the integral formula

$$(1.4) \quad \mathbf{t}(k) = \int_{\mathbb{R}^2} e_k(x)q(x)m(x, k) dx$$

where $m(x, k) = e^{-ikx}\psi(x, k)$ and $e_k(x) = \exp(i(kx + \bar{k}\bar{x}))$. Given \mathbf{t} one can recover $m(x, k)$ (and hence $m(x, 0) = \sqrt{\sigma(x)}$) by solving the $\bar{\partial}$ problem

$$(1.5) \quad \begin{aligned} \bar{\partial}_k m(x, k) &= \frac{\mathbf{t}(k)}{4\pi\bar{k}} e_{-k}(x)\overline{m(x, k)} \\ \lim_{|k| \rightarrow \infty} (m(x, k) - 1) &= 0 \end{aligned}$$

where $\bar{\partial}_k = (1/2)(\partial_{k_1} + i\partial_{k_2})$.

The key point is that the scattering transform of q can be determined directly from the Dirichlet-to-Neumann operator. Because q has compact support, one can

reduce (1.3) and (1.4) to the boundary integral equations

$$(1.6) \quad \psi|_{\partial\Omega} = e^{ikx}|_{\partial\Omega} - S_k(\Lambda_q - \Lambda_0)(\psi|_{\partial\Omega}).$$

$$(1.7) \quad \mathbf{t}(k) = \int_{\partial\Omega} e^{i\bar{k}\bar{x}}(\Lambda_q - \Lambda_0)(\psi|_{\partial\Omega}) ds$$

Here S_k is convolution with the Fadeev Green's function on $\partial\Omega$, an integral operator described in Section 2. The operator Λ_q is the Dirichlet to Neumann operator for the *Schrödinger* problem

$$(1.8) \quad \begin{aligned} (-\Delta + q)\psi &= 0, \\ \psi|_{\partial\Omega} &= f. \end{aligned}$$

The boundary integral equations (1.6) were first introduced by R. Novikov [10]. The operator Λ_0 is the Dirichlet-to-Neumann operator for harmonic functions on Ω , corresponding to $q(x) \equiv 0$ and $\sigma(x) \equiv 1$. Given \mathbf{t} , one can then solve the $\bar{\partial}$ -problem (1.5) and recover σ from

$$\sigma(x) = m(x, 0)^2.$$

In this paper we show that the integral equations (1.6) are still uniquely solvable for the Dirichlet-to-Neumann operator of a positive, essentially bounded conductivity with strictly positive essential lower bound. Moreover, we identify the resulting scattering transform as a natural analogue of Nachman's scattering transform which is, in fact, a limit of scattering transforms obtained through monotone approximation by smooth functions. A key ingredient in our analysis is the Beltrami equation of Astala-Päivärinta and the associated scattering transform, which provides a way of identifying the 'scattering transform' that arises from the limit of Nachman's equations.

To describe our results, we first recall a standard reduction due to Nachman [9, Section 6]. Without loss, we may assume that Ω is the unit disc \mathbb{D} and that $\sigma(x) \equiv 1$ in a neighborhood of \mathbb{D} . We make the second assumption more precise:

$$(1.9) \quad \text{There is an } r_1 \in (0, 1) \text{ so that } \sigma(x) = 1 \text{ for } |x| \geq r_1.$$

Next we describe the Astala-Päivärinta scattering transform which provides the context in which our convergence result can be understood. Given a positive conductivity σ with $\sigma(x) \geq c > 0$ a.e., the Beltrami coefficient associated to σ is $\mu = (1 - \sigma)/(1 + \sigma)$ and satisfies $|\mu(x)| \leq \kappa < 1$. Moreover, μ has compact support since $\sigma(x) = 1$ outside a compact set. For any real solution $u \in H^1(\mathbb{D})$ of (1.1), there exists a real-valued function $v \in H^1(\mathbb{D})$, called the σ -harmonic conjugate of u , so that $f = u + iv$ solves the Beltrami equation

$$(1.10) \quad \bar{\partial}f = \mu\partial f$$

where $\partial = (1/2)(\partial_{x_1} - i\partial_{x_2})$ and $\bar{\partial} = (1/2)(\partial_{x_1} + i\partial_{x_2})$ are the operators of differentiation with respect to x and \bar{x} . Astala and Päivärinta show that the Beltrami equation (1.10) admits CGO solutions which define a scattering transform analogous to \mathbf{t} which remains well-defined under the weaker assumption that $\mu \in L^\infty(\Omega)$.

Theorem 1.1. [3, Theorem 4.2] *Let $\mu \in L^\infty(\mathbb{D})$ with $\|\mu\|_\infty \leq \kappa < 1$. For each $k \in \mathbb{C}$ and each $p \in (2, 1 + \kappa^{-1})$, there exists a unique solution $f_\mu \in W_{\text{loc}}^{1,p}(\mathbb{R}^2)$ of (1.10) of the form $f_\mu = e^{ikx} M_\mu(x, k)$ where $M_\mu(x, k) - 1 \in W^{1,p}(\mathbb{R}^2)$.*

We refer to M_μ as the normalized CGO solution of (1.10) and denote by $M_{\pm\mu}$ the normalized solutions corresponding to μ and $-\mu$. The associated scattering transform τ_μ is given by

$$(1.11) \quad \overline{\tau_\mu(k)} = \frac{1}{2\pi} \int \bar{\partial}_x (M_\mu(x, k) - M_{-\mu}(x, k)) dx$$

