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Neighbours Weights Matrices

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Key words: k -nearest neighbours, Model specification, Spatial J -test, Weights Matrix

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Practical Proposals for Specifying k -Nearest Neighbours Weights Matrices

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Abstract

In this article we introduce and evaluate testing procedures for specifying the number k of nearest neighbours in the weights matrix of spatial econometric models. The spatial J -test is used for specification search. Two testing procedures are suggested: an increasing neighbours testing procedure and a decreasing neighbours testing procedure. Simulations show that the increasing neighbours testing procedures can be used in large samples to determine k . The decreasing neighbours testing procedure is found to have low power, and is not recommended for use in practice. An empirical example involving house price data is provided to show how to use the testing procedures with real data.

Keywords: k -nearest neighbours, Model specification, Spatial J -test, Weights matrix.

1 Introduction

The weights matrix plays an important role in spatial econometric models. It expresses the assumed spatial structure of the variables in the model. Ideally, the weights matrix should be based on theory rather than on spatial patterns

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found in the data. However, the theory may be subjectively interpreted, and may result in several equally plausible specifications of the weights matrix.

The motivation for the present paper is the lack of formal justification for the choice of the weights matrix in spatial econometric models. Focusing on k -nearest neighbours weights matrices, testing strategies for specifying the weights matrix are proposed.

The model we assume is a general spatial autoregressive model

$$\begin{aligned}\mathbf{y} &= \mathbf{X}\boldsymbol{\beta} + \rho\mathbf{W}\mathbf{y} + \mathbf{u}, \\ \mathbf{u} &= \lambda\mathbf{M}\mathbf{u} + \boldsymbol{\varepsilon}.\end{aligned}\tag{1}$$

In this model the variation in the $n \times 1$ dependent variable vector $\mathbf{y}$ is explained by the exogenous variables in the $n \times m$ matrix $\mathbf{X}$ and an $n \times 1$ spatial lag $\mathbf{W}\mathbf{y}$ (to be defined below). The $n \times 1$ error term $\mathbf{u}$ is spatially autocorrelated through the $n \times 1$ spatial lag $\mathbf{M}\mathbf{u}$ (to be defined below), and $\boldsymbol{\varepsilon}$ is an $n \times 1$ vector of independent and identically distributed (IID) error terms with mean 0 and variance σ_ε^2 . Here $\boldsymbol{\beta}$ is an $m \times 1$ vector of regression coefficients, and ρ and λ are scalar spatial autoregressive parameters. Two special cases are obtained if either $\rho = 0$ or $\lambda = 0$. If $\rho = 0$ the model is called a spatial error model, and if $\lambda = 0$ the model is referred to as a spatial lag model. The number of observations, or sample size, is n .

The structure of spatial dependence is formulated in the weights matrices $\mathbf{W}$ and $\mathbf{M}$. In practice it is often assumed that $\mathbf{W} = \mathbf{M}$. The weights matrix is then given by

$$\mathbf{W} = \begin{pmatrix} w_{11} & \cdots & w_{1n} \\ \vdots & \ddots & \vdots \\ w_{n1} & \cdots & w_{nn} \end{pmatrix},\tag{2}$$

which is an $n \times n$ matrix. In the sequel $\mathbf{W}$ will be specialised to a k -nearest neighbours weights matrix. In this type of weights matrix, every unit is assigned k nearest neighbours. Equal weights are given to all neighbours, and the weights matrix is row-normalised. Therefore every non-zero element of $\mathbf{W}$ is equal to $1/k$. The spatial lags $\mathbf{W}\mathbf{y}$ and $\mathbf{M}\mathbf{u}$ become $n \times 1$ vectors of weighted averages of neighbouring observations.

Suppose we desire to compare models with different weights matrices. Then a problem is that there are no formal tests for significant differences between the log likelihood values of the models. The reason is that spatial models with different weights matrices are non-nested (LeSage and Pace

2009, Section 6.2.2). Kelejian (2008) suggests a spatial J -test to test a spatial model against one or more non-nested alternative models. The test is carried out by generalised spatial two-stages least squares (2SLS) estimation (Kelejian and Prucha 1998) and generalised method of moments (GMM) estimation (Kelejian and Prucha 1999) of the null model, the alternative models and an augmented model containing the null model and the predictive power of the alternative models. In simulation studies (Piras and Lozano-Gracia 2011, and Burridge and Fingleton 2010) it is found that the spatial J -test can be used to discriminate between spatial models with different types of weights matrices, for instance contiguity, inverse-distance and k -nearest neighbours weights matrices.

In this paper the spatial J -test is used for specification search in spatial econometric models with k -nearest neighbours weights matrices. Our setup is different from that in previous papers because in our case, both the null and alternative models have k -nearest neighbours weights matrices, but the value of k is unknown. To our knowledge no research exists addressing the question of specifying k . We introduce and examine two approaches for finding the number k : an increasing neighbours testing procedure and a decreasing neighbours testing procedure. The properties of the spatial J -test are studied in the context of the testing procedures. The results show that the increasing neighbours testing procedure can be used in large samples to determine k . The decreasing neighbours testing procedure is found to have low power, and is not recommended for use in practice. An empirical example involving house price data from Stockholm, Sweden, is provided to show how to use the testing procedures with real data. The empirical illustration also contains a comparison with the Bayesian approach to model selection.

The paper is organised as follows. The next section describes the spatial J -test. The third section deals with specification search in k -nearest neighbours spatial econometric models, and introduces the testing procedures. The fourth section contains a simulation study and the fifth section demonstrates the use of the testing procedures with real data. The last section concludes. Appendix A contains the proof that the increasing neighbours testing procedure is consistent. Appendix B contains details about the implementation of a bootstrap spatial J -test in the simulations.

2 Spatial J -Test

The J -test, introduced in a seminal paper of Davidson and MacKinnon (1981), is a test for non-nested hypotheses. The main idea of the J -test is to estimate a model which consists of the null model and the predictive power of the alternative models. Then the significance of the additional term is tested.

Kelejian (2008) suggests a spatial J -test to test a spatial model against one or more non-nested alternative models. Following Kelejian, the spatial model of the null hypothesis is

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \rho\mathbf{W}\mathbf{y} + \mathbf{u} \quad (3)$$

$$= \mathbf{Z}\boldsymbol{\gamma} + \mathbf{u},$$

$$\mathbf{u} = \lambda\mathbf{M}\mathbf{u} + \boldsymbol{\varepsilon}, \quad (4)$$

where $\mathbf{Z} = (\mathbf{X}, \mathbf{W}\mathbf{y})$ and $\boldsymbol{\gamma} = (\boldsymbol{\beta}', \rho)'$. The vectors $\mathbf{y}, \mathbf{u}$ and $\boldsymbol{\varepsilon}$, the matrices $\mathbf{X}, \mathbf{W}$ and $\mathbf{M}$, the parameter vector $\boldsymbol{\beta}$, and the scalar parameters ρ and λ are as before. The sample size is n .

The alternative models are of the form

$$\mathbf{y} = \mathbf{X}_i\boldsymbol{\beta}_i + \rho_i\mathbf{W}_i\mathbf{y} + \mathbf{u}_i \quad (5)$$

$$= \mathbf{Z}_i\boldsymbol{\gamma}_i + \mathbf{u}_i,$$

$$\mathbf{u}_i = \lambda_i\mathbf{M}_i\mathbf{u}_i + \boldsymbol{\varepsilon}_i, \quad i = 1, \dots, G, \quad (6)$$

where $\mathbf{X}_i$ is an $n \times m_i$ matrix of exogenous variables in the i th alternative model, and so on, and G is a finite constant equal to the number of alternatives to the null model. Here we consider a special case mentioned by Kelejian, namely the case where $G = 1$, $\mathbf{X} = \mathbf{X}_1$, $\mathbf{W} = \mathbf{M}$, $\mathbf{W}_1 = \mathbf{M}_1$ but $\mathbf{W} \neq \mathbf{W}_1$, i.e. the test is employed to the specification of the weights matrix.

