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Borel* Sets in the Generalized Baire Space and Infinitary Languages

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Abstract

We start by giving a survey to the theory of $\text{Borel}^*(\kappa)$ sets in the generalised Baire space $\text{Baire}(\kappa) = \kappa^\kappa$. In particular we look at the relation of this complexity class to other complexity classes which we denote by $\text{Borel}(\kappa)$, $\Delta_1^1(\kappa)$ and $\Sigma_1^1(\kappa)$ and the connections between $\text{Borel}^*(\kappa)$ sets and the infinitely deep language $M_{\kappa+\kappa}$. In the end of the paper we will prove the consistency of $\text{Borel}^*(\kappa) \neq \Sigma_1^1(\kappa)$.

Key words: descriptive complexity, generalised Baire space, infinitary languages

2012 MSC: 03E47, 03-02

Among many classification problems studied in mathematics, the classification of the subsets of the reals according to their topological complexity is very classical. It is also very useful: On the one hand in many branches of mathematics all subsets of the reals that one really comes across are of relatively low complexity. On the other hand e.g. the axioms of ZFC can prove properties for these simple sets that it cannot prove for arbitrary sets. Of many such examples let us mention the following two. The Continuum Hypothesis holds for the Borel sets (i.e. each Borel set is either countable or of the same size as the continuum) while ZFC does not prove this for arbitrary subsets of the reals; and all Σ_1^1 -sets are Lebesgue measurable but ZFC also proves the existence of a non-measurable set. For the definitions see below.

This classification of the subsets of the reals can also be used to classify various other mathematical objects. Let L be a fixed countable vocabulary. Then every real

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r can be seen as a code for an L -structure $\mathcal{A}_r$ with the set of natural numbers as the universe so that every such structure also has a code (not necessarily unique), see Section 1 for details. Then one can classify L -theories T (not necessarily first-order) according to the complexity of the set $\text{ISO}(T, \omega)$ which consist of all the pairs (r, q) of reals such that $\mathcal{A}_r$ and $\mathcal{A}_q$ are isomorphic models of T . This is a much studied classification, but since this classification captures only countable models of the theories, it is very different from e.g. the classification of the first-order theories given in [She00].

Let DLO be the theory of dense linear orderings without end points. Then in Shelah's classification DLO is a very complicated theory, but because DLO is ω -categorical, $\text{ISO}(\text{DLO}, \omega)$ is very simple: it is Borel and of a very low rank. On the other hand in [Koe11] it was shown that there is an ω -stable NDOP theory T of depth 2 such that $\text{ISO}(T, \omega)$ is not Borel. In Shelah's classification ω -stable NDOP theories of depth 2 are considered very simple.

Besides the general interest in the uncountable, considerations like the one above, suggest that it may make sense to try to generalise the complexity notions to larger models. For technical reasons, the classical theory is usually not developed in the space of real numbers but in the Baire space (or Cantor space). The Baire space is not homeomorphic with the reals but on the level of Borel sets, it is very close to the reals (they are Borel-isomorphic). Another benefit from working with the notion of the Baire space is that there is a very natural way of generalising it to the uncountable: Suppose that κ is an infinite cardinal such that $\kappa^{<\kappa} = \kappa$. Note that ω satisfies this assumption. (Some work has been done also without this assumption, but in general it is not even clear what the right notion of a Borel set is, if this assumption is dropped.) Now we let the generalised Baire space, $\text{Baire}(\kappa)$, to be the set of all functions $f: \kappa \rightarrow \kappa$. We make this into a topological space by letting the basic open sets be the sets $N_\eta = \{f \in \text{Baire}(\kappa) \mid \eta \subseteq f\}$, where η is a function from some ordinal $\alpha < \kappa$ to κ . From basic open sets we get the family of Borel sets, $\text{Borel}(\kappa)$, by closing the collection of all basic open sets under the unions and intersections of size $\leq \kappa$. Notice that since $\kappa^{<\kappa} = \kappa$, open sets are $\text{Borel}(\kappa)$, the complements of basic open sets are open and thus by an easy induction one can also see that the collection of $\text{Borel}(\kappa)$ sets is closed under complements. Note in addition, that if X is a finite product of copies of $\text{Baire}(\kappa)$ equipped with the product topology, then X is homeomorphic with $\text{Baire}(\kappa)$ giving us the notion of a $\text{Borel}(\kappa)$ set also to these spaces. Alternatively one could let the sets $N_{(\eta_i)_{i < n}} = \{(f_i)_{i < n} \in \text{Baire}(\kappa)^n \mid \eta_i \subseteq f_i \ \forall i < n\}$ be the basic open sets, where for some $\alpha < \kappa$ for all $i < n$, η_i is a function from α to κ , and then proceed as in the case of the space $\text{Baire}(\kappa)$. Of course, $\text{Baire}(\omega)$ is the usual Baire space and $\text{Borel}(\omega)$ is the usual family of all Borel sets.

As in the case of the reals, for a theory T the collection $\text{ISO}(T, \kappa)$ can be formed. Now, for uncountable κ , the classification of theories that we get from descriptive

set theoretic analysis is much closer to that of Shelah's than in the case $\kappa = \omega$. For example for suitable κ and countable first-order theories T , T is shallow and superstable with NDOP and NOTOP if and only if $\text{ISO}(T, \kappa)$ is $\text{Borel}(\kappa)$ (in particular, $\text{ISO}(T, \kappa)$ is $\text{Borel}(\kappa)$ for the theory T constructed in [Koe11] and $\text{ISO}(\text{DLO}, \kappa)$ is not $\text{Borel}(\kappa)$, in fact not even $\Delta_1^1(\kappa)$, see Section 1). For more on such questions, see [FHK14] and [HK15].

In [Bla81] the author observed that Borel sets can be equivalently defined using games. In [MV93] this was generalised to get $\text{Borel}^*(\kappa)$ sets. The idea behind this generalisation followed the lines of [Kar84] where she generalised the logics $L_{\omega+\omega}$ to $M_{\kappa+\kappa}$ via semantic games, see Section 2. In turn, this generalisation was preceded by rather similar generalisations of [Vau73] and [HM76, Mak74] as well as [HR76]. The topic of this paper are $\text{Borel}^*(\kappa)$ sets and their relationship to the infinitary language $M_{\kappa+\kappa}$. In the first two sections we give a survey on the existing theory – in particular we sketch a proof for the fact that in Gödel's L , $\text{Borel}^*(\kappa) = \Sigma_1^1(\kappa)$ for all uncountable regular κ , and so in L , also $\text{ISO}(\text{DLO}, \kappa)$ is $\text{Borel}^*(\kappa)$. Then in the third section we prove the consistency of $\text{Borel}^*(\kappa) \neq \Sigma_1^1(\kappa)$ (for uncountable κ with $\kappa^{<\kappa} = \kappa$); in fact we will show that it is consistent that $\text{ISO}(\text{DLO}, \kappa)$ is not $\text{Borel}^*(\kappa)$.

1 Borel* Versus Some Other Complexity Classes

Throughout this paper, we assume that κ is an infinite cardinal and $\kappa^{<\kappa} = \kappa$. Note that from this it follows that κ is regular.

The following definition of $\text{Borel}^*(\kappa)$ sets is from [Bla81] in the case $\kappa = \omega$ and from [MV93] in the case of uncountable κ .

In this paper a *tree* is a partial order with a root isomorphic to $(P, <)$ where $P \subset \kappa^{<\kappa}$ and $p < q \iff p \subset q$; thus the trees “grow upwards”. A maximal element of a tree is a *leaf*.

1.1 Definition. Let $\lambda, \mu \leq \kappa$ be cardinals.

- (i) A tree t is a μ, λ -tree (or $\mu\lambda$ -tree without comma) if it does not contain chains of length λ and its cardinality is at most μ . It is *closed* if every chain has a unique supremum; in particular every branch ends with a leaf.
- (ii) A pair (t, f) is a $\text{Borel}_\lambda^*(\kappa)$ -code if t is a closed κ^+, λ -tree and f is a function with domain t such that if $x \in t$ is a leaf, then $f(x)$ is a basic open set and otherwise $f(x) \in \{\cup, \cap\}$.
- (iii) For an element $\eta \in \text{Baire}(\kappa)$ and a $\text{Borel}_\lambda^*(\kappa)$ -code (t, f) , the Borel^* -game $B^*(\eta, (t, f))$ is played as follows. There are two players, I and II. The game starts from the root of t . At each move, if the game is at a node $x \in t$ and $f(x) = \cap$, then I chooses an immediate successor y of x and the game continues

from this y . If $f(x) = \cup$, then Π makes the choice. At limits the game continues from the (unique) supremum of the previous moves. Finally, if x is a leaf and $f(x)$ is a basic open set, then the game ends, and Π wins iff $\eta \in f(x)$.