If conductivities are smooth, one has [4]

$$(1.12) \quad \mathbf{t}(k) = -4\pi i \bar{k} \tau_\mu(k).$$

Our first result concerns solvability of the Nachman integral equations for a non-smooth conductivity. Observe that, under our assumption (1.9), a solution ψ of (1.8) generates a solution of the boundary value problem (1.1) via $u(x) = \sigma(x)^{-1/2} \psi(x)$, and the Dirichlet-to-Neumann operators for (1.8) and (1.1) are in fact *identical*. Thus, under the assumption (1.9), we can recast (1.6) and (1.7) in terms of the Dirichlet-to-Neumann operators for the original conductivity problem, taking Ω to be the unit disc \mathbb{D} .

$$(1.13) \quad \psi|_{\partial\mathbb{D}} = e^{ikx}|_{\partial\mathbb{D}} - S_k(\Lambda_\sigma - \Lambda_1)(\psi|_{\partial\mathbb{D}}),$$

$$(1.14) \quad \mathbf{t}(k) = \int_{\partial\mathbb{D}} e^{i\bar{k}\bar{x}} (\Lambda_\sigma - \Lambda_1)(\psi|_{\partial\mathbb{D}}) ds$$

where by abuse of notation we write Λ_1 , the Dirichlet-to-Neumann operator corresponding to (1.1) with $\sigma(x) = 1$, instead of Λ_0 , the Dirichlet-to-Neumann operator corresponding to (1.8) with $q = 0$, which are the same thing. Our first result is that (1.13) is uniquely solvable for $\sigma \in L^\infty$ with strictly positive essential infimum and any k .

Theorem 1.2. *Let $\sigma \in L^\infty(\mathbb{D})$ with $\sigma(x) \geq c$ for a fixed $c > 0$, and suppose that (1.9) holds. For each $k \in \mathbb{C}$, there exists a unique $g \in H^{1/2}(\partial\mathbb{D})$ so that*

$$g = e^{ikx} - S_k(\Lambda_\sigma - \Lambda_1)g.$$

As we will see, (1.9) implies that $\Lambda_\sigma - \Lambda_1$ is smoothing even though σ may be nonsmooth. One can then mimic Nachman's original argument from Fredholm theory to prove the unique solvability. We will show that the "scattering transform" generated by (1.14) is a natural limit of smooth approximations, and remains related to the Astala-Päiväranta scattering transform τ by (1.12), even though the Schrödinger problem now involves a distribution potential.

To make this connection, we consider approximation of $\sigma \in L^\infty$ by smooth conductivities. In particular, suppose that σ is a fixed conductivity obeying (1.9) with strictly positive essential infimum, and that $\{\sigma_n\}$ is a sequence of smooth conductivities in \mathbb{D} obeying

- (i) There is a fixed $r_1 \in (0, 1)$ so that $\sigma_n(x) = 1$ for $|x| \geq r_1$ and for all n ,
- (ii) There is a fixed $c > 0$ so that $\sigma_n(x) \geq c$ for a.e. $x \in \mathbb{D}$ and for all n ,
- (iii) For a.e. x , $\sigma_n(x)$ is monotone nondecreasing with $\sigma_n(x) \rightarrow \sigma(x)$ as $n \rightarrow \infty$.

Theorem 1.3. *Suppose that $\{\sigma_n\}$ obeys (i)–(iii), and denote by \mathbf{t}_n (resp. \mathbf{t}) the scattering transform for σ_n (resp. σ) obtained from (1.13)–(1.14). Then $\mathbf{t}_n \rightarrow \mathbf{t}$ pointwise. Moreover, \mathbf{t} is related to the Astala-Päiväranta scattering transform τ for σ by (1.12).*

We will prove Theorem 1.3 by studying convergence of the operators $\Lambda_{\sigma_n} - \Lambda_1$ to $\Lambda_\sigma - \Lambda_1$ as $n \rightarrow \infty$. An important ingredient in the proof will be the fact that the operators $\Lambda_{\sigma_n} - \Lambda_1$ are *uniformly* compact in a sense to be made precise, so that weak convergence (which is relatively easy to prove) can be “upgraded” to norm convergence.

It is then natural to ask, whether, on the other hand, a sequence of cutoff scattering transforms converging to the “true” scattering transform of a singular conductivity produces a convergent reconstruction. This question is much harder because the truncated scattering transforms, by analogy with the Fourier transform, generate approximate conductivities that are *not* identically 1 outside a compact set. This means that the analysis of Astala and Päivärinta, which exploits the compact support of the Beltrami coefficient μ , must be considerably extended. We will return to this analysis in a subsequent paper.

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2. PRELIMINARIES

Here and in what follows, we use the notation $f \lesssim_c g$ to mean that $f \leq Cg$ where the implied constant C depends on the quantities c .

2.1. H^s Spaces, Fourier Basis, Harmonic extensions. An L^2 function $f \in L^2(\partial\mathbb{D})$ admits a Fourier series expansion $f(\theta) \sim \sum_n b_n \varphi_n(\theta)$, where

$$\varphi_n(\theta) = \frac{1}{\sqrt{2\pi}} e^{in\theta}.$$

The equation

$$(2.1) \quad (P_j f)(\theta) = \sum_{|n| \leq j} b_n \varphi_n(\theta)$$

for $j \in \mathbb{N}$ defines a finite-rank projection. For $s \in \mathbb{R}$, we denote by $H^s(\partial\mathbb{D})$ the completion of $C^\infty(\partial\mathbb{D})$ in the norm

$$\|f\|_{H^s(\partial\mathbb{D})} = \left(\sum_{n=-\infty}^{\infty} (1+|n|)^{2s} |b_n|^2 \right)^{1/2}.$$

It is easy to see that the embedding

$$(2.2) \quad H^s(\partial\mathbb{D}) \hookrightarrow H^{s'}(\partial\mathbb{D})$$

is compact provided $s > s'$.