The J -test is described in the appendix of Kelejian, and our discussion is based on Kelejian. To derive the J -statistic, assuming $\mathbf{W} = \mathbf{M}$, first solve (4) to obtain $\mathbf{u} = (\mathbf{I}_n - \lambda\mathbf{W})^{-1}\boldsymbol{\varepsilon}$. Insert the solution into (3), and pre-multiply the expression by $(\mathbf{I}_n - \lambda\mathbf{W})$, to get

$$(\mathbf{I}_n - \lambda\mathbf{W})\mathbf{y} = (\mathbf{I}_n - \lambda\mathbf{W})\mathbf{Z}\boldsymbol{\gamma} + \boldsymbol{\varepsilon}.$$

Define $\mathbf{y}(\lambda) = (\mathbf{I}_n - \lambda\mathbf{W})\mathbf{y}$ and $\mathbf{Z}(\lambda) = (\mathbf{I}_n - \lambda\mathbf{W})\mathbf{Z}$, and write

$$\mathbf{y}(\lambda) = \mathbf{Z}(\lambda)\boldsymbol{\gamma} + \boldsymbol{\varepsilon}. \quad (7)$$

Likewise for the alternative model (5) and (6), define $\mathbf{Z}_1(\lambda_1) = (\mathbf{I}_n - \lambda_1 \mathbf{W}_1) \mathbf{Z}_1$. Let $\hat{\gamma}_1$ be an estimate of γ_1 in (5). Add the predictive power $\mathbf{Z}_1(\lambda_1) \hat{\gamma}_1$ of the alternative model to the null model to get:

$$\mathbf{y}(\lambda) = \mathbf{Z}(\lambda) \boldsymbol{\gamma} + \alpha_1 [\mathbf{Z}_1(\lambda_1) \hat{\gamma}_1] + \boldsymbol{\varepsilon}. \quad (8)$$

The parameter α_1 is a scalar. Given that the null model is true, $\alpha_1 = 0$. Insert $\mathbf{Z}_1(\lambda_1) = (\mathbf{I}_n - \lambda_1 \mathbf{W}_1) \mathbf{Z}_1$ into (8), and define $\phi_1 = -\alpha_1 \lambda_1$, to get the augmented model

$$\mathbf{y}(\lambda) = \mathbf{Z}(\lambda) \boldsymbol{\gamma} + \alpha_1 [\mathbf{Z}_1 \hat{\gamma}_1] + \phi_1 [\mathbf{W}_1 \mathbf{Z}_1 \hat{\gamma}_1] + \boldsymbol{\varepsilon}. \quad (9)$$

Let $\boldsymbol{\delta} = (\alpha_1, \phi_1)'$. The test of the null model against the alternative model is a test of $\boldsymbol{\delta} = \mathbf{0}$.

The estimation of the null, alternative and augmented models is done using the procedure suggested in Kelejian. The procedure is based on applying the generalised spatial two-stage least squares (2SLS) procedure of Kelejian and Prucha (1998), and generalised method of moments (GMM) procedure of Kelejian and Prucha (1999). See Kelejian for details.

The Wald test of $\boldsymbol{\delta} = \mathbf{0}$ against $\boldsymbol{\delta} \neq \mathbf{0}$ at the significance level α rejects the null model if

$$J = \hat{\boldsymbol{\delta}}' \hat{\mathbf{V}}_{\hat{\boldsymbol{\delta}}}^{-1} \hat{\boldsymbol{\delta}} > \chi_{1-a}^2(2), \quad (10)$$

where $\hat{\boldsymbol{\delta}}$ is an estimate of $\boldsymbol{\delta}$ and $\hat{\mathbf{V}}_{\hat{\boldsymbol{\delta}}}$ is an estimate of the covariance matrix of $\hat{\boldsymbol{\delta}}$.

We remark that in the augmented model (9) the parameter λ_1 in the alternative model is estimated as a free parameter. The degrees of freedom of the test are therefore 2 rather than 1.

We finally point out that Burrridge and Fingleton (2010) introduce a bootstrap version of the spatial J -test.

3 Specification Search in Spatial Econometric Models

In the spatial econometric literature testing alternative model specifications, or different structural forms, has received an excessive attention (LeSage and Pace 2009). For example, Florax et al. (2003) focus on discriminating between spatial lag models and spatial error models.

In this article a different route is taken. We take the structural form of the general spatial econometric model with k -nearest neighbours weights matrix $\mathbf{W}$ as given. The number k is usually unknown, and must be inferred from the data. We concentrate specification search on the value k in the weights matrix. LeSage and Pace (2009, p. 162) suggest to compare the log likelihood values of models with different weights matrices in an informal way. Because models with different values of k are generally not nested, it is not possible to use twice the difference between the values of the log likelihood functions as a formal test for statistically significant differences between the models. In the general spatial econometric model the number of parameters is fixed, and is the same in models with different values of k . Consequently, information criteria cannot be used.

In order to illustrate the use of the spatial J -test, assume that we want to compare two models which are non-nested, say $\mathcal{M}_1$ and $\mathcal{M}_2$. We may conduct two J -tests by testing $\mathcal{M}_1$ against $\mathcal{M}_2$, and $\mathcal{M}_2$ against $\mathcal{M}_1$. The four possible outcomes of the two tests are given by the following table:

	$H_0 : \mathcal{M}_2, H_1 : \mathcal{M}_1$	
$H_0 : \mathcal{M}_1, H_1 : \mathcal{M}_2$	Not rejected	Rejected
Not rejected	Both models acceptable	$\mathcal{M}_1$ acceptable, $\mathcal{M}_2$ not
Rejected	$\mathcal{M}_2$ acceptable, $\mathcal{M}_1$ not	Neither model acceptable

Generally speaking, rejection of a null model can mean two things. The alternative model has statistically significant predictive power, or neither of the models are acceptable. If the null model is not rejected, the reason is either that the null model is preferred over the alternative model, or that both models are acceptable.

We now turn to using the spatial J -test for specification search. We focus on k -nearest neighbours weights matrices when the sample size is large. The objective is to determine the number k . We introduce two approaches, which we call the increasing neighbours testing procedure and the decreasing neighbours testing procedure.

Suppose that we want to specify the number k of nearest neighbours in a general spatial econometric model estimated on a sample with n observations. One approach is to start with a model where the number of neighbours in the weights matrix is small, and test this model against models with more neighbours in the weights matrix, to see if there is statistically significant predictive power when the number of neighbours is gradually increased. This is

the increasing neighbours testing procedure. An alternative is a decreasing neighbours approach, in which we start with a null model where the number of neighbours in the weights matrix is large, and test this model against models with fewer neighbours in the weights matrix, to see if there is statistically significant predictive power when the number of neighbours is gradually decreased.

3.1 Increasing Neighbours Testing Procedure

In the increasing neighbours testing procedure we start with a null model $\mathcal{M}_p$, which has the smallest number p of k -nearest neighbours that we entertain. Test this model against the model $\mathcal{M}_{p+1}$ by a spatial J -test. Assume that the null model is rejected. The conclusion is that the alternative model has statistically significant predictive power, or that neither model is acceptable. We therefore add another neighbour to the weights matrix of the null and alternative models, and continue by testing the null model $\mathcal{M}_{p+1}$ against the alternative model $\mathcal{M}_{p+2}$. As long as the null model is rejected, the alternative model becomes the null model in the next step, and the testing procedure is continued. Assume that finally when the null model is $\mathcal{M}_{p+r}$, the J -test does not reject. The outcome of the test in the last step entails that a model with an additional neighbour, i.e. $p + r + 1$ neighbours, has no additional predictive power. The model $\mathcal{M}_{p+r}$ is therefore preferred over the model $\mathcal{M}_{p+r+1}$, or both $\mathcal{M}_{p+r}$ and $\mathcal{M}_{p+r+1}$ are acceptable. The model $\mathcal{M}_{p+r}$ is also preferred over the model $\mathcal{M}_{p+r-1}$, because in the previous step the test rejected $\mathcal{M}_{p+r-1}$ against $\mathcal{M}_{p+r}$. The outcome of the testing procedure is that the number of neighbours, or the value of k , is $k = p + r$.