- (iv) We say that $X \subseteq \text{Baire}(\kappa)$ is a $\text{Borel}^*_\lambda(\kappa)$ set if there is a $\text{Borel}^*_\lambda(\kappa)$ -code (t, f) such that for all $\eta \in \text{Baire}(\kappa)$, $\eta \in X$ iff Π has a winning strategy in the game $B^*(\eta, (t, f))$.
- (v) We abbreviate $\text{Borel}^*_\kappa(\kappa)$ by $\text{Borel}^*(\kappa)$.
- (vi) We write $\text{Borel}^*(\kappa)$ also for the family of all $\text{Borel}^*(\kappa)$ sets. And we will do the same with the other complexity classes.

The observation by D. Blackwell, mentioned in the introduction, generalises immediately to the following.

1.2 Lemma. $\text{Borel}(\kappa) = \text{Borel}^*_\omega(\kappa)$. In particular, $\text{Borel}(\omega) = \text{Borel}^*(\omega)$ and $\text{Borel}(\kappa) \subseteq \text{Borel}^*(\kappa)$.

Proof. “ $\subseteq$ ” is an easy induction on $\text{Borel}(\kappa)$ sets and “ $\supseteq$ ” is an easy induction on the rank of the (well-founded) trees in $\text{Borel}^*_\omega(\kappa)$ -codes. $\square$

- 1.3 Definition.** (i) $X \subseteq \text{Baire}(\kappa)$ is $\Sigma^1_1(\kappa)$ if it is the first projection $\text{pr}_1(Y)$ of some closed $Y \subseteq \text{Baire}(\kappa) \times \text{Baire}(\kappa)$.
- (ii) $X \subseteq \text{Baire}(\kappa)$ is $\Delta^1_1(\kappa)$ if both X and $\text{Baire}(\kappa) \setminus X$ are $\Sigma^1_1(\kappa)$.

1.4 Remark. As mentioned in the introduction, for every $n < \omega$, the spaces $\text{Baire}(\kappa)^n$ are homeomorphic to each other, so (i) extends to all of them and is equivalent to saying that $X \subset \text{Baire}(\kappa)^n$ is $\Sigma^1_1(\kappa)$ if it is a projection of a closed $C \subset \text{Baire}(\kappa)^m$, for some $m > n$.

In the Lemmas and Theorems below, we show that

$$\text{Borel}(\kappa) \subsetneq \Delta^1_1(\kappa) \subseteq \text{Borel}^*(\kappa) \subseteq \Sigma^1_1(\kappa).$$

All these inclusions were established in [MV93] and can be proved in ZFC under $\kappa^{<\kappa} = \kappa$. The first inclusion is proper [FHK14, Thm 18], but it is undecidable in ZFC whether or not the last inclusion is proper (Theorems 1.14 and 3.1; these results are new) and it remains an open problem whether or not it is consistent that the inclusion $\Delta^1_1(\kappa) \subseteq \text{Borel}^*(\kappa)$ is not proper (Open Question 1.9). However, it can be shown in ZFC that the inclusion $\Delta^1_1(\kappa) \subsetneq \Sigma^1_1(\kappa)$ is proper, see Lemma 1.12 and the original source [FHK14, Thm 18].

- 1.5 Lemma.** (i) $\text{Borel}(\kappa) \subseteq \Delta^1_1(\kappa) \subseteq \Sigma^1_1(\kappa)$.
- (ii) If $X \subseteq \text{Baire}(\kappa)$ is the first projection of some $\Sigma^1_1(\kappa)$ set $Y \subseteq \text{Baire}(\kappa) \times \text{Baire}(\kappa)$, then it is $\Sigma^1_1(\kappa)$. In particular, projections of $\text{Borel}(\kappa)$ sets are $\Sigma^1_1(\kappa)$.

Proof. (i) The second inclusion is trivial, so we prove the first. Since the class of $\Delta_1^1(\kappa)$ sets is closed under complements, by De Morgan's laws, it is enough to show that the class of $\Delta_1^1(\kappa)$ sets is closed under intersections of size $\leq \kappa$. For this, it is enough to show that the class of $\Sigma_1^1(\kappa)$ sets is closed under intersections and unions of size $\leq \kappa$. The proof of this is the same as for classical analytic sets for $\kappa = \omega$. An alternative proof can be obtained by combining the facts that $\text{Borel} \subset \text{Borel}^*$ and Borel sets are closed under complement with Lemma 1.12.(i) below (we thank the referee for pointing this out).

(ii) Now Y is $\Sigma_1^1(\kappa)$, so by Remark 1.4 it is a projection of some closed set $C \subset \text{Baire}(\kappa)^m$ and so X is the projection of C as well, so the claim follows by applying Remark 1.4 again. $\square$

Next we look at the relations between $\text{Borel}^*(\kappa)$ and other complexity classes. The following theorem (for $\kappa > \omega$) and especially the clever proof we give, are from [MV93]:

1.6 Theorem. $\Delta_1^1(\kappa) \subseteq \text{Borel}^*(\kappa)$.

Proof. Let $A \subseteq \text{Baire}(\kappa)$ be a $\Delta_1^1(\kappa)$ set. We need to find a $\text{Borel}^*(\kappa)$ -code for it.

Let $C, D \subseteq \text{Baire}(\kappa) \times \text{Baire}(\kappa)$ be closed sets such that $\text{pr}_1(C) = A$ and $\text{pr}_1(D) = \text{Baire}(\kappa) \setminus A$.

For closed $B \subseteq \text{Baire}(\kappa) \times \text{Baire}(\kappa)$ by $t(B)$ we denote the set of all pairs $(\xi \upharpoonright \alpha, \eta \upharpoonright \alpha)$ such that $(\xi, \eta) \in B$ and $\alpha < \kappa$. For $\xi \in \text{Baire}(\kappa)$, by $t(\xi, B)$ we mean the set of all $\eta : \alpha \rightarrow \kappa, \alpha < \kappa$, such that $(\xi \upharpoonright \alpha, \eta \upharpoonright \alpha) \in t(B)$ and we order $t(\xi, B)$ by the subset relation. Then $T(\xi, B)$ is a tree. B is closed and therefore we have

(*) $\xi \in \text{pr}_1(B)$ iff $t(\xi, B)$ contains a branch of length κ .

Thus, since $\text{pr}_1(C)$ and $\text{pr}_1(D)$ form a partition of $\text{Baire}(\kappa)$, we have

(**) for all $\xi \in \text{Baire}(\kappa)$, exactly one of $t(\xi, C)$ and $t(\xi, D)$ contains a branch of length κ .

For trees t_0 and t_1 , we write $t_0 \leq t_1$ if there is an order preserving $g : t_0 \rightarrow t_1$ (we do not require that g is one-to-one). Note that $t_0 \leq t_1$ iff player II has a winning strategy in the following game $O(t_0, t_1)$: At each move α , first I chooses an element $t_\alpha \in t_0$ and then II chooses an element $u_\alpha \in t_1$. For all $\alpha < \beta$ those elements must satisfy $t_\alpha < u_\alpha < t_\beta$. The player who breaks that rule first, loses.

Now let us look at the tree t' which consists of triples (ξ, η, δ) such that $(\xi, \eta) \in t(C)$ and $(\xi, \delta) \in t(D)$. The ordering is the obvious one: $(\xi, \eta, \delta) \leq (\xi', \eta', \delta')$ if $\xi \subseteq \xi', \eta \subseteq \eta'$ and $\delta \subseteq \delta'$. By (**), t' is a κ^+, κ -tree (in particular, it does not contain a branch of length κ). Now let t be any κ^+, κ -tree such that $t'' \not\leq t'$ (e.g. the tree of all downwards closed chains of t').

Then by (*), for all $\xi \in \text{Baire}(\kappa)$, $t'' \leq t(\xi, C)$ iff $\xi \in \text{pr}_1(C)$ i.e. iff Π has a winning strategy in $O(t'', t(\xi, C))$. Now it is easy to find a $\text{Borel}^*(\kappa)$ -code (t, f) such that for all $\xi \in \text{Baire}(\kappa)$, the game $B^*(\xi, (t, f))$ simulates the game $O(t'', t(\xi, C))$. Then (t, f) is a $\text{Borel}^*(\kappa)$ -code for A . $\square$

1.7 Corollary. $\text{Borel}(\omega) = \Delta_1^1(\omega) = \text{Borel}^*(\omega)$. $\square$

Neither of the identities in Corollary 1.7 above can be proved in the case $\kappa > \omega$ (at least not in ZFC). We start with a straightforward one which was observed in [FHK14]:

1.8 Lemma. *If $\kappa > \omega$, then $\text{Borel}(\kappa) \neq \Delta_1^1(\kappa)$.*

Proof. Recall that by Lemma 1.2 $\text{Borel}(\kappa) = \text{Borel}_\omega^*(\kappa)$. Now choose any reasonable coding of $\text{Borel}_\omega^*(\kappa)$ -codes (t, f) to functions $\eta : \kappa \rightarrow \kappa$ so that if we write (t_η, f_η) for the pair coded by η , every $\text{Borel}_\omega^*(\kappa)$ -code (t, f) is (t_η, f_η) for some η and the set of those η which code a $\text{Borel}_\omega^*(\kappa)$ -code is closed, or at least $\text{Borel}(\kappa)$ (here we need that $\kappa > \omega$, because the property that t_η is well-founded is not $\text{Borel}(\kappa)$ if $\kappa = \omega$). Also choose a coding for strategies of I in the games $B^*(\xi, (t, f))$. In both cases ‘almost’ any coding works – excluding the pathological ones.