The harmonic extension of $f \in L^2(\partial\mathbb{D})$ to \mathbb{D} is given by

$$u(r, \theta) = \sum_{n=-\infty}^{\infty} r^{|n|} b_n \varphi_n(\theta).$$

It is easy to see that for any $r_1 \in (0, 1)$, the estimate

$$(2.3) \quad \|u\|_{L^2(|x| < r_1)} \lesssim_{m, r_1} \|f\|_{H^{-m}}$$

holds for the harmonic extension.

2.2. Faddeev's Green's Function and the operator S_k . The Faddeev Green's function is the convolution kernel $G_k(x - y)$ where

$$(2.4) \quad G_k(x) = \frac{e^{ikx}}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{e^{ix \cdot \xi}}{|\xi|^2 + 2k(\xi_1 + i\xi_2)} d\xi$$

where $x \cdot \xi = x_1\xi_1 + x_2\xi_2$. This is the natural Green's function for the elliptic problem (1.3). Writing $G_k(x) = e^{ikx}g_k(x)$ we see that $g_k(x)$ differs from the Green's function $G_0(x) = -(2\pi)^{-1} \log|x|$ of the Laplacian by a function which is smooth and harmonic on all of \mathbb{R}^2 and, in particular, is regular at 0 (see, for example, [11, Section 3.1] for further discussion and estimates).

In what follows we will assume $\Omega \subset \mathbb{R}^2$ is bounded and simply connected with smooth boundary (since our application is to $\Omega = \mathbb{D}$) even though these assertions are known in greater generality. In the reduction of (1.3) to the boundary integral equation (1.6), the operator S_k is the corresponding single layer

$$(2.5) \quad (S_k f)(x) = \int_{\partial\Omega} G_k(x - y)f(y) dy.$$

For $p \in (1, \infty)$ and any $f \in L^p(\partial\Omega)$, the function $S_k f$ is smooth and harmonic on $\mathbb{R}^2 \setminus \partial\Omega$. Moreover, since the convolution kernel G_k is at most logarithmically singular, $S_k f$ restricts to a well-defined function on $\partial\Omega$. When restricted to $\partial\Omega$,

$$(2.6) \quad S_k : H^s(\partial\Omega) \rightarrow H^{s+1}(\partial\Omega), \quad s \in [-1, 0]$$

(see [9, Lemma 7.1]), even if Ω only has Lipschitz boundary.

It follows from the form of $G_k(x)$ and classical potential theory that, if $\nu(x)$ is the unit normal to $\partial\Omega$ at $x \in \partial\Omega$, the identities

$$(2.7) \quad \lim_{\substack{z \rightarrow x \\ z \in \mathbb{R}^2 \setminus \Omega}} \langle \nu(x), (\nabla S_k f)(z) \rangle = - \left(\frac{1}{2}I - S_k \right) f(x)$$

$$(2.8) \quad \lim_{\substack{z \rightarrow x \\ z \in \Omega}} \langle \nu(x), (\nabla S_k f)(z) \rangle = - \left(\frac{1}{2}I + S_k \right) f(x)$$

hold

2.3. Alessandrini Identity. We will make extensive use of the following identity [1] which is an easy consequence of Green's theorem. Suppose that u solves (1.1) and that $v \in H^1(\Omega)$ with boundary trace $g \in H^{1/2}(\partial\Omega)$. Then

$$(2.9) \quad \langle g, \Lambda_\sigma f \rangle = \int_{\Omega} \sigma(x)(\nabla u)(x) \cdot (\nabla v)(x) dx$$

where $\langle g, h \rangle$ denotes the dual pairing of $g \in H^{1/2}(\partial\Omega)$ with $h \in H^{-1/2}(\partial\Omega)$.

2.4. A Priori Estimates and Uniqueness Theorems. We'll need the following results from [2] which we state here for the reader's convenience. First, the following *a priori* estimate on solutions of Beltrami's equation to analyze convergence of CGO solutions to the Beltrami equations assuming that the Beltrami coefficients converge pointwise.

Theorem 2.1. [2, Theorem 5.4.2] *Let $f \in W_{loc}^{1,q}(\Omega)$, for some $q \in (q_\kappa, p_\kappa) = (1 + \kappa, 1 + \frac{1}{\kappa})$, satisfy the distortion inequality*

$$|\bar{\partial}f| \leq \kappa |\partial f|$$

for almost every $x \in \Omega$. Then $f \in W_{loc}^{1,p}(\Omega)$ for every $p \in (q_\kappa, p_\kappa)$. In particular, f is continuous, and for every $s \in (q_\kappa, p_\kappa)$, the critical interval, we have the Caccioppoli estimate

$$(2.10) \quad \|\eta \nabla f\|_s \leq C_s(k) \|f \nabla \eta\|_s$$

whenever η is a compactly supported Lipschitz function in Ω .

The following uniqueness theorem for CGO solutions of the conductivity equation will help establish the unique solvability of the integral equation (1.13).

Theorem 2.2. [2, Corollary 18.1.2] *Suppose that $\sigma, 1/\sigma \in L^\infty(\mathbb{D})$ and that $\sigma(x) \equiv 1$ for $|x| \geq 1$. Then the equation $\nabla \cdot (\sigma \nabla u) = 0$ admits a unique weak solution $u \in W_{loc}^{1,2}(\mathbb{C})$ such that*

$$(2.11) \quad \lim_{|x| \rightarrow \infty} (e^{-ikx} u(x, k) - 1) = 0.$$

3. BOUNDARY INTEGRAL EQUATION

In this section we prove Theorem 1.2. Our strategy is to show that the integral operator

$$T_k := S_k(\Lambda_\sigma - \Lambda_1)$$

is compact on $H^{1/2}(\partial\mathbb{D})$ and then mimic Nachman's argument in [9, Section 8] to show that $I + T_k$ is injective. The following simple lemma reduces the compactness statement to interior elliptic estimates plus the property (2.3) of harmonic extensions.

