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INVERSE PROBLEMS FOR ELLIPTIC EQUATIONS WITH POWER TYPE NONLINEARITIES

MATTI LASSAS, TONY LIIMATAINEN, YI-HSUAN LIN, AND MIKKO SALO

Dedicated to the memory of Yaroslav Kurylev

ABSTRACT. We introduce a method for solving Calderón type inverse problems for semilinear equations with power type nonlinearities. The method is based on higher order linearizations, and it allows one to solve inverse problems for certain nonlinear equations in cases where the solution for a corresponding linear equation is not known. Assuming the knowledge of a nonlinear Dirichlet-to-Neumann map, we determine both a potential and a conformal manifold simultaneously in dimension 2, and a potential on transversally anisotropic manifolds in dimensions $n \geq 3$. In the Euclidean case, we show that one can solve the Calderón problem for certain semilinear equations in a surprisingly simple way without using complex geometrical optics solutions.

Keywords. Inverse boundary value problem, Calderón problem, semilinear equation, Riemannian manifold, transversally anisotropic.

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1. INTRODUCTION

In this paper we study inverse boundary value problems for nonlinear elliptic equations. A standard example of inverse problems for linear elliptic equations is the problem introduced by Calderón [Cal80], where the objective is to determine the electrical conductivity of a medium by making voltage and current measurements on its boundary. It is closely related to the problem of determining an unknown potential q in a Schrödinger operator $\Delta + q$ from boundary measurements, first solved in [SU87] in dimensions $n \geq 3$. There is an extensive theory concerning inverse boundary value problems for linear elliptic equations, and we refer to [Uh109] for a survey.

It is also natural to consider analogous inverse problems under nonlinear settings. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^∞ boundary, and consider the reaction-diffusion equation

$$\partial_t w - \Delta w = a(x, w) \text{ in } \Omega \times \{t > 0\}.$$

Equations of this type arise in the modelling of chemical reactions, population dynamics and pattern formation [Vol14]. Examples include the Fisher, Kolmogorov or logistic diffusion equations with quadratic nonlinearity (i.e. $a(x, w)$ is quadratic

in w), the Newell-Whitehead-Segel equation with cubic nonlinearity, and equations in combustion involving polynomial or exponential nonlinearities.

A stationary solution $w(x, t) = u(x)$ satisfies the elliptic equation

$$\Delta u + a(x, u) = 0 \text{ in } \Omega.$$

The Dirichlet problem for this equation is related to maintaining a temperature (or concentration or population) f on the boundary. The boundary measurements for such an equation, provided that it is well-posed for some class of boundary values, may be encoded by a Dirichlet-to-Neumann map (DN map) Λ_q , which maps the boundary value f to the flux $\Lambda_q(f) = \partial_\nu u|_{\partial\Omega}$ of the corresponding equilibrium state across the boundary.

In fact, inverse problems for nonlinear elliptic equations have also been widely studied. A standard method, introduced in [Isa93] in the parabolic case, is to show that the first linearization of the nonlinear DN map is actually the DN map of a linear equation, and to use the theory of inverse problems for linear equations. For the semilinear Schrödinger equation $\Delta u + a(x, u) = 0$, the problem of recovering the potential $a(x, u)$ was studied in [IS94, Sun10] in dimensions $n \geq 3$, and in [IN95, Sun10, IY13] when $n = 2$. In addition, inverse problems have been studied for quasilinear elliptic equations [Sun96, SU97, KN02, LW07, MU18], the degenerate elliptic p -Laplace equation [SZ12, BHKS18], and the fractional semilinear Schrödinger equation [LL19]. Certain Calderón type inverse problems for quasilinear equations on Riemannian manifolds were recently considered in [LLS18]. We refer to the survey articles [Sun05, Uhl09] for further details on inverse problems for nonlinear elliptic equations.

Inverse problems have also been studied for hyperbolic equations with various nonlinearities. Many of the works mentioned above rely on a solution to a related inverse problem for a linear equation. This is in contrast to the study of inverse problems for nonlinear hyperbolic equations, where it has been realized that the nonlinearity can actually be beneficial in solving inverse problems.

By using the nonlinearity as a tool, some still unsolved inverse problems for hyperbolic linear equations have been solved for their nonlinear counterparts. For the scalar wave equation with a quadratic nonlinearity, Kurylev-Lassas-Uhlmann [KLU18] proved that local measurements determine the global topology, differentiable structure and the conformal class of the metric g on a globally hyperbolic 4-dimensional Lorentzian manifold. The authors of [LUW18] studied inverse problems for general semilinear wave equations on Lorentzian manifolds, and in [LUW17] they studied analogous problem for the Einstein-Maxwell equations. For more inverse problems of nonlinear hyperbolic equations, we refer readers to [CLOP19, dHUW18, KLOU14, WZ19] and references there in.

In this work we introduce a method which uses nonlinearity as a tool that helps in solving inverse problems for certain nonlinear elliptic equations. The method is based on *higher order linearizations* of the DN map, and essentially amounts to using sources with several parameters and obtaining new linearized equations after differentiating with respect to these parameters. We demonstrate the scope of the method by solving Calderón type problems for three mathematical models.

The first model is the Calderón problem for a semilinear Schrödinger equation with quadratic nonlinearity,

$$(1.1) \quad \Delta u + qu^2 = 0 \text{ in } \Omega \subset \mathbb{R}^n,$$

where $q \in C^\infty(\overline{\Omega})$ and $n \geq 2$. The solution to a related inverse problem with $a(x, u)$ in place of qu^2 is known under assumptions like $\partial_u a(x, u) \leq 0$ [IS94, IN95, Sun10]. Theorem 1.1 proves uniqueness for the nonlinearity qu^2 , which appears to be a

new result. The method applies to more general models, but we begin with the operator (1.1) in order to introduce our approach in the simplest possible setting.

The second new result is Theorem 1.2, where we simultaneously determine the metric, the manifold and the potential up to gauge symmetry from the knowledge of the DN map of a semilinear Schrödinger equation on two-dimensional Riemannian surfaces. The analogous result for a linear Schrödinger equation is not known in this generality. Here we use nonlinearity to simultaneously determine the topology and the conformal structure of the Riemannian surface, as well as the potential, up to a natural gauge transformation.

The third result, Theorem 1.3, is the recovery of the potential q from the knowledge of the DN map of a Schrödinger operator with nonlinearity of the form qu^m , $m \geq 3$, on transversally anisotropic manifolds in dimensions $n \geq 3$. Transversally anisotropic manifolds are product type manifolds which appear in several works related to the anisotropic Calderón problem. Again, the solution to the analogous inverse problem for a linear equation is not known in this generality. Existing results will be discussed in more detail later in this introduction.

Let us introduce the mathematical setting for this article. We will denote by (M, g) a compact Riemannian manifold with C^∞ boundary ∂M , where $\dim(M) = n$, $n \geq 2$. For example, one could have $M = \overline{\Omega}$ where Ω is a bounded C^∞ domain in \mathbb{R}^n , and g could be the Euclidean metric. Let $q \in C^\infty(M)$. We will consider semilinear elliptic equations of the form

$$(1.2) \quad \begin{cases} \Delta_g u + qu^m = 0 & \text{in } M, \\ u = f & \text{on } \partial M, \end{cases}$$

where

$$m \in \mathbb{N} \text{ and } m \geq 2.$$

Here Δ_g is the Laplace-Beltrami operator, given in local coordinates by

$$\Delta_g u = \frac{1}{\det(g)^{1/2}} \sum_{a,b=1}^n \frac{\partial}{\partial x_a} \left(\det(g)^{1/2} g^{ab} \frac{\partial u}{\partial x_b} \right),$$

where $g = (g_{ab}(x))$ and $(g^{ab}(x)) = g^{-1}$.

We will show that the Dirichlet problem (1.2) has a unique small solution u for sufficiently small boundary data $f \in C^s(\partial M)$, where $s > 1$ with $s \notin \mathbb{N}$. More precisely this means that there is $\delta > 0$ such that whenever $\|f\|_{C^s(\partial M)} \leq \delta$, there is a unique solution u_f to (1.2) with sufficiently small $C^s(M)$ norm (see Section 2 for more details on well-posedness). We will call u_f the unique small solution. Here C^s is the standard Hölder space for $s > 1$ with $s \notin \mathbb{N}$ (often written as $C^{k,\alpha}$ if $s = k + \alpha$ where $k \in \mathbb{Z}$ and $0 < \alpha < 1$), see e.g. [Tay11, Section 13.8]. Hence, the DN map is defined by using the unique small solution in a following way:

$$\Lambda_{M,g,q} : C^s(\partial M) \rightarrow C^{s-1}(\partial M), \quad f \mapsto \partial_\nu u_f|_{\partial M},$$

where ∂_ν denotes the normal derivative on the boundary ∂M .

As a warm-up, we begin with a theorem that illustrates our method in a simple setting. This theorem is in \mathbb{R}^n for $n \geq 2$, where Δ_g is the Euclidean Laplacian and $M = \overline{\Omega}$ with Ω a bounded smooth domain in \mathbb{R}^n .

Theorem 1.1 (Global uniqueness for a quadratic nonlinearity). *Let $n \geq 2$, and let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^∞ boundary $\partial\Omega$. Let $q_1, q_2 \in C^\infty(\overline{\Omega})$. Assume*

the DN maps Λ_{q_j} for the equations

$$(1.3) \quad \begin{cases} \Delta u + q_j u^2 = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega, \end{cases}$$

for $j = 1, 2$ satisfy

$$\Lambda_{q_1}(f) = \Lambda_{q_2}(f)$$

for all $f \in C^s(\partial\Omega)$ with $\|f\|_{C^s(\partial M)} < \delta$, where $\delta > 0$ is any sufficiently small number. Then $q_1 = q_2$ in Ω .

We will offer a detailed proof of Theorem 1.1 in Section 3, but let us briefly discuss the idea how to prove the theorem by using the method of *higher order linearization*. The second order linearization of the nonlinear DN map has already been used in the works [Sun96, SU97] related to nonlinear equations with matrix coefficients. First and second order linearizations were also used in in [KN02] for a nonlinear conductivity equation (see also [CNV19]). Under certain assumptions on the nonlinearity, by using the second order linearization, they can recover quadratic parts of the nonlinearity (see [KN02, Theorem 1.2 and Theorem 1.3]). In this work, we use similar ideas but obtain interesting new phenomena for related nonlinear inverse problems.

For the equation (1.3) with quadratic nonlinearity, the first order linearization of the nonlinear DN map Λ_q , linearized at the zero boundary value, is just the DN map for the standard Laplace equation:

$$(D\Lambda_q)_0 : C^s(\partial\Omega) \rightarrow C^{s-1}(\partial\Omega), \quad f \mapsto \partial_\nu v_f|_{\partial\Omega},$$

where v_f is the unique solution of $\Delta v_f = 0$ in Ω with $v_f|_{\partial\Omega} = f$. Thus the first linearization does not carry any information about the unknown potential q . However, for a quadratic nonlinearity the *second linearization* $(D^2\Lambda_q)_0$, which is a symmetric bilinear map from $C^s(\partial M) \times C^s(\partial M)$ to $C^{s-1}(\partial M)$, turns out to be very useful: it is characterized by the identity (see (2.7))

$$\int_{\partial\Omega} (D^2\Lambda_q)_0(f_1, f_2) f_3 \, dS = -2 \int_{\Omega} q v_{f_1} v_{f_2} v_{f_3} \, dx$$

where v_{f_j} is the harmonic function with boundary value f_j . Thus we have the implications

$$\begin{aligned} \Lambda_{q_1}(f) &= \Lambda_{q_2}(f) \text{ for small } f \\ \implies (D^2\Lambda_{q_1})_0 &= (D^2\Lambda_{q_2})_0 \\ \implies \int_{\Omega} (q_1 - q_2) v_1 v_2 v_3 \, dx &= 0 \end{aligned}$$

for any functions $v_1, v_2, v_3 \in C^s(\overline{\Omega})$ that are harmonic in Ω .