Now, for non-pathological codings, it is easy to see that the set of all (ξ, η, δ) such that $\xi, \eta, \delta \in \text{Baire}(\kappa)$, η codes a $\text{Borel}_\omega^*(\kappa)$ -code and δ codes a winning strategy of I in the game $B^*(\xi, (t_\eta, f_\eta))$ is $\text{Borel}(\kappa)$. But then by the Gale-Stewart theorem (and Lemma 1.5 (ii)), if we let A be the set of pairs (ξ, η) such that $\xi, \eta \in \text{Baire}(\kappa)$, η codes a $\text{Borel}_\omega^*(\kappa)$ -code and ξ is not in the $\text{Borel}(\kappa)$ set coded by (t_η, f_η) , then A is $\Sigma_1^1(\kappa)$. Similarly one can see that the complement of A is also $\Sigma_1^1(\kappa)$ and thus A is $\Delta_1^1(\kappa)$. But then also $B = \{(\eta, \eta) \mid (\eta, \eta) \in A\}$ is $\Delta_1^1(\kappa)$, since $B = \Delta \cap A$ where $\Delta = \{(\eta, \eta) \mid \eta \in \text{Baire}(\kappa)\}$ is clearly closed (and thus $\Delta_1^1(\kappa)$, see the proof of Lemma 1.5). However, B can not be $\text{Borel}(\kappa)$ because obviously it can not have a $\text{Borel}_\omega^*(\kappa)$ -code (this is the usual Cantor style diagonalisation, see the proof of Lemma 1.12 (ii) where we do everything in more detail). $\square$

However, the other identity, namely $\Delta_1^1(\kappa) = \text{Borel}^*(\kappa)$, is more complicated. In fact,

1.9 Open Question. Is $\Delta_1^1(\kappa) = \text{Borel}^*(\kappa)$ together with $\kappa = \kappa^{<\kappa} > \omega$ consistent?

This question is related to Question 2.7. As can be understood from Section 2, there is a connection between $\text{Borel}^*(\kappa)$ sets and classes of models definable in the language $M_{\kappa+\kappa}$. However the connection is not as close as one might think e.g. they are different in L , see the discussion after Open Question 2.7.

In any case, one can try to use the intuition provided by the theory of the language $M_{\kappa+\kappa}$ to understand $\text{Borel}^*(\kappa)$ sets and this intuition suggests that the answer to 1.9 is no: it seems very unlikely that $\text{Borel}^*(\kappa)$ could be closed under

taking complements (which it would be, if $\Delta_1^1(\kappa) = \text{Borel}^*(\kappa)$), because $M_{\kappa+\kappa}$ is not closed under the negation as is shown in an unpublished manuscript by T. Huuskonen from the 90's. But often the proofs from the theory of $M_{\kappa+\kappa}$ do not work in the context of $\text{Borel}^*(\kappa)$ and this is also the case with Huuskonen's proof: one of the problems in using it in the context of $\text{Borel}^*(\kappa)$, is that the models which witness that the sentence does not have a negation in $M_{\kappa+\kappa}$ are necessarily of size $> \kappa$.

The question of the consistency of $\Delta_1^1(\kappa) \neq \text{Borel}^*(\kappa)$ is easier to handle. In fact, for every uncountable regular κ , $\Delta_1^1(\kappa) \neq \text{Borel}^*(\kappa)$ in L (see Lemma 1.12 (ii) and Theorem 1.14) and the same holds for an uncountable κ with $\kappa^{<\kappa} = \kappa$ also in the model we construct in Section 3. As a preparation, let us look at the way of seeing this.

- 1.10 Definition.** (i) We let $\text{CUB}_\omega(\kappa)$ be the set of all $\eta \in \text{Baire}(\kappa)$ such that the set $\{\alpha < \kappa \mid \eta(\alpha) > 0\}$ contains an ω -cub set i.e. an unbounded set $X \subseteq \kappa$ which is ω -closed i.e. if $\alpha_i \in X$ for all $i < \omega$, then $\cup_{i < \omega} \alpha_i \in X$.
- (ii) A set $X \subseteq \text{Baire}(\eta)$ is co-meagre if it is an intersection of κ many dense and open subsets of $\text{Baire}(\kappa)$.
- (iii) $Y \subseteq \text{Baire}(\kappa)$ has the property of Baire if there are an open set U and a co-meagre set X such that $Y \cap X = U \cap X$.

In [Hal96] it was shown that the classical result that $\text{Borel}(\omega)$ sets have the property of Baire generalises to $\text{Borel}(\kappa)$ for uncountable $\kappa = \kappa^{<\kappa}$.

The following lemma can be found in [FHK14]; item (iii) was independently known also to P. Lücke and P. Schlicht.

1.11 Lemma. *Suppose $\kappa > \omega$.*

- (i) $\text{CUB}_\omega(\kappa)$ is $\text{Borel}^*(\kappa)$.
- (ii) $\text{CUB}_\omega(\kappa)$ does not have the property of Baire.
- (iii) It is consistent that every $\Delta_1^1(\kappa)$ set has the property of Baire and at the same time $\kappa = \kappa^{<\kappa} > \omega$.
- (iv) It is consistent that $\kappa = \kappa^{<\kappa} > \omega$ and $\Delta_1^1(\kappa) \neq \text{Borel}^*(\kappa)$.

Proof. (i) It is easy to see that the set $A_\eta = \{\alpha < \kappa \mid \eta(\alpha) > 0\}$ contains an ω -cub set iff the player II has a winning strategy in the game $CG_\omega(A_\eta)$: the game lasts ω moves. At each move $n < \omega$, first the player I chooses an ordinal $\alpha_n \in \kappa$ and then II chooses an ordinal $\beta_n \in \kappa$ such that $\beta_n > \alpha_n$. In the end II wins if $\cup_{n < \omega} \beta_n \in A_\eta$.

But now one just needs to find a $\text{Borel}^*(\kappa)$ -code (t, f) such that the Borel^* game $B^*(\eta, (t, f))$ “simulates” the game $CG_\omega(A_\eta)$. To do this, let t be the tree of all increasing sequences of κ of length $\leq \omega$ and if $l \in t$ is a leaf, then $f(l) = \{\xi \in \kappa^\kappa \mid \xi(\sup(l) \neq 0)\}$ and $f(s) = \cup$ if $|s|$ is odd and $f(s) = \cap$ otherwise. Then II has a winning strategy in $CG_\omega(A_\eta)$ if and only if she has one in $B^*(\eta, (t, f))$; for details see [FHK14, Thm 49(1)].

(ii) Suppose U is open and X_i , $i < \kappa$, are open and dense. We need to show that $\text{CUB}_\omega(\kappa) \cap X \neq U \cap X$ where $X = \bigcap_{i < \kappa} X_i$. We assume that $U \neq \emptyset$, the other case is similar. Now choose an increasing sequence $\eta_i: \alpha_i \rightarrow \kappa$, $\alpha_i < \kappa$, so that

- (a) if $i = 0$, then let η_i be such that $N_{\eta_i} \subseteq U$ (for N_{η_i} , see the introduction),
- (b) if $i = j + 1$, then let η_i be such that it extends η_j and $N_{\eta_j} \subseteq X_j$,
- (c) if i is limit, then let $\eta_i = (\bigcup_{j < i} \eta_j) \cup \{(\bigcup_{j < i} \alpha_j, 0)\}$.

Now if we let $\eta = \bigcup_{i < \kappa} \eta_i$, $\eta \in X \cap U$ but $\eta \notin \text{CUB}_\omega(\kappa)$.

(iii) The statement is forced by adding κ^+ many Cohen subsets to κ , for details see [FHK14].

(iv) Immediate by (i)-(iii). □

Let us now turn to the relations between the class $\Sigma_1^1(\kappa)$ and the other complexity classes studied above. The proof of the following lemma is a straightforward generalisation from the case $\kappa = \omega$ and in the case $\kappa = \omega$, the item 1.12 (ii) is the famous result of M. Suslin from [Sus17].

1.12 Lemma. (i) $\text{Borel}^*(\kappa) \subseteq \Sigma_1^1(\kappa)$.

(ii) $\Delta_1^1(\kappa) \neq \Sigma_1^1(\kappa)$.

Proof. (i) Let (t, f) be a $\text{Borel}^*(\kappa)$ -code. As in the proof of Lemma 1.8 one can quite freely choose the way of coding strategies of player II in the game $B^*(\xi, (t, f))$ to functions $\eta: \kappa \rightarrow \kappa$ and find out that the set of those pairs $(\xi, \eta) \in \text{Baire}(\kappa) \times \text{Baire}(\kappa)$ for which η codes a winning strategy of II in the game $B^*(\xi, (t, f))$ is closed. And thus the set with the $\text{Borel}^*(\kappa)$ -code (t, f) is $\Sigma_1^1(\kappa)$.