Lemma 3.1. *For any f and g belonging to $H^{1/2}(\partial\mathbb{D})$, the identity*

$$(3.1) \quad \langle g, (\Lambda_\sigma - \Lambda_1)f \rangle = \int_{\mathbb{D}} (\sigma - 1) \nabla v \cdot \nabla u \, dx$$

holds, where u solves (1.1) and v is the harmonic extension of g to \mathbb{D} and $\langle g, h \rangle$ denotes the dual pairing of $g \in H^{1/2}(\partial\mathbb{D})$ with $h \in H^{-1/2}(\partial\mathbb{D})$.

Proof. Let w be the harmonic extension of f to \mathbb{D} . It follows from Alessandrini's identity (2.9) that

$$\begin{aligned} \langle g, (\Lambda_\sigma - \Lambda_1)f \rangle &= \int_{\mathbb{D}} \sigma \nabla v \cdot \nabla u \, dx - \int_{\mathbb{D}} \nabla v \cdot \nabla w \, dx \\ &= \int_{\mathbb{D}} (\sigma - 1) \nabla v \cdot \nabla u \, dx + \int_{\mathbb{D}} \nabla v \cdot \nabla (u - w) \, dx \end{aligned}$$

The second term vanishes since v is harmonic and $(u - w)|_{\partial\mathbb{D}} = 0$. \square

Next, we note the following interior elliptic estimate.

Lemma 3.2. *Suppose that σ satisfies (1.9), let $f \in H^{1/2}(\partial\mathbb{D})$, and let u denote the unique solution of (1.1) for the given f . For any $m > 0$, the estimate*

$$(3.2) \quad \|\nabla u\|_{L^2(|x| < r_1)} \lesssim \|f\|_{H^{-m}(\partial\mathbb{D})}$$

holds, where the implied constant depends only on $m, r_1, \text{ess inf } \sigma$, and $\text{ess sup } \sigma$.

Proof. As before, let w be the harmonic extension of f into \mathbb{D} . Let r_1 be the radius defined in (1.9), and let $0 < r_1 < r_2 < 1$. Choose $\chi \in C^\infty(\overline{\mathbb{D}})$ so that

$$(3.3) \quad \chi(x) = \begin{cases} 0, & 0 \leq |x| \leq r_1 \\ 1, & r_2 \leq |x| \leq 1 \end{cases}$$

Let $h(x) = \chi(x)w(x)$. Note that h has support where $\sigma(x) = 1$. We compute

$$\begin{aligned} \nabla \cdot (\sigma \nabla(u - h)) &= \nabla \cdot (\sigma \nabla u) - \nabla \cdot (\sigma \nabla(h)) \\ &= -(\Delta \chi)w - 2\nabla \chi \cdot \nabla w \end{aligned}$$

By construction, we know $(u - h)|_{\partial \mathbb{D}} = 0$.

The unique solution $v \in H_0^1(\mathbb{D})$ of

$$\nabla \cdot (\sigma \nabla v) = g$$

obeys the bound

$$\|\nabla v\|_{L^2(\mathbb{D})} \lesssim \|g\|_{L^2(\mathbb{D})}$$

where the implied constants depend only on $\text{ess inf } \sigma$ and $\text{ess sup } \sigma$. Hence

$$\|\nabla u\|_{L^2(|x| < r_1)} = \|\nabla(u - w)\|_{L^2(|x| < r_1)} \lesssim \|-(\Delta \chi)h - 2\nabla \chi \cdot \nabla h\|_{L^2(\mathbb{D})}$$

We obtain the desired estimate using (2.3). \square

Next, we prove an operator bound on $(\Lambda_\sigma - \Lambda_1)$ with a uniformity that will be useful later.

Lemma 3.3. *Let $\sigma \in L^\infty(\mathbb{D})$ with $\sigma(x) \geq c > 0$ a.e. for some constant c . Suppose, moreover, that σ obeys (1.9). Then for any $m > 0$, the operator $(\Lambda_\sigma - \Lambda_1)$ is bounded from $H^{-m}(\partial \mathbb{D})$ to $H^m(\partial \mathbb{D})$ with constants depending only on r_1 , m , $\text{ess inf } \sigma$, and $\text{ess sup } \sigma$.*

Proof. We will begin with $f, g \in H^{1/2}(\partial \mathbb{D})$ and show that the pairing

$$|\langle g, (\Lambda_\sigma - \Lambda_1)f \rangle|$$

can be bounded in terms of $\|f\|_{H^{-m}}$ and $\|g\|_{H^{-m}}$. Then a density argument will establish the lemma.