The last statement is very close to the linearized Calderón problem for a *linear* Schrödinger equation (the difference is that here one has the product of three harmonic functions, instead of two). Choosing v_1 and v_2 to be harmonic exponentials as in the work of Calderón [Cal80], and choosing $v_3 \equiv 1$, shows that the Fourier transform of $q_1 - q_2$ vanishes and hence $q_1 = q_2$. Thus, somewhat strikingly, we can solve a Calderón type inverse problem for the nonlinear equation $\Delta u + qu^2 = 0$ in a much simpler way than for the linear equation $\Delta u + qu = 0$ (the latter requires complex geometrical optics solutions as in [SU87]). The method also provides extremely simple reconstruction of the potential q , see Corollary 3.1.

We also mention that the second order linearization can be described as

$$(D^2\Lambda_q)_0(f_1, f_2) = \partial_{\epsilon_1} \partial_{\epsilon_2} u_{\epsilon_1 f_1 + \epsilon_2 f_2}|_{\epsilon_1 = \epsilon_2 = 0}.$$

That is, one considers boundary data

$$f = \epsilon_1 f_1 + \epsilon_2 f_2 \in C^s(\partial\Omega),$$

where ϵ_1, ϵ_2 are sufficiently small parameters, and takes the mixed derivative

$$\left. \frac{\partial}{\partial \epsilon_1} \frac{\partial}{\partial \epsilon_2} \right|_{\epsilon_1 = \epsilon_2 = 0}$$

of the equation (1.3). This idea is similar to the recent works on inverse problems for nonlinear hyperbolic equations mentioned above, and it yields the equations

$$(1.4) \quad \Delta w_j = -2q_j v_{f_1} v_{f_2},$$

for $j = 1, 2$, where $w_j = \left. \frac{\partial}{\partial \epsilon_1} \frac{\partial}{\partial \epsilon_2} \right|_{\epsilon_1 = \epsilon_2 = 0} u_j$ and v_{f_j} are harmonic functions, i.e. solutions to the linearized equation $\Delta v = 0$. Taking the mixed derivative of the DN maps yields (see Section 2)

$$\partial_\nu w_1 = \partial_\nu w_2 \text{ on } \partial\Omega.$$

Subtracting the equations (1.4) for $j = 1, 2$ and integrating the resulting equation against the harmonic function v_{f_3} yields the desired formula

$$\int_{\Omega} (q_1 - q_2) v_{f_1} v_{f_2} v_{f_3} dx = 0$$

which was mentioned in the discussion above.

We move on to describe our next result. By using higher order linearizations we prove the following *simultaneous recovery* on a two-dimensional Riemannian surface.

Theorem 1.2 (Simultaneous recovery of metric and potential). *Let (M_1, g_1) and (M_2, g_2) be two compact connected manifolds with mutual C^∞ boundary ∂M , where $\dim(M_1) = \dim(M_2) = 2$ and $m \geq 2$. Let Λ_{M_j, g_j, q_j} be the DN maps of*

$$(1.5) \quad \Delta_{g_j} u + q_j u^m = 0 \text{ in } M_j$$

for $j = 1, 2$. Let $s > 1$ with $s \notin \mathbb{N}$ and assume that

$$\Lambda_{M_1, g_1, q_1} f = \Lambda_{M_2, g_2, q_2} f \text{ on } \partial M,$$

for any $f \in C^s(\partial M)$ with $\|f\|_{C^s(\partial M)} \leq \delta$, where $\delta > 0$ is sufficiently small. Then:

- (1) *There exists a conformal diffeomorphism $J : M_1 \rightarrow M_2$ and a positive smooth function σ such that*

$$\sigma J^* g_2 = g_1, \text{ with } J|_{\partial M} = \text{Id} \text{ and } \sigma|_{\partial M} = 1.$$

- (2) *Moreover, one can also recover the potential up to a natural gauge invariance in the sense that*

$$\sigma q_1 = q_2 \circ J \text{ in } M_1.$$

We see that the conformal factor σ (and also the diffeomorphism J) couples to the potential. This is due to the *gauge symmetry* of the inverse problem:

$$\Lambda_{M_1, \sigma J^* g, \sigma^{-1} J^* q} = \Lambda_{J(M_1), g, q}$$

where J and σ satisfy boundary conditions as above. For the linear equation $\Delta_g u + qu = 0$, an analogous result has been proved when M is a domain in \mathbb{R}^2 with a Riemannian metric [IUY12], when M is a manifold and the potentials are zero [LU01], and when the manifold M is a priori known [GT11]. The recovery of properties of both the manifold and potential is stated as an open question in [GT13], where further references to two-dimensional results are given. The proof of Theorem 1.2 uses the first linearization of the DN map to recover the metric and

the manifold up to a conformal transformation. Then the second linearization is used to recover the potential on a single fixed manifold (up to the gauge symmetry).

The final new result in this article is to consider inverse problems for the semi-linear Schrödinger equation on *transversally anisotropic* manifold. Let us recall the definition of a transversally anisotropic manifold.

Definition 1.1. *Let (M, g) be a compact oriented manifold with a C^∞ boundary and with $\dim M \geq 3$. (M, g) is called transversally anisotropic if $(M, g) \subset\subset (T, g)$, where $T = \mathbb{R} \times M_0$ and $g = e \oplus g_0$. Here (\mathbb{R}, e) is the Euclidean line and (M_0, g_0) is $(n - 1)$ -dimensional compact manifold with a smooth boundary.*

For more details of inverse problems in transversally anisotropic geometries for linear equations, we refer readers to [FKLS16, FKL+17].

We prove the following.

Theorem 1.3. *Let (M, g) be a transversally anisotropic manifold, let $q_j \in C^\infty(M)$, and let Λ_{q_j} be the DN maps for the equations*

$$\Delta_g u + q_j u^m = 0 \text{ in } M$$

for $j = 1, 2$, where we assume that

$$m \in \mathbb{N}, \quad m \geq 3.$$

If the DN maps satisfy

$$\Lambda_{q_1}(f) = \Lambda_{q_2}(f)$$

for all sufficiently small f , then $q_1 = q_2$ in M .

The higher order linearization method in this case reduces the proof of Theorem 1.3 to showing for any $m \geq 3$ that the identity

$$(1.6) \quad \int_M f v_1 \cdots v_{m+1} dV = 0$$

holding for any $v_j \in C^\infty(M)$ with $\Delta_g v_j = 0$ in M , implies $f \equiv 0$. Thus we prove that the products of at least four harmonic functions on a transversally anisotropic manifold form a complete set. The main point is that the argument works for arbitrary transversally anisotropic manifolds without any restriction on the transversal geometry.

The solution to the analogous inverse problem for a linear equation $\Delta_g u + qu = 0$ on transversally anisotropic manifolds is only known under the additional assumption that the transversal manifold (M_0, g_0) has injective geodesic X-ray transform [FKLS16]. In the linearized version of that problem, the identity (1.6) only holds for $m = 1$ and one needs to prove that products of pairs of harmonic functions form a complete set. In [FKLS16] this is done by using complex geometrical optics solutions that concentrate near two-dimensional surfaces that are translates of geodesics on M_0 . Using products of such solutions and their complex conjugates recovers certain integrals over geodesics in M_0 , but does not yield pointwise information. In [FKL+17] products of solutions concentrating near two intersecting geodesics were used instead to recover microlocal information in the linearized inverse problem. The products are supported near finitely many points in M_0 , but there is oscillation that prevents recovering more information. We also mention [GST18] that deals with the linearized problem on certain complex manifolds.

The idea behind the proof of Theorem 1.3 is that since one can use products of at least four harmonic functions, we can use solutions related to two intersecting geodesics on M_0 as well as their complex conjugates. The product of these four solutions is supported near finitely many points in M_0 and the product does not have high oscillations. This allows one to recover the potential completely.

We mention that the aim of this paper is not to work in the highest possible generality or to provide an extensive list of all possible applications of the higher order linearization method. For example, it is clear that the method applies to certain more general nonlinearities and less regular coefficients. These are left to forthcoming works. Here we have included applications that illustrate the power of the higher order linearization method.

Finally, we mention that before submitting this paper we became aware of an upcoming preprint of Ali Feizmohammadi and Lauri Oksanen, which simultaneously and independently proves a result similar to Theorem 1.3, and we agreed with them to publish the preprints of the results at the same time on the same preprint server.

The paper is organized as follows. In Section 2, we lay out the basic properties for semilinear elliptic equations that we use. This includes the well-posedness of the Dirichlet problem and higher order linearizations of the DN map. We use the higher order linearization approach to prove Theorem 1.1 in Section 3, Theorem 1.2 in Section 4, and Theorem 1.3 in Section 5, respectively.

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2. PRELIMINARIES

In this section, we prove well-posedness of the Dirichlet problem for semilinear elliptic equations with small boundary data, and study higher order linearizations of the DN map. We state the first result for a general nonlinearity satisfying two conditions: the first ensures that $u \equiv 0$ is a solution, and the second states that the equation linearized at $u \equiv 0$ is well-posed.

Proposition 2.1 (Well-posedness). *Let (M, g) be a compact Riemannian manifold with C^∞ boundary ∂M and let Q be the semilinear elliptic operator*

$$Q(u) := \Delta_g u + a(x, u),$$

where $a \in C^\infty(M \times \mathbb{R})$ satisfies the following two conditions:

$$(2.1) \quad a(x, 0) = 0,$$

$$(2.2) \quad \text{The map } v \mapsto \Delta_g v + \partial_u a(\cdot, 0)v \text{ is injective on } H_0^1(M).$$

Let $s > 1$ with $s \notin \mathbb{Z}$. There exist $\delta, C > 0$ such that for any f in the set

$$U_\delta := \{h \in C^s(\partial M); \|h\|_{C^s(\partial M)} < \delta\},$$

there is a solution $u = u_f$ of

$$(2.3) \quad \begin{cases} \Delta_g u + a(x, u) = 0 & \text{in } M, \\ u = f & \text{on } \partial M, \end{cases}$$

which satisfies

$$\|u\|_{C^s(M)} \leq C\|f\|_{C^s(\partial M)}.$$

The solution u_f is unique within the class $\{w \in C^s(M); \|w\|_{C^s(M)} \leq C\delta\}$, and if $f \in C^\infty(\partial M)$, then $u_f \in C^\infty(M)$. Moreover, there are C^∞ maps

$$\begin{aligned} S: U_\delta &\rightarrow C^s(M), & f &\mapsto u_f, \\ \Lambda: U_\delta &\rightarrow C^{s-1}(\partial M), & f &\mapsto \partial_\nu u_f|_{\partial M}. \end{aligned}$$

Proof. We prove the existence of solutions by using the implicit function theorem in Banach spaces [RR06, Theorem 10.6]. Let

$$X = C^s(\partial M), \quad Y = C^s(M), \quad Z = C^{s-2}(M) \times C^s(\partial M).$$

Consider the map

$$F : X \times Y \rightarrow Z, \quad F(f, u) = (Q(u), u|_{\partial M} - f).$$

We wish to show that F indeed maps to Z and is a C^∞ map. Note that since a is smooth and since $C^s(M)$ is an algebra under pointwise multiplication, the map

$$u \mapsto a(x, u)$$

takes $C^s(M)$ to $C^s(M)$, and if $\|u\|_{C^s(M)} \leq K$ then $\|a(x, u)\|_{C^s(M)} \leq C(a, s, K)$ (these facts follow from a local coordinate computation). Thus F is well defined. If $u, v \in C^s(M)$ we use the Taylor formula

$$a(x, u + v) = \sum_{j=0}^m \frac{\partial_u^j a(x, u)}{j!} v^j + \int_0^1 \frac{\partial_u^{m+1} a(x, u + tv)}{m!} v^{m+1} (1-t)^m dt.$$

Since $C^s(M)$ is an algebra, we have that when $\|v\|_{C^s(M)} \leq 1$ one has

$$\left\| a(x, u + v) - \sum_{j=0}^m \frac{\partial_u^j a(x, u)}{j!} v^j \right\|_{C^s(M)} \leq C_{m,a,u} \|v\|_{C^s(M)}^{m+1}.$$

This shows that $u \mapsto a(x, u)$ is a C^∞ map $C^s(M) \rightarrow C^s(M)$. Since the other parts of F are linear, F is a C^∞ map in the standard sense of [RR06, Definition 10.2].