An alternative would be to start by testing $\mathcal{M}_p$ against the union $\bigcup_{i=p+1}^q \mathcal{M}_i$, where q is the largest number of neighbours that we entertain. In practice this may lead to problems with multicollinearity in the estimation of the augmented model, because alternative ordered k -nearest neighbours spatial econometric models are close to each other.

In each test in the increasing neighbours testing procedure, the weights matrix of the null model is underspecified, and the alternative model which adds an additional neighbour to the weights matrix has predictive power, until the last test when the null model is the true model. Because the alternative model has predictive power, the spatial J -statistic $J_{p+r} \xrightarrow{P} \infty$, as $n \rightarrow \infty$. Therefore all false null models are rejected with probability 1 for n large enough. In the last test the size of the spatial J -test is equal to

the nominal significance level α for n large enough. The increasing neighbours testing procedure therefore gives a consistent way of determining k . In Appendix A the asymptotic properties of the increasing neighbours testing procedure are derived.

In practice the first value of k tested is $k = 1$. In finite samples it is possible to increase the power of the testing procedure by starting with a value of k larger than $k = 1$.

3.2 Decreasing Neighbours Testing Procedure

In the decreasing neighbours testing procedure we start with a null model $\mathcal{M}_q$, which has the largest number q of k -nearest neighbours that we entertain. Test this model against the model $\mathcal{M}_{q-1}$ by a spatial J -test. Assume that the null model is rejected. Continue by testing the null model $\mathcal{M}_{q-1}$ against the alternative model $\mathcal{M}_{q-2}$. As long as the null model is rejected, the alternative model becomes the null model in the next step, and the testing procedure is continued. Assume that finally when the null model is $\mathcal{M}_{q-s}$, the J -test does not reject. The outcome of the test in the last step entails that the model $\mathcal{M}_{q-s}$ is preferred over the model $\mathcal{M}_{q-s-1}$, or both $\mathcal{M}_{q-s}$ and $\mathcal{M}_{q-s-1}$ are acceptable. The model $\mathcal{M}_{q-s}$ is also preferred over the model $\mathcal{M}_{q-s+1}$, because in the previous step the test rejected $\mathcal{M}_{q-s+1}$ against $\mathcal{M}_{q-s}$. The outcome of the testing procedure is that the number of neighbours, or the value of k , is $k = q - s$.

In each test in the decreasing neighbours testing procedure, the weights matrix of the null model and the alternative model are overspecified, until the last test when the alternative model is the true model. But there is no guarantee that the alternative model which relaxes a false neighbour has predictive power. Thus we cannot prove that the decreasing neighbours testing procedure is consistent. In fact, simulations reported in Section 4.2 indicate that the decreasing neighbours testing procedure is not consistent.

4 Simulation Study

This section contains a simulation study of the spatial J -test and bootstrap spatial J -test when they are used to distinguish between spatial models with different k -nearest neighbours weights matrices. We compare the performances of the increasing and decreasing neighbours approaches to specifica-

tion search. The focus is on the rejection probabilities of the spatial J -test and bootstrap spatial J -test in the sequences of tests conducted in the testing procedures.

In small samples the spatial J -test is found to be oversized. We therefore consider a bootstrap version of the spatial J -test (Burridge and Fingleton 2010). Details about the implementation of the bootstrap spatial J -test are provided in Appendix B.

4.1 Simulation Design

The simulation design follows suggestions in Pace et al. (2011). The level of spatial dependence, the regressors and the signal-to-noise ratio are chosen to be typical of the values found in spatial economic data, especially house price data used in the empirical illustration in Section 5. For example, in Gerkman (2011) using house price data from Helsinki, Finland, the estimate of ρ is 0.547 and the value of R^2 is 0.799.

Data are generated from the following data-generation process (DGP):

$$\mathbf{y} = (\mathbf{I}_n - \rho \mathbf{W})^{-1} \mathbf{X} \boldsymbol{\beta} + (\mathbf{I}_n - \rho \mathbf{W})^{-1} (\mathbf{I}_n - \lambda \mathbf{W})^{-1} \boldsymbol{\varepsilon}, \quad \boldsymbol{\varepsilon} \sim N(\mathbf{0}, \sigma_\varepsilon^2 \mathbf{I}_n). \quad (11)$$

This DGP is the general spatial econometric model in (3) and (4). The spatial characteristics of the dependent variable $\mathbf{y}$ are governed by the spatial weights matrix $\mathbf{W}$, which is an $n \times n$ matrix, and the spatial autoregressive parameters ρ and λ . For simplicity we let the weights matrix in the spatial lag and the errors be the same, i.e. $\mathbf{W} = \mathbf{M}$. The weights matrix $\mathbf{W}$ is of k -nearest neighbours type. When observation i is a neighbour to observation j , $w_{ij} = 1/k$, otherwise $w_{ij} = 0$. The diagonal elements are $w_{ii} = 0$. The weights matrix $\mathbf{W}$ is row-normalised.

The null model in the DGP has $k = 8$ nearest neighbours. The values for the spatial autoregressive parameters ρ and λ are 0.2, 0.5 and 0.8. The values of ρ and λ are chosen in order to represent small, medium and large positive spatial autocorrelation.

The matrix of regressors $\mathbf{X}$ is an $n \times m$ matrix. The columns of $\mathbf{X}$, or the regressors $\mathbf{x}_i$, are generated as $\mathbf{x}_i \sim N(\mathbf{0}, \mathbf{I}_n)$, $i = 1, \dots, m$. The number of regressors is $m = 5$. The regression coefficients are set to 1. The vector $\boldsymbol{\beta}$ is therefore an $m \times 1$ vector of ones.

Following Pace et al. (2011), we define the signal-to-noise ratio, denoted

Table 1: Signal-to-noise ratio R^2 in the DGP with $k = 8$ nearest neighbours for different values of ρ and λ in the simulations. The value of R^2 is set to 0.5 when $\rho = \lambda = 0$. The values of R^2 are simulated conditional on the explanatory variables for $n = 1000$.

ρ	0.2	0.2	0.2	0.5	0.5	0.5	0.8	0.8	0.8
λ	0.2	0.5	0.8	0.2	0.5	0.8	0.2	0.5	0.8
R^2	0.679	0.699	0.776	0.722	0.757	0.855	0.849	0.887	0.955

by R^2 , as follows:

$$R^2 = 1 - \sigma_\varepsilon^2 \frac{\text{tr}((\mathbf{I}_n - \lambda \mathbf{W})^{-1}(\mathbf{I}_n - \rho \mathbf{W})^{-2}(\mathbf{I}_n - \lambda \mathbf{W})^{-1}))}{\boldsymbol{\beta}' \mathbf{X}' (\mathbf{I}_n - \rho \mathbf{W})^{-2} \mathbf{X} \boldsymbol{\beta}}. \quad (12)$$

In the simulation experiments the value of σ_ε^2 is chosen so that R^2 equals a given signal-to-noise level. The value of R^2 depends in addition on ρ and λ . We set R^2 equal to 0.1, 0.5 and 0.9 in the model with $\rho = \lambda = 0$, which are the values recommended by Pace et al. (2011). The value 0.1 represents low, the value 0.5 moderate and the value 0.9 high signal-to-noise level.

Table 1 reports the signal to noise ratio R^2 for the different values of ρ and λ in the simulations, conditional on the explanatory variables for $n = 1000$. Note that when $\rho = \lambda = 0$, R^2 measures the signal from $\mathbf{X}$, or the regressor; when either $\rho \neq 0$ or $\lambda \neq 0$, R^2 includes the signal from the spatial dependence in $\mathbf{y}$ and $\mathbf{u}$. To save space, we only report the results for $R^2 = 0.5$ (additional results for the values $R^2 = 0.1$ and 0.9 are available upon request).