Let v be a harmonic extension of g into \mathbb{D} . Then by Lemma 3.1 we obtain

$$\begin{aligned} |(g, (\Lambda_\sigma - \Lambda_1)f)| &= \left| \int_{\mathbb{D}} (\sigma - 1) \nabla v \cdot \nabla u \, dx \right| \\ &\lesssim_\sigma \|\nabla u\|_{L^2(|x| < r_1)} \|\nabla v\|_{L^2(|x| < r_1)} \\ &\lesssim_{\sigma, r_1, m} \|f\|_{H^{-m}} \|g\|_{H^{-m}} \end{aligned}$$

where we used Lemma 3.2 to estimate $\|\nabla u\|_{L^2(|x| < r_1)}$ and we used (2.3) again to estimate $\|\nabla v\|_{L^2(|x| < r_1)}$. The implied constants depend only on $\text{ess inf } \sigma$ and $\text{ess sup } \sigma$. \square

It now follows from Lemma 3.3 and the compact embedding (2.2) that T_k is compact as an operator from $H^{1/2}(\partial \mathbb{D})$ to $H^{-1/2}(\partial \mathbb{D})$. Thus, to show that (1.13) is uniquely solvable, it suffices by Fredholm theory to show that the only vector

$g \in H^{1/2}(\partial\mathbb{D})$ with $g = -T_k g$ is the zero vector. We will show that any such g generates a global solution to the problem

$$(3.4) \quad \begin{aligned} \nabla \cdot (\sigma \nabla u) &= 0, \\ \lim_{|x| \rightarrow \infty} e^{-ikx} u(x, k) &= 0. \end{aligned}$$

We will then appeal to Theorem 2.2 to conclude that $g = 0$.

Proof of Theorem 1.2. We follow the proof of Theorem 5 in [9, Section 7]. Fix $k \in \mathbb{C}$, suppose that $g \in H^{1/2}(\partial\mathbb{D})$ satisfies $T_k g = -g$, let $h = (\Lambda_\sigma - \Lambda_1)g$ and let $v = S_k h$ on $\mathbb{R}^2 \setminus \partial\mathbb{D}$. The function v is harmonic on $\mathbb{R}^2 \setminus \partial\mathbb{D}$ and continuous across $\partial\mathbb{D}$. Thus, if v_+ and v_- are the respective boundary values of v from $\mathbb{R}^2 \setminus \partial\mathbb{D}$ and from $\partial\mathbb{D}$, $v_+ = v_- = g$. It follows from (2.7)–(2.8) and the fact that $g = -T_k g$ that

$$(3.5) \quad \frac{\partial v_+}{\partial \nu} - \frac{\partial v_-}{\partial \nu} = h = \Lambda_\sigma g - \Lambda_1 g.$$

Since $\partial v_- / \partial \nu = \Lambda_1 g$, we conclude that $\partial v_+ / \partial \nu = \Lambda_\sigma g$. Now define

$$u(x) = \begin{cases} v(x), & x \in \mathbb{R}^2 \setminus \Omega \\ u_i(x), & x \in \Omega \end{cases}$$

where u_i is the unique solution to the problem

$$\nabla \cdot (\nabla u_i) = 0, \quad u|_{\partial\mathbb{D}} = g.$$

In this case $u_+ = u_-$ and $\partial u_+ / \partial \nu = \partial u_- / \partial \nu$, so u extends to a solution of (3.4) as claimed. It now follows from Theorem 2.2 that $u = 0$. Since g is the boundary trace of u , we conclude that $g = 0$. \square

4. CONVERGENCE OF SCATTERING TRANSFORMS

In this section we prove Theorem 1.3 in two steps. First, we show that the Dirichlet-to-Neumann operators Λ_{σ_n} associated to the sequence $\{\sigma_n\}$ converge in norm to Λ_σ . We then use this fact to conclude that the corresponding scattering transforms converge. The second step uses Astala-Päivärinta's scattering transform to identify the limit.

We begin with a simple result on weak convergence that exploits Alessandrini's identity and convergence of positive quadratic forms.

Lemma 4.1. *Suppose that $\{\sigma_n\}$ is a sequence of positive $L^\infty(\mathbb{D})$ obeying conditions (i)–(iii) of Theorem 1.3. Then $\Lambda_{\sigma_n} \rightarrow \Lambda_\sigma$ in the weak operator topology on $\mathcal{L}(H^{1/2}(\partial\mathbb{D}), H^{-1/2}(\partial\mathbb{D}))$.*

Proof. For any σ , it follows from (2.9) that Λ_σ defines a positive quadratic form

$$\langle f, \Lambda_\sigma f \rangle = \int_{\mathbb{D}} \sigma |\nabla u|^2 dx$$

on $H^{1/2}(\partial\mathbb{D})$. Moreover, by monotone convergence, the quadratic forms $\Lambda_\sigma - \Lambda_{\sigma_n}$ are nonnegative for all n . If we can show that

$$(4.1) \quad \lim_{n \rightarrow \infty} \langle f, (\Lambda_\sigma - \Lambda_{\sigma_n}) f \rangle = 0$$

it will then follow by polarization that $\Lambda_{\sigma_n} \rightarrow \Lambda_\sigma$ in the weak operator topology. But

$$(4.2) \quad \langle f, (\Lambda_\sigma - \Lambda_{\sigma_n}) f \rangle = \int_{\mathbb{D}} (\sigma - \sigma_n) |\nabla u|^2 dx + \int_{\mathbb{D}} \sigma_n (|\nabla u|^2 - |\nabla u_n|^2) dx.$$

The first right-hand term in (4.2) goes to zero by monotone convergence. Since the $\{\sigma_n\}$ are uniformly bounded, it suffices to show that $\nabla u_n \rightarrow \nabla u$ in L^2 . A straightforward computation shows that

$$0 = \nabla \cdot (\sigma_n \nabla (u_n - u)) + \nabla \cdot ((\sigma_n - \sigma) \nabla u).$$

Multiplying through by $v_n = u_n - u$ and integrating over \mathbb{D} , we obtain

$$(4.3) \quad \int_{\mathbb{D}} \sigma_n |\nabla v_n|^2 dx = - \int_{\mathbb{D}} (\sigma_n - \sigma) \nabla v_n \cdot \nabla u dx.$$