Note that $F(0, 0) = 0$ by (2.1). The linearization of F at $(0, 0)$ in the u -variable is

$$D_u F|_{(0,0)}(v) = (\Delta_g v + \partial_u a(x, 0)v, v|_{\partial M}).$$

This is a homeomorphism $Y \rightarrow Z$ by (2.2). To see this, let $(w, \phi) \in Z = C^{s-2}(M) \times C^s(\partial M)$, and consider the Dirichlet problem

$$\begin{cases} (\Delta_g + \partial_u a(x, 0))v = w & \text{in } M, \\ v = \phi & \text{on } \partial M. \end{cases}$$

Solutions are unique by (2.2), and using the Fredholm alternative and Schauder estimates there is a unique solution $v \in Y = C^s(M)$ (see e.g. [Tay11, Exercise 1 in Section 13.8]). Thus the implicit function theorem in Banach spaces [RR06, Theorem 10.6 and Remark 10.5] yields that there is $\delta > 0$ and an open ball $U_\delta = B_X(0, \delta) \subset X$ and a C^∞ map $S : U_\delta \rightarrow Y$ such that whenever $\|f\|_{C^s(\partial M)} \leq \delta$ we have

$$F(f, S(f)) = (0, 0).$$

Since S is Lipschitz continuous and $S(0) = 0$, $u = S(f)$ satisfies

$$\|u\|_{C^s(M)} \leq C\|f\|_{C^s(\partial M)}.$$

Moreover, by redefining δ if necessary $u = S(f)$ is the only solution to $F(f, u) = (0, 0)$ whenever $\|f\|_{C^s(\partial M)} \leq \delta$ and $\|u\|_{C^s(M)} \leq C\delta$. We have proven the existence of unique small solutions of the Dirichlet problem (2.3) and the fact that the solution operator $S : U_\delta \rightarrow C^s(M)$ is a C^∞ map. Since the normal derivative is a linear map $C^s(M) \rightarrow C^{s-1}(\partial M)$, it follows that also Λ is a well defined C^∞ map $U_\delta \rightarrow C^{s-1}(\partial M)$. \square

We now specialize to a power type nonlinearity, for which the higher order linearizations of the DN map will be particularly simple. The next proposition justifies the formal calculation that we may differentiate the equation

$$(2.4) \quad \Delta_g u + q(x)u^m = 0 \text{ in } M, \quad u_f|_{\partial M} = \epsilon_1 f_1 + \cdots + \epsilon_m f_m$$

in the ϵ_j variables to have equations corresponding to first and m th linearizations,

$$\Delta_g v_{f_k} = 0 \text{ and } \Delta_g w = -(m!)q v_{f_1} \cdots v_{f_m}.$$

The normal derivative of w is the m th linearization of the DN map of (2.4). In the proposition, we write

$$(D^k f)_x(y_1, \dots, y_k)$$

to denote the k th derivative at x of a mapping f between Banach spaces, considered as a symmetric k -linear form acting on (y_1, \dots, y_k) . We refer to [Hor85, Section 1.1], where the notation $f^{(k)}(x; y_1, \dots, y_k)$ is used instead of $(D^k f)_x(y_1, \dots, y_k)$.

Proposition 2.2. *Let $q \in C^\infty(M)$, and let Λ_q be the DN map for the semilinear equation*

$$(2.5) \quad \Delta_g u + q(x)u^m = 0 \text{ in } M,$$

where

$$m \in \mathbb{N} \text{ and } m \geq 2.$$

For $f \in C^s(\partial M)$, let v_f be the solution of the Laplace equation

$$(2.6) \quad \Delta_g v_f = 0 \text{ in } M, \quad v_f|_{\partial M} = f.$$

The first linearization $(D\Lambda_q)_0$ of Λ_q at $f = 0$ is the DN map of the Laplace equation:

$$(D\Lambda_q)_0 : C^s(\partial M) \rightarrow C^{s-1}(\partial M), \quad f \mapsto \partial_\nu v_f|_{\partial M}.$$

The higher order linearizations $(D^j \Lambda_q)_0$ are identically zero for $2 \leq j \leq m-1$.

The m -th linearization $(D^m \Lambda_q)_0$ of Λ_q at $f = 0$ is characterized by the following identity: for any $f_1, \dots, f_{m+1} \in C^s(\partial M)$ one has

$$(2.7) \quad \int_{\partial M} (D^m \Lambda_q)_0(f_1, \dots, f_m) f_{m+1} dS = -(m!) \int_M q v_{f_1} \cdots v_{f_{m+1}} dV$$

here each v_{f_k} , $k = 1, \dots, m+1$, is a solution to (2.6) with boundary value $f = f_k$.

Proof. The nonlinearity $a(x, u) = q(x)u^m$ satisfies the conditions in Proposition 2.1, and thus the DN map $\Lambda_q = \partial_\nu S|_{\partial M}$ is well defined for small data. Here $S : f \mapsto u_f$ is the solution operator for the Dirichlet problem of the equation (2.5). To compute the derivatives of Λ_q at 0, it is enough to consider derivatives of S . Writing $f := \epsilon_1 f_1 + \dots + \epsilon_k f_k$, the function $u_f \in C^s(M)$ depends smoothly on $\epsilon_1, \dots, \epsilon_k$ since S is smooth. Applying $\partial_{\epsilon_1} \cdots \partial_{\epsilon_k}|_{\epsilon_1=\dots=\epsilon_k=0}$ to the Taylor formula (see e.g. [Hor85, Section 1.1])

$$S(f) = \sum_{j=0}^k \frac{(D^j S)_0(f, \dots, f)}{j!} + \int_0^1 \frac{(D^{k+1} S)_{tf}(f, \dots, f)}{k!} (1-t)^k dt$$

implies that $(D^k S)_0$ may be computed using the formula

$$(D^k S)_0(f_1, \dots, f_k) = \partial_{\epsilon_1} \cdots \partial_{\epsilon_k} u_f|_{\epsilon_1=\dots=\epsilon_k=0}.$$

Moreover, since u_f is smooth in the ϵ_j variables and Δ_g is linear, we may differentiate the equation

$$(2.8) \quad \Delta_g u_f + q(x)u_f^m = 0, \quad u_f|_{\partial M} = f$$

freely in the ϵ_j variables.

Let first $k = 1$, so that $u = u_{\epsilon_1 f_1}$. Since $u_0 = 0$ and $m \geq 2$, the derivative of (2.8) in ϵ_1 evaluated at $\epsilon_1 = 0$ satisfies

$$\Delta_g(\partial_{\epsilon_1} u_f|_{\epsilon_1=0}) = 0, \quad \partial_{\epsilon_1} u_f|_{\partial M} = f_1.$$

Thus the first linearization of the map S at $f = 0$ is

$$(DS)_0(f_1) = \partial_{\epsilon_1} u_{\epsilon_1 f_1}|_{\epsilon_1=0} = v_{f_1},$$

where v_{f_1} satisfies (2.6) with $f = f_1$.

For $2 \leq k \leq m-1$, applying $\partial_{\epsilon_1} \cdots \partial_{\epsilon_k}|_{\epsilon_1=\dots=\epsilon_k=0}$ to (2.8) gives that

$$\Delta_g(\partial_{\epsilon_1} \cdots \partial_{\epsilon_k} u_f|_{\epsilon_1=\dots=\epsilon_k=0}) = 0, \quad \partial_{\epsilon_1} \cdots \partial_{\epsilon_k} u_f|_{\partial M} = 0,$$

since $\partial_{\epsilon_1} \cdots \partial_{\epsilon_k}(q(x)u_f^m)$ is a sum of terms containing positive powers of u_f , which are equal to zero when $f = 0$. Uniqueness of solutions for the Laplace equation implies that

$$(D^j S)_0(f_1, \dots, f_k) = 0, \quad 2 \leq k \leq m-1.$$

When $k = m$, the only term in $\partial_{\epsilon_1} \cdots \partial_{\epsilon_m}(q(x)u_f^m)$ which does not contain a positive power of u_f is $q(x)(m!)(\partial_{\epsilon_1} u_f) \cdots (\partial_{\epsilon_m} u_f)$. This is the only nonzero term after setting $\epsilon_1 = \dots = \epsilon_m = 0$, and thus the function

$$w := (D^m S)_0(f_1, \dots, f_m) = \partial_{\epsilon_1} \cdots \partial_{\epsilon_m} u_f|_{\epsilon_1=\dots=\epsilon_m=0}$$

solves

$$(2.9) \quad \Delta_g w = -q(x)(m!)v_{f_1} \cdots v_{f_m} \text{ in } M$$

with zero Dirichlet boundary values.

By linearity one has

$$(D^k \Lambda_q)_0 = \partial_\nu (D^k S)_0|_{\partial M}.$$

The claims for $(D^k \Lambda_q)_0$ when $1 \leq k \leq m-1$ follow immediately. For $k = m$ we observe that $(D^m \Lambda_q)_0(f_1, \dots, f_m) = \partial_\nu w|_{\partial M}$ satisfies

$$\int_{\partial M} (\partial_\nu w) f_{m+1} dS = \int_M (\langle dw, dv_{f_{m+1}} \rangle_g + (\Delta_g w) v_{f_{m+1}}) dV.$$

The integral of $\langle dw, dv_{f_{m+1}} \rangle_g$ vanishes since $w|_{\partial M} = 0$ and $v_{f_{m+1}}$ is harmonic. The proposition follows by using (2.9). \square

3. PROOF OF THEOREM 1.1

In this section, we use the higher order linearization approach (in fact, the second order linearization of the DN map) to prove Theorem 1.1.

Proof of Theorem 1.1. We could use Proposition 2.2 to have the integral equation (3.6) below directly, even for the product of three harmonic functions instead of two (this is a stronger statement since one can always take the third harmonic function to be constant). The theorem would follow from this by using harmonic exponentials. However, we choose to give a direct hands-on approach that describes how to use the method.