(ii) Here we give the easiest proof i.e. we diagonalise, but we will return to this question after this proof. Let us fix a coding for open sets of $\text{Baire}(\kappa) \times \text{Baire}(\kappa)$: fix a one-to-one and onto function $\pi: \kappa \rightarrow B$, where B is the set of all pairs (f, g) functions $f, g: \alpha \rightarrow \kappa$, $\alpha < \kappa$. Then we think of $\eta \in \text{Baire}(\kappa)$ as the code of the open set $U_\eta = \bigcup_{\alpha < \kappa} N_{\eta(\alpha)}$, see the alternative way of defining the topology on $\text{Baire}(\kappa) \times \text{Baire}(\kappa)$ in the introduction. Now every open set has a (non-unique) code and every $\eta \in \text{Baire}(\kappa)$ codes some open set. Now every $\eta \in \text{Baire}(\kappa)$ is also a code for a $\Sigma_1^1(\kappa)$ set, namely to the set A_η which consists of those $\xi \in \text{Baire}(\kappa)$ such that for some $\delta \in \text{Baire}(\kappa)$, $(\xi, \delta) \notin U_\eta$. Notice that now every $\Sigma_1^1(\kappa)$ set has a code.

Now let A be the set of those $\eta \in \text{Baire}(\kappa)$ such that $\eta \in A_\eta$. It is easy to see that the set $B = \{(\eta, \delta) \in \text{Baire}(\kappa) \times \text{Baire}(\kappa) \mid (\eta, \delta) \notin U_\eta\}$ is closed and thus $A = \text{pr}_1(B)$ is $\Sigma_1^1(\kappa)$. This set A is not $\Delta_1^1(\kappa)$ because if it is, then $C = \text{Baire}(\kappa) \setminus A$ has a code η which means that $\eta \in C$ iff $\eta \in A_\eta$ iff $\eta \in A$ iff $\eta \notin C$, a contradiction. □

There are also more concrete examples of $\Sigma_1^1(\kappa)$ sets that are not $\Delta_1^1(\kappa)$: Fix a vocabulary L so that it consists of one binary predicate symbol $\leq$ (for simplicity) and fix also a one-to-one and onto function $\pi: \kappa^2 \rightarrow \kappa$. Then we let every $\eta \in \text{Baire}(\kappa)$

code the following L -structure $\mathcal{A}_\eta$: The universe of $\mathcal{A}_\eta$ is κ and for all $(x, y) \in \kappa^2$, the pair (x, y) is in the interpretation of $\leq$ if $\eta(\pi(x, y)) \geq 1$. Notice that now every L -structure with universe κ has a code (not unique). Then, as in the introduction, we let $\text{ISO}(\text{DLO}, \kappa)$ consists of those pairs $(\xi, \eta) \in \text{Baire}(\kappa) \times \text{Baire}(\kappa)$ such that $\mathcal{A}_\xi$ and $\mathcal{A}_\eta$ are isomorphic models of the theory DLO. Clearly, $\text{ISO}(\text{DLO}, \kappa)$ is $\Sigma_1^1(\kappa)$.

By strengthening the methods behind the proof of Theorem 1.6 and using results from [HT91], it was shown in [MV93], that

1.13 Fact. *If $\kappa > \omega$, then $\text{ISO}(\text{DLO}, \kappa)$ is not $\Delta_1^1(\kappa)$.*

In fact, this holds for a large class of first-order theories, see [FHK14, MV93]. For more on these questions, see [FHK14, HK15].

We finish this section with the following result from [FHK14]:

1.14 Theorem. *If $V = L$ and $\kappa > \omega$ is regular, then $\text{Borel}^*(\kappa) = \Sigma_1^1(\kappa)$.*

Proof. Let $A \subseteq \text{Baire}(\kappa)$ be $\Sigma_1^1(\kappa)$. We need to find a $\text{Borel}^*(\kappa)$ -code for it. Let f, g be functions with domain κ such that

- (α) for all $i < \kappa$, there is $\gamma < \kappa$ such that both $f(i)$ and $g(i)$ are functions from γ to κ ,
- (β) A is the first projection of the set

$$(\text{Baire}(\kappa) \times \text{Baire}(\kappa)) \setminus \bigcup_{i < \kappa} N_{(f(i), g(i))}.$$

Let $\varphi(x, y, z, w, u)$ be the formula of set theory which says that

- (a) x and y are functions from z to z ,
- (b) for all $i \in z$, either $w(i)$ is not a (proper) subset of y or $u(i)$ is not a (proper) subset of x (i.e. for all $i \in z$ either for all $j \in z$, $(i, y \upharpoonright j) \notin w$ or for all $j \in z$, $(i, x \upharpoonright j) \notin u$).

Let $\theta = \kappa^{++}$. Now for all $\xi \in \text{Baire}(\kappa)$, $\xi \in A$ iff $L_\theta \models \exists x \varphi(x, \xi, \kappa, f, g)$. Notice also that φ is very absolute.

Let T be (e.g.) the theory of L_θ and for all $\xi \in \text{Baire}(\kappa)$, let C_ξ be the set of all $\alpha < \kappa$ such that there is $\beta > \alpha$ with the following properties:

- (i) α is regular in L_β ,
- (ii) L_β is a model of T ,
- (iii) $L_\beta \models \exists x \varphi(x, \xi \upharpoonright \alpha, \alpha, f \upharpoonright \alpha, g \upharpoonright \alpha)$.

Notice that whether $\alpha \in C_\xi$ or not, depends only on $\xi \upharpoonright \alpha$.

1.14.1 Claim. *(See also [FHK14, Thm 18(3)] and [HK15, Lemma 1.9]) For all $\xi \in \text{Baire}(\kappa)$, $\xi \in A$ iff C_ξ contains an ω -cub set (see Definition 1.10).*

Proof of Claim 1.14.1. “ $\Rightarrow$ ”: Suppose $\xi \in A$. For all $\alpha < \kappa$, let $SH(\alpha \cup \{\xi, \kappa, f, g\})$ be the Skolem closure of the set $\alpha \cup \{\xi, \kappa, f, g\}$ under the definable Skolem functions in L_θ (among the realisations, the Skolem functions choose the least one in the definable well-ordering of L). Let D be the set of those $\alpha < \kappa$ such that $SH(\alpha \cup \{\xi, \kappa, f, g\}) \cap \kappa = \alpha$. It is routine to check that D contains an ω -cub set, in fact it is closed and unbounded. But $D \subseteq C_\xi$, because if $\alpha \in D$, then the Mostowski collapse of $SH(\alpha \cup \{\xi, \kappa, f, g\})$ is L_β for some β and this β witnesses that $\alpha \in C_\xi$.

“ $\Leftarrow$ ”: Suppose C_ξ contains an ω -cub set C . For a contradiction, suppose that $\xi \notin A$ i.e. $L_\theta \models \neg \exists x \varphi(x, \xi, \kappa, f, g)$. Following the idea from the above, let $D \subseteq \kappa$ be the set of those $\alpha < \kappa$ such that $SH(\alpha \cup \{\xi, \kappa, f, g, C\}) \cap \kappa = \alpha$. Again D is closed and unbounded and if $\alpha \in D$ is of cofinality ω , then $\alpha \in C$ (because $C \cap \alpha$ is unbounded in α and C is ω -closed).

Let α be the least limit point of D . Then $\alpha \in C \subseteq C_\xi$ and $\alpha \cap D$ has order type ω . Let β^* be such that L_{β^*} is the Mostowski collapse of $SH(\alpha \cup \{\xi, \kappa, f, g, C\})$ and let β witness the fact that $\alpha \in C_\xi$. Since

$$L_\beta \models \exists x \varphi(x, \xi \restriction \alpha, \alpha, f \restriction \alpha, g \restriction \alpha) \text{ but } L_{\beta^*} \models \neg \exists x \varphi(x, \xi \restriction \alpha, \alpha, f \restriction \alpha, g \restriction \alpha),$$

$\beta > \beta^*$ (the element that witnesses the truth of the existential claim can not be in L_{β^*}) and since $L_\beta \models T$, β is also a limit ordinal. Thus since $D \cap \alpha$ is definable from Skolem functions and truth in L_{β^*} which are definable in L_{β^*+2} , $D \cap \alpha \in L_\beta$. Since the order type of $D \cap \alpha$ is ω and $L_\beta \models T$, it is easy to see that L_β thinks that α has cofinality ω . This is a contradiction since by the definition of C_ξ , L_β should think that α is regular. $\square_{\text{Claim 1.14.1}}$

Now to find the required $\text{Borel}^*(\kappa)$ -code for A it is enough to find a $\text{Borel}^*(\kappa)$ -code (t, h) such that the game $B^*(\xi, (t, h))$ simulates the game $CG_\omega(C_\xi)$. This is easy (recall that the question of whether $\alpha \in C_\xi$ or not depends only on $\xi \restriction \alpha$). $\square$

2 Topological Complexity Classes and $M_{\kappa+\kappa}$

The complexity hierarchy of subsets of $\text{Baire}(\kappa)$ is reflected by the definability hierarchy in model theory. Fix a coding of models of size κ into elements of $\text{Baire}(\kappa)$ via some well-behaved coding $\eta \mapsto \mathcal{A}_\eta$ (for example as the one defined in Section 1 in connection with Fact 1.13). We say that $B \subset \text{Baire}(\kappa)$ is *closed under isomorphism*, if $\eta \in B$ implies $\xi \in B$ for all ξ with $\mathcal{A}_\eta \cong \mathcal{A}_\xi$ and *definable in the logic L* , if there exists a sentence $\varphi \in L$ such that $B = \{\eta \mid \mathcal{A}_\eta \models \varphi\}$. Obviously, if L is any reasonable logic and B is definable in L , then B is closed under isomorphism.