In the sequence of tests conducted in the increasing neighbours testing procedure $k = 1$ to 8, and in the decreasing neighbours testing procedure $k = 15$ to 8. The value $k = 8$ in the null model gives the size of the test, and the other values of k give the power of the test.

The sample size is $n = 100, 200, 500, 1000$ and 2500. The sample sizes $n = 100, 200$ and 500 are regarded as small. The sample size $n = 1000$ is no longer considered a small sample size, and the sample size $n = 2500$ is considered large (Pace et al. 2011).

The number of replications is 1000. Integers are randomly drawn from a uniform $[1, 10000]^2$ distribution to represent coordinates of the observations on the dependent variable $\mathbf{y}$. The weights matrices are constructed from the

coordinates in the plane. For each observation k nearest neighbours are found based on Euclidian distances. The coordinates and the vector of errors ε are drawn 1000 times, but the matrix of regressors $\mathbf{X}$ is kept constant across replications.

The simulations are done in Matlab, and routines of the Spatial Econometric Toolbox (version 2010) of LeSage are used in the estimations. The spatial J -test and bootstrap spatial J -test are programmed in Matlab following Kelejian (2008), and use estimation by generalised spatial two-stage least squares (2SLS) and generalised method of moments (GMM).

4.2 Results

The rejection probabilities of the spatial J -test and the bootstrap spatial J -test are reported in Table 3–7. The notation J^* is used with the bootstrap spatial J -test. The nominal significance level is 5%.

Numerical problems were encountered in estimating the augmented model (9) with the null model $\mathcal{M}_3$ and the predictive power of model $\mathcal{M}_4$. The models $\mathcal{M}_3$ and $\mathcal{M}_4$ are close to each other, as shown in Table 2, which reports the simulated log likelihood values of the models for $n = 1000$. The simulated rejection probabilities of the spatial J -test of $\mathcal{M}_3$ against $\mathcal{M}_4$ are not reliable, and are therefore not reported (indicated by a '.' in Tables 3–7). The numerical problems were exacerbated in the case of the bootstrap spatial J -test.

For the increasing neighbours testing procedure the results are reported for all combinations of ρ and λ , whereas for the decreasing neighbours testing procedure the results are reported only for selected values of ρ and λ , in order to save space. The last column of the tables shows the rejection probabilities of model $\mathcal{M}_8$ against $\mathcal{M}_9$ in the increasing neighbours testing procedure, and against model $\mathcal{M}_7$ in the decreasing neighbours testing procedure, and gives size of the tests. The remaining entries in the tables are estimates of power. The results show that the decreasing neighbours testing procedure has low power for all sample sizes. The low rejection probabilities for large values of n indicate that the decreasing neighbours testing procedure is not consistent. We therefore do not further comment on the results of the decreasing neighbours testing procedure.

Table 3 shows the result for the smallest sample size $n = 100$. If we define the size of the spatial J -test as the maximum rejection probability over the parameter space of the null model in the DGP, then the size of the

Table 2: Simulated values of the log likelihood function for $n = 1000$.

Model	Log lik	Δ Log lik
$\mathcal{M}_1$	-1756.83	
$\mathcal{M}_2$	-1702.55	54.281
$\mathcal{M}_3$	-1638.24	64.310
$\mathcal{M}_4$	-1638.26	0.017
$\mathcal{M}_5$	-1615.15	22.708
$\mathcal{M}_6$	-1596.37	19.176
$\mathcal{M}_7$	-1579.12	17.257
$\mathcal{M}_8$	-1564.70	14.416
$\mathcal{M}_9$	-1576.97	-12.682
$\mathcal{M}_{10}$	-1588.60	-11.634
$\mathcal{M}_{11}$	-1599.78	-11.181
$\mathcal{M}_{12}$	-1610.51	-10.723
$\mathcal{M}_{13}$	-1620.72	-10.208
$\mathcal{M}_{14}$	-1630.40	-9.684
$\mathcal{M}_{15}$	-1639.52	-9.123

test is at least 21%. The rejection probabilities in the last column of Table 3 vary between 9.4% for $\rho = 0.8$ and $\lambda = 0.2$, and 21.4% for $\rho = 0.2$ and $\lambda = 0.8$. The bootstrap corrects the size of the test to 8.6%. The power of the spatial J -test is low for all values of ρ and λ . The bootstrap test reduces the rejection probabilities further. For example, when $\rho = 0.5$ and $\lambda = 0.5$, and model $\mathcal{M}_4$ is tested against model $\mathcal{M}_5$, the rejection probability of the J -test is 30% and the rejection probability of the bootstrap J -test is 23%. The rejection probabilities increase with ρ and λ , which is what we would expect.

Table 4 shows the result for $n = 200$. The size of the spatial J -test is at least 30%, and the size of the bootstrap spatial J -test is 9.7%. In the increasing neighbours testing procedure the power of the test increases with the sample size, although the power of the individual J -tests is low. For example, when $\rho = 0.5$ and $\lambda = 0.5$, and model $\mathcal{M}_4$ is tested against model $\mathcal{M}_5$, the rejection probability of the J -test is 42% and the rejection probability of the bootstrap J -test is 34%.

Table 5 shows the result for $n = 500$. The size of the spatial J -test is at least 21%, and the size of the bootstrap spatial J -test is 8.6%. The power

Table 3: Rejection probabilities of the spatial J -test and bootstrap spatial J -test when $n = 100$.

ρ, λ	Increasing neighbours testing procedure								
	H_0	$\mathcal{M}_1$	$\mathcal{M}_2$	$\mathcal{M}_3$	$\mathcal{M}_4$	$\mathcal{M}_5$	$\mathcal{M}_6$	$\mathcal{M}_7$	$\mathcal{M}_8$
	H_1	$\mathcal{M}_2$	$\mathcal{M}_3$	$\mathcal{M}_4$	$\mathcal{M}_5$	$\mathcal{M}_6$	$\mathcal{M}_7$	$\mathcal{M}_8$	$\mathcal{M}_9$
0.2, 0.2	J	0.104	0.134	.	0.113	0.126	0.108	0.121	0.095
	J^*	0.077	0.120	.	0.083	0.086	0.080	0.082	0.069
0.2, 0.5	J	0.153	0.188	.	0.185	0.176	0.196	0.200	0.164
	J^*	0.101	0.143	.	0.133	0.110	0.119	0.108	0.086
0.2, 0.8	J	0.223	0.355	.	0.338	0.328	0.323	0.320	0.214
	J^*	0.147	0.241	.	0.207	0.198	0.200	0.179	0.077
0.5, 0.2	J	0.194	0.315	.	0.214	0.202	0.165	0.138	0.105
	J^*	0.144	0.258	.	0.178	0.149	0.114	0.085	0.053
0.5, 0.5	J	0.284	0.368	.	0.295	0.253	0.214	0.201	0.130
	J^*	0.199	0.313	.	0.234	0.186	0.115	0.099	0.061
0.5, 0.8	J	0.333	0.438	.	0.415	0.380	0.346	0.293	0.155
	J^*	0.217	0.323	.	0.277	0.246	0.195	0.140	0.049
0.8, 0.2	J	0.366	0.526	.	0.408	0.295	0.222	0.170	0.094
	J^*	0.254	0.458	.	0.300	0.244	0.167	0.081	0.042
0.8, 0.5	J	0.365	0.500	.	0.403	0.338	0.239	0.200	0.116
	J^*	0.267	0.424	.	0.264	0.209	0.125	0.076	0.029
0.8, 0.8	J	0.467	0.568	.	0.527	0.460	0.339	0.243	0.113
	J^*	0.342	0.494	.	0.395	0.327	0.205	0.110	0.047

ρ, λ	Decreasing neighbours testing procedure								
	H_0	$\mathcal{M}_{15}$	$\mathcal{M}_{14}$	$\mathcal{M}_{13}$	$\mathcal{M}_{12}$	$\mathcal{M}_{11}$	$\mathcal{M}_{10}$	$\mathcal{M}_9$	$\mathcal{M}_8$
	H_1	$\mathcal{M}_{14}$	$\mathcal{M}_{13}$	$\mathcal{M}_{12}$	$\mathcal{M}_{11}$	$\mathcal{M}_{10}$	$\mathcal{M}_9$	$\mathcal{M}_8$	$\mathcal{M}_7$
0.2, 0.5	J	0.197	0.209	0.210	0.182	0.176	0.175	0.175	0.147
	J^*	0.118	0.120	0.103	0.095	0.106	0.098	0.085	0.079
0.5, 0.5	J	0.169	0.169	0.188	0.194	0.189	0.149	0.186	0.110
	J^*	0.085	0.084	0.095	0.089	0.095	0.069	0.088	0.048
0.8, 0.5	J	0.174	0.177	0.170	0.174	0.156	0.189	0.175	0.098
	J^*	0.110	0.097	0.091	0.092	0.083	0.080	0.066	0.032

Table 4: Rejection probabilities of the spatial J -test and bootstrap spatial J -test when $n = 200$.