Let ϵ_1, ϵ_2 be sufficiently small numbers and let $f_1, f_2 \in C^\infty(\partial M)$. Let the function $u_j := u_j(x; \epsilon_1, \epsilon_2) \in C^s(M)$ be the unique small solution of

$$(3.1) \quad \begin{cases} \Delta u_j + q_j u_j^2 = 0 & \text{in } \Omega, \\ u_j = \epsilon_1 f_1 + \epsilon_2 f_2 & \text{on } \partial\Omega, \end{cases}$$

for $j = 1, 2$ provided by Proposition 2.2. Let us differentiate (3.1) with respect to ϵ_ℓ so that

$$(3.2) \quad \begin{cases} \Delta \left(\frac{\partial}{\partial \epsilon_\ell} u_j \right) + 2q_j u_j \left(\frac{\partial}{\partial \epsilon_\ell} u_j \right) = 0 & \text{in } \Omega, \\ \frac{\partial}{\partial \epsilon_\ell} u_j = f_\ell & \text{on } \partial\Omega. \end{cases}$$

Inserting $\epsilon_1 = \epsilon_2 = 0$ into (3.2), shows that

$$\Delta v_j^{(\ell)} = 0 \text{ in } \Omega \text{ with } v_j^{(\ell)} = f_\ell \text{ on } \partial\Omega,$$

where

$$v_j^{(\ell)}(x) = \left. \frac{\partial}{\partial \epsilon_\ell} \right|_{\epsilon_1=\epsilon_2=0} u_j(x; \epsilon_1, \epsilon_2).$$

Here we used $u_j(x; 0, 0) \equiv 0$. The functions v_j^ℓ are just harmonic functions defined in Ω with boundary data $f_\ell|_{\partial\Omega}$. By uniqueness of the Dirichlet problem for the Laplacian we have that

$$(3.3) \quad v^{(\ell)} := v_1^{(\ell)} = v_2^{(\ell)} \text{ in } \Omega \text{ for } \ell = 1, 2.$$

Next, let us differentiate (3.2) with respect to ϵ_k for $k \neq \ell$. Then we have that

$$(3.4) \quad \begin{cases} \Delta \left(\frac{\partial^2}{\partial \epsilon_1 \partial \epsilon_2} u_j \right) + 2q_j u_j \left(\frac{\partial^2}{\partial \epsilon_1 \partial \epsilon_2} u_j \right) + 2q_j \left(\frac{\partial u_j}{\partial \epsilon_1} \right) \left(\frac{\partial u_j}{\partial \epsilon_2} \right) = 0 & \text{in } \Omega, \\ \frac{\partial^2}{\partial \epsilon_1 \partial \epsilon_2} u_j = 0 & \text{on } \partial\Omega. \end{cases}$$

Again, evaluating at $\epsilon_1 = \epsilon_2 = 0$, the equation (3.4) becomes

$$(3.5) \quad \begin{cases} \Delta w_j + 2q_j v^{(1)} v^{(2)} = 0 & \text{in } \Omega, \\ w_j = 0 & \text{on } \partial\Omega, \end{cases}$$

where $w_j(x) = \left(\frac{\partial^2}{\partial \epsilon_1 \partial \epsilon_2} u_j \right)(x; 0, 0)$ and we used $u_j(x; 0, 0) \equiv 0$ for $j = 1, 2$ again. By using the fact that $\Lambda_{q_1}(\epsilon_1 f_1 + \epsilon_2 f_2) = \Lambda_{q_2}(\epsilon_1 f_1 + \epsilon_2 f_2)$ for small ϵ_1, ϵ_2 , we have

$$\partial_\nu u_1|_{\partial\Omega} = \partial_\nu u_2|_{\partial\Omega},$$

and applying $\partial_{\epsilon_1} \partial_{\epsilon_2}|_{\epsilon_1=\epsilon_2=0}$ to this identity gives that

$$\partial_\nu w_1|_{\partial\Omega} = \partial_\nu w_2|_{\partial\Omega}.$$

Thus, by integrating the equation (3.5) over Ω (i.e. integrating against the harmonic function $v^{(3)} = 1$) and by using integration by parts we have

$$(3.6) \quad 0 = \int_{\partial\Omega} (\partial_\nu w_1 - \partial_\nu w_2) dS = \int_{\Omega} \Delta(w_1 - w_2) dx = 2 \int_{\Omega} (q_2 - q_1) v^{(1)} v^{(2)} dx$$

where $v^{(1)}$ and $v^{(2)}$ are defined in (3.3). Therefore, by choosing f_1 and f_2 as the boundary values of the Calderón's exponential solutions [Cal80],

$$(3.7) \quad v^{(1)}(x) := \exp((k + i\xi) \cdot x), \quad v^{(2)}(x) := \exp((-k + i\xi) \cdot x),$$

where $k, \xi \in \mathbb{R}^n$, $k \perp \xi$ and $|k| = |\xi|$, we obtain that the Fourier transformation of the difference $q_2 - q_1$ is zero. Thus $q_1 = q_2$. \square

In the proof above we did not need to construct special solutions for an elliptic equation with unknown coefficients, such as complex geometrical optics solutions. The linearization technique allowed us to simply use known harmonic functions. This fact gives an extremely simple reconstruction in the setting of Theorem 1.1.

Corollary 3.1. *Let $n \geq 2$, and let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^∞ boundary $\partial\Omega$. Assume that $q \in C^\infty(\overline{\Omega})$, and let Λ_q be the DN map for the equation*

$$\Delta u + qu^2 = 0 \text{ in } \Omega.$$

Then

$$(3.8) \quad \widehat{q}(-2\xi) = -\frac{1}{2} \int_{\partial\Omega} \frac{\partial^2}{\partial \epsilon_1 \partial \epsilon_2} \Big|_{\epsilon_1=\epsilon_2=0} \Lambda_q(\epsilon_1 f_1 + \epsilon_2 f_2) dS,$$

where f_1 and f_2 are the boundary values of the exponential solutions (3.7) and \widehat{q} stands for the Fourier transform of q .

Proof. The proof of the reconstruction can be directly read from (3.6) in the proof of Theorem 1.1. \square

We end of this section with a remark about the stability of the reconstruction formula in Corollary 3.1.

Remark 3.2. From the above reconstruction formula, we adapt the same notations in Section 2, then we can simply rewrite the formula (3.8) as

$$\widehat{q}_j(-2\xi) = -\frac{1}{2} \int_{\partial\Omega} (D^2\Lambda_{q_j})_0(f_1, f_2) dS, \text{ for } j = 1, 2.$$

Subtracting these two formulas gives that

$$(\widehat{q}_1 - \widehat{q}_2)(-2\xi) = -\frac{1}{2} \int_{\partial\Omega} ((D^2\Lambda_{q_1})_0 - (D^2\Lambda_{q_2})_0)(f_1, f_2) dS.$$

Now, we assume that $\|D^k(\Lambda_{q_1} - \Lambda_{q_2})_0\|_*$ is sufficiently small for $k = 0, 1, 2$, and $\|q_j\|_{H^1(\Omega)} \leq R$ for $j = 1, 2$, where

$$\|T\|_* = \sup_{\|f_1\|_{C^s(\partial\Omega)} = \dots = \|f_k\|_{C^s(\partial\Omega)} = 1} \|T(f_1, \dots, f_k)\|_{C^{s-1}(\partial\Omega)}$$

for a bounded k -linear form $T : C^s(\partial\Omega) \times \dots \times C^s(\partial\Omega) \rightarrow C^{s-1}(\partial\Omega)$. Next, by taking harmonic functions $v_{f_1} = v^{(1)}$, $v_{f_2} = v^{(2)}$ in Ω , where $v^{(1)}, v^{(2)}$ are the functions defined in (3.7), one can obtain that

$$(3.9) \quad \|q_1 - q_2\|_{L^2} \leq \omega(\|D^2(\Lambda_{q_1} - \Lambda_{q_2})_0\|_*),$$

where $\omega(t)$ is a modulus of continuity satisfying, for some $C = C(R)$,

$$\omega(t) \leq C |\log t|^{-\frac{2}{n+2}}, \quad 0 < t < \frac{1}{e}.$$

One can directly prove the logarithmic stability (3.9) by using standard arguments in stability for the Calderón problem, for example, see [Sal08, Section 4].

4. SIMULTANEOUS RECOVERY ON TWO-DIMENSIONAL RIEMANNIAN SURFACES

We use the higher order linearization approach to simultaneously recover, from the DN map, the conformal class of a Riemannian surface and the potential of a semilinear Schrödinger operator up to the gauge symmetry. We use first order linearization to recover the conformal class of the manifold by using the result [LU01] (see also [LLS18] for a recent alternative proof). Then by using the result [GT11] we recover the potential on the known conformal manifold (up to gauge).

Proof of Theorem 1.2. The proof is divided into two steps. We first recover the manifold and the conformal class of the metric. After that we recover the potential on a known manifold up to the gauge.

Step 1. Recovering the conformal manifold.

Note first that

$$\begin{aligned} \Lambda_{q_1}(f) &= \Lambda_{q_2}(f) \text{ for small } f \\ \implies (D\Lambda_{q_1})_0 &= (D\Lambda_{q_2})_0. \end{aligned}$$

By Proposition 2.2, $(D\Lambda_{q_j})_0$ is the DN map for the Laplace equation on (M, g_j) , i.e. we have that the DN maps of the following Dirichlet problems

$$\begin{cases} \Delta_{g_j} v_j = 0 & \text{in } M_j, \\ v_j = f & \text{on } \partial M \end{cases}$$

agree. Thus by using [LLS18, Theorem 5.1], one can determine the manifold and the Riemannian metric up to a conformal transformation. That is, there exists a conformal C^∞ diffeomorphism J such that $\sigma J^* g_2 = g_1$ with $J|_{\partial M} = \text{Id}$, for some smooth positive function $\sigma \in C^\infty(M_1)$ with $\sigma|_{\partial M} = 1$. This completes the first part of the proof.

Step 2. Recovering the potential.

Let us make a change of coordinates to pass from the equation (1.5) on (M_2, g_2) onto the manifold (M_1, g_1) . We denote

$$\tilde{q}_2 = \sigma^{-1} q_2 \circ J \equiv \sigma^{-1} J^* q_2.$$

Let $f \in C^s(\partial M)$ be small and let u_2 be the solution to

$$\Delta_{g_2} u_2 + q_2 u_2^m = 0 \text{ in } M_2 \text{ with } u_2 = f \text{ on } \partial M.$$

Let us denote

$$\tilde{u}_2 := J^* u_2 \equiv u_2 \circ J.$$

Then \tilde{u}_2 solves

$$\begin{aligned} \Delta_{g_1} \tilde{u}_2 + \tilde{q}_2 (\tilde{u}_2)^m &= \Delta_{\sigma J^* g_2} \tilde{u}_2 + \tilde{q}_2 (\tilde{u}_2)^m \\ &= \sigma^{-1} \Delta_{J^* g_2} \tilde{u}_2 + \sigma^{-1} (J^* q_2) (\tilde{u}_2)^m \\ &= \sigma^{-1} J^* (\Delta_{g_2} u_2) + \sigma^{-1} (J^* q_2) (J^* u_2)^m \\ &= \sigma^{-1} [J^* (\Delta_{g_2} u_2) + (J^* q_2) (J^* u_2)^m]. \end{aligned}$$

Here we used the conformal invariance of the Laplace-Beltrami operator in dimension 2 in the second equality. In the third equality, the coordinate invariance of Laplace-Beltrami operator was used. Since u_2 solves $\Delta_{g_2} u_2 + q_2 u_2^m = 0$ in M_2 , we trivially have that $J^* (\Delta_{g_2} u_2) + (q_2 \circ J) (J^* u_2)^m = 0$ in M_1 . Consequently, we have that

$$(4.1) \quad \begin{cases} \Delta_{g_1} \tilde{u}_2 + \tilde{q}_2 (\tilde{u}_2)^m = 0 & \text{in } M_1, \\ \tilde{u}_2 = f & \text{on } \partial M, \end{cases}$$

since $\sigma|_{\partial M} = 1$ and $J|_{\partial M} = \text{Id}$.