2.1 Theorem. *Suppose $B \subset \text{Baire}(\kappa)$ is closed under isomorphism. Then it is $\text{Borel}(\kappa)$ if and only if it is definable in $L_{\kappa+\kappa}$.*

When $\kappa = \omega$, this result is known as the Lopez-Escobar theorem (see e.g. [Kec94]) and for $\kappa = \omega_1$ it has been proved by R. Vaught under CH, see [Vau73]. Vaught's proof generalises to any infinite $\kappa = \kappa^{<\kappa}$.

The following definition is due to [Kar84]:

2.2 Definition. Let λ and κ be cardinals. The language $M_{\lambda\kappa}$ is then defined to be the set of pairs $(t, \mathcal{L})$ consisting of a closed λ, κ -tree t (see Definition 1.1) and a labeling function

$$\mathcal{L}: t \rightarrow a \cup \{\wedge, \vee\} \cup \{\exists x_i \mid i < \kappa\} \cup \{\forall x_i \mid i < \kappa\}$$

where a is the set of basic formulas, i.e. atomic and negated atomic formulas. The labeling $\mathcal{L}$ also satisfies the following conditions:

- (i) If $x \in t$ is a leaf, then $\mathcal{L}(x) \in a$.
- (ii) If $x \in t$ has exactly one immediate successor then $\mathcal{L}(x)$ is either $\exists x_i$ or $\forall x_i$ for some $i < \kappa$.
- (iii) Otherwise $\mathcal{L}(x) \in \{\vee, \wedge\}$.
- (iv) If $x < y$, $\mathcal{L}(x) \in \{\exists x_i, \forall x_i\}$ and $\mathcal{L}(y) \in \{\exists x_j, \forall x_j\}$, then $i \neq j$.

The truth of $M_{\lambda\kappa}$ is defined in terms of a semantic game. Let $(t, \mathcal{L})$ be a sentence and let $\mathcal{A}$ be a model. In the semantic game $S(\varphi, \mathcal{A}) = S(t, \mathcal{L}, \mathcal{A})$ for $M_{\lambda\kappa}$ the players start at the root of t and climb up one step at a time. Suppose that they are at the element $x \in t$. If $\mathcal{L}(x) = \vee$, then player II chooses an immediate successor of x , if $\mathcal{L}(x) = \wedge$, then player I chooses an immediate successor of x . If $\mathcal{L}(x) = \forall x_i$ then player I picks an element $a_i \in \mathcal{A}$ and if $\mathcal{L}(x) = \exists x_i$ then player II picks $a_i \in \mathcal{A}$ and they move to the immediate successor of x . If they come to a limit, they move to the unique supremum. If x is a maximal element of t , then they plug the elements a_i in place of the corresponding free variables in the basic formula $\mathcal{L}(x)$ and if the resulting sentence is true, then player II wins. $\mathcal{A} \models (t, \mathcal{L})$ if and only if II has a winning strategy in the semantic game.

One immediately sees some similarity with the definition of the $\text{Borel}^*(\kappa)$ sets and that maybe there is some hope to prove a result similar to Theorem 2.1. Employing this intuition, the following was shown in [FHK14] (the key idea is due to S. Coskey and P. Schlicht):

2.3 Theorem. *If $B \subset \text{Baire}(\kappa)$ is $\text{Borel}^*(\kappa)$ and closed under isomorphism, then it is definable in $\Sigma_1^1(M_{\kappa+\kappa})$.*

The converse of 2.3 is consistent:

2.4 Theorem ($V = L$). *Let $\kappa > \omega$ be regular. If $B \subset \text{Baire}(\kappa)$ is definable in $\Sigma_1^1(M_{\kappa+\kappa})$, then B is $\text{Borel}^*(\kappa)$.*

Proof. By Theorem 1.14, if B is $\Sigma_1^1(\kappa)$, then it is $\text{Borel}^*(\kappa)$, so we have to show that B is $\Sigma_1^1(\kappa)$ whenever it is definable in $\Sigma_1^1(M_{\kappa+\kappa})$. But if B is definable by a formula $\exists R\varphi(R)$ where φ is in $M_{\kappa+\kappa}$ and R is a second order variable, then B is the projection of a set definable in $M_{\kappa+\kappa}$ via the formula φ in the vocabulary extended by $\{R\}$. Thus the result follows from Theorem 2.5 below and Lemma 1.12. $\square$

2.5 Theorem. *If $B \subset \text{Baire}(\kappa)$ is definable in $M_{\kappa+\kappa}$, then it is $\text{Borel}^*(\kappa)$.*

Proof. Given a sequence $\bar{a} = (a_0, \dots, a_n)$ of ordinals below κ and a basic formula $\varphi(\bar{a})$ (atomic or negated atomic, as in Definition 2.2), let $N(\varphi(\bar{a}))$ be the set of all η such that $\mathcal{A}_\eta \models \varphi(\bar{a})$. Clearly $N(\varphi(\bar{a}))$ is an open set.

Let t be a tree and $\mathcal{L}$ a labeling function such that $(t, \mathcal{L})$ is a sentence in $M_{\kappa+\kappa}$. Let t^* consist of functions f such that $\text{dom } f$ is a downward closed linear sub-order of t with a maximal element, and $\text{ran } f$ is κ and if $x \in \text{dom } f$, but $\mathcal{L}(x) \notin \{\exists x_i \mid i < \kappa\} \cup \{\forall x_i \mid i < \kappa\}$, then $f(x) = 0$. Order t^* by $f <_{t^*} g \iff f \subset g$. If f is a leaf of t^* , then $\text{dom } f$ is a branch and there is a maximal element $x \in \text{dom } f$ which is also a maximal element in t . Let $A = \{i < \kappa \mid \exists y \in \text{dom } f (\mathcal{L}(y)) \in \{\exists x_i, \forall x_i\}\}$. Then for each $i \in A$, let α_i be the ordinal such that $f(y) = \alpha_i$ where y is the unique element of $\text{dom } f$ such that $\mathcal{L}(y) \in \{\exists x_i, \forall x_i\}$. Then let $h(f) = N(\mathcal{L}(x)((\alpha_i)_{i \in A}))$, where $\varphi((\alpha_i)_{i \in A})$ is the sentence obtained from the formula φ by replacing the free variable x_i with α_i whenever x_i occurs (if ever). Note that this $h(f)$ is not necessarily a basic open set, but note that in the definition of $\text{Borel}^*(\kappa)$ sets, basic open sets can be replaced by any open sets (even any Borel sets) and obtain an equivalent definition. If $\max \text{dom } f$ is not a leaf, then let $h(f) = \cup$, if $\mathcal{L}(\max \text{dom } f) \in \{\forall\} \cup \{\exists x_i \mid i < \kappa\}$ and $h(f) = \cap$ otherwise. Then (t^*, h) is a $\text{Borel}^*(\kappa)$ -code for the set defined by $(t, \mathcal{L})$. $\square$

A *dual* of a formula of $M_{\kappa+\kappa}$ is obtained by switching all conjunctions to disjunctions, existential quantifiers to universal quantifiers and vice versa and the basic formulas to their first-order negations. A formula is *determined* if either the formula or its dual holds in every model. In a similar way define a dual of a $\text{Borel}^*(\kappa)$ set and determined $\text{Borel}^*(\kappa)$ set. Applying a separation theorem of [MV93] that any two disjoint $\Sigma_1^1(\kappa)$ -sets can be separated by a $\text{Borel}^*(\kappa)$ set and its dual (a stronger version of Theorem 1.6 above) and a separation theorem of [Tuu92] which says that every two inconsistent $\Sigma_1^1(M_{\kappa+\kappa})$ -sentences can be separated by an $M_{\kappa+\kappa}$ -sentence and its dual, we have a corollary:

2.6 Corollary. *The following are equivalent for a set $D \subset \text{Baire}(\kappa)$:* p

- $D \subset \text{Baire}(\kappa)$ is $\Delta_1^1(\kappa)$ and closed under isomorphism,
- both D and $\text{Baire}(\kappa) \setminus D$ are definable in $M_{\kappa+\kappa}$,
- D is definable by a determined $M_{\kappa+\kappa}$ -formula,
- D is a determined $\text{Borel}^*(\kappa)$ set.

However, the converse of 2.5 is not known to be consistent:

2.7 Open Question. Is it consistent that the sets $B \subset \text{Baire}(\kappa)$ definable in $M_{\kappa+\kappa}$ are precisely the $\text{Borel}^*(\kappa)$ sets closed under isomorphism?