ρ, λ	Increasing neighbours testing procedure								
	H_0	$\mathcal{M}_1$	$\mathcal{M}_2$	$\mathcal{M}_3$	$\mathcal{M}_4$	$\mathcal{M}_5$	$\mathcal{M}_6$	$\mathcal{M}_7$	$\mathcal{M}_8$
	H_1	$\mathcal{M}_2$	$\mathcal{M}_3$	$\mathcal{M}_4$	$\mathcal{M}_5$	$\mathcal{M}_6$	$\mathcal{M}_7$	$\mathcal{M}_8$	$\mathcal{M}_9$
0.2, 0.2	J	0.112	0.130	.	0.114	0.135	0.112	0.109	0.089
	J^*	0.084	0.097	.	0.102	0.111	0.090	0.076	0.058
0.2, 0.5	J	0.133	0.191	.	0.187	0.206	0.173	0.179	0.157
	J^*	0.110	0.166	.	0.125	0.126	0.097	0.099	0.065
0.2, 0.8	J	0.229	0.328	.	0.396	0.396	0.402	0.403	0.303
	J^*	0.154	0.242	.	0.266	0.261	0.211	0.187	0.097
0.5, 0.2	J	0.241	0.381	.	0.339	0.279	0.211	0.151	0.096
	J^*	0.206	0.332	.	0.290	0.246	0.150	0.107	0.041
0.5, 0.5	J	0.277	0.401	.	0.418	0.399	0.295	0.224	0.176
	J^*	0.218	0.378	.	0.336	0.241	0.146	0.072	0.059
0.5, 0.8	J	0.324	0.496	.	0.554	0.508	0.442	0.378	0.197
	J^*	0.203	0.374	.	0.397	0.302	0.209	0.146	0.051
0.8, 0.2	J	0.445	0.697	.	0.701	0.574	0.352	0.192	0.086
	J^*	0.392	0.605	.	0.650	0.484	0.280	0.123	0.036
0.8, 0.5	J	0.437	0.641	.	0.643	0.549	0.387	0.221	0.109
	J^*	0.374	0.554	.	0.482	0.364	0.178	0.075	0.018
0.8, 0.8	J	0.484	0.661	.	0.671	0.616	0.484	0.315	0.102
	J^*	0.346	0.450	.	0.454	0.380	0.280	0.139	0.020

ρ, λ	Decreasing neighbours testing procedure								
	H_0	$\mathcal{M}_{15}$	$\mathcal{M}_{14}$	$\mathcal{M}_{13}$	$\mathcal{M}_{12}$	$\mathcal{M}_{11}$	$\mathcal{M}_{10}$	$\mathcal{M}_9$	$\mathcal{M}_8$
	H_1	$\mathcal{M}_{14}$	$\mathcal{M}_{13}$	$\mathcal{M}_{12}$	$\mathcal{M}_{11}$	$\mathcal{M}_{10}$	$\mathcal{M}_9$	$\mathcal{M}_8$	$\mathcal{M}_7$
0.2, 0.5	J	0.190	0.190	0.173	0.169	0.154	0.148	0.138	0.106
	J^*	0.096	0.101	0.075	0.080	0.081	0.073	0.062	0.064
0.5, 0.5	J	0.219	0.209	0.188	0.187	0.181	0.191	0.180	0.137
	J^*	0.103	0.083	0.068	0.083	0.081	0.063	0.065	0.057
0.8, 0.5	J	0.162	0.167	0.158	0.161	0.158	0.141	0.140	0.087
	J^*	0.0850	0.067	0.057	0.070	0.053	0.049	0.027	0.020

of the spatial J -test and bootstrap spatial J -test is higher when $n = 500$ compared to $n = 100, 200$. For example, when $\rho = 0.5$ and $\lambda = 0.5$, and model $\mathcal{M}_4$ is tested against model $\mathcal{M}_5$, the rejection probability of the J -test is 58% and the rejection probability of the bootstrap J -test is 49%.

Table 6 shows the results for $n = 1000$, which is no longer considered a small sample size. The size of the spatial J -test is at least 12%, and the size of the bootstrap spatial J -test is 7.2%. When ρ and λ are intermediate or large, we find that the power of the test is high in comparison to the small sample sizes. For example, when $\rho = 0.5$ and $\lambda = 0.5$, and model $\mathcal{M}_4$ is tested against model $\mathcal{M}_5$, the rejection probability of the J -test is 74% and the rejection probability of the bootstrap J -test is 73%.

Table 7 finally shows the results for the large sample size $n = 2500$. The rejection probabilities of the spatial J -test in the last column of the table are close to the nominal level 5%, except at the boundary of the parameter space. The rejection probabilities of the bootstrap spatial J -test are therefore not reported. For intermediate and large values of ρ and λ , the power of the tests approach one. For example, when $\rho = 0.5$ and $\lambda = 0.5$, and model $\mathcal{M}_4$ is tested against model $\mathcal{M}_5$, the rejection probability of the J -test is 97%.

The simulation results in Table 3–7 provide clear evidence of the following main features of the testing procedures:

1. The increasing neighbours testing procedure can be used in large samples, say $n = 1000$ and $n = 2500$, to determine k . The decreasing neighbours testing procedure has low power, and is not recommended for use in practice.
2. The performance of the increasing neighbours testing procedure depends on the strength of the spatial dependence in the data. The increasing neighbours testing procedure has high power in large samples when ρ (and λ) are intermediate or large. For small values of ρ (and λ), the spatial signal in the data is insufficient to determine k .
3. The bootstrap is useful for correcting the size of the spatial J -test in small samples, but does not solve the problem with low power of the test in small samples.

Table 5: Rejection probabilities of the spatial J -test and bootstrap spatial J -test when $n = 500$.