Next, let u_1 solve the nonlinear equation (1.5) on (M_1, g_1) with potential q_1 and boundary value f . We show that there holds

$$(4.2) \quad \partial_{\nu_1} u_1 = \partial_{\nu_1} \tilde{u}_2 \text{ on } \partial M,$$

by the assumption that $\Lambda_{M_1, g_1, q_1} = \Lambda_{M_2, g_2, q_2}$ for small data. Since $\Lambda_{M_1, g_1, q_1} = \Lambda_{M_2, g_2, q_2}$, it follows that if $u_1 = u_2 = f$ on ∂M , then

$$\partial_{\nu_1} u_1 = \partial_{\nu_2} u_2 \text{ on } \partial M.$$

We calculate

$$\partial_{\nu_2} u_2 = \nu_2 \cdot du_2 = \nu_2 \cdot d(u_2 \circ J \circ J^{-1}) = (J_*^{-1} \nu_2) \cdot d\tilde{u}_2 = \nu_1 \cdot d\tilde{u}_2 = \partial_{\nu_1} \tilde{u}_2.$$

Here \cdot denotes the canonical pairing between vectors $\nu_2 \cdot du_2 = \nu_2^k \partial_k u$ (with Einstein summation implied), and we used the facts that J is diffeomorphic conformal mapping with $J|_{\partial M} = \text{Id}$ and $\sigma|_{\partial M} = 1$. Thus we have (4.2) and consequently

$$\Lambda_{M_1, g_1, q_1}(f) = \tilde{\Lambda}_{M_1, g_1, \tilde{q}_2}(f) \text{ for small } f,$$

where $\tilde{\Lambda}_{M_1, g_1, \tilde{q}_2}$ stands for the DN map of the Dirichlet problem (4.1).

By Proposition 2.2, we have

$$\begin{aligned} \Lambda_{M_1, g_1, q_1}(f) &= \tilde{\Lambda}_{M_1, g_1, \tilde{q}_2}(f) \text{ for small } f \\ \implies (D^2 \Lambda_{M_1, g_1, q_1})_0 &= (D^2 \tilde{\Lambda}_{M_1, g_1, \tilde{q}_2})_0 \\ \implies \int_{M_1} (q_1 - \tilde{q}_2) v_1 v_2 v_3 dV &= 0 \end{aligned}$$

where $v_j \in C^s(M_1)$ are harmonic functions in M_1 . Choosing $v_3 = 1$, we get

$$\int_{M_1} (q_1 - \tilde{q}_2) v_1 v_2 dV = 0$$

for any harmonic functions v_1 and v_2 in M_1 . Choosing v_j to be complex geometrical optics solutions constructed in [GT11] (see the proof of Proposition 5.1 in that paper, actually since v_j are harmonic Carleman estimates are not needed and the construction in [GST18] would suffice), it follows that

$$q_1 = \tilde{q}_2 \text{ in } M_1.$$

This concludes the proof. \square

5. GLOBAL UNIQUENESS ON TRANSVERSALLY ANISOTROPIC MANIFOLDS

We will prove the following result, whose proof is based on the construction in [FKLS16] of harmonic functions on transversally anisotropic manifolds that concentrate near certain two dimensional surfaces.

Proposition 5.1. *Let (M, g) be a transversally anisotropic manifold and assume that $m \geq 4$. If $f \in C^1(M)$ satisfies*

$$(5.1) \quad \int_M f u_1 \cdots u_m dV = 0$$

for any $u_j \in C^\infty(M)$ with $\Delta_g u_j = 0$ in M , then $f \equiv 0$.

Theorem 1.3 follows immediately from this proposition:

Proof of Theorem 1.3. Let Λ_{q_j} be the DN map for the equation $\Delta_g u + q u^m = 0$ in M . If $\Lambda_{q_1}(f) = \Lambda_{q_2}(f)$ for small f , then $(D^m \Lambda_{q_1})_0 = (D^m \Lambda_{q_2})_0$ and thus by Proposition 2.2 one has

$$\int_M (q_1 - q_2) v_1 \cdots v_{m+1} dV = 0$$

where $v_j \in C^s(M)$ are harmonic functions in M . Since $m \geq 3$, it follows from Proposition 5.1 that $q_1 = q_2$. \square

Before we prove Proposition 5.1 in the general case, let us explain the main idea of the proof in a simplified setting. Consider the situation where each point $y_0 \in M_0$ on the transversal manifold M_0 is an intersection point of two distinct nontangential geodesics γ, η (depending on y_0), which have no other intersection points. Assume also that γ and η do not self-intersect.

As in [FKLS16, Section 2], we choose special harmonic functions u_1, u_2, u_3 and u_4 of the form

$$\begin{aligned} u_1 &= e^{-(\tau+i\lambda)x_1} (\tilde{v}_{\tau+i\lambda} + r_1), & u_2 &= \overline{e^{(\tau+i\lambda)x_1} (\tilde{v}_{\tau+i\lambda} + r_2)}, \\ u_3 &= e^{-\tau x_1} (\tilde{w}_\tau + r_3), & u_4 &= \overline{e^{\tau x_1} (\tilde{w}_\tau + r_4)}. \end{aligned}$$

Here $\tilde{v}_{\tau+i\lambda}, \tilde{w}_\tau$ are Gaussian beam quasimodes on M_0 concentrating near γ and η , respectively, as in [FKLS16, Proposition 3.1], and the functions $r_\ell, 1 \leq \ell \leq 4$, are small correction terms. Here x_1 is the variable of the Euclidean direction.

Since $\tilde{v}_{\tau+i\lambda}$ and \tilde{w}_τ are supported near γ and η , respectively, and since γ and η only intersect at y_0 , the product $u_1 u_2 u_3 u_4$ concentrates near the point y_0 . Thus by using these solutions $u_\ell, 1 \leq \ell \leq 4$, it follows by the assumption (5.1) that

$$(5.2) \quad \int_{B_\delta(y_0)} \hat{f}(2\lambda, \cdot) e^{2i\tau \text{Im}(\Phi+\Psi)} A dV_{g_0} = \mathcal{O}(\tau^{-R}),$$

where $R > 0$ can be chosen arbitrarily large, $B_\delta(y_0)$ is a geodesic ball of radius $\delta > 0$ in M_0 , and \hat{f} denotes the partial Fourier transform of f with respect to x_1 . Here Φ and Ψ are the phase functions of the Gaussian beams corresponding to γ and η , respectively, and A is an amplitude function, which is nonzero at the intersection point $y_0 \in M_0$.

The point here is the following. The Hessians of $\text{Im}(\Phi)$ and $\text{Im}(\Psi)$ at y_0 are positive definite in directions orthogonal to γ and η , respectively. In fact, the Hessians of $\text{Im}(\Phi)$ and $\text{Im}(\Psi)$ are also nonnegative definite and their gradients vanish on γ and η , respectively. Since γ and η intersect at y_0 , the sum $\text{Im}(\Phi + \Psi)$ is positive definite at y_0 and $d(\Phi + \Psi)(y_0) = 0$. By multiplying (5.2) by $\tau^{1/2}$, it follows by taking $\tau \rightarrow \infty$ that

$$c\hat{f}(2\lambda, y_0) = 0,$$

for some constant $c \neq 0$. Since $\lambda \in \mathbb{R}$ is arbitrary and $y_0 \in M_0$ is also arbitrary, this proves the proposition in this special case. For details we refer to the following proof.

To deal with a transversally anisotropic manifold, we need to consider the (in most cases rare) possibility where γ and η may intersect at many different points and may have self-intersections. This makes the proof much more technical, and will be achieved by introducing additional parameters in the above construction and by using the auxiliary Lemma 5.2 below.

Now, we turn to prove Proposition 5.1 in the general case.

Proof of Proposition 5.1. We do the proof in several steps.

Step 1. Preparation.

By taking $u_j = 1$ for $j \geq 5$, it is sufficient to prove the result when $m = 4$. We assume that $M \subset \mathbb{R} \times M_0$ with $g = e \oplus g_0$. The dimension of M is denoted by n , and so $\dim(M_0) = n - 1$.

We may enlarge M_0 so that it has strictly convex boundary (first embed M_0 in some closed manifold M_1 , remove a small geodesic ball from $M_1 \setminus M_0$ and glue a part with strictly convex boundary near the removed part). By [Sal17, Lemma 3.1], there is a set E of zero measure in (M_0, g_0) so that all points in $M_0 \setminus E$ lie on some nontangential geodesic between boundary points. Fix a point $y_0 \in M_0 \setminus E$ and a direction $v_0 \in S_{y_0}M_0$ so that the geodesic $\gamma : [0, T] \rightarrow M_0$ through (y_0, v_0) is a nontangential geodesic between boundary points. Without loss of generality, we may assume that γ is a unit speed geodesic (i.e. $|\dot{\gamma}| = 1$). The property of a geodesic being nontangential is not changed under small perturbations. Therefore, we may find $w_0 \in S_{y_0}M_0$ close to v_0 so that $w_0 \neq v_0$ and the unit speed geodesic $\eta : [0, S] \rightarrow M_0$ through (y_0, w_0) is also a nontangential geodesic between boundary points. We may arrange so that the geodesics γ and η are distinct and are not reverses of each other (in fact, γ can only self-intersect at y_0 finitely many times [KS13, Lemma 7.2], and it is enough choose w_0 near v_0 that is different from the corresponding finitely many tangent vectors of γ and their negatives).

We next show that two distinct geodesics γ and η that are not the reverse of each other can intersect only finitely many times. Assume the opposite, that there are infinitely many intersection points $\{p_k\}_{k \in \mathbb{N}}$ and intersection times $\{t_k\}_{k \in \mathbb{N}}, \{s_k\}_{k \in \mathbb{N}}$ satisfying

$$\gamma(t_k) = p_k = \eta(s_k), \text{ for all } k \in \mathbb{N}.$$

Since M is compact, $t_k \in [0, T]$ and $s_k \in [0, S]$, by passing to subsequences and using continuity of unit speed geodesics γ, η , we may assume that $\gamma(t_k) \rightarrow \gamma(t_0) = p_0$ and $\eta(s_k) \rightarrow \eta(s_0) = p_0$ as $k \rightarrow \infty$, for some $t_0 \in [0, T]$, $s_0 \in [0, S]$ and $p_0 \in M$.

In addition, we denote the tangent vectors $V_\gamma := \dot{\gamma}(t_0)$ and $V_\eta := \dot{\eta}(s_0)$. By using the continuity of $\dot{\gamma}(t)$, $\dot{\eta}(s)$ and the compactness of the unit sphere, we have (by passing to subsequences again) that

$$\lim_{k \rightarrow \infty} \dot{\gamma}(t_k) = V_\gamma \text{ and } \lim_{k \rightarrow \infty} \dot{\eta}(s_k) = V_\eta.$$

Now, it is clear that $V_\gamma \neq \pm V_\eta$, by using the fact that γ and η are distinct and not reverses. The injectivity radius at p_0 is positive. However, since γ and η intersect in all geodesic balls $B_\epsilon(p_0)$ for any $\epsilon > 0$, this is a contradiction. This shows that two different nontangential geodesics can only intersect finitely many times.

Step 2. Choice of the harmonic functions u_j .