Since σ_n is bounded below by a fixed positive constant c independent of n , we can use the Cauchy-Schwarz inequality to conclude that

$$\frac{c}{2} \int_{\mathbb{D}} |\nabla v_n|^2 dx \leq \frac{1}{2c} \int_{\mathbb{D}} (\sigma - \sigma_n) |\nabla u|^2 dx$$

and conclude that $\nabla u_n \rightarrow \nabla u$ in L^2 by monotone convergence. \square

From Lemma 3.3 we obtain the following uniform approximation property for the operators

$$(4.4) \quad A_n := \Lambda_{\sigma_n} - \Lambda_1.$$

Lemma 4.2. *Suppose that $\{\sigma_n\}$ is a sequence of conductivities obeying hypotheses (i)–(iii) of Theorem 1.3, and let A_n be defined as in (4.4). Given any $\varepsilon > 0$ there is a $k \in \mathbb{N}$ independent of n so that*

$$\|(I - P_k)A_n\|_{H^{1/2} \rightarrow H^{-1/2}} < \varepsilon, \quad \|A_n(I - P_k)\|_{H^{1/2} \rightarrow H^{-1/2}} < \varepsilon,$$

Proof. From Lemma 3.3 we have the uniform operator bound $\|A_n\|_{H^m \rightarrow H^{-m}} \lesssim_m 1$ since the σ_n have uniformly bounded essential infima and suprema and all obey (1.9). If A'_n denotes the Banach space adjoint of A_n , we have the same bound on A'_n by duality. The second bound is equivalent to the bound

$$\|(I - P_k)A'_n\|_{H^{1/2} \rightarrow H^{-1/2}} < \varepsilon$$

by duality, so we'll only prove the first bound. We write

$$\begin{aligned} \|(I - P_k)A_n\|_{H^{1/2} \rightarrow H^{-1/2}} &\leq \|(I - P_k)\|_{H^m \rightarrow H^{1/2}} \|A_n\|_{H^{-m} \rightarrow H^m} \\ &\lesssim_m k^{1/2-m} \end{aligned}$$

with constants uniform in n . \square

Now let $A = \Lambda_\sigma - \Lambda_1$ where $\sigma_n \rightarrow \sigma$.

Proposition 4.3. *Suppose that $\{\sigma_n\}$ satisfies hypotheses (i)–(iii) of Theorem 1.3. Then $A_n \rightarrow A$ in the norm topology on the bounded operators from $H^{1/2}$ to $H^{-1/2}$.*

Proof. Write

$$(4.5) \quad A_n - A = P_k(A_n - A)P_k + (I - P_k)(A_n - A) + (A_n - A)(I - P_k).$$

Since A is a fixed compact operator, we can choose $N \in \mathbb{N}$ so $\|(I - P_k)A\|_{H^{1/2} \rightarrow H^{-1/2}}$ and $\|A(I - P_k)\|_{H^{1/2} \rightarrow H^{-1/2}}$ are small for any $k \geq N$. Combining this observation with Proposition 4.3, we can choose $k \in \mathbb{N}$, uniformly in n , so that the first and

third right-hand terms of (4.5) are small uniformly in n . The middle term vanishes for any fixed k and $n \rightarrow \infty$ by Lemma 4.1. \square

As an easy consequence:

Proposition 4.4. *Fix $k \in \mathbb{C}$. Suppose that $\{\sigma_n\}$ is a sequence obeying hypotheses (i)–(iii) of Theorem 1.3, and denote by $g_n(\cdot, k)$ and $g(\cdot, k)$ the respective solutions of (1.13) corresponding to σ_n and σ . Then, for each fixed k , $g_n \rightarrow g$ in $H^{1/2}(\partial\mathbb{D})$. Moreover, the scattering transforms \mathbf{t}_n of σ_n converge pointwise to given by (1.14).*

Proof. By a slight abuse of notation, denote by T_n the operator $S_k(\Lambda_{\sigma_n} - \Lambda_1)$ and by T the operator $S_k(\Lambda_\sigma - \Lambda_1)$. It follows from (2.6) that $T_n \rightarrow T$ in $\mathcal{L}(H^{1/2}, H^{1/2})$. Since

$$g_n = (I - T_n)^{-1} (e^{ikx}|_{\partial\mathbb{D}}), \quad g = (I - T)^{-1} (e^{ikx}|_{\partial\mathbb{D}}),$$

it follows from the second resolvent identity that $g_n \rightarrow g$ in $H^{1/2}(\partial\mathbb{D})$. Convergence of \mathbf{t}_n to \mathbf{t} follows from the norm convergence of g_n to g and of $\Lambda_{\sigma_n} - \Lambda_1$ to $\Lambda_\sigma - \Lambda_1$. \square

In the remainder of this section, we will identify what \mathbf{t} actually is. In order to do so we need to prove a convergence theorem for the Astala-Päiväranta scattering transforms τ_n of the Beltrami coefficients $\mu_n = (1 - \sigma_n)/(1 + \sigma_n)$ to the transform τ of σ that is of some interest in itself.

Proposition 4.5. *Suppose that $\{\mu_n\}$ is a sequence of Beltrami coefficients with $0 \leq \mu_n(x) \leq \kappa$ for a.e. x and $0 \leq \kappa < 1$. Suppose further that $\mu_n(x) \rightarrow \mu(x)$ where $\mu \in L^\infty(\mathbb{D})$ has the same properties. Finally, fix $k \in \mathbb{C}$ and let $M_{\pm\mu_n}(x, k)$ be the normalized CGO solution for the Beltrami equation (1.10) with Beltrami coefficients $\pm\mu_n$, and let $M_{\pm\mu}$ be the normalized CGO solution for $\pm\mu$. Then, for a single choice of sign, $M_{\pm\mu_n} - 1 \rightarrow M_{\pm\mu} - 1$ weakly in $W^{1,p}(\mathbb{R}^2)$ for any $p \in (2, 1 + \kappa^{-1})$.*

We will prove Proposition 4.5 in several steps. First we show how to conclude the proof of Theorem 1.3 given its result.