ρ, λ	Increasing neighbours testing procedure								
	H_0	$\mathcal{M}_1$	$\mathcal{M}_2$	$\mathcal{M}_3$	$\mathcal{M}_4$	$\mathcal{M}_5$	$\mathcal{M}_6$	$\mathcal{M}_7$	$\mathcal{M}_8$
	H_1	$\mathcal{M}_2$	$\mathcal{M}_3$	$\mathcal{M}_4$	$\mathcal{M}_5$	$\mathcal{M}_6$	$\mathcal{M}_7$	$\mathcal{M}_8$	$\mathcal{M}_9$
0.2, 0.2	J	0.115	0.138	.	0.155	0.137	0.118	0.083	0.062
	J^*	0.104	0.137	.	0.116	0.129	0.079	0.058	0.052
0.2, 0.5	J	0.141	0.186	.	0.209	0.190	0.169	0.118	0.087
	J^*	0.127	0.162	.	0.152	0.138	0.128	0.076	0.063
0.2, 0.8	J	0.193	0.265	.	0.314	0.310	0.282	0.283	0.207
	J^*	0.153	0.247	.	0.219	0.202	0.155	0.163	0.086
0.5, 0.2	J	0.293	0.542	.	0.571	0.440	0.294	0.147	0.065
	J^*	0.293	0.507	.	0.509	0.419	0.262	0.106	0.041
0.5, 0.5	J	0.309	0.538	.	0.579	0.465	0.333	0.194	0.093
	J^*	0.250	0.480	.	0.491	0.362	0.247	0.116	0.054
0.5, 0.8	J	0.325	0.533	.	0.573	0.545	0.444	0.329	0.184
	J^*	0.278	0.489	.	0.459	0.402	0.314	0.170	0.059
0.8, 0.2	J	0.679	0.934	.	0.951	0.891	0.658	0.267	0.075
	J^*	0.675	0.930	.	0.935	0.840	0.599	0.214	0.052
0.8, 0.5	J	0.621	0.841	.	0.888	0.806	0.586	0.291	0.089
	J^*	0.559	0.823	.	0.843	0.722	0.454	0.149	0.024
0.8, 0.8	J	0.554	0.758	.	0.797	0.721	0.559	0.296	0.074
	J^*	0.371	0.640	.	0.513	0.365	0.179	0.094	0.006

ρ, λ	Decreasing neighbours testing procedure								
	H_0	$\mathcal{M}_{15}$	$\mathcal{M}_{14}$	$\mathcal{M}_{13}$	$\mathcal{M}_{12}$	$\mathcal{M}_{11}$	$\mathcal{M}_{10}$	$\mathcal{M}_9$	$\mathcal{M}_8$
	H_1	$\mathcal{M}_{14}$	$\mathcal{M}_{13}$	$\mathcal{M}_{12}$	$\mathcal{M}_{11}$	$\mathcal{M}_{10}$	$\mathcal{M}_9$	$\mathcal{M}_8$	$\mathcal{M}_7$
0.2, 0.5		0.134	0.126	0.132	0.128	0.116	0.117	0.082	0.074
		0.063	0.064	0.049	0.063	0.055	0.083	0.058	0.053
0.5, 0.5		0.200	0.178	0.176	0.155	0.152	0.137	0.112	0.099
		0.081	0.064	0.075	0.061	0.064	0.074	0.062	0.058
0.8, 0.5		0.185	0.174	0.169	0.168	0.164	0.137	0.117	0.079
		0.048	0.056	0.049	0.042	0.055	0.048	0.050	0.035

Table 6: Rejection probabilities of the spatial J -test and bootstrap spatial J -test when $n = 1000$.

ρ, λ	Increasing neighbours testing procedure								
	H_0	$\mathcal{M}_1$	$\mathcal{M}_2$	$\mathcal{M}_3$	$\mathcal{M}_4$	$\mathcal{M}_5$	$\mathcal{M}_6$	$\mathcal{M}_7$	$\mathcal{M}_8$
	H_1	$\mathcal{M}_2$	$\mathcal{M}_3$	$\mathcal{M}_4$	$\mathcal{M}_5$	$\mathcal{M}_6$	$\mathcal{M}_7$	$\mathcal{M}_8$	$\mathcal{M}_9$
0.2, 0.2	J	0.113	0.209	.	0.216	0.165	0.116	0.087	0.059
	J^*	0.128	0.206	.	0.223	0.145	0.098	0.076	0.043
0.2, 0.5	J	0.131	0.244	.	0.246	0.184	0.146	0.107	0.075
	J^*	0.136	0.220	.	0.253	0.166	0.108	0.081	0.044
0.2, 0.8	J	0.171	0.254	.	0.285	0.231	0.185	0.145	0.117
	J^*	0.142	0.231	.	0.222	0.193	0.139	0.112	0.072
0.5, 0.2	J	0.419	0.758	.	0.825	0.691	0.415	0.174	0.059
	J^*	0.409	0.754	.	0.830	0.626	0.393	0.154	0.041
0.5, 0.5	J	0.416	0.702	.	0.743	0.624	0.406	0.215	0.074
	J^*	0.428	0.689	.	0.729	0.574	0.347	0.163	0.040
0.5, 0.8	J	0.401	0.622	.	0.671	0.573	0.436	0.256	0.104
	J^*	0.327	0.570	.	0.603	0.497	0.327	0.144	0.031
0.8, 0.2	J	0.877	0.986	.	1.000	0.989	0.887	0.432	0.076
	J^*	0.852	0.986	.	1.000	0.985	0.862	0.361	0.040
0.8, 0.5	J	0.835	0.974	.	0.981	0.971	0.823	0.414	0.081
	J^*	0.800	0.971	.	0.978	0.939	0.753	0.336	0.048
0.8, 0.8	J	0.671	0.867	.	0.888	0.835	0.638	0.298	0.055
	J^*	0.546	0.774	.	0.785	0.618	0.310	0.069	0.002

ρ, λ	Decreasing neighbours testing procedure								
	H_0	$\mathcal{M}_{15}$	$\mathcal{M}_{14}$	$\mathcal{M}_{13}$	$\mathcal{M}_{12}$	$\mathcal{M}_{11}$	$\mathcal{M}_{10}$	$\mathcal{M}_9$	$\mathcal{M}_8$
	H_1	$\mathcal{M}_{14}$	$\mathcal{M}_{13}$	$\mathcal{M}_{12}$	$\mathcal{M}_{11}$	$\mathcal{M}_{10}$	$\mathcal{M}_9$	$\mathcal{M}_8$	$\mathcal{M}_7$
0.2, 0.5	J	0.119	0.133	0.126	0.099	0.103	0.099	0.089	0.064
	J^*	0.085	0.073	0.072	0.071	0.056	0.068	0.065	0.048
0.5, 0.5	J	0.199	0.203	0.187	0.171	0.157	0.135	0.107	0.070
	J^*	0.126	0.124	0.124	0.116	0.099	0.079	0.079	0.046
0.8, 0.5	J	0.228	0.202	0.195	0.178	0.157	0.142	0.093	0.071
	J^*	0.083	0.086	0.110	0.100	0.076	0.079	0.072	0.037

Table 7: Rejection probabilities of the spatial J -test when $n = 2500$.

ρ, λ	Increasing neighbours testing procedure								
	H_0	$\mathcal{M}_1$	$\mathcal{M}_2$	$\mathcal{M}_3$	$\mathcal{M}_4$	$\mathcal{M}_5$	$\mathcal{M}_6$	$\mathcal{M}_7$	$\mathcal{M}_8$
	H_1	$\mathcal{M}_2$	$\mathcal{M}_3$	$\mathcal{M}_4$	$\mathcal{M}_5$	$\mathcal{M}_6$	$\mathcal{M}_7$	$\mathcal{M}_8$	$\mathcal{M}_9$
0.2, 0.2	J	0.150	0.338	.	0.380	0.285	0.180	0.080	0.043
0.2, 0.5	J	0.179	0.333	.	0.364	0.277	0.183	0.091	0.053
0.2, 0.8	J	0.197	0.294	.	0.319	0.268	0.201	0.132	0.084
0.5, 0.2	J	0.703	0.988	.	0.994	0.966	0.750	0.292	0.042
0.5, 0.5	J	0.684	0.934	.	0.973	0.905	0.665	0.267	0.051
0.5, 0.8	J	0.570	0.816	.	0.859	0.768	0.553	0.263	0.065
0.8, 0.2	J	0.994	1.000	.	1.000	1.000	0.997	0.749	0.041
0.8, 0.5	J	0.987	1.000	.	1.000	0.999	0.992	0.653	0.050
0.8, 0.8	J	0.898	0.984	.	0.993	0.988	0.927	0.543	0.058
ρ, λ	Decreasing neighbours testing procedure								
	H_0	$\mathcal{M}_{15}$	$\mathcal{M}_{14}$	$\mathcal{M}_{13}$	$\mathcal{M}_{12}$	$\mathcal{M}_{11}$	$\mathcal{M}_{10}$	$\mathcal{M}_9$	$\mathcal{M}_8$
	H_1	$\mathcal{M}_{14}$	$\mathcal{M}_{13}$	$\mathcal{M}_{12}$	$\mathcal{M}_{11}$	$\mathcal{M}_{10}$	$\mathcal{M}_9$	$\mathcal{M}_8$	$\mathcal{M}_7$
0.2, 0.5	J	0.124	0.119	0.092	0.099	0.084	0.075	0.081	0.057
0.5, 0.5	J	0.314	0.282	0.268	0.248	0.183	0.139	0.102	0.058
0.8, 0.5	J	0.340	0.331	0.325	0.305	0.267	0.195	0.108	0.048