We denote the points of $\mathbb{R} \oplus M_0$ by (x_1, x') . We now use the argument in [FKLS16, Proposition 2.2] and choose solutions

$$u_1 = e^{-L(\tau+i\lambda)x_1}(\tilde{v}_{L(\tau+i\lambda)}(x') + r_1), \quad u_2 = \overline{e^{L(\tau+i\lambda)x_1}(\tilde{v}_{L(\tau+i\lambda)}(x') + r_2)},$$

of $\Delta_g u_j = 0$ in M , where $\tau \geq 1$ is sufficiently large, $L \geq 1$ is an additional large parameter that will be fixed later, and $\lambda \in \mathbb{C}$ is fixed,

$$\tilde{v}_{L(\tau+i\lambda)} := \tau^{-\frac{n-2}{8}} v_{L(\tau+i\lambda)}(x')$$

where $v_{L(\tau+i\lambda)}$ is the Gaussian beam quasimode concentrating near the geodesic γ in M_0 constructed in [FKLS16, Proposition 3.1], and r_j are remainder terms satisfying

$$\|r_j\|_{H^k(M)} = O(\tau^{-R})$$

as $\tau \rightarrow \infty$ where $k, R > 0$ can be chosen arbitrarily large.

There are three differences in the above construction compared with [FKLS16, Proposition 2.2]. The first difference is that λ is complex, but the construction of $v_{\tau+i\lambda}$ goes through without changes for complex λ . The second difference is the factor $\tau^{-\frac{n-2}{8}}$ in front of $v_{L(\tau+i\lambda)}$. The argument in [FKLS16, Proposition 3.1] implies that

$$\|v_{L(\tau+i\lambda)}\|_{L^2(M_0)} = O(1), \quad \|v_{L(\tau+i\lambda)}\|_{L^4(M_0)} = O((L\tau)^{\frac{n-2}{8}}).$$

Hence the main term in the solutions u_1 and u_2 is normalized so that

$$\|\tilde{v}_{L(\tau+i\lambda)}\|_{L^4(M_0)} = O(1).$$

This normalization will be appropriate for dealing with products of four solutions. The third difference is the decay for the error terms r_j , which is better than the decay stated in [FKLS16, Proposition 2.2] and can be justified as follows. By the argument in [FKLS16, Proposition 2.2], writing $s = \tau + i\lambda$, the function r_1 is obtained by solving the equation

$$e^{sx_1} \Delta_g(e^{-sx_1} r_1) = \tau^{-\frac{n-2}{8}} (-\Delta_{g_0} - s^2) v_s \text{ in } M.$$

By using a Carleman estimate shifted to a negative semiclassical Sobolev space H_{scl}^{-k} , with $h = \tau^{-1}$ as the semiclassical parameter, and using the corresponding solvability result in H_{scl}^k (see e.g. [FKSU09, Lemma 4.3 and Proposition 4.4]), there is a solution $r_1 \in H^k(M)$ with

$$(5.3) \quad \|r_1\|_{H_{\text{scl}}^k(M)} \leq \frac{C}{\tau} \|(-\Delta_{g_0} - s^2) v_s\|_{H_{\text{scl}}^k(M)}.$$

By [FKLS16, Proposition 3.1], one has $\|(-\Delta_{g_0} - s^2) v_s\|_{L^2(M_0)} = O(\tau^{-K})$ for some arbitrarily large K . However, by looking at the formula for $f = (-\Delta_{g_0} - s^2) v_s$ given after [FKLS16, equation (3.4)] we see that taking k derivatives of f has the effect of bringing at most k powers of s to the front of the expression, or reducing the degree of vanishing of h_j on Γ by at most k . In effect this means that $\|(-\Delta_{g_0} - s^2) v_s\|_{H_{\text{scl}}^k(M_0)} = O(\tau^{-K})$, leading to

$$(5.4) \quad \|(-\Delta_{g_0} - s^2) v_s\|_{H_{\text{scl}}^k(M)} = O(\tau^{-K})$$

where K can be chosen arbitrarily large. The required decay

$$\|r_1\|_{H^k(M)} \leq \tau^k \|r_1\|_{H_{\text{scI}}^k(M)} = O(\tau^{-R})$$

follows by combining (5.3) and (5.4) after choosing K large enough, and the same estimate for r_2 follows analogously. It follows from this discussion, after taking $k > n/2$ and using the Sobolev embedding $H^k(M) \subset L^\infty(M)$, that

$$(5.5) \quad u_1 u_2 = e^{-2i\text{Re}(L\lambda)x_1} |\tilde{v}_{L(\tau+i\lambda)}(x')|^2 + O_{L^2(M)}((L\tau)^{-R}).$$

We now repeat the previous construction for the geodesic η and choose solutions

$$u_3 = e^{-(\tau+i\mu)x_1} (\tilde{w}_{\tau+i\mu}(x') + r_3), \quad u_4 = \overline{e^{(\tau+i\mu)x_1} (\tilde{w}_{\tau+i\mu}(x') + r_4)},$$

where $\mu \in \mathbb{C}$ is fixed, $\|r_j\|_{L^2(M)} = O(\tau^{-R})$ as $\tau \rightarrow \infty$, and

$$\tilde{w}_{\tau+i\mu} := \tau^{-\frac{n-2}{8}} w_{\tau+i\mu}(x')$$

where $w_{\tau+i\mu}$ is the Gaussian beam quasimode concentrating near η constructed in [FKLS16, Proposition 3.1] so that

$$\|\tilde{w}_{\tau+i\mu}\|_{L^4(M_0)} = O(1) \text{ as } \tau \rightarrow \infty.$$

Similarly as for $u_1 u_2$, one has

$$(5.6) \quad u_3 u_4 = e^{-2i\text{Re}(\mu)x_1} |\tilde{w}_{\tau+i\mu}(x')|^2 + O_{L^2(M)}(\tau^{-R}).$$

Step 3. The integral of f against $u_1 u_2 u_3 u_4$.

By the assumption that f integrates to zero against products of four harmonic functions, we have

$$\int_M f u_1 u_2 u_3 u_4 dV = 0.$$

Using (5.5) and (5.6), we have

$$\begin{aligned} & \int_M f u_1 u_2 u_3 u_4 dV \\ &= \int_M f(x_1, x') e^{-2i\text{Re}(L\lambda+\mu)x_1} |\tilde{v}_{L(\tau+i\lambda)}(x')|^2 |\tilde{w}_{\tau+i\mu}(x')|^2 dV + O(\tau^{-R}). \end{aligned}$$

If we extend f by zero to $\mathbb{R} \times M_0$ and denote the partial Fourier transform of f in the x_1 variable by $\hat{f}(\lambda, x')$, the previous identity becomes

$$\int_M f u_1 u_2 u_3 u_4 dV = \int_{M_0} \hat{f}(2\text{Re}(L\lambda + \mu), \cdot) |\tilde{v}_{L(\tau+i\lambda)}|^2 |\tilde{w}_{\tau+i\mu}|^2 dV_{g_0} + O(\tau^{-R}).$$

Note that $\tilde{v}_{L(\tau+i\lambda)}$ and $\tilde{w}_{\tau+i\mu}$ can be chosen to be supported in arbitrarily small but fixed neighborhoods of γ and η , respectively. Thus if p_1, \dots, p_N are the distinct points in M_0 where γ intersects η , then the last integral over M_0 is actually over $U_1 \cup \dots \cup U_N$ where U_r is a small neighborhood of p_r in M_0 .

We will use the abbreviation

$$F(x') = F_{\text{Re}(L\lambda+\mu)}(x') := \hat{f}(2\text{Re}(L\lambda + \mu), x').$$

Note for later purposes that F is independent of $\text{Im}(\lambda)$ and $\text{Im}(\mu)$ and that

$$\|F\|_{C^1(M_0)} \lesssim \|f\|_{C^1(M)}.$$

Combining the above facts, we have

$$(5.7) \quad \sum_{r=1}^N \tau^{\frac{1}{2}} \int_{U_r} F |\tilde{v}_{L(\tau+i\lambda)}|^2 |\tilde{w}_{\tau+i\mu}|^2 dV_{g_0} = o(1)$$

as $\tau \rightarrow \infty$. It will be shown below that with the normalizing factor $\tau^{\frac{1}{2}}$, the left hand side has a nontrivial limit as $\tau \rightarrow \infty$.

Step 4. Analysis of the integrals in (5.7).

Fix now p to be one of the intersection points p_r and let $U = U_r$. We consider the integral

$$(5.8) \quad \int_U F |\tilde{v}_{L(\tau+i\lambda)}|^2 |\tilde{w}_{\tau+i\mu}|^2 dV_{g_0}.$$

By the construction in [FKLS16, Proposition 3.1], in U the quasimode $\tilde{v}_{\tau+i\lambda}$ is a finite sum

$$\tilde{v}_{L(\tau+i\lambda)}|_U = v^{(1)} + \dots + v^{(P)}$$

where $t_1 < \dots < t_P$ are the times in $[0, T]$ when $\gamma(t_j) = p$, each $v^{(j)}$ has the form

$$v^{(j)} = \tau^{\frac{n-2}{8}} e^{iL(\tau+i\lambda)\Phi_j} a_j$$

where each $\Phi = \Phi_j$ is a smooth complex function in U satisfying, for t close to t_j ,

$$\Phi(\gamma(t)) = t, \quad \nabla\Phi(\gamma(t)) = \dot{\gamma}(t), \quad \text{Im}(\nabla^2\Phi(\gamma(t))) \geq 0, \quad \text{Im}(\nabla^2\Phi)|_{\dot{\gamma}(t)^\perp} > 0,$$

and each a_j is a smooth function in U of the form

$$a_j(t, y) = (a_{0,j}(t, y) + O(\tau^{-1}))\chi(y/\delta')$$

where (t, y) are Fermi coordinates for γ for t close to t_j , $a_{0,j}(t, 0)$ is a nonvanishing function independent of τ and λ , χ is a smooth cutoff function supported in the unit ball, and $\delta' > 0$ is a fixed number that can be taken very small. Note that as opposed to [FKLS16], there is no power of τ in the definition of a_j . By the argument in [FKLS16, formula (3.5) and before] one also has, as $\tau \rightarrow \infty$,

$$(5.9) \quad \|v^{(j)}\|_{L^4(U)} = O(1), \quad \|v^{(j)}\|_{L^4(U \cap \partial M_0)} = O(1).$$

In a similar way, $\tilde{w}_{\tau+i\mu}$ in U is a finite sum

$$\tilde{w}_{\tau+i\mu}|_U = w^{(1)} + \dots + w^{(Q)}$$

where $s_1 < \dots < s_Q$ are the times in $[0, S]$ when $\eta(s_k) = p$, each $w^{(k)}$ has the form

$$w^{(k)} = \tau^{\frac{n-2}{8}} e^{i(\tau+i\mu)\Psi_k} b_k$$

where each $\Psi = \Psi_k$ satisfies, for s close to t_k ,

$$\Psi(\eta(s)) = s, \quad \nabla\Psi(\eta(s)) = \dot{\eta}(s), \quad \text{Im}(\nabla^2\Psi(\eta(s))) \geq 0, \quad \text{Im}(\nabla^2\Psi)|_{\dot{\eta}(s)^\perp} > 0,$$

and b_j has a similar form as a_j but is supported near η .