Proof of Theorem 1.3, given Proposition 4.5. Proposition 4.5 and (1.11) show that $\tau_{\mu_n} \rightarrow \tau$ pointwise as $n \rightarrow \infty$ since the integral in (1.11) may be regarded as integrating the derivatives of $M_{\pm\mu_n}$ (which, by (1.10), are supported in the unit disc) against a smooth, compactly supported function which is identically 1 in a neighborhood of \mathbb{D} . Since τ_n converges pointwise to τ and $\mathbf{t}_n(k) = -4\pi i \bar{k} \tau_{\mu_n}(k)$, we conclude that $\mathbf{t}(k) = -4\pi i \bar{k} \tau(k)$. \square

To establish the weak convergence, we first need a uniform bound on $M_{\pm\mu_n} - 1$ in $W^{1,p}(\mathbb{R}^2)$.

Lemma 4.6. *Suppose that $\{\mu_n\}$ is a sequence of Beltrami coefficients obeying the hypothesis of Proposition 4.5, and let $M_n = M_{\mu_n}$. Then there exists a constant C such that*

$$(4.6) \quad \sup_n \|M_n - 1\|_{W^{1,p}(\mathbb{R}^2)} < C.$$

Proof. Let $c_n = \|M_n - 1\|_{W^{1,p}(\mathbb{R}^2)}$. If $c_n \rightarrow +\infty$ as $n \rightarrow \infty$, set $v_n = c_n^{-1}(M_n - 1)$. Since $\{v_n\}$ is bounded in $W^{1,p}$, by passing to a subsequence we can assume that $\{v_n\}$ has a weak limit, v . Note that $\|v_n\|_{W^{1,p}(\mathbb{R}^2)} = 1$.

We first claim that, if such a limit exists, it is nonzero. Suppose, on the other hand, that $v_n \rightarrow 0$ weakly in $W^{1,p}(\mathbb{R}^2)$. It follows from the Rellich-Kondrachov Theorem that $v_n \rightarrow 0$ in $L^p_{\text{loc}}(\mathbb{R}^2)$. A short computation shows that

$$(4.7) \quad \bar{\partial}v_n = \frac{\mu_n}{c_n} \bar{\partial}e_k + \mu_n \overline{\partial(e_k v_n)}$$

and, since $v_n \in W^{1,p}(\mathbb{R}^2)$ we may invert the $\bar{\partial}$ operator using the Cauchy transform and use standard estimates on the Cauchy transform (see, for example, [2, Theorem 4.3.8]) to conclude that

$$(4.8) \quad \|v_n\|_{L^p(\mathbb{R}^2)} \lesssim_p \left\| \frac{\mu_n}{c_n} \bar{\partial}e_k \right\|_{L^{2p/(2+p)}(\mathbb{R}^2)} + \left\| \mu_n \overline{\partial(e_k v_n)} \right\|_{L^{2p/(2+p)}(\mathbb{R}^2)} \\ + \left\| \mu_n \overline{e_k \partial v_n} \right\|_{L^{2p/(2+p)}(\mathbb{R}^2)}.$$

The first right-hand term in (4.8) clearly goes to zero as $n \rightarrow \infty$ since $c_n \rightarrow \infty$. The function in the second term is supported in \mathbb{D} owing to the factor μ_n and therefore also converges to zero since $v_n \rightarrow 0$ in $L^p_{\text{loc}}(\mathbb{R}^2)$. The function in the third term is again supported in \mathbb{D} and, using a version of the Caccioppoli inequality adapted to the v_n 's (see Lemma 4.7 below), we have $\|\partial v_n\|_{L^p(\mathbb{D})} \lesssim \|v_n\|_{L^p(2\mathbb{D})} + \mathcal{O}(c_n^{-1})$, which shows that the third term also goes to zero as $n \rightarrow \infty$. Thus, $v_n \rightarrow 0$ in $L^p(\mathbb{R}^2)$. Applying Lemma 4.7 to the compactly supported function $\bar{\partial}v_n$ shows that, also $\|\bar{\partial}v_n\|_{L^p} \rightarrow 0$ as $n \rightarrow \infty$, contradicting the fact that $\|v_n\|_{W^{1,p}(\mathbb{R}^2)} = 1$ for all n . From this contradiction we conclude that $\{v_n\}$ has a nonzero limit, again assuming that $c_n \rightarrow \infty$.

Next, we show that the limit function v is a weak solution of the equation $\bar{\partial}v = \mu \overline{\partial(e_k v)}$. For $\varphi \in C_0^\infty(\mathbb{R}^2)$ we compute from (4.7)

$$(\varphi, v_n) = c_n^{-1}(\varphi, \mu_n \bar{\partial}e_k) + (\varphi, \mu_n \overline{\partial(e_k v_n)}).$$

where $(f, g) = \int fg$. It is easy to see that the first right-hand term vanishes as $n \rightarrow \infty$. In the second term,

$$(\varphi, \mu_n \overline{\partial(e_k v_n)}) = (\bar{\partial}(e_{-k})\mu\varphi, \bar{v}_n) + (e_k\mu\varphi, \overline{\partial v_n}) + (\varphi(\mu_n - \mu), \overline{\partial(e_k v_n)}) \\ \rightarrow (\bar{\partial}(e_{-k})\mu\varphi, \bar{v}) + (e_k\mu\varphi, \overline{\partial v})$$

since $\|v_n\|_{W^{1,p}} = 1$ and $v_n \rightarrow v$ in L^p_{loc} . It follows that v is a weak solution of $\bar{\partial}v = \mu \overline{\partial(e_k v)}$ with $\|v\|_{W^{1,p}} \leq 1$.