5 Empirical Illustration

In spatial econometric models of house prices, k -nearest neighbours weights matrices are commonly used. Assume that a buyer has chosen a certain neighbourhood where he or she intends to purchase a house. The buyer will then compare houses that are for sale in that area. The price he or she is willing to pay for a particular house is affected by the price level in the neighbourhood. Also the seller's expectation of the price is affected by the selling prices of other houses in the neighbourhood. The seller will be satisfied with a price that is at least as good as the average in the neighbourhood at the moment of trade. The average price in the neighbourhood, on the other hand, is affected by the attributes of the neighbourhood, like available services, safety, or simply good reputation. The selling prices of houses will therefore be spatially dependent.

In this section we illustrate the use of the testing procedures for determining the number k of nearest neighbours in house price data. The data consist of 1377 transactions of single-family houses between January 2000 and May 2001 in the county of Stockholm, Sweden. The data were analysed by Wilhelmsson (2002), and include the selling price, spatial coordinates and, in addition, information about the size of the house in square metres, as well as other characteristics. See Wilhelmsson for a detailed description of the data.

We assume the general spatial econometric model used in the simulations, and demonstrate how the testing strategies are applied in practice. The results in Wilhelmsson are in favour of a spatial error model. Therefore we also apply specification search to the spatial error model. Because the sample size is large, the bootstrap spatial J -test is not used in the empirical illustration. In addition, the outcome of the specification search is compared with the results of a Bayesian model comparison. For the Bayesian model comparison routines of the Spatial Econometric Toolbox (version 2010) of LeSage are used.

We begin by applying the increasing neighbours testing procedure, and start with $k = 1$. First the model $\mathcal{M}_1$ is tested against $\mathcal{M}_2$, and we find that it is rejected. In the next step $\mathcal{M}_2$ is tested against $\mathcal{M}_3$, and so on. The procedure is continued until the first non-rejected null model. Table 8 shows that this happens when $\mathcal{M}_{16}$ is tested against $\mathcal{M}_{17}$. The value of the J -statistic is $J_{16} = 4.27$, which is smaller than the critical value of 5.99 in a $\chi^2(2)$ distribution. The outcome of the increasing neighbours testing

Table 8: Spatial J -statistics in the general spatial econometric model for the Swedish house price data.

Increasing neighbours testing procedure								
H_0	$\mathcal{M}_1$	$\mathcal{M}_2$	$\mathcal{M}_3$	$\mathcal{M}_4$	$\mathcal{M}_5$	$\mathcal{M}_6$	$\mathcal{M}_7$	$\mathcal{M}_8$
H_1	$\mathcal{M}_2$	$\mathcal{M}_3$	$\mathcal{M}_4$	$\mathcal{M}_5$	$\mathcal{M}_6$	$\mathcal{M}_7$	$\mathcal{M}_8$	$\mathcal{M}_9$
J	18.30	35.27	46.44	53.42	71.26	78.29	78.83	85.65
H_0	$\mathcal{M}_9$	$\mathcal{M}_{10}$	$\mathcal{M}_{11}$	$\mathcal{M}_{12}$	$\mathcal{M}_{13}$	$\mathcal{M}_{14}$	$\mathcal{M}_{15}$	$\mathcal{M}_{16}$
H_1	$\mathcal{M}_{10}$	$\mathcal{M}_{11}$	$\mathcal{M}_{12}$	$\mathcal{M}_{13}$	$\mathcal{M}_{14}$	$\mathcal{M}_{15}$	$\mathcal{M}_{16}$	$\mathcal{M}_{17}$
J	76.21	82.18	84.80	87.22	70.26	54.81	30.07	4.27

Decreasing neighbours testing procedure				
H_0	$\mathcal{M}_{24}$	$\mathcal{M}_{23}$	$\mathcal{M}_{22}$	$\mathcal{M}_{21}$
H_1	$\mathcal{M}_{23}$	$\mathcal{M}_{22}$	$\mathcal{M}_{21}$	$\mathcal{M}_{20}$
J	7.85	14.82	12.64	4.35

procedure is $k = 16$. When the decreasing neighbours testing procedure is applied, we choose a large value of k , in this case $k = 24$. First the model $\mathcal{M}_{24}$ is tested against $\mathcal{M}_{23}$, and we find that it is rejected. In the next step $\mathcal{M}_{23}$ is tested against $\mathcal{M}_{22}$, and so on. The procedure is continued until the first non-rejected null model. Table 8 shows that this happens when $\mathcal{M}_{21}$ is tested against $\mathcal{M}_{20}$. The value of the J -statistic is $J_{21} = 4.35$, which is smaller than the critical value of 5.99. The outcome of the decreasing neighbours testing procedure is $k = 21$. The outcomes of the testing procedures are not the same. Table 8 shows that all J -statistics in the increasing neighbours testing procedure are large, until the last step when it is small. In comparison, when the decreasing neighbours testing procedure is applied, all J -statistics are small compared to the J -statistics in the increasing neighbours testing procedure. Based on the simulation results in Section 4.2, which show that the increasing neighbours testing procedure is more powerful, we select a weights matrix that has $k = 16$ nearest neighbours.

In Wilhelmsson the general spatial econometric model is not among the models considered. His results are in favour of a spatial error model

$$\begin{aligned} \mathbf{y} &= \mathbf{X}\boldsymbol{\beta} + \mathbf{u}, \\ \mathbf{u} &= \rho \mathbf{W}\mathbf{u} + \boldsymbol{\varepsilon}. \end{aligned} \tag{13}$$

In this model a k -nearest neighbours weights matrix with $k = 4$ is used.

Table 9: Spatial J -tstatistics in the spatial error model for the Swedish house price data.

Increasing neighbours testing procedure								
H_0	$\mathcal{M}_1$	$\mathcal{M}_2$	$\mathcal{M}_3$	$\mathcal{M}_4$	$\mathcal{M}_5$	$\mathcal{M}_6$	$\mathcal{M}_7$	$\mathcal{M}_8$
H_1	$\mathcal{M}_2$	$\mathcal{M}_3$	$\mathcal{M}_4$	$\mathcal{M}_5$	$\mathcal{M}_6$	$\mathcal{M}_7$	$\mathcal{M}_8$	$\mathcal{M}_9$
J	97.38	101.34	93.65	87.09	80.99	76.68	72.31	72.76
H_0	$\mathcal{M}_9$	$\mathcal{M}_{10}$	$\mathcal{M}_{11}$	$\mathcal{M}_{12}$	$\mathcal{M}_{13}$	$\mathcal{M}_{14}$		
H_1	$\mathcal{M}_{10}$	$\mathcal{M}_{11}$	$\mathcal{M}_{12}$	$\mathcal{M}_{13}$	$\mathcal{M}_{14}$	$\mathcal{M}_{15}$		
J	71.16	54.61	21.72	9.95	7.66	5.17		
Decreasing neighbours testing procedure								
H_0	$\mathcal{M}_{24}$	$\mathcal{M}_{23}$	$\mathcal{M}_{22}$	$\mathcal{M}_{21}$	$\mathcal{M}_{20}$	$\mathcal{M}_{19}$	$\mathcal{M}_{18}$	$\mathcal{M}_{17}$
H_1	$\mathcal{M}_{23}$	$\mathcal{M}_{22}$	$\mathcal{M}_{21}$	$\mathcal{M}_{20}$	$\mathcal{M}_{19}$	$\mathcal{M}_{18}$	$\mathcal{M}_{17}$	$\mathcal{M}_{16}$
J	3.36	0.49	0.54	0.73	5.58	3.76	0.48	1.40
H_0	$\mathcal{M}_{16}$	$\mathcal{M}_{15}$						
H_1	$\mathcal{M}_{15}$	$\mathcal{M}_{14}$						
J	2.54	5.10						

No motivation is given for the specific choice of k . We therefore apply the increasing neighbours testing procedure to the data in order to formally specify k . The results are reported in Table 9. The outcome of the increasing neighbours testing procedure is $k = 14$. The decreasing neighbours testing procedure does not work for these data, since we find that the null model is always accepted. Based on the increasing neighbours testing procedure, we select a weights matrix that has $k = 14$ nearest neighbours.