The negation holds in L by Theorem 2.4, because provably there is a $\Sigma_1^1(L_{\omega\omega})$ -sentence which expresses a property which is not expressible in $M_{\kappa+\kappa}$, not even on models of size κ . (The property is the following: the models consist of two distinct linear orderings and the sentence says that the linear orderings are isomorphic.)

At least one source of difficulty here seems to be the following difference between the definitions of $\text{Borel}^*(\kappa)$ -codes and $M_{\kappa+\kappa}$ -sentences: in a $\text{Borel}^*(\kappa)$ -code (t, h) , the attachment h of open sets to the leaves, can be completely arbitrary, but in a $M_{\kappa+\kappa}$ -sentence $(t, \mathcal{L})$, the truth value of the basic formula $\mathcal{L}(x)$, for a leaf x , depends in a continuous way on the moves that the players have chosen during the game (namely which interpretations they have chosen for the quantifiers).

3 Consistency of $\text{Borel}^*(\kappa) \neq \Sigma_1^1(\kappa)$

3.1 Theorem (ZFC). *It is consistent that $\text{ISO}(\text{DLO}, \kappa)$ is not $\text{Borel}^*(\kappa)$ and at the same time $\Delta_1^1(\kappa) \subsetneq \text{Borel}^*(\kappa)$ and $\kappa^{<\kappa} = \kappa$.*

Proof. We start from a model in which $\kappa^+ = 2^\kappa$ and $\kappa^{<\kappa} = \kappa > \omega$ (for instance from L) and force the statement with a $<\kappa$ -closed, κ^+ -c.c. forcing. Given a code (t, h) of a $\text{Borel}^*(\kappa)$ subset of $\text{Baire}(\kappa) \times \text{Baire}(\kappa)$, we will design a forcing p.o. $\mathbb{R}(t, h)$ such that $\mathbb{R}(t, h) \Vdash B(\check{t}, \check{h}) \neq \text{ISO}(\text{DLO}, \kappa)$, where $B(t, h)$ is the $\text{Borel}^*(\kappa)$ set coded by (t, h) . By iterating this forcing we shall kill all possible $\text{Borel}^*(\kappa)$ -code candidates for $\text{ISO}(\text{DLO}, \kappa)$. By combining this forcing with the Cohen forcing $2^{<\kappa}$, we will be able to show, using methods from [FHK14], that in the generic extension also $\Delta_1^1(\kappa) \subsetneq \text{Borel}^*(\kappa)$.

Given trees t, t^* , let us define the game $H(t, t^*)$. At the γ :th move, player I picks a pair $(a_\gamma, b_\gamma) \in t \times t^*$ and then player II picks an element $c_\gamma \in t^*$. The rules declare the following. If $\gamma < \gamma'$, then we must have $b_\gamma < c_\gamma < b_{\gamma'}$ and $a_\gamma < a_{\gamma'}$. The first player who breaks the rules has lost the game.

We will first find for each κ^+ - κ -tree t a $<\kappa$ -closed κ^+ -c.c. forcing $\mathbb{P}(t)$ such that $\mathbb{P}(t) \Vdash \exists t^*(\text{II} \uparrow H(\check{t}, t^*))$. The order $\mathbb{P}(t)$ will consist of triples (P, U, f) , where intuitively, P approximates t^* , U cuts the branches of t^* and f approximates the winning strategy of II in $H(t, t^*)$. We require (P, U, f) to satisfy the following:

- P1 $P \subset \kappa^{<\kappa}$ is closed downward,
- P2 $U \subset \kappa^{<\kappa}$ is an antichain,
- P3 If $q \in U$, then $\text{dom } q$ is a limit ordinal and $\forall p \in P (p \not\supset q)$,

- P4 f is a function with $\text{dom } f \subset (t \times P)^{<\alpha}$ for some $\alpha < \kappa$ and $\text{ran } f \subset P$,
- P5 If $p = ((a_i, b_i))_{i < \beta} \in \text{dom } f$, then p is strictly increasing in the coordinatewise ordering of $t \times P$ and $b_i < f((a_i, b_i))_{i < \beta}$ for all $i < \beta$.
- P6 If $p, q \in \text{dom } f$, $\text{dom } p = \text{dom } q = \alpha + 1$, $p \neq q$ and $p \restriction \alpha = q \restriction \alpha$, then $f(p)$ and $f(q)$ are incomparable.
- P7 If $p \restriction \beta \in \text{dom } f$ for some $p \in (t \times P)^{<\kappa}$ and all $\beta < \alpha = \text{dom } p$, then $\bigcup_{\beta < \alpha} f(p \restriction \beta) \notin U$.

The order on $\mathbb{P}(t)$ we define as follows: $(P, U, f) < (P', U', f')$, if

- O1 $P \subset P'$, $U \subset U'$ and $f \subset f'$,
- O2 if $p \in \text{dom } f' \setminus \text{dom } f$, then $f'(p) > \alpha$, where α is the smallest ordinal such that $P \cup U \cup \text{ran } f \subset \alpha^{<\alpha}$. Call this α the *rank* of (P, U, f) and denote $\alpha = \text{rank}(P, U, f)$.

Next we show that $\mathbb{P}(t)$ is as wanted.

3.1.1 Claim. $\mathbb{P}(t)$ is $<\kappa$ -closed.

Proof of Claim 3.1.1. Suppose $(p_\beta)_{\beta < \alpha}$, $p_\beta = (P_\beta, U_\beta, f_\beta)$, is an increasing sequence of conditions of limit length $\alpha < \kappa$. Then let

$$p_\alpha = (P_\alpha, U_\alpha, f_\alpha) = \left(\bigcup_{\beta < \alpha} P_\beta, \bigcup_{\beta < \alpha} U_\beta, \bigcup_{\beta < \alpha} f_\beta \right)$$

and let us show that $p_\alpha \in \mathbb{P}(t)$ and $p_\alpha > p_\beta$ for all $\beta < \alpha$. To check that $p_\alpha \in \mathbb{P}(t)$, note that all conditions except P7 are local and easy to check. For the condition P7, suppose that $p \restriction \beta \in \text{dom } f_\alpha$ for all $\beta < \text{dom } p$ and assume for a contradiction that $\bigcup_{\beta < \text{dom } p} f_\alpha(p \restriction \beta) \in U_\alpha$. But then $\bigcup_{\beta < \text{dom } p} f_\alpha(p \restriction \beta) \in U_\gamma$ for some $\gamma < \alpha$. This means by O2, that the values of $f_{\gamma+1}$ are above $f_\alpha(p \restriction \beta)$ for all $\beta < \text{dom } p$ which is a contradiction unless $\bigcup_{\beta < \text{dom } p} f_\alpha(p \restriction \beta) = \bigcup_{\beta < \text{dom } p} f_\gamma(p \restriction \beta)$. But the latter is a contradiction with P7 applied to p_γ . □ Claim 3.1.1

Let G be $\mathbb{P}(t)$ -generic and let

$$t^* = \bigcup \{P \mid (P, U, f) \in G \text{ for some } U, f\}.$$

3.1.2 Claim. In the $\mathbb{P}(t)$ -generic extension t^* is a $\kappa^+ \kappa$ -tree.

Proof of Claim 3.1.2. We must show that there are no branches of length κ . Suppose on contrary that b is a branch and let $\dot{b}$ be the $\mathbb{P}(t)$ -name for b . Suppose $p_0 = (P_0, U_0, f_0)$ forces that $\dot{b}$ is a branch and suppose $(P_1, U_1, f_1) = p_1 > p_0$. By induction define $p_{\alpha+1} = (P_{\alpha+1}, U_{\alpha+1}, f_{\alpha+1})$ assuming that $p_\alpha = (P_\alpha, U_\alpha, f_\alpha)$ is already defined, such that $p_{\alpha+1}$ decides $\dot{b}$ up to $\text{rank}(p_\alpha)$. Suppose α is a limit and that p_β has been defined for $\beta < \alpha$ and for every $\beta < \alpha$, $p_{\beta+1}$ has evaluated $\dot{b}$ up to β , from which it follows that it has been evaluated up to α in fact. Denote this evaluated branch by e_α . If $\cup e_\alpha \subset \text{ran } f$, then just continue: let $p_\alpha = \sup_{\beta < \alpha} p_\beta$ which is well defined by Claim 3.1.1. Otherwise let $U_\alpha = \bigcup_{\beta < \alpha} U_\beta \cup \{\dot{b} \upharpoonright \alpha\}$, $f_\alpha = \bigcup_{\beta < \alpha} f_\beta$ and $P_\alpha = \bigcup_{\beta < \alpha} P_\beta$: then $p_\alpha = (P_\alpha, U_\alpha, f_\alpha)$ marks an end to the branch $\dot{b} \upharpoonright \alpha$ which is a contradiction, because $p_\alpha > p_\beta$ for $\beta < \alpha$ (P7 is satisfied, because $\cup e_\alpha \not\subset \text{ran } f$). So we need to show that this process terminates, i.e. the “otherwise”-part of the previous sentence is satisfied at some point. If it does not terminate, then we obtain a branch in $\text{ran } f$, but f is a strategy in the game and by the property P6, this branch determines a branch in t which is a contradiction, because t is $\kappa^+ \kappa$ -tree. $\square$ Claim 3.1.2