Thus, assuming that $\|M_n - 1\|_{W^{1,p}(\mathbb{R}^2)}$ is not bounded, we have constructed a nonzero solution $v \in W^{1,p}(\mathbb{R}^2)$ of the equation $\bar{\partial}v = \mu \overline{\partial(e_k v)}$. However, this violates the uniqueness of the normalized CGO solution for Beltrami coefficient μ proved in [3, Theorem 4.2], a contradiction. We conclude that $\|M_n - 1\|_{W^{1,p}(\mathbb{R}^2)}$ is bounded uniformly in n . \square

To complete the proof of Lemma 4.6, we need to establish the a priori bounds on the sequence v_n constructed in its proof. To do so, we will need the *a priori* estimate for solutions of the Beltrami equation from Theorem 2.1.

Lemma 4.7. *Suppose that v_n is a sequence of functions as constructed in the proof of Lemma 4.6. Then, the estimate*

$$\|\partial v_n\|_{L^p(\mathbb{D})} \lesssim c_n^{-1} + \|v\|_{L^p(2\mathbb{D})}$$

where the implied constants are independent of n .

Proof. By construction, the function $f = e^{ikz}(c_n v_n + 1)$ satisfies the Beltrami equation

$$\bar{\partial}f = \mu_n \bar{\partial}f$$

and hence satisfies the distortion inequality. Thus by Theorem 2.1, for a compactly supported smooth function η we can write

$$(4.9) \quad \|\eta \partial f\|_p \leq C_{p,\kappa} \|f \nabla \eta\|_p$$

Note that by the triangle inequality, we have

$$(4.10) \quad \begin{aligned} \|\eta \partial f\|_p = \|\eta \partial(e^{ikz}(c_n v_n + 1))\|_p &\geq c_n \|\eta e^{ikz} \partial v_n\|_p \\ &\quad - |k| \|\eta e^{ikz}(c_n v_n + 1)\|_p \end{aligned}$$

which enables us to write

$$(4.11) \quad \begin{aligned} c_n \|\eta e^{ikz} \partial v_n\|_p &\leq \|\eta \partial(e^{ikz}(c_n v_n + 1))\|_p \\ &\quad + |k| \|\eta e^{ikz}\|_p + c_n |k| \|\eta e^{ikz} v_n\|_p \end{aligned}$$

Next, we apply (4.9) to obtain

$$(4.12) \quad \begin{aligned} c_n \|\eta e^{ikz} \partial v_n\|_p &\lesssim c_n \|e^{ikz}(\nabla \eta) v_n\|_p + \|e^{ikz} \nabla \eta\|_p \\ &\quad + |k| \|\eta e^{ikz}\|_p + c_n |k| \|\eta e^{ikz} v_n\|_p. \end{aligned}$$

Thus we conclude

$$(4.13) \quad \begin{aligned} \|\eta e^{ikz} \partial v_n\|_p &\lesssim \|e^{ikz}(\nabla \eta) v_n\|_p + |k| \|\eta e^{ikz} v_n\|_p \\ &\quad + \frac{1}{c_n} \left(\|e^{ikz} \nabla \eta\|_p + |k| \|\eta e^{ikz}\|_p \right) \end{aligned}$$

To obtain the desired estimate, we choose η supported on the disk of radius 2 so that $\eta = e^{-ikz}$ in \mathbb{D} . \square

We can now give the proof of Proposition 4.5 and thereby complete the proof of Theorem 1.3.

Proof of Proposition 4.5. By Lemma 4.6, the sequences $\{M_{\pm\mu_n} - 1\}$ for either choice of sign are bounded in $W^{1,p}(\mathbb{R}^2)$. We will take a single choice of sign, the + sign, and write M_n for M_{μ_n} and M for M_μ from now on. The sequence $\{M_n - 1\}$ has a weak limit point in $W^{1,p}(\mathbb{R}^2)$ which we denote by $M^\sharp - 1$. By the Rellich-Kondrachov theorem, $M_n - 1$ converges in $L^p_{\text{loc}}(\mathbb{R}^2)$ to $M^\sharp - 1$. We wish to show that

$$(4.14) \quad \begin{aligned} \bar{\partial}M^\sharp &= \mu \bar{\partial}(e_k M^\sharp), \\ M^\sharp - 1 &\in W^{1,p}(\mathbb{R}^2) \end{aligned}$$

since we can then conclude that $M^\sharp - 1$ is nonzero (as the PDE does not admit the solution $M^\sharp = 1$) and that $M^\sharp = M$ since the PDE is uniquely solvable for $M^\sharp - 1 \in W^{1,p}(\mathbb{R}^2)$.

From $\bar{\partial}M_n = \mu_n \bar{\partial}(e_k M_n)$ we conclude that for any $\varphi \in C_0^\infty(\mathbb{R}^2)$,

$$\begin{aligned} -(\bar{\partial}\varphi, M_n) &= \left(\varphi, \mu_n \bar{\partial}(e_k M_n) \right) \\ &= \left(\varphi, \mu \bar{\partial}(e_k M_n) \right) + \left((\mu_n - \mu), \bar{\partial}(e_k M_n) \right) \end{aligned}$$

The second right-hand term vanishes as $n \rightarrow \infty$ by dominated convergence since $\mu_n - \mu$ is supported in \mathbb{D} while $\{\partial(e_k M_n)\}$ is uniformly bounded in $L^p(\mathbb{R}^2)$. Weak convergence of derivatives allows us to conclude that (4.14) holds. \square

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