We conclude this empirical example by a Bayesian model comparison. All models $\mathcal{M}_1$ to $\mathcal{M}_{24}$ are estimated with uninformative priors, which result in estimates similar to maximum likelihood estimates (LeSage and Pace 2009). In Table 10 we see that the preferred model is a spatial error model with $k = 13$ nearest neighbours. This result is almost the same as the outcome of the increasing neighbours testing procedure. We therefore conclude that the weights matrix with $k = 4$ used by Wilhelmsson is underspecified.

We finally note that if the estimates of spatial error models with different k -nearest neighbours weights matrices are compared, $k = 13$ and $k = 14$ give almost identical parameter estimates, whereas $k = 4$ gives different

Table 10: Model probabilities for selected spatial error models for the Swedish house price data. The probabilities for models not shown are < 0.001 .

Model	$\mathcal{M}_{10}$	$\mathcal{M}_{11}$	$\mathcal{M}_{12}$	$\mathcal{M}_{13}$	$\mathcal{M}_{14}$
Model Probability	0.000	0.056	0.195	0.749	0.000

parameter estimates (not reported).

6 Conclusions

In this article we have introduced and evaluated testing procedures for specifying the number k of nearest neighbours in the weights matrix of spatial econometric models. The spatial J -test is used for specification search. Two testing procedures based on it are suggested: an increasing neighbours testing procedure and a decreasing neighbours testing procedure. The testing procedures give formal justification for the choice of k , something which has been lacking in the classical spatial econometric literature.

In the increasing neighbours testing procedure we begin with a null model where k is small. In the sequence of tests k is always smaller in the null model than in the alternative model. The spatial J -test rejects if the alternative model which contains more neighbours in the weights matrix has predictive power. In the decreasing neighbours testing procedure we begin with a null model where k is large. In the sequence of tests k is always larger in the null model than in the alternative model. The spatial J -test rejects if the null model imposes false neighbours, and the alternative model which relaxes a false neighbours has predictive power.

Simulation results show that the power of the spatial J -test is prominently higher against alternative models that add a neighbour that has predictive power than against alternative models that relax a false neighbour. The increasing neighbours testing procedures can be used in large samples to determine k . The decreasing neighbours testing procedure is found to have low power, and is not recommended for use in practice.

The paper includes an empirical example which demonstrates the use of the testing procedures with real data on house prices. The value of k is found to be larger than the values commonly used in spatial econometric models of house prices. A Bayesian model comparison exercise yields similar results.

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A Increasing Neighbours Testing Procedure

This appendix contains the proof that the increasing neighbours testing procedure is consistent.

In the increasing neighbours testing procedure the spatial J -test rejects the null model $\mathcal{M}_{p+r}$ against the alternative model $\mathcal{M}_{p+r+1}$ if the weights matrix of $\mathcal{M}_{p+r}$ is underspecified and $\mathcal{M}_{p+r+1}$ which contains an additional neighbour has predictive power.

Proposition 1 *In the increasing neighbours testing procedure the test that rejects $\mathcal{M}_{p+r}$ against $\mathcal{M}_{p+r+1}$ if $J_{p+r} > \chi^2_{1-\alpha}(2)$ has asymptotic size α and asymptotic power 1.*

Proof of Proposition 1. We assume Assumptions A1–A7 in the Appendix of Kelejian (2008). It now follows that if the alternative model has predictive power, then $J_{p+r} \xrightarrow{P} \infty$, as $n \rightarrow \infty$, because the spatial J -test is consistent. Thus

$$\lim_{n \rightarrow \infty} P\{J_p > \chi^2_{1-\alpha}, \dots, J_k > \chi^2_{1-\alpha}\} = \lim_{n \rightarrow \infty} P\{J_k > \chi^2_{1-\alpha}\} = \alpha.$$

On the other hand, $J_p, \dots, J_{p+r} \xrightarrow{P} \infty$, if $p + r < k$, which shows that

$$\lim_{n \rightarrow \infty} P\{J_p > \chi^2_{1-\alpha}, \dots, J_{p+r} > \chi^2_{1-\alpha}\} = 1.$$

■

B Bootstrap Spatial J -Test

This appendix contains the details about the implementation of the bootstrap spatial J -test, and the simulation of size and power of the bootstrap test using a procedure suggested by Davidson and MacKinnon (2006).

We denote bootstrap quantities by a '*'. For generating samples of bootstrap observations $\mathbf{y}^*$, we use a semiparametric bootstrap algorithm.

Algorithm 2 (Bootstrap spatial J -test) 1. *Estimate the general spatial econometric model (1) to obtain the estimates $\hat{\rho}$, $\hat{\lambda}$ and $\hat{\beta}$.*

2. *Check whether $|\hat{\rho}| < 1$ and $|\hat{\lambda}| < 1$.*

3. *If both conditions are satisfied, generate bootstrap observations $\mathbf{y}^*$ from*

$$\begin{aligned}\mathbf{y}^* &= \mathbf{X}\hat{\beta} + \hat{\rho}\mathbf{W}\mathbf{y}^* + \mathbf{u}^*, \\ \mathbf{u}^* &= \hat{\lambda}\mathbf{M}\mathbf{u}^* + \boldsymbol{\varepsilon}^*,\end{aligned}\tag{B1}$$

where $\boldsymbol{\varepsilon}^$ is obtained by resampling the rescaled residuals $(n/(n-m))^{1/2}\hat{\boldsymbol{\varepsilon}}$, and m is the number of estimated parameters.*

If the number of bootstrap replications is B , repeated application of Algorithm 1 gives B bootstrap samples $\mathbf{y}_j^*$, which are used to compute bootstrap J -statistics J_j^* . The empirical distribution of the J_j^* is used to approximate the distribution of J under the null hypothesis. The bootstrap critical value at the significance level α is given by the $1 - \alpha$ quantile of the J_j^* .

Let $\hat{J}$ denote the realised value of the J -statistic. For a test at significance level α , we reject the null hypothesis if $\hat{J}$ is larger than the bootstrap critical value.

The usual procedure for estimating the size and power of a bootstrap spatial J -test requires computing $M(B + 1)$ J -statistics. Davidson and MacKinnon (2006) propose a procedure for estimating the size and power of a bootstrap test which substantially reduces the computational burden. For each replication $j = 1, \dots, M$, compute the J -statistic $\hat{J}_j$ and generate one bootstrap J -statistic J_j^* . Compute $J^*(1 - \alpha)$, the $1 - \alpha$ quantile of the J_j^* . The approximate rejection probability of the bootstrap test is then estimated by

$$\frac{1}{M} \sum_{j=1}^M I(\hat{J}_j > J^*(1 - \alpha)).\tag{B2}$$

See Davidson and MacKinnon (2006) for more details. The amount of computation required to estimate (B2) is $2M$ J -statistics, which is much smaller than $M(B + 1)$. Davidson and MacKinnon (2006) establish the validity of (B2) under the assumption of independence between the bootstrap DGP and the test statistic. For the J -test, the bootstrap DGP and the test statistic are asymptotically independent.

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