Let G be $\mathbb{P}(t)$ -generic and let

$$g = \bigcup \{f \mid (P, U, f) \in G \text{ for some } P, U\}.$$

3.1.3 Claim. *In the $\mathbb{P}(t)$ -generic extension, g is a winning strategy of player II in $H(t, t^*)$.*

Proof of Claim 3.1.3. If s is a strategy of I, let $\dot{s}$ be a name for s and let $\dot{g}$ be a name for g . We will show that $\mathbb{P}(t)$ forces that $\dot{g}$ beats $\dot{s}$. It is enough to show that II can always follow the rules, so suppose they have played α moves and suppose that $p \in \mathbb{P}(t)$ decides the game $s * g$ (the game in which those strategies are used) up to the move α . Find a $q > p$ which decides the next move given by g . By definition of $\mathbb{P}(t)$ this will follow the rules. Essential here is that since $\mathbb{P}(t)$ is closed, every play of length $< \kappa$ is already in the ground model. $\square$ Claim 3.1.3

3.1.4 Claim. *Denote by $\dot{t}^*$ a $\mathbb{P}(t)$ -name for t^* defined by $\dot{t}^* = \{(\check{p}, q) \mid q \in \mathbb{P}(t), q = (P, U, f) \text{ and } p \in P\}$. The forcing $\mathbb{P}(t) * \dot{t}^*$ contains a dense sub-order $\mathbb{R}$ which is $< \kappa$ -closed.*

Proof of Claim 3.1.4. By definition $(q, \rho) \leq (q', \rho')$, if $q \leq q'$ and $q' \Vdash \rho \leq \rho'$. It is easy to see that the suborder $\mathbb{R}'$ of $\mathbb{P}(t) * \dot{t}^*$ consisting of the pairs $(q, \check{p})$ such that $(\check{p}, q) \in \dot{t}^*$ is dense. Let $\mathbb{R}$ be the subset of $\mathbb{R}'$ consisting of those $(q, \check{p})$ for which $\text{dom}(p) \geq \sup\{\text{dom } \eta \mid \eta \in U_q\}$ where $q = (P_q, U_q, f_q)$ (*). It is again easy to see that $\mathbb{R}$ is dense.

Suppose $(q_i, \check{p}_i)_{i < \alpha}$ is an increasing sequence in $\mathbb{R}$ of length $\alpha < \kappa$. Let $q_\alpha = \sup_{i < \alpha} q_i$ in $\mathbb{P}(t)$ and $p_\alpha = \bigcup_{i < \alpha} p_i$. Then q_α is of the form (P, U, f) and by (*) it is

possible to extend P to P' such that $p_\alpha \in P'$ and $q'_\alpha = (P', U, f)$ is still in $\mathbb{P}(t)$. But then $(q'_\alpha, \check{p}_\alpha) \in \mathbb{R}$. $\square$ Claim 3.1.4

3.1.5 Claim. *For each (t, h) there exists a κ^+ -c.c. $<\kappa$ -closed forcing $\mathbb{R}(t, h)$ such that in the $\mathbb{R}(t, h)$ -generic extension $\text{ISO}(\text{DLO}, \kappa)$ is not the $\text{Borel}^*(\kappa)$ set coded by (t, h) .*

Proof of Claim 3.1.5. If $\mathbb{P}(t)$ forces that, let $\mathbb{R}(t, h) = \mathbb{P}(t)$. Otherwise let $\mathbb{R}(t, h)$ be the dense sub-order of $\mathbb{P}(t) * t^*$ given by Claim 3.1.4. Let us show that this works. It is sufficient to show that $\mathbb{P}(t) * t^*$ forces the statement. Let us work in the $\mathbb{P}(t)$ -generic extension $V[G]$. Let $\eta, \xi \in 2^\kappa$ be such that $\mathcal{A}_\eta$ and $\mathcal{A}_\xi$ are non-isomorphic models of DLO, but $\Pi \upharpoonright \text{EF}_{t^*}(\mathcal{A}_\eta, \mathcal{A}_\xi)$. These can be found by [HT91]. Since $\mathbb{P}(t)$ didn't force the statement, the pair (η, ξ) is not in the set coded by (t, h) . Now forcing with t^* adds a branch to t^* and since t^* can be embedded into the tree of partial isomorphisms between $\mathcal{A}_\eta$ and $\mathcal{A}_\xi$ via the winning strategy of Π in $\text{EF}_{t^*}(\mathcal{A}_\eta, \mathcal{A}_\xi)$, it adds a branch also to that tree, and so $\mathcal{A}_\eta$ and $\mathcal{A}_\xi$ are isomorphic in $V[G][G_0]$, where G is $\mathbb{P}(t)$ -generic over V and G_0 is t^* -generic over $V[G]$. Next we show, that in $V[G][G_0]$, (η, ξ) is not in the $\text{Borel}^*(\kappa)$ set coded by (t, h) .

Towards contradiction assume that $V[G][G_0] \models (\eta, \xi) \in B(t, h)$ and let us show that then $V[G] \models (\eta, \xi) \in B(t, h)$, which is a contradiction. Let σ be a winning strategy of player Π in $V[G][G_0]$ in $B^*((\eta, \xi), (t, h))$, as in the definition of $\text{Borel}^*(\kappa)$, and let $\dot{\sigma}$ be a name for σ . Let us show how Π has to play to win $B^*((\eta, \xi), (t, h))$ in $V[G]$. For that, let g be a winning strategy of player Π in $H_t(t^*)$ which exists in $V[G]$ by Claim 3.1.3.

Assume that a_0 is the first move of I in $B^*((\eta, \xi), (t, h))$. Player Π finds a condition c_0 in t^* which decides $\dot{\sigma}$ far enough to give an answer b_0 to that move. Player Π answers in $B^*((\eta, \xi), (t, h))$ with b_0 and at the same time imagines that (b_0, c_0) is the first move of I in $H_t(t^*)$ and replies using g in this imaginary game by $d_0 > c_0$. Suppose that the players have played $(a_i, b_i)_{i < \alpha}$ in $B^*((\eta, \xi), (t, h))$ so that a_i are the moves of player I and b_i are the moves of player Π . At the same time player Π has constructed a sequence $(c_i, d_i)_{i < \alpha}$ using the imaginary game. Next player I picks a_α in $B^*((\eta, \xi), (t, h))$. Player Π solves $\dot{\sigma}$ by a condition $c_\alpha > \sup_{\beta < \alpha} d_\beta$ so that she obtains an answer b_α and again imagines that (b_α, c_α) is just the next move of I in $H_t(t^*)$ and picks d_α using g . In this way the players will climb up a branch $b \subset t$ with the basic open set $h(b)$ in the end. By definition $h(b) = N_p$ for some $p \in 2^{<\kappa}$ in V , and neither $\mathbb{P}$ nor $\mathbb{P} * t^*$ adds small subsets (Claims 3.1.1 and 3.1.4), so $h(b)^V = h(b)^{V[G]} = h(b)^{V[G][G_0]}$. Now since σ was winning in $V[G][G_0]$, the above described strategy is winning in $V[G]$. $\square$ Claim 3.1.5

Thus, for a code (t, h) we have constructed a forcing $\mathbb{R}(t, h)$ which forces that

$$\text{ISO}(\text{DLO}, \kappa) \neq B(t, h).$$

Using this fact, we will define a $< \kappa$ -support iterated forcing $\mathbb{Q}$ of length κ^+ such that in the $\mathbb{Q}$ -generic extension there are no pairs (t, h) such that $\text{ISO}(\text{DLO}, \kappa) = B(t, h)$ *at all* which means that $\text{ISO}(\text{DLO}, \kappa)$ is not $\text{Borel}^*(\kappa)$ and moreover $\mathbb{Q} \Vdash \Delta_1^1(\kappa) \subsetneq \text{Borel}^*(\kappa)$

Let $s: \kappa^+ \rightarrow \kappa^+ \times \kappa^+$ be onto such that $s_2(\alpha) < \alpha$ for $\alpha < \kappa^+$ where $s(\alpha) = (s_1(\alpha), s_2(\alpha))$. Define the $< \kappa$ -support iterated forcing construction (see [Kun80, Ch. VIII])

$$(\mathbb{P}_\beta, \rho_\beta)_{\beta < \kappa^+} \text{ along with a sequence } \sigma(\alpha, \beta)$$

as follows. For each $\beta < \kappa^+$, let $\{\sigma(\alpha, \beta) \mid \alpha < \kappa^+\}$ be the enumeration of all $\mathbb{P}_\beta$ -names for codes for $\text{Borel}^*(\kappa)$ sets and ρ_β is a $\mathbb{P}_\beta$ -name for the Cohen forcing $\mathbb{C} = 2^{<\kappa}$, if β is odd (of the form $\alpha + 2n + 1$ with α a limit and $n < \omega$) and ρ_β is a $\mathbb{P}_\beta$ -name for $\mathbb{R}(\dot{i}, \dot{h})$ with $(\dot{i}, \dot{h}) = \sigma(s(\beta))$, if β is even.