Inserting the formulas for $\tilde{v}_{L(\tau+i\lambda)}$ and $\tilde{w}_{\tau+i\mu}$ in (5.8) yields that

$$(5.10) \quad \tau^{\frac{1}{2}} \int_U F |\tilde{v}_{L(\tau+i\lambda)}|^2 |\tilde{w}_{\tau+i\mu}|^2 dV_{g_0} = \sum_{j,k,l,m} I_{jklm}$$

where

$$\begin{aligned} I_{jklm} &= \tau^{\frac{1}{2}} \int_U F v^{(j)} \overline{v^{(k)}} w^{(l)} \overline{w^{(m)}} dV_{g_0} \\ &= \tau^{\frac{n-1}{2}} \int_U e^{i\tau \Xi_{jklm}} A_{jklm} F dV_{g_0} \end{aligned}$$

where

$$\begin{aligned} \Xi_{jklm} &:= L\Phi_j - L\overline{\Phi_k} + \Psi_l - \overline{\Psi_m}, \\ A_{jklm} &:= e^{-L\lambda\Phi_j - L\overline{\lambda}\overline{\Phi_k}} e^{-\mu\Psi_l - \mu\overline{\Psi_m}} a_j \overline{a_k} b_l \overline{b_m}. \end{aligned}$$

We will next analyze the integrals I_{jklm} and show that the only nontrivial contributions as $\tau \rightarrow \infty$ come from the terms where $d\Xi_{jklm}(p) = 0$. After this, we will fix the parameters so that $d\Xi_{jklm}(p) = 0$ will happen only when $j = k$ and $l = m$.

Step 5. Evaluation of I_{jklm} when $d\Xi_{jklm}(p) = 0$.

Let j, k, l, m be such that $\Xi = \Xi_{jklm}$ satisfies $d\Xi(p) = 0$, and write

$$B := FA_{jklm}.$$

Writing z for the geodesic normal coordinates in (M_0, g_0) with origin at p , the phase function Ξ has the Taylor expansion

$$\Xi(z) = \Xi(0) + \frac{1}{2}Hz \cdot z + O(|z|^3).$$

Here $\Xi(0) = L(t_j - t_k) + s_l - s_m$ and $H = H_{jklm} = (\partial_{z_a z_b} \Xi)_{a,b}$ is the Hessian of Ξ in the z coordinates. Note that the imaginary parts of Hessians of $\Phi_j, \Phi_k, \Psi_l, \Psi_m$ at p are all positive semidefinite. Moreover, they are positive definite in the codimension one subspaces $\dot{\gamma}(t_j)^\perp, \dot{\gamma}(t_k)^\perp, \dot{\eta}(s_l)^\perp, \dot{\eta}(s_m)^\perp$, respectively. Thus it follows that $\text{Im}(H) = \text{Im}(\nabla^2(L(\Phi_j + \Phi_k) + \Psi_l + \Psi_m))|_p$ is positive semidefinite, and moreover it is positive definite since the above codimension one subspaces span the whole tangent space at p . The last fact holds since $\dot{\gamma}(t_j) \neq \pm \dot{\eta}(s_l)$, which follows because the geodesics γ and η are distinct and one is not the reverse of the other. Finally, since $\text{Im}(H)$ is positive definite, by choosing U small one has $|z| \leq \delta$ in U where δ is very small, which implies that

$$|e^{i\tau(\frac{1}{2}Hz \cdot z + O(|z|^3))}| \leq e^{-c\tau|z|^2}$$

for some $c > 0$. This shows that one may indeed use dominated convergence in the argument below.

One has

$$\begin{aligned} I_{jklm} &= \tau^{\frac{n-1}{2}} \int_U e^{i\tau\Xi} B dV_{g_0} \\ &= \tau^{\frac{n-1}{2}} e^{i\tau\Xi(0)} \int_{\mathbb{R}^{n-1}} e^{i\tau(\frac{1}{2}Hz \cdot z + O(|z|^3))} B(z) |g_0(z)|^{1/2} dz. \end{aligned}$$

We will change variables $z \rightarrow \tau^{-1/2}z$, which brings a Jacobian $\tau^{-\frac{n-1}{2}}$ that cancels the power of τ in front. Note that one has $|g_0(z/\tau^{1/2})| \rightarrow 1$ and

$$B(z/\tau^{1/2}) \rightarrow F(p) e^{-L\lambda t_j - L\bar{\lambda} t_k} e^{-\mu s_l - \bar{\mu} s_m} a_j(p) \overline{a_k(p)} b_l(p) \overline{b_m(p)}$$

as $\tau \rightarrow \infty$. Combining these facts and using dominated convergence gives that

$$I_{jklm} = e^{i\tau(L(t_j - t_k) + s_l - s_m)} c_{jklm} F(p) e^{-L\lambda t_j - L\bar{\lambda} t_k} e^{-\mu s_l - \bar{\mu} s_m} + o(1)$$

where

$$c_{jklm} = a_j(p) \overline{a_k(p)} b_l(p) \overline{b_m(p)} \int_{\mathbb{R}^{n-1}} e^{\frac{i}{2} H_{jklm} z \cdot z} dz.$$

The last integral is finite since $\text{Im}(H_{jklm})$ is positive definite. For later purposes we observe that H_{jjll} is purely imaginary, hence

$$c_{jjll} = |a_j(p)|^2 |b_l(p)|^2 \int_{\mathbb{R}^{n-1}} e^{-\frac{1}{2} \text{Im}(H_{jjll}) z \cdot z} dz$$

where the last integral is positive. In particular, we have

$$I_{jjll} = c_{jjll} F(p) e^{-2L\text{Re}(\lambda)t_j} e^{-2\text{Re}(\mu)s_l} + o(1)$$

where $c_{jjll} > 0$.

Step 6. Evaluation of I_{jklm} when $d\Xi_{jklm}(p) \neq 0$.

Write $\varphi = \text{Re}(\Xi_{jklm})$. Since $d\Phi_j(p), d\Psi_l(p)$ etc are real, we have $d\varphi(p) \neq 0$, and I_{jklm} may be written as

$$I_{jklm} = \tau^{\frac{1}{2}} \int_U e^{i\tau\varphi} F \check{v}^{(j)} \overline{\check{v}^{(k)}} \check{w}^{(l)} \overline{\check{w}^{(m)}} dV_{g_0}$$

where

$$\check{v}^{(j)} = \tau^{\frac{n-2}{8}} e^{-L\tau \text{Im}(\Phi_j) - L\lambda\Phi_j} a_j, \quad \check{w}^{(l)} = \tau^{\frac{n-2}{8}} e^{-\tau \text{Im}(\Psi_l) - \mu\Psi_l} b_l.$$

We wish to use a non-stationary phase argument as in [FKLS16, end of proof Proposition 3.1]. Write

$$e^{i\tau\varphi} = \frac{1}{i\tau} P(e^{i\tau\varphi}), \quad Pw = \langle |d\varphi|^{-2} d\varphi, dw \rangle,$$

where we assume that U has been chosen so small that $d\varphi$ is nonvanishing in U . Since F is C^1 , we may integrate by parts to obtain

$$(5.11) \quad \begin{aligned} I_{jklm} &= \frac{1}{i\tau^{\frac{1}{2}}} \int_U e^{i\tau\varphi} P^t \left[F \check{v}^{(j)} \overline{\check{v}^{(k)}} \check{w}^{(l)} \overline{\check{w}^{(m)}} \right] dV_{g_0} \\ &\quad + \frac{1}{i\tau^{\frac{1}{2}}} \int_{U \cap \partial M} \frac{\partial_\nu \varphi}{|d\varphi|^2} e^{i\tau\varphi} F \check{v}^{(j)} \overline{\check{v}^{(k)}} \check{w}^{(l)} \overline{\check{w}^{(m)}} dS \end{aligned}$$

where the boundary term only appears if $p \in \partial M$.

The boundary term in (5.11) goes to zero as $\tau \rightarrow \infty$ since $\|\check{v}^{(j)}\|_{L^4(U \cap \partial M)} \lesssim 1$ by (5.9) etc. In the integral over U , if the derivative hits F one can estimate

$$\begin{aligned} &\left| \frac{1}{i\tau^{\frac{1}{2}}} \int_U e^{i\tau\varphi} (P^t F) \check{v}^{(j)} \overline{\check{v}^{(k)}} \check{w}^{(l)} \overline{\check{w}^{(m)}} dV_{g_0} \right| \\ &\lesssim \tau^{-1/2} \|F\|_{C^1(M)} \|\check{v}^{(j)} \overline{\check{v}^{(k)}} \check{w}^{(l)} \overline{\check{w}^{(m)}}\|_{L^1(M_0)} \end{aligned}$$

which goes to zero as $\tau \rightarrow \infty$ since $\|\check{v}^{(j)}\|_{L^4(M_0)} \lesssim 1$ etc. The worst behaviour in τ in the integral over U occurs when a derivative hits $e^{-L\tau \text{Im}(\Phi_j)}$ or $e^{-\tau \text{Im}(\Psi_l)}$ etc. This brings a factor like $\tau d(\text{Im}(\Phi_j))$ into the integrand, and since $d(\text{Im}(\Phi_j))$ vanishes on γ one can choose new coordinates $z = (z', z_{n-1})$ near 0 so that $|d(\text{Im}(\Phi_j))| \lesssim |z'|$. Thus the integral that one needs to estimate looks like

$$\frac{1}{i\tau^{\frac{1}{2}}} \int_U e^{i\tau\varphi} F [\tau d(\text{Im}(\Phi_j))] \check{v}^{(j)} \overline{\check{v}^{(k)}} \check{w}^{(l)} \overline{\check{w}^{(m)}} dV_{g_0}.$$

Evaluating this integral as in Step 5, and using that the change of variables $z \rightarrow \tau^{-1/2}z$ together with $|d(\text{Im}(\Phi_j))| \lesssim |z'|$ brings an additional factor $\tau^{-1/2}$, shows that this kind of integral is $O(\tau^{-1/2})$. This concludes the proof that

$$d\Xi_{jklm}(p) \neq 0 \implies \lim_{\tau \rightarrow \infty} I_{jklm} = 0.$$

Step 7. Evaluation of (5.7).

Recall from Step 3 that p_1, \dots, p_N were the distinct intersection points of γ and η and that U_r were small neighborhoods of p_r . As in Step 4, for each r with $1 \leq r \leq N$ let $t_1^{(r)} < \dots < t_{P_r}^{(r)}$ be the times in $[0, T]$ when $\gamma(t_j^{(r)}) = p_r$, and let $s_1^{(r)} < \dots < s_{Q_r}^{(r)}$ be the times in $[0, S]$ when $\eta(s_j^{(r)}) = p_r$.

Going back to (5.7) and using (5.10), we have

$$\sum_{r=1}^N \sum_{j,k=1}^{P_r} \sum_{l,m=1}^{Q_r} I_{jklm}^{(r)} = o(1)$$

as $\tau \rightarrow \infty$, where

$$I_{jklm}^{(r)} = \tau^{\frac{n-1}{2}} \int_{U_r} e^{i\tau \Xi_{jklm}^{(r)}} A_{jklm}^{(r)} F dV_{g_0}$$

where $\Xi_{jklm}^{(r)}$ and $A_{jklm}^{(r)}$ are defined in U_r as in Step 4.