It is easily seen that $\mathbb{P}_\gamma$ is $< \kappa$ -closed and has the κ^+ -c.c. for all $\gamma \leq \kappa^+$. We claim that $\mathbb{Q} = \mathbb{P}_{\kappa^+}$ forces that $\text{ISO}(\text{DLO}, \kappa)$ is not $\text{Borel}^*(\kappa)$. Let G be $\mathbb{P}_{\kappa^+}$ -generic and let $G_\gamma = "G \cap \mathbb{P}_\gamma"$ for every $\gamma < \kappa$. Then G_γ is $\mathbb{P}_\gamma$ -generic.

Suppose that in $V[G]$, $\text{ISO}(\text{DLO}, \kappa) = B(t, h)$ for some (t, h) . By [Kun80, Theorem VIII.5.14], there is $\delta < \kappa^+$ such that $(t, h) \in V[G_\delta]$. Let δ_0 be the smallest such δ .

Now there exists $\sigma(\gamma, \delta_0)$, a $\mathbb{P}_{\delta_0}$ -name for (t, h) . By the definition of s , there exists an even $\delta > \delta_0$ with $s(\delta) = (\gamma, \delta_0)$. Thus

$$\mathbb{P}_{\delta+1} \Vdash "\sigma(\gamma, \delta_0) \text{ is not a } \text{Borel}^*(\kappa)\text{-code for } \text{ISO}(\text{DLO}, \kappa)",$$

i.e. $V[G_{\delta+1}] \models B(t, h) \neq \text{ISO}(\text{DLO}, \kappa)$. We want to show that this holds also in $V[G]$. In $V[G_{\delta+1}]$ define

$$\mathbb{P}^{\delta+1} = \{(p_i)_{i < \kappa^+} \in \mathbb{P}_{\kappa^+} \mid (p_i)_{i < \delta+1} \in G_{\delta+1}\}.$$

Then $\mathbb{P}^{\delta+1}$ has κ^+ -c.c. and is $< \kappa$ -closed because at each stage of the iteration the forcings have these properties and the iteration has $< \kappa$ -support. Assume that $G^{\delta+1}$ is $\mathbb{P}^{\delta+1}$ -generic over $V[G_{\delta+1}]$. We will show that in $V[G_{\delta+1}][G^{\delta+1}]$ we have $B(t, h) \neq \text{ISO}(\text{DLO}, \kappa)$. On the other hand $V[G] = V[G_{\delta+1}][G^{\delta+1}]$ for some $G^{\delta+1}$, so this finishes the proof of the part of the theorem concerning $\text{ISO}(\text{DLO}, \kappa)$.

There are two cases. First assume that there are η and ξ in $V[G_{\delta+1}]$ such that $\mathcal{A}_\eta$ and $\mathcal{A}_\xi$ are isomorphic linear orders and $V[G_{\delta+1}] \models (\eta, \xi) \notin B(t, h)$. Then in $V[G_{\delta+1}][G^{\delta+1}]$, we have still that $\mathcal{A}_\eta$ and $\mathcal{A}_\xi$ are isomorphic, but $(\eta, \xi) \notin B(t, h)$: $\mathbb{P}^{\delta+1}$ does not add small sets and it does not add a winning strategy of Π in the game $B^*((\eta, \xi), (t, h))$, because otherwise we would obtain a winning strategy already in $V[G_{\delta+1}]$ using $< \kappa$ -closedness of $\mathbb{P}^{\delta+1}$ in an argument similar to the one in the end of the proof of Claim 3.1.5.

The other case is that there are η and ξ in $V[G_{\delta+1}]$ such that $\mathcal{A}_\eta$ and $\mathcal{A}_\xi$ are non-isomorphic linear orders and $V[G_{\delta+1}] \models (\eta, \xi) \in B(t, h)$. Now dually to the

first case, the winning strategy of Π in $B^*((\eta, \xi), (t, h))$ remains a winning strategy, because otherwise we would be able to beat it already in $V[G_{\delta+1}]$ using the closedness of $\mathbb{P}^{\delta+1}$. On the other hand $\mathcal{A}_\eta$ and $\mathcal{A}_\xi$ do not become isomorphic, because that would add a winning strategy of Π in $\text{EF}_\kappa(\mathcal{A}_\eta, \mathcal{A}_\xi)$ which is impossible by the same argument.

Now we are left to show that $\Delta_1^1(\kappa) \neq \text{Borel}^*(\kappa)$ in the generic extension by $\mathbb{Q}$. The κ^+ -long $<\kappa$ -support iteration of the Cohen forcing $\mathbb{C}$ yields a model in which $\Delta_1^1(\kappa)$ sets have the property of Baire and the same proof works in this case, because in our iteration every other step was $\mathbb{C}$. But this in turn implies that $\Delta_1^1(\kappa) \subsetneq \text{Borel}^*(\kappa)$, see Lemma 1.11 above. $\square$

Remark. Apart from DLO this can be done for any theory T which is unclassifiable and not strictly stable. The reason is that the crucial property of DLO is that we can find $\eta, \xi \in 2^\kappa$ such that $\mathcal{A}_\eta$ and $\mathcal{A}_\xi$ are non-isomorphic models of DLO, but $\Pi \upharpoonright \text{EF}_{t^*}(\mathcal{A}_\eta, \mathcal{A}_\xi)$ (see the proof of Claim 3.1.5 above) and in [HT91] this property was shown to hold for any non-strictly stable theory. This, extended version of the theorem, is used in [HKM15] to prove a non-reducibility result.

The following answers a question asked in [FHK14]:

3.2 Corollary. *It is consistent that $\Delta_1^1(\kappa) \subsetneq \text{Borel}^*(\kappa) \subsetneq \Sigma_1^1(\kappa)$ and $\kappa^{<\kappa} = \kappa > \omega$.*

Proof. $\text{ISO}(\text{DLO}, \kappa)$ is $\Sigma_1^1(\kappa)$, so the result follows from Theorem 3.1. $\square$

References

- [Bla81] D. Blackwell. Borel sets via games. *Ann. Probab.*, 9(2):321–322, 1981.
- [FHK14] S.D. Friedman, T. Hyttinen, and V. Kulikov. *Generalized Descriptive Set Theory and Classification Theory*, volume 230 of *Memoirs of the American Mathematical Society*. American Mathematical Society, 2014.
- [Hal96] A. Halko. Negligible subsets of the generalized Baire space $\omega_1^{\omega_1}$. *Annales Academiæ Scientiarum Fennicæ. Mathematica. Dissertationes.*, 107, 1996.
- [HK15] T. Hyttinen and H. Kulikov. On Σ_1^1 -complete equivalence relations on the generalized Baire space. *Mathematical Logic Quarterly*, 61(1-2):66–81, 2015.
- [HKM15] T. Hyttinen, V. Kulikov, and M. Moreno. A Borel-reducibility counterpart of the Shelah’s Main Gap Theorem. 2015. manuscript.

- [HM76] V. Harnik and M. Makkai. Applications of Vaught sentences and the covering theorem. *J. Symb. Logic*, 41:171–187, 1976.
- [HR76] J. Hintikka and V. Rantala. A new approach to infinitary languages. *Ann. Math. Logic*, 10:95–115, 1976.
- [HT91] T. Hyttinen and H. Tuuri. Constructing strongly equivalent nonisomorphic models. *Annals of Pure and Applied Logic*, 52(3):203–248, June 1991.
- [Kar84] M. Karttunen. Model theory for infinitely deep languages. *Annals Acadamiae Scientiarum Fennicae Series A I Mathematica Dissertationes*, 50, 1984.
- [Kec94] A. S. Kechris. *Classical Descriptive Set Theory*, volume 156. Springer-Verlag, Graduate Texts in Mathematics, 1994.
- [Koe11] M. Koerwien. A complicated ω -stable depth 2 theory. *J. Symb. Log.*, 76(1):47–65, 2011.
- [Kun80] K. Kunen. *Set Theory / An Introduction to Independence Proofs*. North Holland Publishing Company, 1980.
- [Mak74] M. Makkai. Generalizing vaught sentences from ω to strong cofinality ω . *Fund. Math.*, 82:105–119, 1974.
- [MV93] A. Mekler and J. Väänänen. Trees and Π_1^1 -subsets of ${}^{\omega_1}\omega_1$. *The Journal of Symbolic Logic*, 58(3):1052–1070, September 1993.
- [She00] S. Shelah. *Classification Theory, Revised Edition*. North Holland Publishing Company, 2000.
- [Sus17] M. Suslin. Sur une définition des ensembles mesurables b sans nombres transfinis. *Comptes Rendus de l'Académie des Sciences. Paris*, 164:88–91, 1917.
- [Tuu92] H. Tuuri. Relative separation theorems for $L_{\kappa^+\kappa}$. *Notre Dame Journal of Formal Logic*, 33(3):383–401, 1992.
- [Vau73] R. Vaught. Descriptive set theory in $L_{\omega_1\omega}$. *Gambridge summer school in mathematical logic, edited by A. Mathias and H. Rogers*, 337:574–598, 1973.