The integrals $I_{jklm}^{(r)}$ were evaluated in Steps 5 and 6. If we define

$$c_{jklm}^{(r)} := \begin{cases} a_j^{(r)}(p_r) \overline{a_k^{(r)}(p_r)} b_l^{(r)}(p_r) \overline{b_m^{(r)}(p_r)} \int_{\mathbb{R}^{n-1}} e^{\frac{i}{2} H_{jklm}^{(r)} z \cdot z} dz, & d\Xi_{jklm}^{(r)}(p_r) = 0, \\ 0, & \text{otherwise,} \end{cases}$$

then we get from Steps 5 and 6 that

$$(5.12) \quad \sum_{r=1}^N \sum_{j,k=1}^{P_r} \sum_{l,m=1}^{Q_r} e^{i\tau [L(t_j^{(r)} - t_k^{(r)}) + s_l^{(r)} - s_m^{(r)}]} c_{jklm}^{(r)} F(p_r) e^{-L(\lambda t_j^{(r)} + \bar{\lambda} t_k^{(r)})} e^{-\mu s_l^{(r)} - \bar{\mu} s_m^{(r)}} = o(1)$$

as $\tau \rightarrow \infty$. In the above formula $F(p_r) = F_{\text{Re}(\lambda+\mu)}(p_r)$.

Step 7. Choosing L so that $d\Xi_{jklm}^{(r)}(p_r) = 0$ only when $j = k$ and $l = m$.

Next, we choose $L \in \mathbb{N}$ large enough, but fixed, so that

$$d\Xi_{jklm}(p_r) \neq 0$$

for all $1 \leq r \leq N$ unless $j = k$ and $l = m$. We have

$$\begin{aligned} \nabla \Xi_{jklm}(p_r) &= L \nabla \Phi_j - L \nabla \bar{\Phi}_k + \nabla \Psi_l - \nabla \bar{\Psi}_m \\ &= L(\dot{\gamma}(t_j^{(r)}) - \dot{\gamma}(t_k^{(r)})) + \dot{\eta}(s_l^{(r)}) - \dot{\eta}(s_m^{(r)}). \end{aligned}$$

Since the geodesic γ is transversal and thus it is not closed geodesic, we have

$$\dot{\gamma}(t_j^{(r)}) - \dot{\gamma}(t_k^{(r)}) \neq 0$$

for all $j \neq k$ for all r with $1 \leq r \leq N$. Let

$$\alpha = \min \left\{ |\dot{\gamma}(t_j^{(r)}) - \dot{\gamma}(t_k^{(r)})| : 1 \leq r \leq N, 1 \leq j, k \leq P_r \right\} > 0$$

and

$$\beta = \max \left\{ |\dot{\eta}(s_l^{(r)}) - \dot{\eta}(s_m^{(r)})| : 1 \leq r \leq N, 1 \leq l, m \leq Q_r \right\}.$$

We choose

$$L \geq \frac{\beta + 1}{\alpha},$$

then we have

$$\begin{aligned} |L(\dot{\gamma}(t_j^{(r)}) - \dot{\gamma}(t_k^{(r)})) + \dot{\eta}(s_l^{(r)}) - \dot{\eta}(s_m^{(r)})| &\geq |L(\dot{\gamma}(t_j) - \dot{\gamma}(t_k))| - |\dot{\eta}(s_l) - \dot{\eta}(s_m)| \\ &\geq L\alpha - \beta \geq 1 > 0. \end{aligned}$$

Thus we have found that for any $L \geq \frac{\beta+1}{\alpha}$, one has

$$d\Xi_{jklm}(p_r) \neq 0$$

if $j \neq k$.

For $1 \leq r \leq N$, assume then that $j = k$ and $l \neq m$. Then we have

$$\nabla \Xi_{jklm}(p_r) = L(\dot{\gamma}(t_j^{(r)}) - \dot{\gamma}(t_j^{(r)})) + \dot{\eta}(s_l^{(r)}) - \dot{\eta}(s_m^{(r)}) \neq 0,$$

since η is transversal and $l \neq m$.

In conclusion, the only case when $d\Xi_{jklm}^{(r)}(p_r) = 0$ is $j = k$ and $l = m$.

Step 8. Conclusion of the proof.

Going back to (5.12) and using the result in Step 7, and taking $\tau \rightarrow \infty$, we have

$$(5.13) \quad \sum_{r=1}^N \sum_{j=1}^{P_r} \sum_{l=1}^{Q_r} c_{jjl}^{(r)} F_{\operatorname{Re}(L\lambda+\mu)}(p_r) e^{-Lt_j^{(r)}(\lambda+\bar{\lambda})} e^{s_l^{(r)}(-\mu-\bar{\mu})} = 0,$$

where we have used that $e^{i\tau[L(t_j^{(r)}-t_k^{(r)})+s_l^{(r)}-s_m^{(r)}]} = 1$, when $j = k$ and $l = m$.

Let $\lambda \in \mathbb{R}$ and choose μ so that

$$2L\lambda + 2\mu = 2\lambda,$$

which is equivalent to

$$\mu = (1 - L)\lambda \in \mathbb{R}.$$

Then (5.13) reads

$$(5.14) \quad \sum_{r=1}^N \sum_{j=1}^{P_r} \sum_{l=1}^{Q_r} c_{jjl}^{(r)} F_\lambda(p_r) e^{-2\lambda[L(t_j^{(r)}-s_l^{(r)})+s_l^{(r)}]} = 0,$$

We define two sets of real numbers as follows:

$$\mathcal{Q}_1 = \cup_{r,r'=1}^N \cup_{j,k=1}^{P_r} \{t_j^{(r)} - t_k^{(r')}\}, \quad \mathcal{Q}_2 = \cup_{r,r'=1}^N \cup_{l,m=1}^{Q_r} \{s_l^{(r)} - s_m^{(r')}\},$$

and

$$\tilde{\alpha} = \min_{d_1 \in \mathcal{Q}_1, d_2 \in \mathcal{Q}_2, d_1 \neq d_2} |d_1 - d_2|, \quad \tilde{\beta} = \max_{d \in \mathcal{Q}_2} |d|.$$

Finally, we redefine the number L in Step 7 as

$$L = \max \left\{ \frac{\tilde{\beta}}{\tilde{\alpha}} + 1, \frac{\beta + 1}{\alpha} \right\},$$

where α, β are the numbers given in Step 7.

Let $(r_1, j_1, l_1) \neq (r_2, j_2, l_2)$, then we want to show that

$$(5.15) \quad L(t_{j_1}^{(r_1)} - s_{l_1}^{(r_1)}) + s_{l_1}^{(r_1)} \neq L(t_{j_2}^{(r_2)} - s_{l_2}^{(r_2)}) + s_{l_2}^{(r_2)}.$$

Let us set

$$d_1 = t_{j_1}^{(r_1)} - t_{j_2}^{(r_2)} \text{ and } d_2 = s_{l_1}^{(r_1)} - s_{l_2}^{(r_2)}.$$

We have following cases:

(a) For the case $d_1 = d_2$. Suppose that

$$(5.16) \quad L(t_{j_1}^{(r_1)} - s_{l_1}^{(r_1)}) + s_{l_1}^{(r_1)} = L(t_{j_2}^{(r_2)} - s_{l_2}^{(r_2)}) + s_{l_2}^{(r_2)},$$

then we have $L(d_1 - d_2) + d_2 = 0$. It follows that $s_{l_1}^{(r_1)} = s_{l_2}^{(r_2)}$. Thus $l_1 = l_2$ and $r_1 = r_2$. Since $d_1 = d_2$, we also have $t_{j_1}^{(r_1)} = t_{j_2}^{(r_2)}$. Thus, $j_1 = j_2$ holds, which leads to a contradiction to $(r_1, j_1, l_1) \neq (r_2, j_2, l_2)$. Thus we must have (5.15).

(b) For the case $d_1 \neq d_2$. If (5.16) holds, we have

$$L = \frac{d_2}{d_1 - d_2}.$$

However, this cannot be true since

$$L = \frac{\tilde{\beta}}{\tilde{\alpha}} + 1 > \frac{d_2}{d_1 - d_2}.$$

Thus again we have (5.15).

We now go back to (5.14) and use Lemma 5.2 below together with the condition (5.15) and the fact that $F_\lambda(p_r) = \hat{f}(2\lambda, p_r)$. This yields that

$$c_{jll}^{(r)} \hat{f}(2\lambda, p_r) = 0$$

for all (r, j, l) . In Step 5 we proved that $c_{jll}^{(r)} > 0$ for all (r, j, l) , showing that $\hat{f}(2\lambda, p_r) = 0$ for all $\lambda \in \mathbb{R}$ and all r . Since the point y_0 in Step 1 was one of the p_r , it follows that $f(x_1, y_0) = 0$ for all $x_1 \in \mathbb{R}$. By Step 1 this is true for almost every y_0 in (M_0, g_0) , and by the continuity of f one gets that $f \equiv 0$ as required. \square

Lemma 5.2. *Let f_1, \dots, f_N be compactly supported distributions in \mathbb{R} such that for some distinct real numbers a_1, \dots, a_N one has*

$$\sum_{j=1}^N \hat{f}_j(\lambda) e^{a_j \lambda} = 0, \quad \lambda \in \mathbb{R}.$$

Then $f_1 = \dots = f_N = 0$.

Proof. Suppose without loss of generality that $a_1 > a_2 > \dots > a_N$. Then

$$\hat{f}_1(\lambda) = - \sum_{j=2}^N e^{-(a_1 - a_j)\lambda} \hat{f}_j(\lambda).$$

By the Paley-Wiener-Schwartz theorem there are $C, M > 0$ so that

$$|\hat{f}_j(\lambda)| \leq C(1 + |\lambda|)^M, \quad \lambda \in \mathbb{R}.$$

Write $\delta = a_1 - a_2 > 0$. Since $a_1 - a_j \geq \delta$ for $j \geq 2$, it follows that

$$|\hat{f}_1(\lambda)| \leq \begin{cases} C(1 + |\lambda|)^M, & \lambda \leq 0, \\ C(1 + \lambda)^M e^{-\delta\lambda}, & \lambda \geq 0. \end{cases}$$

However, no nontrivial compactly supported distribution f_1 can have the above decay for its Fourier transform. To see this, note that

$$e^{\epsilon\lambda} \hat{f}_1(\lambda) \in \mathcal{S}'(\mathbb{R}), \quad 0 \leq \epsilon < \delta.$$

Thus using [Hor85, Theorem 7.4.2] there exists an analytic function U in $\{0 < \text{Im}(t) < \delta\}$ so that the Fourier transform of $e^{\epsilon\lambda} \hat{f}_1(\lambda)$ is $U(\cdot + i\epsilon)$. By [Hor85, Remark after Theorem 7.4.3] the limit of $U(\cdot + i\epsilon)$ in $\mathcal{S}'(\mathbb{R})$ as $\epsilon \rightarrow 0$ is the Fourier transform of $\hat{f}_1(\lambda)$, i.e. $2\pi f_1(-\cdot)$. Fix some interval $I \subset \mathbb{R}$ that is outside the support of $f_1(-\cdot)$, and consider the rectangle $Z = I \times (0, \delta)$. For ϵ close to 0, one has

$$|U(t + i\epsilon)| \leq \|e^{\epsilon\lambda} \hat{f}_1(\lambda)\|_{L^1} \lesssim \|(1 + |\lambda|)^M e^{\epsilon\lambda}\|_{L^1(\mathbb{R}_-)} + 1 \lesssim \epsilon^{-M}.$$

Since the limit of $U(\cdot + i\epsilon)$ in $\mathcal{D}'(I)$ is $2\pi f_1(-\cdot)|_I = 0$, by [Hor85, Theorem 3.1.15] one has $U = 0$ in Z . Now U is analytic, so $U = 0$ in $\{0 < \text{Im}(t) < \delta\}$ and $f_1 = 0$. Repeating this argument gives that $f_2 = \dots = f_N = 0$. \square

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